On Parametric Linear System Solving

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Abstract. Parametric linear systems are linear systems of equations in which some symbolic parameters, that is, symbols that are not considered to be candidates for elimination or solution in the course of analyzing the problem, appear in the coefficients of the system. In this work we assume that the symbolic parameters appear polynomially in the coefficients and that the only variables to be solved for are those of the linear system. The consistency of the system and expression of the solutions may vary depending on the values of the parameters. It is well-known that it is possible to specify a covering set of regimes, each of which is a semi-algebraic condition on the parameters together with a solution description valid under that condition.

We provide a method of solution that requires time polynomial in the matrix dimension and the degrees of the polynomials when there are up to three parameters. In previous methods the number of regimes needed is exponential in the system dimension and polynomial degree of the parameters. Our approach exploits the Hermite and Smith normal forms that may be computed when the system coefficient domain is mapped to the univariate polynomial domain over suitably constructed fields. Our approach effectively identifies *intrinsic singularities* and *ramification points* where the algebraic and geometric structure of the matrix changes.

Parametric eigenvalue problems can be addressed as well: simply treat λ as a parameter in addition to those in $\mathbf{A}$ and solve the parametric system $(\lambda\mathbf{I} - \mathbf{A})\mathbf{u} = 0$. The algebraic conditions on λ required for a nontrivial nullspace define the eigenvalues. We do not directly address the problem of computing the Jordan form, but our approach allows the construction of the algebraic and geometric eigenvalue multiplicities revealed by the Frobenius form, which is a key step in the construction of the Jordan form of a matrix.

Keywords: Hermite form · Smith form · Frobenius form · Parametric linear systems.

1 Introduction

Speaking in simple generalities, we say that symbolic computation is concerned with mathematical equations that contain *symbols*; symbols are used both for *variables*, which are typically to be solved for, and *parameters*, which are typically carried through and appear in the *solutions*, which are then interpreted as formulae: that is, objects that can be further studied, perhaps by varying the parameters. One prominent early researcher said that the difference between symbolic and numeric computation was merely a matter of *when* numerical values were inserted into the parameters: before the computation meant you were going to do things numerically, and after the computation meant you had done symbolic computation. The words “parameters” and “variables” are therefore not precisely descriptive, and can often be used interchangeably. Indeed as a matter of practice, polynomial equations can often be taken to have one subset of its symbols taken as variables rather than any other subset in quite strategic fashion: it may be better to solve for x as a function of y than to solve for y as a function of x.

In this paper we are concerned with systems of equations containing several symbols, some of which we take to be variables, and all the rest as parameters. More, we restrict our attention to problems in which the *variables* appear only linearly. Parameters are allowed to appear polynomially, of whatever degree.

Parametric linear systems (PLS) arise in many contexts, for instance in the analysis of the stability of equilibria in dynamical systems models such as occur in mathematical biology and other areas. Understanding the different potential kinds of dynamical behavior can be important for model selection as well as analysis. Another important area of interest is the role of parametric linear systems in dealing with the stability of the equilibria of parametric autonomous system of ordinary differential equations (see [25] and [11]). One particularly famous example is the Lotka-Volterra system which arises naturally from predator-prey equations. See also [24] and [23]. Other examples of the use of parametric linear system from science and engineering includes their application in computing the characteristic solutions for differential equations [8], dealing with colored Petri nets [13] and in operations research and engineering [9], [17], [21], [31]. Some problems in robotics [2] and certain modelling problems in mathematical biology, see e.g. [29], also can benefit from the ability to effectively solve PLS.

After some discussion of prior comprehensive solving work in Section 2, we proceed with formal problem and solution definitions for parametric linear systems (PLS) in section 3. Our primary tool for solving these is by way of *comprehensive triangular Smith normal form (CTSNF)*, which is introduced in section 4. The following section reduces PLS to CTSNF and section 6 describes the solution of CTSNF problems for the case of up to three parameters.

An application that seems at first to be of only theoretical interest is the computation of the *matrix logarithm*, or indeed any of several other matrix functions such as matrix square root. We briefly discuss this example in more detail with a pair of small matrices in Section 7.2. We also give other examples in section 7.

2 Prior Work

Interest in computation of the solution of PLS dates back to the beginning of symbolic computation. For instance, one of the first things users have requested of computer algebra systems is the explicit form of the inverse of a matrix containing only symbolic entries[4]: the user is then typically quite dissatisfied at the complexity of the answer if the dimension is greater than, say, 3. Of course, the determinant itself, which must appear in such an answer, has a factorial number of terms in it, and thus growth in the size of the answer must be more than exponential. Therefore the complexity of any algorithm to solve PLS must be at least exponential in the number of parameters.

An interesting pair of papers addressing the case of only one parameter is [1] and [15]. These papers assume full rank of the linear system—and thus compute the "generic" case when in fact there are isolated values of the parameter for which the rank drops—and use rational interpolation of the numerical solutions of specialized linear systems to recover this generic solution.

Many authors have sought comprehensive solutions—by which is meant complete coverage of all parametric regimes—through various means. One of the first serious methods was the matrix-minor based approach of William Sit [25], which enables practical solution of many problems of interest. Recently, the problem of computing the Jordan form of a parametric matrix once the Frobenius form is known has been attacked by using Regular Chains [4] and this has been moderately successful in practice. Simple methods and heuristics for linear systems containing parameters continue to generate interest, even when Regular Chains are used, such as in [3].

Other authors such as [30], [18], [20], [16], and [19] have tackled the even more difficult problem of computing the comprehensive solution of systems of *polynomial* equations containing parameters, and of course their methods can be applied to the linear equations being considered here.

By restricting our attention in this paper to linear problems and to those of three parameters or fewer we are able to guarantee better worst case performance (polynomially many solution regimes) and hope to provide better efficiency in many instances than is possible using those general-purpose approaches.

3 Definitions and Notation

Let F be a field and $Y = (y_1, \ldots, y_s)$ a list of parameters. Then $F[Y]$ is the ring of polynomials and $F(Y)$ is the field of rational functions in Y. For each tuple $a = (a_1, \ldots, a_s)$ in F^s, evaluation at a is a mapping $F[Y] \to F$. We will extend this mapping componentwise to polynomials, vectors, matrices, and sets thereof over $F[Y]$. We will use the Householder convention, typesetting matrices in upper case bold, e.g. $\mathbf{A}$, and lower case bold for vectors, e.g. $\mathbf{b}$.

[4] This is merely an anecdote, but one of the present authors attests that this really has happened.

