

Memoir on the vibratory movement of an elliptical membrane

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Imagine a membrane stretched equally in all directions, and whose outline, invariably fixed, is an ellipse. Our goal, in this Memoir, is to determine by analysis all the circumstances of its vibratory movement; we calculate the shape and the position of nodal lines and the corresponding sound. But these movements are subject to certain general laws which may be defined without the help of analysis.

When the elliptical membrane is vibrated, there are two systems of nodal lines which are, one of ellipses, the other of hyperbolas, and all these curves of second order have the same foci as the ellipse of the contour.

All these vibratory movements can be divided in two kinds. In one of these kinds, the major axis remains fixed and forms a nodal line, and if we consider two symmetrical points with respect to the major axis, their movements are equal and in opposite directions. In the other kind, on the contrary, the ends of the major axis located between the foci and the vertices form bellies [*ventres*] of vibration, while the part located between the two foci offers a minimum of vibration, so that if we take a point M on the line segment of the foci, and a very close point on a perpendicular at M , the amplitude of the vibration is less for the first than for the second point; if we consider any two points of the membrane, symmetrical relative to the major axis, their movements are equal and the same direction.

Let us define a hyperbolic line as the two branches of a hyperbola terminated at the major axis which have the same asymptote, so that a hyperbola counts as two hyperbolic lines; but if one of the axes of the membrane is stationary, it will be counted for a single hyperbolic nodal line. So the movements of the two kinds can be grouped two by two in a very remarkable way. Indeed, to a number a of elliptical nodal lines and to a number b of hyperbolic nodal lines corresponds a vibrational state of each kind. Now, although

these vibrational states differ both by the two systems of nodal lines and by the resulting sound, they nevertheless merge into the circular membrane to give, as lines of nodes, a concentric circles and b diameters, which divide them into equal parts. We understand from this that if the eccentricity is very small, the sounds of these two vibratory states will differ very little.

We must put aside the case where there are no hyperbolic nodal lines; because the movement cannot be of the second kind, and there is only one vibrational state that produces a nodal ellipses.

The vibratory movement of a membrane enclosed between two confocal ellipses, all of whose points are perfectly fixed, is also subject to very simple laws.

The nodal lines of this membrane are still ellipses and portions of hyperbola branches which have the same foci as the two ellipses of the contours. And there are still two kinds of vibratory movements: in one, the portions of the major axis enclosed between the two contours are nodes; in the other, bellies of vibration. But when we study the vibrational states of the two kinds which give for nodes a ellipses and b hyperbolic lines, we find, if the number b is large enough and the eccentricity is not very large, that the sound is almost the same, as well as the arrangement of the nodal ellipses. Now, the two sounds differing excessively little, we know that in experience we will produce the two vibrational states together, and, in the resulting movement, the arrangement of the b hyperbolic nodal lines can vary in infinite ways.

Mr. Bourget gave the theory of the circular membrane (Annals of the école Normale, t. III) and did the experiments necessary to verify it; he found sounds a little higher than indicated by the calculation.

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1. Let us consider a flat, homogeneous membrane, equally stretched in all directions, and whose contour is invariably fixed. Let us trace in the plane of this membrane two axes of arbitrary rectangular coordinates, Ox and Oy , and let us run a z -axis perpendicular to this plane. If we communicate a vibratory movement to this membrane, a point on its surface whose coordinates are x , y and $z = 0$ will experience a normal displacement w governed by the equation

$$(a) \quad \frac{d^2 w}{dt^2} = m^2 \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right),$$

where m^2 denotes the ratio of the tension to the density of the membrane.[*]

*Theory of Elasticity of Mr. Lamé, IXth Lesson.

And we have to integrate this equation, by imposing the condition that w is zero on the contour.

We must assume in this Memoir that this outline is an ellipse. But we will first take it to be circular and present very succinctly the solution of this particular case, which will sometimes be useful to us as a means of comparison.

Circular membrane.

2. Place the origin of the coordinates in the center of the circle and pass from the rectilinear coordinates x and y to polar coordinates r and α by the formulas

$$x = r \cos \alpha, \quad y = r \sin \alpha,$$

arbitrarily taking the direction of the polar axis.

Equation (a) becomes

$$\frac{d^2 w}{dt^2} = m^2 \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} + \frac{1}{r^2} \frac{d^2 w}{d\alpha^2} \right),$$

and if we set $w = u \sin 2\lambda mt$, we have

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{1}{r^2} \frac{d^2 u}{d\alpha^2} = -4\lambda^2 u.$$

Let $u = PQ$, then by designating by P a function of α and by Q a function of r we will have an equation which can be written

$$\frac{r^2}{Q} \frac{d^2 Q}{dr^2} + \frac{r}{Q} \frac{dQ}{dr} + 4\lambda^2 r^2 = -\frac{1}{P} \frac{d^2 P}{d\alpha^2};$$

as the first member depends only on r and the second only on α , they are equal to the same constant n^2 , and we have

$$(1) \quad \frac{d^2 P}{d\alpha^2} + n^2 P = 0,$$

$$(2) \quad r^2 \frac{d^2 Q}{dr^2} + r \frac{dQ}{dr} - (n^2 - 4\lambda^2 r^2) Q = 0.$$

We have taken for the constant a positive quantity, in order to obtain for P the periodic function

$$P = A \cos n\alpha + B \sin n\alpha,$$

and, so that P does not change when we replace α by $\alpha + 2\pi$, n must be an integer.

If we integrate equation (2) by series, we obtain the two particular solutions:

$$(3) \quad \left\{ \begin{array}{l} Q = Cr^n \left[1 - \frac{(\lambda r)^2}{1(n+1)} + \frac{(\lambda r)^4}{1 \cdot 2(n+1)(n+2)} \right. \\ \left. - \frac{(\lambda r)^6}{1 \cdot 2 \cdot 3(n+1)(n+2)(n+3)} + \dots \right], \end{array} \right.$$

$$(4) \quad \left\{ \begin{array}{l} Q' = C'r^{-n} \left[1 + \frac{(\lambda r)^2}{1(n-1)} + \frac{(\lambda r)^4}{1 \cdot 2(n-1)(n-2)} \right. \\ \left. + \frac{(\lambda r)^6}{1 \cdot 2 \cdot 3(n-1)(n-2)(n-3)} + \dots \right], \end{array} \right.$$

the second of which is deduced from the first by the change of n into $-n$. If we sum them up, we get the general solution; But as obviously the vibratory movement must remain finite in the center of the circle, and that Q' becomes infinite for $r = 0$, we must confine ourselves to take for Q the first particular solution, which we will carry into

$$(5) \quad u = PQ, \quad w = u \sin 2\lambda mt.$$

Finally, for w to be a possible solution, Q must be zero along the contour circle $r = h$, and λ is determined by the equation

$$1 - \frac{(\lambda h)^2}{1(n+1)} + \frac{(\lambda h)^4}{1 \cdot 2(n+1)(n+2)} - \dots = 0.$$

Let us put down this equation

$$(6) \quad 1 - \frac{\tau^2}{1(n+1)} + \frac{\tau^4}{1 \cdot 2(n+1)(n+2)} - \frac{\tau^6}{1 \cdot 2 \cdot 3(n+1)(n+2)(n+3)} + \dots = 0;$$

it has an infinity of roots $\tau_1, \tau_2, \tau_3, \dots$, which we will suppose to be arranged in increasing order of magnitude, and λ can take any of the values

$$\lambda_1 = \frac{\tau_1}{h}, \quad \lambda_2 = \frac{\tau_2}{h}, \quad \lambda_3 = \frac{\tau_3}{h}, \dots$$

Thus formula (5) represents an infinity of vibrational movements, which depend on n and λ ; n is susceptible to all integer values, and to each value of n corresponds an infinity of values of λ .

3. Consider one of these vibrational states and see what the nodal lines are. The vibrational movement satisfies the equation

$$(b) \quad w = (A \cos n\alpha + B \sin n\alpha) Q \sin 2\lambda mt;$$

n and λ are known, and the pitch, or the number of vibrations that occur per unit time, is $N = \frac{\lambda m}{\pi}$. To get the rows of nodes, we will let $w = 0$, which we can satisfy, whatever let t , by letting

$$(7) \quad A \cos n\alpha + B \sin n\alpha = 0,$$

or by letting

$$(8) \quad Q = 0.$$

From equation (7) we draw $\tan n\alpha = -\frac{A}{B}$: therefore if we designate by $n\alpha$, the smallest of the arcs whose tangent is $-\frac{A}{B}$ and by k any integer, w is zero for

$$\alpha = \alpha_1 + \frac{k\pi}{n},$$

and therefore we have for nodal lines n diameters which divide the circumference of the circle into equal parts.

Moving on to equation (8), we first notice that Q is zero at the center of the membrane, unless n is zero because of the factor r'' , and then it is zero for different values of r which are

$$r = \frac{\tau_1}{\lambda}, \frac{\tau_2}{\lambda}, \frac{\tau_3}{\lambda}, \dots;$$

these are the rays of the nodal circles which have the same center as the membrane.

The number of these values of r for which Q is zero is infinite; but we must reject all those that are larger than the radius of the membrane. When in formula (b) we gave ourselves the value of n , λ is susceptible of an infinity of values $\frac{\tau_1}{h}, \frac{\tau_2}{h}, \dots$; suppose that the one we adopted is the s^{th} ,

$$\lambda_s = \frac{\tau_s}{h};$$

then the nodal circles with the number of $s - 1$ will have as radii

$$\frac{\tau_1}{\lambda_s}, \frac{\tau_2}{\lambda_s}, \dots, \frac{\tau_{s-1}}{\lambda_s}.$$

We see from the above that in examining these vibrational movements there are no other difficulties in calculation than finding the roots of equation (b), in which the whole number n varies. Mr. Bourget gave, in his Memoir, a method for easily calculating the roots of this equation, and he gave the numerical values of these first roots for $n = 0, 1, 2, \dots, 7$.

The vibrational movements represented by formula (b) are called *simple* movements, and these are the ones that are observed by experience. Finally any vibratory movement that one can imagine is the superposition of a finite or infinite number of these simple movements.

Passage from rectilinear coordinates to coordinates of the ellipse.

4. Denote by A the semi-major axis of the elliptical membrane, and by c the half-distance between the foci; take as axes of x and y the axes of symmetry of the ellipse; then adopt a second system of coordinates determined by the ellipses and hyperbolas which have the same foci as the contour of the membrane.

Any one of these ellipses is given by the equation

$$(1) \quad \frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - c^2} = 1,$$

in which ρ is $> c$, and if we let

$$\rho = c \frac{e^\beta + e^{-\beta}}{2}, \quad \rho' = \sqrt{\rho^2 - c^2} = c \frac{e^\beta - e^{-\beta}}{2},$$

ρ and ρ' are the semi-major axis and the semi-minor axis of this ellipse, and β is what Mr. Lamé calls the *thermometric parameter* (*On inverse functions of the transcendent*, 1st Lesson).

Any of the confocal hyperbolas has the equation

$$(2) \quad \frac{x^2}{\nu^2} - \frac{y^2}{c^2 - \nu^2} = 1,$$

where ν is $< c$, and if we let

$$\nu = c \cos \alpha, \quad \nu' = \sqrt{c^2 - \nu^2} = c \sin \alpha :$$

ν and ν' are the half-axes of this hyperbola, and α its thermometric parameter.

We pass from the coordinates x and y to the coordinates ν and ρ or α and β by means of the formulas

$$(3) \quad \begin{cases} x = \frac{\rho\nu}{c} = c \frac{e^\beta + e^{-\beta}}{2} \cos \alpha, \\ y = \frac{\rho'\nu'}{c} = c \frac{e^\beta - e^{-\beta}}{2} \sin \alpha, \end{cases}$$

that we deduce from equations (1) and (2). If we wanted to have formulas that could apply immediately to the circle, we would adopt

$$(4) \quad x = \rho \cos \alpha, \quad y = \rho' \sin \alpha.$$

Let M be a point which comes from the intersection of the ellipse $\beta = \beta_1$, and the hyperbola $\nu = \nu_1$. Extend the ordinate of point M until it meets in N with the circle described on the major axis. We see from equations (4)

that the angle α is equal to the angle made by the ray led from center to point N with the x -axis, and as this angle has for its cosine $\frac{v}{c}$ it is also the one made with the x -axis by the asymptote to the hyperbola branch which contains the point M , and the ray led from the center to the point N is this asymptote.

It also follows from formulas (4), that we will obtain all the points of the plane by supposing ρ and ρ' positive, and causing ρ' to vary from 0 to ∞ , and α from 0 to 2π .

When the coordinates are thus varied, the equation $\beta = \text{const.}$ represents an entire ellipse, but $\alpha = \text{const.}$ does not represent any more than one of the four branches of the hyperbola ended at the transverse axis, and the whole hyperbola is given by the four equations

$$\alpha = \alpha_1, \quad \alpha = \pi - \alpha_1, \quad \alpha = \pi + \alpha_1, \quad \alpha = 2\pi - \alpha_1,$$

which are those of the four branches. We assume in what follows that β is positive; however not only is this assumption not essential, but we will have occasion to recognize in the sequel that it can be useful to vary the sign of this coordinate.

It is also good to consider the limit positions of these ellipses and these hyperbolas; for $\beta = 0$, the ellipse is reduced to the line segment which joins the foci F and F' ; the equation $\alpha = 0$ represents the line Fx bounded at F and unbounded in the direction of positive x , $\alpha = \pi$ represents the line $F'x'$ unbounded in the direction of negative x ; finally $\alpha = \frac{\pi}{2}$ determines the entire positive y axis, and $\alpha = \frac{3\pi}{2}$ the negative part of the y axis.

5. Let us take the equation

$$(5) \quad m^2 \left(\frac{d^2 w}{dx^2} + \frac{d^2 w}{dy^2} \right) = \frac{d^2 w}{dt^2},$$

which by the substitution of

$$w = u \sin 2\lambda mt$$

becomes

$$(6) \quad \frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = -4\lambda^2 u,$$

and substitute for x and y the coordinates α and β .

To simplify, let

$$E(\beta) = \frac{e^\beta + e^{-\beta}}{2}, \quad \mathcal{E}(\beta) = \frac{e^\beta - e^{-\beta}}{2}$$

and

$$H^2 = E^2(\beta) \sin^2 \alpha + \mathcal{E}^2(\beta) \cos^2 \alpha = E^2(\beta) - \cos^2 \alpha.$$

we have (IInd Lesson of curvilinear coordinates of Mr. Lamé)

$$\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} = \left[\left(\frac{d\alpha}{dx} \right)^2 + \left(\frac{d\alpha}{dy} \right)^2 \right] \left(\frac{d^2 u}{d\alpha^2} + \frac{d^2 u}{d\beta^2} \right)$$

and by differentiating the equations (3), we find

$$\frac{d\beta}{dx} = -\frac{d\alpha}{dy} = \frac{\mathcal{E}(\beta) \cos \alpha}{cH^2}, \quad \frac{d\alpha}{dx} = \frac{d\beta}{dy} = \frac{-E(\beta) \sin \alpha}{cH^2},$$

$$\left(\frac{d\alpha}{dx} \right)^2 + \left(\frac{d\alpha}{dy} \right)^2 = \frac{1}{c^2 H^2},$$

and we have, instead of equation (6),

$$\frac{1}{c^2 H^2} \left(\frac{d^2 u}{d\alpha^2} + \frac{d^2 u}{d\beta^2} \right) = -4\lambda^2 u.$$

Let

$$u = PQ,$$

and regard P as a function of α only and Q as a function of β only, and we will have, instead of the previous equation,

$$\frac{d^2 P}{d\alpha^2} Q + P \frac{d^2 Q}{d\beta^2} = -4\lambda^2 c^2 [E^2(\beta) - \cos^2 \alpha]$$

or

$$-\frac{1}{P} \frac{d^2 P}{d\alpha^2} + 4\lambda^2 c^2 \cos^2 \alpha = +\frac{1}{Q} \frac{d^2 Q}{d\beta^2} + 4\lambda^2 c^2 E^2(\beta).$$

Since the first member can only contain α , and the second only β , they are equal to the same constant N ; so that we have, instead of a partial difference equation, two second order differential equations

$$\frac{d^2 P}{d\alpha^2} + (N - 4\lambda^2 c^2 \cos^2 \alpha) P = 0,$$

$$\frac{d^2 Q}{d\beta^2} - [N - 4\lambda^2 c^2 E^2(\beta)] Q = 0,$$

The first of these equations is suitable for the circular membrane, if we make $c = 0$, and we know that we must then take for the constant N the square of an integer; it does not follow that the same thing takes place here; because we do not see that the equation does not depend on λc ; but we are sure that if the constant depends on this quantity, it reduces at least to the square of an integer for $c = 0$.

