Paul's Online Notes

Section 14.5 : Lagrange Multipliers - Practice Problems Solutions

Back to Problem List

5. Find the maximum and minimum values of $f(x, y, z) = 3x^2 + y$ subject to the constraints $4x - 3y = 9$ and $x^2 + z^2 = 9$.

Hide Solution ▾

Before proceeding with the problem let's note that the second constraint is the sum of two terms that are squared (and hence positive). Therefore, the largest possible range of x is $-3 \leq x \leq 3$ (the largest values would occur if $z = 0$). We'll get a similar range for z.

Now, the first constraint is not the sum of two (or more) positive numbers. However, we've already established that x is restricted to $-3 \leq x \leq 3$ and this will give $-7 \leq y \leq 1$ as the largest possible range of y's. Note that we can easily get this range by acknowledging that the first constraint is just a line and so the extreme values of y will correspond to the extreme values of x.

So, because we now know that our answers must occur in these bounded ranges by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Hide Step 2 ▾

The first step here is to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 6x &= 4\lambda + 2x\mu \\ 1 &= -3\lambda \\ 0 &= 2z\mu \\ 4x - 3y &= 9 \\ x^2 + z^2 &= 9 \end{aligned}$$

Hide Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be "easy" and which will be "hard" until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you've ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren't really as "bad" as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system we get a "freebie" to start off with. Notice that from the second equation we quickly can see that $\lambda = -\frac{1}{3}$ regardless of any of the values of the other variables in the system.

Hide Step 4

Next, from the third equation we can see that we have either $z = 0$ or $\mu = 0$, So, we have 2 possibilities to look at. Let's take a look at $z = 0$ first.

In this case we can go straight to the second constraint to get,

$$x^2 = 9 \qquad \rightarrow \qquad x = \pm 3$$

We can in turn plug each of these possibilities into the first constraint to get values for y.

$$\begin{aligned} x = -3 &\quad : \quad -12 - 3y = 9 \quad \rightarrow \quad y = -7 \\ x = 3 &\quad : \quad 12 - 3y = 9 \quad \rightarrow \quad y = 1 \end{aligned}$$

Okay, from this step we have two possible absolute extrema.

$$(-3, -7, 0) \qquad (3, 1, 0)$$

Hide Step 5

Now let's go back and take a look at what happens if $\mu = 0$. If we plug this into the first equation in our system (and recalling that we also know that $\lambda = -\frac{1}{3}$) we get,

$$6x = -\frac{4}{3} \qquad \to \qquad x = -\frac{2}{9}$$

We can plug this into each of our constraints to get values of y (from the first constraint) and z (form the second constraint). Here is that work,

$$4\left(-\frac{2}{9}\right) - 3y = 9 \qquad \to \qquad y = -\frac{89}{27}$$

$$\left(-\frac{2}{9}\right)^2 + z^2 = 9 \qquad \to \qquad z = \pm\frac{5\sqrt{29}}{9}$$

This leads to two more potential absolute extrema.

$$\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right) \qquad \left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right)$$

Hide Step 6

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f(-3,-7,0) = 20 \quad f(3,1,0) = 28 \quad f\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right) = -\frac{85}{27} \quad f\left(-\frac{2}{9}, \cdot\right.$$

The absolute maximum is then 28 which occurs at $(3,1,0)$. The absolute minimum is $-\frac{85}{27}$ which occurs at $\left(-\frac{2}{9}, -\frac{89}{27}, -\frac{5\sqrt{29}}{9}\right)$ and $\left(-\frac{2}{9}, -\frac{89}{27}, \frac{5\sqrt{29}}{9}\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that, in this case, the value of the absolute extrema (as opposed to the location) did not actually depend on the value of z in any way as the function we were optimizing in this problem did not depend on z. This will happen sometimes and we shouldn't get too worried about it when it does.

Note however that we still need the values of z for the location of the absolute extrema. We need the values of z for the location because the points that give the absolute extrema are also required to satisfy the constraint and the second constraint in our problem does involve z's!

 Page Last Modified : 11/16/2022