

# The Theodorus Variation

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## Recommended Reference Format:

Ewan Brinkman, Robert M. Corless, and Veselin Jungić. 2021. The Theodorus Variation. *Maple Trans.* V, I, Article XXXXX (2021), 17 pages. <https://doi.org/10.5206/mt.vViLXXXXX>

## 1 Introduction

In this note we study what Susan Gerofsky, a Canadian mathematics educator, in [9] calls the *reverse Wurzelschnecke*. “Wurzelschnecke” is German for “root snail”; this will make sense very soon. This object, this “root snail backward,” is closely related to the Spiral of Theodorus, an ancient figure recently given that name in [7, p. 33], that still attracts attention of mathematicians [8] and artists [13] alike.

To construct the Spiral of Theodorus, Figure 2, we start with the equilateral right triangle  $\triangle OT_1T_2$  with  $|\overline{OT_1}| = |\overline{T_1T_2}| = 1$ . The hypotenuse is therefore of length  $\sqrt{2}$ . The point  $T_n$ , for  $n \geq 3$ , is determined so that  $\angle OT_{n-1}T_n$  is the right angle and that  $|\overline{T_{n-1}T_n}| = 1$ . The side lengths of  $\triangle OT_{n-1}T_n$  are  $\sqrt{n-1}$ , 1, and  $\sqrt{n} = \sqrt{(\sqrt{n-1})^2 + 1^2}$ . See Figure 1. Observe that the spiral in Figure 2 is (arbitrarily) oriented counterclockwise.

It is a straightforward observation<sup>1</sup> that  $|\overline{OT_n}| = \sqrt{n}$ . This can be proved by mathematical induction and the Pythagorean theorem, for instance. This is the reason that the Spiral of Theodorus is sometimes called the “square root spiral.”

Theodorus of Cyrene [12] lived in the fifth century BCE and was “mentioned by Proclus along with Hippocrates as a celebrated geometer.” [10, p. 202]

Sir Thomas Heath (1861 –1940), a British classicist, in his book *A History of Greek Mathematics* writes:

In Plato’s *Theaetetus* we have the story of Theodorus lecturing on surds and proving separately, for the square root of every non square number from 3 to 17, that it is incommensurable with 1. [10, p. 22]

<sup>1</sup>That is, straightforward for practiced geometers and other mathematicians. The *point* of doing this with elementary students who may just be learning the Pythagorean theorem is to get them to appreciate this fact. Even to get the students to appreciate that the shorthand  $\sqrt{n}$  applies to *all* of the triangles in turn, starting from  $n = 2$ ,  $n = 3$ , and so on, is important.

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<https://doi.org/10.5206/mt.vViLXXXXX>

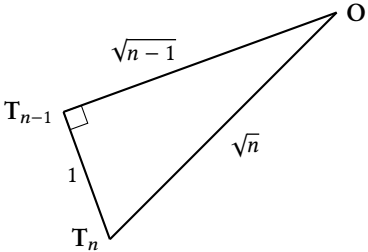


Fig. 1. The right triangle  $\triangle OT_{n-1}T_n$ , which has one side of unit length, and the others of length  $\sqrt{n-1}$  and  $\sqrt{n}$ . Pythagoras' theorem is seen to hold:  $(\sqrt{n-1})^2 + (1)^2 = (\sqrt{n})^2$ .

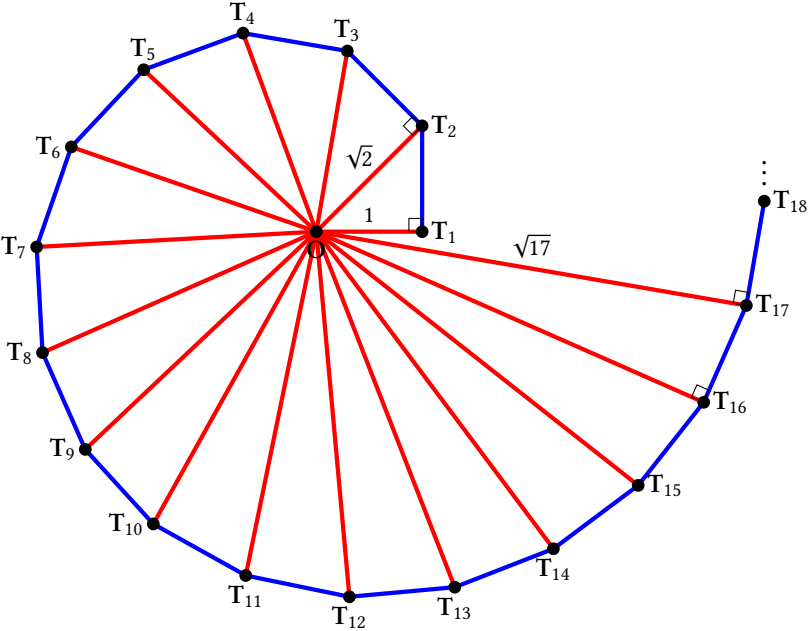


Fig. 2. The *Spiral of Theodorus*, built by placing the basic right triangles with the base of each subsequent triangle on the hypotenuse of the previous triangle. By construction, these are the same length.

Later in the text, Heath explains:

Plato gives no hint as to how Theodorus proved the propositions attributed to him, namely that  $\sqrt{3}, \sqrt{5}, \dots, \sqrt{17}$  are all incommensurable with 1; there is therefore a wide field open for speculation, and several conjectures have been put forward. [10, p. 204]

The modern reader is probably familiar with the argument that the identity  $p^2 = n \cdot q^2$ , for  $n, p$  and  $q$  being integers, implies, from unique factorization, that every prime divisor of  $n$  appears to an even power, and hence  $n$  is a perfect square.

We observe that this “modern” proof is practically the same *reductio ad absurdum* argument that Pythagoreans used to prove that  $\sqrt{2}$  is irrational. Keeping in mind that Theodorus’s proof appeared several decades after the discovery of the irrationality of  $\sqrt{2}$ , it is a natural question to ask about the value of Theodorus’s approach.

