Evolutionary Multiobjective Optimization in Dynamic Environments: A Set of Novel Benchmark Functions

Subhodip Biswas, Swagatam Das, Ponnuthurai N. Suganthan and Carlos A. Coello Coello

Table 1: Summary of proposed benchmark set: Unconstrained Dynamic Functions (Fs)

Function		**Number of objectives**	**Pareto Set**		**Pareto Front**		
			Nature	**Variation**	**Nature**	**Shift**	**Angular shift / Curvature variation**
*F*1		2	Trigonometric	Horizontal shift	Continuous	Diagonal	No change
*F*2	**UDF1**	2	Trigonometric	Vertical shift	Continuous	Diagonal	No change
*F*3	**UDF2**	2	Polynomial	Curvature change +Vertical shift	Continuous	Diagonal	No change
*F*4		2	Trigonometric	No change	Discrete	Diagonal	No change
*F*5	**UDF3**	2	Trigonometric	No change	Discrete	Diagonal	No change
*F*6		3	Trigonometric	No change	Continuous	Radial	No change
*F*7	**UDF4**	2	Trigonometric	Horizontal shift	Continuous	No change	Angular shift + Curvature variation
*F*8		2	Trigonometric	Vertical shift	Continuous	No change	Angular shift + Curvature variation
*F*9	**UDF5**	2	Polynomial	Curvature change +Vertical shift	Continuous	No change	Angular shift + Curvature variation
*F*10	**UDF6**	2	Trigonometric	No change	Discrete	Diagonal	Angular shift only
*F*11		2	Trigonometric	No change	Discrete	Diagonal	Angular shift only
*F*12	**UDF7**	3	Trigonometric	No change	Continuous	Radial +shift of the center	No change
*F*13	**UDF8**	2	Trigonometric	Random Vertical or Horizontal shift	Continuous	Random Diagonal Shift	Random Angular Shift or Curvature variation
*F*14	**UDF9**	2	Polynomial	Random Curvature change or Vertical shift	Continuous	Random Diagonal Shift	Random Angular Shift or Curvature variation

APPENDIX A: Dynamic test problems (all objective functions are to be minimized)

As before, we assume:

$G(t)=\sin(0.5\pi t)$, $t=\frac{1}{n_s}\left\lfloor\frac{\tau}{T}\right\rfloor$, $M(t)=0.5+|G(t)|$, $H(t)=0.5+|G(t)|$, $R(t)=1+|G(t)|$ and $K(t)=\lceil nG(t)\rceil$

1) *F*1

$$f_1=x_1+\frac{2}{|J_1|}\sum_{j\in J_1}[x_j-\sin(6\pi x_1+(j+K(t))\frac{\pi}{n})]^2+|G(t)|$$

$$f_2=1-x_1+\frac{2}{|J_2|}\sum_{j\in J_2}[x_j-\sin(6\pi x_1+(j+K(t))\frac{\pi}{n})]^2+|G(t)|,$$

where $J_1=\{j\,|\,j$ is odd and $2\le j\le n\}$ and $J_2=\{j\,|\,j$ is even and $2\le j\le n\}$.

- Search space: $[0,1]\times[-1,1]^{n-1}$, where n is the number of dimensions of the search space.

- Optimal PF $f_2 = 1-(f_1 - |G(t)|) + |G(t)|, 0 + |G(t)| \le f_1 \le 1 + |G(t)|$
- Optimal PS $x_j = \sin(6\pi x_1 + (j + K(t))\frac{\pi}{n}), j = 2,...,n; 0 \le x_1 \le 1.$

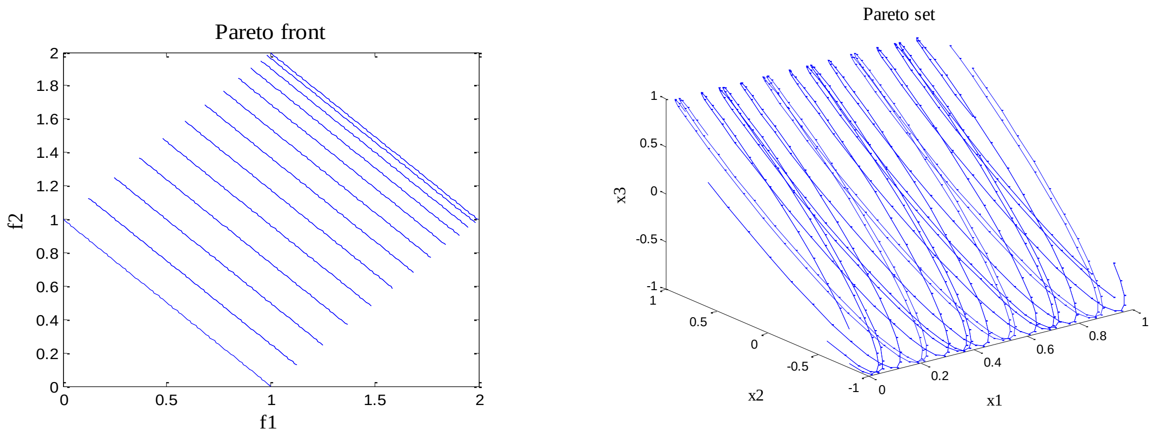

Fig. A1: PF and PS for F1

2) *F*2

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - \sin(6\pi x_1 + j\frac{\pi}{n}) - G(t)]^2 + |G(t)|$$

