Analysis versus Algebra in Symbolic Computation

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Maple Transactions, ORCCA,
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Announcement: Maple Transactions

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We welcome expositions on topics of interest to the Maple community, including in computer-assisted research in mathematics, education, and applications. Student papers especially welcome.

Announcement: The History of Symbolic Computation Project

The goals of this project include

- to document for posterity, using contemporary records and living memory, the history of our subject
- to provide a framework for metastudies of computational mathematics
- to provide a framework for examining connections to other disciplines
- to examine the impact of our field on society in general

Immediate plans

- Discuss at ISSAC the era 1965–1975 (paper accepted; link goes to arxiv version)
- Discuss our own memories and opinions on the history of the subject at the 2025 ACA meeting in Heraklion [We are here now]
- Gather materials, including interviews of pioneers, for a curated collection [we are thinking “book” but it might be several volumes or it might have videos or dynamic material] of relevant historical observations.

This is not a one-person project. If you wish to join the team, send me an email.

About this session

We have speakers from several (but not all) countries; from several (but not all) research areas in symbolic computation. This morning's session has myself, Lihong Zhi, and Jürgen Gerhard. This afternoon has Stephen Watt, Michael Wester, and David Jeffrey. Thursday morning has Michael Monagan, Tateaki Sasaki, and concludes with Arthur Norman.

Welcome to a huge project.

And now for my talk, the first in the session.

An early beginning

The search for formulaic answers via algebra has an ancient history.

- The Antikythera Mechanism was found not far from Heraklion
- Don Ramon Llull invented a machine that computed with symbols (theological symbols, mind)
- Ada Lovelace thought of using Babbage's difference engine for symbolic computation

The goal was formulæ, not numbers.

Formulæ may be flawed

That such formulæ may have lacunæ where they do not apply was perhaps first noted explicitly by Cauchy in his 1821 *Cours d'Analyse* (English translation presented at the MacTutor site):

> *We must even note that they suggest that algebraic formulas have an unlimited generality, whereas in fact the majority of these formulas are valid only under certain conditions and for certain values of the quantities they contain. By determining these conditions and these values, and by fixing precisely the sense of all the notations I use, I make all uncertainty disappear.*

Enter digital computers

- Early generations of software performed nearly all transformations by taking an "algebraic approach" and not considering "analytic" issues
- Continuity matters, both for numerical and for symbolic computation
- Algebraic manipulation needs more analysis than one might think, particularly for simplification

The square root bug

The meaning of the symbol $\sqrt{x}$ is now settled by consensus (even if some of us do *not* like the consensus). It means $\exp(\ln(x)/2)$ where $\ln(x)$ is the (again, settled by consensus) principal branch of the logarithm with its argument θ in $-\pi < \theta \leq \pi$.

Under that consensus, $\sqrt{x^2}$ now simplifies to $\mathrm{csgn}(x)x$ where the function csgn is $+1$ in the right half plane and on the imaginary axis in the top half plane, undefined at 0 and -1 everywhere else.

Simplifying $\sqrt{x^2}$ to x can happen correctly if assumptions about x are made. But without rules and assumptions, replacing $\sqrt{x^2}$ with x is an oversimplification.

Assumptions were put into Maple in about 1992; other languages had them earlier. Macsyma would famously ask questions before simplifying, for instance.

Continuity

A function $f : A \to \mathbb{R}$ or to $\mathbb{C}$ is *continuous* (abbreviated *cts*) at a point $x = a \in A$ if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\,, \tag{1}$$

and all quantities in the equation make sense [e.g. $f(a)$ exists]. This explicitly uses a distance function, a *metric*.

This is already problematic on computers. Mostly this issue is ignored (I would appreciate references to the contrary).

A simple example

Solve $ax = a$ for x; from the paper "Well, it isn't quite that simple" (Corless & Jeffrey (1992)). The solution is $x = 1$, unless $a = 0$. The solution set is not cts at $a = 0$. This is also known as the "specialization problem." A lot of work has been done on this, especially in the context of symbolic linear algebra. For instance, William Sit's paper "An algorithm for solving parametric linear systems." Journal of Symbolic Computation 13, no. 4 (1992): 353-394.

Many papers from ORCCA on the subject as well, e.g. Olagunju, Moreno Maza and Jeffrey, SYNASC 2024. Most recently for me, CASC 2020 with Rafiee Sevyeri and Saunders. Theorist tried to do this in the 1990s.

This simple example illustrates the heart of eigenvalue problems: $\mathbf{A}(a)\mathbf{x} = \lambda\mathbf{x}$ will have nontrivial difficulties exactly at these points of discontinuity.

