Math 320: Chapter 1

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1 Common sets

- Natural numbers $\mathbb{N}$ (Rudin uses $\mathbb{J}$)
  - Set notation $\{1, 2, 3, 4, \ldots\}$
  - Closed under addition and multiplication
  - Not closed under subtraction (might get a negative number)
- Integers $\mathbb{Z}$
  - Set notation $\ldots - 3, -2, -1, 0, 1, 2, 3, 4, \ldots$
  - Closed under subtraction and addition
  - But not under division
- Rationals $\mathbb{Q}$
  - Set notation: $\left\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$
    * Where $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ iff $m_1 * n_2 = m_2 * n_1$
    * Our intuitive idea of division
  - Alternative we could define $\mathbb{Q}$ in a more straightforward way
    * Define $\mathbb{Q}$ as the set of ordered pairs $\{(m, n) : m \in \mathbb{Z}, n \in \mathbb{N}\}$
    * We also need to make sure the equivalent fractions are accounted for
    * where (m_1, n_1) is equivalent to (m_2, n_2) (ie $(m_1, n_1) \sim (m_2, n_2)$) if $m_1 n_2 = m_2 n_1$
  - Closed under addition, subtraction, multiplication, and division (if divisor is non-zero)
  - Question: Are the rationals sufficient for everything we want to do in real analysis (in calculus)?
    * Can we do things in calculus?
    * Can we take limits? (and have them work properly?)
  - Nope! (note: the name of this course is **real** analysis, not **rational** analysis

– The problem is that the rationals have holes, they're not "filled in all the way" like the reals are

Example: Holes in the rationals

(Rudin 1.1a) We want to show that there's a number we can't reach in the rationals ($\sqrt{2}$). $\nexists p \in \mathbb{Q} : p^2 = 2$

Proof:

By contradiction: Suppose that $\exists p \in \mathbb{Q} : p^2 = 2$. Then since p is rational, we can write $p = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

- WLOG we may suppose that m and n are not both even (because if not then we could just divide both by 2 and have a new m and n and repeat until this statement is true)
  - Q: How can we be sure that this dividing by 2 will eventually end? (More explicitly, how many times do we need to do it?)
- We have $2 = p^2 = \frac{m^2}{n^2}$

$$2 = \frac{m^2}{n^2} \tag{1}$$

$$m^2 = 2n^2 \tag{2}$$

- So m must be even
- Ie $m = 2k$ for some odd integer k

$$2n^2 = m^2 \tag{3}$$

$$2n^2 = (2k)^2 \tag{4}$$

$$n^2 = 2k^2 \tag{5}$$

- So n must be even, but we said that one of them had to be odd.
- Contradiction!

Example:

(Rudin 1.1b)

- Let $A = \left\{p \in \mathbb{Q} : p > 0, p^2 < 2\right\}$
- Let $B = \left\{p \in \mathbb{Q} : p > 0, p^2 > 2\right\}$
- Consider the area where the two sets meet. We know that they meet at $\sqrt{2}$, which is not a rational.
- So we know that the set of rationals has little holes in it, like $\sqrt{2}$ which make us unable to use limits, which are important for real analysis

Then:

- $\forall p \in A, \exists q \in A : p < q$ (A does not have a largest element)
- Similarly B does not have a smallest element ($\forall p \in B, \exists q \in B : q < p$

Proof:

First one

- Consider arbitrary $p \in A$. Try $q = \frac{2p+2}{2+p}$
- Check $q \in \mathbb{Q}$, the denominator is non-zero and rational, so the result is rational.
- Check $q > 0$, both numerator and denominator are positive
- Check $q^2 < 2$.

$$q^2 = \frac{(2p+2)^2}{(2+p)^2} \tag{6}$$

$$= \frac{2(p^2-2)}{(p+2)^2} + 2 \tag{7}$$

- The fraction is less than zero, because the numerator is negative ($p^2 < 2$ because $p \in A$), so $q^2 < 2$)
- Check that $p < q$. Well $q = p + \frac{2-p^2}{2+p}$, which is positive, so $q > p$

Second one

- Exercise (Try the exact same choice of q)

Q: Where did that q come from? (Think calculus)

Q: Why do we care so much that there is $q^2 = 2$, but don't care that there isn't a $q^2 = -1$?