For the most part, for such objects over $F[Y]$, we know Y from context and write $\mathbf{A}$ rather than $\mathbf{A}(Y)$, but write $\mathbf{A}(a)$ for the evaluation at $Y = a$.

For a set of polynomials, S, we will denote by $V(S)$ the variety of the ideal generated by S. This is the set of tuples a such that $f(a) = \{0\}$, for all $f \in S$. We will be concerned with pairs N, Z of polynomial sets, $N, Z \subset F[Y]$, defining a semialgebraic set in F^s consisting of those tuples a that evaluate to nonzero on N and to zero on Z. By a slight abuse of notation, we call this semialgebraic set $V(N, Z) = V(Z) \setminus V(N)$. Our inputs are polynomial in the parameters but the output coefficients in general are rational functions. The evaluation mapping extends partially to $F(Y)$: For a rational function $n(Y)/d(Y)$ in lowest terms (n and d relatively prime), the image $n(a)/d(a)$ is well defined so long as $d(a) \neq 0$.

Definition 3.1. *The data for a* **parametric linear system (PLS) problem** *is matrix* $\mathbf{A}$ *and right hand side vector* $\mathbf{b}$ *over* $F[Y]$*, together with a semialgebraic constraint,* $V(N, Z)$*, with* $N, Z \subset F[Y]$*. Only of interest are those parameter value tuples in* $V(N, Z)$*, i.e., on which the polynomials in* N *are nonzero and the polynomials in* Z *are zero.*

For the PLS problem $(\mathbf{A}, \mathbf{b}, N, Z)$*, a* **solution regime** *is a tuple* $(\mathbf{u}, \mathbf{B}, N', Z')$*, with coefficients of* $\mathbf{u}$ *and* $\mathbf{B}$ *in* $F(Y)$*, such that, for all* $a \in V(N', Z')$*,* $\mathbf{u}(a)$ *is a solution vector and* $\mathbf{B}(a)$ *is a matrix whose columns form a nullspace basis for* $\mathbf{A}(a)$*.*

A **PLS solution** *is a set of solution regimes that covers* $V(N, Z)$*, which means, for PLS solution* $\{(\mathbf{u}_i, \mathbf{B}_i, N_i, Z_i) | i \in 1, \ldots, k\}$*, every parameter value assignment that satisfies the problem semialgebraic constraint* N, Z *also satisfies at least one regime semialgebraic constraint* N_i, Z_i*. In other words* $V(N, Z) \subset \cup_{i=1}^{k} V(N_i, Z_i)$*.*

We call entries that *must* occur in any Z in the solution an *intrinsic* restriction, or singularity. We call the differing sets $V(N_i, Z_i)$ that may occur in covers of $V(N, Z)$ the *ramifications* of the cover.

We next give an example that illustrates the PLS definition and also sketches the prior approach to PLS given by William Sit in [25]. If, for $\mathbf{M}$ of size $r \times r$, $\mathbf{A}$ is $\begin{bmatrix} \mathbf{M} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$, and conformally $\mathbf{b} = \begin{bmatrix} \mathbf{c} & \mathbf{d} \end{bmatrix}^T$, then a solution $\mathbf{u} = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}^T$ satisfies

$$\mathbf{M}\mathbf{v} + \mathbf{B}\mathbf{w} = \mathbf{c} \tag{3.1}$$

and

$$\mathbf{C}\mathbf{v} + \mathbf{D}\mathbf{w} = \mathbf{d}\,. \tag{3.2}$$

Under the condition that $\det(\mathbf{M})$ is nonzero and all larger minors of $\mathbf{A}$ are zero, equation (3.1) can be solved with specific solution $\mathbf{w} = 0$ and $\mathbf{v} = \mathbf{M}^{-1}\mathbf{c}$. Provided the system is consistent (equation (3.2) holds), we have the regime

$$(\begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}^T, \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{B} \\ \mathbf{I} \end{bmatrix}, N, Z),$$

where $N = \{\det(\mathbf{M})\}$ and $Z = \{\text{all } (i+1) \times (i+1) \text{ minors of } \mathbf{A}\})$. Call solution regimes of this type *minor defined* regimes.

Since an $n \times n$ matrix has $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ minors, there are exponentially many minor defined regimes. However, some of these regimes may not be solutions due to inconsistency or it may be possible to combine several regimes into one. For instance if $\det(\mathbf{M})$ is a constant, and $\mathbf{b} = 0$, then all rank r solutions are covered by this one regime. Sit [25] has made a thorough study of minor defined regimes and their simplifications.

Another approach is to base solution regimes on the pivot choices in an LU decomposition. The simplest thing to do is to leave it to the user, although one has to also inform the user through a proviso when this might be necessary [6]. That is, provide the generic answer, but also provide a description of the set N. A more sophisticated approach is developed in [4, 3] using the theory of regular chains and its implementation in Maple [18] to manage the algebraic conditions. For example a given matrix entry may be used as a pivot, with validity dependent on adding the polynomial to the non-zero part, N, of the semialgebraic set. For a comprehensive solution the case that that entry is zero must also be pursued. In the worst case, this leads to a tree of zero/nonzero choices of depth n and branching factor n.

4 Triangular Smith forms and degree bounds

In this paper we take a different approach, with the solution regimes arising from Hermite normal forms, of which triangular Smith forms are a special case. We give a system of solution regimes of polynomial size in the matrix dimension, n, and polynomial degree, d. Each regime is computed in polynomial time and the regime count is exponential only in the number of parameters. To use Hermite forms we will need to work over a principal ideal domain such as, for parameters x, y, $F(y)[x]$. We will restrict our input matrix to be polynomial in the parameters. This first lemma shows it is not a severe constraint.

Lemma 4.1. *Let $(\mathbf{A}, \mathbf{b}, N, Z)$ be a well defined PLS over field $F(Y)$, for parameter set Y, with $\mathbf{A} \in F(Y)^{m \times n}$ and $\mathbf{b} \in F(Y)^m$ with numerator and denominator degrees bounded by d in each parameter of Y. Well defined means that denominators of $\mathbf{A}, \mathbf{B}$ are in N. The problem is equivalent (same solutions) to one in which the entries of the matrix and vector are polynomial in the parameters Y, the dimension is the same, and the degrees are bounded by nd.*

Proof. Because the PLS is well defined, it is specified by N that all denominator factors of $\mathbf{A}(a), \mathbf{b}(a)$ are nonzero for $a \in V(N, Z)$. Let $\mathbf{L}$ be a diagonal matrix with the i-th diagonal entry being the least common multiple (lcm) of the denominators in row i of $\mathbf{A}$, $\mathbf{b}$. These lcms also evaluate to nonzero on $V(N, Z)$. It follows that $L(a)\mathbf{A}(a)\mathbf{u}(a) = \mathbf{L}(a)\mathbf{b}(a)$ if and only if $\mathbf{A}(a)\mathbf{u}(a) = \mathbf{b}(a)$. Thus the PLS $(\mathbf{LA}, \mathbf{Lb}, V(N, Z))$ is equivalent and its matrix and vector have polynomial entries of degrees bounded by nd.

