Optimal Backward Error and the Leaky Bucket

or, Variations on a Theme of Euler

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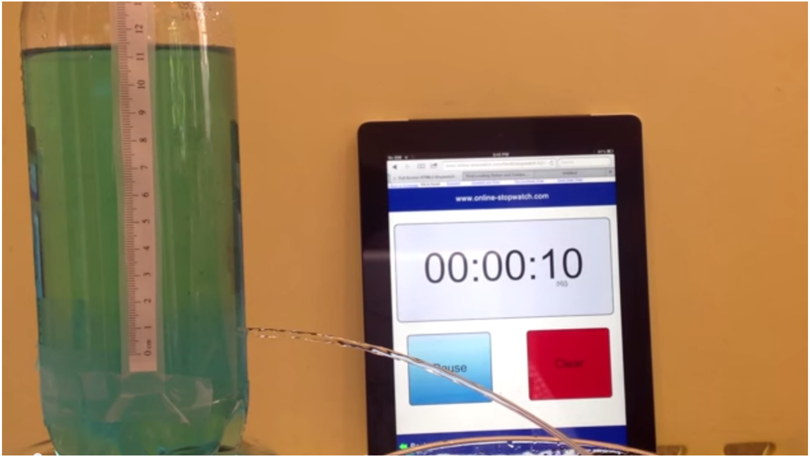
SIMIODE

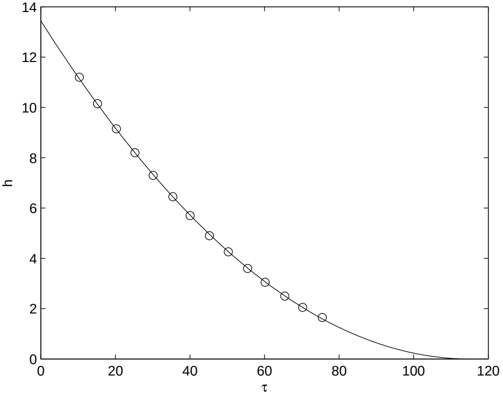
Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations



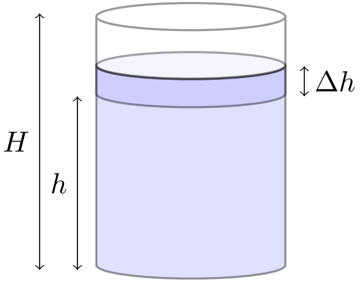
https://www.youtube.com/watch?v=6cUuNUjlaPI

Data collection: Torricelli's Law

τ (s)	$\mathbf{h}$ (cm)
10.306	11.20
15.184	10.15
20.265	9.15
25.241	8.20
30.117	7.30
35.402	6.45
40.026	5.70
45.205	4.90
50.286	4.26
55.472	3.60
60.144	3.05
65.425	2.50
70.202	2.05
75.489	1.65



Derivation of the equation
(Hubbard & West, later Blasone et al., EJP, 2015).



fluid density ρ

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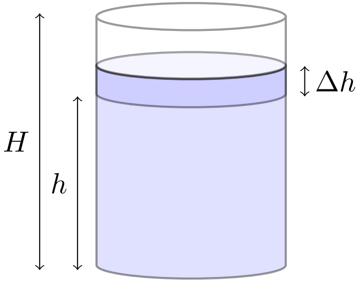

fluid density ρ

We are neglecting:

- friction/dissipation
- change in momentum of the fluid in the bucket

$$\frac{1}{2}(\rho A h)\left[\left(\frac{dh}{d\tau}+\Delta\frac{dh}{d\tau}\right)^2-\left(\frac{dh}{d\tau}\right)^2\right]$$

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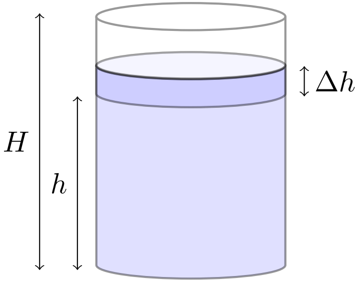

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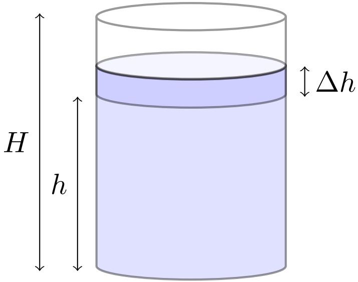


fluid density ρ

In time, $\Delta\tau$, water level falls Δh.