$$f_2 = 1 - x_1 + \frac{2}{|J_2|}\sum_{j\in J_2}[x_j - \sin(6\pi x_1 + j\frac{\pi}{n}) - G(t)]^2 + |G(t)|,$$

where $J_1 = \{j \mid j \text{ is odd and } 2 \le j \le n\}$ and $J_2 = \{j \mid j \text{ is even and } 2 \le j \le n\}$.

- Search space: $[0,1]\times[-2,2]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF: $f_2 = 1-(f_1 - |G(t)|) + |G(t)|$, $0 + |G(t)| \le f_1 \le 1 + |G(t)|$
- optimal PS: $x_j = \sin(6\pi x_1 + j\frac{\pi}{n}) + G(t), j = 2,...,n, \quad 0 \le x_1 \le 1.$

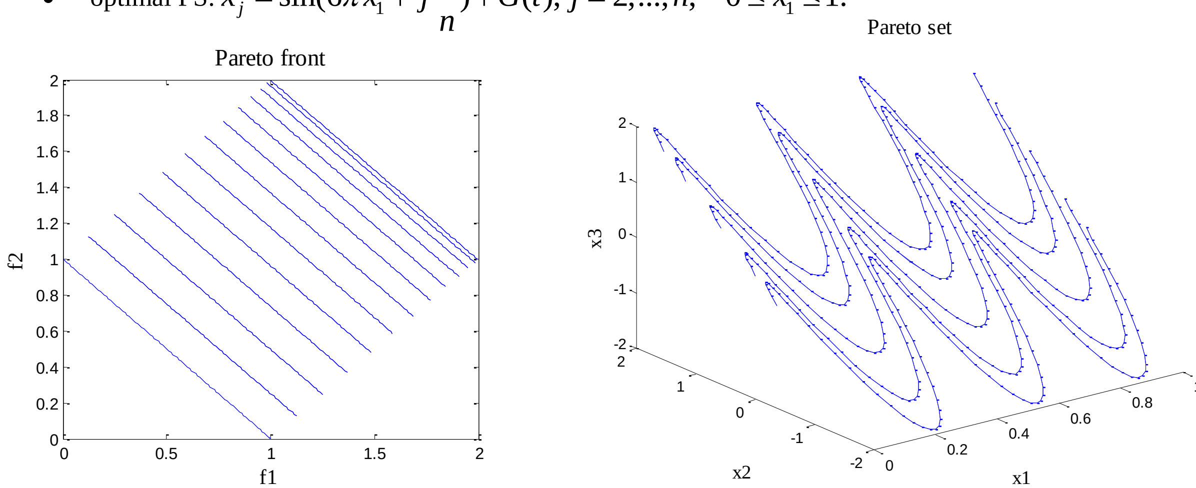

Fig. A2: PF and PS for F2

3) *F3*

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j \in J_1}\left[x_j - x_1^{0.5\left(2+\frac{3(j-2)}{n-2}+G(t)\right)} - G(t)\right]^2 + \left|G(t)\right|$$

$$f_2 = 1 - x_1 + \frac{2}{|J_2|}\sum_{j \in J_2}\left[x_1^{0.5\left(2+\frac{3(j-2)}{n-2}+G(t)\right)} - G(t)\right]^2 + \left|G(t)\right|,$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$.

- Search space: $[0,1]\times[-1,2]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF: $f_2 = 1 - (f_1 - \left|G(t)\right|) + \left|G(t)\right|$, $0 + \left|G(t)\right| \le f_1 \le 1 + \left|G(t)\right|$
- Optimal PS: $x_j = x_1^{0.5\left(2+\frac{3(j-2)}{n-2}+G(t)\right)} + G(t), \quad j = 2,\ldots,n, \quad 0 \le x_1 \le 1.$

