Bohemian Matrix Geometry

Robert M. Corless
Joint work with many people, including
George Labahn, Leili Rafiee Sevyeri, and Dan Piponi

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ORCCA, Western University & University of Waterloo, Canada

Announcement: Maple Transactions

Maple Transactions

an open access journal with no page charges

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We welcome expositions on topics of interest to the Maple community, including in computer-assisted research in mathematics, education, and applications. Student papers especially welcome.

Rhapsodizing about Bohemian Matrices

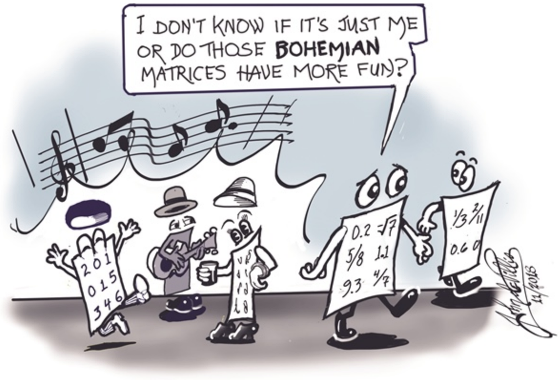

Figure 1: A cartoon by mathematician John de Pillis (UC Riverside), which appeared in Nick Higham's column in SIAM News

Bohemian Matrices

A family of matrices is called “Bohemian” if all entries are all from a single finite population P. The name comes from BOunded HEight Matrix of Integers. See bohemianmatrices.com for instances.

See also the London Mathematical Society Newsletter, November 2020, page 16.

Such matrices have been studied for quite a long time (e.g. by Olga Taussky–Todd), though the name “Bohemian” only dates to 2015. See also the Wikipedia entry at `https://en.wikipedia.org/wiki/Bohemian_matrices`.

Why we're interested

- Allows to study common properties of discrete random matrices
- investigate extreme possibilities by exhaustive computation
- New look at some old problems (e.g. Hadamard conjecture)
- Original motivation: software testing! Found examples where *`eig`* failed to converge
- Found a bug in Maple's *`fsolve`* (now fixed)

Other uses

NJH has used Bohemian matrices as a class to optimize over to look for improved lower bounds on such things as the growth factor in matrix factoring; Laureano Gonzalez-Vega has looked at *correlation matrices*. Matthew Lettington (Cardiff) looks at magic squares with these tools. David R. Nelson (Harvard) uses ideas like these to study non-Hermitian quantum mechanics. (Thanks to Nick Trefethen for making this connection)

We have used this idea to understand some things about simple matrix structures, such as upper Hessenberg and upper Hessenberg Toeplitz matrices.

And, hey, we have a Bohemian Matrix Calendar.

George Pólya

Bohemian matrices are both a *generalization* and a *specialization* in the senses expounded by George Pólya: they generalize (e.g.) Bernoulli matrices or binary matrices, and specialize general matrices.

It ought not to be surprising that they can be connected to Algebra, Analysis, Geometry, Probability, Combinatorics, Number Theory, Optimization, and Algorithms (at least).

Density of eigenvalues in $\mathbb{C}$

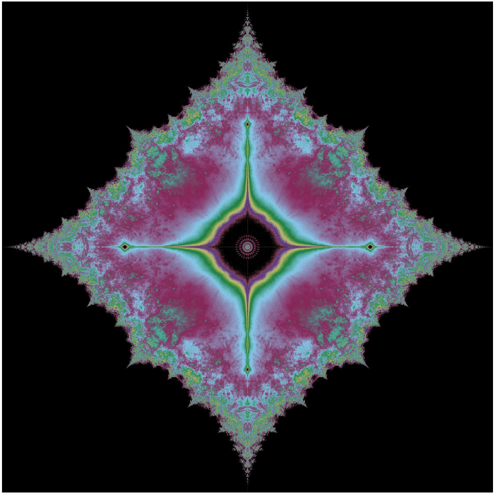

Figure 2: Density plot of eigenvalues of all $2^{30} = 1{,}073{,}741{,}824$ skew-symmetric tridiagonal matrices of dimension 31 with population $\{1, i\}$. Hotter colours correspond to higher density. Picture by Aaron Asner.