Suppose that we know one of the values of N , and that we have found values of P and Q , which satisfy these two equations; then the formula

$$w = PQ \sin 2\lambda mt$$

represents a possible vibratory movement of the membrane, if we determine λ by the condition that Q is zero for the value of β relative to the contour.

On the determination of the constant N .

6. The first question we must ask ourselves is therefore to determine the constant N . Now, for the expression of w to be admitted, it is necessary that when we change α to $\alpha + 2\pi$, w remains the same, since w will continue to give the displacement of the same point of the membrane; thus P is necessarily a periodic function whose period is 2π , and this condition must determine the constant N .

Let us recall the results obtained by Sturm on second order linear differential equations. Any such equation can be put in the form

$$(1) \quad \frac{d\left(L \frac{dy}{dx}\right)}{dx} + Gy = 0,$$

L and G being two functions of x . Let us imagine that G also contains a parameter h , and give it an increment δh ; then G will take the value $G + \delta G$, and y will change to the function y_1 , which satisfies the equation

$$(2) \quad \frac{d\left(L \frac{dy_1}{dx}\right)}{dx} + (G + \delta G)y_1 = 0,$$

Multiply the equations (1) and (2) by y_1 and y , and subtract, we obtain

$$y_1 \frac{d}{dx} \left(L \frac{dy}{dx} \right) - y \frac{d}{dx} \left(L \frac{dy_1}{dx} \right) - \delta G y y_1 = 0.$$

Multiply by dx , and integrate between the limits x_0 and X ; we will have

$$\int_{x_0}^X y_1 \frac{d}{dx} \left(L \frac{dy}{dx} \right) dx - \int_{x_0}^X y \frac{d}{dx} \left(L \frac{dy_1}{dx} \right) dx - \int_{x_0}^X \delta G y y_1 dx = 0;$$

apply integration by parts to the first two terms, and if δh is infinitely small, the increase of y will be too, and we will have, by replacing y_1 , by $y + \delta y$,

$$(A) \quad \left\{ \begin{array}{l} \left[L \left(\frac{dy}{dx} \delta y - y \delta \frac{dy}{dx} \right) \right]_X - \left[L \left(\frac{dy}{dx} \delta y - y \delta \frac{dy}{dx} \right) \right]_{x_0} \\ - \int_{x_0}^X y^2 \delta G dx = 0 \end{array} \right.$$

Sturm then supposes that the quantity $L \frac{dx}{dy} : y$ undergoes for $x = x_0$, by the increase of h , a variation of a given sign; for the research we are proposing, imagine instead that y is zero or maximum or minimum for $x = x_0$, and whatever the value of the parameter h . So for $x = x_0$, you have to make either $y = 0$, $\delta y = 0$ or $\frac{dy}{dx} = 0$, and $\delta \frac{dy}{dx} = 0$; in both cases, the second bracket is zero, and it remains

$$(3) \quad \left[L \left(\frac{dy}{dx} \delta y - y \delta \frac{dy}{dx} \right) \right]_X = \int_{x_0}^X y^2 \delta G dx;$$

the first member is therefore of the same sign as the increase that G takes by the variation of the parameter.

We can now recognize if the roots in x of the equation $y_1 = 0$ are smaller or larger than those in the equation $y = 0$. y is a function of x and h , and if for $x = X$ we have

$$y = 0,$$

as soon as we give h an increase δh , y will no longer be zero, unless we also give x an increase δx such that we have

$$\frac{dy}{dx} \delta x + \frac{dy}{dh} \delta h = 0,$$

or

$$\frac{dy}{dx} \delta x + \delta y = 0,$$

and therefore the root undergoes an increase equal to

$$(4) \quad \delta x = - \frac{\delta y}{\frac{dy}{dx}}.$$

Suppose L is positive; y being zero for $x = X$, it follows from the formula (3) that if δG is positive, $\frac{dy}{dx} \delta y$ is also positive, and, as a result of the formula (4), that δx is negative; so the roots of $y_1 = 0$ are smaller than those of $y = 0$, and similarly we see that if δG is negative, the roots of $y_1 = 0$ are larger than those of $y = 0$.

If we then imagine that we give to the parameter h , no longer an infinitely small increase, but a finite increase, and that h increases from h to h_1 , if at the same time G increases throughout this interval, or decreasing throughout, the previous conclusions, relating to the roots of $y = 0$ and $y_1 = 0$, are applicable, as can be recognized by dividing the interval from h to h_1 into infinitely small parts.

These considerations are due to Sturm (Journal of M. Liouville, 1st series, t. I, p. 106); but we will show how they can be used to recognize if a function is periodic, and we will get new results.

7. Let us first try to assume that N is the square of an integer g^2 , and, substituting the letter h for λc , P is given by the equation

$$(5) \quad \frac{d^2 P}{d\alpha^2} + (g^2 - 4h^2 \cos^2 \alpha) P = 0.$$

It is very easy to recognize that the general solution of a linear differential equation of the second order, such as the preceding ones, can ordinarily be divided into two particular solutions, one of which is zero, and the other of which is maximum or minimum for the value zero given to the variable. So let

$$P = P_1 + P_2,$$

P_1 being a solution which becomes zero [*annule*] for $\alpha = 0$ and P_2 a solution which is maximum or minimum for this value.

These are the two functions P_1 and P_2 that we will examine. In the case where h becomes zero, they satisfy the equation

$$(6) \quad \frac{d^2 P'}{d\alpha^2} + g^2 P' = 0,$$

and P_1 is reduced to $A \sin g\alpha$, P_2 to $B \cos g\alpha$.

P_1 becomes zero for $\alpha = 0$, like $P' = A \sin g\alpha$; however the coefficient of P' , in equation (6), is always greater than that of P in equation (5); it therefore follows from what we have seen above that the roots of $P' = 0$ are smaller than those of $P = 0$; the roots of $P' = 0$ are from 0 to 2π ,

$$0, \quad \frac{\pi}{g}, \quad \frac{2\pi}{g}, \dots, \quad \frac{(2g-1)\pi}{g}.$$

Let us give to h an excessively small value, and divide, from the x -axis, a circumference which has its center at the origin into $2g$ equal parts, then lead to the points of division the rays Oa, Ob, Oc, \dots ; the roots of $P' = 0$ are equal to the angles aOb, aOc, \dots , and the roots of $P_1 = 0$ are larger and represented by the angles aOb', aOc', \dots . But when after a turn on the circumference we return to the point a , P' again becomes zero at the value $\alpha = 2\pi$, while P_1 , which is zero for $\alpha = 0$, is not for $\alpha = 2\pi$, but for a slightly larger value. P_1 does not therefore take the same value when we increase the arc α by a circumference.

To demonstrate that P_2 , taken from equation (5), does not have 2π for period, apply the formula (3), replacing x by α , L and G by 1 and $g^2 - 4h^2 \cos^2 \alpha$, y by P_2 , x_0 and X by 0 and 2π , and we will have

$$\left(\frac{dP_2}{d\alpha} \delta P_2 - P_2 \delta \frac{dP_2}{d\alpha} \right)_{2\pi} + 4 (2h\delta h + \delta h^2) \int_0^{2\pi} P_2^2 \cos^2 \alpha d\alpha = 0,$$

a formula where we do not have to take account of δh^2 when h is zero. Suppose we vary h from zero to the infinitely small value δh ; P_2 , for $h = 0$, reduces to $B \cos g\alpha$, and it is then maximum for $\alpha = 2\pi$ as for $\alpha = 0$; so, by making $h = 0$ in this formula, $\frac{dP_2}{d\alpha}$ becomes zero, and we have

$$\left(P_2 \delta \frac{dP_2}{d\alpha}\right)_{2\pi} = 4(\delta h)^2 \int_0^{2\pi} P_2^2 \cos^2 \alpha d\alpha;$$

the second member is essentially positive, so $\delta \frac{dP_2}{d\alpha}$ is not zero for $\alpha = 2\pi$, or $\frac{dP_2}{d\alpha}$ is not zero for $\alpha = 2\pi$ when we make $h = \delta h$; so finally P_2 , which is maximum for $\alpha = 0$, is not for $\alpha = 2\pi$, and the function is not periodic.

If we take for the constant

$$N = g^2 4 + h^2.$$

still designating by g an integer, the equation which gives P would become

$$\frac{d^2 P}{d\alpha^2} + (g^2 + 4h^2 \sin^2 \alpha) P = 0.$$

By defining the particular solutions P_1 and P_2 as above, we will recognize that the roots of $P_1 = 0$ and of $P_2 = 0$ are smaller than those of $A \sin g\alpha = 0$ and of $B \cos g\alpha = 0$, and we will also demonstrate, as before, that P_1 and P_2 are not periodic functions.

8. Finally, let us take for N the expression

$$N = g^2 + 2h^2,$$

we have the equation

$$\frac{d^2 P}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha) P = 0,$$

and we will demonstrate that P is then a periodic function if h is excessively small, that is to say if we can neglect the powers of h^2 greater than the first.*]

If we consider only the solutions which are zero, or maximum or minimum for $\alpha = 0$, and which we have designated by P_1 and P_2 , we have, according to (3), the equation

$$(7) \quad \frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} = -2(2h\delta h + \delta h^2) \int_0^\alpha P^2 \cos 2\alpha d\alpha.$$

*Nevertheless, if $g = 1$, take $N = 1 + h^2$ or $= 1 + 3h^2$, depending on whether it is P_1 or P_2 .

Instead of allowing just any h , let us take it equal to zero, and give it the increase δh ; then apply formula (7) by making $\alpha = 2\pi$. If it is P_1 that we are considering, we will have

$$P_1^2 \cos 2\alpha = \sin^2 g\alpha \cos 2\alpha = \frac{\cos 2\alpha}{2} - \frac{\cos 2(g+1)\alpha + \cos 2(g-1)\alpha}{4},$$

and if g is not equal to 1, we will have

$$\int_0^{\frac{\pi}{2}} P_1^2 \cos 2\alpha \, d\alpha = 0.$$

If we consider P_2 , we have

$$P_2^2 \cos 2\alpha = \cos^2 g\alpha \cos 2\alpha = \frac{\cos 2\alpha}{2} - \frac{\cos 2(g+1)\alpha + \cos 2(g-1)\alpha}{4},$$

and if g is not equal to 1, we still have

$$\int_0^{\frac{\pi}{2}} P_2^2 \cos 2\alpha \, d\alpha = 0;$$

so equation (7) is reduced in both cases to

$$(8) \quad \left(\frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} \right) \frac{\pi}{2} = 0;$$

but for $h = 0$, P_1 and P_2 become $\sin g\alpha$ and $\cos g\alpha$, and one of the two functions is zero and the other maximum for $\alpha = \frac{\pi}{2}$; now from this formula we conclude that if P is zero for $\alpha = \frac{\pi}{2}$, δP is too, and that if $\frac{dP}{d\alpha}$ is zero for $\alpha = \frac{\pi}{2}$, $\delta \frac{dP}{d\alpha}$ is zero at the same time. So, for a very small value of h and for $\alpha = \frac{\pi}{2}$, P_1 is zero or maximum like $\sin g\alpha$, and P_2 is zero or maximum like $\cos g\alpha$.

Suppose now that h is no longer excessively small, but has some value; and, setting $N = R + 2h^2$, consider the equation

$$(9) \quad \frac{d^2 P}{d\alpha^2} + (R - 2h^2 \cos 2\alpha) P = 0,$$

in which R depends on h , and is reduced to the square g^2 of an integer for $h = 0$; by applying the equation (3), we have

$$\frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} = \int_0^\alpha P^2 (\delta R - 4h\delta h \cos 2\alpha) \, d\alpha;$$

so imagine that we know how to determine the constant R so that the integral

$$\int_0^{\frac{\pi}{2}} P^2 (\delta R - 4h\delta h \cos 2\alpha) \, d\alpha$$

is zero, regardless of h : the property we just obtained when h is very small takes place for any value of h , because we will still have the equation (8), and P will be zero or maximum for $\alpha = \frac{\pi}{2}$, depending on whether $\sin g\alpha$ and $\cos g\alpha$, to which it reduces for $h = 0$, is zero or maximum.

Note now that nothing indicates that, for the same value of g , the constant R is the same in the functions P_1 , and P_2 ; it is indeed different, and the general solution of equation (9) cannot be periodic. To fix the ideas, let us choose the constant so that we will have

$$\int_0^{\frac{\pi}{2}} P_1^2 (\delta R - 4h \cos 2\alpha) d\alpha = 0,$$

and I say that the function P_1 will take the same values or equal values and of opposite sign in each quadrant, so that α being between 0 and $\frac{\pi}{2}$ the four quantities

$$P_1(\alpha), \quad P_1(\pi - \alpha), \quad P_1(\pi + \alpha), \quad P_1(2\pi - \alpha)$$

are equal, except for the sign, and that P_1 is periodic.

As we will have occasion to see later, if we set $\nu = \cos \alpha$, the general solution of the differential equation which gives P is the sum of two particular solutions which develop thus

$$P' = A_0 + A_1\nu^2 + A_2\nu^4 + A_3\nu^6 + \dots,$$

$$P'' = B\nu + B_1\nu^3 + B_2\nu^5 + B_3\nu^7 + \dots$$

The first is maximum and the second zero for $\nu = 0$, wherefore $\alpha = \frac{\pi}{2}$.

It follows from this that P_1 is equal to P' or P'' , depending on whether it is zero or maximum for $\alpha = \frac{\pi}{2}$; however, if we change ν to $-\nu$ or α to $\pi - \alpha$, P' remains constant and P'' changes sign only; therefore P_1 remains the same, except for the sign, when we replace α by $\pi - \alpha$.

To go to the third quadrant, we note that the general solution of P can be divided into two solutions, one of which is even, and the other of which is odd, according to the powers of $\nu' = \sin \alpha$; P_1 , which is zero for $\alpha = \pi$, is equal to the odd solution in ν' , and we conclude

$$P_1(\pi + \alpha) = -P_1(\pi - \alpha), \quad P_1(\pi + \alpha) = \pm P_1(\alpha).$$

Finally, the values of P_1 in the fourth quadrant can be obtained in the same way. So the function P_1 takes the same values down to the sign in each quadrant, and behaves in sign changes like $\sin g\alpha$, and it has 2π for period.

If we determine the constant R so that we have

$$\int_0^{\frac{\pi}{2}} P_2^2 \left(\frac{dR}{dh} - 4h \cos 2\alpha \right) d\alpha = 0,$$

we arrive at similar conclusions for P_2 , which behaves in the passage from one quadrant to the next like $\cos g\alpha$. It is therefore only necessary to study the functions P_1 and P_2 between the limits $\alpha = 0$ and $\alpha = \frac{\pi}{2}$.

9. If we first regard h as very small, the constant R is reduced to almost g^2 , and the differential equation to

$$\frac{d^2 P}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha) P = 0.$$

If we give h the increase δh , any root of

$$P_1 = 0 \quad \text{or of} \quad P_2 = 0$$

undergoes a variation whose value is

$$\delta\alpha = -\delta P : \frac{dP}{d\alpha}.$$

We have the general formula

$$(10) \quad \left\{ \begin{array}{l} \frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} \\ = \left(\frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} \right)_a - 2(2h\delta h + \delta h^2) \int_a^\alpha P^2 \cos 2\alpha d\alpha. \end{array} \right.$$

Let us make $a = 0$; suppose that P represents P_1 or P_2 , and that α is a root of $P = 0$, enclosed between 0 and $\frac{\pi}{4}$, the preceding equation becomes

$$\frac{dP}{d\alpha} \delta P = -2(2h\delta h + \delta h^2) \int_0^\alpha P^2 \cos 2\alpha d\alpha.$$

All the elements of the integral are positive; therefore $\frac{dP}{d\alpha} \delta P$ is negative and the variation of the roots positive, when we give to h the increase δh .