Heath discusses this question in some length by presenting a few hypothesis about the nature of Theodorus’s proof. One of those hypotheses, that Heath attributes to Friedrich Hultsch (1833 – 1906), a German classical philologist, was that Theodorus actually, to use a modern term, visualized Pythagoreans’ *reductio ad absurdum* argument. [10, pp. 204 – 205]

Fast forward to the 21st century, Jonathan Borwein (1953 – 2016), a Canadian mathematician, as an example of experimental mathematic accessible at the high school level, in multiple publications used an image that depicts the fact that an assumption that  $\sqrt{2}$  is rational leads to a contradiction. See, for example, [1, 2, 3, 4].

In the spirit of experimental mathematics<sup>2</sup>, Borwein (and his coauthor) advise the reader to basically go along the Spiral of Theodorus<sup>3</sup> and creates visual proof(s) that nonperfect squares do not have rational square roots.

For more about connections between the Spiral of Theodorus and a technique that Borwein called *minimal configurations* [4], the technique that emphasizes the visual side of the fact that the magnitudes  $\sqrt{n}$ , where  $n \in \mathbb{N}$  is a nonperfect square, and 1 are incommensurable, see [9, 11].

## 2 A continuous version

Philip J. Davis (1923 – 2018), an American applied mathematician, has in [7, p. 38] the following, most mysterious, formula:

$$T(a) = \prod_{k \geq 1} \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{k} + a - 1}. \quad (1)$$

[Davis actually uses  $a$  instead of  $a - 1$ , there; we make this shift to accommodate the notation in this paper.] Here  $i$  is the square root of  $-1$ . This is an *infinite* product, but note that if the variable  $a$  happens to be a positive integer, then something interesting happens, namely, most of the terms in the product cancel out. Let us do the computation explicitly if  $a = 4$ . Note that the infinite product must be defined as a limit, say

$$T(a) = \lim_{N \rightarrow \infty} \prod_{k=1}^N \frac{1 + i/\sqrt{k}}{1 + i/\sqrt{k} + a - 1}. \quad (2)$$

Letting  $a = 4$  and writing the first few terms out explicitly we have

$$\begin{aligned} T(4) &= \lim_{N \rightarrow \infty} \left( \frac{1 + i/\sqrt{1}}{1 + i/\sqrt{1} + 3} \right) \left( \frac{1 + i/\sqrt{2}}{1 + i/\sqrt{2} + 3} \right) \left( \frac{1 + i/\sqrt{3}}{1 + i/\sqrt{3} + 3} \right) \left( \frac{1 + i/\sqrt{4}}{1 + i/\sqrt{4} + 3} \right) \cdots \left( \frac{1 + i/\sqrt{N}}{1 + i/\sqrt{N} + 3} \right) \\ &= \lim_{N \rightarrow \infty} \frac{(1 + i/\sqrt{1})(1 + i/\sqrt{2})(1 + i/\sqrt{3})}{(1 + i/\sqrt{N+1})(1 + i/\sqrt{N+2})(1 + i/\sqrt{N+3})}, \end{aligned} \quad (3)$$

because all of the terms except the first 3 in the numerator and the last 3 in the denominator cancel, or “telescope.” This limit can therefore be seen to exist, and equal  $(1 + i/\sqrt{1})(1 + i/\sqrt{2})(1 + i/\sqrt{3})$ . A

<sup>2</sup>To use technology to gain insight and intuition, to visualize mathematical principles, to discover new relationships, and to test possibly false conjectures.

<sup>3</sup>Borwein never explicitly mentions the Spiral of Theodorus

similar thing happens if  $a$  is any positive integer; at the end, we are left with a finite product of complex numbers, and indeed  $T(n+1) = (1 + i/\sqrt{n})T(n)$ , and  $T(1) = 1$ .

Great, you say. What has this to do with the spiral of Theodorus? The connection is as follows: embed the spiral in the complex plane, with the origin at  $z = 0 + 0i$ . Then vertex  $T_1 = (1, 0) = 1 + 0i$ , while  $T_2 = (1, 1) = 1 + i$ . Note that our *function* has  $T(1) = 1$  and  $T(2) = 1 + i/\sqrt{1} = 1 + i$ .

Now think of the multiplication of complex numbers: as Richard Feynman (1918 – 1988), an American theoretical physicist, observed famously, to a physicist complex multiplication is very simple: you multiply the magnitudes, and you add the angles. The angle of the complex number  $1 + i/\sqrt{n}$  has  $\tan \phi_n = 1/\sqrt{n}$ , which is exactly the central angle of the  $n$ th Theodorus triangle; and the magnitude of  $T(2)$  is  $|T(2)| = \sqrt{2}$  which is the length of the hypotenuse of the first Theodorus triangle (and the base of the second). Since  $T(n+1) = (1 + i/\sqrt{n})T(n)$  we have

$$\begin{aligned} |T(n+1)| &= \sqrt{1 + \frac{1}{n}} |T(n)| \\ &= \frac{\sqrt{n+1}}{\sqrt{n}} |T(n)| \end{aligned} \quad (4)$$

and since  $|T(1)| = 1$  we have by induction that  $|T(n)| = \sqrt{n}$ . Notice that this is exactly the length  $|\overline{OT}_n|$ .

What we have just proved is that the complex curve  $T(a)$  parameterized by the real parameter  $a$  exactly interpolates the Theodorus spiral (provided, of course, that the infinite product defined when  $a$  is not an integer actually exists; we don't do this here, because it demands knowledge of the convergence of infinite products which is not usually part of the standard curriculum at least until later; but the proof that it does exist, which is intelligible assuming the reader does have that background, is in [7]). We do, however, show that this works in Maple, in section 3. This analysis is very similar to Euler's interpolation of the gamma function, which indeed this is modelled on; see [5] or, better yet, the magnificent (Chauvenet prize-winning) paper [6].

