A Fractal Eigenvector

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talk intended for Clarkson University, Potsdam NY
Joint work with Neil Calkin, Eunice Chan, David Jeffrey, and Piers Lawrence
These slides available at rcorless.github.io

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Mandelbrot polynomials and Matrices

1. Piers Lawrence & RMC, The Largest Root of the Mandelbrot Polynomials (Jonfest proceedings, 2013)
2. Bini and Robol's MPSolve paper (JCAM 2014) (version 1 was 2000, Bini & Fiorentino)
3. Neil J Calkin, Eunice Chan, & RMC, Some Facts and Conjectures about Mandelbrot Polynomials (Maple Transactions 2021)
4. Neil Calkin et al, A Fractal Eigenvector (American Math Monthly 2022)
5. Eunice Y.S. Chan, A comparison of solution methods for Mandelbrot-like polynomials, Masters Thesis, 2016

Piers Lawrence had the fundamental idea which opened the door to these results.

Related papers

1. Eunice Chan & RMC, A New Kind of Companion Matrix (ELA 2017)
2. Eunice Chan & RMC, Minimal Height Companion Matrices for Euclid Polynomials (Math. Comput. Sci. 2019)
3. Eunice Chan et al, Algebraic Linearizations (LAA 2019)
4. Eunice Chan, RMC, & Leili Rafiee Sevyeri, Generalized Standard Triples (ELA 2021)

NB: There is also a strongly related paper from 2017 by Robol, Vandebril, and Van Dooren.

Bohemian matrices

Another related thread of work: Bohemian Matrices

1. cover image: London Mathematical Society Newsletter, November 2020, page 16 (RMC, NJ Higham, & SE Thornton)
2. Upper H–berg and Toeplitz Bohemians (Chan et al, 2020, LAA)
3. What can we learn from Bohemian matrices? (RMC, 2021)
4. Skew-symmetric tridiagonal Bohemian matrices (RMC 2021 Maple Transactions)
5. Computational Discovery on Jupyter (chapter 4) (an OER by Neil Calkin, Eunice Chan, and RMC 2022, and to be a SIAM book :)

The picture I want to get to

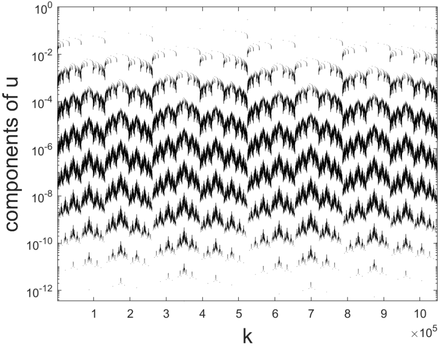


Figure 1: A Fractal Eigenvector

We begin the story with concrete instances.

Put $M_1 = [1]$, and

$$M_2 = \begin{bmatrix} M_1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & M_1 \end{bmatrix} . \tag{1}$$

The transpose of the right-eigenvector corresponding to the dominant eigenvalue $\lambda \approx 1.755$ is (approximately)

$$\begin{bmatrix} 1.0 & 0.5698 & 0.7549 \end{bmatrix} . \tag{2}$$

We can plot the components against the index into the vector. (I learned to do this in engineering vibration class).

A boring plot

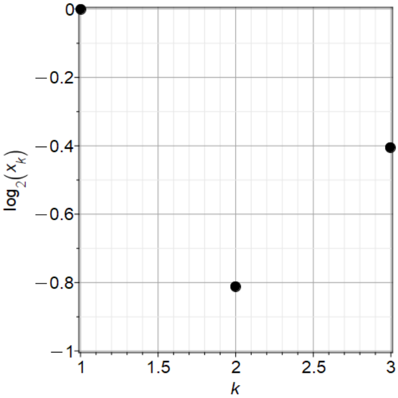


Figure 2: The eigenvector [1.0, 0.5698, 0.7549] plotted on a log scale.

A digraph

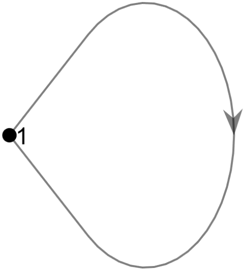


Figure 3: A digraph for $M_1 = [1]$, interpreting it as an adjacency matrix: vertex 1 has an arc connecting it to itself.

