Revisiting the discharge time of a cylindrical leaking bucket

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Abstract

In [1] we find an exploration of a new mathematical model of the flow in a leaking bucket. The model is derived using the non-steady Bernoulli's Principle, and results in a more sophisticated model than the simple ordinary differential equation $\dot{y} = -\sqrt{y}$ derived using the steady Bernoulli's Principle. The simpler model goes sometimes by the name of Torricelli's Law and is very well studied; indeed it is a favourite example in many textbooks. This present paper provides an alternative derivation of the model from [1] that uses an energy balance, and carefully lays out some numerical issues omitted from the treatment in [1]. We also provide an analytic solution in terms of ${}_2F_1$ hypergeometric functions, which, while possibly unfamiliar to the student, are available to them via computer algebra systems. Even before that solution, an intermediate equation $dy/d\tau = -\sqrt{y - y^\beta}$ is derived, which already explains the similarity of the solutions to the more sophisticated model to the ones from the simple Torricelli's Law. This paper is intended for use with students with a bit more maturity and experience than was the intended audience for [1] so it can be regarded as a potential follow-up in a later year of the program, although we recommend attention to the numerical issues even on the first encounter.

1 Introduction

The flow in a leaky bucket is of perennial didactic interest, starting from the simple model $\dot{y} = -\sqrt{y}$ derived from either the steady Bernoulli principle or more prosaically from an energy balance. In [1] we find a derivation of the following more sophisticated model

$$h\frac{d^2h}{d\tau^2} - \frac{1}{2}\beta\Big(\frac{dh}{d\tau}\Big)^2 + gh = 0 \tag{1}$$

with $h(0) = H$, $h'(0) = 0$, using the non-steady Bernoulli principle. The model is then solved numerically and the results compared to experiments carried out by high-school students. There is a lot to like and respect in the paper [1], but there are (we believe) some important omissions: first, the paper [1] does not describe precisely how equation (1) was solved numerically, and this turns out to matter (as we will see) and not just because of the responsibility of scientific authors to ensure reproducibility by others "skilled in the art". There are further gaps, also.

This present paper fills in some of the gaps in [1], and provides a new model: equation (1) is shown to be equivalent to

$$\frac{dy}{d\tau} = -\sqrt{y - y^\beta} \tag{2}$$

with a modified initial condition, and since typically $\beta \gg 1$ and $0 \le y < 1$ this demonstrates the similarity of the solutions to those provided by Torricelli's law. We also provide an analytic solution in terms of a ${}_2F_1$ hypergeometric function.

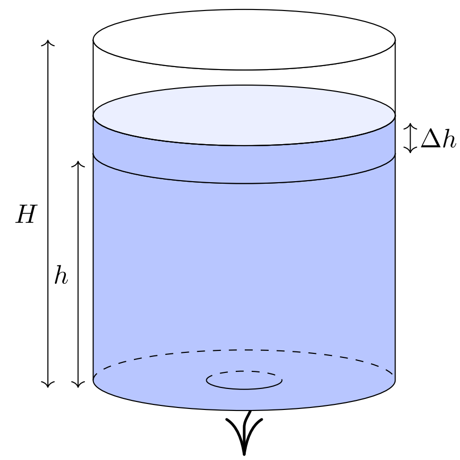
Much of what we present here is intended for students at a somewhat higher level than the discussion in [1] was, say at about second year of university, and so would be most suitable for a follow-on lab, perhaps. We feel that the point about reproducible science, however, can be usefully made even in the first encounter.

We encountered [1] while writing [2], which explores methods for understanding the reliability of numerical methods. Much of the introductory material in the next section is taken nearly verbatim from there, in order to make this present paper self-contained.

2 Torricelli's Law and the Leaky Bucket

Quite a few textbooks use Torricelli's Law, often derived using Bernoulli's Law for steady flow, to write down a differential equation modelling the height of fluid in a bucket with a hole in the bottom. For example, the beautiful book [3]—really one of our favourite textbooks of all time—has an extensive treatment in their section 4.2 "Uniqueness: The Leaking Bucket Versus Radioactive Decay".

