Teaching Linear Algebra in a Mechanized Environment

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These slides available at
https://rcorless.github.io/TeachingLinearAlgebraMechanized__Environment.pdf

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Editor-in-Chief, Maple Transactions

Announcement: Maple Transactions

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Context

- Teaching *Computational Mathematics* is increasingly important (Data Science, Visualization, Machine Learning, . . .)
- This is *difficult* because computational mathematics involves several things at once: mathematics, programming, complexity, and numerical stability, because of the *compromises* needed for efficiency
- New tools, methods, and topics are arising continually
- **Incorporating new things means removing old things** because we have only finite time to teach, and the students are learning other things as well
- **Active Learning** is by now recognized as being by far the most effective way to learn
- This talk will focus on Linear Algebra.

Is technology disheartening?

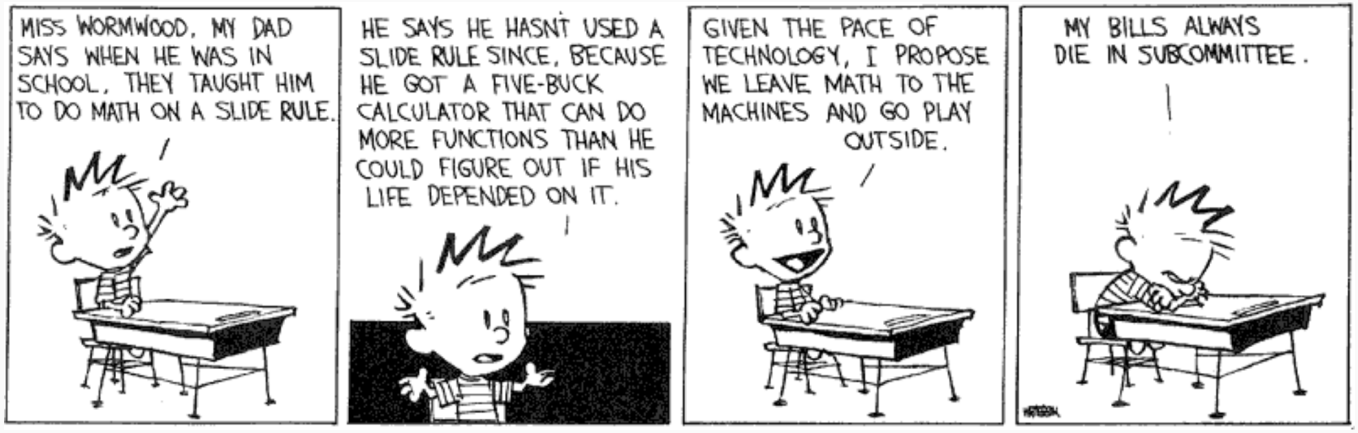

Figure 1: Hard to compete with a machine

Isn’t technology taken into account for education already?

No, at least not everywhere.

There have been arguments for at least fifty years about this. Maybe the best paper on the subject is Bruno Buchberger’s 1990 paper “Should Students Learn Integration Rules?” [Link] where he summarizes a resolution of previous fierce discussions with colleagues into what is known now as the “White Box/Black Box Principle”. [I will give details shortly.]

But not everyone knows this. Let’s look first at a **recent bad example** (from my own University)

How not to do it

We find the following question from a 2022 Math 1600 exam at Western (no calculators or cell phones allowed):

Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{1}$$

This question is obsolete in at least two ways. First, "nearly anything that can be done with the matrix inverse can be done without it." Second, who inverts 3 by 3 matrices by hand nowadays?

Maybe a better question: Does this matrix factor $\mathbf{A} = \mathbf{LU}$ without pivoting? **[can't tell without doing it]** Matrix factorings are *far* more useful than matrix inverses.

Linear Algebra I

> *"Linear algebra is the first course where the student encounters algebra, analysis, and geometry all together at once."*

—William (Velvel) Kahan,
to RMC at the 4th SIAM Linear Algebra Conference in Minneapolis 1991

See Kahan's paper Mathematics Written in Sand [Link] for an early and prescient view of the use of computational environments as "computational laboratories."

He envisaged the student as being an active explorer.

Challenges from floating-point arithmetic

- “Admit, for instance, the existence of a minimum magnitude, and you will find that the minimum which you have introduced, small as it is, causes the greatest truths of mathematics to totter.” — Aristotle, “On the Heavens”
- Floats are *not associative*: $a + (b + c) \neq (a + b) + c$ necessarily. For instance $-M + (M + 1) = 0$ while $(-M + M) + 1 = 1$ if $M = 3.14 \cdot 10^{17}$
- This (and other features such as UINT32s) break students' models of how the world works
- We have to enable students to deal with floats and UINT32s
- One has to rethink proofs in these contexts.

Proofs in a first Linear Algebra course

Here are some things one might like to prove in the first course.

- Cramer's Rule
- that the Normal Equations $\mathbf{A}^H\mathbf{A}\mathbf{x} = \mathbf{A}^H\mathbf{b}$ give the least-squares solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$
- the matrix determinantal lemma
  $\det(\mathbf{A} + \mathbf{u}\mathbf{v}^H) = (1 + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}) \det(\mathbf{A})$ under certain conditions

[Link] Keith Devlin: What is a Mathematical Proof? "Proofs are stories that convince suitably qualified others that a certain statement is true."