Another digraph

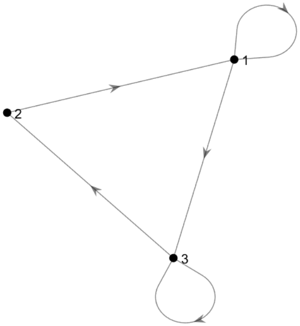


Figure 4: A digraph for $M_2 = \begin{bmatrix} M_1 & 0 & \color{red}{1} \\ \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & M_1 \end{bmatrix}$. Vertex 1 is connected to itself and to vertex 3. Vertex 3 is also connected to itself. Vertex 3 is connected to vertex 2, which is connected to vertex 1.

Another matrix

$$M_3 = \begin{bmatrix} M_2 & 0 & e_1 e_3^T \\ e_3^T & 0 & 0 \\ 0 & e_1 & M_2 \end{bmatrix}. \tag{3}$$

This matrix is 7 by 7. All its entries are either 0 or 1. $e_3^T = [0, 0, 1]$ and $e_1 = [1, 0, 0]^T$ are elementary vectors.

The next one will be 15 by 15, M_4 built from two copies of M_3 and some "red glue" made up of elementary vectors of the correct dimension.

Explicitly here is M_3

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{4}$$

Another digraph

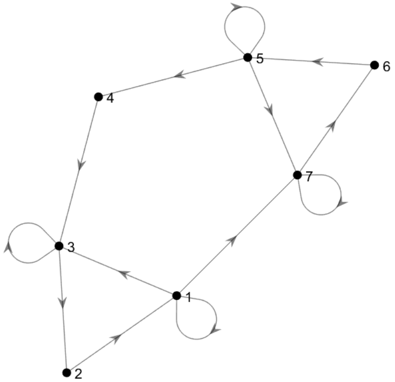


Figure 5: A digraph for M_3

Now, some polynomials

Define $p_{k+1}(c) = \det(cI + M_k)$ for $k \geq 1$. Notice the plus sign, and the "off-by-one" index (drat it). Then $p_2(c) = c + 1$,
$p_3(c) = c^3 + 2c^2 + c + 1 = cp_2^2(c) + 1$,
$p_4(c) = c^7 + 4c^6 + 6c^5 + 6c^4 + 5c^3 + 2c^2 + c + 1$ which is, yes, $cp_3^2(c) + 1$.

Theorem

$$p_{k+1}(c) = cp_k^2(c) + 1 \tag{5}$$

for $k \geq 2$. More, we may take $p_0(c) = 0$ and $p_1(c) = 1$.

Proof Sketch (thx D. E. Knuth)

- Show by induction that $c\mathbf{I} + \mathbf{M}_k$ is unit upper Hessenberg[1] and of dimension $d_k = 2^k - 1$.
- Notice that the $1, d_k$ entry, the top right corner, is 1
- Determinant is linear in the first row.
- Write the determinant as the sum of the determinant of a block lower triangular matrix and a unit upper Hessenberg matrix with entry 1 in the top right corner
- Notice that the determinant of the block diagonal matrix is $p_{k+1}(c) \cdot c \cdot p_{k+1}(c)$
- Notice that the determinant of the other matrix is 1.

[1] Common terminology in numerical analysis. Is this terminology known to you? I'd like to take an informal survey. I know that not every mathematician knows this term.

With fewer words

$$
\begin{aligned}
\det(cI + M_{k+1}) = \det & \begin{bmatrix} cI + M_k & 0 & 0 \\ e_d^T & c & 0 \\ 0 & e_1 & cI + M_k \end{bmatrix} \\
+ \det & \begin{bmatrix} Z_k & 0 & 1 \\ e_d^T & c & 0 \\ 0 & e_1 & cI + M_k \end{bmatrix}
\end{aligned}
$$

where Z_k is the same as $cI + M_k$ except with its top row replaced by zeros. So

