HW 2

Problem 1

(a) $\lim_{n\to\infty} \frac{3^n}{2^n} \Rightarrow \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$. $f(n) = 3^n$, $g(n) = 2^n$

Since limit of the ratio of 3^n and 2^n is greater than 0, that implies $3^n = \Omega(2^n)$ by limit theorem. Thus $3^n \neq O(2^n)$

(b) Let $f(n) = 2n$ and $g(n) = n$. $\lim_{n\to\infty} \frac{2n}{n} = 2$. By limit theorem, $2n = \Theta(n)$, thus $2n = O(n)$.

Now $\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^{(2n-n)} = \lim_{n\to\infty} 2^n = \infty$.

By limit theorem, $2^{2n} = \Omega(2^n)$, thus the statement, "If $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$" is false.

c) $\lim_{n\to\infty} \frac{f(n)^3}{g(n)^3} \Rightarrow \lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right)^3 \Rightarrow \left(\lim_{n\to\infty} \frac{f(n)}{g(n)}\right)^3$

assuming $f(n) = O(g(n))$ then $0 \leq \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.

Since the limit of the ratio of $f(n)$ and $g(n)$ is less than infinity, raising it to the 3rd power means it will still be less than infinity so $f(n)^3 = O(g(n)^3)$. Hence if $f(n) = O(g(n))$, then $f(n)^3 = O(g(n)^3)$.

Problem 2

(a) $\lim\limits_{n\to\infty} \frac{f(n)}{dg(n)} \Rightarrow \frac{1}{d} \lim\limits_{n\to\infty} \frac{f(n)}{g(n)}$. If $f(n) = O(g(n))$ then $\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} \in [0, \infty)$. If d is any constant >0, then $\frac{1}{d} \lim\limits_{n\to\infty} \frac{f(n)}{g(n)}$ will also be $[0, \infty)$ which implies $f(n) = O(d \cdot g(n))$. Thus $f(n) = O(d\,g(n))$ iff $f(n) = O(g(n))$.

(b) Let $h_{min} = \min(h_1(n), h_2(n))$, $h_{max} = \max(h_1(n), h_2(n))$, and $h_{min} = O(h_{max})$.

Then $h_{min} \le c \cdot h_{max}$ for some $n > n_0$ by $O()$ def.

$h_{min} + h_{max} \le c \cdot h_{max} + h_{max}$

$h_{min} + h_{max} \le (c+1)\, h_{max}$

$h_1(n) + h_2(n) \le c_1\, h_{max}$ for some $n > n_0$

$\therefore h_1(n) + h_2(n) = O(\max(h_1(n), h_2(n)))$

Since $f(n) + g(n) = O(h_1(n) + h_2(n))$ then by transitive property $f(n) + g(n) = O(\max(h_1(n), h_2(n)))$.

(c) Let b and a be arbitrary positive constants.

Let $\log_a n = O(\log_b n)$.

$\lim\limits_{n\to\infty} \frac{\log_a n}{\log_b n} \Rightarrow \lim\limits_{n\to\infty} \frac{\frac{\log_c n}{\log_c a}}{\frac{\log_c n}{\log_c b}} \Rightarrow \lim\limits_{n\to\infty} \frac{\log_c b}{\log_c a} = \frac{\log_c b}{\log_c a}$

for some constant $c > 0$ & $\neq 1$. This is a positive constant meaning it is in $[0, \infty)$ which by limit theorem implies $\log_a n = O(\log_b n)$ for any positive constants b and a.

Problem 3

(a) The algorithm has a runtime of $n^3 - n^n$ since it iterates through n elements, $n-1$ times, n times.

$n^3 - n^2 = O(n^3)$ since $\lim_{n\to\infty} \frac{n^3 - n^2}{n^3} \Rightarrow \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{1} = 1$,

which by limit theorem implies $n^3 - n^2 = \Theta(n^3)$

which also means $n^3 - n^2 = O(n^3)$

(b) Store A[1] in B[1,1]

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for i ← 1 to n do
    for j ← i+1 to n do
        add A[j] and B[i, j-1]
        Store result in B[i, j-1]
    end for
end for
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The run time is $2(n^2 - n)$ since 2 elements are being added $n-1$ times, n times.

Problem 4

(a) $\lim_{n\to\infty} \frac{5\sqrt{n} + \log_2 n + 3}{100 \log_2 n + 25} \Rightarrow \lim_{n\to\infty} \frac{\frac{5\sqrt{n}}{\log_2 n} + 1 + \cancel{\frac{3}{\log_2 n}}^{\to 0}}{100 + \cancel{\frac{25}{\log_2 n}}^{\to 0}} = \infty$

1. no 2. yes 3. no 4. A_2

(b) $\lim_{n\to\infty} \frac{5\sqrt{n} + \ln n}{\sqrt{n} \cdot \log_2 n} \Rightarrow \lim_{n\to\infty} \frac{5 + \cancel{\frac{\ln n}{\sqrt{n}}}^{\to 0}}{\log_2 n} = 0$

1. yes 2. no 3. no 4. A_1

(c) $\lim_{n\to\infty} \frac{5 \log_{10} n - 20}{\frac{1}{2} \log_2 n} \Rightarrow \lim_{n\to\infty} \frac{5 \cdot \frac{\log_2 n}{\log_2 10} - 20}{\frac{1}{2} \log_2 n} \Rightarrow \lim_{n\to\infty} \frac{\frac{10}{\log_2 10} - \cancel{\frac{40}{\log_2 n}}^{\to 0}}{1}$

1. yes 2. yes 3. yes 4. A_2

(d) $\lim_{n\to\infty} \frac{10 n^{2/4}}{\sqrt{n} \log_2 n} \Rightarrow \lim_{n\to\infty} \frac{10 n^{1/2}}{\log_2 n} = \infty$

1. no 2. yes 3. no 4. A_2.