Retiring the third law of black hole thermodynamics

Ryan Unger

Princeton University

Zürich PDE & mathematical physics seminar
December 2022

Classical thermodynamics

Classical thermodynamics

Law	Thermodynamics
Zeroth	T constant throughout body in thermal equilibrium

Classical thermodynamics

Law	Thermodynamics
Zeroth	T constant throughout body in thermal equilibrium
First	$dE = TdS + \text{work terms}\,(-PdV)$

Classical thermodynamics

Law	Thermodynamics
Zeroth	T constant throughout body in thermal equilibrium
First	$dE = TdS + \text{work terms}\,(PdV)$
Second	$\delta S \geq 0$ in any process

Classical thermodynamics

Law	Thermodynamics
Zeroth	T constant throughout body in thermal equilibrium
First	$dE = TdS + \text{work terms}\,(PdV)$
Second	$\delta S \geq 0$ in any process
Third	Impossible to achieve $T = 0$ in finite time

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

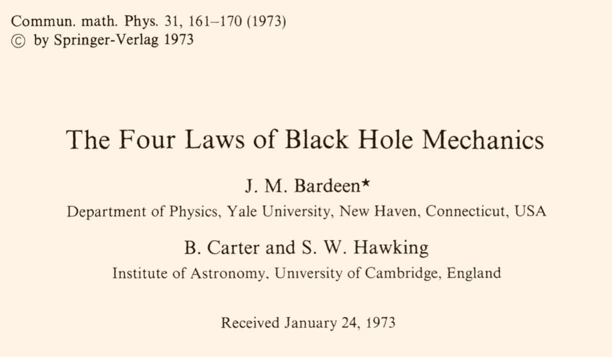
J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between black hole dynamics and classical thermodynamics.

Black hole thermodynamics


Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between black hole dynamics and classical thermodynamics.

- What is a black hole?
- What is the analogue of temperature T?
- What is the analogue of entropy S?

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \tag{EFE}$$

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \tag{EFE}$$

Example: Minkowski space. $\mathcal{M}^{3+1} = \mathbb{R}^{3+1}_{t,x,y,z}$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

This metric describes the geometry of special relativity with speed of light $c = 1$.

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \tag{EFE}$$

Example: Minkowski space. $\mathcal{M}^{3+1} = \mathbb{R}^{3+1}_{t,x,y,z}$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

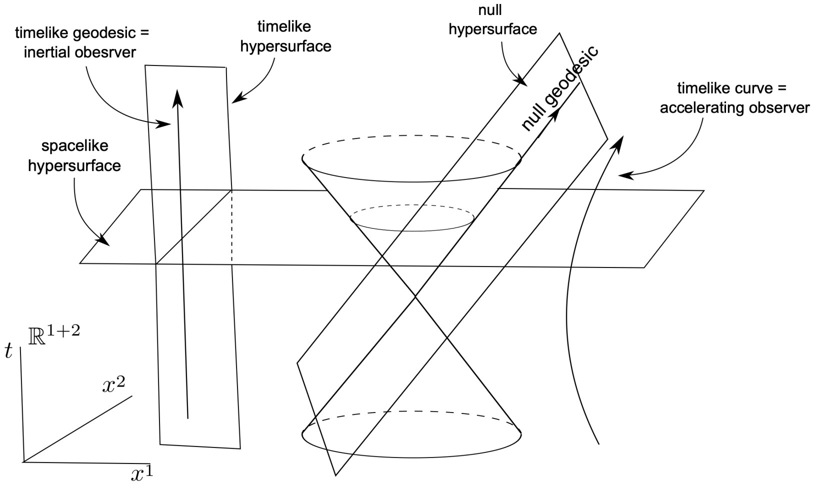
This metric describes the geometry of special relativity with speed of light $c = 1$.

Given a vector $v \in T_p\mathcal{M}$:

- $g(v, v) < 0$, v is *timelike*
- $g(v, v) = 0$, v is *null*
- $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

Crash course on black holes: Geometry of Minkowski space



Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric "at infinity."

Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric "at infinity."

An asymptotically flat spacetime is said to contain a **black hole region**

$$\mathcal{BH} \subset \mathcal{M}$$

if observers and signals originating in $\mathcal{BH}$ cannot reach infinity.

Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric "at infinity."

An asymptotically flat spacetime is said to contain a **black hole region**

$$\mathcal{BH} \subset \mathcal{M}$$

if observers and signals originating in $\mathcal{BH}$ cannot reach infinity.

The **event horizon** of the black hole is its topological boundary

$$\mathcal{H}^+ := \partial(\mathcal{BH}).$$

Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric "at infinity."

An asymptotically flat spacetime is said to contain a **black hole region**

$$\mathcal{BH} \subset \mathcal{M}$$

if observers and signals originating in $\mathcal{BH}$ cannot reach infinity.

The **event horizon** of the black hole is its topological boundary

$$\mathcal{H}^+ := \partial(\mathcal{BH}).$$

The event horizon $\mathcal{H}^+$ is a **null hypersurface**: it is ruled by geodesics with null tangent vectors.

Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric "at infinity."

An asymptotically flat spacetime is said to contain a **black hole region**

$$\mathcal{BH} \subset \mathcal{M}$$

if observers and signals originating in $\mathcal{BH}$ cannot reach infinity.

The **event horizon** of the black hole is its topological boundary

$$\mathcal{H}^+ := \partial(\mathcal{BH}).$$

The event horizon $\mathcal{H}^+$ is a **null hypersurface**: it is ruled by geodesics with null tangent vectors.

Crash course on black holes: The Cauchy problem

- Dynamics in general relativity is properly phrased in terms of the *Cauchy problem*
- *Cauchy data*: induced (Riemannian) geometry on a spacelike hypersurface (Σ^3, g_0, k_0) together with initial data for matter fields

- Dynamics in general relativity is properly phrased in terms of the *Cauchy problem*
- *Cauchy data*: induced (Riemannian) geometry on a spacelike hypersurface (Σ^3, g_0, k_0) together with initial data for matter fields

Theorem (Choquet-Bruhat '52, Choquet-Bruhat–Geroch '69).
Any Cauchy data $(\Sigma^3, g_0, k_0, matter)$ for the Einstein equations coupled to a suitable matter model admits a unique maximal globally hyperbolic development which solves the Einstein field equations

$$Ric(g) - \frac{1}{2}R(g)g = 2T.$$

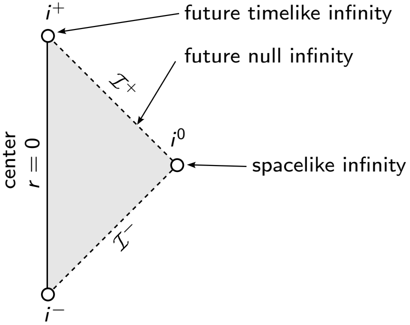
Crash course on black holes: Penrose diagrams

Penrose diagrams: A $(1+1)$-dimensional diagrammatic representation of spacetimes. This is particularly useful for visualizing causality.