Let us make $a = \frac{\pi}{2}$, and suppose that α is now a root of $P = 0$, enclosed between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, we deduce from the same formula

$$\frac{dP}{d\alpha} \delta P = 2(2h\delta h + \delta h^2) \int_0^{\frac{\pi}{2}} P^2 \cos 2\alpha d\alpha.$$

that the second member is negative; so the root increase is still positive.

Suppose for example that it is P_2 , and that g is even; then the function P is maximum like $\cos g\alpha$ for $\alpha = 0$ and $\alpha = \frac{\pi}{2}$. If Ob , Oc , Od , ... make

angles with Ox equal to the roots of the equation $\cos g\alpha = 0$, these lines can represent nodal lines of the circular membrane, and the arcs bc , cd , ... are equal to each other and double the extreme arcs ab and ef of the quadrant af .

Let us consider an ellipse whose eccentricity is very small, and take [*menons*] the asymptotes of the hyperbolic nodal lines Ob' , Oc' , Od' , ...; it follows from what we have shown that the angles aOb' , aOc' , aOd' , ... are respectively larger than aOb , aOc , ... But there is more: the angles $b'Oc'$, $c'Od'$, ... are $> bOc$, cOd , ... in the first half of the quadrant and are less in the second half.

To prove it, let us denote by α_1 and α_2 , two consecutive roots of the equation $P_2 = 0$, and consider the function

$$\Pi = A \sin g(\alpha - \alpha_1),$$

which becomes zero for $\alpha = \alpha_1$; P_2 is not reduced to Π for $h = 0$; but we can imagine a function P which satisfies the differential equation of the second order, and which by the variation of h passes from Π to P_1 while remaining constantly zero for $\alpha = \alpha_1$. So for $\alpha = \alpha_1$, we will have

$$P = 0, \quad \delta P = 0,$$

and by making $\alpha = \alpha_2$, and $a = \alpha_1$ in equation (10), we get

$$\left(\frac{dP}{d\alpha}\delta P\right)_{\alpha=\alpha_1} = -2(2h\delta h + \delta h^2) \int_{\alpha_1}^{\alpha_2} P^2 \cos 2\alpha d\alpha;$$

if α_1 and α_2 are less than $\frac{\pi}{4}$, the integral of the second member is positive, and the first member is negative; so the variation of the root α ,

$$\delta\alpha_2 = -\delta P : \frac{dP}{d\alpha},$$

but this time counted from α_1 , is positive; therefore the interval of the two roots α_1 and α_2 has increased; it is therefore larger than $\frac{2\pi}{g}$. We would see that on the contrary if α_1 and α_2 are between $\frac{\pi}{4}$ and $\frac{\pi}{2}$, the integral is negative, and that the interval between two roots decreases when h increases, while keeping very small values.

10. But whatever the size of h and whatever the direction in which the constant R varies, when we give an infinitely small increase to h , the roots undergo infinitely small modifications, and those which were included between 0 and $\frac{\pi}{2}$ and will remain constant; because Oa and Of are lines where P_2 is maximum and cannot become zero, and that consequently these

roots by changing magnitude cannot cross. Then this property belongs in all cases to the functions P_1 and P_2 , the first of which is zero, and the second maximum for $\alpha = 0$, while one becomes zero and the other is maximum for $\alpha = \frac{\pi}{2}$, according to the parity of g . So, for example, if P_2 becomes zero for $\alpha = \frac{\pi}{2}$, it is impossible that by increasing h a root between 0 and $\frac{\pi}{2}$ crosses the limit $\frac{\pi}{2}$; because if for a value of h a root between 0 and $\frac{\pi}{2}$ becomes equal to $\frac{\pi}{2}$, the equation $P_2 = 0$ would have a double root for $\alpha = \frac{\pi}{2}$; therefore P and $\frac{dP}{d\alpha}$, and consequently the derivatives of all orders, would become zero together for the same value of α ; which is impossible.

Now for $h = 0$ the functions P_1 and P_2 are reduced to $\sin g\alpha$ and $\cos g\alpha$, and become zero g times from 0 to π ; therefore whatever h , the equations $P_1 = 0$ and $P_2 = 0$ also have g roots from 0 to π (admitting among these roots that which would be zero, but not that which would be equal to π).

Development of the functions P_1 and P_2 according to powers of h .

11. To develop according to powers of h the solutions P_1 and P_2 of the equation

$$\frac{d^2 P}{d\alpha^2} + (R - 2h^2 \cos 2\alpha) P = 0$$

that have 2π for period, and the first of which is zero, the second maximum for $\alpha = 0$, we will let

$$R = g^2 + \beta h^4 + \gamma h^6 + \delta h^8 + \dots,$$

by designating by g any integer, and we will seek to determine the coefficients according to the condition that P_1 and P_2 are periodic.

First consider P_2 ; let in the differential equation

$$P = P_2 = \cos g\alpha + h^2 p, \quad R = g^2 + Bh^4,$$

and we will have

$$\frac{d^2 p}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha + Bh^4)p - (2 \cos 2\alpha \cos g\alpha - Bh^2 \cos g\alpha) = 0.$$

Then let

$$p = p + h^2 p_1,$$

and we will have

$$(I) \quad 0 = \frac{d^2 p}{d\alpha^2} + g^2 p - 2 \cos 2\alpha \cos g\alpha,$$

$$(b) \quad \begin{cases} 0 = \frac{d^2 p_1}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha + Bh^4) p_1 \\ \quad + (-2 \cos 2\alpha + Bh^2) p + B \cos g\alpha. \end{cases}$$

To solve equation (I), we will replace $2 \cos 2\alpha \cos g\alpha$ by $\cos(g+2)\alpha + \cos(g-2)\alpha$, and we will let

$$p = a \cos(g+2)\alpha + b \cos g\alpha + c \cos(g-2)\alpha;$$

we find immediately

$$a = \frac{-1}{4(g+1)}, \quad c = \frac{1}{4(g-1)};$$

for b , it is not determined, and indeed P_2 is reduced to $\cos g\alpha$ for $h = 0$; but if we suppose that we have obtained its expression, and that we multiply it by $1 + Bh^2$, this new expression can still represent P_2 , and the coefficient b thereby changes in an arbitrary way.

Since we can give b the value we want, we will make

$$b = 0.$$

In equation (b), we let

$$p_1 = p_1 + h^2 p_2, \quad B = \beta + Ch^2,$$

and we will have the two equations

$$(II) \quad \frac{d^2 p_1}{d\alpha^2} + g^2 p_1 - 2 \cos 2\alpha p + \beta \cos g\alpha = 0,$$

$$(c) \quad \begin{cases} 0 = \frac{d^2 p_2}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha + \beta h^4 + Ch^6) p_2 \\ \quad + (-2 \cos 2\alpha + \beta h^2 + Ch^4) p_1 \\ \quad + (\beta + Ch^2) p + C \cos g\alpha. \end{cases}$$

To solve equation (II), we must replace $2 \cos 2\alpha p$ by its value

$$(1) \quad \begin{aligned} & a \cos(g+4)\alpha + b \cos(g+2)\alpha \\ & + (a+c) \cos g\alpha + b \cos(g-2)\alpha + c \cos(g-4)\alpha, \end{aligned}$$

then substitute for p_1

$$(2) \quad \begin{aligned} p_1 = & d \cos(g+4)\alpha + e \cos(g+2)\alpha \\ & + f \cos g\alpha + h \cos(g-2)\alpha + k \cos(g-4)\alpha, \end{aligned}$$

and we find, by equating to zero the coefficients of the cosines of the different arcs,

$$d = -\frac{a}{8(g+2)}, \quad e = \frac{b}{4(g+1)}, \quad h = \frac{b}{4(g-1)}, \quad k = \frac{c}{8(g-2)},$$

$$\beta = a + c.$$

b is left arbitrary in these formulas; if we assume $b = 0$, we have

$$d = \frac{1}{32(g+1)(g+2)}, \quad e = 0, \quad h = 0, \quad k = \frac{1}{32(g-1)(g-2)},$$

$$\beta = \frac{1}{2(g^2-1)}.$$

The coefficient f remains undetermined, and indeed, if we imagine that we have obtained the expression of P_2 and that we multiply it by $1 + Bh^2 + Ch^4$, we will not only change the coefficient b in an arbitrary manner, but also the coefficient f ; the simplest thing is to make $f = 0$.

In equation (c), let

$$p_2 = p_2 + h^2 p_3, \quad C = \gamma + Dh^2,$$

and we have

$$(III) \quad \frac{d^2 p_2}{d\alpha^2} + g^2 p_2 - 2 \cos 2\alpha p_1 + \gamma \cos g\alpha = 0,$$

$$(d) \quad \begin{cases} 0 = \frac{d^2 p_3}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha + \beta h^4 + Ch^6) p_3 \\ \quad + (-2 \cos 2\alpha + \beta h^2 + Ch^4) p_2 \\ \quad + (\beta + Ch^2) p_1 + Cp + D \cos g\alpha. \end{cases}$$

Let us replace in equation (III) $2 \cos 2\alpha p_1$ by

$$(3) \quad d \cos(g+6)\alpha + e \cos(g+4)\alpha + (d+f) \cos(g+2)\alpha + (e+h) \cos g\alpha \\ + (k+f) \cos(g-2)\alpha + h \cos(g-4)\alpha + k \cos(g-6)\alpha,$$

and let

$$(4) \quad p_2 = l \cos(g+6)\alpha + m \cos(g+4)\alpha + n \cos(g+2)\alpha \\ + \varpi \cos g\alpha + q \cos(g-2)\alpha + r \cos(g-4)\alpha + s \cos(g-6)\alpha;$$

we will have

$$l = \frac{-d}{12(g+3)}, \quad m = \frac{-e}{8(g+2)}, \quad r = \frac{h}{8(g-2)}, \quad s = \frac{k}{12(g-3)},$$

$$n = \frac{-d-f-\beta\alpha}{4(g+1)}, \quad q = \frac{k+f-\beta c}{4(g-1)}, \quad \gamma = e + h - \beta b = 0.$$

These are the expressions of l , m , n , \dots , regardless of the values given to b and f ; and if we suppose them zero, we get

$$l = \frac{-1}{2^7 \cdot 3(g+1)(g+2)(g+3)}, \quad m = 0, \quad r = 0, \quad s = \frac{1}{2^7 \cdot 3(g-1)(g-2)(g-3)},$$

$$n = \frac{-(g^2+4g+7)}{2^7(g+1)^3(g-1)(g+2)}, \quad q = \frac{g^2-4g+7}{2^7(g-1)^3(g+1)(g-2)}, \quad \gamma = 0.$$

ϖ is undetermined, like b and f , and we will make it zero too.

Now that we see what kind of simplification results from the hypothesis of the nullity of arbitrary constants, and that we recognize that it brings about the disappearance of the terms of even rank in p , p_1 , p_2 , etc., immediately make these reductions in the calculations that follow. Let us put in equation (d)

$$p_3 = p_3 + h^2 p_4, \quad D = \delta + Eh^2,$$

we will get the two equations

$$(IV) \quad \frac{d^2 p_3}{d\alpha^2} + g^2 p_3 - 2 \cos 2\alpha p_2 + \beta p_1 + \delta \cos g\alpha = 0,$$

$$(e) \quad \begin{cases} 0 = \frac{d^2 p_4}{d\alpha^2} + (g^2 - 2h^2 \cos 2\alpha + \beta h^4 + Ch^6) p_4 \\ \quad + (-2 \cos 2\alpha + \beta h^2 + Ch^4) p_3 + (\beta + Ch^2) p_2 \\ \quad + Ch^2 p_1 + (\delta + Eh^2) p + E \cos g\alpha. \end{cases}$$

Let us replace in equation (IV) $2 \cos \alpha p_2$ by

$$(5) \quad \begin{aligned} l \cos(g+8)\alpha + (l+n) \cos(g+4)\alpha + (q+n) \cos g\alpha \\ + (q+s) \cos(g-4)\alpha + s \cos(g-8)\alpha, \end{aligned}$$

and let

$$(6) \quad \begin{aligned} p_3 = R_1 \cos(g+8)\alpha + R_2 \cos(g+4)\alpha + R_3 \cos g\alpha \\ + R_4 \cos(g-4)\alpha + R_5 \cos(g-8)\alpha; \end{aligned}$$

we will have

$$R_1 = \frac{-l}{16(g+4)}, \quad R_2 = \frac{-(l+n)+\beta d}{8(g+2)}, \quad R_4 = \frac{q+s-\beta k}{8(g-2)}, \quad R_5 = \frac{s}{16(g-4)},$$

$$\delta = q + n,$$

or, by performing the calculations,

$$\begin{aligned} R_1 &= \frac{1}{2^{11} \cdot 3(g+1)(g+2)(g+3)(g+4)}, & R_5 &= \frac{1}{2^{11} \cdot 3(g-1)(g-2)(g-3)(g-4)}, \\ R_2 &= \frac{g^3+7g^2+20g+20}{2^8 \cdot 3(g+1)^3(g-1)(g+2)^2(g+3)}, & R_5 &= \frac{g^3-7g^2+20g-20}{2^8 \cdot 3(g-1)^3(g+1)(g-2)^2(g-3)}, \\ \delta &= \frac{5g^2+7}{32(g^2-1)^3(g^2-2^2)}; \end{aligned}$$

add to these values $R_3 = 0$.

If we again let

$$p_4 = p_4 + h^2 p_5, \quad E = \varepsilon + Hh^2,$$

p_4 will be given to us by the equation

$$\frac{d^2 p_4}{d\alpha^2} + g^2 p_4 - 2 \cos 2\alpha p_3 + \beta p_2 + \delta p + \varepsilon \cos g\alpha = 0,$$

and we will have

$$(7) \quad \begin{aligned} p_4 &= S_1 \cos(g+10)\alpha + S_2 \cos(g+6)\alpha + S_3 \cos(g+2)\alpha \\ &\quad - S_4 \cos(g-2)\alpha + S_5 \cos(g-6)\alpha + S_6 \cos(g-2)\alpha, \end{aligned}$$

taking

$$\begin{aligned} S_1 &= \frac{-R_1}{4 \cdot 5(g+5)}, & S_2 &= \frac{-R_1-R_2+\beta l}{4 \cdot 3(g+3)}, & S_3 &= \frac{-R_2+\beta n+\delta a}{4(g+1)}, \\ S_4 &= \frac{R_4-\beta q-\delta c}{4(g-1)}, & S_5 &= \frac{R_4+R_5-\beta s}{4 \cdot 3(g-3)}, & S_6 &= \frac{R_5}{4 \cdot 5(g-5)}, \\ \varepsilon &= 0 \end{aligned}$$

or

$$\begin{aligned} S_1 &= \frac{-1}{2^{11} \cdot 3 \cdot 4 \cdot 5(g+1)(g+2)(g+3)(g+4)(g+5)}, & S_6 &= \frac{1}{2^{11} \cdot 3 \cdot 4 \cdot 5(g-1)(g-2)(g-3)(g-4)(g-5)}, \\ S_2 &= \frac{-(g^4+11g^3+49g^2+101g+78)}{2^{13}(g+1)^3(g+2)^2(g+3)^2(g+4)(g-1)}, & S_2 &= \frac{g^4-11g^3+49g^2-101g+78}{2^{13}(g-1)^3(g-2)^2(g-3)^2(g-4)(g+1)}, \\ S_3 &= \frac{-(g^7+7g^6+18g^5+24g^4+63g^3+81g^2+206g+464)}{2^{10} \cdot 3(g+1)^5(g+2)^2(g+3)(g-1)^3(g-2)}, \\ S_4 &= \frac{g^7-7g^6+18g^5-24g^4+63g^3-81g^2+206g-464}{2^{10} \cdot 3(g-1)^5(g-2)^2(g-3)(g+1)^3(g+2)}. \end{aligned}$$

To complete this calculation, note that the coefficient η of h^{12} in the constant has the value

$$\eta = S_3 + S_4,$$

and, replacing S_3 and S_4 with their expressions,

$$\eta = \frac{9g^6 + 22g^4 - 203g^2 - 116}{2^6(g^2 - 1)^5(g^2 - 2)^2(g^2 - 3^2)}.$$

So, by putting in R the values of the first terms, we get

$$(l) \quad \begin{cases} R = g^2 + \frac{1}{2(g^2 - 1)}h^4 + \frac{5g^2 + 7}{32(g^2 - 1)^3(g^2 - 4)}h^8 \\ + \frac{9g^6 + 22g^4 - 203g^2 - 116}{64(g^2 - 1)^5(g^2 - 4)^2(g^2 - 9)}h^{12} + \dots \end{cases}$$

12. It is time to notice that we cannot continue the development of P_2 and the constant R in this way without worrying about the value of the integer g ; because the coefficient of h^4 contains in its denominator the factor $g - 1$, the coefficient of h^8 the factor $g - 2$, the coefficient of h^{12} the factor $g - 3$, and so on; so that whatever the integer taken for g , we will end up finding an infinite term. We must even stop the development of R before meeting an infinite term; because, for a term of the constant to be accepted, the term must itself be of the same order as P_2 .

