Moler’s Law

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I always make this announcement: Maple Transactions

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Who is Cleve Moler?

1) A very nice (and smart!) guy
2) The co-founder of Matlab & The Mathworks, i.e.
    a) An extremely useful piece of math software (tens of millions of users)
    b) A $1.1B/yr company (!)
3) [Link] his Wikipedia page

Maybe “Moler’s Law” is worth paying attention to?

Instead of telling you straight out, I'll sneak up on it

The following occurred on a Grade 11 level Math Contest Exam[1]

If

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2\,, \tag{1}$$

what is x?

I will pause a minute for you to solve this. Try not to use anything

more advanced than a senior high-school student would use.

[1]These can be very tricky, in Canada. Do you have them here, too? This example and the ones that follow will seem pretty "pure" and not have any direct application; that's because I want to work with the simplest examples. They're tricky enough.

Euler and Condorcet

Leonhard Euler worked on this problem, after hearing that the Marquis de Condorcet had as well.

[Link] Leonhard Euler was one of the greatest mathematicians ever

[Link] The Marquis de Condorcet was a mathematician but more importantly one of the greatest political thinkers of the Enlightenment

Two methods of solution

First and simplest: If

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2\,, \tag{2}$$

then

$$x^2 = 2\,, \tag{3}$$

because *the infinite tower is the same.* One fewer power makes no difference.

Therefore $x = \sqrt{2} = 1.41412\ldots$. We ignore the negative root (for now).

The second solution (really the same)

Taking logarithms,

$$\ln\left(x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}\right) = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} \ln x = 2\ln x = \ln 2 \tag{4}$$

so $\ln x = \ln 2/2 = \ln\sqrt{2}$ so $x = \sqrt{2}$ again.

Let's do it again

If

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 4\,, \tag{5}$$

what is x? Again, I will let you solve it.

Oops?

Solving this the same way we get $x^4 = 4$ or
$x = 4^{1/4} = (2^2)^{1/4} = 2^{2/4} = 2^{1/2} = \sqrt{2}$.

Ack! With logarithms? $4 \ln x = \ln 4$ or $x = \sqrt{2}$ again!

$$2 = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}} = 4\,. \tag{6}$$

Did we really just prove that $2 = 4$?

Being careful with definitions

Put $a_0 = 0$ and define

$$a_{n+1} = x^{a_n} \ . \tag{7}$$

Then we can answer questions about the infinite tower by answering questions about what happens to this infinite sequence.

Moler’s Law

> The hardest thing to compute
> is something that doesn’t exist.
>
> —Cleve Moler

There is no x, with 4

There *is* an x, namely $x = \sqrt{2}$, such that the infinite tower (in the precise mathematical sense of $\lim_{n\to\infty} a_n$) is 2. But there *is no such x* which will make the limit 4, unless we start at a different a_0. **I will not** prove that here. It's hard to prove a negativve. **Moler's Law applies.**

Related: "Perron's Paradox: Let N be the largest positive integer. Suppose that $N > 1$. Then $N^2 > N$, a contradiction. Therefore $N = 1$." [Link] Oskar Perron (1880–1975), a German mathematician

Even if you have computed an answer, you have to be careful.

Also related: "Begging the question."

Integrals

Consider

$$\int_{x=0}^{\infty} \frac{dx}{x^4 + e^x} \, . \tag{8}$$

This integral isn't too much stranger than the ones you have seen already. Maybe something like this could occur in practice.

(See the classic 1980 paper "Handheld calculator evaluates integrals" by W. Kahan for an excellent discussion of integration in practice)

For another example, consider

$$\int_{x=0}^{\pi/4} x \tan x \, dx \, . \tag{9}$$

That last one looks as though it might occur on an exam. (It did, at least once.)

Both of those exist

Both of those integrals exist, but for the first one there is no short expression in terms of known functions,[2] while the second one can be expressed only in terms of something you likely haven't heard of before.

$$\int_{x=0}^{\pi/4} x \tan x \, dx = -\frac{\pi \ln(2)}{8} + \frac{\mathit{Catalan}}{2} \,. \tag{10}$$

[Link] Eugène Charles Catalan (1814–1894), a French/Belgian mathematician

Catalan's constant is about 0.9159655942.

The indefinite integral/antiderivative/primitive contains a function you won't have heard of: the *dilogarithm*.

[2] How would you prove such a thing?

Proving that some things are impossible

"Students are taught integration as a process, that starts with f and ends with F. But that process hardly ever succeeds. A compact $F(u)$ is almost always difficult *or impossible* to construct from a given $f(u)$."

