Lab 4 - LDPC Factor Graph Construction

This is the first of the two colab notebooks that form the project.

Here you will construct a **factor graph** for decoding a (regular) **low-density parity-check (LDPC)** code. Although communication systems often refer to messages and codewords, in this lab we model only the transmitted bits x and the channel outputs y. All other steps are either preprocessing (encoding) or postprocessing (reading off the decoded bits).

As the block length N increases, well-designed LDPC codes achieve better decoding performance, while the treewidth of the factor graph grows, making exact inference intractable and motivating approximate methods such as belief propagation.

Learning goals

- Represent an LDPC code as a **factor graph** with:
  - binary **variable nodes** (bits),
  - **parity-check factors** (XOR constraints),
  - **channel factors** connecting each transmitted bit to its channel output.
- Validate the graph:
  - **structurally**: degrees, factor arities, correct wiring,
  - **semantically**: parity-check factors behave correctly on a known satisfying assignment.

You will be given:

- a generator for a *regular* parity-check matrix H (with configurable density),
- a Binary Symmetric Channel (BSC) noise generator.

You will implement:

1. a parity-check factor constructor,
2. a channel factor constructor,
3. the factor-graph construction from H,
4. graph validation checks and visualization,
5. treewidth scaling analysis.

```python
# Colab setup
# !pip -q install pgmpy networkx
```

```python
import numpy as np
import itertools
import networkx as nx
```

```python
from pgmpy.models import FactorGraph
from pgmpy.factors.discrete import DiscreteFactor
```

Factor graphs for Low-Density Parity-Check (LDPC) decoding

Variables and factorization

The task is to transmit N bits over a noisy channel. LDPC codes introduce M parity-check constraints to enable error correction.

The factor graph contains two types of binary variables:

- **transmitted bits**: $x_0, \ldots, x_{N-1}$
- **channel outputs**: $y_0, \ldots, y_{N-1}$

The joint distribution defined by the factor graph factorizes as

$$p(x, y) \;\propto\; \prod_{m=1}^{M} \psi_m(x_{N(m)}) \;\; \prod_{n=0}^{N-1} \phi_n(x_n, y_n),$$

where $N(m)$ denotes the set of transmitted-bit indices that appear in factor ψ_m, and:

- ψ_m are **parity-check factors** enforcing local constraints on transmitted bits,
- ϕ_n are **channel factors** modeling the noise affecting each bit.

Parity-check factors $\psi_m(x_{N(m)})$

Each parity-check factor enforces an even-parity constraint on a small subset of transmitted bits.

Formally, the factor is defined as

$$\psi_m(x_{N(m)}) \;=\; \begin{cases} 1 & \text{if } \sum_{n \in N(m)} x_n \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, each parity-check factor is a **local constraint** that involves only the k transmitted-bit variables connected to it.

Channel factors $\phi_n(x_n, y_n)$

Each channel factor connects a transmitted bit x_n to its corresponding channel output y_n.

We model the channel as a **binary symmetric channel** (**BSC**) with flip probability f, defining

$$\phi_n(x_n, y_n) = p(y_n \mid x_n) = \begin{cases} 1-f & \text{if } y_n = x_n, \\ f & \text{if } y_n \neq x_n. \end{cases}$$

Channel factors are independent across bit positions.

Decoding as inference

In a communication scenario, the channel outputs $y_0, \ldots, y_{N-1}$ are observed.

Decoding consists of inferring the transmitted bits $x_0, \ldots, x_{N-1}$ given these observations, that is, computing the posterior distribution $p(x \mid y)$.

In the next session, you will approximate this posterior using belief propagation on the same factor graph, after introducing evidence on the y variables.

Regular LDPC parity-check matrix generator

A **regular LDPC code** is specified by a sparse parity-check matrix $H \in \{0,1\}^{M \times N}$, which defines the structure of the parity-check factors in the factor graph.