We will reduce PLS to triangular Smith normal form computations. The rest of this section concerns computation of triangular Smith normal form and bounds for the degrees of the form and its unimodular cofactor.

Definition 4.2. *Given field K and variable x, a matrix $\mathbf{H}$ over $K[x]$ is in (reduced)* **Hermite normal form** *if it is upper triangular, its diagonal entries are monic, and, for each column in which the diagonal entry is nonzero, the off-diagonal entries are of lower degree than the diagonal entry. If each diagonal entry of $\mathbf{H}$ exactly divides all those below and to the right, then $\mathbf{H}$ is column equivalent to a diagonal matrix with the same diagonal entries (its Smith normal form). An equivalent condition is that, for each i, the greatest common divisor of the $i \times i$ minors in the leading i columns equals the greatest common divisor of all $i \times i$ minors. Following Storjohann[27, Section 8,Definition 8.2] we call such a Hermite normal form a* **triangular Smith normal form**. *It will be the central tool in our PLS solution.*

For notational simplicity, we've left out the possibility of echelon structure in a Hermite normal form. We will talk of Hermite normal forms only for matrices having leading columns independent up to the rank of the matrix. Every such matrix over $K[x]$ is row equivalent to a unique matrix in Hermite form as defined above. For given $\mathbf{A}$ we have $\mathbf{UA} = \mathbf{H}$, with $\mathbf{U}$ unimodular, i.e. $\det(\mathbf{U}) \in K^*$, and $\mathbf{H}$ in Hermite form. If $\mathbf{A}$ is nonsingular, the unimodular cofactor $\mathbf{U}$ is unique and has determinant $1/c$, where c is the leading coefficient of $\det(\mathbf{A})$. This follows since $\det(\mathbf{U})\det(\mathbf{A}) = \det(\mathbf{H})$, which is monic.

The next definition and lemma concern assurance that Hermite form computation will yield a triangular Smith form.

Definition 4.3. *Call a matrix* **nice** *if its Hermite form is a triangular Smith form (each diagonal entry exactly divides those below and to the right). In particular, a nice matrix has leading columns independent up to the rank.*

There is always a column transform (unimodular matrix $\mathbf{R}$ applied from the right) such that $\mathbf{AR}$ is nice. The following fact, proven in [14] shows that a random transform over F suffices with high probability.

Fact 4.4. Let $\mathbf{A}$ be a $m \times n$ matrix over $K[x]$ of degree in x at most d. Let $\mathbf{R}$ be a unit lower triangular matrix with below diagonal elements chosen from subset S of K uniformly at random. Then $\mathbf{AR}$ is nice over $K[x]$ with probability at least $1 - 4n^3 d/|S|$.

Note that $\deg_x(\mathbf{AR}) = \deg_x(\mathbf{A})$ and, for $K = F(y), \mathbf{A} \in F[y, x]^{m\times n}$ and $S \subset F$ we also have $\deg_y(\mathbf{AR}) = \deg_y(\mathbf{A})$.

We continue with analysis of degree bounds for Hermite forms of matrices, particularly degree bounds for triangular Smith forms of nice matrices. The first result needed is the following fact from [10]. Through the remainder of this paper we will employ "soft O" notation, where, for functions $f, g \in \mathbb{R}^k \to \mathbb{R}$ we write $f = O\tilde{~}(g)$ if and only if $f = O(g \cdot \log^c |g|)$ for some constant $c > 0$.

Fact 4.5. Let F be a field, x, y parameters, and let $\mathbf{A}$ be in $F[y, x]^{n\times n}$, nonsingular, with $\deg_x(\mathbf{A}) \le d$, $\deg_y(\mathbf{A}) \le e$. Over $F(y)[x]$, let $\mathbf{H}$ the unique Hermite form row equivalent to $\mathbf{A}$ and $\mathbf{U}$ be the unique unimodular cofactor such that $\mathbf{UA} = \mathbf{H}$. The coefficients of the entries of $\mathbf{H}$, $\mathbf{U}$ are rational functions of y. Let

Δ be the least common multiple of the denominators of the coefficients in $\mathbf{H}$, $\mathbf{U}$, as expressed in lowest terms.

(a) $\deg_x(\mathbf{U}) \le (n-1)d$ and $\deg_x(\mathbf{H}) \le nd$.
(b) $\deg_y(\mathrm{num}(\mathbf{H})), \deg_y(\mathrm{num}(\mathbf{U})) \le n^2de$ (bounds both numerator and denominator degrees).
(c) $\deg_y(\Delta) \le n^2de$.
(d) $\mathbf{H}$ and $\mathbf{U}$ can be computed in polynomial time: deterministically in $O\tilde{}(n^9d^4e)$ time and Las Vegas probabilistically (never returns incorrect result) in $O\tilde{}(n^7d^3e)$ expected time.

Proof. This is [10, Summary Theorem]. The situation there is more abstract, more involved. We offer this tip to the reader: their $\partial, z, \sigma, \delta$ correspond respectively to our x, y, identity, identity.

Item (c) is not stated explicitly in a theorem of [10] but is evident from the proofs of Theorems 5.2 and 5.6 there. The common denominator is the determinant of a matrix over $K[z]$ of dimension n^2d and with entries of degree in z at most e.

We will generalize this fact to nonsingular and non-square matrices in Theorem 4.6. In that case the unimodular cofactor, $\mathbf{U}$, is not unique and may have arbitrarily large degree entries. The following algorithm is designed to produce a $\mathbf{U}$ with bounded degrees.

Algorithm 1 U, H `= HermiteForm(A)`

Require: Nice matrix $\mathbf{A} \in F[y,x]^{m\times n}$, for field F and parameters x, y.