Following [3] and in contrast to [1] we use an energy balance to derive an equation for the height of fluid in a leaky bucket.



fluid density ρ, velocity v

The bucket is assumed to be cylindrical with cross-sectional area A, with perfectly vertical walls and a perfectly flat bottom, while the hole in the bottom has an area a. Thus the volume of fluid lost out the hole in a time interval $\Delta\tau$ is $a \cdot v(\tau)\Delta\tau$ (if $\Delta\tau$ is small enough that the velocity $v(\tau)$ can be considered constant). At the same time, the volume lost is $A \cdot h'(\tau)\Delta\tau$ where $h'(\tau) = dh/d\tau$; by conservation these are equal, so

$$av(\tau) = A\frac{dh}{d\tau}. \tag{3}$$

The change in potential energy in this time $\Delta\tau$ is $(\rho A\Delta h)gh$ because the mass is the product of ρ—the density of the fluid—with the volume $A\Delta h$, and the change in energy occurs over the height h because the leak is at the bottom. If there is no energy dissipation (we revisited this assumption in [2]) then this must equal the kinetic energy of the fluid flowing out, $\frac{1}{2}[\rho A\Delta h]v^2$, plus the increase in kinetic energy of the (slowly) downward flowing fluid in the bucket itself, which like the dissipation we ignore. Thus $\rho A\Delta hgh = \frac{1}{2}\rho A\Delta hv^2$ or

$$2gh = v^2 = \left(\frac{A}{a}\frac{dh}{d\tau}\right)^2. \tag{4}$$

Thus at last we have

$$\frac{dh}{d\tau} = -\frac{a}{A}\sqrt{2gh} \tag{5}$$

and we take the negative sign because the height is decreasing over time. Let us now nondimensionalize. Put $y = h/H$ where H is the height of the bucket itself. Then $y = 1$ corresponds to a bucket brim full of liquid while $y = 0$ means the bucket is empty. Dividing (5) by H we have

$$\frac{dy}{d\tau} = -\frac{a}{A}\sqrt{\frac{2g}{H}} \cdot \sqrt{y}\,. \tag{6}$$

Now we rescale time. Put

$$t = \frac{a}{A}\sqrt{\frac{2g}{H}}\,\tau \tag{7}$$

and so by the chain rule $\frac{dy}{d\tau} = \frac{dt}{d\tau}\,\frac{dy}{dt}$ and the scale factors cancel leaving

$$\frac{dy}{dt} = -\sqrt{y}\,. \tag{8}$$

The initial condition is thus $y(0) = y_0$, the initial fraction full: $0 \le y_0 \le 1$. We will almost always take $y_0 = 1$.

Hubbard and West [3] use this example to great effect for studying uniqueness of solutions. Given an empty bucket, there's no way to tell when it was full—if it ever was. This fact is well-displayed in the model, as we will see. The reference solution[1] of this separable equation is easy: $y(t) = (\sqrt{y_0} - t/2)^2$.

This solution is valid only so long as $y > 0$. Investigating when $y = 0$ separately we have $\frac{dy}{dt} = 0$ thereafter, so

$$y(t) = \begin{cases} \left(\sqrt{y_0} - t/2\right)^2 & 0 \le t < 2\sqrt{y_0} \\ 0 & 2\sqrt{y_0} \le t \end{cases} \tag{9}$$

and by inspection this is continuous, even differentiable, at $T = 2\sqrt{y_0}$. In dimensional terms, the discharge time when $y_0 = 1$ is predicted to be

$$\tau_{\mathrm{T}} = \frac{A}{a}\sqrt{\frac{H}{2g}}\cdot 2 = \frac{A}{a}\sqrt{\frac{2H}{g}}\,. \tag{10}$$

Notice that $y(t) \ge 0$ always, and of course this is necessary for the $\sqrt{y(t)}$ in equation (8) to be real.