- Each row corresponds to one parity-check factor.
- Each column corresponds to one transmitted-bit variable.
- Each row has exactly k ones (each check involves k bits).
- Each column has exactly j ones (each bit participates in j checks).

For a given row m, the set

$$N(m) = \{\, n : H_{m,n} = 1 \,\}$$

is exactly the set of transmitted bits connected to parity-check factor ψ_m.

The following code generates a random **regular LDPC** parity-check matrix:

- each **column** has weight j,
- each **row** has weight k.

If generation fails (rare for small sizes), re-run the cell or adjust the parameters.

```python
def generate_regular_ldpc_H(N: int, j: int, k: int, seed: int = 0, max_tries: int = 2000) -> np.ndarray:
    """Generate a random (j,k)-regular binary parity-check matrix H of shape (M,N),
    where M = N*j/k must be an integer.

    Notes:
    - This construction is intentionally simple (sufficient for the lab).
    - It may create short cycles; that's fine (Lab 2 is loopy).
    """
    rng = np.random.default_rng(seed)
    if (N * j) % k != 0:
        raise ValueError(f"Need M = N*j/k integer, but N*j={N*j} not divisible by k={k}.")
    M = (N * j) // k

    for _ in range(max_tries):
        # Create j "stubs" per column
        col_stubs = np.repeat(np.arange(N), j)
        rng.shuffle(col_stubs)

        # Assign stubs into M groups of size k (rows)
        H = np.zeros((M, N), dtype=np.uint8)
        ok = True
        for m in range(M):
            cols = col_stubs[m * k : (m + 1) * k]
            # avoid duplicate variable in a single check (keeps row weight exactly k)
            if len(set(cols.tolist())) != k:
                ok = False
                break
            H[m, cols] = 1
        if not ok:
            continue

        # Verify regularity
        if np.all(H.sum(axis=0) == j) and np.all(H.sum(axis=1) == k):
            return H
```

```python
    raise RuntimeError("Failed to construct a regular LDPC matrix. Try a different seed or parameters.")
```

Part 1: Choose code parameters and generate H

Pick modest sizes so you can visualize and test easily. Suggested starting point:

- $N = 24, j = 3, k = 6$ (then $M = \frac{Nj}{k} = 12$)

```python
# TODO: Choose parameters

# 2**(N-M) would be the number of valid codewords, so we want M to be not so large to avoid restricting the code too much.
# On the other hand, we want M to be large enough to have a non-trivial code (not just the whole space). For example, with N=24,

N = 24
j = 3
k = 6
seed_H = 0
M = (N * j) // k
print(f"Generating (j={j}, k={k})-regular LDPC matrix H with N={N} columns and M={M} rows...")

H = generate_regular_ldpc_H(N=N, j=j, k=k, seed=seed_H)
M = H.shape[0]
print(H)

print("H shape:", H.shape)
print("column weights (should all be j):", np.unique(H.sum(axis=0)))
print("row weights (should all be k):", np.unique(H.sum(axis=1)))
```

```
Generating (j=3, k=6)-regular LDPC matrix H with N=24 columns and M=12 rows...
[[1 1 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0]
 [0 0 0 1 0 0 1 1 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 1]
 [0 0 0 0 1 1 0 0 1 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0]
 [0 0 0 0 1 0 0 1 0 0 0 0 0 1 0 0 1 1 0 0 1 0 0 0]
 [0 1 0 0 1 0 1 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 0]
 [0 1 1 0 0 1 0 1 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1 0 0 1 0 0 0 0 1]
```

```
 [0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 0 1 0 1 0 0 1 0 0]
 [0 0 1 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0]
 [0 0 0 0 0 0 1 0 0 1 1 0 0 0 0 1 0 1 0 0 0 0 1 0]
 [1 0 1 1 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1]
 [1 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 0 1 0 1 1 0]]
H shape: (12, 24)
column weights (should all be j): [3]
row weights (should all be k): [6]
```

Part 2: Build factors

2.1 Parity-check factors $\psi_m(x_{N(m)})$

Implement `make_parity_check_factor` which receives a matrix H, a set of variables, and returns a factor defined over the transmitted-bit variables connected to that row.

