Paul's Online Notes

Section 14.5 : Lagrange Multipliers - Practice Problems Solutions

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1. Find the maximum and minimum values of $f(x,y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$.

Hide Solution

Before proceeding with the problem let's note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of x is $-\frac{3}{2} \le x \le \frac{3}{2}$ (the largest values would occur if $y = 0$). Likewise, the largest possible range of y is $-3 \le y \le 3$ (with the largest values occurring if $x = 0$).

Note that, at this point, we don't know if x and/or y will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Hide Step 2

The first actual step in the solution process is then to write down the system of equations we'll need to solve for this problem.

$$\begin{aligned} 162x &= 8x\lambda \\ 2y &= 2y\lambda \\ 4x^2 + y^2 &= 9 \end{aligned}$$

Hide Step 3

For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

In this case, simply because the numbers are a little smaller, let’s start with the second equation. A little rewrite of the equation gives us the following,

$$2y\lambda - 2y = 2y\left(\lambda - 1\right) = 0 \qquad \to \qquad y = 0 \quad \text{or} \quad \lambda = 1$$

Be careful here to not just divide both sides by y to “simplify” the equation. Remember that you can’t divide by anything unless you know for a fact that it won’t ever be zero. In this case we can see that y clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Hide Step 4 ▾

We now have two possibilities from Step 3. Either $y = 0$ or $\lambda = 1$. We’ll need to go through both of these possibilities and see what we get.

Let’s start by assuming that $y = 0$. In this case we can go directly to the constraint to get,

$$4x^2 = 9 \qquad \to \qquad x = \pm\frac{3}{2}$$

Therefore, from this part we get two points that are potential absolute extrema,

$$\left(-\frac{3}{2}, 0\right) \qquad \left(\frac{3}{2}, 0\right)$$

Hide Step 5 ▾

Next, let’s assume that $\lambda = 1$. In this case, we can plug this into the first equation to get,

$$162x = 8x \qquad \to \qquad 154x = 0 \qquad \to \qquad x = 0$$

So, under this assumption we must have $x = 0$. We can now plug this into the constraint to get,

$$y^2 = 9 \qquad \rightarrow \qquad y = \pm 3$$

So, this part gives us two more points that are potential absolute extrema,

$$(0, -3) \qquad (0, 3)$$

Hide Step 6 ▾

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f\left(-\frac{3}{2}, 0\right) = \frac{729}{4} \qquad f\left(\frac{3}{2}, 0\right) = \frac{729}{4} \qquad f(0, -3) = 9 \qquad f(0, 3) = 9$$

The absolute maximum is then $\frac{729}{4} = 182.25$ which occurs at $\left(-\frac{3}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$. The absolute minimum is 9 which occurs at $(0, -3)$ and $(0, 3)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