Nonlinear Sierpinski

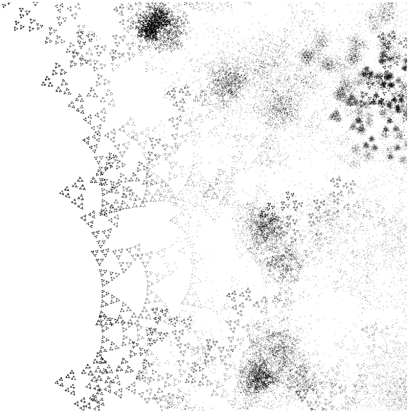

Figure 3: upper Hessenberg Toeplitz, -1 subdiagonal, zero diagonal, population cube roots of unity, dimension $m = 13$, all 531,441 matrices, zoomed in on an edge.

Details for today's talk can be found at
https://arxiv.org/abs/2202.07769

The Maple Workbook that contains the source code partially implementing the Schmidt–Spitzer theorem can be found, together with all the images from our paper and the slides from this talk, at *https://github.com/rcorless/Bohemian-Matrix-Geometry*

Please download those images and look at them on your own devices. That gives higher resolution than this projection does.

(This is "Screen-sharing for in-person lectures" :)

Toeplitz matrix fun

Banded Toeplitz matrices are surprisingly "easy" to understand now (after work of Toeplitz, Szegő, Kac, Widom, Wiener, Schmidt & Spitzer, Böttcher et al., and many others).

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 & 0 \\ a_{-1} & a_0 & a_1 & a_2 & a_3 & 0 \\ a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 \\ 0 & a_{-2} & a_{-1} & a_0 & a_1 & a_2 \\ 0 & 0 & a_{-2} & a_{-1} & a_0 & a_1 \\ 0 & 0 & 0 & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

This matrix has "symbol" $\frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3$. In general (for infinite dimension) it's a Laurent series; for banded matrices, a Laurent polynomial.

Bounds and patterns II

Theorem: (Toeplitz) The eigenvalues of an infinite-dimensional Toeplitz operator are related to* the image of the unit circle under the symbol: $a(e^{i\theta})$.

* Ok so I am not telling the whole story here. Which infinite matrix? And what about winding numbers? And for which class of symbols (functions) is this true for?

Important Note: The eigenvalues of finite-dimensional truncations of Toeplitz matrices do *not* converge to the spectrum of the corresponding infinite-dimensional Toeplitz operators (but their pseudospectra do).

Another Theorem (Schmidt & Spitzer 1963): The eigenvalues of finite-dimensional *banded* Toeplitz matrices converge to semialgebraic curves (that can be determined by a simple algebraic computation) defined by the symbol .

Our theorem, needed to explain the Sierpinski structure

The Schmidt–Spitzer curves (and therefore, we believe, eigenvalues) of finite-dimensional upper Hessenberg Toeplitz matrices converge to analogous computable curves defined by roots of convergent series.

This convergence allows us to explain the Sierpinski-like fractal structures in the Bohemian eigenvalue density plots.

Eigenvalues of One Toeplitz matrix

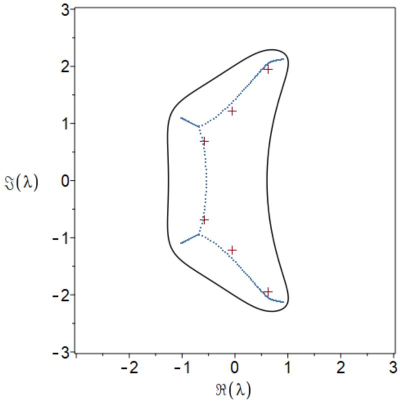

Figure 4: Eigenvalues of a single dimension $m = 6$ upper Hessenberg zero-diagonal Toeplitz matrix with entries from $\{-1,0,1\}$. The black curve is the image of the unit circle under the symbol; the dotted blue curve is the Schmidt–Spitzer curve for the infinite-dimensional banded Toeplitz matrix.