$$\det(cI + M_{k+1}) = p_{k+1}(c) \cdot c \cdot p_{k+1}(c) + 1\,. \tag{6}$$

Eigenvalues of M_{11}, roots of $p_{12}(-z)$

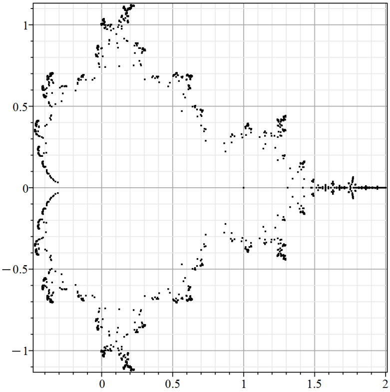


Figure 6: Eigenvalues of M_{11}, which is 1023 by 1023

The first open conjecture

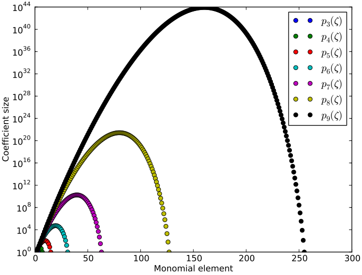


Figure 7: Are these polynomials unimodal? *We have no proof, but we think so.* Log-concave, even, apart from the Catalan section.

I would *really welcome* a proof of this conjecture.

Mandelbrot

Of course these polynomials and matrices are connected to the Mandelbrot iteration: $z_{n+1} = z_n^2 + c$. If $c \neq 0$, divide by c and write $p_n(c) = z_n(c)/c$. Then the polynomials defined by $p_0(c) = 0$ and

$$p_{n+1}(c) = cp_n^2(c) + 1 \tag{7}$$

are called *Mandelbrot polynomials*. The matrices $-M_{n-1}$ (with all entries -1) have as eigenvalues the roots of the Mandelbrot polynomials. The eigenvectors of $\boldsymbol{M}_n$ and $-\boldsymbol{M}_n$ are (of course) the same. Yes, different papers used different indexing. It's a total pain.

The dominant eigenvalue

In 2013 I published a paper with Piers Lawrence containing a result obtained with the help of Neil Calkin which says that, for $n \geq 2$,

$$p_{n+1}(-2 + \frac{3\theta^2 4^{-n}}{2}) = \cos\theta + O(4^{-n}) \,. \tag{8}$$

This gives the asymptotic expansion as $n \to \infty$ for the dominant eigen*value* when $\theta = \pi/2$.

There are unresolved questions there too (I computed a few more terms in the expansion, but I don't know all terms; and I don't know if the series converges, or in which way.)

So the dominant eigenvalue of $\boldsymbol{M}_n$ can be expressed as

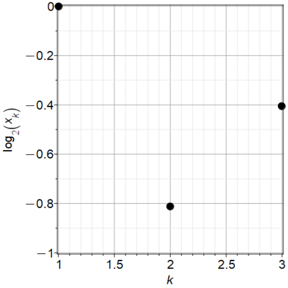
$$\rho_n = 2 - \frac{3\pi^2}{8} 4^{-n} + O(4^{-2n}) \tag{9}$$

which is asymptotically valid as $n \to \infty$. Four subsequent Newton iterations (using the recurrence) are enough to get this eigenvalue to full accuracy.

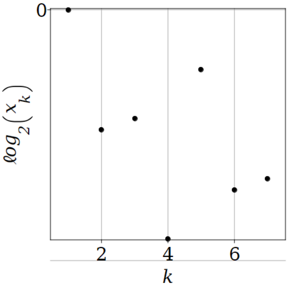
“Of course”[2] the dominant eigenvalue must be simple, real, and positive by the Perron–Frobenius theory. [Which, had we known about them, we could have expected from the digraphs.] More, the corresponding eigenvector must have all positive components.