It should be:

- value = 1 if the parity constraint is satisfied (even parity)
- value = 0 otherwise

Tips:

- With `DiscreteFactor`, you provide a `values` array of length 2^d (reshaped to `[2]*d`), where `d` is the number of variables in that factor.
- You can generate all assignments with `itertools.product([0,1], repeat=d)`.

Example of a parity-check factor for $k = 3$ bits:

x_1	x_2	x_3	ψ_{123}
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

```python
def make_parity_check_factor(var_names):
    """Return a DiscreteFactor enforcing even parity over var_names.
    """
    d = len(var_names)

    # Generate all possible assignments for the variables
    assignments = list(itertools.product([0, 1], repeat=d))

    # Create values array: 1 if even parity (sum is even), 0 otherwise
    values = np.array([1 if sum(assignment) % 2 == 0 else 0
                       for assignment in assignments], dtype=np.float64)

    # Reshape to [2, 2, ..., 2] (d times)
    values = values.reshape([2] * d)

    # Create and return the DiscreteFactor
    return DiscreteFactor(var_names, [2] * d, values) # [2]*d means cardinality of 2 (binary) for each of the d (k) variables

# test make_parity_check_factor
factor = make_parity_check_factor(['X1', 'X2', 'X3', 'X4'])
print("Factor variables:", factor.variables)
print("Factor cardinality:", factor.cardinality)
print(factor)
```

```
Factor variables: ['X1', 'X2', 'X3', 'X4']
Factor cardinality: [2 2 2 2]
+-------+-------+-------+-------+--------------------+
| X1    | X2    | X3    | X4    |   phi(X1,X2,X3,X4) |
+=======+=======+=======+=======+====================+
| X1(0) | X2(0) | X3(0) | X4(0) |             1.0000 |
+-------+-------+-------+-------+--------------------+
| X1(0) | X2(0) | X3(0) | X4(1) |             0.0000 |
+-------+-------+-------+-------+--------------------+
| X1(0) | X2(0) | X3(1) | X4(0) |             0.0000 |
+-------+-------+-------+-------+--------------------+
| X1(0) | X2(0) | X3(1) | X4(1) |             1.0000 |
```

```
+-------+-------+-------+-------+------------------+
| X1(0) | X2(1) | X3(0) | X4(0) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(0) | X2(1) | X3(0) | X4(1) |           1.0000 |
+-------+-------+-------+-------+------------------+
| X1(0) | X2(1) | X3(1) | X4(0) |           1.0000 |
+-------+-------+-------+-------+------------------+
| X1(0) | X2(1) | X3(1) | X4(1) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(0) | X3(0) | X4(0) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(0) | X3(0) | X4(1) |           1.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(0) | X3(1) | X4(0) |           1.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(0) | X3(1) | X4(1) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(1) | X3(0) | X4(0) |           1.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(1) | X3(0) | X4(1) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(1) | X3(1) | X4(0) |           0.0000 |
+-------+-------+-------+-------+------------------+
| X1(1) | X2(1) | X3(1) | X4(1) |           1.0000 |
+-------+-------+-------+-------+------------------+
```