1.1 Ordered Sets

Definition: Set order

An **order** **($<$)** on a set S is a relation st

- Every pair of distinct elements $x, y \in S$, exactly one of $x < y$, $x > y$, $x = y$ is true
- Transitivity: For $x, y, z \in S$, $x < y$ $y < z$ implies $x < z$

Definition: Ordered set

A pair $(S, <)$ of an order and a set (duh)

- Notation: Just S if the order is obvious from the context

Example: Ordered sets

With ordering $x < y$ if $y - x$ is positive

- $S = \mathbb{N}$
- $S = \mathbb{Z}$
- $\mathbb{Q}$, $\mathbb{R}$ etc

Example: English set

S as the set of English words, $<$ is the dictionary order (by letters)

- "a" $<$ "aa" $<$ "ab" $<$ "b"

Notation:

- $x < y$ and $y > x$ are the same
- $x \leq y$ means $x < y$ or $x = y$

Definition: Bounded above

Let S be an ordered set and E be a subset of S ($E \subset S$)

We say E is **bounded above** if there exists an element $\theta \in S : \forall x \in E : x \leq \theta$

θ is an **upper bound(ub)** of E

- Note: This requires a "universal bounding set" or else the idea of being bounded above doesn't make sense

Example:

Let $S = \mathbb{Q}$ with the standard ordering. Let $E = A$ (from the last section, $0 < a$, $a^2 < 2$)

- Then E is bounded above by $p = 2$ (or really any element $p > \sqrt{2}$)

For $p \in E$, check $2 - p$

$$\begin{aligned} 2 - p &= \frac{4 - p^2}{(2 + p)} && (8) \\ &> \frac{4 - 2}{2 + p} && (9) \\ &> 0 && (10) \end{aligned}$$

Example:

Let $S = A$ and $E = A$ (not a proper subset), in this case E is not bounded above

- We know that for any element $\theta \in E$ there exists another element in E that is larger (from last lecture)

Definition: Bounded below

Let S be an ordered set, $E \subset S$. E is **bounded below** if $\exists \beta \in S : \forall x \in E : \beta \leq x$

- β is called a **lower bound (lb)**

1.2 Least upper bounds and greatest lower bounds

Definition: Least upper bound (LUB)

Let S be an ordered set, subset E bounded above. If $\exists \alpha \in S$ such that

- α is a UB for E
- If $\gamma < \alpha$, then γ is NOT a UB for E

Then α is called the **least upper bound (LUB)** or **supremum**

- Notation: $\alpha = sup(E)$

Remark: Why can we say that α is THE supremum? How do we know that it's unique?

Definition: Greatest lower bound (GLB)

Similarly the **greatest lower bound** or **infenum** is the element α (if it exists) st

- α is a LB for E
- $\gamma > \alpha$ then gamma is not a LB for E
- Notation: $\alpha = inf(E)$

Example:

Let $S = \mathbb{Q}$ with the normal ordering. $E = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

- What is the supremum? $sup(E) = 1$
- What is the infenum? $inf(E) = 0$

Things to check:

- Are they rational (in the universal set S)? Well yes
- Are they a UB/LB? Yes
- The hard part: Prove that they are the greatest/least lower/upper bound (todo)

Note:

- E contains its supremum, but not its infenum

Definition: Least upper bound property (LUB property)

An ordered set S has the LUB property if $\forall E \subset S$ where E is not the empty set, and E is bounded above, then E has a least upper bound (in S)

- All subsets of S that are bounded above, have a LUB

There is also a parallel definition for the **greatest upper bound property**.

Example:

Does $\mathbb{Z}$ have the LUB property?

What about $\mathbb{Q}$?

- How would you go about proving it?
  - Not having the statement is more straightforward because you can just find a counter-example
  - Considering arbitrary subsets is more difficult

1. I think the answers are yes and no.

Theorem:

(Rudin 1.11) Let S be an ordered set.

- S has the LUB property $\iff$ S has the GLB property

Proof:

Forward:

Let S be an ordered set with the LUB property. (WTS S has the GLB property).

Let $E \subset S$ with E nonempty and bounded below (assumptions for the GLB property) (wts E has an infenum)

Let $L = \{x \in S : \text{x is a LB for } E\}$. L is non-empty because E is bounded below

If $y \in E$, then y is an UB for L. (Because all elements of L are less than all elements of E). Since E is non-empty, L is bounded above.

So now we have set $L \subset S$ that is non-empty and bounded above, because S has the LUB property, L has supremum α. Claim that this α is the infenum of E.

$\alpha \leq x : \forall x \in E$, hence α is a lower bound for E, so α is an element of L (why?)

- $\forall \gamma \in S$, $\gamma < \alpha$ implies γ is not an upper bound of L.
- Since all values in E are upper bounds of L, $\gamma \notin E$.
- So since being less than α implies it is not in E, it follows that $\forall x \in E : \alpha \leq x$

Since α is the supremum of L, $\alpha \geq \gamma : \forall \gamma \in L$. Since L is the set of all upper bounds of E, we get that α is the greatest lower bound of E

□

Backward: (exercise, should be very similar)

Remark: General proof notes:

- Use the structure and values that we have access to to generate the desired values. Ie use facts about bounded above/below, use the LUB property to get a solid value in S, the supremum of L.
- The fact that we know so little about S and E makes things easier, because there only are so many things you can try to create values from