Walter Gautschi, a Swiss–American mathematician, starts from here and writes in [8] the *complex argument* of  $T(a)$  as the *integral of a sum*, so that

$$T(a) = \sqrt{a} \exp \left( \frac{i}{2} \int_{\alpha=1}^a U(\alpha) d\alpha \right) \quad (5)$$

with

$$U(a) = \sum_{k=1}^{\infty} \frac{1}{(k+a-1)^{\frac{3}{2}} + \sqrt{k+a-1}}. \quad (6)$$

As pointed out in [6] and in [8], this sum converges, but only “painfully slowly”. Taking a million terms only gets you three significant figures of accuracy, if you work naively. This is also true of the *Theodorus constant*, defined as  $U(1)$  or

$$T = \sum_{k \geq 1} \frac{1}{k^{3/2} + k^{1/2}}. \quad (7)$$

...regarding its slope as it crosses the  $0^0$  ray as a fundamental world constant and calling this world constant  $T$  in honor of Theodorus, I wanted to determine  $T$  to about eight or ten figures to the right of the decimal point.

—Philip J. Davis, [7, p. 40]

Davis then goes on to discuss two different methods to do so.

### 3 Some experiments in Maple

However, we can evaluate that very slowly-converging sum in Maple extremely simply. Everything is done for us, internally, by the Levin's  $u$ -transform [14]. [This is similar in effect to the first technique discussed by Davis, namely the use of the Euler-Maclaurin sum formula. But it's built-in, and works very well on this example.]

$$> U := a \rightarrow \text{evalf}(\text{Sum}(\frac{1}{(k+a-1)^{\frac{3}{2}} + (k+a-1)^{\frac{1}{2}}}, k = 1..infinity))$$

$$U := a \mapsto \text{evalf}\left(\sum_{k=1}^{\infty} \frac{1}{(k+a-1)^{\frac{3}{2}} + \sqrt{k+a-1}}\right) \quad (8)$$

> Digits := 100;

$$\text{Digits} := 100 \quad (9)$$

> CodeTools:-Usage(U(1))

```
memory used=11.58MiB, alloc change=0 bytes, cpu time=62.00ms,
real time=48.00ms, gc time=0ns
```

$$\begin{aligned} &1.8600250792211903071806959157171433246665241215234 \\ &51493049199503598342723399921320568838756499614496 \end{aligned} \quad (10)$$

This gives us 100 digits of the Theodorus constant in about a twentieth of a second. But what we need to plot the spiral is really the integral

$$\phi(a) = \frac{1}{2} \int_{\alpha=1}^a U(\alpha) d\alpha, \quad (11)$$

which gives the polar angle  $\phi(a)$  of  $T(a) = \sqrt{a} e^{i\phi(a)}$ . While this is perfectly possible in Maple by using **evalf/Int**, and it is rapid and accurate for small values of  $a$ , as  $a$  increases this takes increasingly more time to do. For  $a = 2$  using 15 Digits the computation takes less than 2 seconds, but for  $a = 16$  this takes about 8 seconds (on a 2021 vintage 1.3GHz Microsoft Surface Pro).

Gautschi goes on in [8] to find a definite integral expression for  $\phi(a)$ , which is both faster and more elegant than the integration of the accelerated sum above, in terms of special functions. In Maple's notation, this is  $\phi(a) = G(a)$  where

$$> \text{epsilon} := t \rightarrow \frac{t}{\exp(t) - 1}$$

$$\epsilon := t \mapsto \frac{t}{e^t - 1} \quad (12)$$

$$> G := \alpha \rightarrow \text{Int}\left(\frac{t^{-\frac{1}{2}} \cdot \text{epsilon}(t) \cdot (1 - \exp(-(\alpha - 1) \cdot t))}{t} \cdot \frac{\exp(-t)}{2 \cdot \text{sqrt}(t)} \cdot \text{erfi}(\text{sqrt}(t)), t = 0..infinity\right)$$

$$G := \alpha \mapsto \int_0^{\infty} \frac{\epsilon(t) \cdot (1 - e^{-(\alpha-1) \cdot t}) \cdot \left(\frac{1}{2}\right) \cdot e^{-t} \cdot \text{erfi}(\sqrt{t})}{t^{\frac{3}{2}} \cdot \sqrt{t}} dt. \quad (13)$$

This clever formulation involving the **erfi** function (also known as Dawson's Integral) is *less computationally expensive* to evaluate, for large  $a$ , than the integral of the accelerated sum above: while the sum takes about a fifth of a second to evaluate (at 15 Digits) at each value of  $a$ , the integral of  $U(a)$  where  $U(a)$  is evaluated by **evalf/Sum** takes about 8 seconds when  $a = 16$ ; whereas the integral of the special function takes only about 2 seconds. Having both methods available in Maple is good: the agreement of the two results leads to increased confidence.