—William Kahan, Handheld Calculator Evaluates Integrals, p. 24.

You can make a start on the algebraic theory of this, which leads to what is known as the Risch Integration Algorithm, by trying to prove that

$$\int_{u=1}^{x} \frac{1}{u}\, du \tag{11}$$

can **not** be expressed as a polynomial in x or as a ratio of polynomials in x.

But this integral exists perfectly well, and is the natural logarithm. What "doesn't exist" in this case is merely an expression in terms of simpler functions.

Similarly, to express

$$\int_{u=1}^{x} \frac{\ln u}{1+u}\, du \tag{12}$$

one needs a new function (in this case, the "dilogarithm.") By Maple,

$$\int_{1}^{x} \frac{\ln(u)}{1+u} du = \frac{\pi^2}{12} + dilog(x+1) + \ln(x)\ln(x+1) \ . \tag{13}$$

Some integrals really don't exist though

Not only is there no compact finite expression for the following one, there is no finite value, either:

$$\int_{u=0}^{x} \frac{1}{u}\, du\ . \tag{14}$$

We will return to this.

That first integral, again

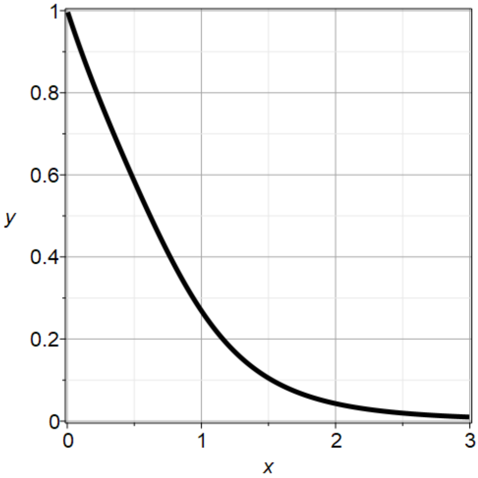


Figure 1: A section of the graph of $f(x) = 1/(x^4 + e^x)$.

Numerical quadrature

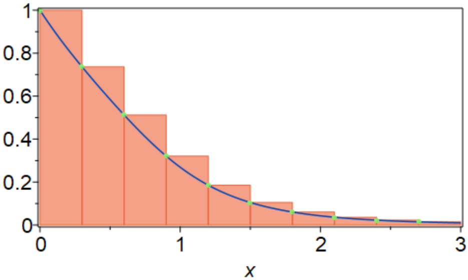
One nice thing about numerical quadrature is that you can give real assurance. Because for this example f is monotonically decreasing[3] the *Left*-hand Riemann sum provides an *upper* bound, while the Right-hand Riemann sum provides a lower bound. Using 1000 terms one can see that

$$0.7407 < \int_{x=0}^{3} \frac{dx}{x^4 + e^x} < 0.7437 \tag{15}$$

and more sophisticated schemes will give tighter bounds. More computer work will give more precision and certainty.

[3] Students are frequently asked to prove such things

Left Riemann sum

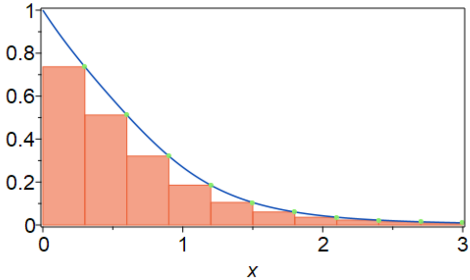


A left Riemann sum approximation of $\int_0^3 f(x)\,dx$,

where $f(x) = \dfrac{1}{x^4 + e^x}$ and the partition is uniform.

The approximate value of the integral is 0.8981612212. Number of subintervals used: 10.

Figure 2:



A right Riemann sum approximation of $\int_0^3 f(x)\,\mathrm{d}x$,

where $f(x) = \dfrac{1}{x^4 + e^x}$ and the partition is uniform.

The approximate value of the integral is
0.6011290048. Number of subintervals used: 10.

Figure 3:

The simpler integral of $1/x$, now

Taking the points $x_k = k/N$ for $1 \leq k \leq N$ we have that, because the right-hand Riemann sum is a lower bound,

$$\int_{u=0}^{1} \frac{1}{u}\, du > \frac{1}{N} \sum_{k=1}^{N} \frac{N}{k} = \sum_{k=1}^{N} \frac{1}{k}\,. \tag{16}$$

Have you seen already a proof that the sum on the right goes to infinity as $N \to \infty$?

