Maple Transactions

Mathieu and other Special Functions in Symbolic Computation Systems

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This paper discusses some of the philosophical and historical underpinnings of the talk "The Mathieu Functions: Computational and Historical Perspectives" given at the Maple Conference 2022. In particular I discuss the role symbolic computation systems play in curating the mathematical knowledge of the 19th century, not just for special functions but also much of the underlying theory that supports expression of that knowledge.

CCS Concepts: • **Computing methodologies** → **Representation of mathematical objects**; **Representation of mathematical functions**.

Additional Key Words and Phrases: Special functions, 19th Century Mathematics, Computer Algebra, Symbolic Computation, Mathematical Knowledge Management

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1 Introduction

> Stigler's Law of Eponymy states that:
> no scientific discovery is named after its original discoverer.
>
> —Actually lots of people said this before Stigler (and he knew it: it's a joke).

The talk "The Mathieu Functions: Computational and Historical Perspectives" was based in large part on the (quite long!) paper [6], which is available open-access. I don't wish to repeat here the things that were written there, except to emphasize again the contribution of Gertrude Blanch, so I will take a larger perspective. The focus of this present paper will be on the notion of special function itself, with perhaps some use of Mathieu functions as examples. In some sense Mathieu functions are one of the "flowers" of the 19th century, being special functions that are *not* representable in terms of so-called hypergeometric functions[1], and they have some interesting features. They retain their utility in modern engineering applications as well.

I do want to keep this paper reasonably short, however, and so what I will omit will be by far more than I include. I will point out some further reading, some of which is delightful and useful, and some of which is merely useful. Anything useless I include had better be at least delightful!

I am writing for an audience of non-specialists[2]: the topic will be mathematical, certainly, and computational, certainly, but I will try to make sure that all definitions are given and that the presentation is self-contained enough to be read by anyone with an interest. In particular, I

[1] I am going to mention hypergeometric functions three times in the text, and never define them. They are important, but I have to draw the line somewhere. Mathieu functions are not hypergeometric functions, although very many (almost all?) known functions are. The study of hypergeometric functions can thus systematize the study of a lot of special functions, but won't help us here.

[2] who hopefully like formulas. There's going to be a few.

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want teachers of mathematics at all levels to feel welcome reading this paper. There are a *lot* of barriers and gates in modern mathematics: long chains of definitions and theorems with obscure mathematicians' names on them, for instance; crazily complicated and conflicting notation, for another. I'll open as many gates and lower as many barriers as I can. I *will* assume a memory of calculus, however: the reader will need both derivatives and antiderivatives. Not how to *do* them, just what they *are* or how they are written.

Symbolic computation systems have helped with lowering barriers, quite remarkably. I once attended a talk on linear algebra where I was completely lost in the first half of the talk, until the speaker started putting Maple code up. At that point I said (quietly to myself) "Oh, so *that's* all he meant!" I will use Maple syntax freely in this paper, not merely to demonstrate that Maple can do things (indeed I'll show some places where it can't), but in order to make the material intelligible.

2 Why learn all this stuff?

Modern computers can do *direct numerical simulation* of an amazing variety of physical, chemical, biological, and economic phenomena, using as basic elements only the operations of addition, subtraction, multiplication, and division. The modern discipline of Scientific Computing is essentially just that.

Nowadays even Scientific Computing is being challenged by certain Artificial Intelligence techniques (which, actually, just boil down to very large-scale linear algebra; so perhaps this is a subset of Scientific Computing). So maybe we don't need elementary or special functions any more? I remember overhearing three of my numerical analysis professors talking about this forty years ago: one of them expressed the opinion that, given (then) modern techniques for solving differential equations, the topic of "special functions" might be dying.

Well, special functions are not dead yet (ironically, and somewhat morbidly, all three of those professors *are*), and indeed several serious recent advances in numerical analysis and Scientific Computing are based on the old specialized knowledge of such functions. I will give references and further readings at the end, but as one example of this for now, the paper [23] is just ten years old and has gathered close to two hundred citations.

Classic works on *elliptic* functions are among the most checked-out books in engineering libraries [19]. Elliptic functions are astonishingly useful (and that book is one of my favourites). I will be unable to *not* talk about elliptic functions here; I hope that you will take something away with you from this paper, and elliptic functions are maybe the best thing that you could take away, out of all the special functions.

Special functions are also valuable in and of themselves, as pieces of the collective masterpiece that is mathematics, built up over the millennia in response to natural challenges and to human desires to explore. In my view, mathematics is comparable to language and to music as one of the greatest creations of the collective human mind.

Using that last approach as a motive for teaching mathematics only works for idealistic students (yes, there are a lot of idealistic young people); the reason society *pays* for the teaching, however, is that mathematics really is useful. Like elliptic functions!

However mathematics is taught, and for whatever reasons, the body of knowledge that is mathematics must be carefully *curated*. It must be organized, conserved where needed or potentially needed, distributed to those who need or want it, and ruthlessly culled when no longer needed. New works must be carefully considered. The good stuff must not be forgotten!

In this paper we're going to look a little bit at the role of symbolic computation systems in the modern curation of mathematical knowledge. Of course the world of mathematics is nothing like this picture of careful curation: it's much more like a free-for-all. So what we'll go through is more a personal view of mine, and I would welcome discussion by submission of further papers to the Maple Transactions[3].

I have some experience with the subject of special functions. I won't give a lengthy self-citation list, but I have published papers and book chapters on special functions since the early nineties, some with eminent authorities indeed, and similarly in several areas such as approximation theory. I designed a poster for the Lambert W function, available for download at https://www.orcca.on.ca/LambertW. I have also taught engineering calculus for many years, and various upper-year courses including courses on numerical analysis, symbolic computation, and, yes, on special functions. Let's see if I can make the story as interesting as it deserves[4].

3 Elementary functions

> The *elementary functions* of the calculus are not "elementary" in the sense of being simple, but instead they are "elementary" in a similar sense to the elementary particles of physics.
>
> —James H. Davenport, footnote on p. 258 in [24].

The *elementary functions* are polynomials, rational functions, $\ln z$, and $\exp z$ (by which is meant e^z where $e = 2.7182\ldots$ is the base of the natural logarithm), combinations thereof, and that's it.

What about the trig functions? Well, both trig functions and inverse trig functions can be defined in terms of exponentials and logs, if we allow complex numbers.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{1}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} . \tag{2}$$

The hyperbolic functions are also elementary:

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \tag{3}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2} . \tag{4}$$

Anything that **cannot** be expressed as some polynomial or algebraic or rational combination of these is considered **not** to be elementary. Proving that something is not elementary turns out to require some rather advanced mathematics; we will touch only on the very simplest case. But for now let's just explore elementary functions.

As a first nonpolynomial instance, consider the curve named in a curious fashion[5] after Maria Gaetana Agnesi, who is widely considered to be the first woman to be recognized in Europe for her work in Mathematics; this was in the 18th century. Her curve is called "The Witch of Agnesi" and was studied before she did, by Fermat and others, so at least Stigler's law is working in someone's favour for once. The "Witch" has the equation

$$y = \frac{8a^3}{x^2 + 4a^2} . \tag{5}$$

[3]This paper is somewhat anomalous. Normally, the invited speakers are particularly solicited to write papers for the Proceedings of the Maple Conference. But in 2022 I was both Editor-in-Chief and an invited speaker, and everyone (including me) forgot to solicit a paper from me. Luckily, I remembered at the last minute, and so wrote myself a nice invitation email, which I gratefully accepted. Also luckily, I have been thinking rather carefully about Mathieu functions—and elliptic functions!—since the conference, because my co-authors and I are using them for a problem in hemodynamics [7].

[4]If I can't, that's on me.

[5]it seems to be a mistranslation of the Latin *versiera*, but I am not sure I believe that.