Ensure: For $K = F(y)$, Unimodular $\mathbf{U} \in K[x]^{m\times m}$ and $\mathbf{H} \in K[x]^{m\times n}$ in triangular Smith form such that $\mathbf{UA} = \mathbf{H}$. The point of the specific method given here is to be able, in Theorem 4.6, to bound $\deg_x(\mathbf{U},\mathbf{H})$ and $\deg_y(\mathbf{U},\mathbf{H})$ (numerators and denominators).

1: Compute $r = \mathrm{rank}(\mathbf{A})$ and nonsingular $\mathbf{U}_0 \in K^{m\times m}$ such that $\mathbf{A} = \mathbf{U}_0\mathbf{A}$ has nonsingular leading $r \times r$ minor. Because $\mathbf{A}$ is nice the first r columns are independent and such $\mathbf{U}_0$ exists. $\mathbf{U}_0$ could be a permutation found via Gaussian elimination, say, or a random unit upper triangular matrix. In the random case, failure to achieve nonsingular leading minor becomes evident in the next step, so that the randomization is Las Vegas.

2: Let $\mathbf{U}_0\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0}_{r\times m-r} \\ \mathbf{A}_3 & \mathbf{I}_{m-r} \end{bmatrix}$. $\mathbf{B}$ is nonsingular. Compute its unique unimodular cofactor $\mathbf{U}_1$ and Hermite form $\mathbf{T} = \mathbf{U}_1\mathbf{B} = \begin{bmatrix} \mathbf{H}_1 & * \\ 0 & * \end{bmatrix}$.
If $\mathbf{H}_1$ is in triangular Smith form, let $\mathbf{H} = \mathbf{U}_1\mathbf{U}_0\mathbf{A} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ 0 & 0 \end{bmatrix}$.
Let $\mathbf{U} = \mathbf{U}_1\mathbf{U}_0$ and return $\mathbf{U}$, $\mathbf{H}$.
Otherwise go back to step 1 and choose a better $\mathbf{U}_0$. With high probability this repetition will not be needed; probability of success increases with each iteration.

Theorem 4.6. *Let F be a field, x, y parameters, and let $\mathbf{A}$ be in $F[y,x]^{m\times n}$ of rank r, $\deg_x(\mathbf{A}) \le d$, and $\deg_y(\mathbf{A}) \le e$. Then, for the triangular Smith form form $\mathbf{UAR} = H$ computed as $\mathbf{U}$, $\mathbf{H}$ =* `HermiteForm`*($\mathbf{AR}$), we have*

(a) Algorithm `HermiteForm` *is (Las Vegas) correct and runs in expected time $\mathrm{O}(m^7 d^3 e)$;*
(b) $\deg_x(\mathbf{U}, \mathbf{H}) \le md$;
(c) $\deg_y(\mathbf{U}, \mathbf{H}) = O\tilde{}(m^2 de)$.

Proof. Let $\mathbf{R}$ be as in Fact 4.4 with $K = F(y)$ and $S \subset F$. If the field F is small, an extension field can be used to provide large enough S.

We apply `HermiteForm` to $\mathbf{AR}$ to obtain $\mathbf{U}$, $\mathbf{H}$, and use the notation of the algorithm in this proof. We see by construction that $\mathbf{B}$ is nonsingular, from which it follows that $\mathbf{U}_1$ and $\mathbf{T}$ are uniquely determined. $\mathbf{B}$ is nice because $\mathbf{A}$ is nice and all j-minors of $\mathbf{B}$ for $j > r$ are either zero or equal to $\det \mathbf{A}_1$. It follows that the leading r columns of $\mathbf{H}$ must be those of $\mathbf{T}$. The lower left $(m-r) \times (n-r)$ block of $\mathbf{H}$ must be zero because $\mathrm{rank}(\mathbf{H}) = \mathrm{rank}(\mathbf{A})$. The leading r rows are independent, and any nontrivial linear combination of those rows would be nonzero in the lower left block. Then $\mathbf{H}$ is in triangular Smith form and left equivalent to $\mathbf{A}$ as required. The runtime is dominated by computation of $\mathbf{U}_1$ and $\mathbf{T}$ for $\mathbf{B}$, so Fact 4.5 provides the bound in (a).

For the degree in x, applying Fact 4.5, we have $\deg_x(\mathbf{U}_1) \le (m-1)d$. Noting that $\mathbf{U}_0$ has degree zero, we have $\deg_x(\mathbf{U}) = \deg_x(\mathbf{U}_1)$ and $\deg_x(\mathbf{H}) = \deg_x(\mathbf{U}) + \deg_x(\mathbf{A}) \le (m-1)d + d = md$.

For the degree in y, note first that the bounds d, e for degrees in $\mathbf{A}$ apply as well to $\mathbf{B}$. We have, by Fact 4.5, that $\deg_y(\mathrm{num}(\mathbf{U}_1)) = O\tilde{}(m^2 de)$ and the same bound for $\deg_y(\mathrm{den}(\mathbf{U}_1))$. For $\mathbf{H}$, note that $\mathrm{num}(\mathbf{H})/\mathrm{den}(\mathbf{H}) = \mathrm{num}(\mathbf{U})\mathbf{A}/\mathrm{den}(\mathbf{U})$ so that and $\deg_y(\mathrm{den}(\mathbf{H})) \le \deg_y(\mathbf{U}) = O\tilde{}(m^2 de)$, and $\deg_y(\mathrm{num}(\mathbf{H})) \le \deg_y(\mathrm{num}(\mathbf{U})A) = O\tilde{}(m^2 de) + e = O\tilde{}(m^2 de)$. □

5 Reduction of PLS to triangular Smith forms

In this section we define the Comprehensive Triangular Smith Normal form problem and solution and show that PLS can be reduced to it. The next section addresses the solution of CTSNF itself.

