Taylor series

rctcwyvrn

August 2020

1 Taylor series (review)

A power series is

$$S(x) = a_0 + a_1x + a_2x^2 \ldots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Say we know all the derivatives of a function $f(x)$, can we determine a power series?

Let $f(x) = a_0 + a_1x + a_2x^2 \ldots$ and solve for a_i.

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots \tag{2}$$
$$f''(x) = 2a_2 + 6a_3x \ldots \tag{3}$$
$$f^n(x) = n!a_n + (n+1)\ldots 2 * a_{n+1} \tag{4}$$

Sub in $x = 0$ to determine all the coefficients a_i

$$f'(0) = a_1 \tag{5}$$
$$f''(0) = 2a_2 \tag{6}$$
$$f^n(0) = n!a_n \tag{7}$$

So filling in the coefficients we get the **Taylor series**

$$f(x) = f(0) + \frac{f'(0)}{1!}x \ldots \frac{f^n(0)}{n!}x^n \ldots.$$

In sum notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n.$$

Definition: Hyperbolic trig

$$\sinh = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} \dots \tag{8}$$

$$\cosh = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots \tag{9}$$

The series are just the sin and cos ones, but with no negative signs

- See notebook for graphical representation of sinh and cosh

Now back to ODE stuff

Example: Generating the derivatives of the solution from the ODE

Given $y' = 2y$, $y(0) = 4)$, we know that $y'' = 2y' = 4y$, continuing on we get that $y^{(n)} = 2^n y$

Now write out the taylor series

$$y(x) = \sum_{n=0}^{\infty} \frac{y^n(0)}{n!} x^n \tag{10}$$

$$= \sum_{n=0}^{\infty} \frac{2^n y(0)}{n!} x^n \tag{11}$$

$$= 4 \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \tag{12}$$

$$= 4e^{2x} \tag{13}$$

And we can check that this is correct by just solving the ODE (which is separable)

But is there a more general way of doing this (getting a Taylor series from a DE)?

Example: Undetermined coefficients

Assume that there is a Taylor series, and figure out the coefficients from the DE.

$$Ly = y' - 2y = 0.$$

Assume y has a power series $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \tag{14}$$

Now substitute them into Ly

$$\begin{aligned} Ly &= y' - 2y = 0 && (15)\\ &= 1a_1x^0 + 2a_2x + 3a_3 + x^2 \ldots - 2(a_0 + a_1x + a_2x^2 \ldots) && (16)\\ &= (a_1 - 2a_0)x^0 + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \ldots = 0 && (17) \end{aligned}$$

This should hold for all values of x, so each of those coefficients should be zero.

- Alternatively we can think of x, $x^2 \ldots$ as linearly independent values, and the sum is equal to zero, which implies that the coefficients are zero

$$\begin{aligned} a_1 &= 2a_0 && (18)\\ a_2 &= \frac{2a_1}{2} = 2^2\frac{a_0}{2} && (19)\\ a_3 &= \frac{2a_2}{3} = 2^3\frac{a_0}{3!} && (20)\\ \ldots & && (21)\\ a_n &= 2^n\frac{a_0}{n!} && (22)\\ & && (23) \end{aligned}$$

So we now have

$$\begin{aligned} y(x) &= a_0(1 + 2x + \ldots \frac{2^n}{n!}x^n && (24)\\ &= a_0e^{2x} && (25) \end{aligned}$$

Try it again, this time with just sum notation instead

Example:

$y = \sum_{n=0}^{\infty} a_n x^n$, so $y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$

$$\begin{aligned} Ly &= y' - 2y = 0 && (26) \\ &= \sum_{n=1}^{\infty} a_n n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n && (27) \\ &= \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m - 2\sum_{m=0}^{\infty} a_m x^m && (28) \\ &= \sum_{m=0}^{\infty} (a_{m+1}(m+1) - 2a_m)x^m && (29) \\ & && (30) \end{aligned}$$

By the same argument as before, all these coefficients should be 0,

$$a_{m+1} = \frac{2a_m}{m+1}.$$

which gives us a recursive formula for the coefficients. Start at $m = 0$

$$\begin{aligned} a_1 &= \frac{2a_0}{1} && (31) \\ a_2 &= \frac{2a_1}{2} = \frac{2^2 a_0}{2} && (32) \\ a_3 &= \frac{2^3 a_0}{3!} && (33) \\ \ldots a_n &= \frac{2^n a_0}{n!} && (34) \end{aligned}$$

And then we end up at the same solution, $y(x) = a_0 e^{2x}$

Now for a more complex example

Example: Ainy Equation

$$Ly = y'' - xy = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$, $y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$

$$\begin{aligned} Ly &= y'' - xy = 0 && (35) \\ &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} && (36) \end{aligned}$$

Since we have the n+1 on the right summation, we want to shift the left sum to match, so set it to $m+1 = n-2$

$$\begin{aligned} &= a_2(2)(1) + \sum_{m=0}^{\infty} a_{m+3}(m+3)(m+2)x^{m+1} - a_m x^{m+1} && (37) \\ &= 2a_2 + \sum_{m=0}^{\infty} (a_{m+3}(m+3)(m+2) - a_m)x^{m+1} && (38) \end{aligned}$$

So from that we can solve the coefficients:

$$\begin{aligned} a_2 &= 0 && (39) \\ a_{m+3} &= \frac{a_m}{(m+3)(m+2)} && (40) \\ m = 0 : a_3 &= \frac{a_0}{3*2} && (41) \\ m = 3 : a_6 &= \frac{a_3}{6*5} = \frac{a_0}{6*5*3*2} && (42) \\ m = 1 : a_4 &= \frac{a_1}{4*3} && (43) \\ m = 4 : a_7 &= \frac{a_4}{7*6} = \frac{a_1}{7*6*4*3} && (44) \\ m = 2 : a_5 &= 0 && (45) \\ m = 5 : a_8 &= 0 && (46) \end{aligned}$$

The general solution is the two groups of coefficients, namely

$$y(x) = a_0(1 + \frac{1}{3*2}x^3 + \frac{1}{6*5*3*2}x^5 \ldots) + a_1(1 + \frac{1}{4*3}x^4 + \frac{1}{7*6*4*3}x^7 \ldots).$$