APPENDIX B

Conversion Matrices

IMPEDANCE MATRIX → SCATTERING MATRIX

The conversion of an impedance matrix into a scattering matrix is obtained from (2.29*b*), which is given as

$$\boldsymbol{S} = \boldsymbol{h}[\boldsymbol{Z} + \boldsymbol{z}]^{-1}[\boldsymbol{Z} - \boldsymbol{z}_*]\boldsymbol{h}_*^{-1}, \tag{B.1}$$

where

$$\boldsymbol{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \qquad \boldsymbol{z} = \begin{bmatrix} Z_{01} & 0 \\ 0 & Z_{02} \end{bmatrix},$$

$$\boldsymbol{h} = \begin{bmatrix} \sqrt{Z_{01}} & 0 \\ 0 & \sqrt{Z_{02}} \end{bmatrix}, \qquad \boldsymbol{h}_*^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{Z_{01}}} & 0 \\ 0 & \dfrac{1}{\sqrt{Z_{02}}} \end{bmatrix}$$

and $\boldsymbol{z} = \boldsymbol{z}_*$ for real Z_{01} and Z_{02}. In more detail it is calculated as

$$S_{11} = \frac{(Z_{22} + Z_{02})(Z_{11} - Z_{01}) - Z_{12}Z_{21}}{\Delta_Z},$$

$$S_{12} = \frac{2\sqrt{Z_{01}}\sqrt{Z_{02}}Z_{21}}{\Delta_Z},$$

$$S_{21} = \frac{2\sqrt{Z_{01}}\sqrt{Z_{02}}Z_{12}}{\Delta_Z},$$

$$S_{22} = \frac{(Z_{11} + Z_{01})(Z_{22} - Z_{02}) - Z_{12}Z_{21}}{\Delta_Z},$$

where $\Delta_Z = (Z_{11} + Z_{01})(Z_{22} - Z_{02}) - Z_{12}Z_{21}$.

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SCATTERING MATRIX → IMPEDANCE MATRIX

The conversion of a scattering matrix into an impedance matrix is obtained by rearranging (B.1) and is given as

$$\boldsymbol{Z} = (\boldsymbol{z}\boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_* + \boldsymbol{z}_*)(\boldsymbol{U}_n - \boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_*)^{-1}, \tag{B.3}$$

where $\boldsymbol{U}$ is an identity matrix of order 2, and

$$\boldsymbol{h}^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{Z_{01}}} & 0 \\ 0 & \dfrac{1}{\sqrt{Z_{02}}} \end{bmatrix} \quad \text{and} \quad \boldsymbol{h}_* = \begin{bmatrix} \sqrt{Z_{01}} & 0 \\ 0 & \sqrt{Z_{02}} \end{bmatrix}$$

for real Z_{01} and Z_{02}. To reduce calculation errors, $\boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_*$ and $(\boldsymbol{U}_n - \boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_*)^{-1}$ are first calculated as

$$\boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_* = \begin{bmatrix} S_{11} & \dfrac{\sqrt{Z_{02}}}{\sqrt{Z_{01}}} S_{12} \\ \dfrac{\sqrt{Z_{01}}}{\sqrt{Z_{02}}} S_{21} & S_{22} \end{bmatrix}, \tag{B.4}$$

$$(\boldsymbol{U}_n - \boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_*)^{-1} = \frac{1}{\Delta_{S_X}} \begin{bmatrix} 1 - S_{22} & \dfrac{\sqrt{Z_{02}} S_{12}}{\sqrt{Z_{01}}} \\ \dfrac{\sqrt{Z_{01}} S_{21}}{\sqrt{Z_{02}}} & 1 - S_{11} \end{bmatrix}, \tag{B.5}$$

where $\Delta_{S_X} = (1 - S_{11})(1 - S_{22}) - S_{12}S_{21}$

The conversion of scattering matrix into the impedance matrix is finally found as

$$\boldsymbol{Z} = (\boldsymbol{z}\boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_* + \boldsymbol{z}_*)(\boldsymbol{U}_n - \boldsymbol{h}^{-1}\boldsymbol{S}\boldsymbol{h}_*)^{-1} =$$
$$\frac{1}{\Delta_{S_X}} \begin{bmatrix} Z_{01}[(1 + S_{11})(1 - S_{22}) + S_{12}S_{21}] & 2\sqrt{Z_{01}}\sqrt{Z_{02}} S_{12} \\ 2\sqrt{Z_{01}}\sqrt{Z_{02}} S_{21} & Z_{02}[(1 + S_{22})(1 - S_{11}) + S_{12}S_{21}] \end{bmatrix}. \tag{B.6}$$

