Rings and ideals

rctcwyvrn

August 2020

1 Rings and ideals

1.1 Motivational metaphors

Term	Groups	Rings
Notation	G	R
Operations	$\cdot$	$+, \times$
Commutativity	abelian only	for us, always
Sub-structure	subgroup	(not discussed)
Kernel	normal subgroup	ideal
Quotient	G/H	R/I

Other motivational notes

- We want to try to generalize the properties we see in $\mathbb{Z}$ to any abelian group where we can also multiply
- So we can talk about ”primes” (irreducible polynomials) in a polynomial ring
- Note: Lots of interesting overlap/re-interpretation of this chapter in alg geo so keep that in mind

1.2 Rings

Definition: Ring

A **ring** is a triple $(R, +, \times)$ such that

- $(R, +)$ is an abelian group, with identity 0_R (notation: Just 0)
- $\times$ is a binary associative operator, with an identity 1_R
- Multiplication distributes over addition

Also

- R is **commutative** if multiplication is commutative
- Abbreviate $(R, +, \times)$ to just R
- For simplicity, assume all rings are commutative for the rest of this chapter
- $+$ and $\times$ are usually just referred to as addition and multiplication

Remark: Some ring facts

- For all $r \in R$, $r \times 0_R = 0_R$ and $r \times (-1_R) = -r$

Example: Typical rings

- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all rings with the usual addition and multiplication.
- The integer mod n are also a ring with the usual operators (call it $\mathbb{Z}/n\mathbb{Z}$

Definition: Zero ring

The **zero ring** is the ring with a single element. Denote with 0. A ring is **nontrivial** if it is not the zero ring

- Note: A ring is trivial iff $0_R = 1_R$ (duh)

Example: Product ring

Given two rings R, S, the **product ring** $(R \times S)$, is defined as you would expect, the operators go to their respective pairs

Example: Polynomial ring

Given any ring R, the **polynomial ring** $R[x]$ is defined as the set of polynomials with coefficients in R:

$$R[x] = \{\}.$$

- I'm too lazy to type it all out but it's defined as you would expect, polynomials of arbitrary order with coefficients in R
- Addition and multiplication are done exactly as expected, distribute n stuff
- Called "R adjoin x"