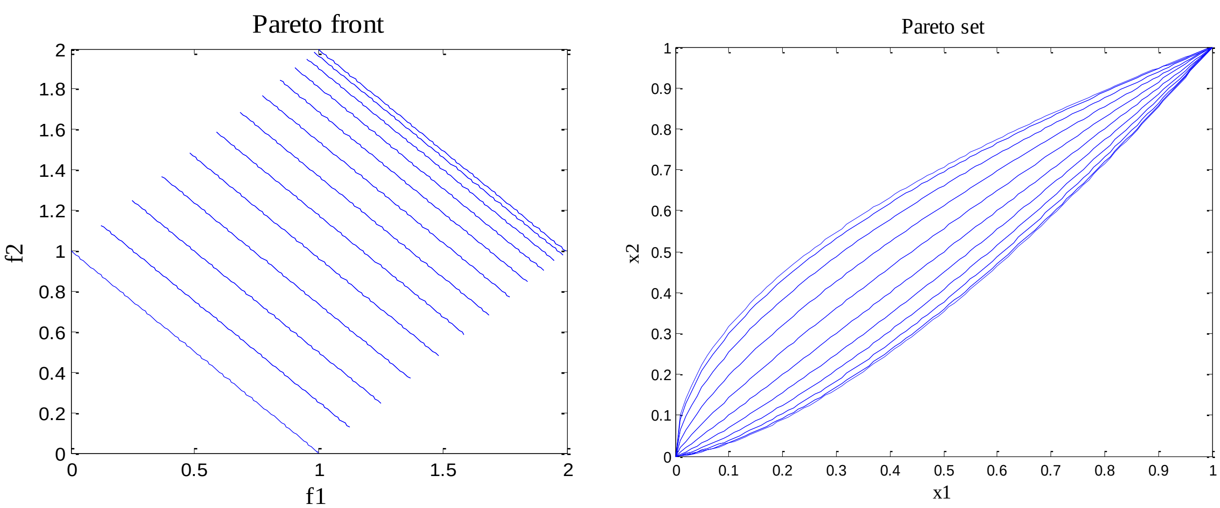

Fig. A3: PF and PS for F3

4) *F4*

$$f_1 = x_1 + \left(\frac{1}{2N} + \varepsilon\right)\left|\sin(2N\pi x_1) - 2N\left|G(t)\right|\right| + \frac{2}{|J_1|}\sum_{j\in J_1}[2y_j^2 - \cos(4\pi y_j) + 1]^2$$

$$f_2 = 1 - x_1 + \left(\frac{1}{2N} + \varepsilon\right)\left|\sin(2N\pi x_1) - 2N\left|G(t)\right|\right| + \frac{2}{|J_2|}\sum_{j\in J_2}[2y_j^2 - \cos(4\pi y_j) + 1]^2$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$, $\varepsilon = 0.1$ and $N = 10$.

$$y_j = x_j - \sin(6\pi x_1 + j\frac{\pi}{n}), \quad j = 2,\ldots,n$$

- Search space: $[0,1]\times[-1,1]^{n-1}$,
  where n is the number of dimensions of the search space.
- Optimal PF: $2N+1$ distinct points:
  $\left(\frac{i}{2N} + \left|G(t)\right|, 1 - \frac{i}{2N} + \left|G(t)\right|\right)$, $i = 0,1,\ldots,2N$
- Optimal PS $x_j = \sin(6\pi x_1 + j\frac{\pi}{n})$, $j = 2,\ldots,n.$ $\quad 0 \le x_1 \le 1.$

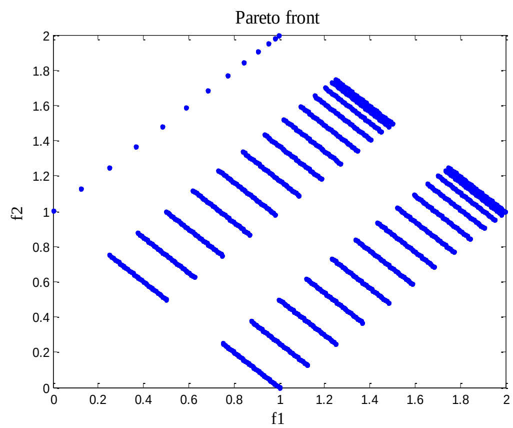

Fig. A4: PF for F4

5) *F5*

$$f_1 = x_1 + \max\{0, (\frac{1}{2N} + \varepsilon)[\sin(2N\pi x_1) - 2N|G(t)|]\} + \frac{2}{|J_1|}[4\sum_{j\in J_1} 2y_j^2 - 2\prod_{j\in J_1}\cos(\frac{20\pi y_j}{\sqrt{j}}) + 2]^2$$

$$f_2 = 1 - x_1 + \max\{0, (\frac{1}{2N} + \varepsilon)[\sin(2N\pi x_1) - 2N|G(t)|]\} + \frac{2}{|J_2|}[4\sum_{j\in J_2} 2y_j^2 - 2\prod_{j\in J_2}\cos(\frac{20\pi y_j}{\sqrt{j}}) + 2]^2$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$, $\varepsilon = 0.1$ and $N = 10$.