[2] I hadn’t known any of this theory before Neil taught me, in 2012

So let's look at some more



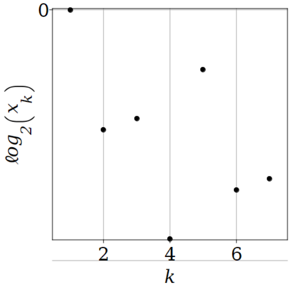
(a) vector for M_2



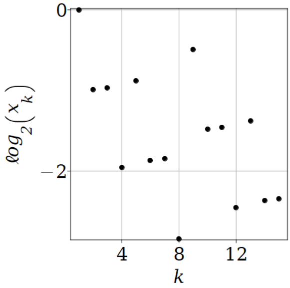
(b) vector for M_3

Figure 8: Comparing the dominant eigenvector for M_2 with that for M_3

So let's look at some more



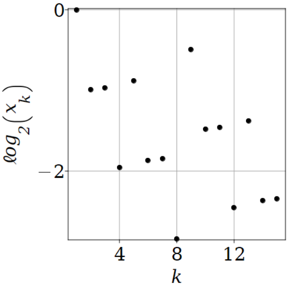
(a) vector for M_3



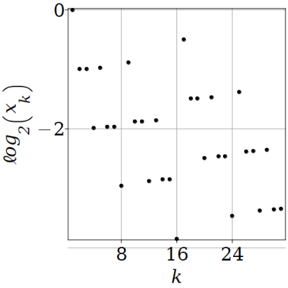
(b) vector for M_4

Figure 9: Comparing the dominant eigenvector for M_3 with that for M_4

So let’s look at some more



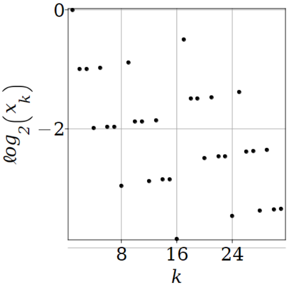
(a) vector for M_4



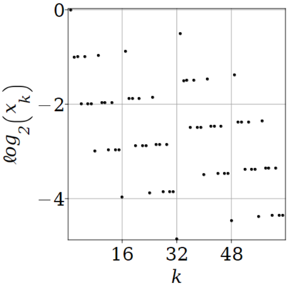
(b) vector for M_5

Figure 10: Comparing the dominant eigenvector for M_4 with that for M_5

So let's look at some more



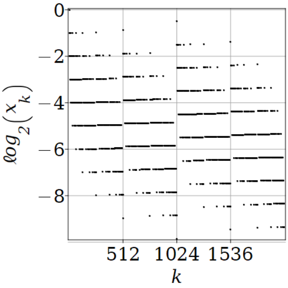
(a) vector for M_5



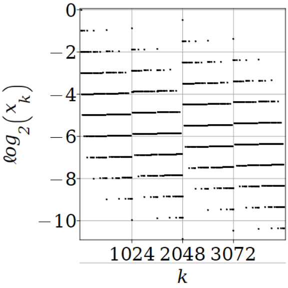
(b) vector for M_6

Figure 11: Comparing the dominant eigenvector for M_5 with that for M_6

So let's look at some more



(a) vector for M_{11}



(b) vector for M_{12}

Figure 12: Comparing the dominant eigenvector for M_{11} with that for M_{12}

The first picture, again

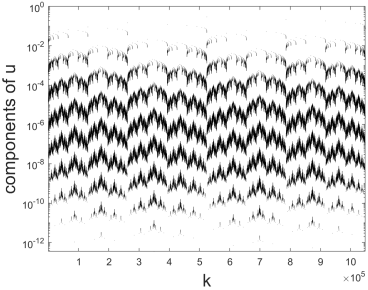


Figure 13: A Fractal Eigenvector again. Notice that there are more than a million components. Still all positive, and plotted on a log scale.

Ok, what's going on here, then?

That's not converging to the picture I showed at the start!

(The one we are analyzing is an easier eigenvector than that of the first picture).

The more difficult eigenvector is the eigenvector of $M_{20}\mathrm{J}_{20}$ where J_n is the dimension $d_n = 2^n - 1$ self-involutory permutation matrix (aka "anti-identity matrix"). This *happens* to have eigenvalues $\mu_k = (-1)^{k-1}\sigma_k$ where the σ_k are the *singular* values of M_n. We'll get to those.

Detecting a pattern

Let's come back to the simpler eigenvector. After a while, one notices that each successive eigenvector seems to contain *two* copies of the previous eigenvector.

Since each matrix contains two copies of the previous matrix, this seems very natural in retrospect.

Well, to guard against pareidolia we should prove something.

