Bohemian Matrix Geometry

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Rhapsodizing about Bohemian Matrices

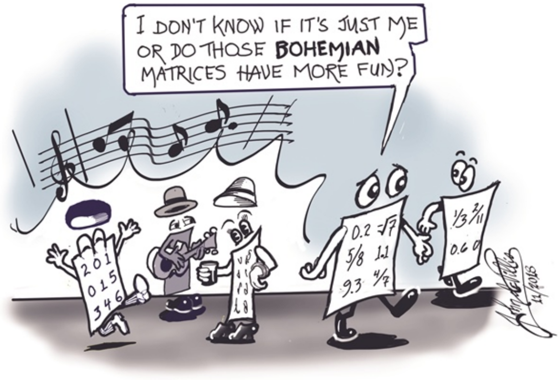

Figure 1: A cartoon by mathematician John de Pillis (UC Riverside), which appeared in Nick Higham's column in SIAM News

Bohemian Matrices

A family of matrices is called "Bohemian" if all entries are all from a single finite population P. The name comes from BOunded HEight Matrix of Integers. See bohemianmatrices.com for instances.

See also the London Mathematical Society Newsletter, November 2020, page 16.

Such matrices have been studied for quite a long time (e.g. by Olga Taussky–Todd), though the name "Bohemian" only dates to 2015. See also the Wikipedia entry at `https://en.wikipedia.org/wiki/Bohemian_matrices`.

Why we're interested

Framing the problem this way allows one to study common properties of discrete random matrices, to investigate extreme possibilities by exhaustive computation, and gives a new look at several old problems (e.g. the Hadamard conjecture).

Our original motivation was simply the construction of test problems for eigenvalue solvers; Steven Thornton has solved several *trillion* eigenvalue problems, and uncovered small instances (10 by 10 matrices with complex entries, 20 by 20 matrices with real entries) for which Matlab's ***`eig`*** routine failed to converge. [Reported to the Mathworks and I think they have fixed it now.]

We also found a bug in *`fsolve`* with this approach (now fixed). [It was a *cubic* polynomial, ow.]

Other uses

Nick Higham has used Bohemian matrices as a class to optimize over to look for improved lower bounds on such things as the growth factor in matrix factoring; Laureano Gonzalez-Vega has looked at *correlation matrices*. Matthew Lettington (Cardiff) looks at magic squares with these tools.

We have used this idea to understand some things about simple matrix structures, such as upper Hessenberg and upper Hessenberg Toeplitz matrices.

And, hey, we have a Bohemian Matrix Calendar.

George Pólya

Bohemian matrices are both a *generalization* and a *specialization* in the senses expounded by George Pólya: they generalize (e.g.) Bernoulli matrices or binary matrices, and specialize general matrices.

It ought not to be surprising that they can be connected to Algebra, Analysis, Geometry, Probability, Combinatorics, Number Theory, Optimization, and Algorithms (at least).

Bohemian Matrix Geometry

The paper for today's talk can be found at *https://arxiv.org/abs/2202.07769*

The Maple Workbook that contains the source code implementing the Schmidt–Spitzer theorem can be found, together with all the images from our paper and the slides from this talk, at *https://github.com/rcorless/Bohemian-Matrix-Geometry*

The "I'm-Not-Doron-Zeilberger" approach: Please download those images and look at them on your own devices. That gives higher resolution than this projection does.

(This is "Screen-sharing for in-person lectures" :)

<show the images in sequence>

Dense/Full Random Matrices

Terence Tao and Van Vu investigated **large** random matrices with entries drawn from discrete real distributions, such as $\{-1,0,1\}$ or $\{-1,1\}$ (Bernoulli matrices) or $\{0,1\}$ (binary matrices).

- They proved that the eigenvalues are (asymptotically) uniformly distributed on a disk of radius $\sqrt{m}$ (real eigenvalues on interval $(-m,m)$ become unlikely in comparison)
- As $m \to \infty$ pictures look similar for $\{-1,0,1\}$ and for $\{-1,1\}$ and for $\{0,1\}$ (mutatis mutandis)

Small-dimension matrices look different to start with, but the "holes fill in" as m increases.