It's our job to train people so that they are qualified.

Programming versus Proof

But in North America, generic students are no longer exposed to proofs in high school. They need motivation when they *do* encounter proofs (usually in the first Linear Algebra course). We contend that **programming can provide such a motivation**. Ed Barbeau (U. Toronto) says that "there should be no proof without doubt." Meaning that if the students don't doubt the statement, proving it is counterproductive.

Having the students write programs, and encounter bugs, can be very motivating. They might be interested in proving their programs correct, after that experience. They should try, by hand. We think that *only then* will the concept of an automatic prover have any meaning for them.

Expanding on that a bit

- A classic proof of existence/uniqueness is somewhat akin to proving that a program terminates.
- What is frequently needed is a proof that when a program terminates, it gives the exact answer to a nearby question. **This involves the original non-mathematical context.**

Buchberger's White Box/Black Box

In his 1990 paper Buchberger outlines the *White Box/Black Box* principle for teaching mathematics in a mechanized environment.

- When learning a particular concept the 1st time, say determinant, the student is not allowed to use the `Determinant` routine
- When using the concept in learning more advanced things (eg Cramer's Rule) they *are* allowed to use it
- Why? People need a certain amount of human action to internalize a concept
- At that point we say the concept has become an **answer** and not a **question**

Sometimes the rule can be broken profitably; one can use Black Box for a while, then open up the box and see what's inside it.

Again, Buchberger is thinking of the student as being *active*.

We have already changed

Because of Wolfram Alpha [Link] homework assignments already have to be different. Then there's Chegg and MathOverflow and Maple Primes and many more. This puts more weight on exams. I am not even going to talk about ChatGPT!

Examining *with* technology is more stressful for the student. We have had lab equipment failures, software failures, and lots more. It's harder to invigilate, too.

But going the no-technology route requires banning cell phones (illegal in some countries). And limits the kind of question you can ask.

A possibly better exam question

Figure 2: Question: Maple gives the output above. Is it correct? Is it correct *and complete*? Does the answer to this question depend on the original (non-mathematical) context of the problem?

A possibly better activity

Take the inductive proof of the modified Gram-Schmidt process[1] and write a program to compute the *QR* factoring of an arbitrary rectangular matrix with complex entries. Prove your program is correct. Test it on matrices with complex entries. Is the output of your program *continuous* with respect to the input data[2]?

[1] I have a YouTube video on this induction for Modified Gram-Schmidt [Link]; one of my more popular videos, in fact.

[2] Probably not, and this is a very hard problem.

Reflections

- A student quotation: “What *good* does [the technology] do? I mean, I *liked* plug-and-chug. Now I have to think about what the answer *means*!”
- The problem of case dissection [link] is *hard*; there can be a combinatorial explosion of cases. The program needs the user to tell it about the assumptions on the variables
- Using technology requires training. (most programming syntax was not at all obvious to me before I learned it)
- We need to re-think our exam questions in the light of student needs, which have changed with the changing environment.

The case of calculus is discussed in more detail in my 2004 paper Computer-Mediated Thinking [link].

But, *which* technology?

One wants to reduce, reuse, and recycle: instead of using a calculus tool in one course and a linear algebra tool in another and a statistical tool in yet another, one wants to present as much of a common front as possible. Jupyter notebooks are very popular just now, and perhaps one should "go with the flow."

Of course, there are many kernels one can connect to Jupyter notebooks: Maple, Mathematica, Matlab, Python, R, Sagemath, and so on.

Let the religious wars recommence...

Mathematical Notions Strengthened by Programming

Several mathematical notions are strengthened by these exercises.

- One can use *mathematical induction* to prove correctness of a program
- Real analysis using ε–δ proofs works well with IEEE floats by means of the IEEE guarantees $\mathrm{fl}(x \operatorname{op} y) = (x \operatorname{op} y)(1+\delta)$ for some $|\delta| \leq \mathbf{u}$ where $\mathbf{u}$ is the *unit roundoff*, 2^{-53} for double precision
- Practice with functions is always useful
- Simply working with visualizations improves people's feel for geometry.

Unintentional Power

Tampering with the first Linear Algebra course will have significant downstream effects. Who will benefit from the changes? Will anyone be harmed? And do we really *have* to change?

Old-fashioned lecturing Linear Algebra teaches students:

- To sit still for 50 minutes
- To take notes (gives practice writing or typing)
- Many other things (maybe gives them practice in following a chain of argument)

But we maintain that the Big Ideas can be taught better with an active technology

- Functions
- Continuity (Carathéodory uses this to define differentiability)
- Limits and Convergence
- Reductionism (cf. Steven Strogatz' *Infinite Powers*)
- maybe, Proof?

In Summary

- The environment in which our students use mathematics has changed
- The mathematics we teach must change with that
- Not only the methods we use to teach mathematics (videos seem extraordinarily popular) but also the **topics must change**
- Students need active training in the responsible use of technology
- That means we have to keep up with the technology
- The reactionaries are winning in some places now, but they should not be allowed to win overall

Thank you

Thank you for listening!

This work was partially supported by NSERC grant RGPIN-2020-06438, and partially supported by the grant PID2020-113192GB-I00 (Mathematical Visualization: Foundations, Algorithms and Applications) from the Spanish MICINN. I also thank CUNEF Universidad for financial support.

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