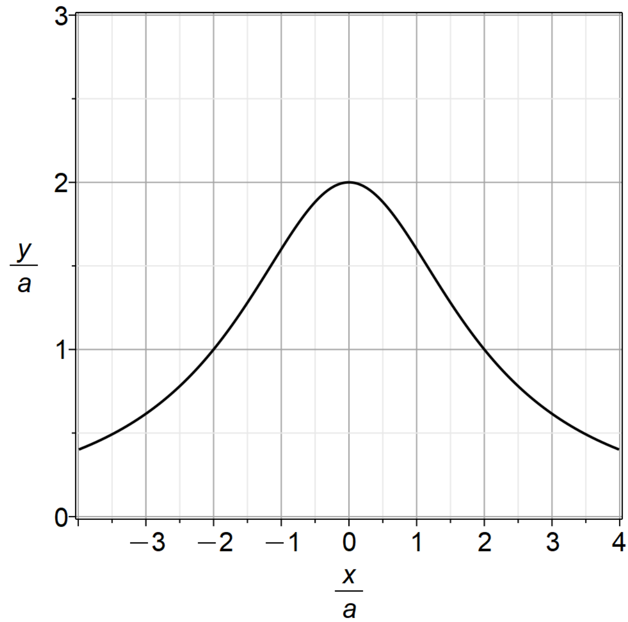


Fig. 1. The curve described by $x^2y = 4a^2(2a - y)$, named after Maria Gaetana Agnesi (1718–1799). Once it has been nondimensionalized, it is a single, simple, rational curve, which is certainly "elementary."

Here a is the radius of a defining circle; for each value of a, one gets a different curve. This can be "nondimensionalized[6]" by dividing both sides by a and rearranging the formula on the right to

$$Y = \frac{8}{X^2 + 4}, \tag{6}$$

where we have conveniently written Y for y/a and X for x/a. Now we can graph Y versus X, label the axes with x/a and y/a, and have all the (nonzero) Witches of Agnesi represented at once.

```
plot( 8/(X^2 + 4), X = -4 .. 4, colour = black,
      view = [-4 .. 4, 0 .. 3], gridlines = true,
      thickness = 2, font = ["Arial", 16],
      labelfont = ["Arial", 16],
      labels = [typeset(x/a), typeset(y/a)],
      axes = boxed );
```

The next step up from polynomial and rational functions are the so-called *algebraic functions*, which involve solving polynomial equations, like $y^2 - x = 0$, which gives the functional inverse of the squaring function. Then there is the problem of solving quadratic equations, cubic equations, or quartic equations, for all of which we have formulae in terms of radicals.

Solving a quintic equation is a more difficult problem, and was only proved *impossible* to solve in general *in terms of radicals* by Ruffini in 1814 and Abel in 1824. But one *can* solve them in terms of *elliptic functions*. And, in a stroke of sleight-of-hand so easy it seems facile to the point of imbecility, they can be solved "in terms of themselves" the way Maple does it, with the **RootOf** function. What

[6]Nondimensionalization is one of the most useful things that I ever learned. I have found it hard to teach, though, and many students really don't like it. I don't know why not.

is the solution of $y^5 + y^4 - 1 = 0$? Maple says,

$$RootOf\left(\_Z^5 + \_Z^4 - 1, index = 1\right) \tag{7}$$

and four more, with index=2, 3, and so on. This is the very definition of "begging the question"! But, if you have means to *compute* the solution to arbitrary accuracy, then don't you *have* a solution? And you can do things with the solution. Call the root α:

```
alias(alpha = RootOf(Z^5 + Z^4 + 1, Z));
simplify(1/(1-alpha));
```

$$\frac{1}{3}\alpha^4 + \frac{2}{3}\alpha^3 + \frac{2}{3}\alpha^2 + \frac{2}{3}\alpha + \frac{2}{3}\,. \tag{8}$$

So this is not begging the question, after all. To the contrary, it's a very powerful idea. The **RootOf** construct gives immediate access to *algebraic* functions, which are on the border of being elementary or not, in Maple.

There's a grey area for elementary functions, here: we allow functional inverses of all the functions listed above to be elementary, but not inverses of *combinations* of these, to be elementary. Instead they get a special name, to wit: *implicitly elementary*. For example, the Lambert W function [10] is not elementary, but it *is* implicitly elementary, being a distinguished solution of the equation

$$we^w - z = 0\,. \tag{9}$$

The solution of this equation is $w = W(z)$. Maple writes instead `LambertW(z)`, because single-letter names are important user "real estate" in the name space. If all solutions are wanted, we say $w = W_k(z)$, while Maple says `solve(w*exp(w)-z,w, allsolutions=true)` is `LambertW(_Z2,z)`. Maple has introduced a symbol here, which we can interrogate: `about(_Z2);` yields

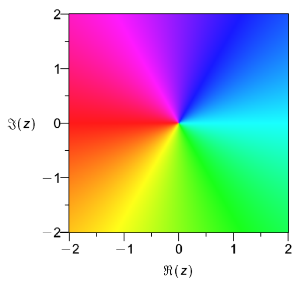
```
Originally _Z2, renamed _Z2~:
  is assumed to be: integer
```

It's more human to write $W_k(z)$ and use the old FORTRAN convention (any variable starting with a letter I through N is an INteger) to indicate to the reader that k is an integer. $W(z)$ without a subscript is shorthand for the principal branch, $W_0(z)$.

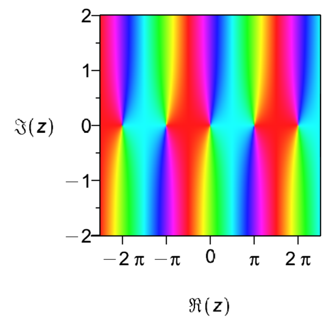
Written the other way around, $z = w \exp w$ is an elementary function of w. This always happens for implicitly elementary functions. So, technically, $W(z)$ is a special function. Let's return to elementary functions, but turn it up a notch and look at complex values.

We plot $\sin z$ in Figure 2b, in a way that might be new to you, by plotting its complex argument (also called the phase) by a particular colour scheme. One can learn to identify important features, such as zeros and singularities, simply by looking for where all the colours meet. This is done in Maple with the command

```
plots[complexplot3d]( sin(z),
  z = -2.5*Pi - 2*I .. 2.5*Pi + 2*I,
  style = surfacecontour,
  contours = [0], orientation = [-90, 0],
  lightmodel = none, font = ["Arial", 24],
  labelfont = ["Arial", 24],
  grid = [200, 200],
  tickmarks = [spacing(Pi), default, default]
  );
```



(a) Phase plot of z, giving the colorwheel



(b) Phase plot of $\sin z$

Fig. 2. One new technological development is that we can do *phase plots* of complex-valued functions. On the left, we have $w = z = x + iy = re^{i\theta}$ pictured. Each ray with constant angle θ has the same colour. If we plot $\sin z$ using the same scheme, we can pick out places where all the colours meet. Those are the zeros of $\sin z$, namely multiples of π. This new visualization technique is remarkably powerful.

The fact that one can plot four-dimensional functions by using colour [29] seems to have caught most mathematicians by surprise. In fact, we're using only three dimensions (two in directional space and one in colour space) and we can plot complex functions using just three dimensions in several ways, but this new colour technique is rather welcome. But complex numbers are full of surprises.

