Lecture 1: A review of ODE techniques

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1   Introduction

What is a differential equation?
A differential equation is an equation that implicitly defines a function by giving a relationship between it and its derivatives

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Definition: ODE

An ordinary nth order differential equation is

$$f(x, y(x), y'(x) \ldots y^{(n)}(x)) = 0.$$

A solution is a function $y(x)$

Definition: Partial differential equation

The most general 2nd order PDE is

$$f(x, y, u(x, y), u_x, u_y, u_{xx}, u_{yy}.$$

A solution is a surface $z = u(x, y)$

Types of equations

1. First order equations

   (a) Separable $\frac{dy}{dx} = P(x)Q(y)$

   (b) Linear $\frac{dy}{dx} + P(x)y = Q(x)$

2. Second order equations

   (a) Constant coefficient $Ly = ay'' + by' + cy = 0$ constants a, b, c

   (b) Cauchy-Euler (Equidimensional equations) $Ly = x^2 y'' + axy' + by = 0$

2 First order equations

2.1 Separable equations

$$\frac{dy}{dx} = P(x)Q(y).$$

We can separate the variables and rewrite it as

$$\int \frac{dy}{Q(y)} = \int P(x)dx.$$

and then integrate as usual

2.2 Linear equations

$$Ly = \frac{dy}{dx} + P(x)y = Q(x).$$

Not separable because of the $P(x)y$

- Recall the product rule $\frac{d}{dx}(F(x)y) = F\frac{dy}{dx} + F'y$

- We can now multiply the ODE by a function F to get it to match the product rule. ie $F' = FP(x)$, which is a separable equation.

$$\ln(F) = \int P(x)dx + C.$$

- We get that $F = e^{\int P(x)dx + c} = Ae^{\int P(x)dx}$. Call it the **integrating factor**

So now multiply everything by the integrating factor

$$FLy = Ae^{\int P(x)dx}\frac{dy}{dx} + Ae^{\int P(x)dx}P(x)y = Ae^{\int P(x)dx} \tag{1}$$

Divide out all the As and integrate the product

$$= \frac{d}{dx}(e^{\int Pdx}y) = e^{\int Pdx}Q(x) \tag{2}$$

Now the equation is separated, so just integrate again and you get the solution (it's gross so I'm not going to write it all out and it's stupid to try and memorize it anyway)

3 Second order constant coefficient linear ODEs

$$Ly = ay'' + by' + cy = 0.$$

What is L? L is a differential operator $L = a\frac{d^2}{dx} + b\frac{d}{dx} + c$ Q: What function has a derivative that differs by a constant? ie $y' = \lambda y$ A: Solve it as a first order linear ODE.

$$y' - \lambda y = 0 \tag{3}$$
$$e^{-\lambda x}y' - \lambda e^{-\lambda x}y = 0 \tag{4}$$
$$e^{-\lambda x}y = C \tag{5}$$
$$y = Ce^{\lambda x} \tag{6}$$

Knowing this, let's guess a solution $y(x,r) = e^{rx}$ (r is a parameter which makes $y(x)$ a solution

$$Ly = (ar^2 + br + c)e^{rx} = 0.$$

So now for this to work, the polynomial must be 0. Solve $\phi(r) = ar^2 + br + c = 0$, use the quadratic formula Cases:

1. Two distinct real roots r_1, r_2, which each has a corresponding solution. We know that the addition of any two solutions and the multiplication of a solution by a constant still makes a solution, so the general solution is

$$y(x) = Ae^{r_1 x} + Be^{r_2 x}.$$

2. One root r. It's pretty straightforward to check that in this case, we have a second solution $y = xe^{rx}$

$$y(x) = Ae^{rx} + Bxe^{rx}.$$

3. Two complex roots c_1, c_2, we can rewrite them as $\lambda \pm i\mu$. So the general solution is

$$y(x) = Ae^{(\lambda+i\mu)x} + Be^{(\lambda-i\mu)x} \tag{7}$$
$$= e^{\lambda x}\left[Ae^{i\mu x} + Be^{-i\mu x}\right] \tag{8}$$
$$= e^{\lambda x}\left[A(\cos(\mu x) + i\sin(\mu x)) + B(\cos(\mu x) - i\sin(\mu x))\right] \tag{9}$$
$$= e^{\lambda x}\left[(A+B)\cos(\mu x) + i(A-B)\sin(\mu x)\right] \tag{10}$$
$$= e^{\lambda x}\left[C\cos(\mu x) + iD\sin(\mu x)\right] \tag{11}$$

4 Cauchy-Euler/Equidimensional equations

$$Ly = x^2 y'' + axy' + by = 0.$$

Why is it called equidimensional? Because each term in the differential equation has the same units (the units of y)

Look for $y : x\frac{dy}{dx} = ry$, that's separable, so solve it

$$\int \frac{1}{y}dy = r\int \frac{1}{x}dx \qquad (12)$$

$$\ln y = r\ln x + c \qquad (13)$$

$$y = x^r + C \qquad (14)$$

Guess that this will be a nice solution, find r st $y(x, r) = x^r$ is a solution.

$$y' = rx^{r-1} \qquad (15)$$

$$y'' = r(r-1)x^{r-2} \qquad (16)$$

Plug into Ly

$$Ly = (r(r-1) + ar + b)x^r \qquad (17)$$

$$\qquad (18)$$

That's a solution if the polynomial is $= 0$

$$r(r-1) + ar + b = 0 \qquad (19)$$

$$r^2 + (a-1)r + b = 0 \qquad (20)$$

$$r_1, r_2 = -\frac{a-1}{2} \pm \frac{\sqrt{(a-1)^2 - 4b}}{2} \qquad (21)$$

Cases

1. 2 distinct real roots. General solution is

$$y(x) = Ax^{r_1} + Bx^{r_2}.$$

2. One double root $r_0 = -\frac{a-1}{2}$. (discriminant $= 0$)

$$Ly = (r(r-1) + ar + b)x^r \qquad (22)$$

$$= ((r + \frac{a-1}{2})^2 - \frac{(a-1)^2 - 4b}{4})x^r \qquad (23)$$

The numerator is just the discriminant, so

$$= (r - (-\frac{a-1}{2}))^2 x^r \qquad (24)$$

$$= (r - r_0)^2 x^r \qquad (25)$$

Now do some black magic and derive wrt r

$$\frac{d}{dr}Ly = 2(r - r_0)x^r + (r - r_0)^2(\ln(x)x^r) \qquad (26)$$

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We see that $r = r_0$ gives us a solution (??)

$$\frac{d}{dr}y(x,r)|_{r=r_0} = x^{r_0}\ln x \tag{27}$$

??? Well that's apparently the other solution. So the general solution is

$$y(x) = Ax_0^r + Bx_0^r \ln x.$$

3. Discriminant < 0, complex pair $\lambda \pm i\mu$

$$y(x) = c_1 x^{\lambda+i\mu} + c_2 x^{\lambda-i\mu} \tag{28}$$
$$= x^\lambda \left[c_1 e^{i\mu\ln x} + c_2 e^{-i\mu\ln x} \right] \tag{29}$$

Use Euler's formula and group terms

$$= x^\lambda \left[(c_1 + c_2)\cos(\mu\ln x) + i(c_1 - c_2)\sin(\mu\ln x) \right] \tag{30}$$
$$= x^\lambda \left[A\cos(\mu\ln x) + B\sin(\mu\ln x) \right] \tag{31}$$