Channel factors $\phi_n(x_n, y_n)$

Implement `make_bsc_channel_factor(x_var, y_var, f)`, a **pairwise factor** over `(x_var, y_var)` defined by a **binary symmetric channel (BSC)** with flip probability f.

```python
def make_bsc_channel_factor(x_var, y_var, f):
    """Pairwise channel factor for a BSC.
    Returns a DiscreteFactor over [x_var, y_var] with cardinalities [2,2].
    """
    return DiscreteFactor(
```

```
        variables=[x_var, y_var],
        cardinality=[2, 2],
        values=[1 - f, f, f, 1 - f]  # P(y|x): P(y=0|x=0)=1-f, P(y=1|x=0)=f, P(y=0|x=1)=f, P(y=1|x=1)=1-f
    )

# test make_bsc_channel_factor
channel_factor = make_bsc_channel_factor('X', 'Y', f=0.1)
print("Channel factor variables:", channel_factor.variables)
print("Channel factor cardinality:", channel_factor.cardinality)
print(channel_factor)
```

```
Channel factor variables: ['X', 'Y']
Channel factor cardinality: [2 2]
+------+------+------------+
| X    | Y    |   phi(X,Y) |
+======+======+============+
| X(0) | Y(0) |     0.9000 |
+------+------+------------+
| X(0) | Y(1) |     0.1000 |
+------+------+------------+
| X(1) | Y(0) |     0.1000 |
+------+------+------------+
| X(1) | Y(1) |     0.9000 |
+------+------+------------+
```

Part 3: Create the factor graph from H

Construct a `pgmpy` `FactorGraph`:

1. add variable nodes: `x0...x{N-1}`
2. add parity-check factors (one per row of H) and connect them
3. (optional today, required later) add likelihood factors and connect them

Implement:

- `build_ldpc_factor_graph(H, y=None, f=None)`

```
def build_ldpc_factor_graph(H: np.ndarray, f: float) -> FactorGraph:
```

```
"""Build an LDPC factor graph with explicit channel-output variables.

Variables:
  - transmitted bits: x0..x{N-1}
  - channel outputs:  y0..y{N-1}

Factors:
  - parity-check factors on x's (from H)
  - channel factors  _n(x_n, y_n) = p(y_n | x_n) for BSC flip prob f

Evidence (a particular received word Y) will be introduced later by clamping y's.
"""
M, N = H.shape
x_vars = [f"x{n}" for n in range(N)]
y_vars = [f"y{n}" for n in range(N)]

G = FactorGraph()

# 1) Add variable nodes
G.add_nodes_from(x_vars)
G.add_nodes_from(y_vars)

# 2) Add parity-check factors (one per row of H) and connect them
for m in range(M):
    var_names_row_m = [x_vars[n] for n in range(N) if H[m, n] == 1]
    parity_factor_m = make_parity_check_factor(var_names_row_m)
    G.add_factors(parity_factor_m)
    for var_name in var_names_row_m:
        G.add_edge(parity_factor_m, var_name)

# 3) Add channel factors  _n(x_n, y_n) and connect them
for n in range(N):
    channel_factor_n = make_bsc_channel_factor(x_vars[n], y_vars[n], f)
    G.add_factors(channel_factor_n)
    G.add_edge(channel_factor_n, x_vars[n])
    G.add_edge(channel_factor_n, y_vars[n])
```

```
    return G

# test build_ldpc_factor_graph
f = 0.1
G_test = build_ldpc_factor_graph(H, f)
print("Factor graph nodes:", G_test.nodes(data=True))
print("Factor graph edges:", G_test.edges())
```

```
Factor graph nodes: [('x0', {'weight': None}), ('x1', {'weight': None}), ('x2', {'weight': None}), ('x3', {'weight': None}), ('x4
Factor graph edges: [('x0', <DiscreteFactor representing phi(x0:2, x1:2, x5:2, x8:2, x20:2, x21:2) at 0x765e9f3a0f40>), ('x0', <D
```

Part 4: Validating the factor graph

We must verify that the factor graph correctly represents the intended LDPC model.

Your task in this part is to **think about validations that make sense**, and to implement some of them.

Examples of questions you may consider include:

Structural checks:

- Does the number and type of factors match the specification of the LDPC code?
- Are parity-check factors connected to the correct subsets of transmitted bits?
- Do variable nodes have the expected degrees (for a regular code)?
- Do parity-check factors behave correctly on a known satisfying assignment?

Semantic checks:

For small N, you can test the parity-check factors exhaustively.