Bernoulli $\{-1,1\}$

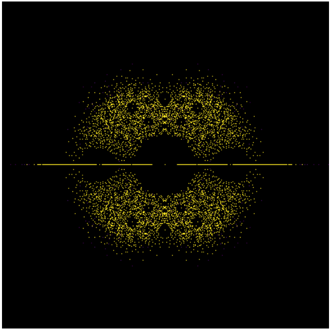

Figure 2: Density plot of all eigenvalues of all $2^{25} = 33{,}554{,}432$ Bernoulli matrices of dimension $m = 5$. There are only 3,150 different characteristic polynomials for this family. The most likely are $\lambda^3(\lambda-1)(\lambda+2)$ and $\lambda^3(\lambda+1)(\lambda-2)$, occurring 336,960 times each. The rarest are $\lambda^5 \pm 5\lambda^4$ and $(\lambda-3)(\lambda+2)^4$ and $(\lambda+3)(\lambda-2)^4$ which occur 16 times each. More than 65% of $m = 5$ Bernoulli matrices are singular. Almost 1.5% have quadruple eigenvalues at zero. **Ask me how I know this.**

Structure: e.g. Skew-symmetric tridiagonal Bohemian matrices

Here is a seven-by-seven example of SSTB[P]. If P has $\#P$ elements, then the number of such matrices is $\#P^6$.

$$\begin{bmatrix} 0 & u_1 & 0 & 0 & 0 & 0 & 0 \\ -u_1 & 0 & u_2 & 0 & 0 & 0 & 0 \\ 0 & -u_2 & 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & -u_3 & 0 & u_4 & 0 & 0 \\ 0 & 0 & 0 & -u_4 & 0 & u_5 & 0 \\ 0 & 0 & 0 & 0 & -u_5 & 0 & u_6 \\ 0 & 0 & 0 & 0 & 0 & -u_6 & 0 \end{bmatrix} \tag{1}$$

A picture

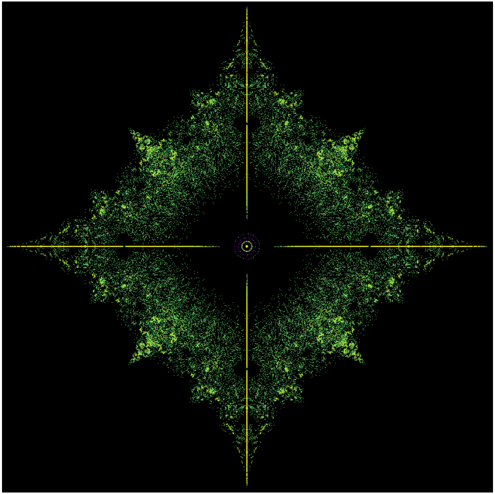

Figure 3: Density of eigenvalues of all $4^{14} = 268{,}435{,}456$ fifteen by fifteen skew-symmetric tridiagonal matrices with population $P = [\pm 1, \pm i]$. Note the "rose" in the middle and its symmetries. Computed in Maple (15 seconds).

Symmetric Matrices

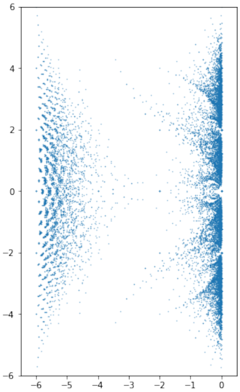

Figure 4: Eigenvalues of all $2^{15} = 32{,}768$ dimension $m = 6$ complex symmetric matrices with entries $-1 \pm i$. They are confined to the box $-m \leq \Re\lambda \leq 0$, $-m \leq \Im\lambda \leq m$. Unlike the previous image, this is just an ordinary Maple plot.

The effect of structure/population

As perhaps should have been obvious to me but wasn't until I tried things out computationally, the *structure* of the matrix strongly affects the eigenvalue density (most obviously, Hermitian matrices have real eigenvalues). Likewise, the population also affects things, though less intelligibly.

For the kinds of matrices and populations Tao & Vu studied, the results are (asymptotically in the large dimension case) disks. For these other structures/populations, what can be said?

Bounds and Patterns I

The Bendixon–Bromwitch–Hirsch Theorem (see Chapter 5, Handbook of Linear Algebra): Write the m-dimensional matrix $\mathbf{A} = \mathbf{H} + i\mathbf{S}$ where $\mathbf{H} = (\mathbf{A} + \mathbf{A}^*)/2$ is the Hermitian part of $\mathbf{A}$ and $\mathbf{S} = (\mathbf{A} - \mathbf{A}^*)/(2i)$ is the skew-Hermitian part of $\mathbf{A}$. Both $\mathbf{H}$ and $\mathbf{S}$ are Hermitian and their eigenvalues μ_k and ν_k are real. Sort their eigenvalues as $\mu_1 \le \mu_2 \le \cdots \le \mu_m$ and $\nu_1 \le \nu_2 \le \cdots \le \nu_m$.