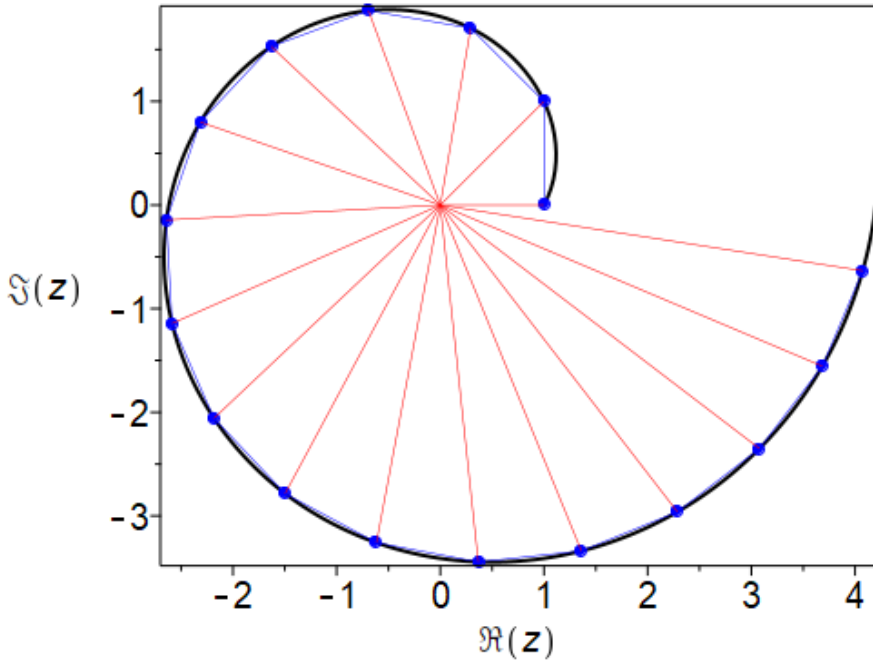


Fig. 3. The continuous version (black curved line) of the Spiral of Theodorus, plotted in Maple.

However, a much faster way to evaluate

$$G(a) = \phi(a) = \frac{1}{2} \int_{\alpha=1}^a U(\alpha) d\alpha \quad (14)$$

for a succession of values, such as you might need to plot the continuous Spiral of Theodorus, is to solve the differential equation  $\phi'(a) = U(a)/2$  numerically, over the range  $1 \leq a \leq A$  where  $A$  is the “return time” of the map, namely the value such that  $\phi(A) = 2\pi$ . This numerical solution takes about 2 seconds in Maple using the default `rkf45` method, but thereafter the value of  $\phi(a)$  is available essentially instantaneously for any  $1 \leq a \leq A$  by evaluating the computed polynomial interpolant. This raises the question of how accurate some piecewise polynomial interpolant of the discrete spiral would be; that is a question we reserve for another day.

By use of `fsolve`, which takes about 5 minutes using the numerical quadrature of the `evalf/Sum` form of  $U(a)$ , we find  $A = 17.6459044714280$  to 15 Digits, with  $\phi(A) = 2\pi$ . This computation is much faster using the Gautschi definite integral, taking less than 30 seconds. The use of the numerical differential equation solver, with event handling, should take only about 2 seconds (essentially the same as just solving the system).

The computations in this section are available at [This Maple Cloud Link](#).

#### 4 Experiments with physical triangles led to a surprise

Gerofsky materialized the Spiral of Theodorus by creating various models of it from materials that included pieces of furniture, cardboard cutouts, and even edible pastry. [9]

During one of her workshops while re-arranging triangles used to build a model of the Spiral of Theodorus, Gerofsky placed the right triangles in *reverse order*, i.e. with the hypotenuse of the subsequent triangle set on the leg of the previous one, rather than on its hypotenuse as it the case with the Spiral of Theodorus. See Figure 4.

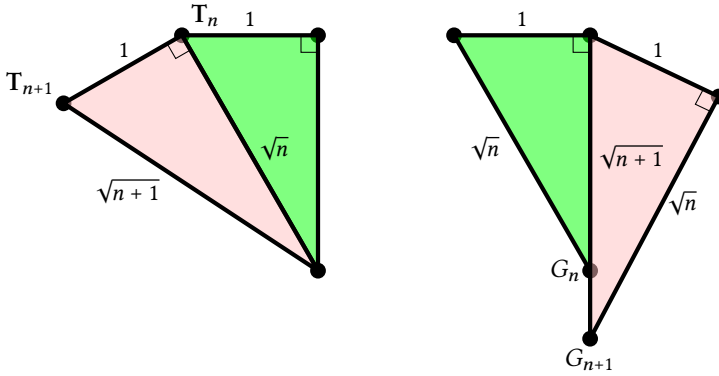


Fig. 4. Two characteristic adjacent right triangles in the Spiral of Theodorus (left) and the reverse Wurzelschnecke (right)

In this new arrangement of triangles, the vertices opposite to the leg of length 1, annotated by  $G_0, G_1, G_2, \dots$  in Figure 5, form a spiral-like object which she called the *reverse Wurzelschnecke*.

In what follows we analyze some of the properties of the reverse Wurzelschnecke, both analytically and computationally. We also provide an elementary proof of the fact conjectured in [9], i.e. we prove that, when  $n$  increases, the points  $G_n$  approach the boundary of a certain circle. We determine the centre and the radius of that circle.

#### 5 Reverse Wurzelschnecke

In this section we draw the reverse Wurzelschnecke in the complex plane and denote its points by  $z_n, n \in \{0, 1, 2, \dots\}$ . For  $n \geq 1$  we denote the other two vertices of the  $z_n$ -determined characteristic triangle by  $w_{n-1}$  and  $w_n$ . See Figure 6.

By construction,

$$|w_n - z_n| = \sqrt{n}, |w_n - w_{n+1}| = 1, |w_{n+1} - z_{n+1}| = \sqrt{n+1}, \text{ and } |w_n - z_{n+1}| = \sqrt{n+2}.$$

