Retiring the third law of black hole thermodynamics

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Oxbridge PDE conference
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joint work with Christoph Kehle (ETH Zürich)

Classical thermodynamics

Law	Thermodynamics

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Zeroth	T constant throughout body in thermal equilibrium

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Second	$dS \geq 0$ in any process
Third	Impossible to achieve $T = 0$ in finite time

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

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Department of Physics, Yale University, New Haven, Connecticut, USA

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- What is a black hole?
- What is the analogue of temperature T?
- What is the analogue of entropy S?

Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

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Example: Minkowski space. $\mathcal{M}^{3+1} = \mathbb{R}^{3+1}_{t,x,y,z}$ and

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Given a vector $v \in T_p\mathcal{M}$:

- $g(v, v) < 0$, v is *timelike*
- $g(v, v) = 0$, v is *null*
- $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

The Reissner–Nordström black hole spacetime

- Mass $M > 0$ and charge $0 \leq |e| \leq M$

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)$$

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- Describes a static, charged, nonrotating black hole

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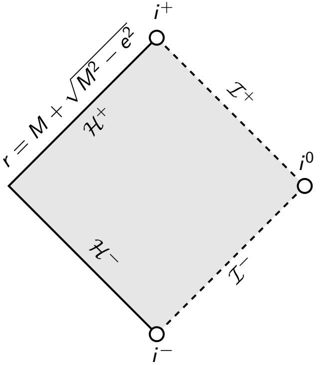

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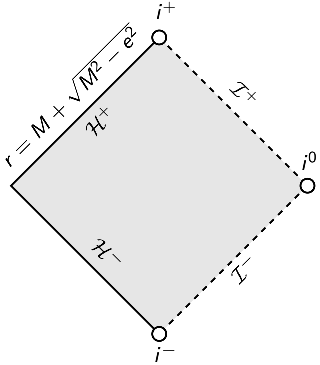

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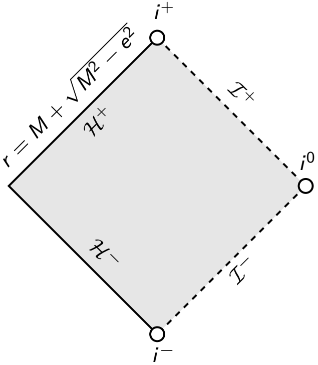

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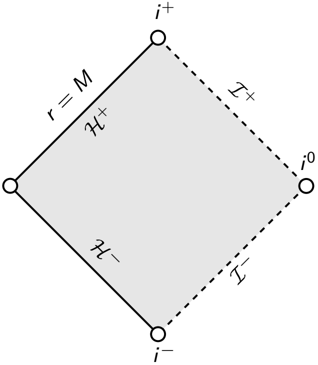

Extremal $|e| = M$

Superextremal $|e| > M$ does not contain a black hole!

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- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Retiring the third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

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Subextremal: $\kappa > 0 \iff T > 0$

Extremal: $\kappa = 0 \iff T = 0$

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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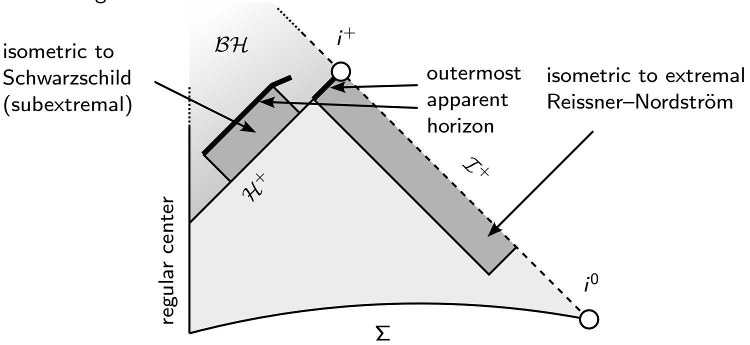
Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