Loss of potential energy is
$(\rho A \Delta h) h g$

$\therefore$ kinetic energy of leaking water
$= \frac{1}{2}(\rho A \Delta v) v^2$

Continuity $\quad \rho A \Delta h = \rho a \Delta v$

$\therefore \frac{dv}{d\tau} = \frac{A}{a}\frac{dh}{d\tau}$

$\therefore gh = \frac{1}{2}v^2 = \frac{1}{2}\left(\frac{A}{a}\right)^2\left(\frac{dh}{d\tau}\right)^2$ or $\left(\frac{dh}{d\tau}\right)^2 = \left(\frac{a}{A}\right)^2 2gh$

$\therefore \frac{dh}{d\tau} = -\left(\frac{a}{A}\right)\sqrt{2g}\sqrt{h}$

Nondimensionalizing: $y = \frac{h}{H} \qquad t = \frac{a}{A}\sqrt{\frac{2g}{H}}\tau$

$$\frac{dy}{dt} = -\sqrt{y} \qquad y(0) = 1 \quad \text{brim full}$$

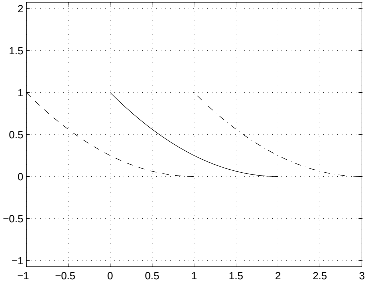
Nondimensionalizing: $y = \frac{h}{H}$ $\qquad t = \frac{a}{A}\sqrt{\frac{2g}{H}}\tau$

$$\frac{dy}{dt} = -\sqrt{y} \qquad y(0) = 1 \quad \text{brim full}$$

Analytical solution:

$$y = \begin{cases} (\sqrt{y_0} - \frac{t}{2})^2 & 0 \le t \le 2\sqrt{y_0} \\ 0 & 2\sqrt{y}_0 \le t \end{cases}$$

- Not Lipschitz continuous at $y = 0$
- Solutions not unique looking backwards "when was the bucket full?"



Forward Euler

$$y_{n+1} = y_n - \Delta t \sqrt{y_n} \qquad y_0 = 1$$

Forward Euler

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Take $\Delta t = 1$.

Then $y_1 = 1 - 1 \cdot \sqrt{1} = 0$ and the bucket empties in one step.

Possibly this Δt is too large.

Forward Euler

$$y_{n+1} = y_n - \Delta t\sqrt{y_n} \qquad y_0 = 1$$

Take $\Delta t = \frac{2}{1+\sqrt{5}} = 0.618\cdots$.

Then $y_1 = 1 - \frac{2}{1+\sqrt{5}}\sqrt{1} = \frac{4}{(1+\sqrt{5})^2}$

and $y_2 = \frac{4}{(1+\sqrt{5})^2} - \frac{2}{1+\sqrt{5}} \cdot \frac{2}{1+\sqrt{5}} = 0.$

So the bucket empties in **two** steps.

Forward Euler

$$y_{n+1} = y_n - \Delta t \sqrt{y_n} \qquad y_0 = 1$$

Take instead $\Delta t = \frac{2}{1+\sqrt{7+2\sqrt{5}}}$.

I claim that the bucket empties in **three** steps.

With $\Delta t = \frac{2}{1+\sqrt{9+2\sqrt{7+2\sqrt{5}}+2\sqrt{5}}}$, the bucket empties in four steps.

Forward Euler

$$y_{n+1} = y_n - \Delta t \sqrt{y_n} \qquad y_0 = 1$$

Now try $\Delta t = 1/2$:

$$y_1 = 1 - \frac{1}{2}\sqrt{1} = \frac{1}{2}$$

$$y_2 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{2}} \doteq 0.146446$$

$$y_3 \doteq -0.04489$$

oops, a negative amount of fluid...

y_4 is complex...