In addition,

$$|z_{n+1} - z_n| = \sqrt{n+2} - \sqrt{n}.$$

PROPOSITION 1. For all  $n \in \mathbb{N}$ ,

$$z_{n+1} = w_n + \sqrt{\frac{n+2}{n}} \cdot (z_n - w_n).$$

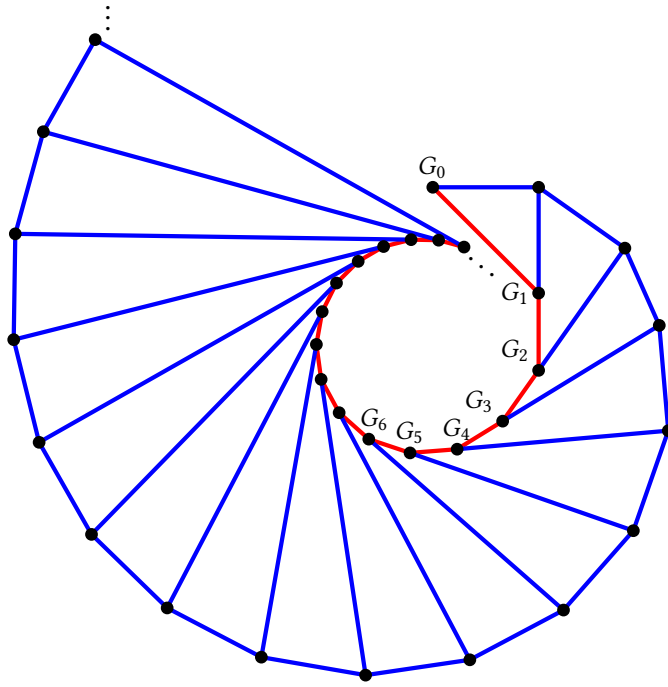


Fig. 5. Reverse Wurzelschnecke

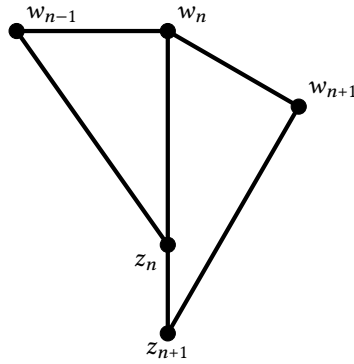


Fig. 6. Reverse Wurzelschnecke: Two characteristic adjacent right triangles in the complex plane

PROOF. Observe that  $z_n$ ,  $w_n$ , and  $z_{n+1}$  are three collinear points. This implies that there is  $t > 1$  such that

$$z_{n+1} = w_n + t(z_n - w_n).$$

Hence

$$t = \frac{z_{n+1} - w_n}{z_n - w_n} \Rightarrow |t| = \frac{|z_{n+1} - w_n|}{|z_n - w_n|} = \sqrt{\frac{n+2}{n}}.$$



Therefore

$$z_{n+1} = w_n + \sqrt{\frac{n+2}{n}} \cdot (z_n - w_n).$$

□

## 6 It is a circle!

We construct the Spiral of Theodorus,  $T_1, T_2, T_3, \dots$ , with the initial point  $O$ . Recall that by construction

$$|\overline{OT_n}| = \sqrt{n}, \quad |\overline{T_n T_{n+1}}| = 1, \quad \overline{OT_n} \perp \overline{T_n T_{n+1}}.$$

Through each point  $T_{n+1}$  we construct the line  $t_n$ ,  $t_n \perp \overline{T_n T_{n+1}}$ . Let

$$S_n = t_n \cap t_{n+1}.$$

See Figure 7.

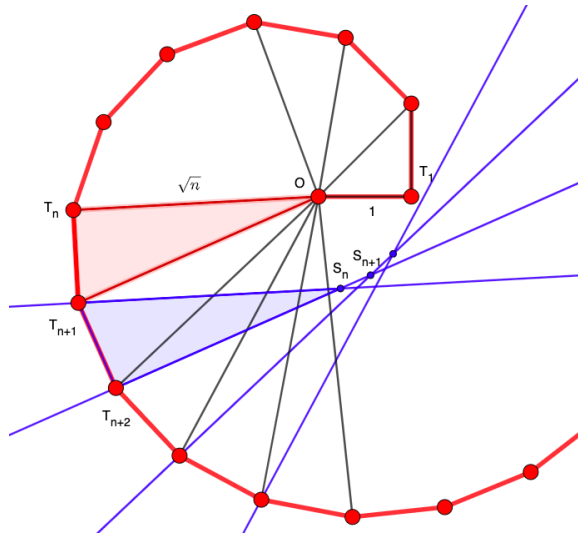


Fig. 7. Two Spirals

**PROPOSITION 2.** *The sequence of points  $\{S_n\}_{n \in \mathbb{N}}$  form the reverse Wurzelschnecke.*

**PROOF.** Observe that

$$\overline{OT_{n+1}} \perp \overline{T_{n+1} T_{n+2}} \quad \text{and} \quad \overline{S_n T_{n+1}} \perp \overline{T_n T_{n+1}}.$$

This means that  $\angle T_n T_{n+1} O$  and  $\angle T_{n+2} T_{n+1} S_n$  are congruent, as two angles with two pairs of mutually perpendicular rays.

It follows that the right triangles  $\triangle T_n T_{n+1} O$  and  $\triangle T_{n+2} T_{n+1} S_n$  are congruent. In particular

$$|\overline{S_n T_{n+2}}| = |\overline{OT_n}| = \sqrt{n}, \quad \text{and} \quad |\overline{S_n T_{n+1}}| = |\overline{OT_{n+1}}| = \sqrt{n+1}.$$

Also, observe that, by construction, the points  $T_{n+2}, S_n, S_{n+1}$  are collinear.

Therefore, the points  $S_1, S_2, S_3, \dots$  form the reverse Wurzelschnecke.

□

PROPOSITION 3. For  $O$  and the sequence  $\{S_n\}_{n \in \mathbb{N}}$  as depicted in Figure 7,

$$\lim_{n \rightarrow \infty} |\overline{OS_n}| = 1.$$

PROOF. We refer to Figures 7 and 8.

We consider the trapezoid  $OT_nT_{n+1}S_n$ . We observe that

$$|\overline{OT_n}| = \sqrt{n}, \quad |\overline{S_nT_{n+1}}| = \sqrt{n+1}, \quad |\overline{T_nT_{n+1}}| = 1$$

and that

$$\angle T_{n+1}T_nO = \angle T_nT_{n+1}S_n = \frac{\pi}{2}.$$

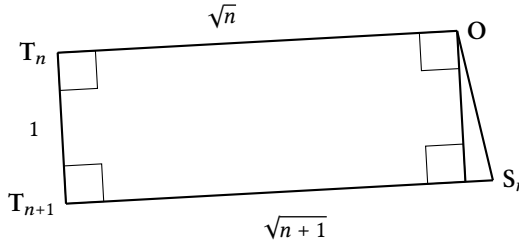


Fig. 8. The trapezoid. The lower line  $T_{n+1}S_n$  is length  $\sqrt{n+1}$  while the upper line  $T_nO$  is length  $\sqrt{n}$ .