All this is fine, and educational. We don't even need numerical methods for this problem. But, for other equations, we do; and we claim it's worthwhile to have a look at what numerical methods do on this nonlinear problem as we do in [2]. Notice one more fact about this equation: the differential equation is *not* Lipschitz continuous at $y = 0$, because $\frac{\partial}{\partial y} - \sqrt{y} = -1/(2\sqrt{y})$ is infinite there; hence there is no finite L such that $|\sqrt{y_1} - \sqrt{y_2}| \le L \cdot |y_1 - y_2|$. This means that the classical existence and uniqueness theorems fail to apply, see e.g theorems 5 and 6, p 28 of [4]. Indeed this is the whole point of Section 4.2 of [3]. Moreover, the hypotheses for some theorems, see e.g. theorem 3.4 in [5], or theorem 203A in [6], about the convergence of numerical methods therefore *also* must fail to hold. Solving an ODE numerically when it's not Lipschitz continuous is an issue!

3 A more complicated model

In [1] we find (1) for the height $h(\tau)$ of fluid in a leaky bucket, where $\beta = (A/a)^2 - 1$, and initial conditions $h(0) = H$ and $h'(0) = 0$. The new initial condition $h'(0) = 0$ is used without comment. The derivation in [1] uses Bernoulli's law for non-steady flow, which seems appropriate for the audience. Here, we simply refine the energy balance argument used to derive Torricelli's Law previously in [2] and in [3] by noting that some of the loss of potential energy goes into accelerating the fluid in the bucket itself, increasing its kinetic energy from $h/2 \cdot \rho A(h'(\tau))^2$ to $1/2 \cdot \rho A h(\dot{h}(\tau + \Delta\tau))^2$. The change, to first order, is $\rho A h(\tau)\dot{h}(\tau)\ddot{h}(\tau)\Delta\tau$, and using $\dot{h}\Delta\tau = \Delta h$ we get $(\rho A \Delta h)h\ddot{h}$. The potential energy term is $\rho A g h \Delta h$ as before. The only other new term, the -1 in the definition of β , is because of the difference in velocity at the outlet from the velocity at the top, but we will see it is not very significant except for small β , which corresponds to relatively large holes.

[1] We use the phrase "reference solution" instead of the more common "exact solution", because starting in [2] we find everything is an "exact solution" of some relevant equation.

Indeed $\beta = 0$ corresponds to $A/a = 1$, *ie* a bucket with no bottom and just gives ballistic flow $\ddot{h} = -g$. Other mathematically but not physically interesting values of β will arise in the solution process, but for now note that a $1cm$ diameter hole in a $10cm$ diameter bucket, which is a relatively large hole, gives $\beta = 10^4 - 1$. Smaller holes will give even larger β, because $\beta \sim (A/a)^2 \propto R^4/r^4$.

Needless to say, equation (1) is still greatly simpler than the true physical situation. The theoretical streamlines used in [1] to derive the equation can be expected to differ from the true flow near the walls, the corners, and along the bottom of the bucket. The model assumptions especially break down as $h \to 0$, when lubrication theory likely gives a better description. Nonetheless we will not criticize this model further on physical grounds. It is, after all, only a model.

We do, however, suggest that the presentation in [1] could be better, in two ways. The first omission is that [1] gives no details on how they solve equation (1), merely stating that equation (1) "needs to be solved numerically". Although many works do not give such details, relying on the reader to be "skilled in the art", this is not reproducible science. In our opinion, people wishing to reproduce results need at least the name of the package used and the parameter values such as error tolerances used. Guidelines can be found now in several places, and they should be acknowledged especially in a didactic example. For instance, Principle 2 in [7], p2 states that "authors should include the data, algorithms, or other information... that would enable one skilled in the art to verify or replicate the claims."

This matters here in this example especially, because of the second omission: it is not noted in [2] that equation (1) is, like Torricelli's Law, not Lipschitz continuous at $h = 0$, and thus numerical methods can be expected to encounter trouble. For instance, using Maple's numerical solvers (originally described in [8] but substantially improved by Allan Wittkopf at Maplesoft) with $H = 1$, $A/a = 100$, $g = 9.8$ and default parameters otherwise,

```
> de := h(t)*(diff(h(t), t, t))-(1/2)*beta*(diff(h(t), t))^2+g*h(t)
> H := 1; beta := 10^4-1; g := 9.8;
> nonstiffsol := dsolve({de, h(0) = H, (D(h))(0) = 0}, h(t), numeric);
> p1 := plots[odeplot](nonstiffsol, 0 .. 50, colour = BLACK, style = POINT):
```

yields "Warning, cannot evaluate the solution further right of 24.560565, maxfun limit exceeded", which suggests that the problem is stiff. Trying with the stiff option,

```
> stiffsol := dsolve({de, h(0) = H, (D(h))(0) = 0}, h(t), numeric, stiff = true);
> p2 := plots[odeplot](stiffsol, 0 .. 50, colour = BLUE);
> plots[display]({p1, p2});
```

yields "Warning, cannot evaluate the solution further right of 45.174001, probably a singularity."