For each assignment $x \in \{0, 1\}^N$ (there are 2^N of them), compute the so-called *syndrome*

$$s = Hx \pmod 2.$$

The parity-check constraints are satisfied if and only if $s = 0$, so the product of all parity-check factors should evaluate to

$$\prod_{m=1}^{M} \psi_m(x_{N(m)}) = \begin{cases} 1 & \text{if } Hx \equiv 0 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

For this semantic test you can ignore the y variables and the channel factors.

```
def validate_ldpc_graph(G, H):
    """Basic validation of an LDPC factor graph.

    This function performs a minimal sanity check using G.check_code().
    Students are encouraged to extend it with additional structural or
    semantic validations.
    """
    assert G.check_model(), "Basic LDPC graph validation failed."

    # Structural checks
    # 1) Does the number and type of factors match the specification?
    parity_factors = [f for f in G.get_factors() if set(f.variables).issubset({f"x{n}" for n in range(H.shape[1])})]
    channel_factors = [f for f in G.get_factors() if set(f.variables).issubset({f"x{n}" for n in range(H.shape[1])} | {f"y{n}" fo
    assert len(parity_factors) == H.shape[0], f"Expected {H.shape[0]} parity-check factors, found {len(parity_factors)}."
    assert len(channel_factors) == H.shape[1], f"Expected {H.shape[1]} channel factors, found {len(channel_factors)}."

    # 2) Are parity factors connected to the correct subsets of the transmited bits?
    for m, parity_factor in enumerate(parity_factors):
        expected_vars = {f"x{n}" for n in range(H.shape[1]) if H[m, n] == 1}
        actual_vars = set(parity_factor.variables)
        assert actual_vars == expected_vars, f"Parity factor {m} connected to {actual_vars}, expected {expected_vars}."

    # 3) Do variable nodes have expected degrees (for a regular LDPC code, each x_var should be connected to j parity factors and
    for n in range(H.shape[1]):
        x_var = f"x{n}"
        y_var = f"y{n}"
        x_neighbors = set(G.neighbors(x_var))
        y_neighbors = set(G.neighbors(y_var))
        assert len(x_neighbors) == j + 1, f"Variable {x_var} should have degree {j+1}, found {len(x_neighbors)}."
```

```python
    assert len(y_neighbors) == 1, f"Variable {y_var} should have degree 1, found {len(y_neighbors)}."

# 4) Do parity-check factors behave correctly on a known satisfying assignment (e.g., all zeros)?
for parity_factor in parity_factors:
    # Create an assignment of all zeros for the variables in this factor
    assignment = {var: 0 for var in parity_factor.variables}
    value = parity_factor.get_value(**assignment)
    assert value == 1.0, f"Parity factor {parity_factor} should evaluate to 1 on all-zero assignment, got {value}."

# 5) Semantic checks
#    For each assignment x, compute the syndrome s = Hx mod 2
#    and verify that the parity-check factors evaluate to 1 if and only if s=0.
N = H.shape[1]
if N <= 16:  # Only feasible for small N (2^16 = 65536 assignments)
    for x_int in range(2**N):
        # Convert integer to binary assignment vector
        x = ((x_int & (1 << np.arange(N))) > 0).astype(np.uint8)

        # Compute syndrome s = Hx mod 2
        syndrome = (H @ x) % 2
        syndrome_is_zero = np.all(syndrome == 0)

        # Compute product of all parity-check factors
        product = 1.0
        for parity_factor in parity_factors:
            # Build assignment dict for this factor's variables
            assignment = {var: x[int(var[1:])] for var in parity_factor.variables}
            product *= parity_factor.get_value(**assignment)

        # Verify: product should be 1 iff syndrome is zero
        expected_product = 1.0 if syndrome_is_zero else 0.0
        assert product == expected_product, (
            f"Semantic check failed for x={x}: "
            f"syndrome={'zero' if syndrome_is_zero else 'nonzero'}, "
            f"expected product={expected_product}, got {product}"
        )
```

```python
        print(f"Semantic check passed for all {2**N} assignments.")
    else:
        print(f"Skipping exhaustive semantic check (N={N} too large, would require {2**N} evaluations).")

    return True

N = 16
j = 3
k = 6
seed_H = 0
H = generate_regular_ldpc_H(N=N, j=j, k=k, seed=seed_H)
G_test = build_ldpc_factor_graph(H, f=0.8)
validate_ldpc_graph(G_test, H)
```

```
Semantic check passed for all 65536 assignments.
```

```
True
```

Part 5: Visualization

Implement code that visualizes a factor graph.