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} |\overline{OS_n}|^2 &= \lim_{n \rightarrow \infty} \left( (\sqrt{n+1} - \sqrt{n})^2 + 1 \right) = \lim_{n \rightarrow \infty} \left( \left( \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \right)^2 + 1 \right) \\ &= \lim_{n \rightarrow \infty} \left( \left( \frac{1}{\sqrt{n+1} + \sqrt{n}} \right)^2 + 1 \right) \\ &= 1. \end{aligned}$$

□

Indeed, the approach to 1 is monotonic, from above. By Maple, we have

$$> (\text{sqrt}(n+1) - \text{sqrt}(n))^2 + 1$$

$$\left( \sqrt{n+1} - \sqrt{n} \right)^2 + 1 \tag{15}$$

$$> \text{asympt}(\text{sqrt}((15)), n)$$

$$1 + \frac{1}{8n} - \frac{9}{128n^2} + \frac{49}{1024n^3} - \frac{1173}{32768n^4} + \frac{7431}{262144n^5} + O\left(\frac{1}{n^6}\right) \tag{16}$$

THEOREM 1. Points  $S_1, S_2, S_3, \dots$  approach the circle with the centre at  $O$  and radius 1.

PROOF. The claim follows from Proposition 3.

□

PROPOSITION 4. The angles  $\psi_n \bmod 2\pi$  of the points  $S_n$  on the inner curve of the reverse spiral are asymptotically equidistributed in the interval  $[0, 2\pi)$ .

PROOF. The angles of the spiral of Theodorus, say  $\theta_n = \sum_{k=2}^n \phi_k$  where  $\phi_n = \arctan(1/\sqrt{n})$  are the internal angles of the triangles, are known to be equidistributed by a theorem established by Edmund Hlawka (1916 – 2009), an Austrian mathematician, in 1980 [7, Historical Supplement H]. Consider the isosceles triangle  $OT_{n+1}S_n$  in Figure 8. The angle  $\angle OS_nT_n$  is  $(\pi - \phi_{n+1})/2$ , so the angle  $\angle T_nOS_n$  is asymptotically  $\pi/2 + O(1/\sqrt{n})$  more than the angle  $\theta_n$  of  $T_n$ , as  $n \rightarrow \infty$ . This asymptotically constant shift does not affect the equidistribution: since the  $\theta_n$  are equidistributed, so are the  $\psi_n$ .  $\square$

## 7 Code Implementation

The Maple code used for the creation of Figure 3 did not have to go very far; only 17 nodes were needed. But if one wants to compute *many* triangles (as we did before we discovered that the inner curve approached a circle), a faster code is needed. Indeed, it was written first, and in Python. The code written in Python can calculate and plot the reverse Wurzelschnecke, as well as show how it overlaps with the Spiral of Theodorus. It can be found on Ewan Brinkman's GitHub [here](#).

### Running the Code

All of the required packages are listed in the file `requirements.txt`. Python version 3 should be used. When run, two options will appear in the terminal. Note that whenever the program ask for input, pressing the enter key will select a default option.

To plot data, type 1 and press enter. The program will look for a data file with the name `DATA_FILE` (set in `settings.py`), in the data folder. Options when plotting are shown in Section 7. Figure 9 shows an example plot of the first 100 triangles.

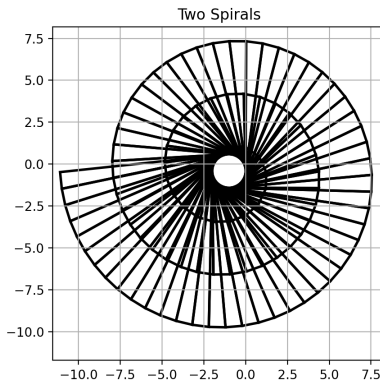


Fig. 9. First 100 triangles of the reverse Wurzelschnecke.

To calculate plotting data, type 2 and press enter. Next, input how many triangles should be calculated. A negative will calculate triangles forever, until the program is interrupted. Finally, enter how often it should save calculated triangles. The default, 1, saves every triangle. A value of 2 will save every other triangle, while entering 1000 will save every thousandth triangle. When saving the calculated data, the program will look for a data file with the name `DATA_FILE` (set in `settings.py`), in the data folder. If no file is found, a new file with the name `DATA_FILE` will be created. Options for calculating triangles are shown in Section 7.

**The Algorithm**

Each triangle is calculated using a previous triangle’s data. Given an  $n$ th triangle, the program will use that triangle’s outside right vertex, rotation and triangle number plus 1 to calculate the next triangle. For example, given the second triangle, the program will use its outside right vertex of  $(\frac{\sqrt{6}}{3}, 1 - \frac{\sqrt{3}}{3})$ , its rotation of  $\arctan \frac{\sqrt{2}}{2}$ , and a triangle number of 3.

First, the new triangle’s inside leg length is calculated by taking the square root of the triangle number. See 7 for using a custom function for calculating triangle side lengths. Next, the new triangle is connected to the previous triangle, without being rotated. Figure 10 shows what the third triangle would look like at this stage.

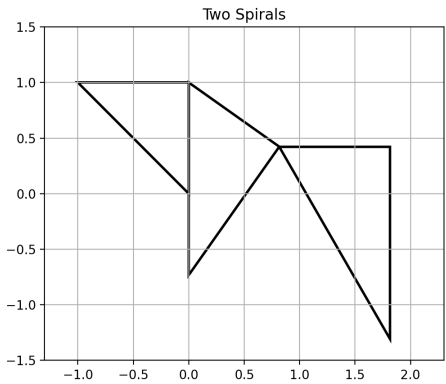


Fig. 10. The third triangle before being rotated.