$$y_j = x_j - \sin(6\pi x_1 + j\frac{\pi}{n}), \quad j = 2,...,n$$

- Search space: $[0,1]\times[-1,1]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF:
    - one disconnected point (0,1)
    - N disconnected parts: $\bigcup_{i=1}^{N}[\frac{2i-1}{2N} + |G(t)|, \frac{2i}{2N} + |G(t)|]$
- Optimal PS: $x_j = \sin(6\pi x_1 + j\frac{\pi}{n}), j = 2,...,n. \quad 0 \le x_1 \le 1.$

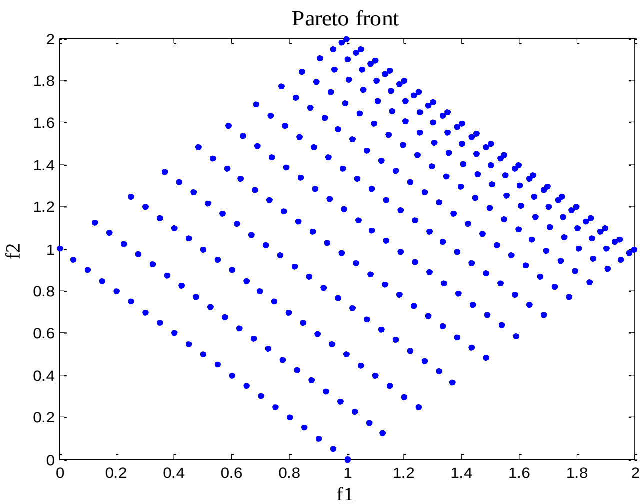

Fig. A5: PF for F5

6) *F*6

$$f_1 = R(t)\cos(0.5\pi x_1)\cos(0.5\pi x_2) + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2$$

$$f_2 = R(t)\cos(0.5\pi x_1)\sin(0.5\pi x_2) + \frac{2}{|J_2|}\sum_{j\in J_2}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2$$

$$f_1 = R(t)\sin(0.5\pi x_1) \qquad + \frac{2}{|J_3|}\sum_{j\in J_3}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2$$

where

$J_1 = \{j \,|\, 3 \le j \le n,$ and $j-1$ is a multiplication of $3\}$,

$J_2 = \{j \,|\, 3 \le j \le n,$ and $j-2$ is a multiplication of $3\}$,

$J_3 = \{j \,|\, 3 \le j \le n,$ and j is a multiplication of $3\}$.

- Search space: $[0,1]^2 \times [-2,2]^{n-2}$, where n is the number of dimensions of the search space.
- Optimal PF: $f_1^2 + f_2^2 + f_3^2 = [R(t)]^2$, $0 \le f_1, f_2, f_3 \le 1$.
- Optimal PS: $x_j = 2x_2\sin(2\pi x_1 + j\frac{\pi}{n}),\ j = 3,\ldots n.$

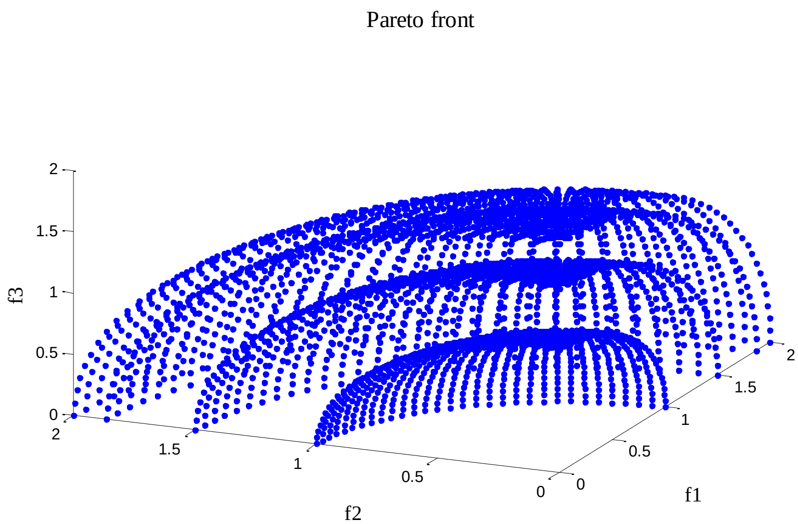

Fig. A6: PF for F6

7) *F7*

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - \sin(6\pi x_1 + (j+K(t))\frac{\pi}{n})]^2$$

$$f_2 = 1 - M(t)(x_1)^{H(t)} + \frac{2}{|J_2|}\sum_{j\in J_2}[x_j - \sin(6\pi x_1 + (j+K(t))\frac{\pi}{n})]^2,$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$.

