Homomorphisms and quotient groups

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1 Homomorphisms and quotient groups

1.1 Generators and group presentations

We can imagine the subgroup generated by x as x being thrown in a box with itself and shook around. So what if we throw in more elements to be shook together?

Definition: Subsets as group generators

The subgroup generated by a subset S of G, $\langle S \rangle$, is the set of finite productive between elements of S and their inverses.

- $S = [a, b]$, then $\langle S \rangle$ is stuff like $abababa$, $a^5b^3ab^2$, $a^{-1}bab^{-100}$
- If $\langle S \rangle = G$, then S is a **set of generators** for G
- Notation: $\mathbb{Z} = \langle 1 \rangle$
- What if we know that x has a special condition? Notation $\mathbb{Z}/100\mathbb{Z} = \langle x | x^{100} = 1 \rangle$
- Basically that the group can be generated by any element that has the given property

Example: $\mathbb{Z}$

$\langle 1 \rangle = \mathbb{Z}$, because each integer can be written as a bunch of 1's or a bunch of -1's

Definition: Group presentation

We can define a group by a set of generators and **relations** between them. The **group presentation** is that expression

- We can then say that two elements of a group are equal iff you can get from one to the other with the relations.

Example: Dihedral group

The group presentation is

$$D_{2n} = \langle r, s | r^n = s^2 = 1, rs = sr^{-1} \rangle .$$

- This defines what the group is by defining the relationships between the generators

Example: Free group

The **free group on n elements** is the group with n generators and no relations.

$$F_n = \langle x_1, x_2, x_3 \ldots \rangle .$$

- Can basically be thought of as arbitrary units being thrown together
- $F_2 = \langle a, b \rangle$ is a bunch of a, b, a^{-1}, b^{-1} thrown together.
- $F_1 = \mathbb{Z}$. Why? Because $\mathbb{Z}$ is just the group made up by adding 1 and -1 to itself a bunch of times

Remark: The same group can have very different presentations, because a generator values can encompass two or more values of another generator set

1.2 Homomorphisms

How can we define relationships between groups that aren't just isomorphisms?

Definition: Homomorphism

For groups $(G, \star)$ and $(H, *)$, A **group homomorphism** is a map $\phi : G \to H$ where $\forall g_1, g_2 \in G$ we have

$$\phi(g_1 \star g_2) = \phi(g_1) * \phi(g_2).$$

- Like a linear map, but over groups instead of vector spaces
- Note the lack of bijection condition, we only need that the group action is respected

Remark: The right way to think about an isomorphism is as a "bijective homomorphism"

Example: Homomorphisms

- All isomorphisms are homomorphisms
- The identity map is a homomorphism
- The **trivial homomorphism** sends everything to 1_H
- From $\mathbb{Z}$ to $\mathbb{Z}/100\mathbb{Z}$ where you just mod everything by 100
- From $\mathbb{Z}$ to itself where you just multiply everything by 10
  - This map is injective, but not surjective
- From permutations S_n to S_{n+1} where you just keep the $n+1$th position constant.
  - Again, injective, but not surjective

Remark: Specifying a homomorphism from $\mathbb{Z} \to G$ is the same as just specifying what the image of 1 is. Because

$$\phi(n) = \phi(1) * \phi(1) \ldots = \phi(1)^n.$$

Remark: The last example shows something important.

To specify a homomorphism $G \to H$, we only have to specify where each generator of G goes. Making sure that the relations are still satisfied (?)

Lemma:

- $G \cong H$ iff there exists homomorphisms st $\phi \circ \psi = id_H$ and $\psi \circ \phi = id_G$
  - Proof: to do later
- Let ϕ be a homomorphism, then $\phi(1_G) = 1_H$ and $\phi(g^{-1}) = \phi(g)^{-1}$
  - Proof for the first one:

$$\phi(g \star 1_G) = \phi(g) * \phi(1_G) \tag{1}$$
$$\phi(g) = \phi(g) * \phi(1_G) \tag{2}$$
$$1_H = \phi(1_G) \tag{3}$$

Definition: Kernel

The **kernel** of a homomorphism is the subset of G that sends values to 1_H

- It also happens to be a (not necessarily proper) subgroup of G, because 1_G is always in the kernel and it is closed
- Notation: $\ker\phi$

