

# Crete summer school spherical symmetry examples sheet

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## 1 The spherically symmetric equations

Let  $g$  be a spherically symmetric Lorentzian metric

$$g = -\Omega^2 dudv + r^2 g_{S^2}, \quad (1.1)$$

on  $\mathcal{M}^{3+1}$  where  $g_{S^2} \doteq d\vartheta^2 + \sin^2 \vartheta d\varphi^2$  is the round metric on the unit sphere. We use the notation  $\not{g} \doteq r^2 g_{S^2}$ .

a) Show that the Christoffel symbols involving null coordinates are given by

$$\Gamma_{uu}^u = \partial_u \log \Omega^2, \quad (1.2)$$

$$\Gamma_{vv}^v = \partial_v \log \Omega^2, \quad (1.3)$$

$$\Gamma_{AB}^u = \frac{2\partial_v r}{\Omega^2 r} \not{g}_{AB}, \quad (1.4)$$

$$\Gamma_{AB}^v = \frac{2\partial_u r}{\Omega^2 r} \not{g}_{AB}, \quad (1.5)$$

$$\Gamma_{Bu}^A = \frac{\partial_u r}{r} \delta_B^A, \quad (1.6)$$

$$\Gamma_{Bv}^A = \frac{\partial_v r}{r} \delta_B^A, \quad (1.7)$$

and the totally spatial Christoffel symbols  $\Gamma_{BC}^A$  are the same as for  $g_{S^2}$  in the coordinates  $(\vartheta, \varphi)$ .

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b) Show that the Ricci tensor is given by

$$R_{uu} = -\frac{2\Omega^2}{r}\partial_u\left(\frac{\partial_u r}{\Omega^2}\right), \quad (1.8)$$

$$R_{vv} = -\frac{2\Omega^2}{r}\partial_v\left(\frac{\partial_v r}{\Omega^2}\right), \quad (1.9)$$

$$R_{uv} = -\partial_u\partial_v\log\Omega^2 - \frac{2}{r}\partial_u\partial_v r, \quad (1.10)$$

$$R_{\vartheta\vartheta} = 1 + \frac{4\partial_u r\partial_v r}{\Omega^2} + \frac{4r}{\Omega^2}\partial_u\partial_v r, \quad (1.11)$$

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta}. \quad (1.12)$$

c) Let  $\phi$  be a *free massless scalar field* on  $(\mathcal{M}, g)$ , i.e., a smooth function  $\phi : \mathcal{M} \rightarrow \mathbb{R}$  which solves the *linear wave equation*

$$\square_g \phi \doteq \nabla^\mu \nabla_\mu \phi = 0. \quad (1.13)$$

If  $\phi$  is also spherically symmetric, i.e.,  $\phi = \phi(u, v)$ , show that

$$\square_g \phi = -\frac{4}{\Omega^2} \left( \frac{\partial_v r \partial_u \phi}{r} + \frac{\partial_u r \partial_v \phi}{r} + \partial_u \partial_v \phi \right). \quad (1.14)$$

d) Define the *energy-momentum tensor* of  $\phi$  by

$$T_{\mu\nu} \doteq \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi. \quad (1.15)$$

Prove that

$$\nabla^\mu T_{\mu\nu} = \square_g \phi \partial_\nu \phi. \quad (1.16)$$

(This part of the exercise does not rely on spherical symmetry.)

e) For a spherically symmetric scalar field, show that

$$T_{uu} = (\partial_u \phi)^2, \quad (1.17)$$

$$T_{vv} = (\partial_v \phi)^2, \quad (1.18)$$

$$T_{uv} = 0, \quad (1.19)$$

$$T_{AB} = \frac{2}{\Omega^2} \partial_u \phi \partial_v \phi g_{AB}. \quad (1.20)$$

f) The *Einstein field equations* for a *self-gravitating massless scalar field* (with zero cosmological constant) read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2T_{\mu\nu} \quad (1.21)$$

with  $T_{\mu\nu}$  as in (1.15). Prove that

$$R_{\mu\nu} = 2\partial_u \phi \partial_v \phi, \quad (1.22)$$

$$\square_g \phi = 0. \quad (1.23)$$

(This part of the exercise does not rely on spherical symmetry.)

g) For a spherically symmetric scalar field, derive the *spherically symmetric Einstein equations*

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r}, \quad (1.24)$$

$$\partial_u \partial_v \log \Omega^2 = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r \partial_v r}{r^2} - 2\partial_u \phi \partial_v \phi, \quad (1.25)$$

$$\partial_u \partial_v \phi = -\frac{\partial_u r \partial_v \phi}{r} - \frac{\partial_v r \partial_u \phi}{r}, \quad (1.26)$$

$$\partial_u \left( \frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2, \quad (1.27)$$

$$\partial_v \left( \frac{\partial_v r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_v \phi)^2. \quad (1.28)$$