It is a surprise to most students to find that (for example) one can solve the equation $y = \sin z$ for z in terms of y using logarithms. It's a bit easier for them to solve the related equation $y = \sinh x$ for x in terms of y because there are no obvious complex numbers present. One uses the definition above: $2y = \exp(x) - \exp(-x) = E - 1/E$ if $E = \exp(x)$; then this quadratic equation can be solved for E: $E^2 - 2yE - 1 = 0$ so

$$e^x = E = y \pm \sqrt{y^2 + 1} \tag{10}$$

and then for real y the choice of sign is determined by the fact that the exponential is always positive, and $\sqrt{y^2 + 1} > |y|$. So we get $x = \ln(y + \sqrt{y^2 + 1})$ as an explicit expression for the functional inverse of $\sinh x$. There's an analogous one for $\arcsin(x)$. Maple knows them both, of course.

```
convert(arcsin(z), ln);
```

$$-\mathrm{I}\ln\left(\mathrm{I}z + \sqrt{-z^2 + 1}\right) \tag{11}$$

Maple uses I for the square root of −1. I have never liked this notation, but one gets used to it.

As an aside, I strongly dislike the notation $\sin^{-1} x$ for the functional inverse of the sine function. So many students are confused by it, and slip into thinking that this is the same thing as the reciprocal $1/\sin x$. My colleague David Jeffrey has invented a much superior and more general notation, which I hope will catch on: if $y = f(x)$ then $x = \operatorname{inv} f(y)$. There's space for a subscript to indicate branches, if $f(x)$ takes on the same value y for differing values of x. The word "arc" works only for trig functions. It's even a stretch to say "arcsinh" and the hyperbolic functions are really just trig functions of a complex argument.

Writing inverse functions in terms of branches gets tricky, and one has to be careful about what are called "branch cuts." I've written at length about that elsewhere, so I won't belabour the point here. David Jeffrey and I have also written about Riemann surfaces, which is a different way to deal with multivalued functions. That method is very popular with pure mathematicians, but doesn't really work computationally: one wants the computer to return one x for a given y. David has written (but not so far published, sorry) about this with the intriguing suggestion of using something he calls "charisma" and that might work. I hope that catches on, too.

3.1 Historical remarks

The trig functions, more-or-less contemporaneously to the solution of a quadratic equation, go back to Babylonian times and ancient astronomy and may date as far back as 3700BCE. People wanted to calculate them in order to use them, and so the history of these functions is inevitably bound up in the history of how to compute them.

A quick aside on solving quadratics, cubics, and quartics: the history of solving the cubic on Wikipedia is pretty interesting reading, and there is a more complicated formula for solving quartics.

One way to compute trig functions is by their Taylor series (more about which in a moment) which were known already by Mahadva of Sangamagrama (c. 1340–c. 1425). This seems quite an egregious case of Stigler's Law, although one might argue that this pre-dates the notion of derivative, even. The formula for Taylor series in general is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \tag{12}$$

and beginning students really have trouble keeping all those symbols in their precise places. Part of the issue is that they don't see $f(a)$ or $f'(a)$ as constants (how could they be? There's a function in there! And if x is a variable, why isn't a?). Practice, practice. Another really big issue is that they do *not* see the right-hand side as being the slightest bit simpler than the left. After all, their calculator usually has a button for $f(x)$ and why would you want to use a more complicated formula? One can sometimes say (which is a "lie to children") that the calculator uses Taylor polynomials to evaluate the function, and that they are learning (something about) how it works internally.

The "Taylor" in the name "Taylor series" was *Brook* Taylor, who published on them in 1715. And in any case Newton certainly knew what they were previously in the 1600s, as well. Then there is Colin Maclaurin who later used them, specialized to expansion about zero, so they are then called Maclaurin series, which also hardly seems fair. Stigler's law is really ticking up its score.

```
series( sin(x), x=Pi/4, 4 );
```

yields

$$\frac{\sqrt{2}}{2} + \frac{1}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right) - \frac{1}{4}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{12}\sqrt{2}\left(x - \frac{\pi}{4}\right)^3 + \mathrm{O}\left(\left(x - \frac{\pi}{4}\right)^4\right). \tag{13}$$

This is first few terms of the Taylor series for $\sin(x)$ about the expansion point $x = \pi/4$, as computed by **series**, one of the oldest Maple commands.

It is relatively new that one can ask for *infinite* power series [26]:

```
convert(sin(x), FormalPowerSeries, x = Pi/4);
```

yields

$$\left(\sum_{n=0}^{\infty} \frac{\sqrt{2}\,(-1)^n \left(x - \frac{\pi}{4}\right)^{2n}}{2\,(2n)\,!}\right) + \left(\sum_{n=0}^{\infty} \frac{\sqrt{2}\,(-1)^n \left(x - \frac{\pi}{4}\right)^{2n+1}}{2\,(2n+1)\,!}\right). \tag{14}$$

I have always thought of this kind of thing as "putting the skills of an Euler, or of a Ramanujan, at the fingertips of the students (or researchers)". This is an example of curation of mathematical knowledge, because the knowledge is put in a form where it can be retrieved or applied to the situation at hand. You can see that quite a bit of knowledge of the sin function has been included in the system: evaluation for complex z, the knowledge of exact special values, and how to compute series.

Let's return to the historical discussion. The exponential function is quite a bit older than trig functions, if one thinks of repeated doubling as an exponential function. The base e was not invented until Napier's time in the 1600s, though. According to the story at https://mathshistory.st-andrews.ac.uk/HistTopics/e/ the importance and the name e only gradually dawned on the mathematicians of that century.

The history of the computation of logarithm is astonishing, and Brigg's monumental work of hand computation of tens of thousands of entries to 14 decimal places just boggles my mind. That logarithms are available on dollar-store calculators nowadays is similarly boggling, but in a different way.

The Bernoullis, and Euler, were responsible for the creation of a great many special functions, especially the Γ function of Euler. Stirling and DeMoivre contributed to series methods for approximating this function, by scandalous use of divergent series.

It is a very interesting exercise to prove mathematically that the logarithm function (in any base) is not a polynomial or a rational function. The methods used for this, now pretty standard in differential algebra, were instrumental in Liouville's work in the 1800s in proving that many functions are not elementary functions. Above, we said that a function was not elementary if one could not write it in terms of that short list of functions. Well, how would you go about proving that? It's not so easy to prove that you *can't* do something! One of the first examples Liouville proved was that

$$y = \int_{t=0}^{x} e^{-t^2}\, dt \tag{15}$$

is not elementary. This integral is proportional to what is called the *error function*, which is quite useful in statistics, in spite of not being "elementary". Liouville's work led in the 20th century to work by Robert Risch, Arthur Norman, Barry Trager and others to the antidifferentiation algorithms implemented in many computer algebra systems[7] including Maple. This is a thread I will pull on later, but for now let's return to the historical notion of *computation* of an elementary function.

Methods for the computation of trig functions and of logarithm were worked out over significant time spans, but Taylor series methods were of very substantial help. All through the 19th century and even in the early 20th century Taylor series methods remained a "serious" contender for the best way to compute a function. Indeed, special functions were regarded as "solved" if one had a convergent Taylor series for them. Gauss heavily used series for the so-called hypergeometric functions in the early 19th century.

But for some functions, better methods were being developed, including Fourier series—which expanded functions not in terms of polynomials, but in terms of the "elementary" trig functions, which themselves were computed by Taylor series methods and refinements, but now considered as *answers* rather than as questions. This shift, from question to answer, occurs repeatedly in the history of mathematics, and is worthy of note. Once something can be computed "sufficiently easily," it can become an answer, I think; although that's not the whole story. But, methods of computation play a significant role in defining what the elementary objects of mathematics are.

The first really interesting new method of computation, though, was the Arithmetic-Geometric Mean (AGM) iteration invented by Gauss in the early 19th century. We will see this in action

[7]This is the first time I have used the term "computer algebra"; previously I used the term "symbolic computation." There is a reason for this, that I will expand on later.

when we look at elliptic functions. The iteration is superficially similar to Archimedes' method of computing π by inscribed polygons and circumscribed polygons (itself a lot of fun to think about and I wish I could take the time, here), but is subtly different. "Timing" turns out to be everything, and the AGM iteration converges astonishingly quickly. This method allows one to compute the so-called *Jacobian elliptic functions*, also invented in the 19th century, (they get their own section down below) extraordinarily rapidly. From that, one gets the fastest known way to compute ordinary trig functions!