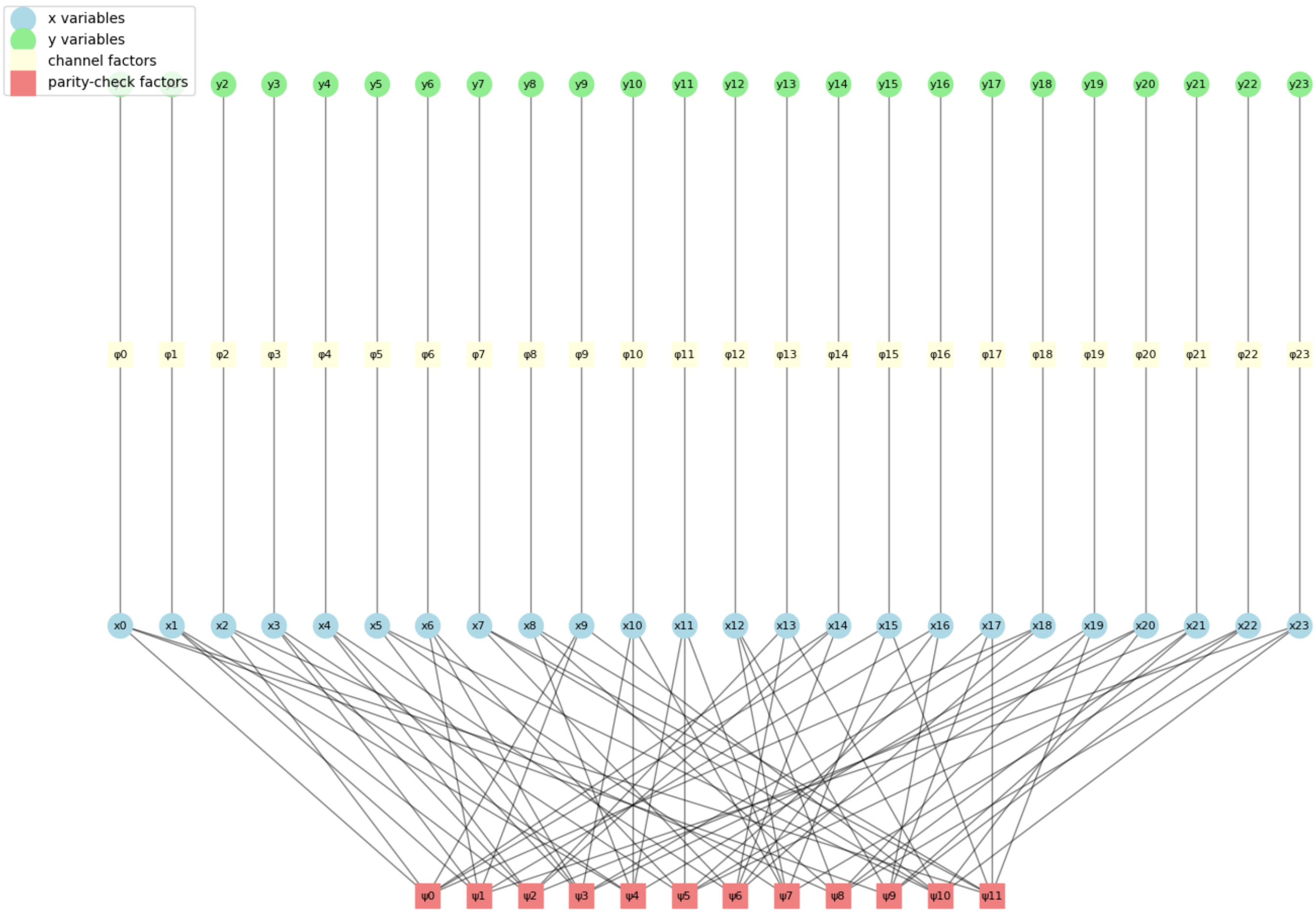
You can use `networkx` and `matplotlib.pyplot`, and all the functions from that library such as `nx.draw_networkx_nodes`

```python
import matplotlib.pyplot as plt
import networkx as nx
from pgmpy.models import FactorGraph

def visualize_factor_graph(G: FactorGraph, max_nodes: int = 200):
    """Visualize an LDPC factor graph with a horizontal layout.

    Rows (top to bottom):
      y variables → channel factors → x variables → parity-check factors
    """
    Hnx = nx.Graph()

    # Separate nodes by type and sort them
    x_vars = sorted([node for node in G.get_variable_nodes() if node.startswith('x')],
```

```
                key=lambda v: int(v[1:]))
y_vars = sorted([node for node in G.get_variable_nodes() if node.startswith('y')],
                key=lambda v: int(v[1:]))
parity_factors = [f for f in G.get_factors() if all(var.startswith('x') for var in f.scope())]
channel_factors = [f for f in G.get_factors() if any(var.startswith('y') for var in f.scope())]

# Sort channel factors by their y variable index
channel_factors = sorted(channel_factors,
                         key=lambda f: int([v for v in f.scope() if v.startswith('y')][0][1:]))

# Add all nodes
Hnx.add_nodes_from(y_vars)
Hnx.add_nodes_from(channel_factors)
Hnx.add_nodes_from(x_vars)
Hnx.add_nodes_from(parity_factors)

# Add edges from the original graph
for edge in G.edges():
  Hnx.add_edge(*edge)

# Create positions: 4 rows (horizontal layout)
pos = {}
n = len(x_vars)
m = len(parity_factors)
row_height = 2

# Row 0 (top): y variables - sorted ascending left to right, lowered to avoid legend overlap
for i, var in enumerate(y_vars):
  pos[var] = (i, -row_height)

# Row 1: channel factors - aligned with corresponding y and x
for i, factor in enumerate(channel_factors):