## 2 The Hawking mass and monotonicities in spherical symmetry

In this problem we again consider a spherically symmetric self-gravitating scalar field.

a) The *Hawking mass* of a spherically symmetric spacetime is defined implicitly by the relation

$$1 - \frac{2m}{r} = g(\nabla r, \nabla r). \quad (2.1)$$

Show that in double null coordinates,

$$m = \frac{r}{2} \left( 1 + \frac{4\partial_u r \partial_v r}{\Omega^2} \right). \quad (2.2)$$

b) Show that in the Schwarzschild spacetime of mass  $M$ , the Hawking mass  $m$  is globally constant and equals  $M$ . *Hint:* Use the definition (2.1).

c) Derive the equations

$$\partial_u m = -\frac{2r^2 \partial_v r}{\Omega^2} (\partial_u \phi)^2, \quad (2.3)$$

$$\partial_v m = -\frac{2r^2 \partial_u r}{\Omega^2} (\partial_v \phi)^2. \quad (2.4)$$

d) Show that if  $\partial_u r(u, v) \leq 0$  (resp.,  $< 0$ ), then  $\partial_u r(u', v) \leq 0$  (resp.,  $< 0$ ) for all  $u' \geq u$ .

e) Show that if  $\partial_v r(u, v) \leq 0$  (resp.,  $< 0$ ), then  $\partial_v r(u, v') \leq 0$  (resp.,  $< 0$ ) for all  $v' \geq v$ .

f) If  $\partial_u r(u, v) \leq 0$ , show that  $\partial_v m(u, v) \geq 0$ .

g) If  $\partial_v r(u, v) \geq 0$ , show that  $\partial_u m(u, v) \leq 0$ .

h) If  $\partial_u r < 0$  at a point  $(u, v)$ , show the following equivalences at  $(u, v)$ :

$$\partial_v r > 0 \iff \frac{2m}{r} < 1, \quad (2.5)$$

$$\partial_v r = 0 \iff \frac{2m}{r} = 1, \quad (2.6)$$

$$\partial_v r < 0 \iff \frac{2m}{r} > 1. \quad (2.7)$$

i) Show that if  $\partial_u r < 0$ , then

$$\partial_u \left( \frac{\Omega^2}{-\partial_u r} \right) \leq 0. \quad (2.8)$$

### 3 The characteristic initial value problem

In this problem, we set up and solve the characteristic initial value problem for the spherically symmetric Einstein-scalar field system (away from the center).

#### 3.1 Definitions and the initial data

Given  $U_0 < U_1$  and  $V_0 < V_1$ , set

$$\mathcal{C}(U_0, U_1, V_0, V_1) \doteq (\{U_0\} \times [V_0, V_1]) \cup ([U_0, U_1] \times \{V_0\}), \quad (3.1)$$

$$\mathcal{R}(U_0, U_1, V_0, V_1) \doteq [U_0, U_1] \times [V_0, V_1], \quad (3.2)$$

so that  $\mathcal{C}$  is the past boundary of  $\mathcal{R}$  when viewed as subsets of  $\mathbb{R}_{u,v}^2$  equipped with the standard Minkowski metric  $-dudv$  and time orientation. A  $C^k$  *characteristic data set* for the spherically symmetric Einstein-scalar field system on  $\mathcal{C}$  consists of continuous functions  $\mathring{r}, \mathring{\Omega}^2, \mathring{\phi} : \mathcal{C} \rightarrow \mathbb{R}$  such that  $\mathring{r}$  and  $\mathring{\Omega}^2$  are strictly positive,  $\mathring{r}$  is  $C^{k+1}$  when restricted to the two intervals in  $\mathcal{C}$ , and  $\mathring{\Omega}^2$  and  $\mathring{\phi}$  are  $C^k$  when restricted to the two intervals. Furthermore, we require that (1.27) and (1.28) hold on  $\mathcal{C}$  for  $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi})$ .

- a) Show that  $\mathring{\Omega}^2$  and  $\mathring{\phi}$ , together with  $\mathring{r}(U_0, V_0)$ ,  $\partial_u \mathring{r}(U_0, V_0)$ , and  $\partial_v \mathring{r}(U_0, V_0)$ , determine a unique characteristic data set on  $(\{U_0\} \times [V_0, V_1']) \cup ([U_0, U_1'] \times \{V_0\})$  if  $V_1' - V_0$  and  $U_1' - U_0$  are sufficiently small.