Crash course on black holes: Penrose diagrams

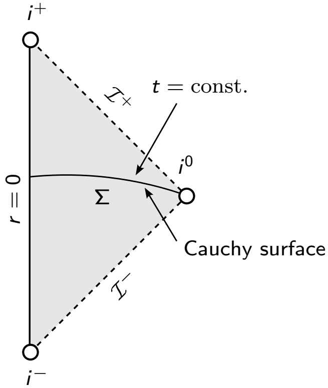
Penrose diagrams: A $(1+1)$-dimensional diagrammatic representation of spacetimes. This is particularly useful for visualizing causality.

- 45 degree lines correspond to null hypersurfaces in $(3+1)$ dimensions
- steeper than 45: timelike
- shallower than 45: spacelike

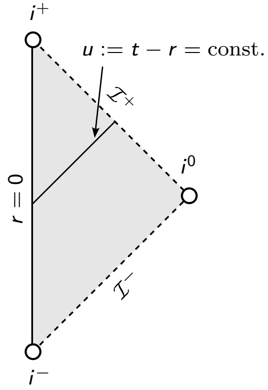


Penrose diagram of Minkowski space

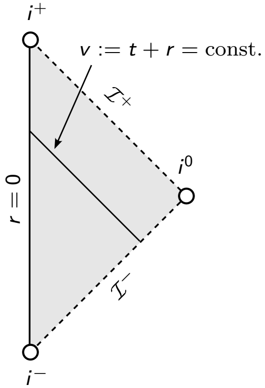
Crash course on black holes: Minkowski space



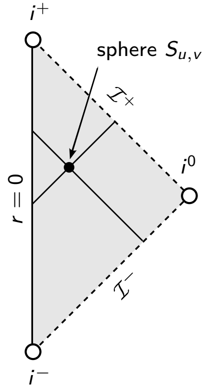
Crash course on black holes: Minkowski space



Crash course on black holes: Minkowski space



Crash course on black holes: Minkowski space



Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

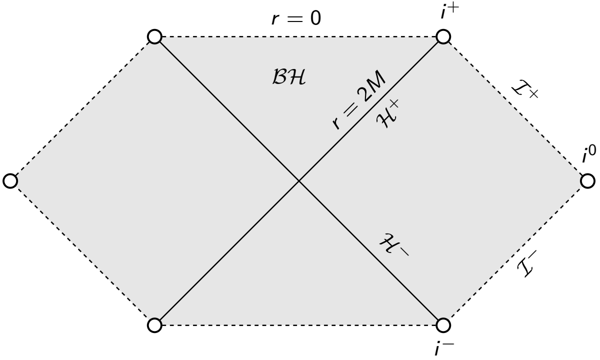
- Solves the **Einstein vacuum** equations
- Describes a static, nonrotating black hole

Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein vacuum** equations
- Describes a static, nonrotating black hole

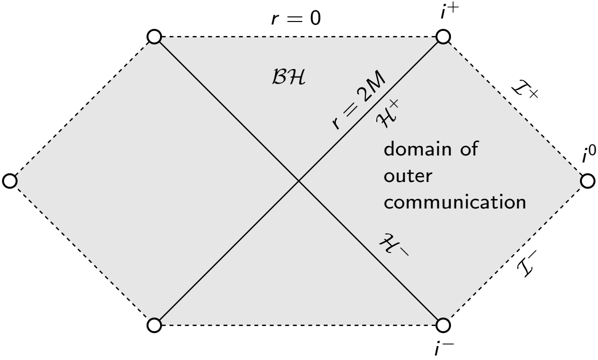


Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- Solves the **Einstein vacuum** equations
- Describes a static, nonrotating black hole

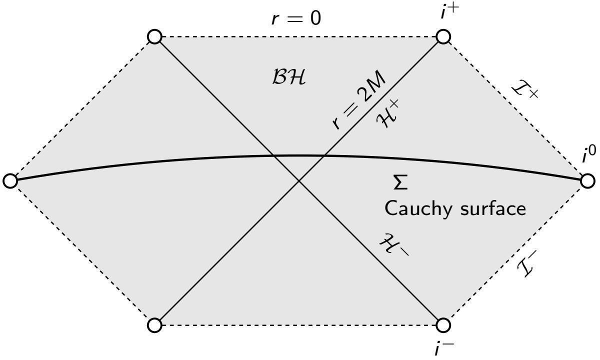


Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein vacuum** equations
- Describes a static, nonrotating black hole



Crash course on black holes: Reissner–Nordström

- Mass $M > 0$ and charge $0 \leq |e| \leq M$

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

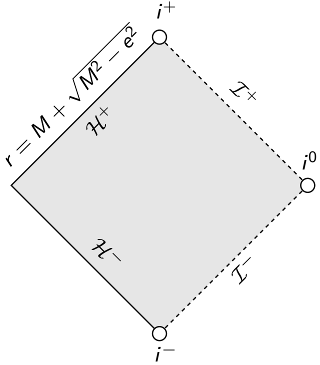
- Solves the **Einstein-Maxwell** equations
- Describes a static, charged, nonrotating black hole
- $e = 0$: Schwarzschild

Crash course on black holes: Reissner–Nordström

- Mass $M > 0$ and charge $0 \le |e| \le M$

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein-Maxwell** equations
- Describes a static, charged, nonrotating black hole
- $e = 0$: Schwarzschild