- Search space: $[0,1]\times[-1,1]^{n-1}$, where n is the number of dimensions of the search space.
- POF $f_2 = 1 - M(t) f_1^{H(t)}, 0 \le f_1 \le 1$
- POS $x_j = \sin(6\pi x_1 + (j+K(t))\frac{\pi}{n}), j = 2,\ldots,n; 0 \le x_1 \le 1.$

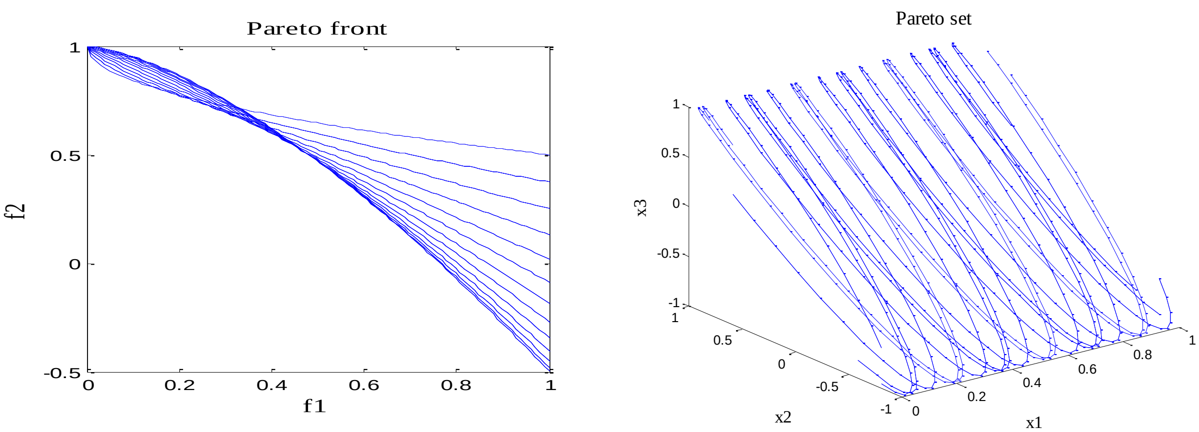

Fig. A7: PF and PS for F7

8) *F*8

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - \sin(6\pi x_1 + j\frac{\pi}{n}) - G(t)]^2$$

$$f_2 = 1 - M(t)x_1^{H(t)} + \frac{2}{|J_2|}\sum_{j\in J_2}[x_j - \sin(6\pi x_1 + j\frac{\pi}{n}) - G(t)]^2,$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$.

- Search space: $[0,1]\times[-2,2]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF $f_2 = 1 - M(t)\, f_1^{H(t)}, 0 \le f_1 \le 1$
- Optimal PS: $x_j = \sin(6\pi x_1 + j\frac{\pi}{n}) + G(t),\ j = 2,...,n, \quad 0 \le x_1 \le 1.$

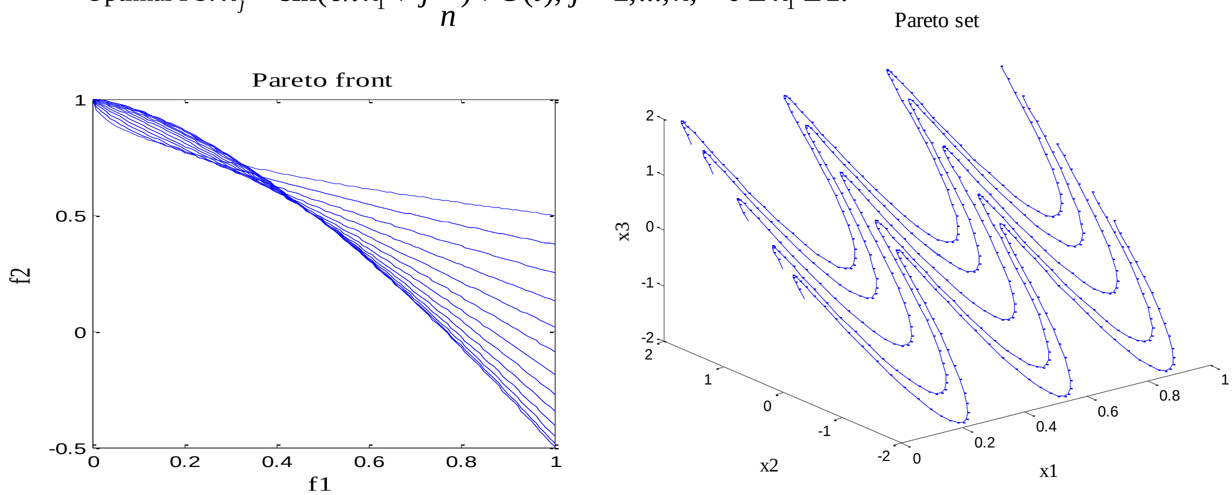

Fig. A8: PF and PS for F8

9) *F*9

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t))} - G(t)]^2$$

$$f_2 = 1 - M(t)x_1^{H(t)} + \frac{2}{|J_2|}\sum_{j\in J_2}[x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t))} - G(t)]^2,$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$.