(Colin Denniston studies complex fluids)

Forward Euler

$$y_{n+1} = y_n - \Delta t \sqrt{y_n} \qquad y_0 = 1$$

In fact, **for almost all choices of** Δt, eventually $y_n < \Delta t^2$ for some n; then $y_{n+1} < 0$ and $y_{n+1} \notin \mathbb{R}$!

Forward Euler **generically** terminates this way for this problem.

Backward Euler is much better:

$$y_{n+1} = y_n - \Delta t \sqrt{y_{n+1}} \quad \Leftrightarrow \quad y_{n+1} = \frac{4y_n^2}{(\Delta t + \sqrt{(\Delta t)^2 + 4y_n})^2}$$

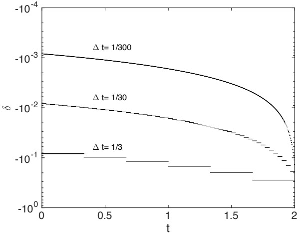
- Always > 0 mathematically
- Ultimately $y_{n+1} \propto y_n^2$ (like error in Newton's method)
- Eventually underflows to zero (e.g. $y_n \doteq 10^{-160}$, $y_{n+1} \equiv 0$).

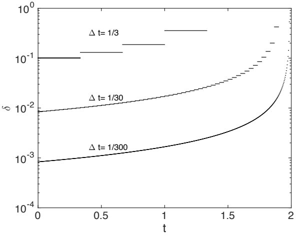
The problem $\dot{y} = -\sqrt{y}$ is indeed stiff for small y.

$$
\begin{aligned}
&\frac{dz}{dt} = -\sqrt{z(t)}(1+\delta(t)) \\
-&\frac{1}{\sqrt{z}}\frac{dz}{dt} - 1 = \delta(t) \\
&\int_{s=t_n}^{t_{n+1}} -\frac{1}{\sqrt{z(s)}}\dot{z}(s)\ ds - \int_{s=t_n}^{t_{n+1}} 1\ ds = \int_{s=t_n}^{t_{n+1}} \delta(s)\ ds \\
&2\sqrt{y_n} - 2\sqrt{y_{n+1}} - \Delta t = \int_{s=t_n}^{t_{n+1}} \delta(s)\ ds \\
&|2\sqrt{y_n} - 2\sqrt{y_{n+1}} - \Delta t| = \Big| \int_{t_n}^{t_{n+1}} \delta(s) ds \Big| \le \Delta t \cdot \|\delta(t)\|_\infty
\end{aligned}
$$

with equality if

$$
\delta_n = \frac{2}{\Delta t}(\sqrt{y_n} - \sqrt{y_{n+1}}) - 1\ .
$$


-10^-4
-10^-3
-10^-2
-10^-1
-10^0
δ
0
0.5
1
1.5
2
t
Δ t= 1/300
Δ t= 1/30
Δ t= 1/3


10^0
10^-1
10^-2
10^-3
10^-4
δ
0
0.5
1
1.5
2
t
Δ t= 1/3
Δ t= 1/30
Δ t= 1/300

The method of modified equations (Griffiths & Sanz-Serna 1986, and also Corless & Fillion 2013)

- Start with a discrete scheme, e.g., $y_{n+1} = y_n - \Delta y \sqrt{y_n}$.
- Interpolate with $z(t)$, the solution of a **functional** equation

$$z(t + \Delta t) = z(t) - \Delta t \sqrt{z(t)}\,.$$

- Convert the functional equation to a singular-perturbed infinite-order ODE by Taylor series:

$$\begin{aligned} z(t) + \dot{z}(t)\Delta t + \frac{1}{2}\ddot{z}(t)(\Delta t)^2 + \cdots &= z(t) - \Delta t\sqrt{z(t)} \\ \dot{z}(t) + \frac{1}{2}\ddot{z}(t)\Delta t + \cdots &= -\sqrt{z(t)} \end{aligned}$$

- Truncate, differentiate & substitute to get a (finite order) modified equation.
- Other series operations, e.g. reciprocation, are useful.