Mathieu's 1868 paper comes right in the middle of all of this explosion of the theory of functions. What he invented in order to understand the "nodal lines" of a vibrating drum with a fixed elliptical rim, are now called Mathieu functions. We will spend some time with them, I promise.

However, my favourite functions are the elliptic functions, and I will have to get them off my chest, first. As another random elliptic function fact, Sofya Kovalevskaya wrote a paper on elliptic integrals in 1874 which she presented to the University of Göttingen as part of her doctoral dissertation; she was the first woman awarded a PhD in mathematics in Europe.

At about the same time, Pafnuty Chebyshev, Charles Hermite, Henri Padé and others were refining the approximation methods of Newton, Cauchy, and others, to start the body of work known as Approximation Theory today. The results of this body of work are in heavy use, behind the scenes, in symbolic computation systems, to support both elementary and special functions.

4 Symbolic computational support for functions

In a modern symbolic computation system, as we have seen already, one wants to be able to manipulate expressions that define functions to make new expressions; one wants to compute series; one wants to differentiate these objects; to integrate them (in finite terms as elementary functions, or to prove that they are not elementary); to evaluate them at a given point or at many given points; and to plot them. One might wish to prove that the functions being considered have certain properties (such as always being positive, or being convex). A modern symbolic computation system has to anticipate all the needs of all possible users.

Of course the existing systems such as Maple are not yet perfect, and there are some hidden "impossibilities" in there—simplification is sometimes impossible, provably so—but even given that, the expectations on the side of a modern user are quite high. But the systems are getting more usable and powerful all the time.

4.1 Historical remarks

Perhaps the history of symbolic computation systems begins with the work of Jean Sammet (1928–2017), who was the first Chair of the Special Interest Group on Symbolic and Algebraic Manipulation. She was the principal designer of FORMAC (FORmula MAnipulation Compiler). This was a preprocessor for FORTRAN which allowed what we call symbolic computation today. One important piece of history is the famous 1972 HAKMEM by Beeler, Gosper, and Schroeppel which contains an amazing number of special function facts. It can be found at https://w3.pppl.gov/~hammett/work/2009/AIM-239-ocr.pdf. Then there was ALTRAN (which was the language Keith Geddes first used in order to teach computer algebra to Waterloo grad students, including me, before we switched to the brand-new system Maple in the middle of the term; this was 1981 I believe). Of course there was REDUCE and Macsyma in there as well.

Mathematica arrived on the scene in 1988. Nowadays there are several systems, including SymPy and SAGE Math. Some systems have come and gone, such as Theorist, which was quite interesting in its attempt to do automatic case analysis, and Derive which survives now only in ghost form on some kinds of calculators, I believe.

All of these systems—all—had some kind of implementation of elementary functions. All of them could evaluate elementary functions; most could differentiate them, and some could integrate them (or could prove that the resulting integrals would not be elementary). The implementation of special functions varied widely. Macsyma had quite a lot, as I recall. The Hewlett-Packard calculator series could do complex arithmetic, knew a lot of special functions including the Gamma function, could integrate and differentiate and compute Taylor series; I miss that calculator. Although there are nice emulators available free these days.

4.2 Real arithmetic

The "real number system" has an unfortunate name. It should have been called the "continuum number system" or some such, because the word "real" has the decidedly hard-to-ignore connotation of "actual." If something is not a "continuum number" then it would be quite easy to believe that it might be some other kind of number. If it's not a "real number" then there's this connotation that it's like Pinocchio, who is not a "real boy." Indeed to the contrary one can wonder if "real numbers" are actually objects that really exist; some of them do, certainly, but others? It gets iffy, out on the borders. But there's no changing the name now. Almost every symbolic computation system has some kind of implementation of something that is intended to act like "real numbers," even though this is technically impossible. We only ever have finite representations and we can only work with numbers that can be constructed or represented somehow; and we don't have enough alphabet to do so. Nonetheless, we do what we can and everyone pretends it's enough. When the computer fails to come back from a computation because of one of the impossibility problems (recognizing zero) we shrug. When floating-point approximate numbers give us grief, we do our best to compensate.

4.3 Complex arithmetic

For me, complex floating-point arithmetic is the natural setting for elementary and special functions. Complex numbers are perfectly "real" meaning actual. Here is the square root of minus one: $(0, 1)$. Complex numbers are just pairs of "continuum" numbers. And functions of complex numbers are some of the simplest functions.

There are lots of discrete functions of course, that is, functions of integers taking integer values, and they are very useful; but for me $\mathbb{C}$ is where it's at. And with computers to take the tedium out, there's no reason to fear the complex plane. So I expect all symbolic computation system implementations of all elementary functions and special functions to work, and work well, over $\mathbb{C}$. Unfortunately, this is not at all the case (even for Maple). Most users just want "real" numbers, so the complex numbers get second billing; and moreover complex-valued functions *are* harder to implement, so the developers have to work harder. But when we get to the Mathieu functions, we will certainly want complex values.

4.4 Polynomials and Rational Functions as Workhorses

Once one has a solid foundation of complex floating-point numbers, one can quickly implement *polynomials* and *rational functions* on top of them. And then one can implement the *square root function*[8], and the *absolute value* function (this one is ridiculously hard: overflow is a real problem for it).

There is a *huge* body of theory of how best to allocate the effort amongst these elementary operations so as to compute the value of the function you want. Chebyshev approximation is widely

[8]This is an example of a function that nearly everyone would consider to be elementary. Indeed, the Greeks could "evaluate" this function by straightedge-and-compass constructions; the cube root cannot be so done.

considered to be the *best*; everyone wants the best. I used a bit of jargon there: Chebyshev approximation means approximating by simpler functions so that the *maximum error* in the approximation is *minimized*. There is also expansion in what are known as *Chebyshev polynomials* (which are maybe my runner-up favourite set of functions), which provide "near" best approximation.

For the Mathieu function code that I implemented and described in my talk, I chose something else, namely what Erik Postma and I called a "blend." A blend is a combination of two Taylor polynomials expanded at either end of an interval. It was invented by Charles Hermite, and its proper name is "two point Hermite interpolational polynomial." Here is the formula: the "grade[9]" $m + n + 1$ polynomial

$$H_{m,n}(s) = \sum_{j=0}^{m} \left[\sum_{k=0}^{m-j} \binom{n+k}{k} s^{k+j} \, (1-s)^{n+1} \right] p_j + \sum_{j=0}^{n} \left[\sum_{k=0}^{n-j} \binom{m+k}{k} s^{m+1} \, (1-s)^{k+j} \right] (-1)^j \, q_j \tag{16}$$

has a Taylor series matching the given $m + 1$ values $p_j = f^{(j)}(0)/j!$ at $s = 0$ and another Taylor series matching the given $n + 1$ values $q_j = f^{(j)}(1)/j!$ at $s = 1$. Putting this in symbolic terms and using a superscript (j) to mean the jth derivative with respect to s gives

$$\frac{H_{m,n}^{(j)}(0)}{j!} = p_j \qquad \text{and} \qquad \frac{H_{m,n}^{(j)}(1)}{j!} = q_j$$

for $0 \le j \le m$ on the left and for $0 \le j \le n$ on the right. These turn out to have decent numerical properties: they can be rapidly (enough) evaluated, and differentiated, and integrated (this was a very nice bonus), and combined with each other. They can be connected pairwise on Taylor polynomials known at the knots of a polygonal path in the complex plane (in which case I call it a "blendstring"—it's really a special kind of piecewise polynomial). The real reason I chose to do this is that if you have a nice differential equation to solve, then you can find Taylor polynomials easily, pretty much everywhere you land. So it is easy to build up a precise, accurate, and convenient representation for the solution of the differential equation. It works well for the Mathieu equation.

