Taylor series

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1 Taylor series (review)

A power series is

$$S(x) = a_0 + a_1x + a_2x^2 \ldots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Say we know all the derivatives of a function $f(x)$, can we determine a power series?

Let $f(x) = a_0 + a_1x + a_2x^2 \ldots$ and solve for a_i.

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots \tag{2}$$
$$f''(x) = 2a_2 + 6a_3x \ldots \tag{3}$$
$$f^n(x) = n!a_n + (n+1)\ldots 2 * a_{n+1} \tag{4}$$

Sub in $x = 0$ to determine all the coefficients a_i

$$f'(0) = a_1 \tag{5}$$
$$f''(0) = 2a_2 \tag{6}$$
$$f^n(0) = n!a_n \tag{7}$$

So filling in the coefficients we get the **Taylor series**

$$f(x) = f(0) + \frac{f'(0)}{1!}x \ldots \frac{f^n(0)}{n!}x^n \ldots.$$

In sum notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n.$$

Definition: Hyperbolic trig

$$\sinh = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots \tag{8}$$
$$\cosh = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots \tag{9}$$

The series are just the sin and cos ones, but with no negative signs

- See notebook for graphical representation of sinh and cosh

Now back to ODE stuff

Example: Generating the derivatives of the solution from the ODE

Given $y' = 2y$, $y(0) = 4)$, we know that $y'' = 2y' = 4y$, continuing on we get that $y^n = 2^n y$

Now write out the taylor series

$$y(x) = \sum_{n=0}^{\infty} \frac{y^n(0)}{n!} x^n \tag{10}$$
$$= \sum_{n=0}^{\infty} \frac{2^n y(0)}{n!} x^n \tag{11}$$
$$= 4 \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \tag{12}$$
$$= 4e^{2x} \tag{13}$$

And we can check that this is correct by just solving the ODE (which is separable)