Setting up for the first theorem

As stated, $\boldsymbol{M}_{n+1}$ has two copies of $\boldsymbol{M}_n$ in it, and is upper Hessenberg so that we may find the eigenvector by solving a unit upper triangular system. In block form, we have

$$\begin{bmatrix} \boldsymbol{M}_n & \boldsymbol{0} & \boldsymbol{e}_1\boldsymbol{e}_{d_n}^T \\ \boldsymbol{e}_{d_n}^T & 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_1 & \boldsymbol{M}_n \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \\ x \end{bmatrix} = \rho_{n+1} \begin{bmatrix} \tilde{x} \\ u \\ x \end{bmatrix} . \tag{10}$$

We are also going to need $C_n(z) = -p_{n+1}(-z) = \det(z\boldsymbol{I} - \boldsymbol{M}_n)$. The indices now match: $C_n(\rho_n) = 0$.

Theorem

The solution to equation (10) *can be constructed recursively as follows. Put* $\mathbf{x}_1(\rho) = [1]$*, a one-vector containing a trivial polynomial in* ρ*. Subsequent vectors of dimension* $2^{n+1} - 1$ *are defined by the following polynomial vector recurrence relation:*

$$\mathbf{x}_{n+1}(\rho_{n+1}) = \begin{bmatrix} \rho_{n+1} C_n(\rho_{n+1}) \mathbf{x}_n(\rho_{n+1}) \\ C_n(\rho_{n+1}) \\ \mathbf{x}_n(\rho_{n+1}) \end{bmatrix}. \tag{11}$$

Notice that $\rho_{n+1} \sim 2 - 3\pi^2/8 \cdot 4^{-n-1}$ is quite close to ρ_n, but is not the same.

So we *almost* have a copy of the previous eigenvector in the lower half of the current eigenvector; and we have a *constant multiple* of that in the top half. Finally, $C_n(\rho_{n+1})$ will simplify, because $C_{n+1}(\rho_{n+1}) = \rho_{n+1} C_n^2(\rho_{n+1}) - 1 = 0$.

Talking about the proof

The proof is about a paragraph long in the paper, and contains no real surprises: just plug the form in and work things through. There is something at the end about the characteristic polynomial being worked out “another way” and that’s not uninteresting, but I would rather examine some consequences first. If people want, we can return to the details of the proof of this theorem, at the end of the talk.

Centrality measures

One meaning for the components of the eigenvector are as a "measure of centrality" or influence. So we can see that the graph components have a kind of fractal-looking centrality.

A big graph

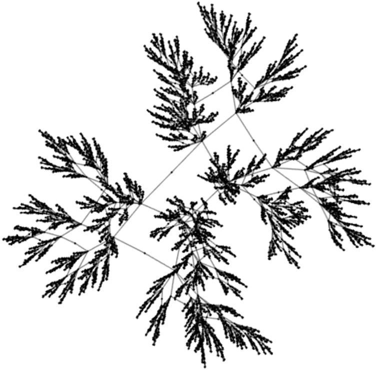


Figure 14: The digraph for M_{13}. A fractal measure of centrality for this doesn't seem surprising!

If we consider just the bottom elements of the eigenvectors, normalized as in the theorem and not in the plots, we see that they appear to *converge* to the sequence $1, 1, 2, 1, 2, 2, 4, 1, 2, 2, 4, 2, 4, 4, 8, \dots$ which is, apparently, *https://oeis.org/A048896*. Powers of two dividing Catalan numbers? Why Catalan numbers?

Mandelbrot and Catalan

The fixed points for the Mandelbrot iteration satisfy $z = cz^2 + 1$, and solving this we get the root we want as:

$$\frac{2}{1 + \sqrt{1 - 4c}} = \sum_{n \geq 0} C_n c^n \tag{12}$$

where the $C_n = \binom{2n}{n}/(n+1)$ are the Catalan numbers. In some sense the Mandelbrot polynomials converge to this generating function in that $z_{n+1}(c)$ has one more term of this series in it than $z_n(c)$ does. In another sense the Mandelbrot polynomials, being of degree $2^n - 1$, mostly are *not* the Catalan generating function (it is that part which is apparently log-concave; the Catalan part is not).

So perhaps this sequence is explainable.