#### 3.2 The proxy system

We will prove local well-posedness for systems of nonlinear wave equations on  $\mathbb{R}_{u,v}^2$  of the form

$$\partial_u \partial_v \Psi = F(\Psi, \partial \Psi), \quad (3.3)$$

where  $\Psi : \mathcal{D} \rightarrow \mathbb{R}^N$ ,  $F : \mathbb{R}^N \times \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$  is smooth, and  $\mathcal{D} \subset \mathbb{R}_{u,v}^2$ .

- a) Show that the spherically symmetric Einstein-scalar field system can be brought into this form if  $r > 0$ , with the variables  $\Psi_1 = \log r$ ,  $\Psi_2 = \log \Omega^2$ , and  $\Psi_3 = \phi$ .
- b) We say that the nonlinearity  $F$  satisfies the *null condition* if there exist functions  $F_0, F_{ij} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  such that

$$F(\Psi, \partial \Psi) = F_0(\Psi) + \sum_{i,j} F_{ij}(\Psi) \partial_u \Psi_i \partial_v \Psi_j. \quad (3.4)$$

Show that the spherically symmetric Einstein-scalar field model satisfies the null condition.

#### 3.3 Uniqueness

A  $C^1$  function  $\Psi : \mathcal{D} \rightarrow \mathbb{R}^N$  is said to be a  $C^1$  *solution* of (3.3) if for any  $\mathcal{R} = \mathcal{R}(U_0, U_1, V_0, V_1) \subset \mathcal{D}$ , the integrated form of (3.3) holds on  $\mathcal{R}$ :

$$\Psi(u, v) = \int_{U_0}^u \int_{V_0}^v F(\Psi, \partial \Psi) dv' du' + \Psi(u, V_0) + \Psi(U_0, v) - \Psi(0, 0) \quad (3.5)$$

for every  $(u, v) \in \mathcal{R}$ . We wish to show:

**Theorem 1.** *Let  $\Psi_1$  and  $\Psi_2$  be two  $C^1$  solutions of (3.3) on  $\mathcal{R}(U_0, U_1, V_0, V_1)$  which agree along  $\mathcal{C}(U_0, U_1, V_0, V_1)$ . Then  $\Psi_1 = \Psi_2$  identically in  $\mathcal{R}$ .*

- a) Show that any classical ( $C^2$ ) solution of (3.3) is a  $C^1$  solution.
- b) Prove the following lemma:

**Lemma 3.1.** *For any constant  $C_{\dagger} > 0$  there exists a constant  $\delta = \delta(C_{\dagger}) > 0$  such that if  $\Psi$  is a  $C^1$   $\mathbb{R}^N$ -valued function on  $\mathcal{R}(U_0, U_1, V_0, V_1)$  with  $0 < U_1 - U_0 < \delta$ ,  $0 < V_1 - V_0 < \delta$ , satisfying*

$$\Psi(u, v) = \int_{U_0}^u \int_{V_0}^v f_1 \cdot \Psi + f_2 \cdot \partial \Psi \, dv' du' \quad (3.6)$$

*for every  $(u, v) \in \mathcal{R}$ , where  $f_1$  and  $f_2$  are continuous  $N \times N$ -matrix valued functions satisfying*

$$\sup_{\mathcal{R}} (|f_1| + |f_2|) \leq C_{\dagger}, \quad (3.7)$$

*then  $\Psi$  vanishes identically in  $\mathcal{R}$ .*

*Hint:* Use (3.6) to directly estimate  $\|\Psi\|_{C^1(\mathcal{R})}$  in terms of itself.

- c) Use Lemma 3.1 to prove Theorem 1. *Hint:* Let  $\Psi \doteq \Psi_2 - \Psi_1$ . Show that  $\Psi$  satisfies (3.6) for an appropriate choice of  $f_1$  and  $f_2$ . Then cover the domain by small rectangles.

### 3.4 Existence in small rectangles

The goal of this section is to prove the following:

**Theorem 2.** *For any  $C_* > 0$  there exists a constant  $\varepsilon_{\text{loc}} > 0$  depending on  $C_*$  and  $F$  with the following property. Let  $\Psi_0$  be a  $C^1$  characteristic data set on  $\mathcal{C}(U_0, U_1, V_0, V_1)$  with  $0 < U_1 - U_0 < \varepsilon_{\text{loc}}$ ,  $0 < V_1 - V_0 < \varepsilon_{\text{loc}}$ , and*

$$\|\mathring{\Psi}\|_{C^1(\mathcal{C})} \leq C_*. \quad (3.8)$$

*Then there exists a unique  $C^1$  solution  $\Psi$  of (3.3) on  $\mathcal{R}(U_0, U_1, V_0, V_1)$  which extends the initial data  $\mathring{\Psi}$ . Moreover, it holds that*