Modified equations for $\dot{y} = -\sqrt{y}$ by

Forward Euler:

$$\left(-\frac{1}{\sqrt{z}} + \frac{\Delta t}{4z}\right)\frac{dz}{dt} = 1$$

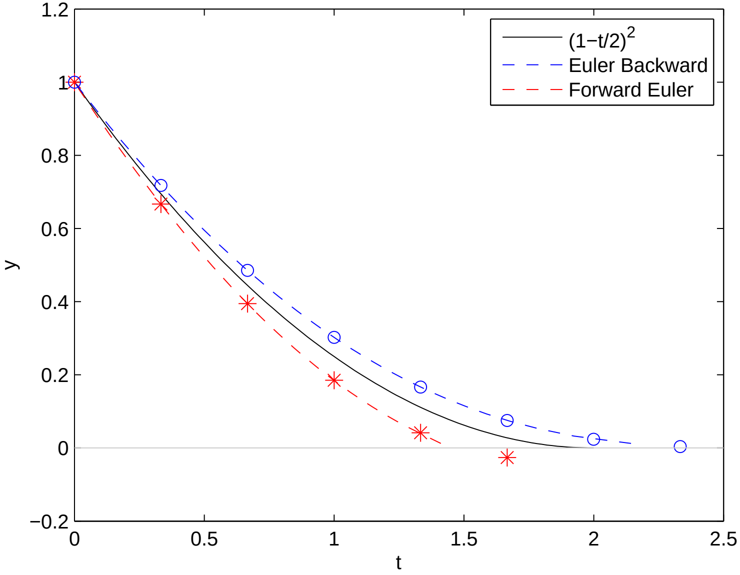
Euler Backward:

$$\left(-\frac{1}{\sqrt{z}} - \frac{\Delta t}{4z}\right)\frac{dz}{dt} = 1$$

Both accurate to $O(\Delta t)^3$

Solutions: $\quad 2\sqrt{y_0} - 2\sqrt{z(t)} \pm \frac{\Delta t}{4}\ln\left(\frac{z(t)}{y_0}\right) = t$

$z(t) \to 0^+ \quad \Leftrightarrow \quad t \to \infty$ if $-$ sign is used.



1.2
1
0.8
0.6
0.4
0.2
0
−0.2
y
0
0.5
1
1.5
2
2.5
t
(1−t/2)^2
Euler Backward
Forward Euler

Back to the physics

Two “refinements” to the model (the first from Blasone et al., EJP, 2015, 36, “Discharge time of a cylindrical leaking bucket”)

$$h\frac{d^2h}{d\tau^2} \quad + \frac{1}{2}\left(1 - \left(\frac{A}{a}\right)^2\right)\left(\frac{dh}{d\tau}\right)^2 + gh = 0$$

correction for difference in velocity top to bottom

correction for accelerating the fluid in the bucket itself

or a crude dissipation model (friction $\propto$ velocity):

$$gh = \frac{1}{2}v^2 - \varepsilon v \quad \Rightarrow \quad -\left(\frac{\sqrt{\mu^2 + y}}{y} + \frac{\mu}{y}\right)\frac{dy}{dt} = 1 \qquad \mu = \frac{\varepsilon}{\sqrt{2gH}}$$

up to $O(\mu^2)$, similar in form to Backward Euler with $\mu = \Delta t/4$.

$$h\ddot{h} - \frac{1}{2}\beta\dot{h}^2 + gh = 0 \qquad h(0) = H, \dot{h} = 0 \quad \text{(unmotivated)}$$

Blasone et al. say "solve this numerically"

(Argh! Irreproducible science)

(Still not Lipschitz cts at $h = 0$)

$$h\ddot{h} - \frac{1}{2}\beta\dot{h}^2 + gh = 0 \qquad h(0) = H, \dot{h} = 0 \quad \text{(unmotivated)}$$

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(Still not Lipschitz cts at $h = 0$)

We can solve this analytically! (by Maple)

Take $y = \frac{h}{H}$ and $t = \sqrt{\frac{2g}{(\beta-1)H}}\tau$.