The top end

The *upper* part of the vector is somehow more surprising: the leading entry is

$$x_{n+1,1}(\rho_{n+1}) = \rho_{n+1}^{n} \prod_{k=1}^{n} C_k(\rho_{n+1}) = \frac{2^{n+1}}{\pi}\left(1 + \widetilde{O}(4^{-n})\right) . \tag{13}$$

Note that $\rho_{n+1}C_k(\rho_{n+1})$ for $k = 1, \ldots, n$ are the nonzero elements of the generated periodic orbit of the Mandelbrot set. We also see Gould's sequence `http://oeis.org/A001316` appearing.

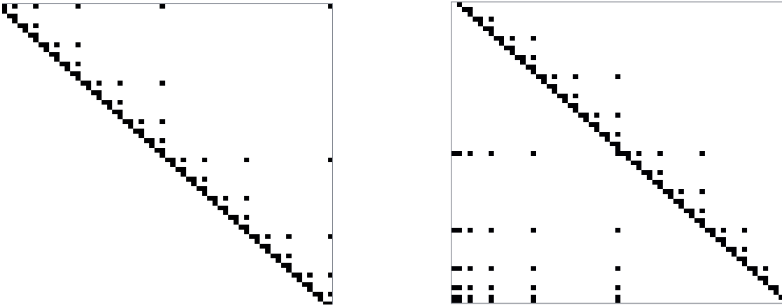
This conjecture was proved by Rhett Robinson in `https://doi.org/10.5206/mt.v1i2.14367`, using Viète's formula. He also managed to connect Gould's sequence to the Catalan numbers, and suggested that the appearance of these sequences could also be explained with Viète's formula.

Rhapsody

If the inverse of a Bohemian matrix is also Bohemian, then we say that the original matrix has *rhapsody*.[3]

[3] This will never not be funny, because Number Theory is the Queen of Mathematics.

Mandelbrot matrices have rhapsody



(a) M_6 (b) M_6^{-1}

Figure 15: The inverse of M_n has entries only -1, 1, and 0. The only proof we have is brutal, and uses the Schur complement twice.

Singular values

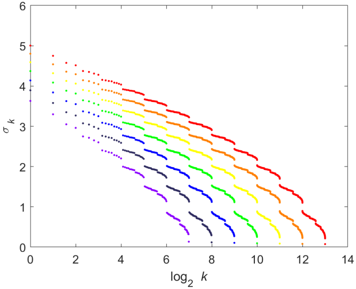


Figure 16: Singular values of M_7 through M_{13}

Singular values are not eigenvalues

Except when they are, of course.

Put $\boldsymbol{S}_n = \boldsymbol{M}_n \boldsymbol{J}$ where $\boldsymbol{J}$ is the anti-identity. Then we can show that $\boldsymbol{S}_n$ is symmetric (but not positive definite) and that its eigenvalues are $(-1)^{k-1}\sigma_k$ where σ_k are the singular values of $\boldsymbol{M}_n$.

Since the entries of $\boldsymbol{S}_n$ are all 1 or zero, its dominant eigenvalue σ_1 is positive and has a positive eigenvector attached to it.

Two things get in the way of using a similar method of analysis: first, we don't have a good asymptotic formula for $\sigma_{1,n}$, and second, the matrix is heavily weighted "in the wings" and so we can't break the eigenvector problem up as neatly as we did for $\boldsymbol{M}_n$.

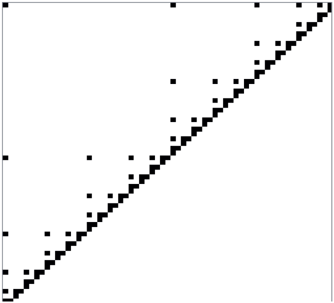
The matrices S_n

If $S_n = M_n J_n$ then we have the following recurrence relation:

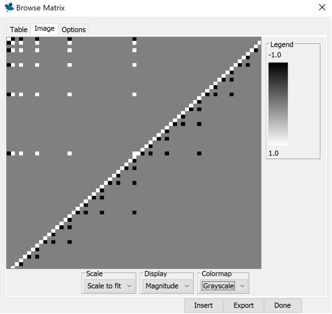
$$S_{n+1} = \begin{bmatrix} \mathbf{e}_1 \mathbf{e}_1^T & \mathbf{0} & S_n \\ \mathbf{0} & 0 & \mathbf{e}_1^T \\ S_n & \mathbf{e}_1 & \mathbf{0} \end{bmatrix} \tag{14}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

Symmetrized Mandelbrot matrices also have rhapsody (of course)



(a) $S_6 = M_6 J$



(b) S_6^{-1}

Figure 17: The inverse of S_n also has entries only -1, 1, and 0.

