

# Math 320: Chapter 1

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## 1 Common sets

- Natural numbers  $\mathbb{N}$  (Rudin uses  $\mathbb{J}$ )
  - Set notation  $\{1, 2, 3, 4, \dots\}$
  - Closed under addition and multiplication
  - Not closed under subtraction (might get a negative number)
- Integers  $\mathbb{Z}$ 
  - Set notation  $\dots - 3, -2, -1, 0, 1, 2, 3, 4, \dots$
  - Closed under subtraction and addition
  - But not under division
- Rationals  $\mathbb{Q}$ 
  - Set notation:  $\{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ 
    - \* Where  $\frac{m_1}{n_1} = \frac{m_2}{n_2}$  iff  $m_1 * n_2 = m_2 * n_1$
    - \* Our intuitive idea of division
  - Alternative we could define  $\mathbb{Q}$  in a more straightforward way
    - \* Define  $\mathbb{Q}$  as the set of ordered pairs  $\{(m, n) : m \in \mathbb{Z}, n \in \mathbb{N}\}$
    - \* We also need to make sure the equivalent fractions are accounted for
      - \* where  $(m_1, n_1)$  is equivalent to  $(m_2, n_2)$  (ie  $(m_1, n_1) \sim (m_2, n_2)$ ) if  $m_1 n_2 = m_2 n_1$
  - Closed under addition, subtraction, multiplication, and division (if divisor is non-zero)
  - Question: Are the rationals sufficient for everything we want to do in real analysis (in calculus)?
    - \* Can we do things in calculus?
    - \* Can we take limits? (and have them work properly?)
  - Nope! (note: the name of this course is **real** analysis, not **rational** analysis)

- The problem is that the rationals have holes, they're not "filled in all the way" like the reals are

**Example:** Holes in the rationals

(Rudin 1.1a) We want to show that there's a number we can't reach in the rationals ( $\sqrt{2}$ ).  $\nexists p \in \mathbb{Q} : p^2 = 2$

**Proof:**

By contradiction: Suppose that  $\exists p \in \mathbb{Q} : p^2 = 2$ . Then since  $p$  is rational, we can write  $p = \frac{m}{n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ .

- WLOG we may suppose that  $m$  and  $n$  are not both even (because if not then we could just divide both by 2 and have a new  $m$  and  $n$  and repeat until this statement is true)

- Q: How can we be sure that this dividing by 2 will eventually end? (More explicitly, how many times do we need to do it?)

- We have  $2 = p^2 = \frac{m^2}{n^2}$

$$2 = \frac{m^2}{n^2} \tag{1}$$

$$m^2 = 2n^2 \tag{2}$$

- So  $m$  must be even
- Ie  $m = 2k$  for some odd integer  $k$

$$2n^2 = m^2 \tag{3}$$

$$2n^2 = (2k)^2 \tag{4}$$

$$n^2 = 2k^2 \tag{5}$$

- So  $n$  must be even, but we said that one of them had to be odd.
- Contradiction!

**Example:**

(Rudin 1.1b)

- Let  $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$
- Let  $B = \{p \in \mathbb{Q} : p > 0, p^2 > 2\}$
- Consider the area where the two sets meet. We know that they meet at  $\sqrt{2}$ , which is not a rational.
- So we know that the set of rationals has little holes in it, like  $\sqrt{2}$  which make us unable to use limits, which are important for real analysis

Then:

- $\forall p \in A, \exists q \in A : p < q$  (A does not have a largest element)
- Similarly  $B$  does not have a smallest element ( $\forall p \in B, \exists q \in B : q < p$ )

**Proof:**

First one

- Consider arbitrary  $p \in A$ . Try  $q = \frac{2p+2}{2+p}$
- Check  $q \in \mathbb{Q}$ , the denominator is non-zero and rational, so the result is rational.
- Check  $q > 0$ , both numerator and denominator are positive
- Check  $q^2 < 2$ .

$$q^2 = \frac{(2p+2)^2}{(2+p)^2} \quad (6)$$

$$= \frac{2(p^2 - 2)}{(p+2)^2} + 2 \quad (7)$$

- The fraction is less than zero, because the numerator is negative ( $p^2 < 2$  because  $p \in A$ ), so  $q^2 < 2$
- Check that  $p < q$ . Well  $q = p + \frac{2-p^2}{2+p}$ , which is positive, so  $q > p$

Second one

- Exercise (Try the exact same choice of  $q$ )

Q: Where did that  $q$  come from? (Think calculus)

Q: Why do we care so much that there is  $q^2 = 2$ , but don't care that there isn't a  $q^2 = -1$ ?

## 1.1 Ordered Sets

**Definition:** Set order

An **order** ( $<$ ) on a set  $S$  is a relation st

- Every pair of distinct elements  $x, y \in S$ , exactly one of  $x < y$ ,  $x > y$ ,  $x = y$  is true
- Transitivity: For  $x, y, z \in S$ ,  $x < y$   $y < z$  implies  $x < z$

**Definition:** Ordered set

A pair  $(S, <)$  of an order and a set (duh)

- Notation: Just  $S$  if the order is obvious from the context

**Example:** Ordered sets

With ordering  $x < y$  if  $y - x$  is positive

- $S = \mathbb{N}$
- $S = \mathbb{Z}$
- $\mathbb{Q}, \mathbb{R}$  etc

**Example:** English set

$S$  as the set of English words,  $<$  is the dictionary order (by letters)

- "a"  $<$  "aa"  $<$  "ab"  $<$  "b"

Notation:

- $x < y$  and  $y > x$  are the same
- $x \leq y$  means  $x < y$  or  $x = y$

**Definition:** Bounded above

Let  $S$  be an ordered set and  $E$  be a subset of  $S$  ( $E \subset S$ )

We say  $E$  is **bounded above** if there exists an element  $\theta \in S : \forall x \in E : x \leq \theta$

$\theta$  is an **upper bound(ub)** of  $E$

- Note: This requires a "universal bounding set" or else the idea of being bounded above doesn't make sense

**Example:**

Let  $S = \mathbb{Q}$  with the standard ordering. Let  $E = A$  (from the last section,  $0 < a, a^2 < 2$ )

- Then  $E$  is bounded above by  $p = 2$  (or really any element  $p > \sqrt{2}$ )

For  $p \in E$ , check  $2 - p$

$$2 - p = \frac{4 - p^2}{(2 + p)} \quad (8)$$

$$> \frac{4 - 2}{2 + p} \quad (9)$$

$$> 0 \quad (10)$$

**Example:**

Let  $S = A$  and  $E = A$  (not a proper subset), in this case  $E$  is not bounded above

- We know that for any element  $\theta \in E$  there exists another element in  $E$  that is larger (from last lecture)

**Definition:** Bounded below

Let  $S$  be an ordered set,  $E \subset S$ .  $E$  is **bounded below** if  $\exists \beta \in S :$   
 $\forall x \in E : \beta \leq x$

- $\beta$  is called a **lower bound (lb)**

## 1.2 Least upper bounds and greatest lower bounds

**Definition:** Least upper bound(LUB)

Let  $S$  be an ordered set, subset  $E$  bounded above. If  $\exists \alpha \in S$  such that

- $\alpha$  is a UB for  $E$
- If  $\gamma < \alpha$ , then  $\gamma$  is NOT a UB for  $E$

Then  $\alpha$  is called the **least upper bound (LUB)** or **supremum**