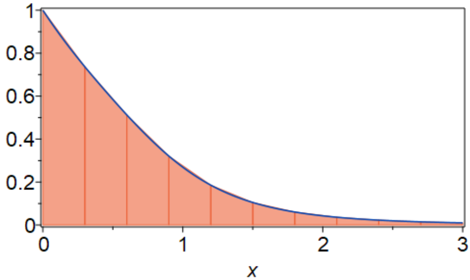
The Trapezoidal Rule

A researcher named *Tai* reinvented the Trapezoidal Rule in 1994, and called it[4] "Tai's model." The paper was published in the journal *Diabetes Care* and one of the referees had already noted that this was not new. Still, the paper has been cited many times. What can we learn from this?

- Diabetes Care needs numerical integration (!)
- Tai's teachers did not teach her the Trapezoidal Rule
- Actually, many teachers do not teach numerical methods

[4] After her parents

Trapezoidal Rule

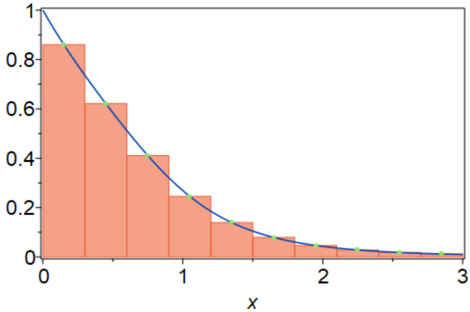


An approximation of $\int_0^3 f(x)\,dx$ using trapezoid rule, where $f(x) = \frac{1}{x^4 + e^x}$ and the partition is uniform.

The approximate value of the integral is 0.7496451130. Number of subintervals used: 10.

Figure 4:

Midpoint Riemann sum



A midpoint Riemann sum approximation of $\int_0^3 f(x)\,dx$, where $f(x) = \dfrac{1}{x^4 + e^x}$ and the partition is uniform. The approximate value of the integral is 0.7385669699. Number of subintervals used: 10.

Figure 5:

Lower and upper bounds

For functions that are convex up, the midpoint rule provides a lower bound while the trapezoidal rule provides an upper bound.

$$0.7207 < \int_0^3 \frac{dx}{x^4 + e^x} < 0.75 \tag{17}$$

with just ten subintervals. Using 1000 as with the Riemann Sum above,

$$0.7422549 < \int_0^3 \frac{dx}{x^4 + e^x} < 0.7422561\,. \tag{18}$$

So the integral is 0.74225^{61}_{49} which is likely more accurate than you need.

This can always be done, for smooth functions.

What about the tail?

$$\int_{x=3}^{\infty} \frac{dx}{x^4 + e^x} < \int_{x=3}^{\infty} \frac{dx}{e^x} = e^{-3} = 0.049 \tag{19}$$

So if we really did want more accuracy for the *infinite* interval we would need to work not over [0,3] but, say, over $[0, 10]$ and the error in chopping the tail would be less than $e^{-10} = 4.5 \times 10^{-6}$.

Looking back

- Students should get training in the technology that they can (will) use in their working lives. It's not easy to learn on your own.
- The technology is always changing. I used Maple for the examples in this talk, but lots of other systems work as well.
- The mathematics is changing too, but less rapidly (the Trapezoidal Rule is maybe 2000 years old).
- Understanding how to use the technology responsibly requires understanding the mathematics

Some harder examples

Consider the following *differential equations*:

- $y' = x^2 + y^2$, with initial condition $y(0) = 1$
- $\dot{x} = x^2 - t^2$, with initial condition $x(0) = -1/2$
- $y' = -\sqrt{y}$, with initial condition $y(0) = 1$ (see [Link] Variations on a theme of Euler)

About differential equations

Differential equations and their generalizations provide one of the main tools for understanding how systems evolve.

- The growth equation $y'(t) = Ky(t)$
- The logistic equation $y'(t) = Ky(t) - Ly^2(t)$
- [Link] The Black–Scholes (partial) differential equation which some people argue *determines* a particular financial reality
- infinitely many more

The ones I showed above are among the simplest. Some equations can be solved exactly in symbols; most must be solved numerically.

Just one of those examples

For $y' = x^2 + y^2$ consider the two related equations

$$u'(x) = 0^2 + u^2(x) \tag{20}$$
$$v'(x) = 1^2 + v^2(x) \tag{21}$$

with $u(0) = y(0) = v(0) = 1$. On the interval $0 < x \leq 1$ we can deduce that $u'(x) < y'(x) < v'(x)$ and hence that $u(x) < y(x) < v(x)$.