Counterexample to the third law: detailed statement of our theorem

There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



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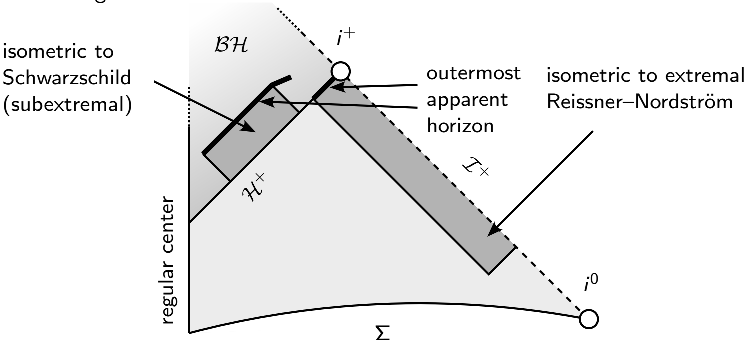

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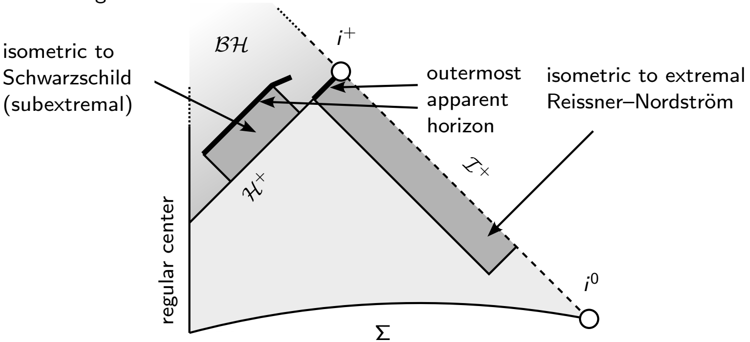

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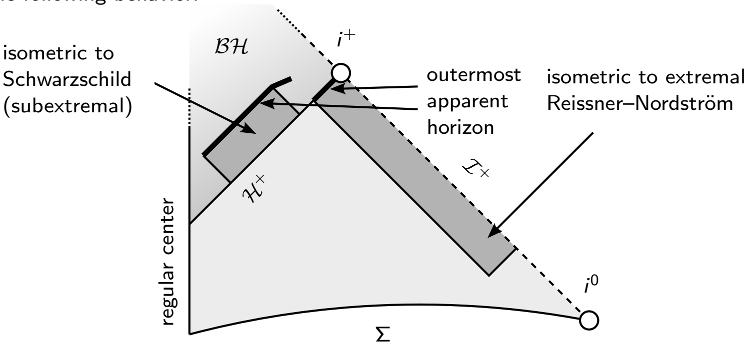

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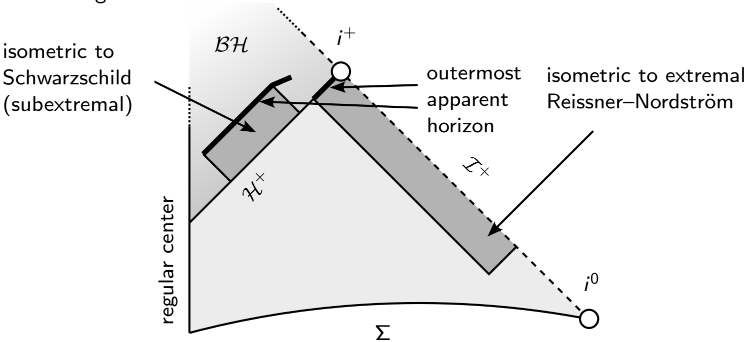