For clarity, consider a special case, that of $g = 4$ for example. The coefficients of h^8 and h^{12} in R keep a finite value but must however be rejected. To recognize this, let us resume the calculation of p_3 , which needs to be modified, because the value of R_5 which appears there becomes infinite.

The expression of $2 \cos 2\alpha p_2$, becomes

$$l \cos 12\alpha + (l + n) \cos 8\alpha + (q + n + s) \cos 4\alpha + q + s,$$

and the terms in $\cos g\alpha$ and $\cos(g - 8)\alpha$ come together in one, in $\cos 4\alpha$.

We will substitute in the equation (IV)

$$p_3 = R_1 \cos 12\alpha + R_2 \cos 8\alpha + R_3 \cos 4\alpha + R_4,$$

to determine the coefficients; but in the result of the substitution, the coefficient of $\cos 4\alpha$ having to be zero, we get

$$\delta = q + n + s;$$

the value of δ must therefore be increased by the quantity s ; we have

$$q + n = \frac{87}{2^7 \cdot 3^4 \cdot 5^3}, \quad s = \frac{1}{2^8 \cdot 3^2},$$

and consequently

$$\delta = \frac{433}{2^8 \cdot 3^3 \cdot 5^3}.$$

We can clearly see how we would continue this development.

If g is > 4 , the first three terms of R are those found in the expression (I); if g is > 6 , we must still take in the development (I) the term in h^{12} , and so on.

We will consider the development of P_2 and the first terms of R when g is less than 4.

If g is zero, the expansion we found for P_2 is applicable, and it is even the only case where we can apply it as far as we want.

If $g = 2$, we obtain, by a special calculation,

$$\begin{aligned}
 P_2 = & \cos 2\alpha + h^2 \left(-\frac{1}{12} \cos 4\alpha + \frac{1}{4} \right) + \frac{h^4}{384} \cos 6\alpha \\
 & - h^6 \left(\frac{1}{23040} \cos 8\alpha + \frac{43}{13824} \cos 4\alpha + \frac{5}{192} \right) \\
 & + h^8 \left(\frac{1}{2211840} \cos 10\alpha + \frac{287}{2211840} \cos 6\alpha \right) \\
 & + h^{10} \left(\frac{-1}{309657600} \cos 12\alpha - \frac{41}{16588800} \cos 8\alpha \right. \\
 & \quad \left. + \frac{21059}{79626240} \cos 4\alpha + \frac{1363}{221184} \right) + \dots; \\
 \text{(A)} \quad R = & 4 + \frac{5}{12} h^4 - \frac{763}{13824} h^8 + \frac{1002419}{79626240} h^{12} + \dots
 \end{aligned}$$

If $g = 4$, we have

$$\begin{aligned}
 P_2 = & \cos 4\alpha + h^2 \left(-\frac{1}{20} \cos 6\alpha + \frac{1}{12} \cos 2\alpha \right) + \frac{h^4}{384} \left(\frac{1}{960} \cos 8\alpha + \frac{1}{192} \right) \\
 & + h^6 \left(\frac{-1}{80640} \cos 10\alpha - \frac{13}{96000} \cos 6\alpha + \frac{11}{17280} \cos 2\alpha \right) \\
 & + h^8 \left(\frac{1}{10321920} \cos 12\alpha + \frac{23}{6048000} \cos 8\alpha - \frac{1}{92160} \right) \\
 & - h^{10} \left(\frac{1}{1857945600} \cos 14\alpha + \frac{53}{1032192000} \cos 10\alpha \right. \\
 & \quad \left. + \frac{4037}{2419200000} \cos 6\alpha + \frac{439}{62208000} \cos 2\alpha \right) + \dots; \\
 \text{(B)} \quad R = & 16 + \frac{1}{30} h^4 + \frac{433}{864000} h^8 - \frac{189983}{21772800000} h^{12} + \dots
 \end{aligned}$$

The expansions (A) and (B) contain only even powers of h^4 , and we will demonstrate below that R enjoys this property whenever g is even. For $g = 1$, we have the following formulas:

$$\begin{aligned}
 P_2 = & \cos \alpha - \frac{h^2}{8} \cos 3\alpha + h^4 \left(\frac{1}{192} \cos 5\alpha - \frac{1}{64} \cos 3\alpha \right) \\
 & - h^6 \left(\frac{1}{9216} \cos 7\alpha - \frac{1}{1152} \cos 5\alpha + \frac{1}{1536} \cos 3\alpha \right) \\
 & + h^8 \left(\frac{1}{737280} \cos 9\alpha - \frac{1}{49152} \cos 7\alpha \right. \\
 & \quad \left. + \frac{1}{24576} \cos 5\alpha + \frac{11}{36864} \cos 3\alpha \right) + \dots; \\
 R = & 1 + h^2 - \frac{1}{8} h^4 - \frac{1}{64} h^6 - \frac{1}{1536} h^8 + \frac{11}{36864} h^{10} + \dots
 \end{aligned}$$

For $g = 3$, we have

$$\begin{aligned}
 P_2 = & \cos 3\alpha + h^2 \left(-\frac{1}{16} \cos 5\alpha + \frac{1}{12} \cos \alpha \right) \\
 & + h^4 \left(\frac{1}{640} \cos 7\alpha + \frac{1}{64} \cos \alpha \right) \\
 & + h^6 \left(\frac{-1}{46080} \cos 9\alpha - \frac{7}{20480} \cos 5\alpha + \frac{1}{768} \cos \alpha \right) \\
 & + h^8 \left(\frac{1}{2^{14} \cdot 3^2 \cdot 5 \cdot 7} \cos 11\alpha - \frac{17}{2^{15} \cdot 3^2 \cdot 5} \cos 7\alpha \right. \\
 & \quad \left. - \frac{1}{2^{14}} \cos 5\alpha - \frac{1}{2^{13}} \cos \alpha \right) + \dots;
 \end{aligned}$$

$$R = 9 + \frac{1}{16}h^4 + \frac{1}{64}h^6 + \frac{59}{61440}h^8 - \frac{3}{16384}h^{10} + \dots$$

15. We now propose to develop P_1 , and as for the same value of g the constant R has a different value in P_1 and P_2 , let us now represent it by R' . So we have the differential equation

$$(m) \quad \frac{d^2 P_1}{d\alpha^2} + (R' - 2h^2 \cos 2\alpha) P_1 = 0,$$

and you have to find the solution that becomes zero for $\alpha = 0$ and choose

$$R' = g^2 + \beta h^4 + \gamma h^6 + \delta h^8 + \varepsilon h^{10} + \eta h^{12} + \dots$$

so that it is periodic. Let

$$P_1 = \sin g\alpha + h^2 p + h^4 p_1 + h^6 p_2 + h^8 p_3 + \dots,$$

and we will have exactly the same calculations as for P_2 , with only the change of the cosines into sines; so we will have

$$\begin{aligned} p &= a \sin(g+2)\alpha + c \sin(g-2)\alpha, \\ p_1 &= d \sin(g+4)\alpha + k \sin(g-4)\alpha, \\ p_2 &= l \sin(g+6)\alpha + n \sin(g+2)\alpha + q \sin(g-2)\alpha \\ &\quad + s \sin(g-6)\alpha, \\ &\dots \end{aligned}$$

and a, c, d, k have the same values as in the expression of P_2 ; so we still have for the constant

$$\begin{aligned} R' &= g^2 + \frac{1}{2(g^2-1)}h^4 + \frac{5g^2+7}{32(g^2-1)^3(g^2-4)}h^8 \\ &\quad + \frac{9g^6+22g^4-203g^2-116}{64(g^2-1)^5(g^2-4)^2(g^2-9)}h^{12} + \dots, \end{aligned}$$

and this development must be stopped at the same term as in R ; then, although the first terms of R and R' are the same, these two constants are not equal, and the two series separate from the term from which we are forced to replace g by its particular value.

14. When we have obtained the value of P_2 for an odd value of g , it is easy to deduce that of P_1 for the same value of g . Indeed, if P_2 is given by the equation

$$(n) \quad \frac{d^2 P}{d\alpha^2} + [R(g^2, h^2) - 2h^2 \cos 2\alpha] P = 0,$$

by changing h^2 to $-h^2$ and α to $\frac{\pi}{2} - \alpha$, we will have a periodic function which will satisfy the equation

$$(p) \quad \frac{d^2 P}{d\alpha^2} + [R(g^2, -h^2) - 2h^2 \cos 2\alpha] P = 0,$$

same form as (m) , and if g is odd, the cosines of P_2 change into sines; so we have the expression of P_1 , and moreover we see that we obtain the constant R' , which suits P_1 , by changing in R h^2 to $-h^2$.

According to this, for $g = 1$ we have

$$\begin{aligned} P_1 &= \sin \alpha - \frac{h^2}{8} \sin 3\alpha + h^4 \left(\frac{1}{192} \sin 5\alpha + \frac{1}{64} \sin 3\alpha \right) \\ &\quad - h^6 \left(\frac{1}{9216} \sin 7\alpha + \frac{1}{1152} \sin 5\alpha + \frac{1}{1536} \sin 3\alpha \right) \\ &\quad + h^8 \left(\frac{1}{737280} \sin 9\alpha + \frac{1}{49152} \sin 7\alpha \right. \\ &\quad \left. + \frac{1}{24576} \sin 5\alpha - \frac{11}{36864} \sin 3\alpha \right) + \dots; \\ R' &= 1 - h^2 - \frac{1}{8} h^4 + \frac{1}{64} h^6 - \frac{1}{1536} h^8 - \frac{11}{36864} h^{10} + \dots \end{aligned}$$

And for $g = 3$ we have

$$\begin{aligned} P_1 &= \sin 3\alpha + h^2 \left(-\frac{1}{16} \sin 5\alpha + \frac{1}{12} \sin \alpha \right) \\ &\quad + h^4 \left(\frac{1}{640} \sin 7\alpha + \frac{1}{64} \sin \alpha \right) \\ &\quad + h^6 \left(\frac{-1}{46080} \sin 9\alpha - \frac{7}{20480} \sin 5\alpha + \frac{1}{768} \sin \alpha \right) \\ &\quad + h^8 \left(\frac{1}{2^{14} \cdot 3^2 \cdot 5 \cdot 7} \sin 11\alpha - \frac{17}{2^5 \cdot 3^2 \cdot 5} \sin 7\alpha \right. \\ &\quad \left. - \frac{1}{2^{14}} \sin 5\alpha + \frac{1}{2^{13}} \cos \alpha \right) + \dots; \end{aligned}$$

$$R' = 9 + \frac{1}{16} h^4 - \frac{1}{64} h^6 + \frac{59}{61440} h^8 + \frac{3}{16384} h^{10} + \dots$$

If g is even and P_2 is given by the equation (n) , changing h^2 to $-h^2$ and α to $\frac{\pi}{2} - \alpha$ in P_2 , we will have a function P which will satisfy the equation (p) ; but the cosines remain cosines in this change; so the new expression still belongs to P_2 , and we conclude

$$R(g^2, -h^2) = R(g^2, h^2).$$

The same reasoning is applicable to P_1 ; therefore, if g is even, P_1 does not change when we replace α by $\frac{\pi}{2} - \alpha$ and h^2 by $-h^2$, and R' contains only fourth powers of h .

By a special calculation, we find, for $g = 2$;

$$\begin{aligned} P_1 &= \sin 2\alpha - \frac{h^2}{12} \sin 4\alpha + \frac{h^4}{384} \sin 6\alpha \\ &\quad + h^6 \left(\frac{-1}{23040} \sin 8\alpha + \frac{5}{13824} \sin 4\alpha \right) \\ &\quad + h^8 \left(\frac{1}{2211840} \sin 10\alpha + \frac{37}{2209140} \sin 6\alpha \right) \\ &\quad + h^{10} \left(\frac{-1}{309657600} \sin 12\alpha + \frac{11}{33177600} \sin 8\alpha \right. \\ &\quad \left. - \frac{289}{79626240} \sin 4\alpha \right) + \dots; \\ R' &= 4 - \frac{1}{12} h^4 + \frac{5}{13824} h^8 - \frac{289}{79626240} h^{12} + \dots \end{aligned}$$

For $g = 4$, we find

$$\begin{aligned}
P_1 = & \sin 4\alpha + h^2 \left(-\frac{1}{20} \sin 6\alpha + \frac{1}{12} \sin 2\alpha \right) + \frac{h^4}{960} \sin 8\alpha \\
& - h^6 \left(\frac{1}{80640} \sin 10\alpha + \frac{13}{96000} \sin 6\alpha - \frac{1}{4320} \sin 2\alpha \right) \\
& + h^8 \left(\frac{1}{10321920} \sin 12\alpha + \frac{23}{6048000} \sin 8\alpha \right) \\
& + h^{10} \left(\frac{1}{1857945600} \sin 14\alpha - \frac{53}{1032192000} \sin 10\alpha \right. \\
& \quad \left. + \frac{293}{2419200000} \sin 6\alpha + \frac{397}{124416000} \sin 2\alpha \right) + \dots; \\
R' = & 16 + \frac{1}{30} h^4 - \frac{317}{864000} h^8 + \frac{4507}{1360800000} h^{12} + \dots
\end{aligned}$$

On the functions Q which must be associated with P_1 and P_2 .

15. We go from the equation

$$(1) \quad \frac{d^2 P}{d\alpha^2} + (R - 2h^2 \cos 2\alpha) P = 0$$

to the one giving Q

$$(2) \quad \frac{d^2 Q}{d\beta^2} - [R - 2h^2 E(2\beta)] Q = 0,$$

by changing α to βi and P to Q , and R has the same value in both; so if in the values of P_1 and P_2 we change α to βi , we will have solutions of the equation (2). Now it is easy to understand that if the membrane folds in a vibratory movement along the lines FA and $F'A'$, included between the foci and the vertices of the major axis, and which have the equations $\alpha = 0$ and $\alpha = \pi$, it must also bend all along the line FF' led between the foci and which is given by $\beta = 0$. So

$$P_1 = \sin g\alpha + h^2 [a \sin(g+2)\alpha + b \sin(g-2)\alpha] + \dots,$$

which becomes zero for $\alpha = 0$ and $\alpha = \pi$, must effectively be associated with the function

$$Q_1 = A \left\{ \frac{e^{\beta g} - e^{-\beta g}}{2} + h^2 \left[a \frac{e^{\beta(g+2)} - e^{-\beta(g+2)}}{2} + b \frac{e^{\beta(g+2)} - e^{-\beta(g+2)}}{2} \right] + \dots \right\},$$

which is deduced from P_1 by changing α to βi , and which becomes zero for $\beta = 0$. Likewise

$$P_2 = \cos g\alpha + h^2 [a \cos(g+2)\alpha + b \cos(g-2)\alpha] + \dots,$$

which is maximum or minimum for $\alpha = 0$, must associate with

$$Q_2 = A \left\{ E(\beta g) + h^2 [a E(\overline{g+2}\beta) + b E(\overline{g-2}\beta)] \right\} + \dots,$$

which is maximum or minimum for $\beta = 0$.

However the expressions of P_1 and P_2 could be convergent, without those of Q_1 and Q_2 being so, for all the values that β can take inside the membrane; but, for the moment, we want to point out the characters of the functions Q_1 and Q_2 rather than to give a means of calculating them.