This second warning message correctly identifies the time when the bucket empties, τ_d. See equation (33).

Other solvers may behave differently. The point is that the presentation in [1] would have been better if they had described what they did, not 'merely' to conform to good standards of practice for reproducible research, but because it matters in this case. The lack of Lipschitz continuity (or the disappearance of the $h\ddot{h}$ term when $h = 0$, making this a singularly-perturbed problem if you like) really is an issue.

Remark: It might seem we are being unfairly harsh in our treatment of [1]. There are good things in that paper, after all, including the involvement of high-school students for data collection, and it is only relatively recently that the community has become energized about reproducible research, namely since about 2003 when the earliest references used in [7] were published. And indeed there are many others who have been guilty of similar omissions—certainly RMC hasn't been as careful in the past as he might have been—but we feel it's time to start encouraging better practice, even or perhaps especially in didactical journals.

It is somewhat ironic that this is all unnecessary, in that equation (1) can be solved analytically as follows. It did not "need" to be solved numerically, after all. Moreover, the solution process gives as an intermediate

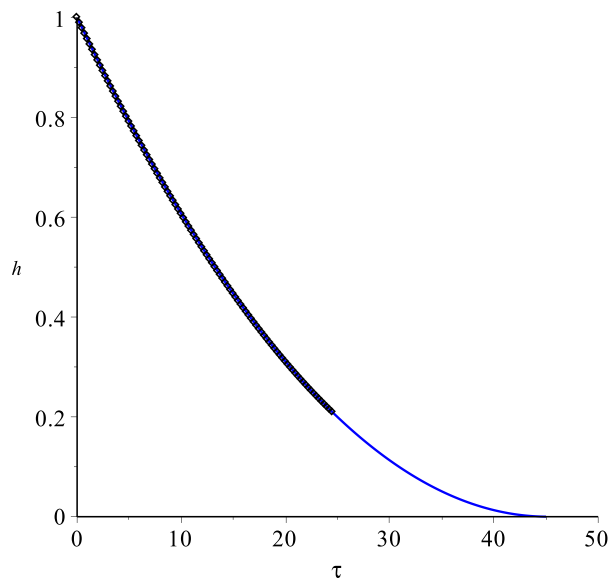


Figure 1: Two numerical solutions of (1) with $h(0) = 1, \dot{h}(0) = 0, g = 9.8, \beta = 9999.0$, and default tolerances by Maple's dsolve with the numeric option. The nonstiff solver (based on RKF45) is cut off by exceeding its work limit, suggesting the problem is stiff. The heavy black line shows how far it got. The stiff solver, which uses a Rosenbrock method, is successful, and correctly reports a singularity when the bucket empties at $\tau \doteq 45.174$.

stage an equation very like Torricelli's Law, that we may think of as a new model, which is the primary reason we include this discussion here.

One may begin the analytical solution by noticing that equation (1) is autonomous, and therefore by using a standard trick (well known to physicists—indeed RMC first learned it in a physics class) we may reduce the second order equation to a sequence of two independent first order equations[2]. The trick is to introduce a new variable, say v, for the velocity and then use the chain rule:

$$\frac{d^2h}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dh}{d\tau}\right) = \frac{d}{d\tau}(v) = \frac{dh}{d\tau}\,\frac{dv}{dh} = v\,\frac{dv}{dh}\,. \tag{11}$$

This trick will not work when $v = 0$, as it is initially here, so we will have to be a bit more careful for small τ. For that, we use Taylor series. It is straightforward to compute the first few terms of the Taylor series solution to (1) valid for small τ: $h(t) = z(t) + \mathcal{O}(\tau^6)$ where the polynomial

$$z(\tau) = H - \frac{1}{2}g\tau^2 + \frac{1}{24}\frac{\beta g^2}{H}\tau^4 \tag{12}$$

has residual

$$\Delta = z\ddot{z} - \frac{1}{2}\beta(\dot{z})^2 + gz = \frac{1}{12}\frac{g^3\beta(2\beta-3)}{H}\tau^4\left(1 - \frac{g\beta}{12H}\tau^2\right) \tag{13}$$

exactly.