- Search space: $[0,1]\times[-1,2]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF $f_2 = 1 - M(t) f_1^{H(t)}, 0 \le f_1 \le 1$
- Optimal PS: $x_j = x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t))} + G(t), \quad j = 2,...,n, \quad 0 \le x_1 \le 1.$

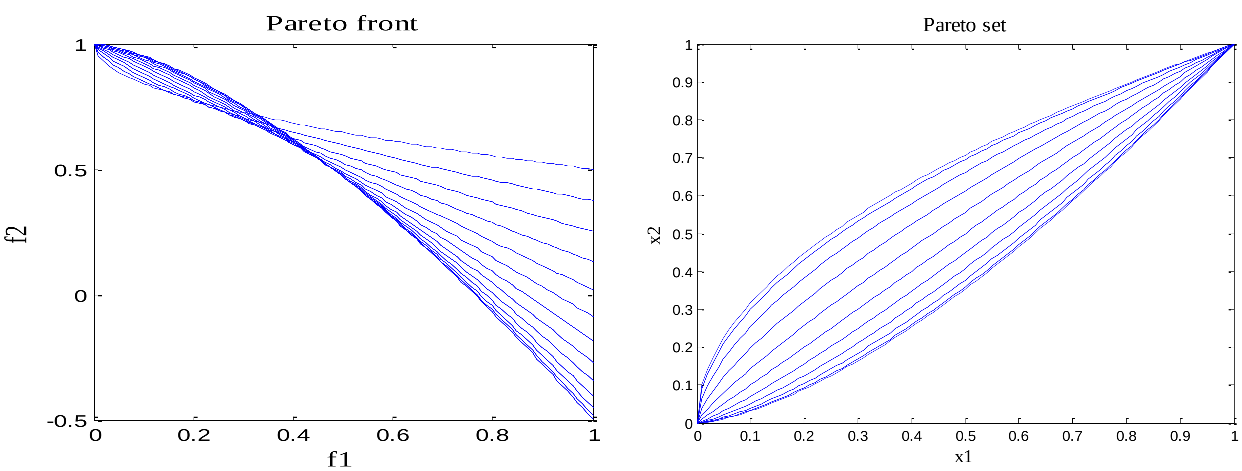

Fig. A9: PF and PS for F9

10) $F10$

$$f_1 = x_1 + (\frac{1}{2N} + \varepsilon)\left|\sin(2N\pi x_1) - 2N\left|G(t)\right|\right| + \frac{2}{|J_1|}\sum_{j\in J_1}[2y_j^2 - \cos(4\pi y_j) + 1]^2$$

$$f_2 = 1 - M(t)x_1 + (\frac{1}{2N} + \varepsilon)\left|\sin(2N\pi x_1) - 2N\left|G(t)\right|\right| + \frac{2}{|J_2|}\sum_{j\in J_2}[2y_j^2 - \cos(4\pi y_j) + 1]^2$$

where $J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$,

$\varepsilon = 0.1$ and $N = 10$.

$$y_j = x_j - \sin(6\pi x_1 + j\frac{\pi}{n}), \quad j = 2,\ldots,n$$

- Search space: $[0,1]\times[-1,1]^{n-1}$,
  where n is the number of dimensions of the search space.
- Optimal PF: $2N+1$ distinct points:
  $(\frac{i}{2N} + \left|G(t)\right|, 1 - \frac{i}{2N}M(t) + \left|G(t)\right|)$, $i = 0,1,\ldots,2N$
- Optimal PS $x_j = \sin(6\pi x_1 + j\frac{\pi}{n}), j = 2,\ldots,n. \quad 0 \le x_1 \le 1.$

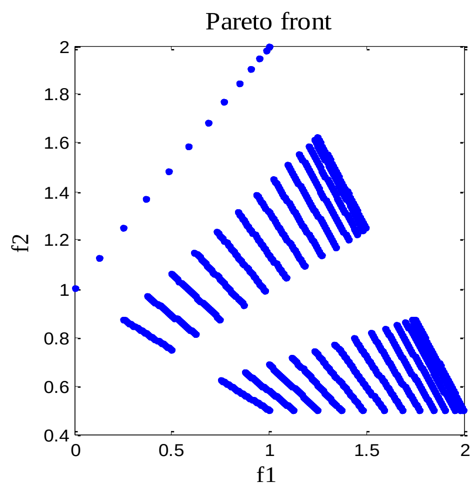

Fig. A10: Optimal PF for F10

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11) *F*11

$$f_1 = x_1 + \max\{0, (\frac{1}{2N} + \varepsilon)[\sin(2N\pi x_1) - 2N|G(t)|]\} + \frac{2}{|J_1|}[4\sum_{j\in J_1} 2y_j^2 - 2\prod_{j\in J_1}\cos(\frac{20\pi y_j}{\sqrt{j}}) + 2]^2$$

where

$$f_2 = 1 - M(t)x_1 + \max\{0, (\frac{1}{2N} + \varepsilon)[\sin(2N\pi x_1) - 2N|G(t)|]\} + \frac{2}{|J_2|}[4\sum_{j\in J_2} 2y_j^2 - 2\prod_{j\in J_2}\cos(\frac{20\pi y_j}{\sqrt{j}}) + 2]^2$$