On the nodal lines.

16. We can already make some reflections on the nature of the nodal lines of an elliptical membrane. We have seen that in a simple vibratory movement the displacement of a point of the membrane is given by the formula

$$w = PQ \sin 2\lambda mt,$$

where P and Q satisfy the two equations (1) and (2) of the previous number, and where λ is determined by the fixity of the contour. And P being a function of α with period 2π , we can only take for it P_1 and P_2 , so that Q_1 and Q_2 being the corresponding values of Q , we have two kinds of solutions given by the formulas

$$w = P_1 Q_1 \sin 2\lambda_1 mt,$$

$$w = P_2 Q_2 \sin 2\lambda_2 mt,$$

and the nodal lines have for equations, in the first kind,

$$P_1 = 0, \quad Q_1 = 0,$$

and, in the second kind,

$$P_2 = 0, \quad Q_2 = 0.$$

In the first kind, the major axis is a nodal line; in the second kind, there is maximum or minimum vibration.

The equations $Q_1 = 0$ and $Q_2 = 0$ give ellipses which have the same foci as the contour of the membrane. The equations $P_1 = 0$ and $P_2 = 0$ determine the asymptotes of the hyperbolic nodal lines which still have the same foci; and the integer g which enters P_1 and P_2 indicates how many times these functions become zero from 0 to π , i.e. the number of hyperbolic nodal lines, in designating by *hyperbolic nodal line* the two branches of a hyperbola terminated at the major axis which have the same asymptote. In this view, a hyperbola is counted for two of these lines; but if the major axis or the minor axis are motionless, they are only counted for a single hyperbolic line.

If g is zero, the movement can only be of the second kind, and there is no hyperbolic line.

If $g = 1$, there is a hyperbolic nodal line only the major axis in the first kind and only the minor axis in the second kind.

If $g = 2$, in the first kind we have for these lines the small and the long axis, and in the second kind a hyperbola.

If $g = 3$, we have for these nodal lines a hyperbola and either the major axis or the minor axis, depending on whether the movement is of the first or of the second kind. And so on.

When the series found for P_1 and P_2 will be rapidly convergent, as we know exactly the number of roots understood from 0 to $\frac{\pi}{2}$, our formulas will be very convenient, and it will be easy to separate these roots by substitutions and to calculate them with the approximation desired by experience.

But if h , which is proportional to the eccentricity of the membrane and the pitch of the sound, is large enough, these series are no longer convergent or are too small to be of convenient use; we can no longer even use the expansions of R and R' according to the powers of h and we are forced to resort to other methods.

Developments of the functions P_1 and P_2 according to powers of $\sin \alpha$ and $\cos \alpha$.

17. If we let

$$\nu = \cos \alpha$$

and we take ν to be variable, the equation

$$(1) \quad \frac{d^2 P}{d\alpha^2} + [R(g^2, h^2) - 2h^2 \cos 2\alpha] P = 0$$

turns into the following :

$$(a) \quad \frac{d^2 P}{d\nu^2} (1 - \nu^2) - \frac{dP}{d\nu} \nu + [R(g^2, h^2) + 2h^2 - 4h^2 \nu^2] P = 0$$

and if we let

$$\nu' = \sin \alpha$$

be variable, it changes to this other :

$$(b) \quad \frac{d^2 P}{d\nu'^2} (1 - \nu'^2) - \frac{dP}{d\nu'} \nu' + [R(g^2, h^2) - 2h^2 + 4h^2 \nu'^2] P = 0.$$

Suppose first g is even: the constant $R(g^2, h^2)$, as we have seen, depends only on the even powers of h^2 ; so we go from the equation (a) to the equation (b) by changing ν to ν' and h^2 to $-h^2$, and we conclude this remarkable

property that, when g is even, P_1 and P_2 are functions of ν which remain constant when we change ν to ν' and h^2 in $-h^2$.*

Suppose g is odd: if P_1 is a solution of equation (1), we know that the value of P_2 corresponding to the same value of g is a solution of the same equation in which we replace $R(g^2, h^2)$ by $R(g^2, -h^2)$; then P_1 also satisfies the two equations (a) and (b), and P_2 to the following two :

$$(c) \quad \frac{d^2 P}{d\nu^2}(1 - \nu^2) - \frac{dP}{d\nu}\nu + [R(g^2, -h^2) + 2h^2 - 4h^2\nu^2] P = 0.$$

$$(d) \quad \frac{d^2 P}{d\nu'^2}(1 - \nu'^2) - \frac{dP}{d\nu'}\nu' + [R(g^2, -h^2) - 2h^2 + 4h^2\nu'^2] P = 0.$$

We go from (a) to (d) or from (c) to (b) by changing ν to ν' and h^2 in $-h^2$; therefore, by the same changes, we pass from the expression of P_1 to that of P_2 , or conversely from that of P_2 to that of P_1 .

Let us consider equation (a) and let, to simplify the writing,

$$R(g^2, h^2) + 2h^2 = m.$$

The general solution of this equation is the sum of two particular solutions, one Π_2 even in ν and the other Π_1 odd. For the function Π_2 , we let

$$(e) \quad \Pi_2 = k_0 + k_1\nu^2 + k_2\nu^4 + k_3\nu^6 + \dots + k_{s-1}\nu^{2s-2} + k_s\nu^{2s} + k_{s+1}\nu^{2s+2} + \dots$$

and we will determine, by substituting in (a), the coefficients

$$k_1 = -\frac{mk_0}{2}, \quad k_2 = \frac{m(m-4)+8h^2}{2 \cdot 3 \cdot 4}k_0, \\ k_3 = \frac{-m(m-4)(m-16)-56h^2m+128h^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}k_0, \dots$$

By equating to zero the coefficient of ν^{2s} in the result of the substitution, we obtain the formula

$$(2) \quad k_{s+1} = \frac{(4s^2-m)k_s+4h^2k_{s-1}}{(2s+1)(2s+2)},$$

which shows how each term is deduced from the previous two.

For the function Π_1 , let

$$(f) \quad \begin{cases} \Pi_1 = a_1\nu + a_2\nu^3 + a_3\nu^5 + a_4\nu^7 + \dots \\ \quad + a_{s-1}\nu^{2s-3} + a_s\nu^{2s-1} + a_{s+1}\nu^{2s+1} + \dots, \end{cases}$$

*Trans. note: The h appearing in the original has been corrected to the h^2 appearing here.

and we will have for the coefficients of the first terms*

$$a_2 = -\frac{m-1}{2 \cdot 3} a_1, \quad a_3 = \frac{(m-1)(m-9)+24h^2}{2 \cdot 3 \cdot 4 \cdot 5} a_1,$$

$$a_4 = \frac{-(m-1)(m-9)(m-25)+(170-26m)4h^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} a_1, \dots,$$

and each term is deduced from the two preceding ones by the relation

$$(3) \quad a_{s+1} = \frac{[(2s-1)^2-m]a_s+4h^2s_{s-1}}{2s(2s+1)}, \dots$$

Suppose the constant R chosen so that P_1 satisfies the equation (a); P_1 is zero or maximum for $\alpha = \frac{\pi}{2}$, and behaves in this property like $\sin g\alpha$, to which it reduces, to within a constant factor, for $h = 0$; therefore it is zero if g is even, and it is maximum if g is odd. However, for $\nu = 0$ where $\alpha = \frac{\pi}{2}$, Π_1 is zero and Π_2 is maximum; so if g is even P_1 is equal to Π_1 , and if g is odd P_1 is equal to Π_2 , except for a constant factor. Imagine, on the contrary, R is chosen so that P_2 satisfies the equation (a), and we also see that P_2 is zero or maximum for $\alpha = \frac{\pi}{2}$, depending on whether g is odd or even; and we conclude that P_2 is equal, except for a factor, to Π_1 if g is odd, and to Π_2 if g is even.

18. Let us come to the equation (b), and let

$$R - 2h^2 = m';$$

the general solution of this equation is also the sum of two particular solutions, one of which is even and the other odd in ν' . If R is chosen so that P_2 is solution of this equation, then P_2 , which is maximum for $\nu' = 0$, merges with the particular solution which enjoys this property, and we have

$$(e') \quad P_2 = k'_0 + k'_1\nu'^2 + k'_2\nu'^4 + \dots + k'_{s-1}\nu'^{2s-2} + k'_s\nu'^{2s} + k'_{s+1}\nu'^{2s+2} + \dots,$$

an expression where k'_0, k'_1, k'_2, \dots are deduced from k_0, k_1, k_2, \dots , by changing h^2 to $-h^2$. So we have[†]

$$k'_1 = -\frac{m'k'_0}{1 \cdot 2}, \quad k'_2 = \frac{m'(m'-4)-8h^2}{1 \cdot 2 \cdot 3 \cdot 4} k'_0, \dots,$$

and the rule which links the coefficients of three consecutive terms together is

$$(4) \quad k'_{s+1} = \frac{(4s^2-m')k'_s-4h^2k'_{s-1}}{(2s+1)(2s+2)}.$$

*Trans. note: corrected the original by including a multiplicative factor of a_1 missing in the second displayed equation below.

[†]Trans. note: I have corrected the subscript of the second equation to read k'_2 rather than k'_1 .

If, on the contrary, R is chosen so that P_1 is a solution of equation (1), P_1 , having to become zero for $\nu' = 0$, merges with the odd solution in ν' of (b), and we have

$$(f') \quad P_1 = a'_1 \nu' + a'_2 \nu'^3 + a'_3 \nu'^5 + \dots + a'_{s-1} \nu'^{2s-3} + a'_s \nu'^{2s-1} + a'_{s+1} \nu'^{2s+1} + \dots$$

a'_1, a'_2, a'_3, \dots being quantities deduced from a_1, a_2, a_3, \dots by changing h^2 to $-h^2$; so we have first

$$a'_2 = -\frac{m'-1}{2 \cdot 3} a'_1, \quad a'_3 = \frac{(m'-1)(m'-9)-24h^2}{2 \cdot 3 \cdot 4 \cdot 5} a'_1, \dots,$$

and then the general formula

$$(5) \quad a'_{s+1} = \frac{[(2s-1)^2 - m']a'_s - 4h^2 a'_{s-1}}{2s(2s+1)}.$$

Here it is very important to note that if we replace R by an arbitrary number, the function (e') or the function (f') is not equal to any of the two functions (e) and (f) ; because the function (f') , for example, which is zero for $\nu' = 0$ or $\alpha = 0$, is neither zero nor maximum for $\alpha = \frac{\pi}{2}$ and cannot therefore be confused with either of the two functions (e) and (f) . It is only if R is determined so that equation (1) has a solution of period 2π , that, this solution having to be zero or maximum for $\alpha = 0$ and $\alpha = \frac{\pi}{2}$, one of the two expressions (e') and (f') , which is equal to it, is identical, except for a factor, to one of the two expressions (e) and (f) .

For example, suppose g even, and, consequently, P_2 is equal to the product of Π_2 by a constant A , if R has been taken correctly. By considering the values of these functions for $\alpha = 45^\circ, 30^\circ, 60^\circ$, we obtain the formulas

$$A \left(k_0 + \frac{k_1}{2} + \frac{k_2}{4} + \frac{k_3}{16} + \dots \right) = k'_0 + \frac{k'_1}{2} + \frac{k'_2}{4} + \dots,$$

$$A \left(k_0 + k_1 \frac{3}{4} + k_2 \frac{9}{16} + \dots \right) = k'_0 + \frac{k'_1}{4} + \frac{k'_2}{16} + \dots,$$

$$\frac{1}{A} \left(k'_0 + k'_1 \frac{3}{4} + k'_2 \frac{9}{16} + \dots \right) = k_0 + \frac{k_1}{4} + \frac{k_2}{16} + \dots,$$

each of which can determine the factor A .

We have so far accepted that R was known; but if it is not, by eliminating A between two of these formulas, we will obtain an equation whose two members will be the products of two series, which will contain only the unknown R and may be used to determine it.

It is extremely easy to recognize that the series $(e), (f), (e'), (f')$ are convergent, as long as ν or ν' is < 1 . Or more generally the series

$$k_0 + k_1 x + k_2 x^2 + \dots + k_n x^n + k_{n+1} x^{n+1} + \dots,$$

in which x is < 1 , and of which three consecutive coefficients are linked by the relation

$$(7) \quad k_{s+1} = A_s k_s + a_s k_{s-1};$$

moreover the limit of A_s , when s grows indefinitely, is less than unity or it is at most equal, and the limit of a_s is zero; then the series is convergent.

Indeed, the limit of the ratio $\frac{k_{s+1}}{k_s}$ is equal to the limit τ of A_s when s grows indefinitely; therefore the limit of the relation of a term to the previous in the series is τx , a number < 1 , and it is convergent. Now the relations (2), (3), (4), (5), satisfying the same conditions as the relation (7), the series are convergent as long as ν and ν' are < 1 , i.e. whatever α may be.

On the contrary, if ν and ν' were > 1 , these series would be divergent.

However to have well convergent series, we will prefer the formulas (e') and (f') when α will be between 0 and 45 degrees, and the formulas (e) and (f) when α will be between 45 and 90 degrees.

19. It is easy to see that, from a sufficiently distant term, all the following terms have the same sign in the four series (e), (f), (e'), (f') but we will also study the number of variations of the latter two. If we first assume $h = 0$, R and m' are reduced to g^2 , and the two series (e') and (f') to

$$P_2 = k'_0 \left[1 - \frac{g^2}{2} \nu'^2 + \frac{g^2(g^2-4)}{2 \cdot 3 \cdot 4} \nu'^4 - \frac{g^2(g^2-4)(g^2-16)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \nu'^6 + \dots \right],$$

$$P_1 = \frac{a'_1}{g} \left[g \nu' - \frac{g(g-1)}{2 \cdot 3} \nu'^3 + \frac{g(g^2-1)(g^2-9)}{2 \cdot 3 \cdot 4 \cdot 5} \nu'^5 - \dots \right],$$

except for factors close to the values of $\cos g\alpha$ and $\sin g\alpha$.

If we make successively

$$g = 1, 2, 3, \dots,$$

the factor in square brackets of P_2 becomes*

$$\begin{aligned} \cos \alpha &= 1 - \frac{1}{1 \cdot 2} \nu'^2 - \frac{3}{1 \cdot 2 \cdot 3 \cdot 4} \nu'^4 - \frac{3 \cdot 15}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \nu'^6 - \dots, \\ \cos 2\alpha &= 1 - 2\nu'^2 \\ \cos 3\alpha &= 1 - \frac{3^2}{1 \cdot 3} \nu'^2 + \frac{3^2(3^2-2^2)}{1 \cdot 2 \cdot 3 \cdot 4} \nu'^4 + \frac{3^2(3^2-2^2)(4^2-3^2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \nu'^6 + \dots, \\ \cos 4\alpha &= 1 - \frac{4^2}{1 \cdot 2} \nu'^2 + \frac{4^2(4^2-2^2)}{1 \cdot 2 \cdot 3 \cdot 4} \nu'^4, \\ &\dots \end{aligned}$$

*Trans. note: I added a factor of ν'^4 in the fourth equation, which seems to fit, but this needs to be verified correct. RMC: This looks correct to me now.

The series which gives $\cos \alpha$ has a single variation and a single positive root in ν' , $\nu' = \sin \frac{\pi}{2}$; $\cos 2\alpha$ also only has a variation and a positive root $\nu' = \sin \frac{\pi}{4}$; $\cos 3\alpha$ has two variations and two positive roots, $\nu' = \sin \frac{\pi}{6}$ and $\sin \frac{\pi}{2}$; $\cos 4\alpha$ has two variations and two positive roots, $\nu' = \sin \frac{\pi}{8}$ and $\sin \frac{3\pi}{8}$. And in general the series which gives $\cos g\alpha$ by means of ν' has as many variations as the equation $\cos g\alpha = 0$ has positive roots in ν' .

We also recognize that the series which expresses $\sin g\alpha$ has a number of variations equal to the number of its roots in ν' .

So when h is zero, P_1 and P_2 expressed by ν' have the same number of variations as positive roots, and we will show that this property subsists for any value of h .