  pos[factor] = (i, -2 * row_height)

# Row 2: x variables - aligned with corresponding y and channel factors
for i, var in enumerate(x_vars):
```

```python
  pos[var] = (i, -3 * row_height)

# Row 3 (bottom): parity-check factors - centered
parity_start_x = (n - m) / 2  # Center the parity factors
for i, factor in enumerate(parity_factors):
  pos[factor] = (parity_start_x + i, -4 * row_height)

# Draw the graph
fig, ax = plt.subplots(figsize=(max(12, n * 0.6), 10))

# Draw nodes
nx.draw_networkx_nodes(Hnx, pos, nodelist=x_vars, node_color='lightblue',
             node_size=300, label='x variables', ax=ax)
nx.draw_networkx_nodes(Hnx, pos, nodelist=y_vars, node_color='lightgreen',
             node_size=300, label='y variables', ax=ax)
nx.draw_networkx_nodes(Hnx, pos, nodelist=channel_factors, node_color='lightyellow',
             node_shape='s', node_size=300, label='channel factors', ax=ax)
nx.draw_networkx_nodes(Hnx, pos, nodelist=parity_factors, node_color='lightcoral',
             node_shape='s', node_size=300, label='parity-check factors', ax=ax)

# Draw edges
nx.draw_networkx_edges(Hnx, pos, ax=ax, alpha=0.5)

# Create labels: use variable names for x/y, simple labels for factors
labels = {}
for var in x_vars + y_vars:
    labels[var] = var
for i, factor in enumerate(channel_factors):
    labels[factor] = f'{i}'
for i, factor in enumerate(parity_factors):
    labels[factor] = f'{i}'

nx.draw_networkx_labels(Hnx, pos, labels=labels, font_size=8, ax=ax)

ax.legend(loc='upper left')
ax.axis('off')
```

```
    plt.tight_layout()
    plt.show()


H = generate_regular_ldpc_H(N=24, j=3, k=6, seed=1)
G_test = build_ldpc_factor_graph(H, f=0.8)
visualize_factor_graph(G_test)
```