Definition 5.1. *For field F, parameters $Y = (y_1, \ldots, y_s)$, F_Y is a* **parameterized extension** *of F if $F_Y = F_s$, the top of a tower of extensions $F_0 = F, F_1, \ldots, F_s$ where, for $i \in 1, \ldots, s$, each F_i is either $F_{i-1}(y_i)$ (rational functions) or $F_{i-1}[y_i]/\langle f_i \rangle$, for f_i irreducible in y_i over F_{i-1} (algebraic extension). When a solution regime to a PLS or CTSNF problem is over a parameterized extension F_Y, the irreducible polynomials involved in defining the extension tower for F_Y will be in the constraint set Z of polynomials that must evaluate to zero.*

*A comprehensive triangular Smith normal form problem (***CTSNF problem***) is a triple $(\mathbf{A}, N, Z)$ of a matrix $\mathbf{A}$ over $F[Y, x]$ and polynomial sets $N, Z \subset F[Y, x]$, so that $V(N, Z)$ constrains the range of desired parameter values as in the PLS problem.*

For CTSNF problem $(\mathbf{A}, N, Z)$ *over* $F[Y,x]$*, a* **triangular Smith regime** *is of the form* $(\mathbf{U}, \mathbf{H}, \mathbf{R}, N', Z')$*, with* $\mathbf{U}$*,* $\mathbf{H}$ *over* $F_Y[x]$*, where* F_Y *is a parameterized extension of* F *and any polynomials defining algebraic extensions in the tower are in* Z'*, such that on all* $a \in V(N', Z')$*,* $H(a)$ *is in triangular Smith form over* $F(a)[x]$*,* $\mathbf{U}(a)$ *is unimodular in* x*,* $\mathbf{R}$ *is nonsingular over* F*, and* $\mathbf{U}(a)\mathbf{A}(a)\mathbf{R} = \mathbf{H}(a)$*.*

A **CTSNF solution** *is a list* $\{(\mathbf{U}_i, \mathbf{H}_i, \mathbf{R}_i, N_i, Z_i) | i \in 1, \ldots, k\}$*, of* **triangular Smith regimes** *that cover* $V(N, Z)$*, which is to say* $V(N,Z) \subset \cup\{V(N_i, Z_i) | i \in 1, \ldots, k\}$*.*

The goal in this section is to reduce the PLS problem to the CTSNF problem. The first step is to show it suffices to consider PLS with a matrix already in triangular Smith form. The second step is to show each CTSNF solution regime generates a set of PLS solution regimes.

Lemma 5.2. *Given a parameterized field* F_Y *and matrix* $\mathbf{A}$ *over* $F[Y,x]$*, let* $\mathbf{H}$ *be a triangular Smith form of* $\mathbf{A}$ *over* $F_Y[x]$*, with* $\mathbf{U}$ *unimodular over* $F_Y[x]$*, and* $\mathbf{R}$ *nonsingular over* F *such that* $\mathbf{UAR} = \mathbf{H}$*. PLS problem* $(\mathbf{A}, \mathbf{b}, N, Z)$ *over* $F[Y,x]$ *has solution regimes* $(\mathbf{u}_1, \mathbf{B}_1, N_1, Z_1), \ldots, (\mathbf{u}_s, \mathbf{B}_s, N_s, Z_s)$ *if and only if PLS problem* $(\mathbf{H}, \mathbf{Ub}, N, Z)$ *has solution regimes* $(\mathbf{R}^{-1}\mathbf{u}_1, \mathbf{R}^{-1}\mathbf{B}_1, N_1, Z_1), \ldots, (\mathbf{R}^{-1}\mathbf{u}_s, \mathbf{R}^{-1}\mathbf{B}_s, N_s, Z_s)$*.*

Proof. Under evaluation at any $a \in V(N,Z)$, $\mathbf{U}(a)$ is unimodular and $\mathbf{R}$ is unchanged and nonsingular. Thus the following are equivalent.

1. $\mathbf{A}(a)\mathbf{u}(a) = \mathbf{b}(a)$.
2. $\mathbf{U}(a)\mathbf{A}(a)\mathbf{u}(a) = \mathbf{U}(a)\mathbf{b}(a)$.
3. $(\mathbf{U}(a)\mathbf{A}(a)\mathbf{R})(\mathbf{R}^{-1}\mathbf{u}(a)) = \mathbf{U}(a)\mathbf{b}(a)$.

□

Then we have the following algorithm to solve a PLS with the matrix already in triangular Smith form. For simplicity we assume a square matrix, the rectangular case being a straightforward extension.

Algorithm 2 `TriangularSmithPLS`

Require: PLS problem $(\mathbf{H}, \mathbf{b}, N, Z)$, with $\mathbf{H} \in K_Y[x]^{n\times n}$ and $\mathbf{b} \in K_Y[x]^n$, where F_Y is a parameterized extension for parameter list Y and x is an additional parameter, with $N, Z \subset F[Y]$ and $\mathbf{H}$ in triangular Smith form.

Ensure: S, the corresponding PLS solution (a list of regimes).

1: For any polynomial $s(x)$ let $\mathrm{sqfr}(s)$ denote the square-free part. Let s_i denote the i-th diagonal entry of $\mathbf{H}$, and define $s_0 = 1, s_{n+1} = 0$. Then, for $i \in 0, \ldots, n$, define $f_i = \mathrm{sqfr}(s_{i+1})/\mathrm{sqfr}(s_i)$. Let $\mathcal{I}$ be the set of indices such that f_i has positive degree or is zero.

2: For each $r \in \mathcal{I}$ include in the output S the regime $R = (\mathbf{u}, \mathbf{B}, V)$, where $\mathbf{u} = (\mathbf{H}_r^{-1}\mathbf{b}_r, 0_{n-r})$, with $\mathbf{H}_r$ the leading $r \times r$ submatrix of $\mathbf{H}$ and $\mathbf{b}_r = (b_1, \ldots, b_r)$ and $\mathbf{B} = (\mathbf{e}_{r+1}, \ldots, \mathbf{e}_n)$. Here $\mathbf{e}_i$ denotes the i-th column of the identity matrix.

3: Return S.

Lemma 5.3. *Algorithm* TriangularSmithPLS *is correct and generates at most* $\sqrt{2d}$ *regimes, where* $d = \deg(\det(\mathbf{H}))$.

Proof. Note that, for each row k, the diagonal entry s_k divides all other entries in the row. Then $\mathbf{H}$ has rank r just in case $s_r \neq 0$ and $s_{r+1} = 0$, i.e., in the cases determined in step 1 of the algorithm. The addition of s_r to N and f_r to Z ensures rank r and invertibility of $\mathbf{H}_r$. For all evaluation points $a \in V(N, Z)$ satisfying those two additional conditions, the last $n-r$ rows of $\mathbf{H}$ are zero. Hence the nullspace $\mathbf{B}$ is correctly the last $n - r$ columns of $\mathbf{I}_n$. For such evaluation points, the system will be consistent if and only if the corresponding right hand side entries are zero, hence the addition of $b_{r+1}, \ldots, b_n$ to Z. □

Algorithm 3 `PLSviaCTSNF`

Require: A PLS problem $(\mathbf{A}, \mathbf{b}, N, Z)$ over $F[Y, x]$, for parameter list Y and additional parameter x.