Suppose, for example, that it is P_2 ; we have between the coefficients of three consecutive terms of

$$P_2 = k'_0 + k'_1\nu'^2 + k'_2\nu'^4 + \dots$$

the relation

$$k'_{s+1} = \frac{(4s^2 - m')k'_s - 4h^2k'_{s-1}}{(2s+1)(2s+2)},$$

and imagine that we increase the quantity h . It follows from this formula that for any value of h (except zero), two consecutive coefficients cannot become zero. Indeed if k'_s , and k'_{s+1} , were zero, k'_{s-1} so would be, then for the same reason k'_{s-2} , and so on, so that all of the terms in the series would become zero.

Secondly, if the coefficient of one of the terms becomes zero for a certain value of h , the coefficients of the two terms which surround it are of opposite sign, as we see by the same formula.

It follows from this that, while h increases, P_2 cannot acquire or lose any variation, and that it consequently has the same number as for $h = 0$. But, as we have shown (n°10), P_2 Always becomes zero the same number of times $\alpha = 0$ to $\alpha = \frac{\pi}{2}$, regardless of h ; so finally the equation

$$P_2(\nu') = 0$$

has precisely as many real, positive and < 1 roots as it has variations.

20. This property separates the roots of this equation. First consider an algebraic equation

$$f(x) = 0,$$

which has as many positive roots as variations; let us form the sequence of derivatives of $f(x)$

$$(A) \quad f(x), f'(x), f''(x), \dots,$$

which for $x = 0$ has the same signs as the series of coefficients of $f(x)$; it is easy to prove that, while one increases x , it is impossible that this series never gains variations; but that when $f(x)$ becomes zero, a variation from the first to the second term is lost. So if the equation $f(x) = 0$ has as many variations as there are positive roots, like the sequence (A) for $x = 0$ has this number of variations, that when x increases, it loses one each time $f(x)$ becomes zero, and it cannot gain one, it can only lose one when x passes through a root of the equation $f(x) = 0$, and counting the number of variations of the sequence (A) for $x = a$ and $x = b$, and making the difference, we have precisely the number of roots between a and b .

All these reasonings are applicable to the equation $P_2(\nu') = 0$, formed of a sequence of an infinite number of terms, and we will have to examine the infinite sequence

$$(B) \quad P_2(\nu'), \quad \frac{dP_2}{d\nu'}, \quad \frac{d^2P_2}{d\nu'^2}, \quad \frac{d^3P_2}{d\nu'^3}, \dots,$$

the first two will be obtained by the sequence

$$P_2 = k'_0 + k'_1\nu'^2 + k'_2\nu'^4 + \dots,$$

$$\frac{dP_2}{d\nu'} = k'_1 2\nu' + k'_2 4\nu'^3 + \dots;$$

then we will calculate the following derivatives using the equation

$$\frac{d^2P}{d\nu'^2}(1 - \nu'^2) - \frac{dP}{d\nu'}\nu' + (R - 2h^2 + 4h^2\nu'^2)P = 0$$

and those that we deduce by differentiation.

The infinite number of terms in the sequence (B) does not offer any trouble; because we know how many roots $P_2(\nu') = 0$ has between 0 and 1; it has as many as $\cos g\alpha = 0$ between 0 and $\frac{\pi}{2}$: it is $\frac{g}{2}$ or $\frac{g+1}{2}$, depending on whether g is even or odd; $P_2(\nu')$ will have as many variations; we will therefore calculate only a number n of terms of the series which gives P_2 , sufficient to count all the variations there, and it follows from the first principles of algebra that the sequence (B) will have no variations beyond its first n terms when we give ν' a positive value.

21. Everything we just said about P_2 can be repeated for P_1 . We can now get a more precise idea of the constant R . This quantity, for $h = 0$, is reduced to the square of an integer: it is therefore positive when h is very small; but we will demonstrate that the constant R relative to P_2 is not only positive, but also greater than $2h^2$, whatever h .

We have just examined the changes in the number of variations of the sequence (B) when we vary ν' from 0 to 1; but as P_1 and P_2 are one an

odd function in ν' and the other an even function, it follows that the series (B) has the same property between -1 and 0 it has between 0 and 1 , and consequently if we increase ν' from -1 to $+1$, a variation is lost only each time ν' goes through a root of $P_2(\nu') = 0$. This happens for $P_1(\nu')$.

P_2 is given by the equation *

$$(n) \quad \frac{d^2 P_2}{d\nu'^2} (1 - \nu'^2) = \frac{dP_2}{d\nu'} \nu' + (2h^2 - 4h^2 \nu'^2 - R) P_2;$$

let us make $\nu' = 0$, we then have $\frac{dP_2}{d\nu'} = 0$, and therefore P_2 and $\frac{d^2 P_2}{d\nu'^2}$ are of opposite sign; because if they were of the same sign, letting ν' grow from a very small negative quantity to a very small positive quantity, $\frac{dP_2}{d\nu'}$, becoming zero, will pass from a sign opposite to that of P_2 to a similar sign, and the series would lose two variations, while it should not lose any. So the coefficient of P_2 is negative, and we have

$$2h^2 - R < 0,$$

or $R > 2h^2$.

Let us denote, as we have already done, by R' the constant R when it belongs to the function P_1 , and we will demonstrate that it is $> -2h^2$ whenever the integer g is > 1 . Indeed, $\frac{dP_1}{d\nu'}$ necessarily becomes zero for a value of ν' ranging from 0 to 1 , and, for this value, P_1 , and $\frac{d^2 P_1}{d\nu'^2}$ are of opposite sign, and since P_1 satisfies the equation (n), when we replace R by R' , we have

$$2h^2 - 4h^2 \nu'^2 - R' < 0$$

or

$$R' > 2h^2(1 - 2\nu'^2),$$

and even more so R' is $> 2h^2$.

It is good to notice that the equation (n) ceases to be applicable to the limit $\nu' = 1$, for this reason that ν' , being the sine of a determined angle, cannot take values greater than unity. And indeed, for $\nu' = 1$, P or $\frac{dP}{d\alpha}$ is zero. Suppose it is P_1 , it follows from this equation that $\frac{dP}{d\nu'}$ would be zero, and therefore also $\frac{dP}{d\alpha}$, according to equality

$$\frac{dP}{d\alpha} = \frac{dP}{d\nu'} \cos \alpha;$$

which is impossible.

Differential equations which determine the function Q.

*Trans. note: corrected the missing '2' in the second derivative.

22. We know that Q is given by the equation

$$\frac{d^2 Q}{d\beta^2} - [R - 2h^2 E(2\beta)] = 0$$

and if we let

$$\rho = c \frac{e^\beta + e^{-\beta}}{2}, \quad \rho' = c \frac{e^\beta - e^{-\beta}}{2},$$

and we take ρ and ρ' for variables, we get the two equations

$$(1) \quad \frac{d^2 Q}{d\rho^2} (\rho^2 - c^2) + \frac{dQ}{d\rho} \rho + (4\lambda^2 \rho^2 - R - 2h^2) Q = 0,$$

$$(2) \quad \frac{d^2 Q}{d\rho'^2} (\rho'^2 - c^2) + \frac{dQ}{d\rho'} \rho' + (4\lambda^2 \rho'^2 - R - 2h^2) Q = 0.$$

When we make $c = 0$ in these equations, they are reduced to one, relating to the circular membrane; the two semi-axes ρ and ρ' of any of the ellipses confocal to the membrane changing into the radius r of a circle, we have

$$(3) \quad \frac{d^2 Q}{dr^2} r^2 + \frac{dQ}{dr} r + (4\lambda^2 r^2 - g^2) Q = 0,$$

an equation found at n°2. We have seen that its general solution is the sum of two particular solutions, one of which becomes infinite for $r = 0$. It must be explained how the solution of the circle can be deduced from that of the ellipse.

For this, first suppose λ is zero in the equations (1), (2) and (3); they become, by noting that R is reduced to g^2 for the hypothesis $\lambda = 0$, which gives $h = 0$,

$$(1') \quad \frac{d^2 Q}{d\rho^2} (\rho^2 - c^2) + \frac{dQ}{d\rho} \rho - g^2 Q = 0,$$

$$(2') \quad \frac{d^2 Q}{d\rho'^2} (\rho'^2 - c^2) + \frac{dQ}{d\rho'} \rho' - g^2 Q = 0,$$

$$(3') \quad \frac{d^2 Q}{dr^2} r^2 + \frac{dQ}{dr} r - g^2 Q = 0.$$

The general integral of (3') is

$$Q = Ar^g + Br^{-g},$$

and becomes infinite for $r = 0$, that is to say at the center of the circle. But the integrals of the two equations (1') and (2') are

$$Q = A \left(\rho + \sqrt{\rho^2 - c^2} \right)^g + \frac{B}{\left(\rho + \sqrt{\rho^2 - c^2} \right)^g},$$

$$Q = A \left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g + \frac{B}{\left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g},$$

and we see that the second part of their expression is not infinite for $\rho = c$ where $\rho' = 0$, except in the case where c is zero.

Thus the general solution of the equations (1) and (2), when we make $\lambda = 0$, contains two arbitrary constants and does not become infinite for $\rho' = 0$.

We have seen that the displacement of a point on the membrane is represented by

$$w = Au \sin 2\lambda mt, \quad u = PQ,$$

and for the same value of g , P can become zero or be maximum for $\alpha = 0$, and has two expressions, P_1 and P_2 , which correspond to two different values of the constant R , which only become identical for $h = 0$; from there, for u , two expressions,

$$u = P_1 Q_1, \quad u = P_2 Q_2,$$

in which Q_1 is a value of Q which becomes zero for $\beta = 0$, and Q_2 a value which becomes maximum for this value of β . Let us see what Q_1 and Q_2 reduce to when we make $\lambda = 0$.

Let us say that

$$Q = A \left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g + B \left(\rho' + \sqrt{\rho'^2 + c^2} \right)^{-g}$$

is zero for $\rho' = 0$, and we will have $\frac{B}{A} = -c^{2g}$; so

$$Q_1 = A \left[\left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g - \frac{c^{2g}}{\left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g} \right];$$

and if we express that $\frac{dQ}{d\rho'}$ is zero for $\rho' = 0$, we have $\frac{B}{A} = c^{2g}$; so

$$Q_2 = A \left[\left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g + \frac{c^{2g}}{\left(\rho' + \sqrt{\rho'^2 + c^2} \right)^g} \right].$$

Finally, if we make $c = 0$, the two values of Q_1 and Q_2 for the same value of the integer g are identical.

These explanations were useful in helping to understand how the theory of the circular membrane is enclosed in that of the elliptical membrane; because it is obvious that what we have just found when λ is zero, is true for any value of λ .

25. Let us return to equations (1) and (2). If we let

$$\frac{\rho}{c} = u,$$

we will have, instead of the equation (1),

$$\frac{d^2 Q}{du^2}(u^2 - 1) + \frac{dQ}{du} + (4h^2 u^2 - R - 2h^2)Q = 0,$$

and this equation is deduced from that which gives P by means of ν by only the change from P to Q and from ν to u . Now we have seen that P_2 is given by the series

$$P_2 = k_0 + k_1 \nu^2 + k_2 \nu^4 + \dots,$$

and P_1 by this other

$$P_1 = a_1 \nu + a_2 \nu^3 + a_3 \nu^5 + \dots,$$

$k_0, k_1, k_2, \dots, a_1, a_2, a_3, \dots$ having the values calculated at n°17; so the corresponding values of Q are*

$$Q_2 = k_0 + k_1 \frac{\rho^2}{c^2} + k_2 \frac{\rho^4}{c^4} + \dots,$$

$$Q_1 = a_1 \frac{\rho}{c} + a_2 \frac{\rho^3}{c^3} + a_3 \frac{\rho^5}{c^5} + \dots$$

However, these series cannot be used; because they are never convergent if ρ is $> c$; which always takes place in our problem.

But let

$$\frac{\rho'}{c'} = u',$$

instead of the equation (2)

$$\frac{d^2 Q}{du'^2}(u'^2 + 1) + \frac{dQ}{du'}u' + (4h^2 u'^2 - R + 2h^2)Q = 0,$$

which is deduced from the equation which gives P by means of ν' (n°17), by changing ν' to $u'\sqrt{-1}$. So Q_2 and Q_1 will be given by the formulas

$$Q_2 = k'_0 - k'_1 \frac{\rho'^2}{c'^2} + k'_2 \frac{\rho'^4}{c'^4} - k'_3 \frac{\rho'^6}{c'^6} + \dots,$$

$$Q_1 = a'_1 \frac{\rho'}{c'} - a'_2 \frac{\rho'^3}{c'^3} + a'_3 \frac{\rho'^5}{c'^5} - \dots$$

We know that the series which give P_1 and P_2 by means of ν' are convergent as long as ν' is < 1 ; the previous ones are deduced by the change

*Trans. note: I believe that the second equation should be for Q_1 rather than Q , a correction I have made.

from ν' to $\frac{\rho'}{c}\sqrt{-1}$; it therefore follows from the convergence circle theorem that these series are convergent, as long as ρ' is $< c$. These series will be very convenient, if we consider a very-eccentric membrane so that ρ' is much smaller than c .

When ρ' is $> c$, we can usually get Q with enough approximation as follows. Let

$$z = \frac{\rho + \sqrt{\rho^2 - c^2}}{2} = \frac{ce^\beta}{2},$$

and taking z to be variable, we will have

$$z^2 \frac{d^2 Q}{dz^2} + z \frac{dQ}{dz} + \left[4\lambda^2 z^2 \left(1 + \frac{c^4}{16z^4} \right) - R \right] Q = 0.$$

However, if $\rho' = \sqrt{\rho^2 - c^2}$ is $> c$, we will have

$$\rho > c\sqrt{2}, \quad \frac{c}{z} < 2(\sqrt{2} - 1),$$

$$\frac{c^4}{16z^4} < 17 - 12\sqrt{2} \quad \text{or} \quad < 0.03056 \dots$$

So if in the previous differential equation, we reduce the factor of $4\lambda^2 z^2$ to unity, the one that will result will make Q known in general with some approximation. Now the equation then takes the form that we found for the circle

$$z^2 \frac{d^2 Q}{dz^2} + z \frac{dQ}{dz} + (4\lambda^2 z^2 - R)Q = 0,$$

and by setting $R = n^2$, we will have for approximate value the expression

$$Q = Cz^n \left[1 - \frac{(\lambda z)^2}{1(n+1)} + \frac{(\lambda z)^4}{1 \cdot 2(n+1)(n+2)} - \frac{(\lambda z)^6}{1 \cdot 2 \cdot 3(n+1)(n+2)(n+3)} + \dots \right],$$

where n however is no longer an integer as in the case of the circle, but depends on h . Besides, we will give another way to develop Q later.

Developments of Q_1 and Q_2 .