Proposition: Kernel determines injectivity

ϕ is injective if and only if $\ker\phi = \{1_G\}$

Example: Kernels

- The kernel of an isomorphism is just 1_G
- The kernel of the trivial homomorphism (sending everything to 1_H is all of G (duh)
- The kernel of the map from Z to the cyclic group of size 100 is $100\mathbb{Z}$, namely all the integer multiples of 100. (Because the mod 100 of the map sends all of them to 0, the identity for the cyclic group
- $\phi : \mathbb{Z} \to G$ by $n \mapsto g^n$. The kernel then depends on g
  - If ord$g = \infty$, then the kernel is just 1
  - If ord$g = a$, then the kernel is $a\mathbb{Z} = \ldots a^{-2}, a^{-1}, 1, a, a^2 \ldots$

Remark: The image of a homomorphism forms a subgroup as well

1.3 Cosets and modding out

Here's the idea:

- Consider a surjective homomorphism $\phi : G \to Q$ that is not injective (kerϕ is non-trivial), what can we say about it?
- Consider a related case, $f : \mathbb{Z} \to \mathbb{Z}/100\mathbb{Z}$, the kernel is $100\mathbb{Z}$
  - We also then know that $f(x) = f(g + x)\ \forall g \in \ker\phi$
  - This basically means that f doesn't really care about elements of the subgroup $100Z$, it's **indifferent**.
  - Similarly, notice that for $N = 100\mathbb{Z}$,

$$N = \{\ldots -200, -100, 0, 100, 200 \ldots\} \tag{4}$$
$$1 + N = \{\ldots -199, -99, 1, 101, 201 \ldots\} \tag{5}$$
$$\ldots \tag{6}$$
$$99 + N = \{\ldots -101, -1, 99, 199, 299 \ldots\} \tag{7}$$

  - The image for each of those sets is the same, $\text{img}(g + N) = \{g\}$

Definition: Quotient groups

Let $\phi : G \to Q$ be a surjective homomorphism with kernel N (subgroup of G)

We claim that in this case, Q should be thought of as the **quotient** of G by N

- Notation: G/N

Remark: We can think of Q as the group whose elements are represented by the sets, ie for the $\mathbb{Z}/100\mathbb{Z}$ homomorphism, the elements of Q can be thought of as these sets

$$N = \{\ldots -200, -100, 0, 100, 200 \ldots\} \tag{8}$$
$$1 + N = \{\ldots -199, -99, 1, 101, 201 \ldots\} \tag{9}$$
$$\ldots \tag{10}$$
$$99 + N = \{\ldots -101, -1, 99, 199, 299 \ldots\} \tag{11}$$

- Note how there are exactly 100 of these sets, just like Q (which is $\mathbb{Z}/100\mathbb{Z}$ remember)
- If the homomorphism had been an isomorphism, then each one of those sets would have one value, and there would be exactly as many elements as the cardinality of G

Remark: We can also define an equivalence relation $\sim_N$ on G where $x \sim_N y$ iff $\phi(x) = \phi(y)$, ie they belong to the same set $a + N$

Definition: Left coset

Let H be any subgroup of G. A set of the form gH is called a **left coset** of H

Remark: $g_1 N$ is often equal to $g_2 N$ even if $g_1 \neq g_2$. ie $g_1 = 3$ and $g_2 = 103$ for the $\mathbb{Z}/100\mathbb{Z}$ group

Remark: Given cosets g_1H and g_2H, $x \mapsto g_2g_1^{-1}x$ is a bijection from $g_1H \to g_2H$ (Note this means all cosets have the same cardinality)

Remark: Elements of the quotient group Q are naturally identified with left cosets of the divisor group, N

- This is just a formalization of what was mentioned before, that we can think of the elements of the quotient group Q as those sets (which turned out to be the left cosets of N)

Definition: Normal groups

A subgroup N of G is **normal** if it is the kernel of some homomorphism.

- Notation: $N \trianglelefteq G$
- Equivalent definitions of a normal subgroup are
  - That the subgroup is closed under conjugation from G, ie $gng^{-1} \in N \ \forall g \in G$ and $n \in N$
  - That the cosets of N form a group (the quotient group)

Definition: Quotient groups again

Let $N \trianglelefteq G$, then the **quotient group**, denoted G/N (read: "G mod N") is defined as follows

- The elements of G/N are left cosets of N
- Define the product of two cosets as such
  - Recall that each coset corresponds to one value in G
  - Take the product of the cosets using those representatives
    - Let $C_1 = g_1N$ and $C_2 = g_2N$.
    - Then $C_1 \cdot C_2$ should be the coset g_1g_2H (the coset that contains g_1g_2)
- By this definition, G/N is isomorphic to Q (the old definition of a quotient group)