We see that this is $\mathcal{O}(\tau^4)$ and can be made as small as we like by restricting our attention to a small enough initial interval $0 \le \tau \le T$ with $T = \mathcal{O}(\beta^{-1/2})$, say.

Maple commands to do this are

[2]According to [9], page 140, this trick is due to Riccati 1712 [10].

```
> de := h(t)*(diff(h(t),t,t))-(1/2)*beta*(diff(h(t),t))^2+g*h(t);
> dsolve({de,D(h)(0) = 0, h(0) = H}, h(t), series);
```

Maple commands to pick out $z(t)$ and compute the residual are

```
> z:= convert(rhs(%), polynom)
> residual:= factor(eval(de, h(t) = z))
```

Remark: If $\beta = 3/2\,, \Delta \equiv 0$ and we have found an exact solution. We do not pursue this here. In fact we henceforth assume $\beta > 3/2$.

Remark: If τ is small enough, $h(\tau) \sim H - 1/2 \cdot g\tau^2$ which is ballistic fall. This somehow accords with intuition, but in fact the initial fall of a real fluid would depend on just how the hole is opened. We now have that $h(\tau) = H - 1/2 \cdot g\tau^2 + \mathcal{O}(\tau^4)$ and $v = -g\tau + \mathcal{O}(\tau^3)$ initially. This will allow us to take care of the actual initial state at $\tau = 0$ while using $d^2h/d\tau^2 = v\, dv/dh$. Using this in equation (1) we get

$$h \cdot v\,\frac{dv}{dh} - \frac{1}{2}\beta v^2 + gh = 0 \tag{14}$$

which is a nonlinear first order equation. Noticing that $v\dfrac{dv}{dh} = \dfrac{1}{2}\dfrac{dv^2}{dh}$ we put $u = v^2$, getting

$$\frac{1}{2}h\frac{du}{dh} - \frac{1}{2}\beta u + gh \tag{15}$$

which is *linear* in u. Using the integrating factor $2h^{-\beta-1}$ we get

$$h^{-\beta}\frac{du}{dh} - \beta h^{-\beta-1}u = -2gh^{-\beta} \tag{16}$$

or

$$\frac{d}{dh}(h^{-\beta}u) = -2gh^{-\beta}\,. \tag{17}$$

Integrating this from $\tau = 0\,(h = H)$ to $\tau = \tau\,(h = h)$ and using $u = v^2 = 0$ at $\tau = 0\,(h = H)$ by continuity we get

$$h^{-\beta}u = -2g\left(\frac{h^{1-\beta}}{1-\beta} - \frac{H^{1-\beta}}{1-\beta}\right) \tag{18}$$

(note $\beta > 3/2 > 1$). Solving for u,

$$u = \frac{2g}{\beta - 1}\left(h - \left(\frac{h}{H}\right)^{\beta} H\right). \tag{19}$$

Now

$$u = (v)^2 = \left(\frac{dh}{d\tau}\right)^2 = \left(H\frac{dy}{d\tau}\right)^2 \tag{20}$$

if $y = h/H$, so

$$H^2\left(\frac{dy}{d\tau}\right)^2 = \frac{2gH}{\beta - 1}(y - y^{\beta})\,. \tag{21}$$

Putting

$$t = \sqrt{\frac{2g}{H(\beta - 1)}}\,\tau \tag{22}$$

(which is a slightly different scaling than we used for Torricelli's Law) we get

$$\left(\frac{dy}{dt}\right)^2 = y - y^{\beta}\,. \tag{23}$$

Taking the negative square root, we get an equation very like (8):

$$\frac{dy}{dt} = -\sqrt{y - y^\beta}\,. \tag{24}$$

This equation is valid for $t > 0$, and the rescaled Taylor series is

$$y = \frac{h}{H} = \frac{(H - \frac{1}{2}g\tau^2 + \mathcal{O}(\tau^4))}{H} = 1 - \frac{(\beta-1)}{4}t^2 + \mathcal{O}(t^4)\,. \tag{25}$$

Remark: This equation is not Lipschitz continuous when $y = 0$, but it also is not Lipschitz continuous if $y = 1$, either. In particular the solution to the initial value problem $\dot{y} = -\sqrt{y - y^\beta}$, $y(0) = 1$ is not unique; one can have $y = 1$, which is spurious, or $y = 1 - (\beta - 1)/4 \cdot t^2 + \mathcal{O}(t^4)$ as above.