$$\|\Psi\|_{C^1(\mathcal{R})} \leq 10C_*. \quad (3.9)$$

The theorem is proved by constructing the solution  $\Psi$  as the limit of an iteration scheme. Set  $\Psi_1 = 0$  on  $\mathcal{R}$  and, for  $n \geq 2$ , let  $\Psi_n$  solve the linear inhomogeneous wave equation

$$\partial_u \partial_v \Psi_n = F(\Psi_{n-1}, \partial \Psi_{n-1}), \quad (3.10)$$

$$\Psi_n|_{\mathcal{C}} = \mathring{\Psi}. \quad (3.11)$$

- Find an explicit recursive formula for  $\Psi_n(u, v)$  using the method of characteristics.
- Use this formula to show that  $\|\Psi_n\|_{C^1(\mathcal{R})} \leq 10C_*$  if  $\varepsilon_{\text{loc}}$  is chosen sufficiently small.
- Show that

$$\|\Psi_n - \Psi_{n-1}\|_{C^1(\mathcal{R})} \leq \frac{1}{2} \|\Psi_{n-1} - \Psi_{n-2}\|_{C^1(\mathcal{R})} \quad (3.12)$$

for  $\varepsilon_{\text{loc}}$  sufficiently small.

- Conclude that  $\Psi_n$  converges to the desired unique  $C^1$  solution  $\Psi$ . *Hint:* Show that  $\Psi_n$  is a Cauchy sequence in  $C^1$ .

### 3.5 Higher regularity

In fact, Theorem 2 can be upgraded to the following:

**Theorem 3.** *Let  $k \geq 2$ . For any  $C_* > 0$  there exists constants  $C_1, C_2, \dots, C_k > 0$  depending on  $C_*$  and  $F$  with the following property. Let  $\varepsilon_{\text{loc}}(C_*, F) > 0$  be as in Theorem 2. Let  $\mathring{\Psi}$  be a  $C^k$  characteristic data set on  $\mathcal{C}(U_0, U_1, V_0, V_1)$  with  $0 < U_1 - U_0 < \varepsilon_{\text{loc}}$ ,  $0 < V_1 - V_0 < \varepsilon_{\text{loc}}$ , and*

$$\|\mathring{\Psi}_0\|_{C^1(\mathcal{C})} \leq C_*. \quad (3.13)$$

*Then there exists a unique classical  $C^k$  solution  $\Psi$  of (3.3) on  $\mathcal{R}(U_0, U_1, V_0, V_1)$  which extends the initial data  $\mathring{\Psi}$ . Moreover, it holds that*

$$\|\Psi\|_{C^k(\mathcal{R})} \leq C_k \|\mathring{\Psi}\|_{C^k(\mathcal{C})} \quad (3.14)$$

*for all  $k$ .*

Note that the size of the region on which  $\Psi$  exists depends only on the  $C^1$  norm of the initial data. The easiest way to prove this result is to directly argue that the iterates  $\Psi_n$  in the proof of Theorem 2 are bounded in Cauchy in  $C^k$ .

*Proof of boundedness for  $k = 2$ .* We claim that there exists constants  $\hat{C}_2, \tilde{C}_2$  such that

$$|\partial_u^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 v}, \quad (3.15)$$

$$|\partial_v^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 u} \quad (3.16)$$

on  $\mathcal{R}$  for every  $n$ . Indeed, differentiating (3.10) in  $u$ , we find

$$\partial_v(\partial_u^2 \Psi_n) = f_1 + f_2 \partial_u^2 \Psi_{n-1}, \quad (3.17)$$

where  $f_1$  and  $f_2$  are uniformly bounded functions by the  $C^1$  estimate for  $\Psi_{n-1}$ . By integrating this and choosing  $\hat{C}_2, \tilde{C}_2$  sufficiently large, (3.15) is easily established by induction. (Note that we used (3.10) again to eliminate the mixed term  $\partial_u \partial_v \Psi_{n-1}$  that could have appeared.) The estimate (3.16) is obtained similarly by differentiating (3.10) in  $v$ . By commuting the equation further, one can show (3.9) for  $k = 2$ .  $\square$

a) Generalize this argument to arbitrary  $k$ .

b) Is it true that  $\Psi_n$  is Cauchy in  $C^k$ ?

### 3.6 Existence in thin slabs

The region of existence in Theorems 2 and 3 is a small rectangle. If the nonlinearity  $F$  satisfies the null condition (3.4), then this local existence result can be upgraded to include a full neighborhood of the bifurcate characteristic hypersurface  $\mathcal{C}$ .