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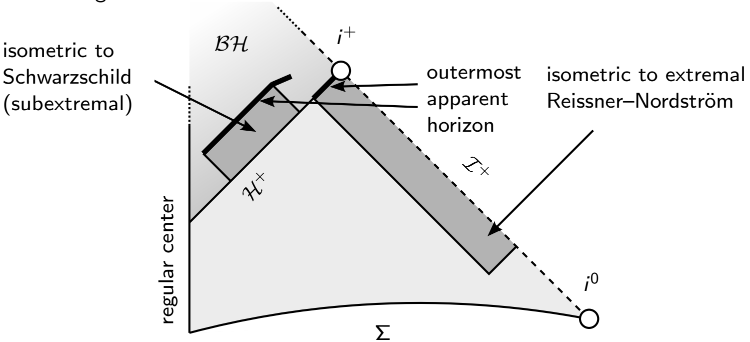

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- Model satisfies the dominant energy condition.

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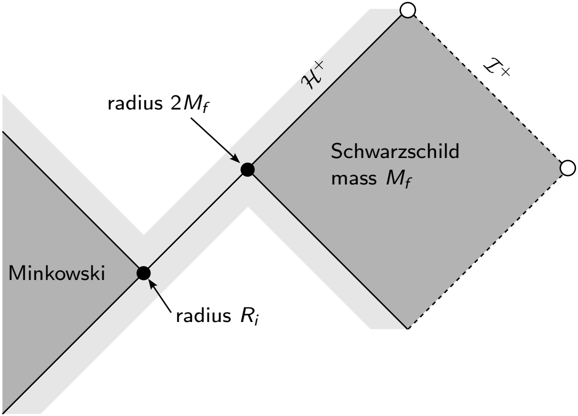
- Spherical symmetry!
- Scalar field ϕ degree of freedom required to break the rigidity of pure Einstein–Maxwell in spherical symmetry.

Prototype: Minkowski to Schwarzschild gluing



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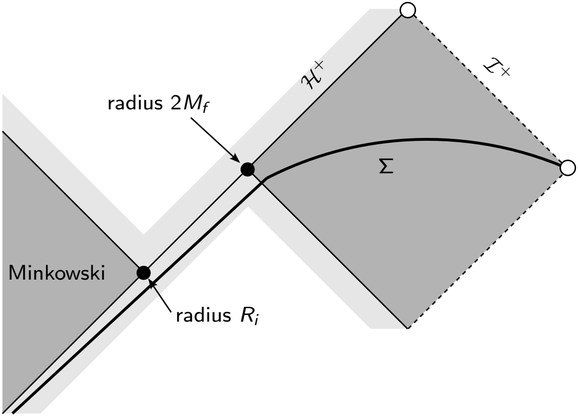

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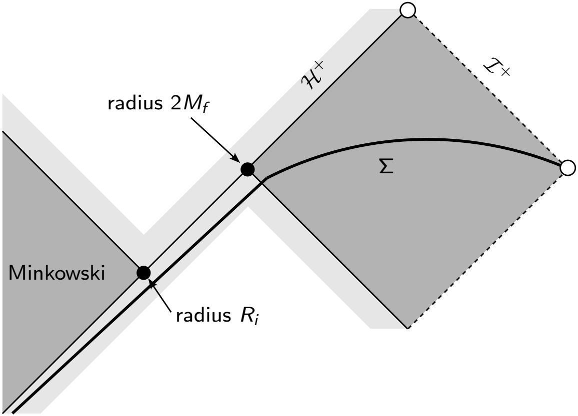

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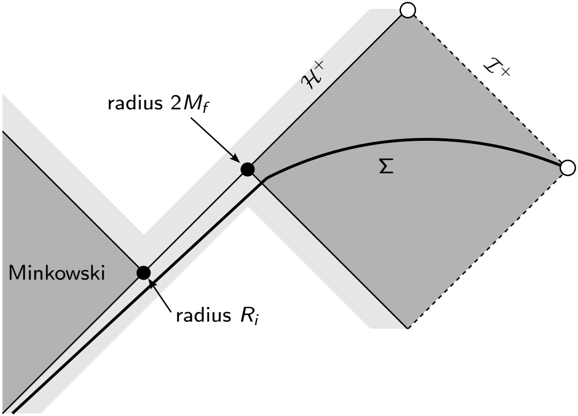