24. The functions* Q_2 and Q_1 satisfy the equation

$$\frac{d^2 Q}{d\beta^2} = \left[R - h^2(e^{2\beta} + e^{-2\beta}) \right] Q;$$

let

$$R - h^2(e^{2\beta} + e^{-2\beta}) = T,$$

*Trans. note: the original reads "The functions Q and Q_1 satisfy...", which I have corrected.

and form the derivatives

$$\frac{dT}{d\beta} = -2h^2(e^{2\beta} - e^{-2\beta}), \quad \frac{d^2T}{d\beta^2} = -2^2h^2(e^{2\beta} + e^{-2\beta}), \dots$$

If we make $\beta = 0$, all the derivatives of odd order become zero, and designating the others by A_2, A_4, \dots , let

$$T = A_0, \quad \frac{d^2T}{d\beta^2} = -2^3h^2 = A_2, \dots, \quad \frac{d^{2i}T}{d\beta^{2i}} = -2^{2i+1}h^2 = A_{2i};$$

then develop Q using the formula

$$Q = Q_0 + \left(\frac{dQ}{d\beta}\right)_0 \beta + \left(\frac{d^2Q}{d\beta^2}\right)_0 \frac{\beta^2}{1 \cdot 2} + \dots,$$

and form the derivatives using the formulas

$$\begin{aligned} \frac{d^2Q}{d\beta^2} &= TQ, & \frac{d^3Q}{d\beta^3} &= Q\frac{dT}{d\beta} + T\frac{dQ}{d\beta}, \\ \frac{d^4Q}{d\beta^4} &= Q\frac{d^2T}{d\beta^2} + 2\frac{dQ}{d\beta}\frac{dT}{d\beta} + T\frac{d^2Q}{d\beta^2}, \\ &\dots\dots\dots \end{aligned}$$

Let's deal with Q_1 first; since it is zero for $\beta = 0$, all the even order derivatives of Q become zero, and the odd order derivatives have the values, by designating the first by B :

$$\begin{aligned} \left(\frac{dQ}{d\beta}\right)_0 &= B, & \left(\frac{d^3Q}{d\beta^3}\right)_0 &= BA_0, & \left(\frac{d^5Q}{d\beta^5}\right)_0 &= B(A_0^2 + 3A_2), \\ \left(\frac{d^7Q}{d\beta^7}\right)_0 &= B\left(A_0^3 + \frac{2 \cdot 3 + 4 \cdot 5}{1 \cdot 2} A_0 A_2 + \frac{2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} A_4\right), \\ \left(\frac{d^9Q}{d\beta^9}\right)_0 &= B\left(A_0^4 + \frac{2 \cdot 3 + 4 \cdot 5 + 6 \cdot 7}{1 \cdot 2} A_0^2 A_2 + \frac{2 \cdot 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} A_0 A_4 \right. \\ &\quad \left. + \frac{2 \cdot 3 \times 6 \cdot 7}{1 \cdot 2 \times 1 \cdot 2} A_2^2 + \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} A_6\right). \\ &\dots\dots\dots \end{aligned}$$

Let us now indicate the general form of the expression of the derivative $\left(\frac{d^{2n+1}Q}{d\beta^{2n+1}}\right)_0$; first if we find a term that contains $\frac{A_k A_l \dots A_t}{\Pi k \cdot \Pi l \dots \Pi t}$ in a factor, it is that we have

$$(k+2) + (l+2) + \dots + (t+2) = 2n.$$

It remains to determine the coefficient of $\frac{A_k A_l \dots}{\Pi k \cdot \Pi l \dots}$; this coefficient is composed of different parts joined by the sign of the addition, and we obtain any one of them in the following way.

Let us write consecutive numbers

$$(A) \quad 2, \quad 3, \quad 4, \quad 5, \dots, \quad 2n-1,$$

then suppose that we put in a parenthesis k consecutive of these numbers, then in a second parenthesis l other consecutive numbers taken from the numbers (A); then put in a third parenthesis m other consecutive numbers, and so on. Let us imagine that these parentheses are separated by at least two of the numbers (A), and always by an even number of numbers (A); finally let us make this restriction again, that if the first parenthesis on the left does not start with the factor 2, there is before it an even number of numbers (A). We will have any part of the coefficient sought, by multiplying between them the products of the numbers contained in each parenthesis.

Let us go to the development of Q_2 . The first derivative of Q_2 , is zero for $\beta = 0$, and we recognize that it is the same for all the derivatives of odd order, and by representing by D the value of Q_2 for $\beta = 0$, we have for even order derivatives

$$\begin{aligned} \left(\frac{d^2 Q}{d\beta^2}\right)_0 &= A_0 D, \quad \left(\frac{d^4 Q}{d\beta^4}\right)_0 = D \left(A_0 \frac{1 \cdot 2}{1 \cdot 2} A_2\right), \\ \left(\frac{d^6 Q}{d\beta^6}\right)_0 &= D \left(A_0^2 + \frac{1 \cdot 2 + 3 \cdot 4}{1 \cdot 2} A_0 A_2 + A_4\right), \\ \left(\frac{d^8 Q}{d\beta^8}\right)_0 &= D \left(A_0^3 + \frac{1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6}{1 \cdot 2} A_0^2 A_2 + \frac{1 \cdot 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} A_0 A_4 + \right. \\ &\quad \left. \frac{1 \cdot 2 \times 5 \cdot 6}{1 \cdot 2 \times 1 \cdot 2} A_2^2 + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} A_6\right). \\ &\dots\dots\dots \end{aligned}$$

Let us indicate the general form of the derivative $\left(\frac{d^{2n} Q}{d\beta^{2n}}\right)$. For the term*

$$M \frac{A_a A_b A_c \dots}{\Pi a \cdot \Pi b \cdot \Pi c \dots},$$

a, b, c, \dots being equal or unequal, we must have

$$(a + 2) + (b + 2) + \dots = 2n;$$

it only remains to give the value of the coefficient M . To this end, let's write the numbers

$$(B) \quad 1, \quad 2, \quad 3, \quad 4, \dots, \quad 2n - 2;$$

this coefficient will be made up of several different parts, each of which will be obtained as follows. Let us put in a parenthesis a consecutive numbers (B), then in a second parenthesis with b other consecutive numbers, and so on. Let us further imagine that these parentheses are separated at least by

*Trans. note: I am not certain this formula is correct, because of the odd spacing in the original; I have added an ellipsis to attempt to correct it.

two numbers, and always by an even number of the numbers (B); finally add this restriction, that if the first parenthesis does not start with the number 1, there is before it an even number of numbers (B). We will have the sought-after part of the coefficient M , by multiplying between them the products of the numbers contained in each parenthesis.

25. We can develop P_1 and P_2 in the same way as the functions Q_1 and Q_2 .

The series obtained for Q_1 and Q_2 can be written by letting

$$M = R - 2h^2,$$

$$Q_1 = B \left[\beta + M \frac{\beta^3}{1 \cdot 2 \cdot 3} + (M^2 - 24h^2) \frac{\beta^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right. \\ \left. + (M^3 - 104h^2M - 160h^2) \frac{\beta^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots \right],$$

$$Q_2 = D \left[1 + M \frac{\beta^2}{1 \cdot 2} + (M^2 - 8h^2) \frac{\beta^4}{1 \cdot 2 \cdot 3 \cdot 4} \right. \\ \left. + (M^3 - 56hM - 32h^3) \frac{\beta^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots \right].$$

By changing β to αi , we have

$$P_1 = B' \left[\alpha - M \frac{\alpha^3}{1 \cdot 2 \cdot 3} + (M^2 - 24h^2) \frac{\alpha^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \right. \\ \left. - (M^3 - 104h^2M - 160h^2) \frac{\alpha^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots \right],$$

$$P_2 = D' \left[1 - M \frac{\alpha^2}{1 \cdot 2} + (M^2 - 8h^2) \frac{\alpha^4}{1 \cdot 2 \cdot 3 \cdot 4} \right. \\ \left. - (M^3 - 56hM - 32h^3) \frac{\alpha^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots \right].$$

The value of R or M must be chosen so that P_1 and P_2 have 2π for period; therefore the values of these two expressions must remain constant, when we replace α with $\alpha + 2\pi$; a very simple way to determine M is to notice that P_1 must become zero for $\alpha = \pi$ as for $\alpha = 0$, and that P_2 must remain the same for these two values of α ; we thus have one of the two equations

$$(a) \quad \begin{cases} \pi - M \frac{\pi^3}{1 \cdot 2 \cdot 3} + (M^2 - 24h^2) \frac{\pi^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ - (M^3 - 104h^2M - 160h^2) \frac{\pi^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots = 0, \end{cases}$$

$$(b) \quad \begin{cases} 1 - M \frac{\pi^2}{1 \cdot 2} + (M^2 - 8h^2) \frac{\pi^4}{1 \cdot 2 \cdot 3 \cdot 4} \\ - (M^3 - 56hM - 32h^3) \frac{\pi^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots = 1. \end{cases}$$

Suppose for example that it is a vibratory movement of the first kind given by the formula

$$w = P_1 Q_1 \sin 2\lambda mt.$$

Let $\beta = \vartheta$ be the equation of the contour which is fixed, M and h will be provided by (a) and the equation

$$(c) \quad \begin{cases} \vartheta - M \frac{\vartheta^3}{1 \cdot 2 \cdot 3} + (M^2 - 24h^2) \frac{\vartheta^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ + (M^3 - 104h^2 M - 160h^2) \frac{\vartheta^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots = 0, \end{cases}$$

If we especially have in mind the comparison of theory with experience, we can proceed as follows. After having produced experimentally a vibratory state of the membrane, one will note the pitch of the sound, and, consequently, the value of $\lambda = \frac{h}{c}$. Then the equation (a) will contain only the unknown M , and it will remain to verify that M and $h = \lambda c$ satisfy (c).

The previous expressions of P_2 and Q_2 allow to recognize that the parts of the major axis located between the foci and the neighboring vertices produce vibrations of maximum amplitude and the part located between the foci of vibrations of minimum amplitude. Indeed, the value of M entering it is positive; because we have demonstrated (n°21) that R is $> 24h^2$. Let us therefore take on the long axis between the focus and the neighboring vertex a point n for which α is zero; consider a very similar point n' on the confocal ellipse which passes through n ; β is the same for these two points and α is zero for n , very small for n' ; therefore the vibratory displacement is greater for n than for n' .

Let us take a point m on the line FF' which joins the foci, and also a point m' very close on the confocal hyperbola which passes by m ; α is the same for m and m' , β is zero for m , very small for m' ; therefore the magnitude of the vibration is smaller in m than in m' .

We have new expressions of P_1 and P_2 by changing in the previous values of P_1 or P_2 according to the parity of g (n°14) α to $\frac{\pi}{2} - \alpha$, h^2 to $-h^2$ (consequently M to $R + 2h^2$), and we easily conclude that the minor axis of the membrane is immobile or at maximum vibration.

Annular membrane.

26. In the two equations

$$\rho = c \frac{e^\beta + e^{-\beta}}{2},$$

$$\frac{d^2 Q}{d\beta^2} - [R - 2h^2 E(2\beta)] Q = 0,$$

make

$$\beta = \varepsilon - l \frac{c}{2a},$$

and we will have the two other equations

$$\rho = a(e^\varepsilon + qe^{-\varepsilon}),$$

$$\frac{d^2 Q}{d\varepsilon^2} - [R - f^2(e^{2\varepsilon} + qe^{-2\varepsilon})]Q = 0,$$

by letting

$$2\lambda a = f, \quad \frac{c^2}{4a^2} = q,$$

and the last two have this advantage over the first that they apply immediately to the circle by making $q = 0$.

Suppose that Q is zero on the ellipse $\rho = \varrho$, let us determine a so that ε is zero on this ellipse, we will have to let

$$a + \frac{c^2}{4a} = \varrho,$$

from which

$$a = \frac{\varrho}{2} \pm \frac{\sqrt{\varrho^2 - c^2}}{2}.$$

Imagine a ring-shaped membrane whose two fixed edges are confocal ellipses; if we denote by $\rho = \varrho$ the inner contour, the function Q develops as follows:

$$Q = \varepsilon \left(\frac{dQ}{d\varepsilon} \right)_0 + \frac{\varepsilon^2}{1 \cdot 2} \left(\frac{d^2 Q}{d\varepsilon^2} \right)_0 + \frac{\varepsilon^3}{1 \cdot 2 \cdot 3} \left(\frac{d^3 Q}{d\varepsilon^3} \right)_0 + \dots,$$

and it remains to determine the expressions of these derivatives. Although we have here derivatives of even order and others of odd order, we can form them identically as the derivatives of Q_1 with respect to β for $\beta = 0$. However, let us content ourselves with writing the first coefficients of this

series in the most convenient form for numerical calculation:

$$\begin{aligned}
\left(\frac{dQ}{d\varepsilon}\right)_0 &= B, \quad \left(\frac{d^2Q}{d\varepsilon^2}\right)_0 = 0, \quad \frac{1}{B} \left(\frac{d^3Q}{d\varepsilon^3}\right)_0 = -f^2(1+q) + R, \\
\frac{1}{B} \left(\frac{d^4Q}{d\varepsilon^4}\right)_0 &= -4f^2(1-q), \\
\frac{1}{B} \left(\frac{d^5Q}{d\varepsilon^5}\right)_0 &= f^4(1+q)^2 - 2f^2(R+6)(1+q) + R^2, \\
\frac{1}{B} \left(\frac{d^6Q}{d\varepsilon^6}\right)_0 &= 12f^4(1+q)^2 - 4f^2(1-q)(3R+8), \\
\frac{1}{B} \left(\frac{d^7Q}{d\varepsilon^7}\right)_0 &= -f^6(1+q)^3 + f^4[3R(1+q)^2 + 4(23+6q+23q^2)], \\
&\quad -f^2(1+q)(3R^2+52R+80) + R^3, \\
\frac{1}{B} \left(\frac{d^8Q}{d\varepsilon^8}\right)_0 &= -24f^6(1-q)(1+q)^2 + 48f^4(1-q^2)(R+12) \\
&\quad -24f^2(1-q)(R^2+8R+8), \\
\frac{1}{B} \left(\frac{d^9Q}{d\varepsilon^9}\right)_0 &= f^8(1+q)^4 - 4f^6(1+q)[R(1+q)^2 + 86 - 36q + 86q^2] \\
&\quad + f^4[6R^2(1+q)^2 + 32R(15+4q+15q^2) \\
&\quad \quad \quad + 16(201+10q+201q^2)] \\
&\quad -4f^2(1+q)(R^3+34R^2+160R+112) + R^4, \\
\frac{1}{B} \left(\frac{d^{10}Q}{d\varepsilon^{10}}\right)_0 &= 40f^8(1-q)(1+q)^3 \\
&\quad -f^6(1-q)(1+q)^2 \left[120R+3200+640\left(\frac{1-q}{1+q}\right)^2\right] \\
&\quad +f^4(1-q^2)(120R^2+3840R+16704) \\
&\quad -f^2(1-q)(40R^3+640R^2+1984R+1024), \\
\frac{1}{B} \left(\frac{d^{11}Q}{d\varepsilon^{11}}\right)_0 &= -f^{10}(1+q)^5 + 5f^8(1+q)^2[R(1+q)^2 + 184 - 144q + 184q^2] \\
&\quad -f^6(1+q)[10R^2(1+q)^2 + 40R(53-22q+53q^2) \\
&\quad \quad \quad + 36912 - 29216q + 36912q^2] \\
&\quad +f^4[10R(1+q)^2 + R^2(1480+400q+1480q^2) \\
&\quad \quad \quad +R(26896+1440q+26896q^2) \\
&\quad \quad \quad +82624+896q+82624q^2] \\
&\quad -f^2(1+q)(5R^4+280R^3+2656R^2+5824R+2304) + R^5.
\end{aligned}$$

According to whether the constant R is relative to a function P_1 or P_2 , we have for the vibratory movement of the annular membrane

$$w = P_1 Q \sin 2\lambda mt$$

or

$$w = P_2 Q \sin 2\lambda mt,$$

by giving Q the value we just calculated; then we finish determining the movement by calculating the quantity λ according to the condition that Q is zero on the external contour $\rho = A$.

27. Thus there are two kinds of solutions: in one, the portions of the major axis located between the two fixed contours are nodes, and in the other, are vibration bellies, and the nodal lines are still ellipses and portions of confocal hyperbolas.

When it is a full membrane, we have two distinct solutions

$$w = P_1 Q_1 \sin 2\lambda mt, \quad w = P_2 Q_2 \sin 2\lambda mt,$$

and Q_1 and Q_2 have two distinct forms, like P_1 and P_2 . On the contrary, when the membrane is annular, the two functions Q which associate with P_1 and P_2 only differ by the constant R which enters it. It follows from this that if the integer g , which designates the number of hyperbolic nodal lines, is large enough, and especially if at the same time the eccentricity is small enough, the constant R will differ very little for an identical value of g in the two functions P_1 and P_2 , and the two functions Q associated with them will be almost identical. The two corresponding vibratory states will therefore render almost the same sound, and, according to an experiment, they will be superimposed, and the resulting state will be represented by the formula

$$W = (AP_1 + BP_2)Q \sin 2\lambda mt;$$

then the hyperbolic nodal lines will be given by the equation

$$AP_1 + BP_2 = 0$$

and will still be g in number. But previously the same hyperbola provided portions of these four branches; now a hyperbola provides only two portions of branches which have the same asymptote; because the function $AP_1 + BP_2$ is not symmetrical with respect to the axes of the ellipse, but if we change α to $\pi + \alpha$, it remains the same or changes sign depending on whether g is even or odd.