The Bendixson–Bromwitch–Hirsch theorem says that the eigenvalues of $\mathbf{A}$ lie in the box $\mu_1 \le \Re\,\lambda \le \mu_m$ and $\nu_1 \le \Im\,\lambda \le \nu_m$.

This can be used to explain the eigenvalues-in-the-strip for symmetric matrices with entries $-1 \pm i$: $\mathbf{A} = -\mathbf{E} + i\mathbf{S}$, and the eigenvalues-in-the-diamond for skew-symmetric tridiagonal matrices with population $\pm 1, \pm i$.

Toeplitz matrix fun

Toeplitz matrices are *cool*. Banded Toeplitz matrices are surprisingly "easy" to understand now (after work of Toeplitz, Widom, Wiener, Schmidt & Spitzer, and Böttcher et al.).

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ a_{-1} & a_0 & a_1 & a_2 & a_3 & 0 \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 \\ 0 & a_{-2} & a_{-1} & a_0 & a_1 & a_2 \\ 0 & 0 & a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & 0 & 0 & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

This matrix has "symbol" $\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3$. In general (for infinite dimension) it's a Laurent series; for banded matrices, a Laurent polynomial.

Bounds and patterns II

Theorem: (Toeplitz) The eigenvalues of an infinite-dimensional Toeplitz operator are related to* the image of the unit circle under the symbol: $a(e^{i\theta})$.

* Ok so I am not telling the whole story here. Which infinite? And what about winding numbers?

Craziness: The eigenvalues of finite-dimensional truncations of Toeplitz matrices do *not* converge to the spectrum of the corresponding infinite-dimensional Toeplitz operators (but their pseudospectra do).

Another Theorem (Schmidt & Spitzer 1963): The eigenvalues of finite-dimensional *banded* Toeplitz matrices converge to semialgebraic curves defined by the symbol (which can be determined by a simple algebraic computation).

Our theorem

The Schmidt–Spitzer curves (and therefore, we believe, eigenvalues of finite-dimensional upper Hessenberg Toeplitz matrices) converge to analogous computable curves.

This allows us to explain the Sierpinski-like fractal structures in the Bohemian eigenvalue density plots.

Eigenvalues of One Toeplitz matrix

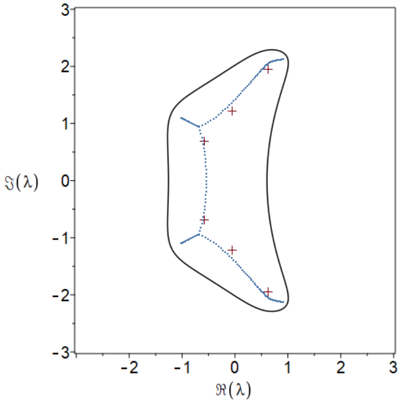

Figure 5: Eigenvalues of a single dimension $m = 6$ upper Hessenberg zero-diagonal Toeplitz matrix with entries from $\{-1,0,1\}$. The black curve is the image of the unit circle under the symbol; the dotted blue curve is the Schmidt–Spitzer curve for the infinite-dimensional banded Toeplitz matrix.

Those semialgebraic curves

The curves are defined by *equal-magnitude* values of the so-called "symbol": $a(z) = a(e^{i\theta}z) = \lambda$. These are Laurent polynomials, so finding the zeros is just univariate polynomial rootfinding of $a(z) - a(e^{i\theta}z) = 0$, given θ. However, λ is in the curve *if and only if* the two equal-magnitude roots are the qth and $q+1$st smallest magnitude roots, where q is the order of the pole in the Laurent polynomial (here $q = 1$). Combinatorics and complex analysis both!

Look at ToeplitzExperiments.maple

What did we prove?

For *upper Hessenberg* matrices, a Laurent *polynomial* symbol

$$a(z) = -\frac{1}{z} + a_0 + a_1 z + \cdots + a_m z^m$$

is not very different to a (finite pole) Laurent *series* because the similarity transform by the diagonal matrix $D = \mathrm{diag}(1, \rho, \rho^2, \ldots)$ shows that the series

$$a(z) = -\frac{\rho}{z} + a_0 + a_1 \frac{z}{\rho} + \cdots + a_m \frac{z^m}{\rho^m} + \cdots$$

converges absolutely and uniformly for $|z| < \rho$ (where $|a_k| \le B$ because Bohemian and so geometric). Everything follows from classical theorems afterwards: the equal-magnitude curves converge. In practice, they converge *rapidly* for the examples we tried.