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The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

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We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

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Question.
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solving $(*)$

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

$$\Box_g \phi = 0 \qquad (*)$$

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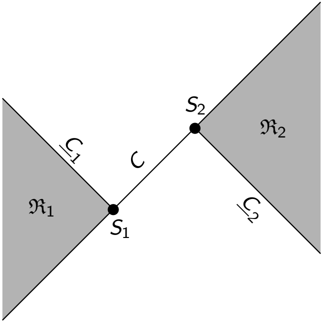
$$\phi|_{\mathfrak{R}_2} = \phi_2\,?$$

Characteristic/null gluing for the linear wave equation



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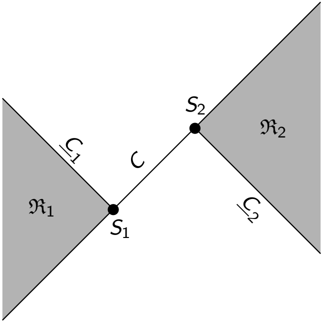

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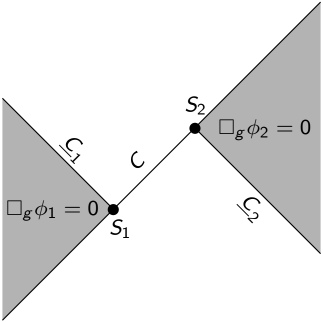

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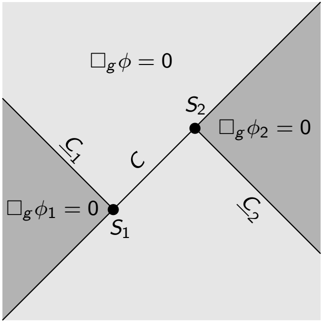

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The characteristic initial value problem

Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

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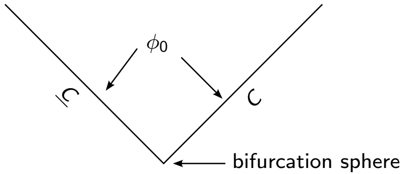

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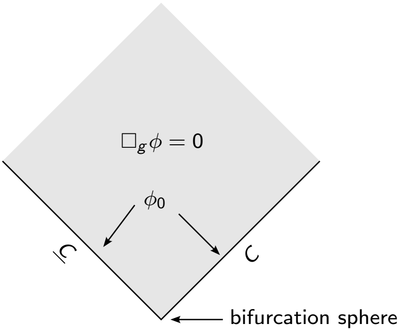

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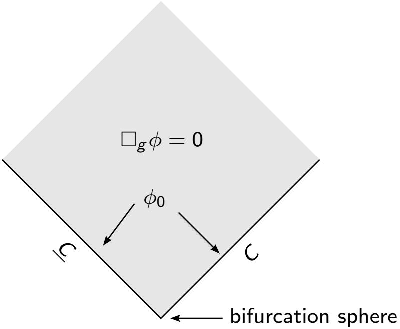

Characteristic data for the wave equation does not involve derivatives.

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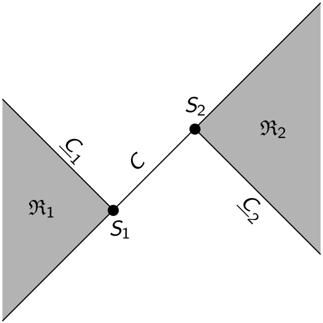
Commuting the wave equation with transverse derivatives gives an inductive system of transport equations for all orders!