Some intuition about quotient groups

- The way they make the most sense to me is thinking of them as "the result of organizing G by N"
- Each coset is a set of values that have some property relating to N, (namely that they're all in g_1N)
- It then makes sense to refer to the resulting groupings of the elements as a collective, instead of by individual elements
- So the quotient group is just the labels on the groupings of values based on how they behave relative to N
- In the case of $\mathbb{Z}/100\mathbb{Z}$ the labels are based on their remainder
- Note that after we mod out, we don't care about the individual elements but just the labels, the representative elements
- The normal condition on the subgroup just ensures that the labels both capture all the values in G and that the cosets form a group themselves

1.4 Proof of Langrange's Theorem

Theorem: Langrange's theorem

Let G be a finite group, H any subgroup. Then $|H|$ divides $|G|$

Proof:

All the cosets of H form a partition (??) of G (though not necessarily a group, since H might not be normal). So if n is the number of cosets, then $n|H| = |G|$ (?? Why does this equal $|G|$ and not just some proportion of it?)

Remark: In general, for finite groups G and normal subgroup H, $|G/N = |G|/|N|$

1.5 Eliminating the homomorphism

Recap: Quotient groups

- The elements of the quotient group G/N are cosets gN
- The group operation is $g_1N \cdot g_2N = (g_1g_2)N$

Where do we use the requirement that N is a normal subgroup in the quotient group definition?

Answer: We don't know what g_1 or g_2 are, so we need to guarantee that the group operation's g_1g_2 will end up at the same coset, no matter which g_1 and g_2 are picked from those respective cosets. **The condition that N must be a normal subgroup gives us this.**

- We get this because N is defined as the kernel of some homomorphismϕ
- Why? I don't really get it... The book says it's because all values in the coset gN have the same value under the homomorphism ϕ

Lemma:

For $\phi : G \to K$ is a homomorphism with H =kerϕ. If $h \in H$ and $g \in G$, then $ghg^{-1} \in H$.

Proof:

Show that $\phi(ghg^{-1}) = 1_k$ (show that it's in the kernel, which is H).

$$\begin{aligned}
\phi(ghg^{-1}) = & && (12)\\
&= \phi(g) * \phi(h) * \phi(g^{-1}) && (13)\\
&= \phi(g) * \phi(g^{-1}) && (14)\\
&= \phi(gg^{-1}) && (15)\\
&= \phi(1_G) && (16)\\
&= 1_K && (17)\\
& && (18)
\end{aligned}$$

Turns out the converse is also true

Theorem: Algebraic condition for normal subgroups

Let H be a subgroup of G, then the following are equivalent

- $H \trianglelefteq G$
- $\forall g \in G$ and $h \in H$, $ghg^{-1} \in H$

Proof:

The last proof showed one direction. For the other one, we need to build a homomorphism with kernel H. We can do this by

- Defining the quotient group directly as the cosets, and verifying that it is valid
- Let our homomorphism be the map from G to our newly built quotient group

For the group operation we need

Claim:

If $g_1' \sim_H g_1$ and $g_2' \sim_H g_2$, then $g_1'g_2' \sim_H g_1g_2$

Short proof:

- $g_1' = g_1h_1$ and $g_2' = g_2h_2$ because they're in the same respective cosets.
- H has the property that $g_2^{-1}h_1g_2$ is some element of H, h_3, then multiply both sides on the left by g_2, leaving $h_1g_2 = g_2h_3$
- So $g_1(h_1g_2)h_2 = g_1(g_2h_3)h_2 = g_1g_2(h_3h_2)$, which is $\sim_H g_1g_2$ because the $h_3h_2 \in H$

This means that our group operation is consistent, no matter which element in the coset we pick, so now define the group operation to be

$$(g_1H) * (g_2H) = (g_1g_2)H.$$

So now we have our G/H group, with a projection map $g \mapsto gH$, which has a kernel H

- Elements of H get sent to hH, which is just H, which is the identity element

□

Example: Modding out a product group

Earlier we had the trivial subgroup of $G \times H$,

$$G' = \{(g, 1_H) | g \in G\}.$$

We can show that

- $G' \trianglelefteq G \times H$
  - Just need to show the $ghg^{-1} \in G'$ condition forall $g \in G \times H$ and $h \in G'$
- Also that $(G \times H)/G' \cong H$
  - Intuition: The different cosets are entirely dependent on the h value in the (g, h) pairs that represent the cosets. ie each h gets it's own coset, with all the values of g
  - $(g, h) \sim_{G'} (1_G, h)$ for any g, h

Remark: Suppose G is abelian. Then all subgroups of G are normal.

Proof:

It's really easy to show the $ghg^{-1} \in H$ condition because we can just reorder it to $gg^{-1}h = 1_G h = h$, which is in H by definition