**Theorem 4.** *For any  $A > 0$ ,  $L > 0$ , and nonlinearity  $F$  satisfying the null condition (3.4), there exists a constant  $\varepsilon_{\text{slab}} = \varepsilon_{\text{slab}}(A, L, F) > 0$  with the following property. Let  $\dot{\Psi}$  be a  $C^k$  characteristic data set on  $\mathcal{C}(U_0, U_1, V_0, V_1)$  with  $0 < U_1 - U_0 < \varepsilon_{\text{slab}}$ ,  $0 < V_0 - V_1 < L$ , and*

$$\|\dot{\Psi}\|_{C^1(\mathcal{C})} \leq A. \quad (3.18)$$

*Then there exists a unique smooth solution of (3.3) on  $\mathcal{R}(U_0, U_1, V_0, V_1)$  which extends the initial data  $\dot{\Psi}$ .*

*The same statement holds for data on  $\mathcal{C}(U_0, U_1, V_0, V_1)$  with  $0 < U_1 - U_0 < L$  and  $0 < V_0 - V_1 < \varepsilon_{\text{slab}}$ .*

a) Prove the following “matrix Grönwall” lemma:

**Lemma 3.2.** *Let  $X, Y : [0, T] \rightarrow \mathbb{R}^N$  be  $C^1$  and satisfy  $X' = Y + MX$ , where  $M : [0, T] \rightarrow \mathbb{R}^{N \times N}$ . Then*

$$|X(T)| \leq \left( |X(0)| + \int_0^T |Y(t)| dt \right) \exp \left( \int_0^T |M(t)| dt \right). \quad (3.19)$$

*Hint:* Consider the equation satisfied by  $x(t) = \sqrt{|X(t)|^2 + \varepsilon^2}$ .

b) I will outline a proof utilizing a bootstrap argument based on the pointwise bounds

$$|\Psi| \leq 10A, \quad (3.20)$$

$$|\partial_u \Psi| \leq 10B, \quad (3.21)$$

$$|\partial_v \Psi| \leq 10A \quad (3.22)$$

and the local existence statement Theorem 2. (Here  $B$  is a large constant to be determined in the course of the proof.) Define the bootstrap set  $\mathcal{A}_{A,B}$  to be the component of

$$\{\tilde{V} \in [V_0, V_1] : \Psi \text{ exists, is } C^\infty, \text{ and satisfies (3.20)–(3.22) on } \mathcal{R}(U_0, U_1, V_0, \tilde{V})\}. \quad (3.23)$$

containing  $V_0$ . The goal is to show that  $\mathcal{A}_{A,B}$  is nonempty, open, and closed for  $B$  sufficiently large and  $\varepsilon_{\text{slab}}$  sufficiently small.

Using Theorem 2, show that if  $B \geq A$  and  $\varepsilon_{\text{slab}}$  is sufficiently small depending on  $A$ , then  $\mathcal{A}_{A,B} \neq \emptyset$ .

- c) Show that  $\mathcal{A}_{A,B}$  is closed.
- d) We separate the proof that  $\mathcal{A}_{A,B}$  is open into two parts. Let  $\tilde{V} \in \mathcal{A}_{A,B}$ . First, show that if  $\varepsilon_{\text{slab}}$  is chosen to be sufficiently small and  $B$  sufficiently large, then the bounds (3.20)–(3.22) hold on  $\mathcal{R}(U_0, U_1, V_0, \tilde{V})$  with “better constants,” i.e.,

$$|\Psi| \leq 2A, \quad (3.24)$$

$$|\partial_u \Psi| \leq 2B, \quad (3.25)$$

$$|\partial_v \Psi| \leq 2A. \quad (3.26)$$

*Hint:* To estimate  $\Psi$  and  $\partial_v \Psi$ , use thinness of the slab in the  $u$ -direction. To estimate  $\partial_u \Psi$ , use the fact that the null condition implies that  $\partial_u \Psi$  satisfies a *linear ODE* in  $v$ . Use Lemma 3.2 to estimate  $|\partial_u \Psi|$ .

- e) Using these “improved” estimates, carry out a continuity argument to show that  $\mathcal{A}_{A,B}$  is open.

### 3.7 Propagation of constraints

We now return to the spherically symmetric Einstein-scalar field system. Using Theorems 2 and 3, we can solve the characteristic initial value problem for the wave equations (1.24)–(1.26). But how do we obtain Raychaudhuri’s equations (1.27) and (1.28)?

- a) Using only (1.24), (1.25), and (1.26), prove the pair of identities

$$\partial_u \left( r \Omega^2 \partial_v \left( \frac{\partial_v r}{\Omega^2} \right) + r^2 (\partial_v \phi)^2 \right) = 0, \quad (3.27)$$

$$\partial_v \left( r \Omega^2 \partial_u \left( \frac{\partial_u r}{\Omega^2} \right) + r^2 (\partial_u \phi)^2 \right) = 0. \quad (3.28)$$

- b) Conclude that (1.27) and (1.28) hold on  $\mathcal{R}$  if they do on  $\mathcal{C}$ .