Ensure: A corresponding PLS solution $S = ((\mathbf{u}_i, \mathbf{B}_i, N_i, Z_i)|i \in 1, \ldots, s)$.

1: Over the ring $F(Y)[x]$, Let T solve the CTSNF problem $(\mathbf{A}, N, Z)$. T is a set of triangular Smith regimes of form $(\mathbf{U}, \mathbf{H}, R, N', Z')$. Let $S = \emptyset$.

2: For each Hermite regime $(\mathbf{U}, \mathbf{H}, R, N', Z')$ in T, using algorithm `TriangularSmithPLS`, solve the PLS problem $(\mathbf{H}, \mathbf{Ub}, N', Z')$. Adjoin to S the solution regimes, adjusted by factor $\mathbf{R}^{-1}$ as in Lemma 5.2.

3: Return S.

Theorem 5.4. *Algorithm 3 is correct.*

Proof. For every parameter evaluation $a \in V(N, Z)$ at least one triangular Smith regime of T in step 2 is valid. Then, by Lemmas 5.3 and 5.2, step 3 produces a PLS regime covering a. □

6 Solving Comprehensive Triangular Smith Normal Form

In view of the reductions of the preceding section, to solve a parametric linear system it remains only to solve a comprehensive triangular Smith form problem. This is difficult in general but we give a method to give a comprehensive solution with polynomially many regimes in the bivariate and trivariate cases.

Theorem 6.1. *Let* $\mathbf{A} \in F[y, x]^{m \times n}$ *of degree* d *in* x *and degree* e *in* y*, and let* N, Z *be polynomial sets defining a semialgebraic constraint on* y*. Then the CTSNF problem* $(\mathbf{A}, N, Z)$ *has a solution of at most* $\mathrm{O}(n^2 de)$ *triangular Smith regimes.*

Proof. If N is nonempty, then at the end of the construction below just adjoin N to the N_* of each solution regime. If Z is nonempty it trivializes the solution to at most one regime: Let $z(y)$ be the greatest common divisor of the polynomials in Z. If z is 1 or is reducible, the condition is unsatisfiable, otherwise return the single triangular Smith regime for $\mathbf{A}$ over $F[y]/\langle z(y)\rangle$. Otherwise construct the solution regimes as follows where we will assume the semialgebraic constraints are empty.

First compute triangular Smith form $\mathbf{U}_0, \mathbf{H}_0, \mathbf{R}_0$ over $F(y)[x]$ such that $\mathbf{A} = \mathbf{U_0 H_0 R_0}$. This will be valid for evaluations that don't zero the denominators (polynomials in y) of $\mathbf{H}_0, \mathbf{U}_0$. So set $N_0 = \mathrm{den}(\mathbf{U}_0, \mathbf{H}_0)$ (or to be the set of irreducible factors that occur in $\mathrm{den}(\mathbf{U}_0, \mathbf{H}_0)$. Set $Z_0 = \emptyset$ to complete the first regime.

Then for each irreducible polynomial $f(y)$ that occurs as a factor in N_0 adjoin the regime $(\mathbf{U}_f, \mathbf{H}_f, \mathbf{R}_f, N_f = N \setminus \{f\}, Z_f = \{f\})$, that comes from computing the triangular Smith form over $(F[y]/\langle f\rangle)[x]$. From the bounds of Theorem 4.6 we have the specified bound on the number of regimes. □

We can proceed in a similar way when there are three parameters, but must address an additional complication that arises.

Theorem 6.2. *Let $\mathbf{A} \in F[z, y, x]^{m\times n}$ of degree d in x and degree e in y, z, and let N, Z be polynomial sets defining a semialgebraic constraint on y and z. Then the CTSNF problem $(\mathbf{A}, V(N, Z))$ has a solution of at most $\mathrm{O}(n^4 d^2 e^2)$ triangular Smith regimes.*

Proof. As in the bivariate case above, we solve the unconstrained case and just adjoin N, Z, if nontrivial, to the semialgebraic condition of each solution regime.

First compute triangular Smith form $\mathbf{U}_0, \mathbf{H}_0, \mathbf{R}_0$ over $F(y, z)[x]$ such that $\mathbf{A} = \mathbf{U}_0\mathbf{H}_0\mathbf{R}_0$. This will be valid for evaluations that don't zero the denominators (polynomials in y, z) of $\mathbf{H}_0, \mathbf{U}_0$. Thus we set $(N_0, Z_0) = (\{\mathrm{den}(\mathbf{U}_0, \mathbf{H}_0)\}, \emptyset)$ to complete the first regime.

Then for each irreducible polynomial f that occurs as a factor in N_0, if y occurs in f, adjoin the regime $(\mathbf{U}_f, \mathbf{H}_f, \mathbf{R}_f, N_f = N \setminus \{f\}, Z_f = \{f\})$, that comes from computing the triangular Smith form over $(F(z)[y]/\langle f\rangle)[x]$. If y doesn't occur in f, interchange the roles of y, z.

In either case we get a solution valid when f is zero and the solution denominator δ_f is nonzero. This denominator is of degree $\mathrm{O}(n^2 de)$ in each of y, z by Theorem 4.6. [It is the new complicating factor arising in the trivariate case.] It is relatively prime to f, so Bézout's theorem [7] in the theory of algebraic curves can be applied: there are at most $\deg(f)\deg(\delta)$ points that are common zeroes of f and δ. We can produce a separate regime for each such (y, z)-point by evaluating $\mathbf{A}$ at the point and computing a triangular Smith form over $F[x]$. Summing over the irreducible f dividing the original denominator in N_0 we have $\mathrm{O}((n^2 de)^2)$ bounding the number of these denominator curve intersection points. □

Corollary 6.3. *For a PLS with $m \times n$ matrix $\mathbf{A}$, $\mathbf{b}$ an m-vector, and with $\deg_x(\mathbf{A}, \mathbf{b}) \le d, \deg_y(\mathbf{A}, \mathbf{b}) \le e, \deg_z(\mathbf{A}, \mathbf{b}) \le e$, we have*

1. *$\mathrm{O}(m^{1.5} d^{0.5})$ regimes in the PLS solution for the univariate case (domain of $\mathbf{A}$, $\mathbf{b}$ is $F[x]$).*

2. $\mathrm{O}(m^{2.5}d^{1.5}e)$ *regimes in the PLS solution for the bivariate case (domain of* $\mathbf{A}$*,* $\mathbf{b}$ *is* $F[x,y]$*).*
3. $\mathrm{O}(m^{4.5}d^{2.5}e^2)$ *regimes in the PLS solution for the trivariate case (domain of* $\mathbf{A}$*,* $\mathbf{b}$ *is* $F[x,y,z]$*).*