Can you solve $u' = u^2$? Or $v' = 1 + v^2$?

Those bounding curves

We have

$$\frac{u'(x)}{u^2(x)} = 1 \tag{22}$$

$$\frac{v'(x)}{1+v^2(x)} = 1 \tag{23}$$

So, integrating both sides with respect to x,

$$\int_{x=0}^{t} \frac{u'(x)}{u^2(x)}\,dx = \int_{x=0}^{t} 1\,dx = t - 0 = t \tag{24}$$

$$\int_{x=0}^{t} \frac{v'(x)}{1+v^2(x)}\,dx = \int_{x=0}^{t} 1\,dx = t - 0 = t \tag{25}$$

Continuing

Or

$$-\frac{1}{u(x)}\bigg|_{x=0}^{t} = t \tag{26}$$

$$\arctan(v(x))|_{x=0}^{t} = t \tag{27}$$

and using $u(0) = v(0) = 1$ we have $1 - 1/u(t) = t$ or $u(t) = 1/(1-t)$ and $\arctan v(t) - \arctan(1) = t$ or $\arctan v(t) = t + \pi/4$ or $v(t) = \tan(t + \pi/4)$.

Both of those functions are singular on $0 \leq t \leq 1$. Therefore $y(x)$ is also singular on that interval.

What will happen if we solve the equation $y'(x) = x^2 + y^2(x)$ with $y(0) = 1$ numerically (naively) on $0 \leq x \leq 1$?

A naive numerical solution

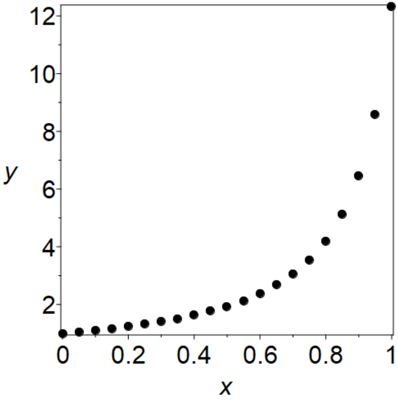


Figure 6: A naive numerical solution of $y' = x^2 + y^2$, $y(0) = 1$. Moler's Law applies.

Why I tell you these things

- When you are out in the real world, you will use power tools
- You need to be trained in the use of the tools
- You will encounter problems more difficult than these examples
- You need to know when to trust the output of your computer tools and when **not** to ("**Computational epistemology**")
- Some of your future tools haven't been invented yet
- You need to develop useful habits of thinking
- Existence (and uniqueness) of solutions is worth money, and sometimes **more** than money

A useful principle

"A good numerical method gives you the exact solution of a nearby problem"

As discussed in "Variations on a theme of Euler" linked previously, one way you can check your solution to your differential equation is to substitute it back into the original equation and see what is left over. If your solution is $Y(x)$ then compute

$$r(x) = Y'(x) - x^2 - Y^2(x) \; . \tag{28}$$

Then $r(x)$ is called the *residual*. If it is zero,[5] then good! But numerically it will usually be small, not zero.

[5] Detecting zero is *provably impossible* in some contexts

Continued

You have therefore solved

$$y'(x) = x^2 + y^2(x) + r(x) \tag{29}$$

exactly. Does *this* equation model your original problem well, if $r(x)$ is small? What is the effect of this change (“perturbation”) of the equation?

(Some problems are sensitive to changes—we say “ill-conditioned”).

These questions are sometimes hard, but they are the right questions to ask in practice.

Another announcement

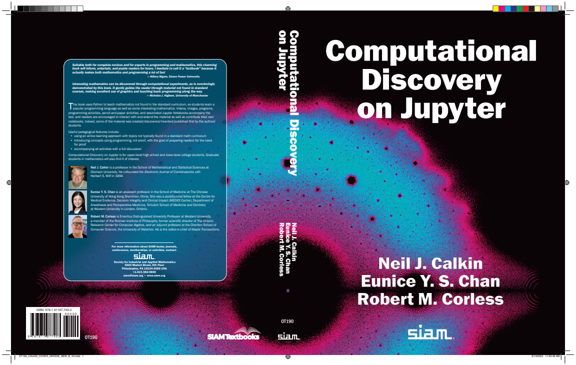


Figure 7: A new book from SIAM: Calkin, Chan, & Corless, “Computational Discovery on Jupyter”, hopefully available November

Thank you

Thank you for listening!