Earlier I complained that Taylor series were hard to teach to students, and that they did not like the complicated formula. One benefit of teaching them about blends might be that they would appreciate the simplicity of Taylor polynomials more! This formula involves *binomial coefficients* instead of just factorials, like Taylor series. And it has two Taylor polynomials hidden in it (one at one end, and one at the other). So, yes, these are more complicated. But they can be, when the degree at each end is just 10, a million times more accurate than Taylor polynomials. So maybe they are worth the pain.

Computer algebra systems don't generally use blends, but choose from other, classical, methods. They might choose piecewise Chebyshev polynomial approximations; or rational approximations. Most of Maple's approximations of special functions are all ad-hoc and special-purpose, although there is a significant interest at Maplesoft in uniformizing the relevant methods. But I say, if you have a function that has special properties, you should maybe take advantage of them.

5 Special Functions

Here's a vague definition for you: *A special function is a useful function that is not elementary.* Clearly there's some room for argument! Bessel functions (invented by Daniel Bernoulli (1700–1782) before

[9]The word "grade" means "degree at most." A polynomial of grade 5 might be degree 0, or 1, or 2, or 3, or 4, or even 5, but not 6 or higher.

the astronomer Bessel (1784–1846) worked on them, go Stigler) are very frequently used, and no-one would argue about them. Similarly with the error function, or the Γ function. But there are "special" functions, even some with famous names attached, that are nothing but curiosities today. For instance, the "versine" function, once useful in navigation, is unlikely to be encountered by anyone today[10]. Let's give some examples.

5.1 The Lambert W function

David Jeffrey and I are in train of writing a book on the Lambert W function. We've been thinking about it for twenty years or more. Since my retirement from Western, I explicitly took it off my list of things to do; the draft is in David's hands. However, that project is the first one "under the water" if I finish my other projects. It's extremely tempting to take it out of the water, and even writing this little subsection on W makes me want to revive the project. Well, let's see.

The Lambert W function is well-known nowadays because it was implemented in Maple from the early days, and because its solution for the tower-of-exponentials problem[11] was listed on the Frequently Asked Question list of the USENET discussion group `sci.math`. Gaston Gonnet used it in one of his papers in the early 1980s, and made a nice connection to the Gamma function as well (which I rediscovered for myself in 2016 when I started writing a paper on the Gamma function with Jon Borwein). So Gaston implemented it in Maple, and the most frequent way people encounter W for the first time is as the result of a call to the solve command.

```
solve( y*exp(y) - z, y );
```

yields `LambertW(z)`. The first question the surprised user asks is "what in the world is Lambert W?" and the help system in Maple is ready to answer that.

I have already mentioned the definition, and the free poster for download (the design of which was funded by Maplesoft and by ETH, courtesy Gaston). So perhaps I will mention now my favourite facts about W.

First is my favourite integral, which Bill Gosper said converged "slambangularly!"

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-v\cot v)^2+v^2}{z+v\csc v\,\mathrm{e}^{-v\cot v}}\,dv\,. \tag{17}$$

This is equal to $W(z)/z$ for all $z > -\exp(-1)$, and the midpoint rule or the trapezoidal rule work *spectrally*[12] well to evaluate the integral. This provides a nice means to compute $W(z)$, although there are better ways yet. In some sense this is analogous to

$$J_0(x)=\int_0^{\pi}\cos(x\sin\theta)\,d\theta \tag{18}$$

which allows one to compute the zeroth order Bessel function $J_0(z)$ if one can numerically evaluate the integral on the right hand side.

Second is that the denominator of that integral presents an interesting nonlinear equation to solve for v once z is given. After all, one would not want to integrate over a singularity! It turns

[10]To the reader who has a special regard for versine and its more useful cousins such as haversine, my apologies. I like them too. But, let's face it. They are museum pieces nowadays. "How to tell you I am not a navigator, without saying that I am not a navigator," maybe, but. . .

[11]If $x^{x^{x^{\cdots}}} = 2$, what is x? Now replace 2 with 4. Oops? What is this tower, in general? See the Lambert W poster linked earlier.

[12]This is a technical term which means that the error diminishes *exponentially* with the amount of effort expended. Put in a little more effort, you get a *lot* less error! The word exponentially has entered the public vocabulary in a sloppy way, but we all got a sharp reminder of what it actually means with the COVID-19 pandemic.

out that all solutions to that equation are found in the following:

$$v_{k,j} = \frac{W_k(z) - W_j(z)}{2i} \,. \tag{19}$$

That is, the solutions of this frighteningly complicated nonlinear equation can all be written as *differences* of various branches of Lambert W.

Solving nonlinear equations is very hard in general; this is one of the most complicated equations that I know that can actually be solved explicitly in terms of "known" (special!) functions.

I leave those two facts for you to ruminate on at your pleasure. You might try computing a few values of branch differences of Lambert W and verifying that those do provide solutions to the equation. For instance,

```
Digits := 30;
eq := z + v*csc(v)*exp(-v*cot(v));
eval(eq, v = (LambertW(2, z) - LambertW(5, z))/(2*I));
eval( %, z=0.12345 );
```

yields a real and imaginary part on the order of 10^{-28}.

You might also try proving that all solutions are of this form. It's not absurdly hard, once you know that it's possible, and it gives practice with Lambert W. It should be within reach for high school students who know about the exponential definitions of the trig functions. He says confidently.

5.2 The Gamma function

The grandparent of all special functions is Euler's Γ function, which *interpolates* the factorial function. The factorial function, of course, is written $n!$ and means $1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$; we have already used it above in the formula for Taylor series. The question Euler wondered about was what could it mean to take $(1/2)!$ or the factorial of any other non-integer. He solved the problem completely.

Euler's definition was as an infinite product, but he also came up with the following integral, valid for $x > 0$:

$$x! := \int_{t=0}^{1} \ln^x\left(\frac{1}{t}\right)\, dt \,. \tag{20}$$

Stirling came up with a lovely (but divergent) asymptotic series for the factorial, which begins as

$$\ln(x!) = (x+\frac{1}{2})\ln(x+\frac{1}{2}) - (x+\frac{1}{2}) + \ln\sqrt{2\pi} + O\left(\frac{1}{x+\frac{1}{2}}\right) \tag{21}$$

Since $\Gamma(x) = (x-1)!$ we have the similar formula

$$\ln\Gamma(x) = (x-\frac{1}{2})\ln(x-\frac{1}{2}) - (x-\frac{1}{2}) + \ln\sqrt{2\pi} + O\left(\frac{1}{x-\frac{1}{2}}\right) \,. \tag{22}$$

This is astonishingly accurate for large enough x, even though divergent, and moreover may be "reverted" to get an approximation for the functional inverse of the Γ function (or, equivalently, of the factorial function). In an entirely unexpected[13] way, the Lambert W function makes another

[13]Well, maybe not *that* unexpected.

appearance: to solve $x = \Gamma(y)$, put $v = x/\sqrt{2\pi}$ and $u = y - 1/2$. The solution process using Lambert W is then clearer (try it!) and the end result is

$$y \approx \frac{1}{2} + \frac{\ln(x/\sqrt{2\pi})}{W\left(\frac{1}{e}\ln(x/\sqrt{2\pi})\right)} . \tag{23}$$

Let's try that out and try to find a y so that $\Gamma(y) = 100$.

```
1/2 + ln(100/sqrt(2*Pi))/LambertW(ln(100/sqrt(2*Pi))*exp(-1));
evalf(%);
```

yields 5.888, and $\Gamma(5.888) = 99.24$. Not bad!

One of the benefits of extending one's vocabulary of functions is that you have greater power of expression. You can say more things, with greater concision. Sometimes you can say things that you did not even know were expressible, before you learned the vocabulary.