## 4 The extension principle away from the center\*

The goal of this exercise is to prove the following:

**Theorem 5.** *Let  $(\mathcal{Q}, r, \Omega^2, \phi)$  be a solution of the spherically symmetric-Einstein scalar field system, where  $\mathcal{Q} \subset \mathbb{R}_{u,v}^2$  is an open set. Suppose that the following hold:*

- i)  $\mathcal{R}' \subset \mathcal{Q}$ , where  $\mathcal{R}' \doteq ([0, U] \times [0, V]) \setminus \{(U, V)\}$  and  $U, V$  are finite,
- ii)  $\partial_u r < 0$  on  $\mathcal{R}'$ , and
- iii)  $\partial_v r \geq 0$  on  $\mathcal{R}'$ .

*Then the solution extends smoothly to a neighborhood of  $(U, V) \in \overline{\mathcal{Q}}$ .*

This theorem says that a “first singularity” in the spherically symmetric Einstein-scalar field model either occurs along the axis  $\Gamma$  (so that no such  $\mathcal{R}'$  exists) or where  $\partial_v r < 0$ . We now sketch the proof as a series of exercises:

- a) Argue using the well-posedness statement from Problem 3 that it suffices to show that  $(r, \Omega^2, \phi)$  are bounded in  $C^1$  on  $\mathcal{R}'$  and  $(r, \Omega^2)$  are bounded below away from zero.
- b) Show that  $r \sim 1$  on  $\mathcal{R}'$ .
- c) Show that  $0 \leq -\Omega^2 / \partial_u r \lesssim 1$  on  $\mathcal{R}'$ . *Hint:* Use Raychaudhuri’s equation.

d) Show that  $|m| \lesssim 1$  on  $\mathcal{R}'$  and hence that

$$\int_{[0,U] \times \{v\}} \frac{-\partial_u r}{\Omega^2} r^2 (\partial_v \phi)^2 dv \lesssim 1 \quad (4.1)$$

for any  $v \in [0, V]$  and

$$\int_{\{u\} \times [0,V]} \frac{\partial_v r}{\Omega^2} r^2 (\partial_u \phi)^2 du \lesssim 1 \quad (4.2)$$

for any  $u \in [0, U]$ . These are fundamental *energy estimates* for the spherically symmetric Einstein-scalar field system.

e) Show that  $|\phi| \lesssim 1$  on  $\mathcal{R}'$ . *Hint:* Use the fundamental theorem in calculus in  $v$  and parts c) and d).

f) Show that the  $r$  wave equation can be written as

$$\partial_v \partial_u r = -\frac{\Omega^2}{2r^2} m. \quad (4.3)$$

g) Multiply and divide (4.3) by  $\partial_u r$  and use the method of integrating factors to estimate  $\partial_u r \sim -1$  on  $\mathcal{R}'$ .

h) Show that  $\partial_v r \lesssim 1$  on  $\mathcal{R}'$ .

i) Show that  $\Omega^2 \lesssim 1$  on  $\mathcal{R}'$ . *Hint:* Multiply c) and g).

j) Derive the equation

$$\partial_u \partial_v (r\phi) = -\frac{\Omega^2 m}{2r^2} \phi \quad (4.4)$$

and complete the argument.

## 5 Formation of trapped surfaces in spherical symmetry\*

The goal of this exercise is to prove the following:

**Theorem 6** (Christodoulou). *Black holes can form dynamically in the spherically symmetric Einstein-scalar field model, starting from data at “past null infinity”: There exists a solution  $(r, \Omega^2, \phi)$  on  $\mathcal{D} \doteq (-\infty, -1] \times [0, \delta]$  (where  $\delta > 0$  is a small parameter) with the following properties:*

1. *The initial ingoing cone is Minkowskian:  $\partial_v r(u, 0) = -\partial_u r(u, 0) = \frac{1}{2}$ ,  $\Omega^2(u, 0) = 1$ , and  $\phi(u, 0) = 0$  for  $u \in (-\infty, -1]$ .*
2. *The initial outgoing cone (formally “ $u = -\infty$ ”) is a portion of null infinity in the sense that  $r(-\infty, v) = \infty$  and  $\partial_v r(\infty, v) > 0$  for  $v \in [0, \delta]$ . (These are to be understood as limiting statements.)*
3. *The solution has no antitrapped surfaces:  $\partial_u r \sim -1$  in  $\mathcal{D}$ .*
4. *The sphere  $(-1, \delta)$  is trapped:  $\partial_v r(-1, \delta) < 0$ .*

*Remark 5.1.* In fact, this theorem holds true for the Einstein vacuum equations, where it necessarily requires a departure from spherical symmetry. The proof strategy given below is essentially an interpretation of Christodoulou’s proof for the Einstein vacuum equations for the spherically symmetric Einstein-scalar field system.