Proof. By Lemma 5.3, each CTSNF regime expands to at most $\sqrt{md}$ PLS regimes. □

7 Normal forms and Eigenproblems

Comprehensive Hermite Normal form and comprehensive Smith Normal form are immediate corollaries of our comprehensive triangular Smith form. For Hermite form, just take the right hand cofactor to be the identity, $\mathbf{R} = \mathbf{I}$, and drop the check for the divisibility condition on the diagonal entries in Algorithm 2. For Smith form one can convert each regime of CTSNF to a Smith regime. Where $\mathbf{UAR} = \mathbf{H}$ with $\mathbf{H}$ a triangular Smith form, perform column operations to obtain $\mathbf{UAV} = \mathbf{S}$ with $\mathbf{S}$ the diagonal of $\mathbf{H}$. In $\mathbf{H}$ the diagonal entries divide the off diagonal entries in the same row. Subtract multiples of the i-th column from the subsequent columns to eliminate the off diagonal entries. Because the diagonal entries are monic, no new denominator factors arise and $\det(\mathbf{V}) = \det(\mathbf{R}) \in F$. Thus when $(\mathbf{U}, \mathbf{H}, \mathbf{R}, N, Z)$ is a valid regime in a CTSNF solution for $\mathbf{A}$, then $(\mathbf{U}, \mathbf{S}, \mathbf{V}, N, Z)$ is a valid regime for Smith normal form.

It is well known that if $\mathbf{A} \in K^{n\times n}$ for field K (that may involve parameters) and λ is an additional variable, then the Smith invariants $s_1, \ldots, s_n$ of $\lambda\mathbf{I} - \mathbf{A}$ are the Frobenius invariants of $\mathbf{A}$ and $\mathbf{A}$ is similar to its Frobenius normal form, $\oplus_{i=1}^{n}\mathbf{C}_{s_i}$, where $\mathbf{C}_s$ denotes the companion matrix of polynomial s. Thus we have comprehensive Frobenius normal form as a corollary of CTSNF, however it is without the similarity transform. It would be interesting to develop a comprehensive Frobenius form with each regimes including a transform.

Parametric eigenvalue problems for $\mathbf{A}$ correspond to PLS for $\lambda\mathbf{I} - \mathbf{A}$ with zero right hand side. Often eigenvalue multiplicity is the concern. The geometric multiplicity is available from the Smith invariants, as for example on the diagonal of a triangular Smith form. Common roots of 2 or more of the invariants expose geometric multiplicity and square-free factorization of the individual invariants exposes algebraic multiplicity. Note that square-free factorization may impose further restrictions on the parameters. Comprehensive treatment of square-free factorization is considered in [18].

7.1 Eigenvalue multiplicity example

The following matrix, due originally to a question on sci.math.num-analysis in 1990 by Kenton K. Yee, is discussed in [5]. We change the notation used there to avoid a clash with other notation used here. The matrix is

$$\mathbf{Y} = \begin{bmatrix} z^{-1} & z^{-1} & z^{-1} & z^{-1} & z^{-1} & z^{-1} & z^{-1} & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & z & z & z & z & z & z & z \end{bmatrix}.$$

One of the original questions was to compute its eigenvectors. Since it contains a symbolic parameter z, this is a parametric eigenvalue problem which we can turn into a parametric linear system, namely to present the nullspace regimes for $\lambda\mathbf{I} - \mathbf{Y}$.

Over $F(z)[\lambda]$, after preconditioning, we get as the triangular Smith form diagonal $(1, 1, 1, 1, 1, \lambda^2 - 1, \lambda^2 - 1, (\lambda^2 - 1)f(\lambda))$, where $f(\lambda) = \lambda^2 - (z + 6 + z^{-1})\lambda + 7$.

Remark 7.1. Without preconditioning, the Hermite form diagonal is instead $(1, 1, 1, 1, \lambda - 1, \lambda^2 - 1, (\lambda^2 - 1), (\lambda + 1)f(\lambda))$.

The denominator of $\mathbf{U}$, $\mathbf{H}$ is a power of z, so the only constraint is $z = 0$ which is already a constraint for the input matrix. We get regimes of rank 5 for $\lambda = \pm 1$, rank 7 for λ being a root of f, and rank 8 for all other λ. In terms of the eigenvalue problem, we get eigenspaces of dimension 3 for each of 1, -1 and of dimension 1 for the two roots of $f(\lambda)$.

To explore algebraic multiplicity, we can examine when f has 1 or -1 as a root. When z is a root of $z^2 + 14\,z + 1$, $f(\lambda)$ factors as $(\lambda - 1)(\lambda - 7)$ and when $z = 1$ we have $f(\lambda) = (\lambda + 1)(\lambda + 7)$. These factorizations may be discovered by taking resultants of f with $\lambda - 1$ or $\lambda + 1$.

7.2 Matrix Logarithm

Theorem 1.28 of [12] states conditions under which the matrix equation $\exp(\mathbf{X}) = \mathbf{A}$ has so-called *primary matrix logarithm* solutions, and under which conditions there are more. If the number of distinct eigenvalues s of $\mathbf{A}$ is *strictly less* than the number p of distinct Jordan blocks of $\mathbf{A}$ (that is, the matrix $\mathbf{A}$ is *derogatory*), then the equation also has so-called *nonprimary* solutions as well, where the branches of logarithms of an eigenvalue λ may be chosen differently in each instance it occurs.

As a simple example of what this means, consider

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \tag{7.1}$$

When we compute its matrix logarithm (for instance using the `MatrixFunction` command in Maple), we find

$$\mathbf{X}_A = \begin{bmatrix} \ln(a) & a^{-1} \\ 0 & \ln(a) \end{bmatrix}. \tag{7.2}$$

This is what we expect, and taking the matrix exponential (a single-valued matrix function) gets us back to $\mathbf{A}$, as expected. However, if instead we consider the derogatory matrix

$$\mathbf{B} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \tag{7.3}$$

then its matrix logarithm as computed by `MatrixFunction` is also derogatory, namely

$$\mathbf{X}_B = \begin{bmatrix} \ln(a) & 0 \\ 0 & \ln(a) \end{bmatrix}. \tag{7.4}$$

Yet there are other solutions as well: if we add $2\pi i$ to the first entry and $-2\pi i$ to the second logarithm, we unsurprisingly find another matrix $\mathbf{X}_C$ which also satisfies $\exp(\mathbf{X}) = \mathbf{B}$. But adding $2\pi i$ to the first entry of $\mathbf{X}_A$ while adding $-2\pi i$ to its second logarithm, we get another matrix

$$\mathbf{X}_D = \begin{bmatrix} \ln(a) + 2\,i\pi & a^{-1} \\ 0 & \ln(a) - 2\,i\pi \end{bmatrix} \tag{7.5}$$

which has the (somewhat surprising) property that $\exp(\mathbf{X}_D) = \mathbf{B}$, not $\mathbf{A}$.