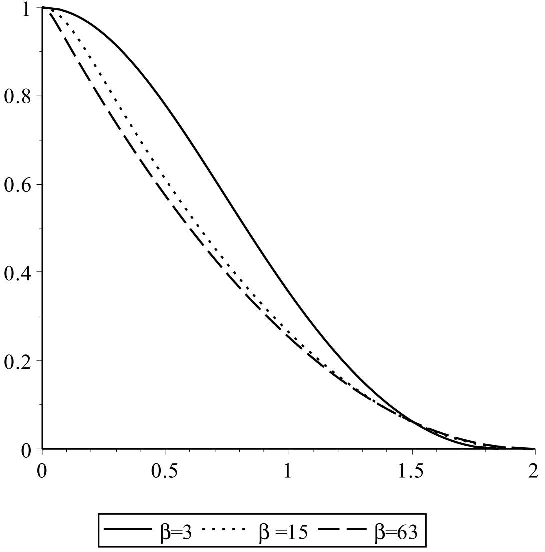
$$\frac{dy}{dt} = -\sqrt{y - y^\beta} \qquad \beta = \left(\frac{A}{a}\right)^2 - 1$$

$$\Rightarrow t + 2\sqrt{y}F\left(\left.\begin{array}{c} \frac{1}{2}, \frac{1}{2(\beta-1)} \\ 1 + \frac{1}{2(\beta-1)} \end{array}\right| y^{\beta-1}\right) - 2F\left(\left.\begin{array}{c} \frac{1}{2}, \frac{1}{2(\beta-1)} \\ 1 + \frac{1}{2(\beta-1)} \end{array}\right| 1\right)$$

Discharge time ($y=0$) at

$$t_e = 2F\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2(\beta-1)} \\ 1+\frac{1}{2(\beta-1)} \end{array} \middle| \; 1\right) \sim 2 + 2\ln 2\frac{1}{\beta} + O(\frac{1}{\beta^2})$$

Typically $\beta \sim 10^4$ish; for $y < 0.999$, $y^\beta < 10^{-11}$ and so $\dot{y} = \sqrt{y - y^\beta}$ is indistinguishable from $\dot{y} = -\sqrt{y}$.



1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
β=3
β =15
β=63

Concluding remarks:

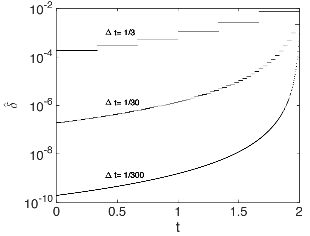
- Can 'close off' modified equations by seeking optimal backward error δ_n such that

$$\left(-\frac{1}{\sqrt{z}} - \frac{\Delta t}{4z}\right)\frac{dz}{dt} = 1+\delta_n(t) \qquad z(t_n) = y_n, z(t_{n+1}) = y_{n+1}$$

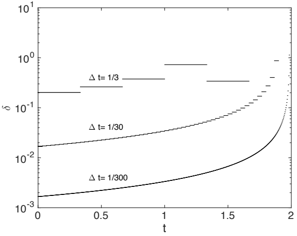
- Numerical solution is the **exact** solution of

$$\left(-\frac{1}{\sqrt{y}} - \frac{\Delta t}{4y}\right)\frac{dy}{dt} = 1 + \delta_n(t) \qquad t_n \leq t \leq t_{n+1}$$

  Typically, δ_n is piecewise constant, $O(\Delta t)^3$ in size.
- The numerical error has been put on the same footing as modelling error.



10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$\hat{\delta}$
Δ t= 1/3
Δ t= 1/30
Δ t= 1/300
0
0.5
1
1.5
2
t



10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
δ
Δ t= 1/3
Δ t= 1/30
Δ t= 1/300
0
0.5
1
1.5
2
t

- Forward Euler introduces "negative dissipation" or energy injection — of course the bucket will empty faster if you shake it.
- The model equations are very bad when y is small. We'd be better with "thin film" theory (uses lubrication theory), so numerical errors here are hardly more significant.
- If adaptive stepsizes used, $\Delta t_n \to 0$ as the solution approaches empty. "That's a feature, not a bug."

Thanks!

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