Next, the outside right vertex and inside vertex of the triangle are rotated clockwise around the vertex connected to the previous triangle. The rotation is the previous triangle’s rotation, plus  $\arctan \frac{\text{outside leg}}{\text{inside leg}}$  of this new triangle. Figure 11 shows what the third triangle would now look like.

This new rotation value will then be passed onto the next triangle, as well as the triangle’s outside right vertex and one more than its triangle number.

Each triangle, an instance of the class `Triangle` from the file `triangles.py`, performs these calculations using the class method `calculate_triangle`.

**Settings**

The constants below are located in the file `settings.py`.

Settings for plotting:

- `SHOW_TRIANGLES`: set to `True` to plot the reverse Wurzelschnecke in full. Set it to `False` to only plot a single vertex of each triangle. The vertex to be plotted can be set using `PLOT_TRIANGLE_POINT` (see below).

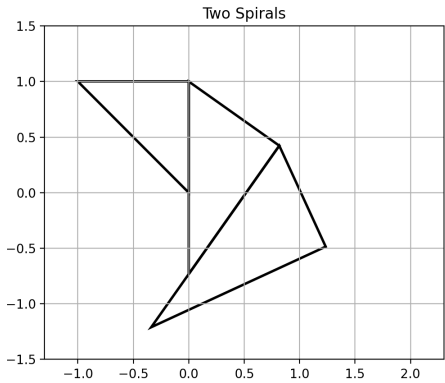


Fig. 11. The third triangle after being rotated.

- `PLOT_TRIANGLE_POINT`: if `SHOW_TRIANGLES` is `False`, this determines which vertex of the triangles is plotted. The options are: "inside" for the inner vertices which form the circle, or either "outside\_left" or "outside\_right" for the other triangles' vertices.
- `ANIMATE_SPIRAL_OF_THEODORUS`: set to `True` to play an animation showing how the Spiral of Theodorus overlaps with the reverse Wurzelschnecke. See more in the section 7.
- `SPIRAL_OF_THEODORUS_AMOUNT`: how many triangles should be used during the animation showing how the Spiral of Theodorus overlaps with the Reverse Wurzelschnecke. Set the parameter `ANIMATE_SPIRAL_OF_THEODORUS` to `True` to play the animation.
- `SHOW_CIRCLE`: set to `True` to plot the circle that the inside vertices of the triangle go around, as well as the circle's center.
- `ANIMATE_PLOT`: if set to `True`, each triangle of the reverse Wurzelschnecke will be plotted with a delay in between each one. The delay can be set with `ANIMATION_INTERVAL`.
- `ANIMATION_INTERVAL`: the amount of milliseconds to pause between each triangle while plotting. This requires `ANIMATE_PLOT` to be `True`.
- `CONNECT_POINTS`: set to `True` to connect each triangle's vertex to the next with a straight line while plotting. `SHOW_TRIANGLES` should be `False` and a vertex should be set with the variable `PLOT_TRIANGLE_POINT`.
- `PLOT_TITLE`: a string used as the title of the plot.
- `COLOUR_PERCENT_DONE`: set to `True` to colour the triangles or triangle vertices based on how many have been plotted before them. For example, if the first 10 triangles are plotted, triangles closer to 1 will be closer to one side of the colour gradient, while triangles closer to 10 will be closer to the other side of the colour gradient.

Settings for calculating triangles:

- `EXACT_VALUES`: set to `True` to use exact values instead of decimal approximations during calculations. Exact values are calculated using `sympy`. Note, using exact values becomes slower as the number of triangles increases. It is recommended to set this to `False`.
- `OUTSIDE_LEG_LENGTH`: the length of the triangles' outside legs. If wanted, this can be changed to a value other than 1.

- `CUSTOM_HYPOTENUSE_FUNCTION`: set to `True` to use a custom function for calculating the hypotenuse of each triangle. The function is called `calculate_hypotenuse` and can be found in the file `utils.py`. For example, instead of taking the square root of the triangle's number plus 1, a cube root could be done instead. The function takes the triangle's number as input (triangle number 1, triangle number 2, etc.), and must return the hypotenuse of the triangle. The triangle's inside leg is then calculated using the hypotenuse. Note, the outside leg length can be set using `OUTSIDE_LEG_LENGTH`.

Settings for the triangle data file:

- `DATA_FILE`: the name of the data file for reading and writing calculated triangles.
- `HEADERS`: the headers of the csv file which stores the calculated triangles.

### Visualizing the Two Spirals Together

In the file `settings.py`, setting the constant `ANIMATE_SPIRAL_OF_THEODORUS` to `True` will play an animation showing how the Spiral of Theodorus (the red lines) can be transformed to overlap with the Reverse Wurzelschnecke (the black triangles). The constant `SPIRAL_OF_THEODORUS_AMOUNT` sets how many triangles should be used for the Spiral of Theodorus. A copy of the animation can also be found on Ewan Brinkman's GitHub [here](#).

During the animation, the reverse Wurzelschnecke does not move. Instead, the Spiral of Theodorus begins by being reflected about the line  $x = 1$ . Next, it is rotated  $\arctan 1$  radians around the point  $(1, 1)$ , before being translated 2 units left.

### Large Calculations

When billions of triangles are calculated with this program, interesting behaviour appears. Figure 12 shows the inside vertices of every one thousandth triangle from the 12.41 billionth triangle to the 12.415 billionth triangle. The vertices are coloured on a gradient from black to red, where black points are closer to the 12.41 billionth triangle and red points are closer to the 12.415 billionth triangle. The shaded red area represents the circle the triangles circle around, with the pink point being that circle's centre.

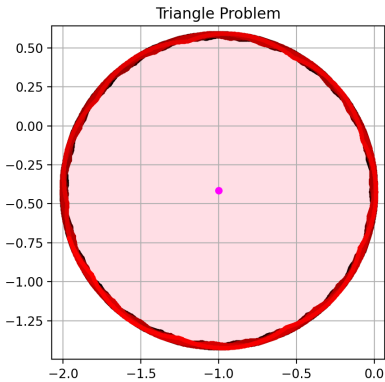


Fig. 12. Inside vertices of triangles near 12.41 billion.