This example demonstrates in a minimal way that the detailed Jordan structure of $\mathbf{A}$ strongly affects the nature of the solutions to the matrix equation $\exp(\mathbf{X}) = \mathbf{A}$. This motivates the ability of code to detect automatically the differing values of the parameters in a matrix that make it derogatory. To explicitly connect this example to CTSNF, consider

$$\mathbf{M} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \tag{7.6}$$

so that $\mathbf{A}$ above is $\mathbf{M}_{b=1}$ and $\mathbf{B} = \mathbf{M}_{b=0}$. The CTSNF applied to $\lambda\mathbf{I} - \mathbf{M}$ produces two regimes, with forms

$$\mathbf{H}_{b\neq 0} = \begin{bmatrix} 1 & \lambda/b \\ 0 & (\lambda - a)^2 \end{bmatrix}, \mathbf{H}_{b=0} = \begin{bmatrix} \lambda - a & 0 \\ 0 & \lambda - a \end{bmatrix}, \tag{7.7}$$

exposing when the logarithms will be linked or distinct. Note that in this case the Frobenius structure equals the Jordan structure.

7.3 Model of infectious disease vaccine effect

Rahman and Zou [22] have made a model of vaccine effect when there are two subpopulations with differing disease susceptibility and vaccination rates. Within this study stability of the model is a function of the eigenvalues of a Jacobian $\mathbf{J}$. Thus we are interested in cases where the following matrix is singular.

$$\mathbf{A} = \lambda\mathbf{I} - \mathbf{J} = \begin{bmatrix} \lambda - w & 0 & -a & -c \\ 0 & \lambda - x & -b & -d \\ 0 & 0 & \lambda - a - y & c \\ 0 & 0 & b & \lambda - d - z \end{bmatrix}.$$

Here w, x are vaccination rates for the two populations, y, z are death rates, a, d are within population transmission rates, and b, c are the between population transmission rates. We have simplified somewhat: for instance a, b, c, d are transmission rates multiplied by other parameters concerning population counts. Stability depends on the positivity of the largest real part of an eigenvalue. For the sake of reducing expression sizes in this example we will arbitrarily set $y = z = 1/10$. For the same reason we will skip right multiplication by an R to achieve triangular Smith form. Hermite form $\mathbf{H}$ of $\lambda\mathbf{I} - \mathbf{J}$ will suffice, revealing the eigenvalues that are wanted.

$$\mathbf{H} = \begin{bmatrix} \lambda + w & 0 & 0 & -(ad - a\lambda - bc - a/10)/c \\ 0 & \lambda + x & 0 & \lambda + 1/10 \\ 0 & 0 & 1 & (d - \lambda - 1/10)/c \\ 0 & 0 & 0 & \lambda^2 + (1/5 - a - d)\lambda + ad - cb - (1/10)d - (1/10)a + 1/100 \end{bmatrix}.$$

The discriminant of the last entry gives the desired information for the application subject to the denominator validity: $c \neq 0$. When $c = 0$ the matrix is already in Hermite form, so again the desired information is provided.

This example illustrates that often more than three parameters can be easily handled. In experiments with this model not reported here, we did encounter cases demanding solution beyond the methods of this paper. On a more positive note, we feel that comprehensive normal form tools could help analyze models like this when larger in scope, for instance modeling 3 or more subpopulations.

7.4 The Kac-Murdock-Szegö example

In [4] we see reported times for computation of the comprehensive Jordan form for matrices of the following form, taken from [28], of dimensions 2 to about 20:

$$\mathrm{KMS}_n = \begin{bmatrix} 1 & -\rho & & & \\ -\rho & \rho^2 + 1 & -\rho & & \\ & \ddots & \ddots & \ddots & \\ & & -\rho & \rho^2 + 1 & -\rho \\ & & & -\rho & 1 \end{bmatrix}. \tag{7.8}$$

This is, apart from the $(1,1)$ entry and the (n,n) entry, a Toeplitz matrix containing one parameter, ρ. The reported times to compute the Jordan form were plotted in [4] on a log scale, and looked as though they were exponentially growing with the dimension, and were reported in that paper as growing exponentially.

The theorem of this paper states instead that polynomial time is possible for this family, because there are only two parameters (ρ and the eigenvalue parameter, say λ). The Hermite forms for these matrices are all (as far as we have computed) trivial, with diagonal all 1 except the final entry which contains the determinant. Thus all the action for the Jordan form must happen with the discriminant of the determinant. Experimentally, the discriminant with respect to λ has degree n^2+n-4 for KMS matrices of dimension $n \geq 2$ (this formula was deduced experimentally by giving a sequence of these degrees to the Online Encyclopedia of Integer Sequences [26]) and each discriminant has a factor $\rho^{n(n-1)}$, leaving a nontrivial factor of degree $2n-4$ growing only linearly with dimension. The case $\rho = 0$ does indeed give a derogatory KMS matrix (the identity matrix). The other factor has at most a linearly-growing number of roots for each of which we expect the Jordan form of the corresponding KMS matrix to have one block of size two and the rest of size one. We therefore see only polynomial cost necessary to compute comprehensive Jordan forms for these matrices, in accord with our theorem.

8 Conclusions

We have shown that using the CTNSF to solve parametric linear systems is of cost polynomial in the dimension of the linear system and polynomial in parameter degree, for problems containing up to three parameters. This shows that polynomially many regimes suffice for problems of this type. To the best of our knowledge, this is the first method to achieve this polynomial worst case.

It remains an open question whether, for linear systems with a fixed number of parameters greater than three, a number of regimes suffices that is polynomial in the input matrix dimension and polynomial degree of the parameters, being exponential only in the number of parameters.

Through experiments with random matrices we have indication that the worst case bounds we give are sharp, though we haven't proven this point. As the examples indicated, many problems will have fewer regimes, and sometimes substantially fewer regimes. We have not investigated the effects of further restrictions of the type of problem, such as to sparse matrices.

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