We see that then the sound and the elliptical nodal lines remaining invariable, the position of the hyperbolic lines which depends on the arbitrary ratio $\frac{B}{A}$ can vary although their number g does not change.

It is obvious that the previous formulas apply to the circular ring by making $q = 0$, $R = g^2$, and they are also suitable for the full elliptical membrane for movement of the first kind

$$w = P_1 Q_1 \sin 2\lambda mt,$$

since it suffices to assume that the fixed interior contour is reduced to the line segment of the focal points, which is the limit of the smallest confocal ellipses. So you have to make $\varrho = c$, $a = \frac{c}{2}$, $q = 1$, and ε is reduced to β .

Elliptical nodal lines.

28. Consider the values of Q_1 and Q_2 given by the equation

$$(1) \quad \frac{d^2 Q}{d\beta^2} + [h^2(e^{2\beta} + e^{-2\beta}) - R]Q = 0,$$

one of which is zero and the other minimum for $\beta = 0$.

Suppose that we increase λ and consequently $h = \lambda c$, since c is fixed: Q_1 and Q_2 vary, but keeping their character at the limit $\beta = 0$. R is a function of h , and let us start by assuming that

$$\frac{dR}{dh} - 4h \quad \text{est} \quad < 0;$$

then, as λ increases, the coefficient of Q in (1) will take larger and larger values; because from the previous inequality we conclude that the derivative of this coefficient with respect to h ,

$$2h(e^{2\beta} + e^{-2\beta}) - \frac{dR}{dh},$$

is > 0 . So the roots of the equation in β

$$Q(\beta, \lambda) = 0$$

decrease in size as λ increases.

Let us denote by $\beta = B$ the parameter of the contour ellipse, on which Q is zero, the equation

$$Q(B, \lambda) = 0$$

determines the number λ . Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be these roots in order of increasing size: λ_i being the i^{th} root, $Q(\beta, \lambda_i)$ is one of the values of Q from our research, and the equation

$$Q(\beta, \lambda_i) = 0$$

will give, by its roots in β , the parameters of the elliptical nodal lines. Now we will prove that this equation has $i - 1$ roots, $\beta_1, \beta_2, \dots, \beta_{i-1}$, less than B , and that consequently the nodal ellipses are equal to $i - 1$ in number.

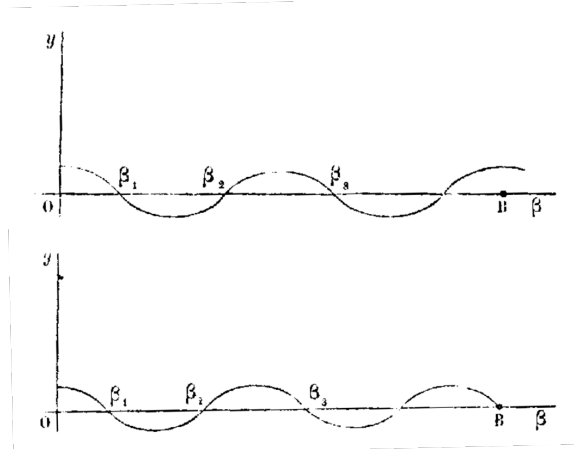
Consider the equation in β ,

$$(a) \quad Q(\beta, \lambda) = 0$$

and represent the curve

$$y = Q(\beta, \lambda),$$

β being taken for abscissa and y for the ordinate, which is zero or maximum for $\beta = 0$. Let i be the number of roots between 0 and B , which are determined by the points $\beta_1, \beta_2, \dots, \beta_i$. Let us increase λ , the points $\beta_1,$



β_2, \dots will approach the origin, the sinuosities of the curve will decrease in amplitude, and for a value λ_{i+1} the curve will pass through the point B . So we have a value of $\lambda = \lambda_{i+1}$ which satisfies the equation in λ

$$(b) \quad Q(B, \lambda) = 0;$$

and if we give a new small increase to λ , the equation (a) will have a new root to the left of B , and will therefore have $i + 1$ roots between 0 and B .

Let us continue to increase λ , the points β_1, β_2, \dots are getting closer to zero again, and, for a value $\lambda = \lambda_{i+2}$, the curve will pass again by the point B ; we will therefore have another new value of λ , λ_{i+2} which satisfies (b), and the equation

$$Q(\beta, \lambda_{i+2} + \varepsilon) = 0$$

where ε is positive and very small, $i + 2$ roots between 0 and B ; the equation

$$Q(\beta, \lambda_{i+2}) = 0$$

itself will have $i + 2$, counting B . Now λ_{i+1} and λ_{i+2} are obviously two consecutive roots of the equation (b) in λ ; we conclude that if λ_{i+1} and λ_{i+2} are two consecutive roots of (b), the equation in β

$$Q(\beta, \lambda_{i+2}) = 0$$

has one more root than

$$Q(\beta, \lambda_{i+1}) = 0$$

between the limits 0 and B . That said, we can easily recognize that $Q(\beta, \lambda_1) = 0$ has no roots between 0 and B ; so $Q(\beta, \lambda_2) = 0$ has one, $Q(\beta, \lambda_3) = 0$ has two, etc., and in general $Q(\beta, \lambda_i) = 0$ has $i - 1$.

The number of hyperbolic nodal lines remaining the same, we see that, as the sound rises, the number of elliptical nodal lines increases. All of the above is based on the existence of the inequality

$$(c) \quad \frac{dR}{dh} - 4h < 0.$$

At n°8, we found the equation

$$\frac{dP}{d\alpha} \delta P - P \delta \frac{dP}{d\alpha} = \delta h \int_0^\alpha P^2 \left(\frac{dR}{dh} - 4h \cos 2\alpha \right) d\alpha;$$

for $\alpha = 0$, the two members are zero; but if we suppose α excessively small, $\frac{dR}{dh} - 4h \cos 2\alpha$ will not change sign between 0 and this value of α ; so the first member will have the same sign as $\frac{dR}{dh} - 4h$.

If this quantity can be positive, the expression

$$\frac{dR}{dh} - 2h(e^{2\beta} + e^{-2\beta})$$

will also be for excessively small values of β ; therefore by taking B sufficiently small, and, consequently, the very-eccentric membrane, the number of elliptical nodal lines would decrease when, the number of hyperbolic nodal lines remaining the same, the pitch of the sounds would increase. As this result does not seem admissible, it seems that the inequality (c) must always be tied-in; however, the above cannot be viewed as a rigorous demonstration.

Most general vibratory movement of the elliptical membrane.

29. We have so far dealt only with simple vibratory movements, which are those which would be most easily produced in experience. We will now assume that we give at all points of a membrane any initial velocities, and determine the vibrational state that will result.

But first let us do some thinking about the signs of the coordinates we use. As we said at n°4, where we let

$$(a) \quad \begin{cases} x = cE(\beta) \cos \alpha, \\ y = c\mathcal{E}(\beta) \sin \alpha, \end{cases}$$

when using the coordinates α and β , we can assume that β is essentially positive, and that α is only susceptible to varying from zero to 2π , or from $-\pi$ to $+\pi$, and despite these restrictions we can represent by these coordinates any point of the plane.

But it follows from the formulas (a), that if we give a negative value to β instead of giving it to α , the point (x, y) remains the same, and therefore also that the coordinates $(-\alpha, -\beta)$ represent the same point as the coordinates (α, β) . So a formula which gives the vibratory movement of the membrane must remain invariable when we replace α and β with $-\alpha$ and $-\beta$: this is what we verify indeed on the two simple solutions that we found

$$w = P_1 Q_1 \sin 2\lambda mt, \quad w = P_2 Q_2 \sin 2\lambda mt,$$

since P_1 and Q_1 are odd functions of α and β , and since P_2 and Q_2 are even functions, and we could have associated the functions Q_1 and Q_2 to functions P_1 and P_2 , according to this condition (n°15).

However, it should be noted that these considerations would not apply to the annular membrane. Indeed, the line drawn between the foci, and which has the equation $\beta = 0$ is no longer located on the surface of the membrane, and if we have expressed that Q is zero for $\beta = \beta_1$ on the inside contour and taken β_1 positive, we can only give β the positive values enclosed between those which suit the two contours.

Returning to the vibratory movement of the full membrane, suppose that the initial speed given at each point of the membrane is expressed by the formula

$$\left(\frac{dw}{dt}\right)_0 = \Phi(\alpha, \beta),$$

in which $\Phi(\alpha, \beta)$ is a function which becomes zero on the contour of the membrane $\beta = \vartheta$, and which, from what we have seen, remains invariable when we replace α and β with $-\alpha$ and $-\beta$. We easily conclude that $\Phi(\alpha, \beta)$

is the sum of two functions $F_1(\alpha, \beta)$, $F_2(\alpha, \beta)$, which, ordered with respect to increasing powers of α and β , are: one of the form

$$F_2 = a + A\alpha^2 + B\beta^2 + C\alpha^4 + D\alpha^2\beta^2 + E\beta^4 + F\alpha^6 + G\alpha^4\beta^2 + \dots$$

even in α and β ; and the other of the form

$$F_1 = A'\alpha\beta + B'\alpha^3\beta + C'\alpha\beta^3 + D'\alpha^5\beta + E'\alpha^3\beta^3 + F'\alpha\beta^5 + G'\alpha^7\beta + \dots,$$

odd in α and odd in β , but even with respect to their set.

After having let

$$\Phi(\alpha, \beta) = F_1(\alpha, \beta) + F_2(\alpha, \beta),$$

let us look at the resulting vibratory movement as the sum of an infinity of simple vibratory movements, the amplitude of which we will propose to determine. Each simple movement of the first or second kind given by the formulas

$$w = aP_1Q_1 \sin 2\lambda mt, \quad w = bP_2Q_2 \sin 2\lambda mt$$

depends first on an integer g , and, this number g once designated, this movement can vary in an infinite number of ways by the number λ , which is susceptible to increasing values $\lambda_1, \lambda_2, \dots, \lambda_i, \dots$, and we will assign them a second index which recalls the number g , and we will replace the two previous formulas by the following two:

$$w = aP_1(g, \lambda_i^g)Q_1(g, \lambda_i^g) \sin 2\lambda_i^g mt,$$

$$w = bP_2(g, \lambda_i'^g)Q_1(g, \lambda_i'^g) \sin 2\lambda_i'^g mt,$$

Then considering a vibratory state composed of an infinity of simple states, we will have

$$w = \sum a_{g, \lambda_i} P_1(g, \lambda_i^g) Q_1(g, \lambda_i^g) \sin 2\lambda_i^g mt \\ + \sum b_{g, \lambda_i'} P_2(g, \lambda_i'^g) Q_2(g, \lambda_i'^g) \sin 2\lambda_i'^g mt$$

and we draw for the initial speed

$$\left(\frac{dw}{dt}\right)_0 = 2m \sum \lambda_i^g a_{g, \lambda_i} P_1(g, \lambda_i^g) Q_1(g, \lambda_i^g) \\ + 2m \sum \lambda_i'^g b_{g, \lambda_i'} P_2(g, \lambda_i'^g) Q_2(g, \lambda_i'^g),$$

expression which must be identified with $\Phi(\alpha, \beta)$; but we will decompose this equality into the following two

$$(1) \quad F_1(\alpha, \beta) = 2m \sum \lambda_i^g a_{g, \lambda_i} P(g, \lambda_i^g) Q(g, \lambda_i^g),$$

$$(2) \quad F_2(\alpha, \beta) = 2m \sum \lambda_i'^g b_{g, \lambda_i'} P(g, \lambda_i'^g) Q(g, \lambda_i'^g).$$

Now consider the four equations

$$(b) \quad \begin{cases} \frac{d^2 Q}{d\beta^2} - [R(g, \lambda c) - 2\lambda^2 c^2 E(2\beta)] Q = 0, \\ \frac{d^2 Q'}{d\beta^2} - [R(g', \lambda' c) - 2\lambda'^2 c^2 E(2\beta)] Q' = 0; \end{cases}$$

$$(c) \quad \begin{cases} \frac{d^2 P}{d\alpha^2} + [R(g, \lambda c) - 2\lambda^2 c^2 \cos 2\alpha] P = 0, \\ \frac{d^2 P'}{d\alpha^2} + [R(g', \lambda' c) - 2\lambda'^2 c^2 \cos 2\alpha] P' = 0. \end{cases}$$

By subtracting the two equations (b) multiplied by Q' and Q , we have

$$0 = Q' \frac{d^2 Q}{d\beta^2} - Q \frac{d^2 Q'}{d\beta^2} + [2(\lambda^2 - \lambda'^2) c^2 E(2\beta) - (R - R')] Q Q';$$

we integrate from $\beta = 0$ to $\beta = \vartheta$, the parameter of the contour, and we will have

$$\begin{aligned} 0 &= \left(Q' \frac{dQ}{d\beta} - Q \frac{dQ'}{d\beta} \right)_{\vartheta} - \left(Q' \frac{dQ}{d\beta} - Q \frac{dQ'}{d\beta} \right)_0 \\ &\quad + 2(\lambda^2 - \lambda'^2) c^2 \int_0^{\vartheta} E(2\beta) Q Q' d\beta - (R - R') \int_0^{\vartheta} Q Q' d\beta. \end{aligned}$$

The first term is zero because Q and Q' are zero for $\beta = \vartheta$; then, if Q and Q' have the character of Q_1 , they are zero for $\beta = 0$, and if they both have the character of Q_2 , their derivatives are zero for $\beta = 0$; so the second term is also zero. We also find

$$\begin{aligned} 0 &= \left(P' \frac{dP}{d\alpha} - P \frac{dP'}{d\alpha} \right)_0^{2\pi} + 2(\lambda^2 - \lambda'^2) c^2 \int_0^{2\pi} P P' \cos 2\alpha d\alpha \\ &\quad - (R - R') \int_0^{2\pi} P P' d\alpha. \end{aligned}$$

whose first part is zero, because P and P' are periodic functions. So we have the two equalities

$$(d) \quad \begin{cases} (R - R') \int_0^{\vartheta} Q Q' d\beta = 2(\lambda^2 - \lambda'^2) c^2 \int_0^{\vartheta} Q Q' E(2\beta) d\beta, \\ 2(\lambda^2 - \lambda'^2) c^2 \int_0^{2\pi} P P' \cos 2\alpha d\alpha = (R - R') \int_0^{2\pi} P P' d\alpha. \end{cases}$$

Multiply these equalities member-to-member, and, dividing by

$$2(R - R')(\lambda^2 - \lambda'^2)c^2$$

we obtain

$$(e) \quad \int_0^\vartheta \int_0^{2\pi} [E(2\beta) - \cos 2\alpha] PP' QQ' d\beta d\alpha = 0.$$

This equality is no longer demonstrated if $\lambda = \lambda'$ or if $R = R'$; it is however still exact, because, if $\lambda = \lambda'$, we will deduce from the equations (d)

$$\int_0^\vartheta QQ' d\beta = 0, \quad \int_0^{2\pi} PP' d\alpha = 0;$$

therefore the two parts of the integral (e) are zero. If $R' = R$, we still see that the two equalities (d) entail (e).

We multiply the two members of equality (1) by

$$P_1(g, \lambda_i^g) Q_1(g, \lambda_i^g) [E(2\beta) - \cos 2\alpha] d\alpha d\beta,$$

and integrate, with respect to α , from 0 to 2π , and, with respect to β , from 0 to ϑ : all the terms will disappear in the second member according to (e), except that which has the coefficient a_{g, λ_i} which is determined. We also have b_{g, λ_i} by means of (2).

We have a similar calculation for the annular membrane fixed between two confocal ellipses; it would be superfluous to insist on it, and even we did the previous calculation only because it required considerations relating to the signs of α and β that are useful to notice.

To return to these signs, let us still imagine that one has to seek the movement of an elliptical membrane from which one removes the two portions cut by a confocal hyperbola, and suppose all the contour is fixed. We will again have a simple vibratory movement represented by the formula

$$w = PQ \sin 2\lambda mt;$$

but P is no longer a periodic function of α . We must vary α only between the limits α_1 , and $\pi - \alpha_1$, relative to the hyperbola of the contour, and we will vary β between the two limits $-\vartheta$ and $+\vartheta$ relative to the two elliptical arcs of the periphery of the membrane.