*Proof.* We will construct the desired solution by a limiting procedure (i.e., sending the initial outgoing cone to  $u = -\infty$ ). Consider a double null rectangle  $\mathcal{R} \doteq [u_0, -1] \times [0, \delta] \subset \mathbb{R}_{u,v}^2$ , where  $u_0 < -1$  and  $\delta > 0$ . We consider a characteristic data set  $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi})$  on  $\mathcal{C}$  (the past boundary of  $\mathcal{R}$ ) chosen as follows: Fix a function  $f \in C_c^\infty(0, 1)$  with  $\|f'\|_{L^2} = 1$  and set

$$\mathring{\phi}(u_0, v) = \frac{\delta^{1/2}}{|u_0|} f\left(\frac{v}{\delta}\right) \quad (5.1)$$



on the initial outgoing cone  $C_{u_0}$ . On the initial ingoing cone  $\underline{C}_0$ , set  $\mathring{\phi}(u, 0) = 0$ . On  $\mathcal{C}$ , set  $\Omega^2 = 1$ . At the bifurcation sphere  $(u_0, 0)$ , set

$$\mathring{r}(u_0, 0) = 1 + \frac{1}{2}|u_0|, \quad \partial_v \mathring{r}(u_0, 0) = \frac{1}{2}, \quad \partial_u \mathring{r}(u_0, 0) = -\frac{1}{2}. \quad (5.2)$$

- a) Show that there exists  $\delta_0 > 0$  such that if  $|u_0|$  is sufficiently large, and  $0 < \delta < \delta_0$ , then the above seed data defines a regular characteristic data set on  $\mathcal{C}$  satisfying the following estimates:

$$|r\phi| \lesssim \delta^{1/2}, \quad (5.3)$$

$$|r^2 \partial_u \phi| \lesssim \delta^{1/2}, \quad (5.4)$$

$$|r \partial_v \phi| \lesssim \delta^{-1/2}, \quad (5.5)$$

$$\partial_u r \sim -1, \quad (5.6)$$

$$\frac{1}{4} \leq \partial_v r \leq \frac{3}{4}, \quad (5.7)$$

$$m(u_0, \delta) = 1 + O(\delta) \quad (5.8)$$

Here, the notation  $x \lesssim y$  means that there exists a constant  $C$ , independent of  $\delta$  and  $u_0$ , but depending possibly on  $f$ , such that  $x \leq Cy$ .

*Hint:* This is easily proved by a bootstrap argument in  $v$  on  $C_{u_0}$  (for example one can try to improve the assumptions  $\frac{1}{2}|u_0| \leq r \leq 2 + \frac{1}{2}|u_0|$  and  $0 \leq \partial_v r \leq 1$  on  $C_{u_0}$ ).

- b) Let  $u_* \in [u_0, -1]$ . Suppose there exists a number  $B > 0$  such that the following bounds hold in  $[u_0, u_*] \times [0, \delta]$ :

$$-2B \leq \partial_u r \leq -\frac{1}{2B}, \quad (5.9)$$

$$|\partial_v r| \leq 2B, \quad (5.10)$$

$$\frac{1}{2B} \leq \Omega^2 \leq 2B. \quad (5.11)$$

We will refer to these estimates as the “bootstrap assumptions.” Use the bootstrap assumptions to infer the following estimates in  $[u_0, u_*] \times [0, \delta]$ :

$$|r - 1 + \frac{1}{2}u| \leq 2B\delta, \quad (5.12)$$

$$|r\phi| \lesssim_B \delta^{1/2}, \quad (5.13)$$

$$|r^2 \partial_u \phi| \lesssim_B \delta^{1/2}, \quad (5.14)$$

$$|r \partial_v \phi| \lesssim_B \delta^{-1/2} \quad (5.15)$$

if  $\delta$  is sufficiently small (independent of  $u_0$ ). Here, the notation  $x \lesssim_B y$  means that there exists a constant  $C$ , independent of  $\delta$  and  $u_0$ , but depending possibly on  $f$  and  $B$ , such that  $x \leq Cy$ .

*Hint:* Write the wave equation in the form  $\partial_u \partial_v (r\phi) = \dots$  and estimate the right-hand side using the bootstrap assumptions. Then integrate in  $u$  and  $v$ , and use the fact that the integral in  $v$  gives a good power of  $\delta$ . Don’t forget to include the initial data (estimated in the previous step) when integrating!

- c) Use the above estimates for the scalar field to show that for  $B$  sufficiently large and  $\delta$  sufficiently small (depending on  $B$ ), the estimates (5.9)–(5.11) hold in  $[u_0, u_*] \times [0, \delta]$  with  $2B$  replaced by  $B$ .