x variables
y variables
channel factors
parity-check factors
y2 y3 y4 y5 y6 y7 y8 y9 y10 y11 y12 y13 y14 y15 y16 y17 y18 y19 y20 y21 y22 y23
φ0 φ1 φ2 φ3 φ4 φ5 φ6 φ7 φ8 φ9 φ10 φ11 φ12 φ13 φ14 φ15 φ16 φ17 φ18 φ19 φ20 φ21 φ22 φ23
x0 x1 x2 x3 x4 x5 x6 x7 x8 x9 x10 x11 x12 x13 x14 x15 x16 x17 x18 x19 x20 x21 x22 x23
ψ0 ψ1 ψ2 ψ3 ψ4 ψ5 ψ6 ψ7 ψ8 ψ9 ψ10 ψ11

Part 6: Treewidth scaling with block length N

To study how inference complexity scales with the block length N, we analyze the **treewidth** of the LDPC code.

We first construct the **primal graph** on the transmitted bits $x_0, \dots, x_{N-1}$: this is an undirected graph in which two bits are connected by an edge if they appear together in any parity-check factor. It is analogous to moralization, but applied to a factor graph rather than a Bayesian network.

Since computing treewidth exactly is NP-hard, we estimate it using a **min-fill heuristic**, which provides an upper bound on the treewidth.

Use `networkx.algorithms.approximation.treewidth_min_fill_in` to estimate the treewidth of the primal graph for increasing values of N, and plot the result. When do you think inference will start being intractable?

```python
from networkx.algorithms.approximation import treewidth_min_fill_in

def ldpc_primal_graph_on_x(G):
    """Build the primal graph on x-variables only.

    Nodes: x0..x{N-1}
    Edge between xi and xj if some parity-check factor contains both.
    """
    # TODO:
    # - create an undirected NetworkX graph
    # - add one node per x-variable
    # - for each parity-check factor, connect all x-variables that appear together
    raise NotImplementedError


def treewidth_scaling_experiment(N_list, j, k, f=0.08, seed0=0):
    """Estimate treewidth scaling for increasing block lengths N.

    Returns a list of tuples (N, estimated_treewidth, num_edges).
    """
    # TODO:
    # - for each N:
    #   - generate a regular LDPC parity-check matrix H
    #   - build the LDPC factor graph
```

```
    #   - construct the primal graph on x-variables
    #   - estimate treewidth using networkx.treewidth_min_fill_in
    # - return the collected results
    raise NotImplementedError

N_list = [12, 18, 24, 30, 36, 42, 50]
j, k = 3, 6
tw_results = treewidth_scaling_experiment(N_list, j=j, k=k, f=0.08, seed0=0)
tw_results
```

```
---------------------------------------------------------------------------
NotImplementedError                       Traceback (most recent call last)
Cell In[12], line 32
     30 N_list = [12, 18, 24, 30, 36, 42, 50]
     31 j, k = 3, 6
---> 32 tw_results = treewidth_scaling_experiment(N_list, j=j, k=k, f=0.08, seed0=0)
     33 tw_results

Cell In[12], line 28, in treewidth_scaling_experiment(N_list, j, k, f, seed0)
     17 """Estimate treewidth scaling for increasing block lengths N.
     18
     19 Returns a list of tuples (N, estimated_treewidth, num_edges).
     20 """
     21 # TODO:
     22 # - for each N:
     23 #   - generate a regular LDPC parity-check matrix H
   (...)
     26 #   - estimate treewidth using networkx.treewidth_min_fill_in
     27 # - return the collected results
---> 28 raise NotImplementedError

NotImplementedError:
```