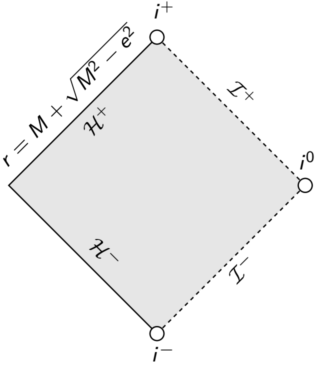
Subextremal $|e| < M$

Crash course on black holes: Reissner–Nordström

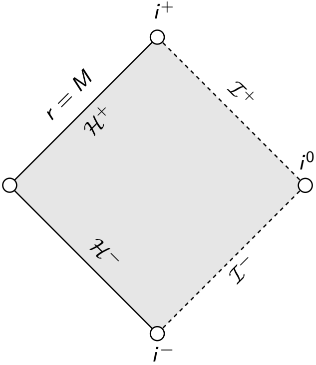
- Mass $M > 0$ and charge $0 \leq |e| \leq M$

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein-Maxwell** equations
- Describes a static, charged, nonrotating black hole
- $e = 0$: Schwarzschild



Subextremal $|e| < M$



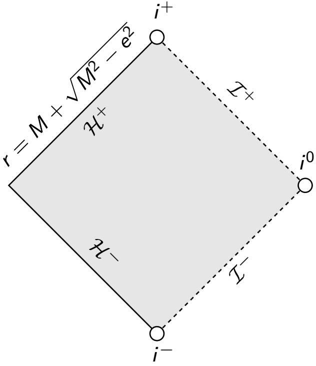
Extremal $|e| = M$

Crash course on black holes: Reissner–Nordström

- Mass $M > 0$ and charge $0 \leq |e| \leq M$

$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein-Maxwell** equations
- Describes a static, charged, nonrotating black hole
- $e = 0$: Schwarzschild



Subextremal $|e| < M$ **Extremal** $|e| = M$

Superextremal $|e| > M$. Does not contain a black hole!

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon
- Associated to a Killing horizon is a function $\kappa \geq 0$, called the **surface gravity**

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon
- Associated to a Killing horizon is a function $\kappa \geq 0$, called the **surface gravity**
- Physically, κ measures the force required to hold a test mass "at the horizon" "from infinity"

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon
- Associated to a Killing horizon is a function $\kappa \geq 0$, called the **surface gravity**
- Physically, κ measures the force required to hold a test mass "at the horizon" "from infinity"
- For Reissner–Nordström,

$$\kappa(g_{M,e}) = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon
- Associated to a Killing horizon is a function $\kappa \geq 0$, called the **surface gravity**
- Physically, κ measures the force required to hold a test mass "at the horizon" "from infinity"
- For Reissner–Nordström,

$$\kappa(g_{M,e}) = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Extremal:** $\kappa = 0$ ($|e| = M$ Reissner–Nordström)
- **Subextremal:** $\kappa > 0$ ($|e| < M$, including Schwarzschild)

Crash course on black holes: Surface gravity and extremality

- The event horizons in these exact solutions are **Killing horizons**: The spacetime admits a Killing field which is null and tangent to the horizon
- Associated to a Killing horizon is a function $\kappa \geq 0$, called the **surface gravity**
- Physically, κ measures the force required to hold a test mass "at the horizon" "from infinity"
- For Reissner–Nordström,

$$\kappa(g_{M,e}) = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Extremal:** $\kappa = 0$ ($|e| = M$ Reissner–Nordström)
- **Subextremal:** $\kappa > 0$ ($|e| < M$, including Schwarzschild)
- The celebrated **Kerr** family also has subextremal and extremal members

Black hole thermodynamics

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	surface gravity $\kappa \not\to 0$ in finite time

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	surface gravity $\kappa \not\to 0$ in finite time

- Proposed by Bardeen–Carter–Hawking '73

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	surface gravity $\kappa \not\to 0$ in finite time

- Proposed by Bardeen–Carter–Hawking '73
- Hawking radiation—interpret $\kappa \propto T$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	surface gravity $\kappa \not\to 0$ in finite time

- Proposed by Bardeen–Carter–Hawking '73
- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$\delta S \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite time	surface gravity $\kappa \not\to 0$ in finite time

- Proposed by Bardeen–Carter–Hawking '73
- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$
- Laws 0, 1, and 2 proved (when suitably interpreted!) by Hawking, Carter, Bardeen–Carter–Hawking, Wald

Retiring the third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

Conjecture (The third law, BCH '73, Israel '86).

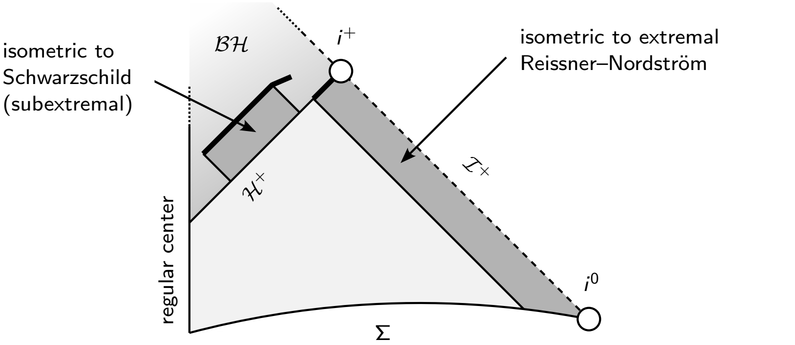
A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

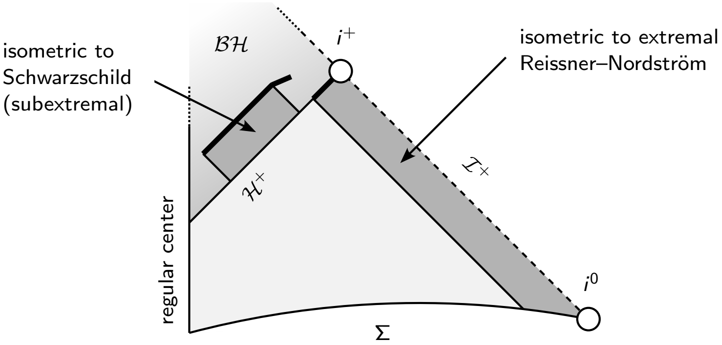
More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:

Retiring the third law



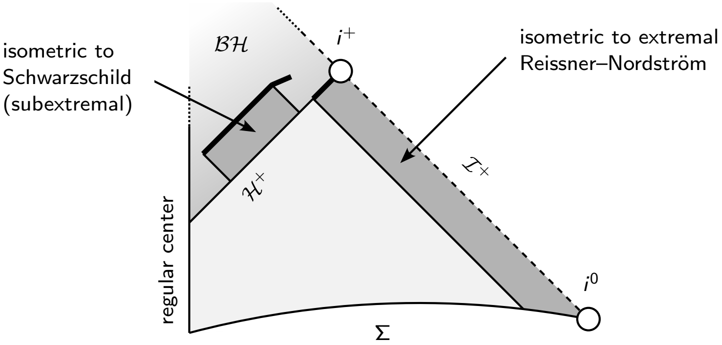
- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse

Retiring the third law



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse
- Forms an exactly Schwarzschild “apparent horizon”
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time

Retiring the third law



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse
- Forms an exactly Schwarzschild "apparent horizon"
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time
- The solution is arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example

The strategy

Steps in the proof:

The strategy

Steps in the proof:

1. Construct spacetime teleologically

The strategy

Steps in the proof:

1. Construct spacetime teleologically
   - Teleological construction allows us to locate the event horizon

The strategy

Steps in the proof:

1. Construct spacetime teleologically
   - Teleological construction allows us to locate the event horizon
2. Extract a Cauchy hypersurface Σ and deduce that the teleologically constructed spacetime arises *dynamically* from initial data

The strategy

Steps in the proof:

1. Construct spacetime teleologically
   - Teleological construction allows us to locate the event horizon
2. Extract a Cauchy hypersurface Σ and deduce that the teleologically constructed spacetime arises *dynamically* from initial data

The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be known to exist, much less be located, *without knowing the entire future of the spacetime*.

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) scalar field ϕ

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) scalar field ϕ
- Equations:

$$
\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) &(1)\\
\nabla^{\mu}F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu\phi}) &(2)\\
g^{\mu\nu}D_\mu D_\nu\phi &= 0 &(3)\\
T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} &(4)\\
T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} &(5)
\end{aligned}
$$

- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ)

Einstein–Maxwell-charged scalar field system

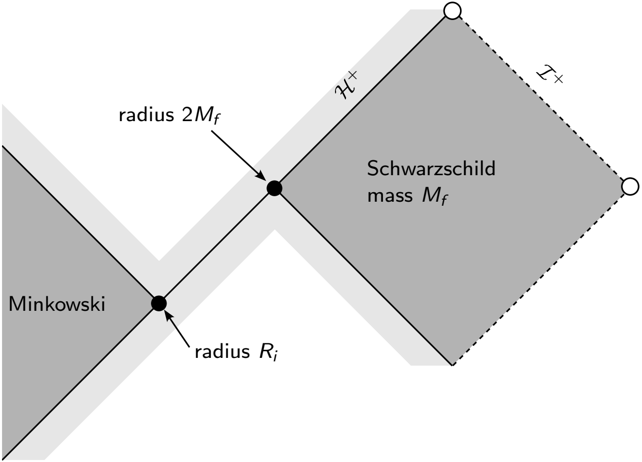
- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) scalar field ϕ
- Equations:

$$
\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) && (1)\\
\nabla^{\mu} F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu \phi}) && (2)\\
g^{\mu\nu} D_\mu D_\nu \phi &= 0 && (3)\\
T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \tfrac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} && (4)\\
T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_\mu \phi \overline{D_\nu \phi}) - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi} && (5)
\end{aligned}
$$

- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ)
- Spherical symmetry!

Prototype: Minkowski to Schwarzschild gluing

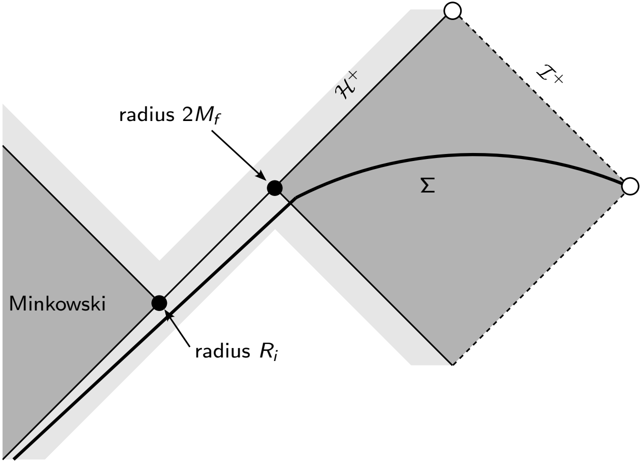
The type of Penrose diagram we want:



We want to **glue** solutions of the Einstein-scalar field system

Prototype: Minkowski to Schwarzschild gluing

The type of Penrose diagram we want:



We want to **glue** solutions of the Einstein-scalar field system

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

$$\Box_g \phi = 0 \qquad (*)$$

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

$$\Box_g \phi = 0 \qquad (*)$$

Question.

Given two domains $\mathfrak{R}_1, \mathfrak{R}_2 \subset \mathcal{M}$ and functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}$$
$$\phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $(*)$

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

$$\Box_g \phi = 0 \qquad (*)$$

Question.

Given two domains $\mathfrak{R}_1, \mathfrak{R}_2 \subset \mathcal{M}$ and functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}$$
$$\phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $(*)$, when does there exist a function

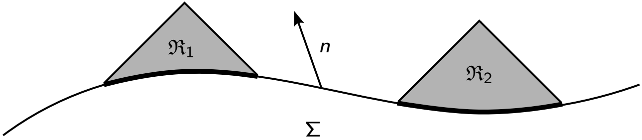
$$\phi : \mathcal{M} \to \mathbb{R}$$

solving $(*)$, such that

$$\phi|_{\mathfrak{R}_1} = \phi_1$$
$$\phi|_{\mathfrak{R}_2} = \phi_2\,?$$

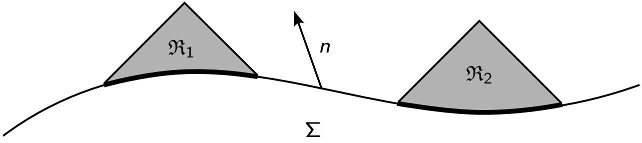
Spacelike gluing for the linear wave equation

Consider first the case of two spacelike separated regions:



Spacelike gluing for the linear wave equation

Consider first the case of two spacelike separated regions:



In this case the gluing problem for $(*)$ is trivial:

Proposition (Cauchy problem for the wave equation).