It appears that the inside vertices have begun to wobble slightly. After more triangles are calculated, this became more apparent. Furthermore, the path the inside vertices trace appear to be shifting away from the centre of the circle that the inside vertices were originally going around. Figures 13 to 17 show some examples. In Figure 17, the inside vertices have even moved away from the circle. [Note: one hypothesis is floating point error. We reran the program multiple times to see if the results were reproducible. For triangle number  $10^8$ , the side length is  $10^4$ ; even using single precision, we would expect a side length accurate to  $10^{-3}$  which would look exact to the human eye. If floating-point error is significant here, it must be because of accumulated errors in the summation of the angles. We have not yet investigated this in detail. ]

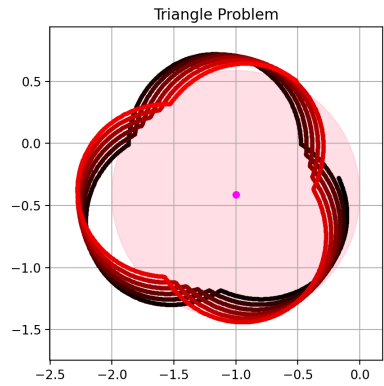


Fig. 13. Inside vertices of every one thousandth triangle from 18.33 billion to 18.335 billion.

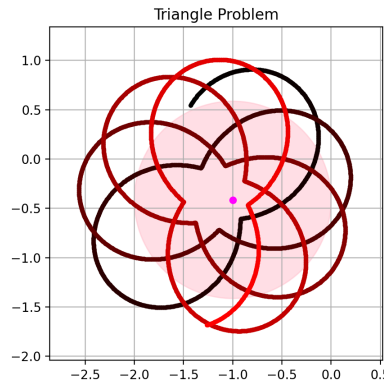


Fig. 14. Inside vertices of every one thousandth triangle from 30.25 billion to 30.255 billion.

## 8 Concluding Remarks

This exploration began with an observation from physical triangles put together in a novel way; then Ewan Brinkman wrote the Python program to compute a large number of triangles, using real

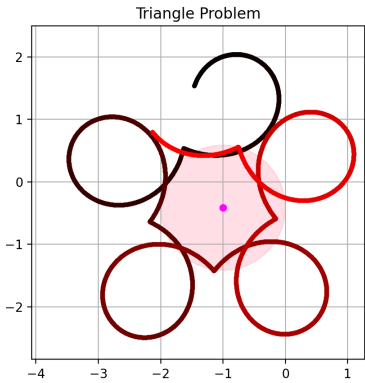


Fig. 15. Inside vertices of every one thousandth triangle from 42.36 billion to 42.365 billion.

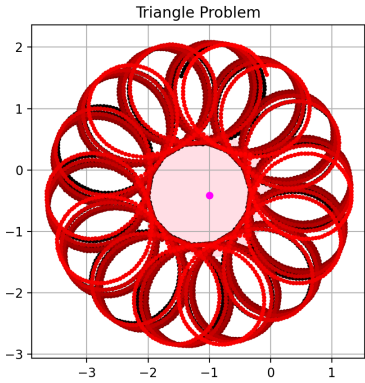


Fig. 16. Inside vertices of every ten thousandth triangle from 42.36 billion to 42.46 billion.

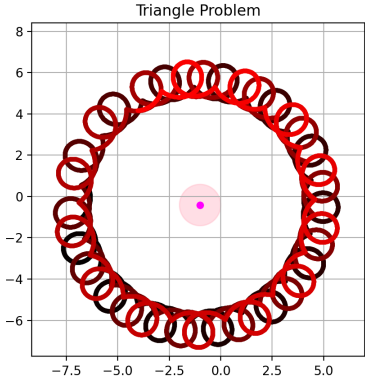


Fig. 17. Inside vertices of every three thousandth triangle from 50 billion to 50.05 billion.



arithmetic and trigonometry to put them together. The first puzzle this computation generated was “what is the true shape of the curve being traced out in the centre?” In this paper, we gave a proof that the shape is a circle, and the points approach it monotonically in radius with radius  $1 + 1/(8n) + O(1/n^2)$ . We have shown that the points are *equidistributed* on that circle, but some of the very large computations carried out with the Python program suggest that the detailed behaviour there might be of interest to numerical analysts; in Figure 17 we see something like “epicycles,” for instance; it’s true that these might be accumulated rounding errors, though. We suspect that *even in spite of the visible regularity* of the deviations that this is the case.

Computation of the location of the  $n$ th point  $T_n$  on the Spiral of Theodorus, even for  $n$  very large, is possible in Maple by use of Levin’s  $u$ -transform to accelerate the summation of angles. Computing the trillionth  $T_n$  takes only milliseconds<sup>4</sup>. Similarly, the location of the  $n$ th point  $S_n$  on the inner curve, approaching the circle, is likewise quickly computable.

After performing these experiments, we connected with the material of Philip J. Davis’ lovely book [7], some of which we summarized in section 2. We then confirmed some of those results by use of Maple in Section 3. Still, there seems lots left to explore. A good place to start would be [7, Supplement B], written by Arie Iserles, a professor of numerical analysis at Cambridge University with extensive interests including dynamical systems; his supplement to Davis’ book generalizes the Theodorus spiral in several ways, including to matrix iterations.

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<sup>4</sup>To 35 significant figures, computed using 50 Digits in Maple (which took only 192 milliseconds), the trillionth point on the spiral of Theodorus is  $T_{10^{12}} = -964121.74674046601468450879463366548 - 265460.46308652576745796389309002521 i$ . The corresponding trillionth inner point, again computed by sequence acceleration in only 28 milliseconds, is  $S_{10^{12}} = 0.26545998102565239710444133249300938 - 0.96412187947069755798057508150144433 i$ .