*Hint:* To estimate  $\partial_v r$ , either use the  $v$ -Raychaudhuri equation or first bound the Hawking mass  $m$  to get an improved estimate on  $|\partial_u \partial_v r|$ .

d) Show that the solution exists in the full rectangle  $[u_0, -1] \times [0, \delta]$  and satisfies

$$|r - 1 + \frac{1}{2}u| \lesssim \delta, \quad (5.16)$$

$$\partial_u r \sim -1, \quad (5.17)$$

$$|\partial_v r| \lesssim 1, \quad (5.18)$$

$$|r\phi| \lesssim \delta^{1/2}, \quad (5.19)$$

$$|r^2 \partial_u \phi| \lesssim \delta^{1/2}, \quad (5.20)$$

$$|r \partial_v \phi| \lesssim \delta^{-1/2}. \quad (5.21)$$

*Hint:* Use a continuity argument: Consider the set  $\mathcal{A}$  consisting of  $u_* \in [u_0, -1]$  such that the solution exists on the rectangle  $[u_0, u_*] \times [0, \delta]$  and satisfies the bootstrap assumptions (5.9)–(5.11). Show that if  $B$  is sufficiently large and  $\delta$  is sufficiently small, then  $\mathcal{A}$  is nonempty, closed, and open.

e) Conclude trapped surface formation as follows: Using the above heirarchy of estimates, compute  $r(-1, \delta)$  and  $m(-1, \delta)$  and show that  $\frac{2m}{r}(-1, \delta) > 1$  for  $\delta$  sufficiently small.

*Hint:* Estimate  $\partial_u m$ .

□

### Extended hints:

b) (5.12) is proved by integrating (5.10). To estimate the scalar field, we write the wave equation as

$$\partial_u \partial_v (r\phi) = \left( -\frac{\Omega^2}{4r^2} - \frac{\partial_u r \partial_v r}{r^2} \right) r\phi, \quad (5.22)$$

which using the bootstrap assumptions implies

$$|\partial_u \partial_v (r\phi)| \lesssim_B \frac{\|r\phi\|_{L^\infty}}{r^2}. \quad (5.23)$$

Integrating in  $u$ , using the estimate for  $\partial_v(r\phi) = \phi \partial_v r + r \partial_v \phi$  obtained from part a) on  $C_{u_0}$ , and the fact that  $r^{-2}$  is integrable in  $u$  on  $[u_0, u_1]$ , we obtain

$$\|\partial_v(r\phi)\|_{L^\infty} \lesssim_B \delta^{-1/2} + \|r\phi\|_{L^\infty}. \quad (5.24)$$

Integrating in  $v$ , we find  $\|r\phi\|_{L^\infty} \lesssim_B \delta^{1/2} + \delta \|r\phi\|_{L^\infty}$ . The second term can be absorbed and we conclude (5.13). Inserting this into (5.24), we conclude (5.15). Integrating (5.22) in  $v$ , we obtain  $|\partial_u(r\phi)| \lesssim_B r^{-2} \delta^{3/2}$ . We then have

$$|r^2 \partial_u \phi| \leq |\partial_u r| |r\phi| + r |\partial_u(r\phi)| \lesssim_B \delta^{1/2}, \quad (5.25)$$

which is (5.14).

c) The bootstrap assumption on  $\Omega^2$  is immediately improved by integrating the wave equation and using smallness in the  $v$  direction. This works similarly for  $\partial_u r$  by integrating the wave equation for  $r$  in  $v$ . To estimate  $\partial_v r$ , we integrate Raychaudhuri's equation<sup>1</sup> in  $v$ :

$$|\Omega^{-2} \partial_v r - \frac{1}{2}| \lesssim \int_0^\delta r \Omega^{-2} (\partial_v \phi)^2 dv' \lesssim 1. \quad (5.26)$$

<sup>1</sup>The use of Raychaudhuri's equation here can be avoided if one is willing to take  $|u_0| \lesssim 1$ . In that case, the estimate can be retrieved by using an integrating factor on the wave equation for  $r$ , but I think this generates terms diverging in  $|u_0|$ .

e) It follows from the first estimate in part d) that  $r(u_1, \delta) = 1 + O(\delta)$  by the definition of  $u_1$ . We then estimate

$$|m(u_1, \delta) - m(u_0, \delta)| \leq \int_{u_0}^{-1} 2\Omega^{-2} r^2 |\partial_v r| (\partial_u \phi)^2 du' \lesssim \delta. \quad (5.27)$$

Combined with  $m(u_0, \delta) = 1 + O(\delta)$ , this implies  $m(u_1, \delta) = 1 + O(\delta)$  and consequently,

$$\frac{2m}{r}(u_1, \delta) = 2 + O(\delta) > 1 \quad (5.28)$$

for  $\delta$  sufficiently small.