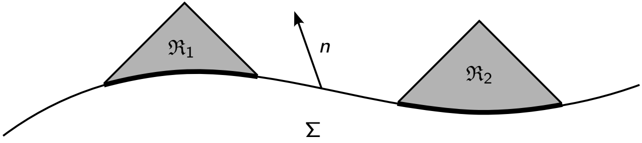
Given a spacelike hypersurface $\Sigma \subset (\mathcal{M}, g)$ and functions $\phi_0, \phi_1 \in C^\infty(\Sigma)$, there exists a unique function $\phi : \mathcal{M} \to \mathbb{R}$ solving

$$\Box_g \phi = 0 \tag{$*$}$$

such that $\phi|_\Sigma = \phi_0$, $n(\phi)|_\Sigma = \phi_1$.

Spacelike gluing for the linear wave equation

Consider first the case of two spacelike separated regions:



In this case the gluing problem for $(*)$ is trivial:

Proposition (Cauchy problem for the wave equation).

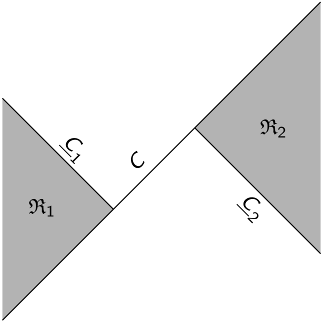
Given a spacelike hypersurface $\Sigma \subset (\mathcal{M}, g)$ and functions $\phi_0, \phi_1 \in C^\infty(\Sigma)$, there exists a unique function $\phi : \mathcal{M} \to \mathbb{R}$ solving

$$\Box_g \phi = 0 \qquad (*)$$

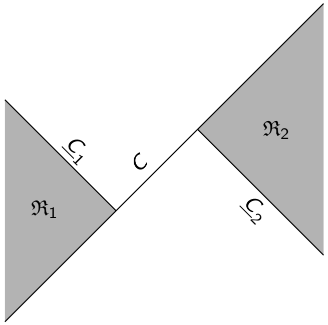
such that $\phi|_\Sigma = \phi_0$, $n(\phi)|_\Sigma = \phi_1$.

Analogue for the Einstein equations is a highly nontrivial problem in elliptic PDE!
(Corvino–Schoen, Chruściel–Delay, Li–Yu, Carlotto–Schoen, Li–Mei, Hintz, ...)

Characteristic/null gluing for the linear wave equation



Characteristic/null gluing for the linear wave equation



Analogy:

- $\mathfrak{R}_1$ is Minkowski
- $\mathfrak{R}_2$ is Schwarzschild
- C will be the event horizon $\mathcal{H}^+$

The characteristic initial value problem

Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

The characteristic initial value problem

Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

$$\Box_g \phi = 0 \qquad (*)$$

and $\phi|_{C \cup \underline{C}} = \phi_0$.

The characteristic initial value problem

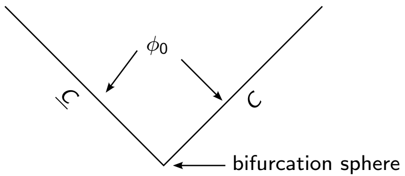
Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

$$\Box_g \phi = 0 \qquad (*)$$

and $\phi|_{C \cup \underline{C}} = \phi_0$.



The characteristic initial value problem

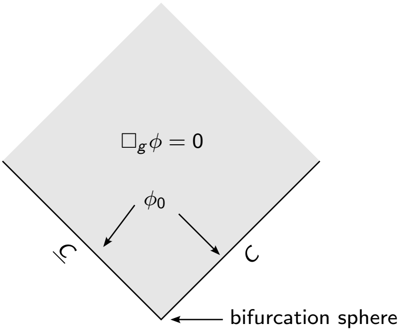
Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

$$\Box_g \phi = 0 \qquad (*)$$

and $\phi|_{C \cup \underline{C}} = \phi_0$.



The characteristic initial value problem

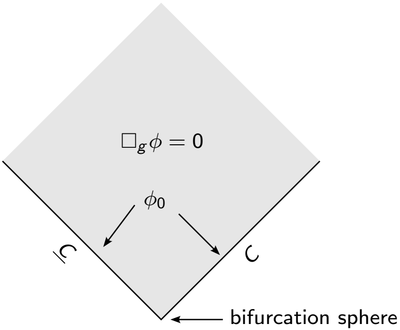
Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

$$\Box_g \phi = 0 \qquad (*)$$

and $\phi|_{C \cup \underline{C}} = \phi_0$.



Characteristic data for the wave equation does not involve derivatives.

The null "constraints"

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface.

The null "constraints"

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface. Example:

- Wave equation on Minkowski $\mathbb{R}^{3+1}$
- $u = t - r$, $v = t + r$ double null coordinates

The null “constraints”

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface. Example:

- Wave equation on Minkowski $\mathbb{R}^{3+1}$
- $u = t - r$, $v = t + r$ double null coordinates
- $\phi = \phi(t, r, \theta^1, \theta^2) = \phi(u, v, \theta^1, \theta^2)$ satisfies

$$\partial_u \partial_v (r\phi) = \not\Delta (r\phi) \qquad (*)$$

The null "constraints"

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface. Example:

- Wave equation on Minkowski $\mathbb{R}^{3+1}$
- $u = t - r$, $v = t + r$ double null coordinates
- $\phi = \phi(t, r, \theta^1, \theta^2) = \phi(u, v, \theta^1, \theta^2)$ satisfies

$$\partial_u \partial_v (r\phi) = \not{\Delta}(r\phi) \tag{$*$}$$

- Integrate $(*)$ on C:

$$\partial_u (r\phi)|_{v_2} - \partial_u (r\phi)|_{v_1} = \int_{v_1}^{v_2} \not{\Delta}(r\phi)\, dv'$$

The null "constraints"

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface. Example:

- Wave equation on Minkowski $\mathbb{R}^{3+1}$
- $u = t - r$, $v = t + r$ double null coordinates
- $\phi = \phi(t, r, \theta^1, \theta^2) = \phi(u, v, \theta^1, \theta^2)$ satisfies

$$\partial_u \partial_v (r\phi) = \Delta\!\!\!/ (r\phi) \qquad (*)$$

- Integrate $(*)$ on C:

$$\partial_u (r\phi)|_{v_2} - \partial_u (r\phi)|_{v_1} = \int_{v_1}^{v_2} \Delta\!\!\!/ (r\phi)\, dv'$$