Back to characteristic gluing



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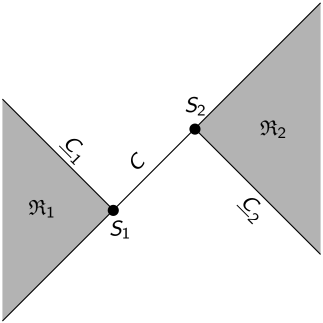

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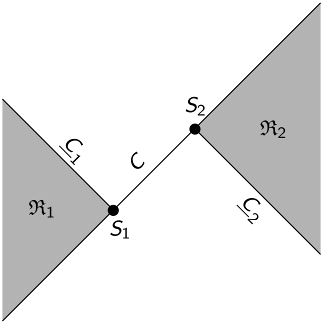

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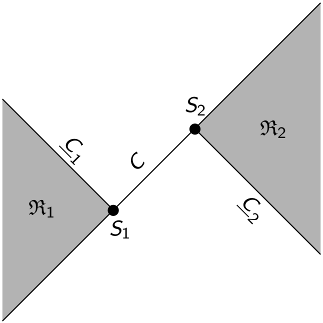

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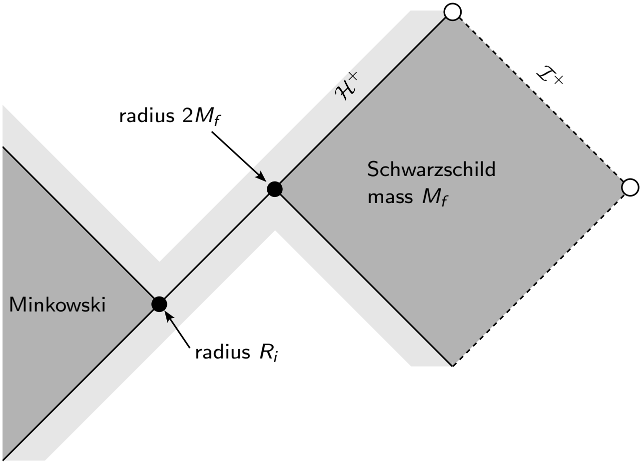
- Initial/boundary value problem for a system of coupled ODEs where you are allowed to (almost) freely prescribe the potential

Minkowski to Schwarzschild gluing



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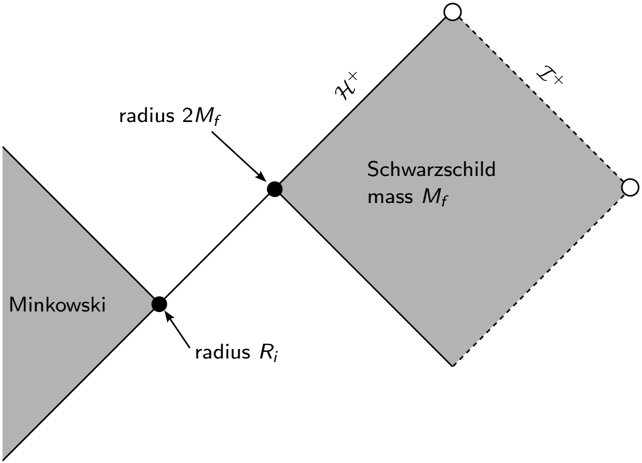

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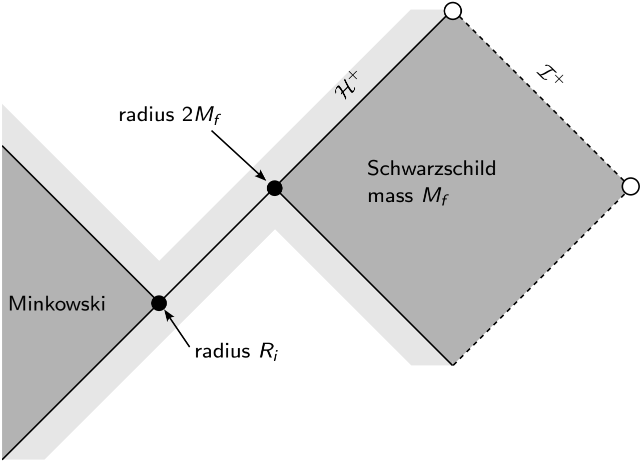

Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$

$$g = -\Omega^2 du\, dv + r^2 g_{S^2}$$

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- Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2 \tag{9}$$

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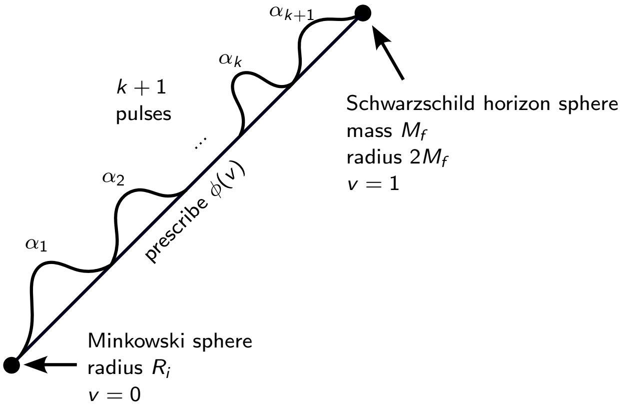
Minkowski to Schwarzschild gluing

Theorem (Kehle–U. ’22).
For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



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  is odd
- **Borsuk–Ulam theorem:** there exists α_* such that

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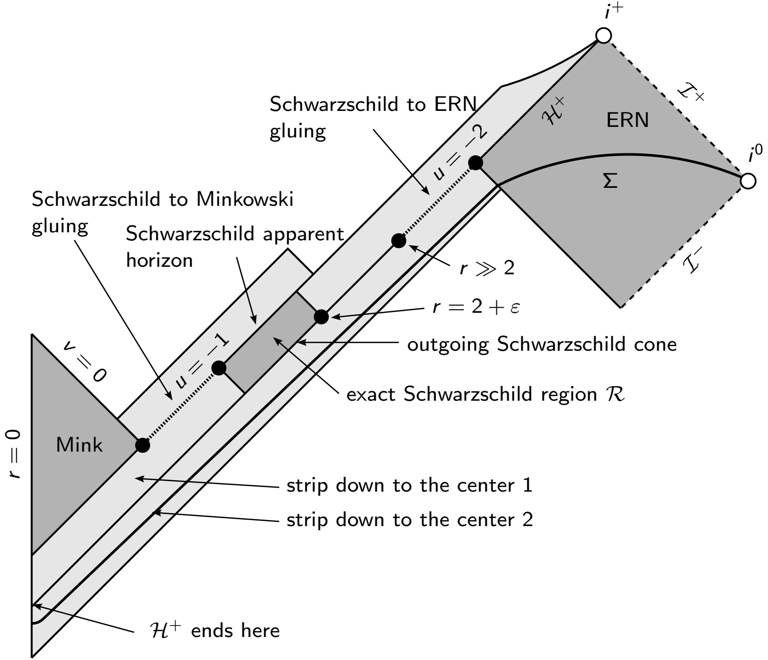
Disproof of the third law



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This would follow from characteristic gluing, in vacuum, of a Minkowski sphere to a Kerr exterior sphere.

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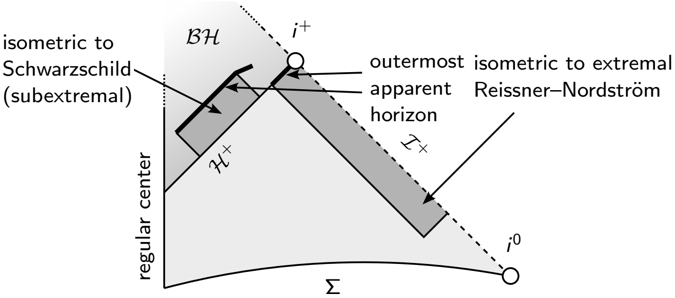

Thank you!