5.3 Elliptic functions

One of the integrals that students have a hard time with in calculus is the following:

$$I(x) = \int_{t=0}^{x} \frac{1}{\sqrt{1-t^2}}\, dt . \tag{24}$$

Maple says, uh, that's inv $\sin(x)$ (actually it uses the notation **arcsin**(x), of course). Of course this is right. That means $I(x) = y$ where $\sin y = x$. So this integral gives the functional inverse of something interesting.

Now let's look at a mildly more complicated integral, namely

$$A(x) = \int_{t=0}^{x} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\, dt . \tag{25}$$

Here $0 \le k \le 1$ is a parameter, called the elliptic modulus. This integral (studied by the great Legendre, and by Abel whom the Abel prize is named for) is *not an elementary function*. It was Liouville who first *proved* in 1835 that this function cannot be expressed in terms of elementary functions [28, p. 126]. It is here that the distinction between "computer algebra systems" and "symbolic computation systems" starts to bite, with Liouville's theory, which began as analysis but was later developed by algebraists such as Ritt into the field of differential algebra that we have today. A modern computer algebra system can *prove* that a given integral is not elementary, by using the Risch algorithm or its refinements. But this proof is essentially algebraic, and ignores some rather important analytical things in the theory of special functions, such as branch cuts. But we will not dwell on this divide, and instead return to the nonelementary integral for $A(x)$ above.

The integral $A(x)$ *can*, however, be expressed as[14]

$$\operatorname{inv}\operatorname{sn}(x,k) \tag{26}$$

the functional inverse of the Jacobian elliptic function $\operatorname{sn}(x,k)$. The integral is therefore what is termed an "elliptic integral." There are three Jacobian elliptic functions, sn, cn, and dn. They correspond to sine, cosine, and the very simple function which is identically 1. In fact, if we define one more function, the Jacobian amplitude function $\operatorname{am}(u,k)$, then

$$\operatorname{sn}(u,k) = \sin(\operatorname{am}(u,k)) \tag{27}$$

$$\operatorname{cn}(u,k) = \cos(\operatorname{am}(u,k)) \tag{28}$$

[14]Maple does not do it this way, however. It gives instead **EllipticF**(x,k), which is much the same thing.

and

$$\frac{d}{du}\mathrm{am}(u,k) = \mathrm{dn}(u,k) \tag{29}$$

```
K := EllipticK(1/sqrt(2));
plot([JacobiSN(u, 1/sqrt(2)), JacobiCN(u, 1/sqrt(2)), JacobiDN(u, 1/sqrt(2))],
      colour = ["Executive␣Blue", "Executive␣Red", black],
      thickness = [1, 1, 1], view = [-4*K .. 4*K, -2 .. 2],
      axes = boxed, font = ["Arial", 16],
      labelfont = ["Arial", 16],
      labels = [u, y], gridlines = true );
```

produces the plot in Figure 3. These are modestly more complicated than trigonometric functions, but *once you get used to them* they are almost indispensable. You do, however, have to get used to them. Maybe that's the real definition of a special function: a non-elementary function that *you are used to.*

Elliptic functions are also incredibly fast to compute to high precision. The key is the AGM iteration: start with two numbers, call them $a_0 > 0$ and $b_0 > 0$. Then compute, for $n = 1, 2$, and so on until $a_n = b_n$ to your desired precision,

$$\begin{aligned} a_n &= \frac{1}{2}\left(a_{n-1} + b_{n-1}\right) \\ b_n &= \sqrt{a_{n-1}b_{n-1}}\,. \end{aligned} \tag{30}$$

This works for complex numbers, too, but let's "keep it real" for the example. Choosing two numbers more or less at random just to show the iteration, take $a_0 = 1$ and, say, $b_0 = 10$. Then convergence in double precision happens in five iterations.

> $Digits := 15;$

$$Digits := 15 \tag{31}$$

> $AGM := (a, b) \rightarrow (\frac{(a+b)}{2}, \mathrm{sqrt}(a \cdot b));$

$$AGM := (a, b) \mapsto \left(\frac{a}{2} + \frac{b}{2}, \sqrt{b \cdot a}\right) \tag{32}$$

> $AGM(1.0, 10.0\,)$

$$5.50000000000000, 3.16227766016838 \tag{33}$$

> $AGM(\%);$

$$4.33113883008419, 4.17043488510804 \tag{34}$$

> $AGM(\%);$

$$4.25078685759612, 4.25002734923307 \tag{35}$$

> $AGM(\%);$

$$4.25040710341460, 4.25040708644996 \tag{36}$$

> $AGM(\%);$

$$4.25040709493228, 4.25040709493228 \tag{37}$$

I want to go on to tell you about *Landen transformations* which are just a different way to organize that iteration, and how Gauss used a very non-intuitive change of variables on an elliptic integral to prove that what the AGM iteration converges to is expressible as an elliptic integral, but this is already too much detail. All I want you to retain from this sketch is that (a) elliptic functions

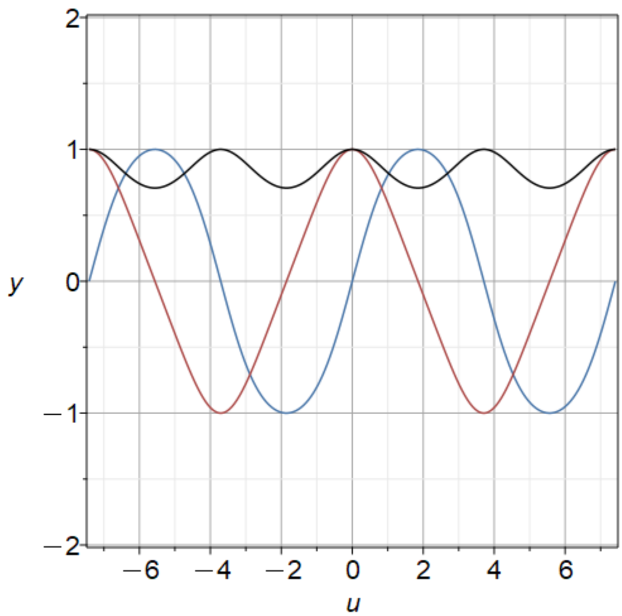


Fig. 3. The Jacobian elliptic functions $\mathrm{sn}(u, k)$ (in blue), $\mathrm{cn}(u, k)$ (in red), and $\mathrm{dn}(u, k)$ (in black) for $k = 1/\sqrt{2}$ and two periods. One sees that sn is like sine (it *is* a sine, just with different argument), cn is like cosine (likewise), and dn is like 1.

are like trig functions and nearly as useful once you get used to them, and (b) they can be computed quickly.

Therefore, they should be thought of as *answers* and not as *questions*. It is this thought that truly answers the question "what is a special function."

5.4 The Mathieu functions

Sixteen pages in, and I am finally getting to the Mathieu functions. Good gravy. Well, I hope that you are still with me. We'll start by reminding you about the trig functions and their connection to a differential equation. They satisfy

$$\frac{\ddot{y}(\theta)}{y(\theta)} = -n^2 \tag{38}$$

where n is an integer. Wait, what?

If we rearrange that we get $\ddot{y}(\theta) + n^2 y(\theta) = 0$ which looks more familiar (where Newton's dot notation is being used here to mean differentiation with respect to θ, and two dots means do it twice), and the solutions are $y(\theta) = A\cos n\theta + B\sin n\theta$, which are periodic with period 2π. This particular way of approaching the trig functions comes from solving the partial differential equation (PDE) known as Laplace's equation on a disk by the technique known as "separation of variables[15]", leading naturally to the *eigenvalues* $-n^2$ which have to be the negatives of squares of integers if the solutions $y(\theta)$ are to be periodic with period 2π. This is a topic of one's second differential equations course, normally; or it might happen in first year physics.

[15]This technique is probably the first one taught for solving PDE. It's quite an old technique, and Mathieu used it explicitly, and credited the physicist Lamé. I don't know who invented it.