- $\Longrightarrow \partial_u \phi|_C$ can be calculated from (1) $\partial_u \phi$ at the bifurcation sphere and (2) $\phi|_C$

The null "constraints"

In any well posed initial value problem, there needs to be a mechanism to compute the full solution on the initial data hypersurface. Example:

- Wave equation on Minkowski $\mathbb{R}^{3+1}$
- $u = t - r$, $v = t + r$ double null coordinates
- $\phi = \phi(t, r, \theta^1, \theta^2) = \phi(u, v, \theta^1, \theta^2)$ satisfies

$$\partial_u \partial_v (r\phi) = \not{\Delta}(r\phi) \qquad (*)$$

- Integrate $(*)$ on C:

$$\partial_u (r\phi)|_{v_2} - \partial_u (r\phi)|_{v_1} = \int_{v_1}^{v_2} \not{\Delta}(r\phi)\, dv'$$

- $\Longrightarrow \partial_u \phi|_C$ can be calculated from (1) $\partial_u \phi$ at the bifurcation sphere and (2) $\phi|_C$
- $(*)$ lets us compute ϕ to arbitrarily high order on C

The null “constraints”

In general, the wave equation can be interpreted as a transport equation for the transverse derivative of ϕ, sourced by ϕ and tangential derivatives (which are a part of the characteristic data)

The null “constraints”

In general, the wave equation can be interpreted as a transport equation for the transverse derivative of ϕ, sourced by ϕ and tangential derivatives (which are a part of the characteristic data)

$$\partial_v \left(\partial_u \phi\right) + (\mathit{known})\partial_u \phi = \mathit{data}$$

The null "constraints"

In general, the wave equation can be interpreted as a transport equation for the transverse derivative of ϕ, sourced by ϕ and tangential derivatives (which are a part of the characteristic data)

$$\partial_v \left(\partial_u \phi\right) + (\mathit{known})\partial_u \phi = \mathit{data}$$

Commuting the wave equation with transverse derivatives gives an inductive system of transport equations for all orders!

The null "constraints"

In general, the wave equation can be interpreted as a transport equation for the transverse derivative of ϕ, sourced by ϕ and tangential derivatives (which are a part of the characteristic data)

$$\partial_v\left(\partial_u\phi\right) + (\mathit{known})\partial_u\phi = \mathit{data}$$

Commuting the wave equation with transverse derivatives gives an inductive system of transport equations for all orders!

These equations have become known as the *null "constraints"* and are very distinct from the "usual" constraint equations for Einstein.

The null “constraints”

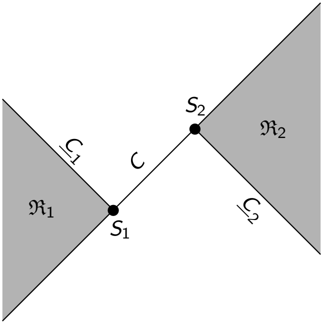
In general, the wave equation can be interpreted as a transport equation for the transverse derivative of ϕ, sourced by ϕ and tangential derivatives (which are a part of the characteristic data)

$$\partial_v\left(\partial_u\phi\right) + (\mathit{known})\partial_u\phi = \mathit{data}$$

Commuting the wave equation with transverse derivatives gives an inductive system of transport equations for all orders!

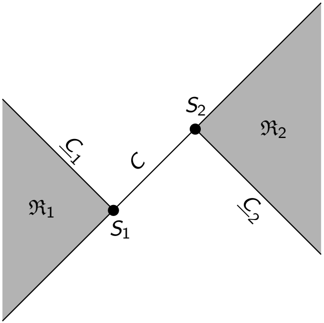
These equations have become known as the *null “constraints”* and are very distinct from the “usual” constraint equations for Einstein. (The Einstein equations in double null gauge also have true constraints which are more reminiscent of the spacelike constraints.)

Back to characteristic gluing



The C^k characteristic gluing problem reduces to:

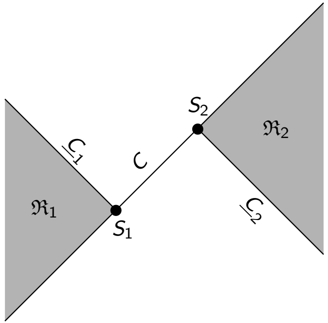
Back to characteristic gluing



The C^k characteristic gluing problem reduces to:

- Given two spheres S_1 and S_2 along an outgoing null cone C
- Given k ingoing and outgoing derivatives of ϕ at S_1 and S_2

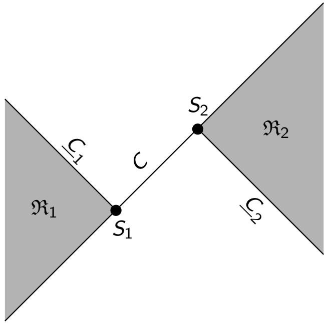
Back to characteristic gluing



The C^k characteristic gluing problem reduces to:

- Given two spheres S_1 and S_2 along an outgoing null cone C
- Given k ingoing and outgoing derivatives of ϕ at S_1 and S_2
- Prescribe ϕ along the part of C between S_1 and S_2 so that the outgoing derivatives agree with the given ones and the solutions of the transport equations for the ingoing derivatives have the specified initial and final values.