The Schmidt–Spitzer semialgebraic curves

The curves are defined by *equal-magnitude* values of the so-called "symbol": $a(z) = a(e^{i\theta}z) = \lambda$. These are Laurent polynomials, so finding the zeros is just univariate polynomial rootfinding of $a(z) - a(e^{i\theta}z) = 0$, given θ. However, λ is in the curve *if and only if* the two equal-magnitude roots are the qth and $q+1$st smallest magnitude roots, where q is the order of the pole in the Laurent polynomial (here $q = 1$). Combinatorics and complex analysis both!

Look at ToeplitzExperiments.maple

What did we prove?

For *upper Hessenberg* matrices, a Laurent *polynomial* symbol

$$a(z) = -\frac{1}{z} + a_0 + a_1 z + \cdots + a_m z^m$$

is not very different to a (finite pole) Laurent *series* because the similarity transform by the diagonal matrix $D = \mathrm{diag}(1, \rho, \rho^2, \ldots)$ shows that the series

$$a(z) = -\frac{\rho}{z} + a_0 + a_1 \frac{z}{\rho} + \cdots + a_m \frac{z^m}{\rho^m} + \cdots$$

converges absolutely and uniformly for $|z| < \rho$ (where $|a_k| \leq B$ because Bohemian and so geometric). Everything follows from classical theorems afterwards: the equal-magnitude curves converge. In practice, they converge *rapidly* for the examples we tried.

What does the theorem explain?

Using that theorem, we can look at upper Hessenberg Toeplitz Bohemian matrices with, say, a population with three elements. Then increasing the dimension by 1 gives us one new term in the symbol—a_m—which can have one of three values; this gives three new eigenvalues for each old eigenvalue, and *moreover these eigenvalues have to lie close to the semialgebraic curve from before*. This explains the "Sierpinski gasket" look of these images.

This is the *first* such explanation of the appearance of a fractal in a Bohemian context.

Unsolved problems

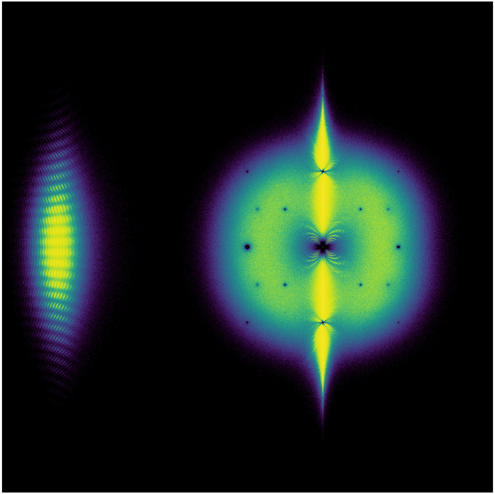

Figure 5: Eigenvalues of 5,000,000 dimension $m = 7$ unstructured matrices with population $-1 \pm i$. Why does this density plot look as it does?

A related unsolved combinatorial problem

m	2^{m^2}	#charpolys	Bernoulli
1	2	2	2
2	16	9	6
3	512	68	28
4	65536	1161	203
5	33,554,432	65,348	3150
6	6.872×10^{10}	??	131,641

Table 1: Number of dense Bohemian matrices with population $-1 \pm i$ and the number of distinct characteristic polynomials. The case $m = 5$ took 7 hours on my Surface Pro (only 40 minutes for the Bernoulli case, though). The case $m = 6$ would have to process 2048 times as many matrices (and larger ones of course). The case $m = 7$ has more than 5.63×10^{14} matrices. See http://oeis.org/A272662.

More on Bohemians

You can find an older version of this talk at a video on my YouTube channel.

You can find a related talk at

"Skew Symmetric Tridiagonal Bohemians"

The (Maple Transactions!) papers that talk refers to are

What can we learn from Bohemian Matrices? https://doi.org/10.5206/mt.v1i1.14039

and

Skew-symmetric tridiagonal Bohemian matrices https://doi.org/10.5206/mt.v1i2.14360

See also chapter 4 of my New Book, *Computational Discovery on Jupyter*, with Neil Calkin and Eunice Chan (Maple version under construction: time permitting I will show you a sample. Uses Jupyter with a Maple kernel, which was new in Maple 2022)!

Thank you

Thank you for listening!

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