Now let's complicate the differential equation *just* a bit:

$$\frac{\ddot{y}(\theta) - (2q\cos 2\theta)y(\theta)}{y(\theta)} = -a \tag{39}$$

where now again the eigenvalue (now called a) again has to be chosen so that $y(\theta)$ is periodic with period 2π. We also have a new parameter in the problem, q. This has to do with the geometry, which (as in the previous collection) is that of an ellipse. Here q is proportional to the distance between the foci of the ellipse. If the ellipse is actually just a circle, then $q = 0$ and we reduce to trig functions. So we somehow feel that the Mathieu functions are (like the Jacobian elliptic functions were) analogous to trig functions.

And, they are. But the eigenvalues are, sadly, more complicated. For small values of q, they are like $a = n^2$, but for larger values they can do different things. This complicates the vocabulary. But, at its most basic, we will have

$$\mathrm{ce}_n(\theta; q) \tag{40}$$

which will be *like* $\cos n\theta$ when q is small. It is *even*, and periodic with period 2π. If moreover the integer $n = 2m$ is itself an even integer, then the function $\mathrm{ce}_{2m}(\theta; q)$ is periodic with period π, just like $\cos(2m\theta)$ is. Similarly,

$$\mathrm{se}_n(\theta; q) \tag{41}$$

is like $\sin n\theta$. Maple knows quite a bit about these functions. It calls them **MathieuCE**`(k,q,theta)` and **MathieuSE**`(k,q,theta)`. This is a bit of an unfortunate clash of notation because the *modified* Mathieu functions are frequently denoted with capital letters, e.g. $\mathrm{Ce}_k(\theta; q)$. Well, Maple's implementation of modified Mathieu functions is hidden away, anyway: they are simply the ordinary Mathieu functions, but with purely imaginary argument.

```
plot([seq(MathieuCE(k, 1.0, theta), k = 0 .. 3)], theta = 0 .. 2*Pi,
     axes = boxed, gridlines = true,
     font = ["Arial", 16], labelfont = ["Arial", 16],
     labels = [theta, typeset(ce[k](theta))],
     tickmarks = [spacing(Pi/2), default] );
```

produces the graph in Figure 4.

Acquiring the vocabulary of the Mathieu functions and internalizing it means learning how the eigenvalues work (sadly, it's complicated) and how to compute them. This is where the great achievement of Gertrude Blanch was. She was the first to compute the *double* eigenvalues systematically.

Once one can compute the eigenvalues reliably, one has to learn how to compute the eigenfunctions, that is, the ce and the se functions. That's not as hard.

Once you have done that, then answering PDE questions with elliptic geometry in terms of Mathieu functions becomes natural. And that, in a nutshell, is what the paper [6] is about.

I will end with a picture. In 2020 I wrote a paper which I put on the arXiv but have never submitted anywhere else [12]. Probably a good home for it would be Maple Transactions, because what it does is use my Mathieu function code to draw the "nodal lines" for elliptic drums, which was the subject of Mathieu's 1868 paper [22]. Using that code, and Maple's graphical capabilities, it's easy to draw the pictures that Mathieu predicted would occur if one sprinkled sand on an elliptical drum and vibrated it at various frequencies. In Figure 5 we go one step farther and plot it in 3d and in colour. For my talk, which you all remember, I started with an animation of a figure like this one, showing the membrane vibrating in that mode.

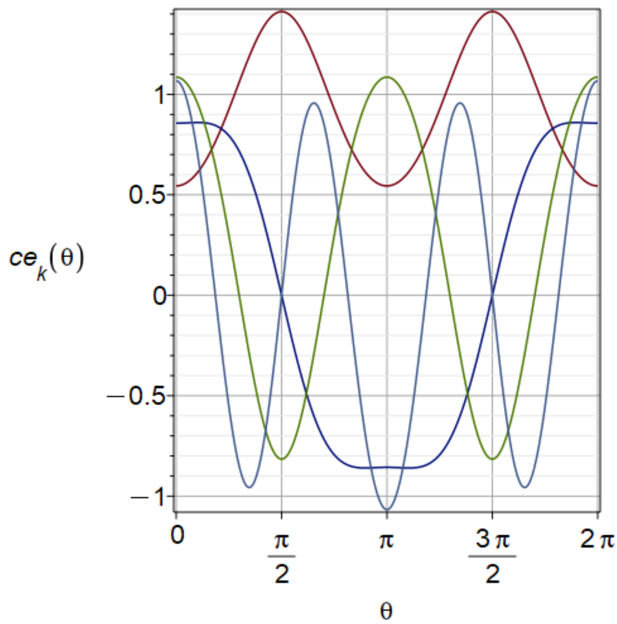


Fig. 4. The first four even Mathieu functions. You can figure out which is which by looking to see which ones have period π, which then must be $k = 0$ and $k = 2$, and the simpler one is $k = 0$; then the other two must be $k = 1$ and $k = 3$ and again the simpler one has $k = 1$.

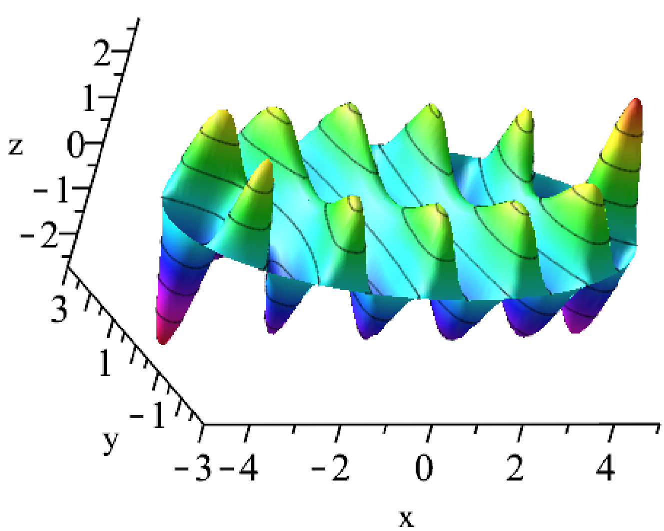


Fig. 5. The opening picture from my talk: the "Whispering" modes in an elliptical gallery. If you stand at one focus, and speak, the sound waves will travel along the edges of the gallery and collect at the other focus, so you can speak one to the other without being overheard. In the picture you can see the larger amplitude waves near the edges.

6 Notes and further references

If you are only going to read one work on the Γ function, then it must be the Chauvenet prize-winning paper "Leonhard Euler's Integral" by Philip J. Davis [14]. The paper is approximately as old as I am, and it reads like a lucid dream. It's in JSTOR, and everyone gets one hundred free papers from JSTOR every month, so you should use one of those free papers up on this one. I cited this paper in my own paper [4] on Γ, written with the late Jon Borwein, and maybe it was my favourite of all we read.

I very much enjoyed Eli Maor's historical book "Trigonometric Delights" [21], so much so that I bought the companion book on the history of e, namely [20]; that's sitting on my office shelf and I have not read it yet, but I am sure it will be excellent.

On elliptic functions, the book "Elliptic Functions and Applications" by D. F. Lawden, which I have cited already [19], is one of my favourite math books of all. It is pellucidly written, with lean dry prose that is *always* on point. You find page after page after page of formulas, sometimes taking up the whole page almost, which looks intimidating until you realize that there is *not a single error or misprint.* Then you fall in love with the book. I have been reading and re-reading this book for more than thirty years now, and I have never found a single mistake. I have implemented many of those said formulas and they have *always* been reliable. What a book! I collect autographs by authors of math books, and I wish I had the author's autograph for this one.