Back to characteristic gluing



The C^k characteristic gluing problem reduces to:

- Given two spheres S_1 and S_2 along an outgoing null cone C
- Given k ingoing and outgoing derivatives of ϕ at S_1 and S_2
- Prescribe ϕ along the part of C between S_1 and S_2 so that the outgoing derivatives agree with the given ones and the solutions of the transport equations for the ingoing derivatives have the specified initial and final values.
- Initial/boundary value problem for a system of coupled ODEs where you are allowed to (almost) freely prescribe the potential

Previous work on characteristic gluing

Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges

Previous work on characteristic gluing

Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges
- Conserved charges arise from **conservation laws** along C (Generalizing the conservation law responsible for the *Aretakis instability*)

Previous work on characteristic gluing

Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges
- Conserved charges arise from **conservation laws** along C (Generalizing the conservation law responsible for the *Aretakis instability*)

Perturbative characteristic gluing for the Einstein vacuum equations by **Aretakis–Czimek–Rodnianski** '21

- Gluing for two data sets close to Minkowski space

Previous work on characteristic gluing

Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges
- Conserved charges arise from **conservation laws** along C (Generalizing the conservation law responsible for the *Aretakis instability*)

Perturbative characteristic gluing for the Einstein vacuum equations by **Aretakis–Czimek–Rodnianski** '21

- Gluing for two data sets close to Minkowski space
- Linearize the Einstein equations about Minkowski, develop procedure to glue the linearized equations (Dafermos–Holzegel–Rodnianski)
- Inverse function theorem

Previous work on characteristic gluing

Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges
- Conserved charges arise from **conservation laws** along C (Generalizing the conservation law responsible for the *Aretakis instability*)

Perturbative characteristic gluing for the Einstein vacuum equations by **Aretakis–Czimek–Rodnianski** '21

- Gluing for two data sets close to Minkowski space
- Linearize the Einstein equations about Minkowski, develop procedure to glue the linearized equations (Dafermos–Holzegel–Rodnianski)
- Inverse function theorem
- Conserved charges arise from kernel of linearized equations and form 10-dimensional obstruction space

Previous work on characteristic gluing

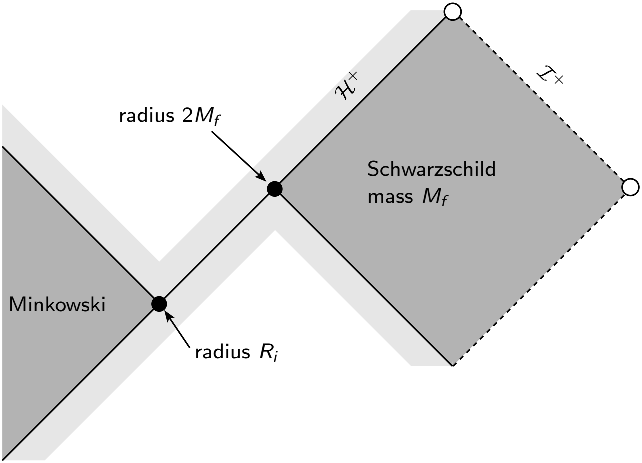
Complete C^1 theory for the linear wave equation obtained by **Aretakis** '17

- Finitely many obstructions: conserved charges
- Conserved charges arise from **conservation laws** along C (Generalizing the conservation law responsible for the *Aretakis instability*)

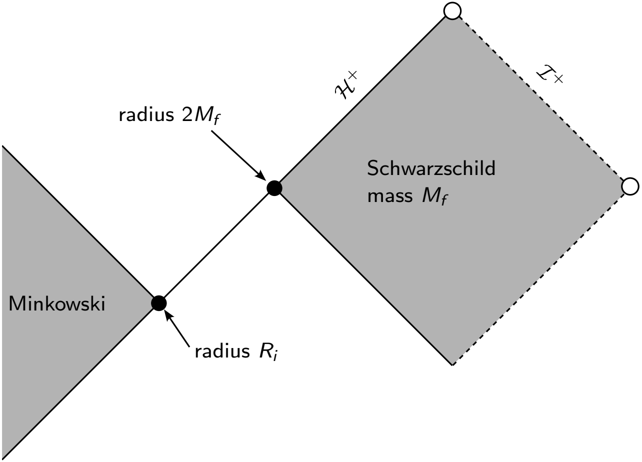
Perturbative characteristic gluing for the Einstein vacuum equations by **Aretakis–Czimek–Rodnianski** '21

- Gluing for two data sets close to Minkowski space
- Linearize the Einstein equations about Minkowski, develop procedure to glue the linearized equations (Dafermos–Holzegel–Rodnianski)
- Inverse function theorem
- Conserved charges arise from kernel of linearized equations and form 10-dimensional obstruction space
- **Czimek–Rodnianski** '22: obstruction free characteristic gluing by adding high-frequency perturbations

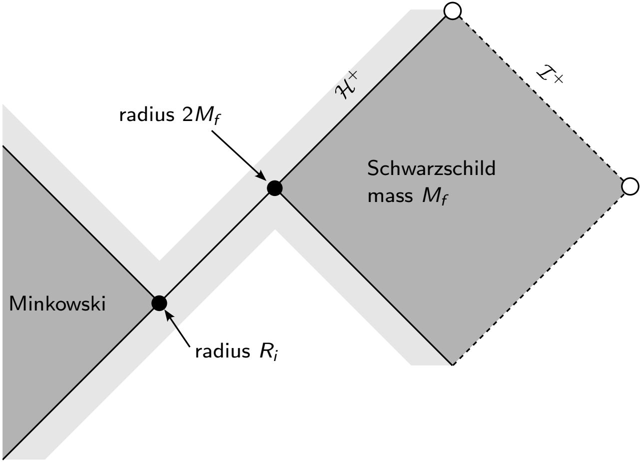
Minkowski to Schwarzschild gluing



Minkowski to Schwarzschild gluing



Minkowski to Schwarzschild gluing



Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$

$$g = -\Omega^2 du\, dv + r^2 g_{S^2}$$

- $\Omega(u, v) > 0$ lapse, $r(u, v) > 0$ area-radius

Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$

$$g = -\Omega^2 du\, dv + r^2 g_{S^2}$$

- $\Omega(u, v) > 0$ lapse, $r(u, v) > 0$ area-radius
- Wave equations

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r} \tag{6}$$

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} \tag{7}$$

$$\partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} + 2\frac{\partial_u r \partial_v r}{r^2} \tag{8}$$

Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$
$$g = -\Omega^2 du\, dv + r^2 g_{S^2}$$
- $\Omega(u, v) > 0$ lapse, $r(u, v) > 0$ area-radius
- Wave equations

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r} \tag{6}$$

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} \tag{7}$$

$$\partial_u \partial_v \log(\Omega^2) = \frac{\Omega^2}{2r^2} + 2\frac{\partial_u r \partial_v r}{r^2} \tag{8}$$