In a typical situation, $\beta > 10^4$ or so. Once the fluid falls a tiny amount, say 0.1% of its initial height, to $y \le 0.999$, then

$$y^\beta \le \left(1 - \frac{10}{10^4}\right)^{10^4} \doteq e^{-10} \doteq 4.5 \cdot 10^{-5} \tag{26}$$

In this situation, the difference between this equation and Torricelli's law is surely negligible, although the time scale, as previously noted, is, slightly different. The scale factor for Torricelli's law is

$$\frac{a}{A}\sqrt{\frac{2g}{H}} \tag{27}$$

whereas here it is

$$\sqrt{\frac{2g}{H(\beta-1)}} = \sqrt{\frac{2g}{H((\frac{A}{a}^2) - 2)}} = \frac{a}{A}\sqrt{\frac{2g}{H}}\left(1 - 2\left(\frac{a}{A}\right)^2\right)^{-\frac{1}{2}} \tag{28}$$

for the model derived in [1]. For $a/A \le 100$, this differs by a relative amount < 0.0001 .

This analysis is already useful. In [1] they find a negligible difference between the numerical solution to equation (1) and the Torricelli's law solution for $\beta \gg 1$. This analysis explains why that could have been expected. We also see more clearly that for small τ the solution is initially flat, in contrast to the Torricelli's law solution, but that this flatness is confined to a small time interval $0 < t < \mathcal{O}(\beta^{-1/2})$.

At this point we don't need more analytical work, but it's satisfying to note that more is possible. Maple can be coaxed into solving equation (24) in terms of hypergeometric functions (specifically using what physicists call ${}_2F_1$ hypergeometric functions). Indeed solution by hand is also possible, by expanding $(y = y^\beta)^{-1/2} = \sqrt{y}(1 - y^{\beta-1})^{-1/2}$ in series and integrating term by term. We get the implicit relation

$$0 = t + 2\sqrt{y}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & \frac{1}{2(\beta-1)} \\ 1 + \frac{1}{2(\beta-1)} & \end{matrix}\,\right|\, y^{\beta-1}\right) - 2\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & \frac{1}{2(\beta-1)} \\ 1 + \frac{1}{2(\beta-1)} & \end{matrix}\,\right|\, 1\right)\,. \tag{29}$$

These can be plotted easily, giving all t values for $0 \le y \le 1$. See figure 2.

The discharge time, when the bucket empties at $y = 0$, is immediately available:

$$t_d = {}_2F_1\left(\left.\begin{matrix}\frac{1}{2} & \frac{1}{2(\beta-1)} \\ 1 + \frac{1}{2(\beta-1)} & \end{matrix}\,\right|\, 1\right)\,. \tag{30}$$