What does the theorem explain?

Using that theorem, we can look at upper Hessenberg Toeplitz Bohemian matrices with, say, a population with three elements. Then increasing the dimension by 1 gives us one new term in the symbol—a_m—which can have one of three values; this gives three new eigenvalues for each old eigenvalue, and *moreover these eigenvalues have to lie close to the semialgebraic curve from before*. This explains the “Sierpinski gasket” look of these images.

Nonlinear Sierpinski

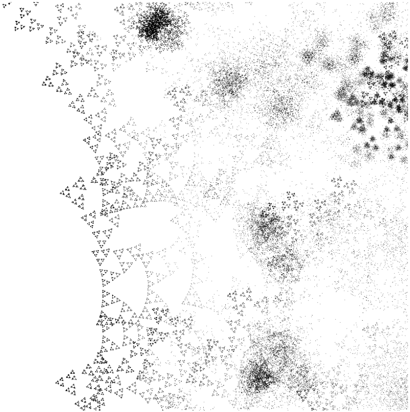

Figure 6: upper Hessenberg Toeplitz, -1 subdiagonal, zero diagonal, population cube roots of unity, dimension $m = 13$, all 531,441 matrices, zoomed in on an edge.

Unsolved problems

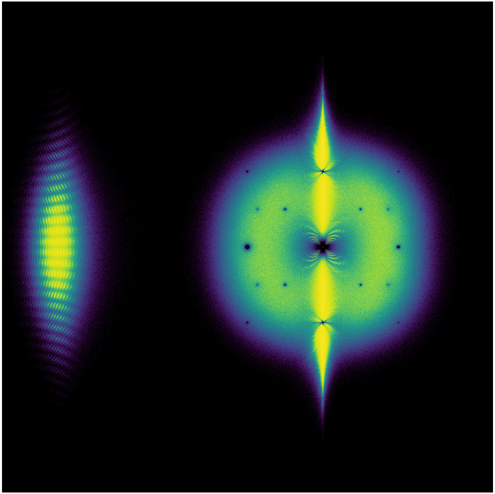

Figure 7: Eigenvalues of 5,000,000 dimension $m = 7$ unstructured matrices with population $-1 \pm i$. Why does this density plot look as it does?

A related unsolved combinatorial problem

m	2^{m^2}	#charpolys	Bernoulli
1	2	2	2
2	16	9	6
3	512	68	28
4	65536	1161	203
5	33,554,432	65,348	3150
6	6.872×10^{10}	??	??

Table 1: Number of dense Bohemian matrices with population $-1 \pm i$ and the number of distinct characteristic polynomials. The case $m = 5$ took 7 hours on my Surface Pro (only 40 minutes for the Bernoulli case, though). The case $m = 6$ would have to process 2048 times as many matrices (and larger ones of course). The case $m = 7$ has more than 5.63×10^{14} matrices.

How are the images made?

We use (mostly) the viridis colour palette by Smith and van der Walt, SciPy 2015. John May has now put this into Maple 2022. This is a sequential map, approximately perceptually uniform, and robust under many kinds of colour blindness.

So what do we do, exactly? We choose a grid, and count the eigenvalues in each pixel; we colourize by the chosen palette using an approximate inversion of the frequency distribution (so that roughly “equal areas” are displayed in each colour interval).

More advanced techniques can be used (this is where Dan Piponi comes in).

More on Bohemians

You can find a version of this talk at a video on my YouTube channel.

You can find a related talk at

"Skew Symmetric Tridiagonal Bohemians"

The (Maple Transactions!) papers that talk refers to are

What can we learn from Bohemian Matrices?
https://doi.org/10.5206/mt.v1i1.14039

and

Skew-symmetric tridiagonal Bohemian matrices
https://doi.org/10.5206/mt.v1i2.14360

See also chapter 4 of my New Book, *Computational Discovery on Jupyter*, with Neil Calkin and Eunice Chan (Maple version under construction: time permitting I will show you a sample. Uses Jupyter with a Maple kernel, which is new for Maple 2022)!

Thank you

Thank you for listening.

Many people were involved in this project since its inception. We thank Alex Brandt, Neil Calkin, Eunice Chan, Mark Giesbrecht, Laureano Gonzalez-Vega, Nick Higham, Ilias Kotsireas, Piers Lawrence, John May, Marc Moreno Maza, Erik Postma, Rafael Sendra, Juana Sendra, and Steven Thornton in particular.

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