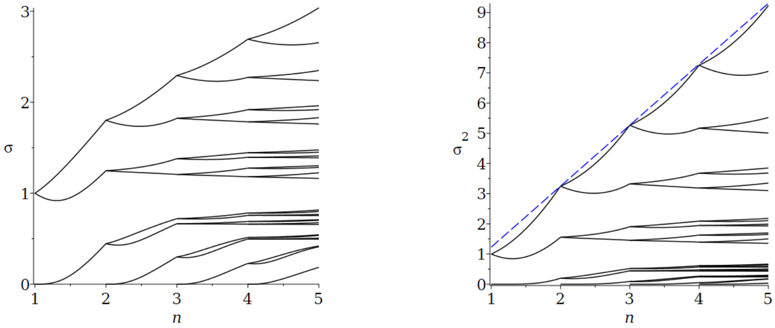
Homotopy continuation

Define the following matrix function:

$$T(s) = \begin{bmatrix} s\boldsymbol{e}_1\boldsymbol{e}_1^T & \mathbf{0} & \boldsymbol{S}_n \\ \mathbf{0} & 0 & s\boldsymbol{e}_1^T \\ \boldsymbol{S}_n & s\boldsymbol{e}_1 & \mathbf{0} \end{bmatrix} \tag{16}$$

At $s = 0$ its distinct eigenvalues will be 0 and $\pm\sigma_{n,k}$. At $s = 1$ its eigenvalues will be $(-1)^{k-1}\sigma_{n+1,k}$ which we want to prove are distinct.

We **conjecture** that the eigenvalues of $T(s)$ do not cross in $0 < s \leq 1$.



(a) Smooth transition to $\sigma_{n,k}$

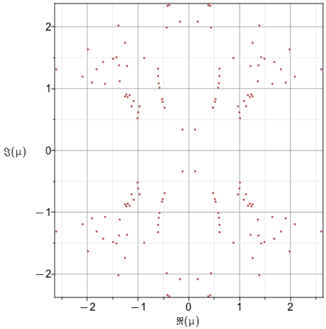
(b) Same graph but with $\sigma_{n,k}^2$.

Figure 18: Absolute values of eigenvalues of $T(s)$ starting from 0 and $\pm\sigma_{n-1,k}$ eventually achieving (in absolute value) the singular values of $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, and so on.

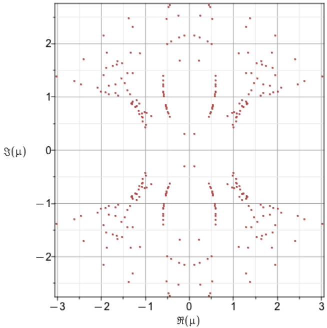
More evidence

Our experiments show that the *discriminant* of the characteristic polynomial with respect to λ is a polynomial in s^2 with positive integer coefficients. If we could prove that, this would prove the conjecture.

Roots of discriminants



(a) Roots of discriminant for $n = 5$



(b) Roots of discriminant for $n = 6$.

Figure 19: No real roots for the discriminant, so no path crossings at all. Unfortunately $n = 7$ is too hard to compute; and we have no proof for general n.

We do not (yet?) have a complete explanation

The dominant *vector* corresponding to the dominant eigenvalue of S_n gives us our original picture. Analyzing S_n^2 along the lines that we did for M_n gets us the closest (the diagonal blocks then give us more information). But I stop here.

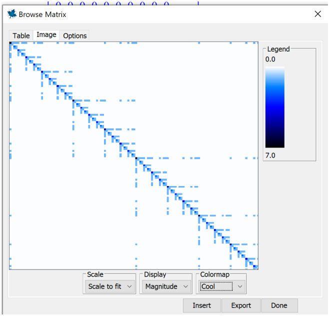


Figure 20: Spying on S_7^2

Thank you!

Happy to take questions!

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