Separate (and somewhat artful) use of Maple can find the simpler expression

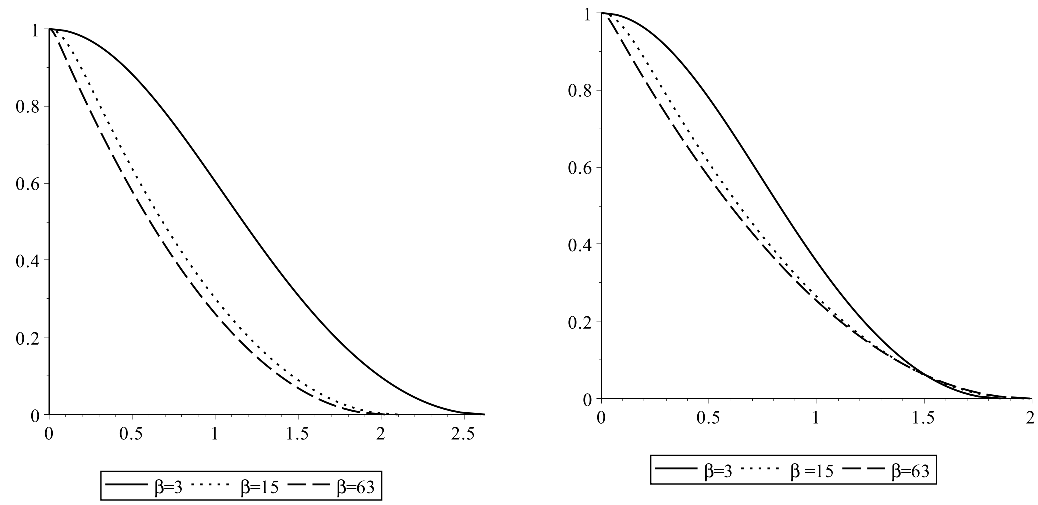


Figure 2: The analytical solution to equation (1). In the right hand graph, we normalize to the Torricelli's law time scale by using equation (36). Maple's routines for evaluating hypergeometric functions work well for $\beta > 1$.

$$t_d = \frac{2\sqrt{\pi}\,\Gamma(1+\frac{1}{2(\beta-1)})}{\Gamma(\frac{1}{2}+\frac{1}{2(\beta-1)})} \tag{31}$$

which can be expanded for large β as

$$t_d = 2 + \frac{2\ln 2}{\beta-1} + \frac{1}{2}\left(2\ln^2 2 - \frac{\pi^2}{6}\right)\cdot\frac{1}{(\beta-1)^2} + \cdots \tag{32}$$

To find the dimensional discharge time, we divide by the scale factor:

$$\tau_d = \sqrt{\frac{H(\beta-1)}{2g}}\left(\frac{2\sqrt{\pi}\,\Gamma(1+\frac{1}{2(\beta-1)})}{\Gamma(\frac{1}{2}+\frac{1}{2(\beta-1)})}\right). \tag{33}$$

These two models have slightly different nondimensional time scales. This must be corrected for when we compare the model predictions. We could change both back to dimensional time, but this unnecessarily reintroduces $\sqrt{2g/H}$; the two models times differ only when the ratio (A/a) is changed. Suppose t_1 is the dimensionless time for equation (24):

$$t_1 = \frac{a}{A}\sqrt{\frac{2g}{H}}\,\tau \tag{34}$$

where τ is real, dimensional time, and suppose t_2 is the dimensionless time in (24):

$$t_2 = \sqrt{\frac{2g}{H(\beta-1)}}\,\tau\,. \tag{35}$$

Then

$$t_2 = \frac{A/a}{\sqrt{\beta-1}}\,t_1 \tag{36}$$

and we can plot all graphs (Torricelli's Law included) on the same scale and compare the predictions. See figure 2. This raises one further comment of [1], namely that when comparing their model with experiment they didn't use the formula $\beta = (A/a)^2 - 1$ but rather choose β so that the discharge time fit the

the experimental data (or alternatively Torricelli's law). This gave what they called an "effective" value of β.

We believe this is less satisfactory, for several reasons. First, these models can't be expected to fit the data well when h is small; there is thus reason to believe the predicted discharge times will be off. The models ought to give good agreement (and they do) when h is not so small. So, fitting the data by choosing β to match the discharge time turns the modelling process into curve fitting. In our opinion, it would have been more satisfactory to see how well the model did with $\beta = \left(\frac{A}{a}\right)^2 - 1$. Indeed equation (1) predicts, for $\beta = 3$, a nondimensional discharge time of $t \doteq 1.85$, earlier by $\sim 7\%$ than the time predicted by Torricelli's law. In fact the model from [1] always predicts $O(1/\beta)$ faster discharge than Torricelli's law does.

4 Conclusion

[1] is actually a very nice paper. Its inclusion of high school students is excellent, for example. Some of this present paper, filling in gaps in [1], would not be expected to be accessible to high school students. However, the numerical issues (Lipschitz continuity) do need a mention; and the analytical solution to $y' = -\sqrt{y - y^\beta}$ is possibly useful.

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