$J_1 = \{j \mid j$ is odd and $2 \le j \le n\}$ and $J_2 = \{j \mid j$ is even and $2 \le j \le n\}$, $\varepsilon = 0.1$ and $N = 10$.

$$y_j = x_j - \sin(6\pi x_1 + j\frac{\pi}{n}), \quad j = 2,\ldots,n$$

- Search space: $[0,1]\times[-1,1]^{n-1}$,
  where n is the number of dimensions of the search space.

- Optimal PF:
  - one disconnected point (0,1)
  - N disconnected parts: $\bigcup_{i=1}^{N}[\frac{2i-1}{2N} + |G(t)|, \frac{2i}{2N}M(t) + |G(t)|]$

- Optimal PS: $x_j = \sin(6\pi x_1 + j\frac{\pi}{n}),\ j = 2,\ldots,n. \quad 0 \le x_1 \le 1.$

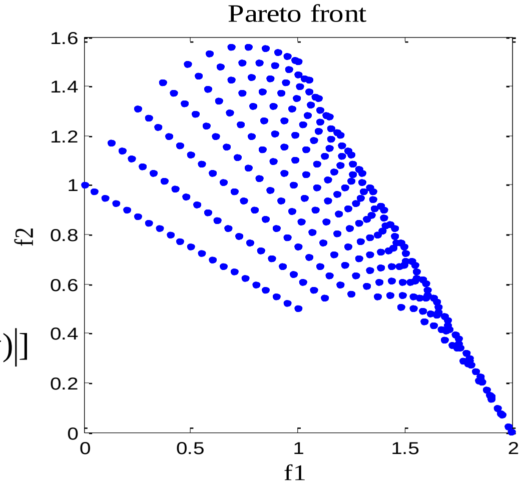

Fig. A11: Optimal PF for F11

12) *F*12

$$f_1 = R(t)\cos(0.5\pi x_1)\cos(0.5\pi x_2) + \frac{2}{|J_1|}\sum_{j\in J_1}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2 + G(t)$$

$$f_2 = R(t)\cos(0.5\pi x_1)\sin(0.5\pi x_2) + \frac{2}{|J_2|}\sum_{j\in J_2}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2 + G(t)$$

$$f_3 = R(t)\sin(0.5\pi x_1) + \frac{2}{|J_3|}\sum_{j\in J_3}[x_j - 2x_2\sin(2\pi x_1 + j\frac{\pi}{n})]^2 + G(t)$$

where $J_1 = \{j \mid 3 \le j \le n,$ and $j-1$ is a multiplication of $3\}$,

$J_2 = \{j \mid 3 \le j \le n,$ and $j-2$ is a multiplication of $3\}$,

$J_3 = \{j \mid 3 \le j \le n,$ and j is a multiplication of $3\}$.

- Search space: $[0,1]^2 \times [-2,2]^{n-2}$, where n is the number of dimensions of the search space.
- Optimal PF: $(f_1 - G(t))^2 + (f_2 - G(t))^2 + (f_3 - G(t))^2 = [R(t)]^2$, $0 \le f_1, f_2, f_3 \le 1$.
- Optimal PS: $x_j = 2x_2\sin(2\pi x_1 + j\frac{\pi}{n}),\ j = 3,\ldots n.$

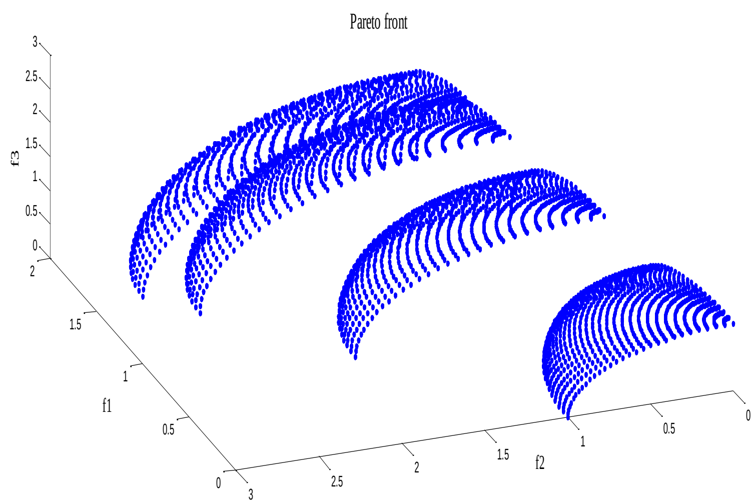

Fig. A12: Optimal PF for F12

13) *F*13

Initialize $t_1 = t_2 = t_3 = t_4 = t_5 = 0$

Let us define t_s as a random variable belonging to the set { t_1, t_2, t_3, t_4, t_5 }, where probability of each of the variables being selected is equal i.e. $P(t_s = t_1) = P(t_s = t_2) = P(t_s = t_3) = P(t_s = t_4) = P(t_s = t_5) = \frac{1}{5}$