One of the best things about that book is its collection of classic applications, such as the harmonic oscillator $y'' + \sin(y) = 0$. Many people are under the impression that this "cannot be solved" analytically, and of course if they mean (under their breath) "in terms of elementary functions" they are right; but the solution is extremely simple in terms of elliptic functions. There is also a direct solution of the general relativity equations for the precession of the orbit of Mercury! And the book *begins* with the heat equation. Elliptic functions are, and always have been, applied mathematics!

Another of my favourite books is the Nobel Prize-Winner Subrahmanyan Chandresekhar's wonderful "Ellipsoidal figures of equilibrium" which studies the shape of planets, using elliptic functions [9]. In looking for the BibTeX entry for that, I just found this minute a "2010" paper[16] by him giving a historical account, which I will read *right now*[17] [8].

The classic paper [5] shows how to compute elementary functions quickly in high precision by using elliptic functions as an intermediate step. See also [13].

Other works on elliptic functions include a paper that I just found today with the amusing title "Elliptic Integrals: the forgotten functions" [16].

General works on special functions include the encyclopedic [1] (Gertrude Blanch wrote the chapter on Mathieu functions). This has been superseded by the online Digital Library of Mathematical Functions, dlmf.nist.gov. Probably the best textbook is [30], although it's old-fashioned in the extreme. It's still *absolutely* worth reading. Then there is the gigantic [2], and hundreds more. For computation of special functions, see the SIAM book by Gil, Segura, and Temme [15]. The classic book [3] uses a lot of special functions, and teaches quite a bit about them through that use.

Orthogonal polynomials are special functions, too (though technically elementary, which is weird). Another one of my "top few" books is *Chebyshev Polynomials* by Rivlin [25]. That was the first of my "autographed" books—I met Ted Rivlin at IBM T.J. Watson Research Center in Yorktown Heights, New York, on my first sabbatical in 1994.

[16]The normally reliable website doi2bib.org got the date wrong! It's 1967, not 2010! I have reported this via Twitter. Let's see if it gets fixed.

[17]Or maybe not. The UWO library is missing that exact volume online. Maybe they have it in print.

Another one of my "top few" is Nick Trefethen's amazing book Approximation Theory and Approximation Practice [27]. Yes, my copy of that one is autographed, too. I was honoured to be asked to review that book for SIAM Review, and if you want to see how I really feel about that book, you can read that review [11].

Then there is Nick Higham's amazing book, Functions of Matrices [18] (yes, yes, autographed and in my "top few"). It turns out that one needs to apply special functions to matrix arguments, as well, in many applications.

Finally, maybe my favourite book of all: "Concrete Mathematics" by Graham, Knuth, and Patashnik [17]. The special functions covered in this book are *mostly* discrete, like the binomials and Stirling numbers and the like, but it also treats the *hypergeometric functions* starting on p. 204. So if you have been frustrated waiting to learn about them, start there! The book was used as an undergraduate textbook, and contains many marginalia contributed by students, some of which are very funny indeed. Readable, beautiful, and useful. If you only choose one book to actually read out of the list I have given you so far, make it this one.

7 Concluding remarks

> It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments.

—Alfred North Whitehead, Introduction to Mathematics, Chapter 5

The central point of this paper, which I hope is clear now, is that special functions are pieces of mathematical vocabulary. Once you get used to them, and think of them as *answers* rather than as *questions*, then you can use them to *think* with. As per Whitehead's famous remark quoted above, this will increase the effectiveness of your thoughts.

One really important new factor nowadays is that we have computers to help. One of the best ways that they can help mathematical thinking is by encoding knowledge of special functions: how to evaluate them, how to differentiate or integrate them, how to combine them with other functions. Even just giving us unambiguous notation to work with is a huge advantage.

It is this curation of 19th century special functions in symbolic computation systems—starting with the work of Gertrude Blanch and the others on the fantastic project which became Abramowitz and Stegun and then the DLMF—which has led to significant revival of interest in special functions. The modern research field of *Mathematical Knowledge Management* starts from here, but then so does a lot of modern research into special functions.

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References

[1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1964.
[2] G. E. Andrews, R. Askey, and R. Roy. *Special Functions*. Cambridge University Press, 1999. Cambridge Books Online.
[3] Carl M Bender, Steven Orszag, and Steven A Orszag. *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory*, volume 1. McGraw–Hill, 1978.
[4] Jonathan M. Borwein and Robert M. Corless. Gamma and factorial in the monthly. *The American Mathematical Monthly*, 125(5):400–424, April 2018.
[5] Richard P Brent. Fast multiple-precision evaluation of elementary functions. *Journal of the ACM (JACM)*, 23(2):242–251, 1976.
[6] Chris Brimacombe, Robert M. Corless, and Mair Zamir. Computation and applications of Mathieu functions: A historical perspective. *SIAM Review*, 63(4):653–720, January 2021.
[7] Chris Brimacombe, Robert M. Corless, and Mair Zamir. Elliptic cross sections in blood flow regulation. *in preparation*, 2023.
[8] S. Chandrasekhar. Ellipsoidal figures of equilibrium—an historical account. *Communications on Pure and Applied Mathematics*, 20(2):251–265, September 1967.
[9] Subrahmanyan Chandrasekhar. *Ellipsoidal figures of equilibrium*. New Haven Yale Univ. Press, later Dover, 1969.
[10] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5(1):329–359, December 1996.
[11] Robert M. Corless. *SIAM Review*, 58(1):159–163, 2016.
[12] Robert M. Corless. Pure tone modes for a 5:3 elliptic drum, 2020.
[13] Annie Cuyt, Brigitte Verdonk, and Haakon Waadeland. Efficient and reliable multiprecision implementation of elementary and special functions. *SIAM Journal on Scientific Computing*, 28(4):1437–1462, January 2006.
[14] Philip J Davis. Leonhard euler's integral: A historical profile of the gamma function: In memoriam: Milton abramowitz. *The American Mathematical Monthly*, 66(10):849–869, 1959.
[15] Amparo Gil, Javier Segura, and Nico M. Temme. *Numerical Methods for Special Functions*. Society for Industrial and Applied Mathematics, January 2007.
[16] R H Good. Elliptic integrals, the forgotten functions. *European Journal of Physics*, 22(2):119–126, February 2001.
[17] Ronald L Graham, Donald E Knuth, and Oren Patashnik. *Concrete mathematics: a foundation for computer science*. Addison–Wesley, 1989.
[18] Nicholas J Higham. *Functions of matrices: theory and computation*. SIAM, 2008.
[19] D F Lawden. *Elliptic functions and applications*. New York: Springer, 1989.
[20] Eli Maor. *E: The Story of a Number*. Princeton paperbacks. Princeton University Press, 1994.
[21] Eli Maor. *Trigonometric Delights*. Princeton Science Library. Princeton University Press, 2013.
[22] Émile Mathieu. Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique. *Journal de mathématiques pures et appliquées*, 13:137–203, 1868.
[23] Sheehan Olver and Alex Townsend. A fast and well-conditioned spectral method. *SIAM Review*, 55(3):462–489, January 2013.
[24] Philippe R. Richard, M. Pilar Vélez, and Steven Van Vaerenbergh, editors. *Mathematics Education in the Age of Artificial Intelligence*. Springer International Publishing, 2022.
[25] Theodore J Rivlin. *Chebyshev polynomials*. Wiley Interscience, 1990.
[26] Bertrand Teguia Tabuguia and Wolfram Koepf. On the representation of non-holonomic power series. *Maple Transactions*, 2(1), September 2022.
[27] Lloyd N Trefethen. *Approximation Theory and Approximation Practice*. SIAM, 2019.
[28] Gerhard Wanner and Ernst Hairer. *Analysis by its History*. Springer-Verlag, 2nd revised printing, 1997.
[29] Elias Wegert. *Visual Complex Functions: An Introduction with Phase Portraits*. Springer Basel, 2012.
[30] Edmund Taylor Whittaker and George N Watson. *A course of modern analysis*. CUP, 1927.

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