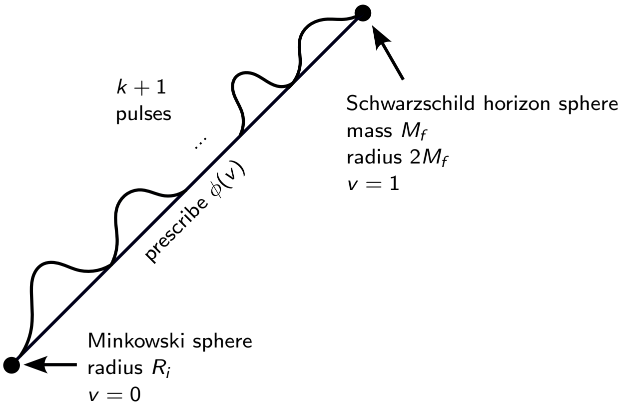
- Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2 \tag{9}$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_v \phi)^2 \tag{10}$$

Minkowski to Schwarzschild gluing

Theorem (Kehle–U. '22).
For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1 \le j \le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1 \le j \le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1\leq j\leq k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0!$)

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1 \le j \le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0$!)
- For each α, generate $r(v)$, $\partial_u r(v)$, $\partial_u \phi_\alpha(v)$, etc.

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1 \le j \le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0$!)
- For each α, generate $r(v)$, $\partial_u r(v)$, $\partial_u \phi_\alpha(v)$, etc.
- The antipodal map $\alpha \mapsto -\alpha$ leaves geometric quantities fixed (r, Ω^2, and derivatives)

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1 \le j \le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0$!)
- For each α, generate $r(v)$, $\partial_u r(v)$, $\partial_u \phi_\alpha(v)$, etc.
- The antipodal map $\alpha \mapsto -\alpha$ leaves geometric quantities fixed (r, Ω^2, and derivatives)
- The antipodal map $\alpha \mapsto -\alpha$ changes the sign of the scalar field and its derivatives

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1\le j\le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0$!)
- For each α, generate $r(v)$, $\partial_u r(v)$, $\partial_u \phi_\alpha(v)$, etc.
- The antipodal map $\alpha \mapsto -\alpha$ leaves geometric quantities fixed (r, Ω^2, and derivatives)
- The antipodal map $\alpha \mapsto -\alpha$ changes the sign of the scalar field and its derivatives
- In particular, the map

$$\alpha \mapsto \Big(\partial_u \phi_\alpha(1), \dots, \partial_u^k \phi_\alpha(1)\Big)$$

Idea of the proof

- Null constraint system becomes a coupled nonlinear system of ODEs sourced by $\phi(v)$
- Scalar field ansatz

$$\phi_\alpha(v) = \sum_{1\le j\le k+1} \alpha_j \chi_j(v), \quad \alpha \in \mathbb{R}^{k+1}$$

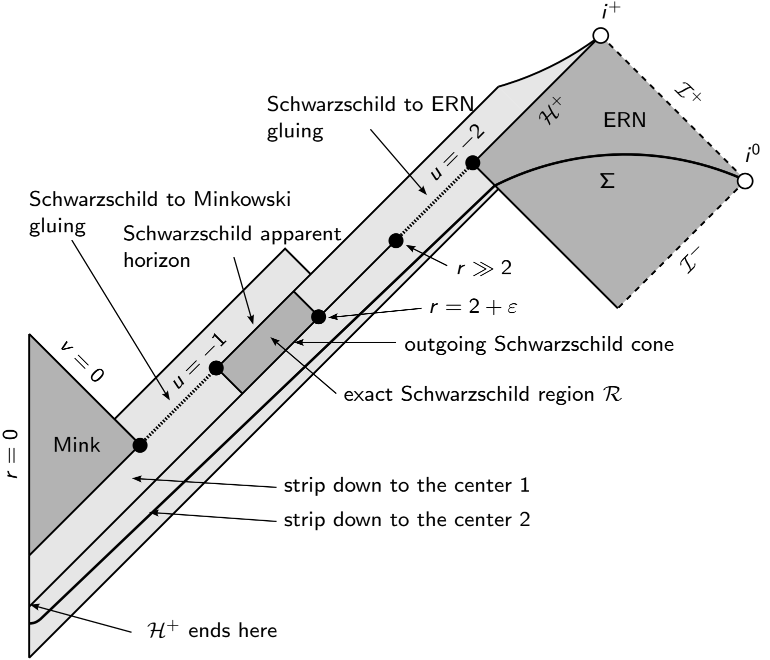
- Initialize $\partial_u \phi_\alpha(0) = \cdots = \partial_u^k \phi_\alpha(0) = 0$
- Initialize $r(v)$ teleologically at $v = 1$ ($\partial_v r(0)$ is a gauge choice, but $\partial_v r(1) = 0$!)
- For each α, generate $r(v)$, $\partial_u r(v)$, $\partial_u \phi_\alpha(v)$, etc.
- The antipodal map $\alpha \mapsto -\alpha$ leaves geometric quantities fixed (r, Ω^2, and derivatives)
- The antipodal map $\alpha \mapsto -\alpha$ changes the sign of the scalar field and its derivatives
- In particular, the map

$$\alpha \mapsto \Big(\partial_u \phi_\alpha(1), \ldots, \partial_u^k \phi_\alpha(1)\Big)$$

  is odd
- **Borsuk–Ulam theorem:** there exists α_* such that

$$\Big(\partial_u \phi_{\alpha_*}(1), \ldots, \partial_u^k \phi_{\alpha_*}(1)\Big) = 0.$$

Disproof of the third law



Future directions

- Critical behavior: Reissner–Nordström with $|e| > M$ does not contain a black hole.

Future directions

- Critical behavior: Reissner–Nordström with $|e| > M$ does not contain a black hole. Can we embed our solutions into a 1-parameter family of spacetimes exhibiting this behavior?

Future directions

- Critical behavior: Reissner–Nordström with $|e| > M$ does not contain a black hole. Can we embed our solutions into a 1-parameter family of spacetimes exhibiting this behavior?
- Event horizon gluing for the Einstein vacuum equations:

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

$$Ric(g) = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, already in vacuum, the "third law of black hole thermodynamics" is false.

Future directions

- Critical behavior: Reissner–Nordström with $|e| > M$ does not contain a black hole. Can we embed our solutions into a 1-parameter family of spacetimes exhibiting this behavior?
- Event horizon gluing for the Einstein vacuum equations:

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

$$Ric(g) = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, <u>*already in vacuum*</u>*, the "third law of black hole thermodynamics" is false.*

Thank you!