for each iteration $\tau = 1 : \max\_gen$

if $t(\tau) \neq t(\tau - 1)$

select t_s randomlyfrom the set { t_1, t_2, t_3, t_4, t_5 },

increase the selected variable t_s by a step of 1

Update f_1, f_2 as per the current values of t_1, t_2, t_3, t_4, t_5

end

end

$$f_1 = x_1 + \frac{2}{|J_1|} \sum_{j \in J_1} [x_j - \sin(6\pi x_1 + (j + K(t_1))\frac{\pi}{n}) - G(t_2)]^2 + |G(t_3)|$$

$$f_2 = 1 - H(t_4) x_1^{H(t_5)} + \frac{2}{|J_2|} \sum_{j \in J_2} [x_j - \sin(6\pi x_1 + (j + K(t_1))\frac{\pi}{n}) - G(t_2)]^2 + |G(t_3)|,$$

where $J_1 = \{ j \mid j$ is odd and $2 \leq j \leq n \}$ and $J_2 = \{ j \mid j$ is even and $2 \leq j \leq n \}$.

- Search space: $[0,1] \times [-1,1]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF $f_2 = 1 - H(t_4)(f_1 - |G(t_3)|)^{H(t_5)} + |G(t_3)|, 0 + |G(t_3)| \leq f_1 \leq 1 + |G(t_3)|$
- Optimal PS $x_j = \sin(6\pi x_1 + (j + K(t_1))\frac{\pi}{n}) + G(t_2),\ j = 2, \ldots, n; 0 \leq x_1 \leq 1.$

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14) $F14$

Initialize $t_1 = t_2 = t_3 = t_4 = t_5 = 0$

Let us define t_s as a random variable belonging to the set { t_1, t_2, t_3, t_4, t_5 }, where probability of each of the variables being selected is equal i.e. $P(t_s = t_1) = P(t_s = t_2) = P(t_s = t_3) = P(t_s = t_4) = P(t_s = t_5) = \frac{1}{5}$

for each iteration $\tau = 1 : \max\_gen$

if $t(\tau) \neq t(\tau - 1)$

Select t_s randomly from the set { t_1, t_2, t_3, t_4, t_5 },

Increase the selected variable t_s by a step of 1

Update f_1, f_2 as per the current values of t_1, t_2, t_3, t_4, t_5

end

end

$$f_1 = x_1 + \frac{2}{|J_1|}\sum_{j \in J_1}[x_j - x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t_1))} - G(t_2)]^2 + |G(t_3)|$$

$$f_2 = 1 - H(t_4)x_1^{H(t_5)} + \frac{2}{|J_2|}\sum_{j \in J_2}[x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t_1))} - G(t_2)]^2 + |G(t_3)|,$$

where $J_1 = \{j \mid j$ is odd and $2 \leq j \leq n\}$ and $J_2 = \{j \mid j$ is even and $2 \leq j \leq n\}$.

- Search space: $[0,1] \times [-2,2]^{n-1}$, where n is the number of dimensions of the search space.
- Optimal PF $f_2 = 1 - H(t_4)(f_1 - |G(t_3)|)^{H(t_5)} + |G(t_3)|, 0 + |G(t_3)| \leq f_1 \leq 1 + |G(t_3)|$
- Optimal PS $x_j = x_1^{0.5(2+\frac{3(j-2)}{n-2}+G(t_1))} + G(t_2), \quad j = 2,...,n, \quad 0 \leq x_1 \leq 1.$

Note that in F13 and F14, several types of dynamic variation of PS and PF have been incorporated to form a pool of changes and at a time only one kind of change is employed. The variables t_1 to t_5 correspond to a single type of variation each. At a particular point of time, only of the variables is selected randomly and gets increased by 1. The objective functions f_1, f_2 are obtained according to updated values of the variables.

N.B.: In all the benchmark functions we have used some common variables. They denote some typical kind of changes in PS or PF as discussed below:

$G(t)$: Vertical or Horizontal shift in PF or PS.

$M(t)$: Angular shift in PF.

$H(t)$: Curvature variation in PF.

$R(t)$: Radius variation in three dimensional PF.

$K(t)$: Phase shifting in trigonometric type of PS.