Continuity is a numerical issue, too

I snuck a second parameter into the eigenvalue problem on the last slide. This means that the eigenvalues λ will depend on the values of the parameter a. Even if $\mathbf{A}(a)$ is analytic in a, this can cause difficulties. In particular, the *order of the eigenvalues can change* if you change a, even by a tiny amount. In Eunice Chan's talk in the Symbolic Linear Algebra session you will see this happen.

$$\int \frac{dt}{t^n} = \frac{t^{n+1} - 1}{n+1} \,. \tag{2}$$

In this form, with the *Kahanian* term $-1/(n+1)$ (see the paper by Corless, Jeffrey, and Stoutemyer in the 2020 Mathematical Gazette), this answer has only a removable discontinuity at $n = -1$.

P.S. David Stoutemyer implemented this in Derive, and teachers hated it.

See Stephanie A. Dick, "Coded Conduct: making MACSYMA users and the automation of mathematics" BJHS Themes, 2020 which examines how much *users* have to change in order to use CAS successfully.

Enter the Constant Problem

A major historical landmark here was Richardson, D., 1969. Some undecidable problems involving elementary functions of a real variable. The Journal of Symbolic Logic, 33(4), pp.514-520.

When integrating, Macsyma does not recognize that the exponent below is -1:

$$x^{\sin(46)-2\sin(23)\cos(23)-1}$$

The output from REDUCE depends on the setting of the "rounded" flag, which can detect a zero divisor. Maple and Mathematica both understand that the exponent is -1. However, it was not difficult to find an example of a more complicated expression that one part of Maple could deduce was -1 but the usual simplifications applied by "int" could not. The existence of such examples is guaranteed by the theorem cited above.

Integration in elementary terms

There is a lot to say, here, with history ranging from Newton and Euler through Liouville and Ritt to Rothstein and Bronstein and more. One of the major successes of symbolic computation is algorithmic solution—apart from the Constant Problem and the issue of branch cuts—of the problem of integration in finite terms. But vexing issues with the continuity of the answers remain. Some progress has been made: see e.g. D.J. Jeffrey, "The importance of being continuous", Math. Magazine 1994, and Lazard, D. and Rioboo, R., 1990. Integration of rational functions: rational computation of the logarithmic part. Journal of Symbolic Computation, 9(2), pp.113-115.

Differential equations, too

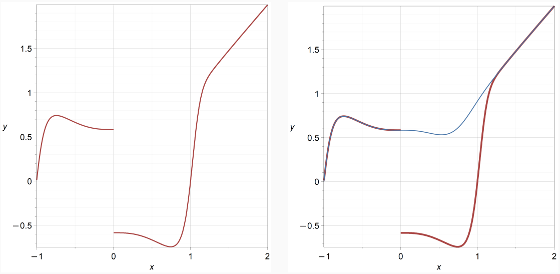
Consider $\varepsilon y' = x^2(x^2 - y^2)$ subject to $y(-1) = 0$. Here $\varepsilon > 0$ is small and this perturbation problem is from a famous book by Bob O'Malley, who used Maple in its solution. The solution via Maple 2025 is explored in a Maple worksheet. In short, Maple produces an answer with a spurious discontinuity at $x = 0$. (Wolfram Alpha gets this one right.) The issue is that the answer contains Bessel functions which have a branch point at 0.

Numerical methods get this right, though they have trouble for small ε.

The solution contains Bessel functions

$$y(x) = \frac{x\left(I_{-\frac{5}{8}}\left(\frac{x^4}{4\varepsilon}\right)K_{\frac{5}{8}}\left(\frac{1}{4\varepsilon}\right) - K_{\frac{5}{8}}\left(\frac{x^4}{4\varepsilon}\right)I_{-\frac{5}{8}}\left(\frac{1}{4\varepsilon}\right)\right)}{K_{\frac{3}{8}}\left(\frac{x^4}{4\varepsilon}\right)I_{-\frac{5}{8}}\left(\frac{1}{4\varepsilon}\right) + I_{\frac{3}{8}}\left(\frac{x^4}{4\varepsilon}\right)K_{\frac{5}{8}}\left(\frac{1}{4\varepsilon}\right)} \tag{3}$$

Plotting the solution



(a) A discontinuous "solution" (b) Jump corrected

Figure 1: left) A discontinuous putative "solution" computed by Maple that contains discontinuous expressions, for $\varepsilon = 1/8$. (right) After analyzing the jump at 0, the "constant" of integration can be adjusted to ensure smoothness.

Final words for today

This paper reviews both progress to date and future prospects in this area. One has to accept that there are unpalatable choices to be made: On the one hand the generation of potentially incorrect results and on the other sometimes prohibitively expensive computations, grossly clumsy presentations of answers or other unfriendly behaviours. Looking at the history and how various systems have responded can help shape an understanding of proper responses to the challenges.

Acknowledgements

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