ADMITTANCE MATRIX → SCATTERING MATRIX

The conversion of an admittance matrix into a scattering matrix is obtained from voltage quantities and is given as

$$\boldsymbol{S} = -\boldsymbol{k}[\boldsymbol{Y} + \boldsymbol{y}]^{-1}[\boldsymbol{Y} - \boldsymbol{y}_*]\boldsymbol{k}_*^{-1}, \tag{B.7}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} Y_{01} & 0 \\ 0 & Y_{02} \end{bmatrix},$$

$$\boldsymbol{k} = \begin{bmatrix} \sqrt{Y_{01}} & 0 \\ 0 & \sqrt{Y_{02}} \end{bmatrix}, \qquad \boldsymbol{k}_*^{-1} = \begin{bmatrix} \dfrac{1}{\sqrt{Y_{01}}} & 0 \\ 0 & \dfrac{1}{\sqrt{Y_{02}}} \end{bmatrix},$$

and $\boldsymbol{y} = \boldsymbol{y}_*$ for real Y_{01} and Y_{02}, $Y_{01} = Z_{01}^{-1}$ and $Y_{02} = Z_{02}^{-1}$. The scattering parameters are again calculated as

$$\boldsymbol{S} = \frac{1}{\Delta_Y} \begin{bmatrix} Y_{01}[(1 - S_{11})(1 + S_{22}) + S_{12}S_{21}] & -2\sqrt{Y_{01}}\sqrt{Y_{02}}S_{12} \\ -2\sqrt{Y_{01}}\sqrt{Y_{02}}S_{21} & Y_{02}[S_{21}S_{12} + (1 - S_{22})(1 + S_{11})] \end{bmatrix} \tag{B.8}$$

where $\Delta_Y = (Y_{11} + Y_{01})(Y_{22} + Y_{02}) - Y_{12}Y_{21}$.

SCATTERING MATRIX → ADMITTANCE MATRIX

The conversion of a scattering matrix into an admittance matrix is obtained by rearranging the matrix in (B.7). To reduce calculation errors, $\boldsymbol{k}^{-1}\boldsymbol{S}\boldsymbol{k}_*$ is first calculated and then its conversion is computed.

$$\boldsymbol{k}^{-1}\boldsymbol{S}\boldsymbol{k}_* = \begin{bmatrix} S_{11} & \dfrac{\sqrt{Y_{02}}}{\sqrt{Y_{01}}}S_{12} \\ \dfrac{\sqrt{Y_{01}}}{\sqrt{Y_{02}}}S_{21} & S_{22} \end{bmatrix}, \tag{B.9}$$

$$\boldsymbol{Y} = (\boldsymbol{y}_* - \boldsymbol{y}\boldsymbol{k}^{-1}\boldsymbol{S}\boldsymbol{k}_*)(\boldsymbol{k}^{-1}\boldsymbol{S}\boldsymbol{k}_* + \boldsymbol{U})^{-1} =$$
$$\frac{1}{\Delta_{S_Y}} \begin{bmatrix} Y_{01}[(1 - S_{11})(1 + S_{22}) + S_{12}S_{21}] & -2\sqrt{Y_{01}}\sqrt{Y_{02}}S_{12} \\ -2\sqrt{Y_{01}}\sqrt{Y_{02}}S_{21} & Y_{02}[S_{21}S_{12} + (1 - S_{22})(1 + S_{11})] \end{bmatrix}, \tag{B.10}$$

where $\Delta_{S_Y} = (1 + S_{11})(1 + S_{22}) - S_{21}S_{12}$

ABCD MATRIX → SCATTERING MATRIX

Since an *ABCD* matrix is not directly convertible into a scattering matrix, the conversion of an *ABCD* matrix into an impedance matrix can be used. From Table 2.1,

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \frac{1}{C}\begin{bmatrix} A & \Delta \\ 1 & D \end{bmatrix}, \tag{B.11}$$

and its conversion yields

$$\boldsymbol{S} = \boldsymbol{h}[\boldsymbol{A_Z} + \boldsymbol{z}]^{-1}[\boldsymbol{A_Z} - \boldsymbol{z_*}]\boldsymbol{h}_*^{-1}, \tag{B.12}$$

where

$$\boldsymbol{A_Z} = \frac{1}{C}\begin{bmatrix} A & \Delta \\ 1 & D \end{bmatrix}.$$

For the final conversion, $[\boldsymbol{A_Z} + \boldsymbol{z}]^{-1}$ is first calculated and the result is

$$[\boldsymbol{A_Z} + \boldsymbol{z}]^{-1} = \frac{C}{(A + CZ_{01})(D + CZ_{02}) - \Delta}\begin{bmatrix} D + CZ_{02} & -\Delta \\ -1 & A + CZ_{01} \end{bmatrix}, \tag{B.13}$$

where $\Delta = AD - BC$. The final results are given as

$$\boldsymbol{S} = \frac{1}{\Delta_{A_Z}}\begin{bmatrix} AZ_{02} + B - CZ_{01}Z_{02} - DZ_{01} & 2\sqrt{Z_{01}}\sqrt{Z_{02}}(AD - BC) \\ 2\sqrt{Z_{01}}\sqrt{Z_{02}} & -AZ_{02} + B - CZ_{01}Z_{02} + DZ_{01} \end{bmatrix}, \tag{B.14}$$

where $\Delta_{A_Z} = AZ_{02} + B + CZ_{01}Z_{02} + DZ_{01}$.