Retiring the third law of black hole thermodynamics

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joint work with Christoph Kehle (ETH Zürich)
arXiv:2211.15742

Outline

1. What is the third law?
2. Main result: the third law is false
3. Geometry and physics of third law violating spacetimes
4. Extremal black hole formation as a critical phenomenon

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

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Fundamentally, these are statements about Einstein gravity coupled to classical matter fields (and are already interesting in vacuum!).

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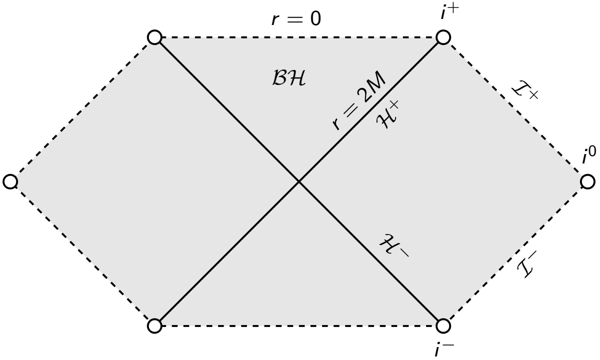
- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Refresher on Schwarzschild



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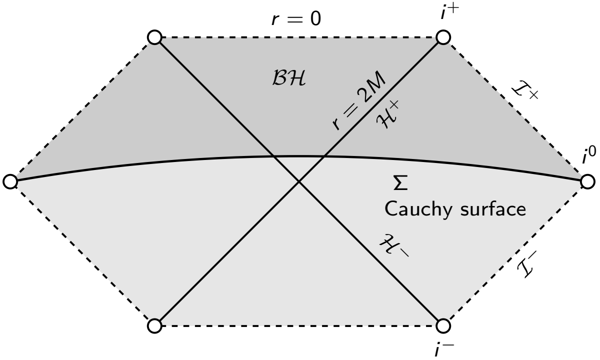

Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface Σ as depicted here.

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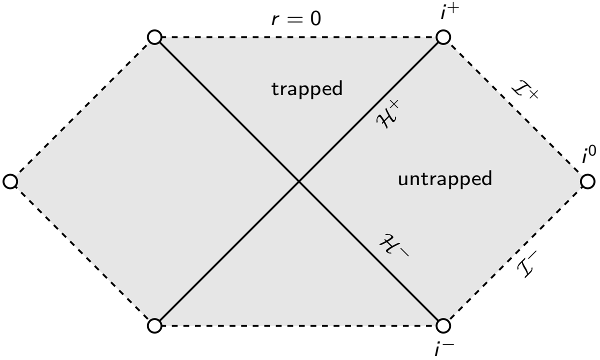

The black hole interior is foliated by **trapped surfaces** (both future null expansions negative).

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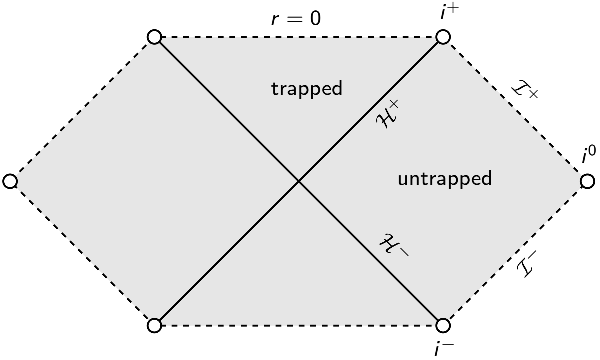

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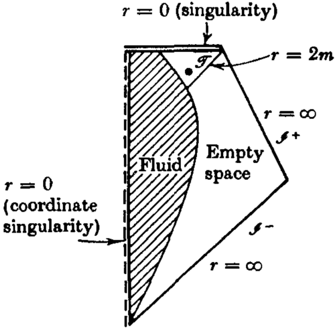
Theorem (Penrose's incompleteness theorem).

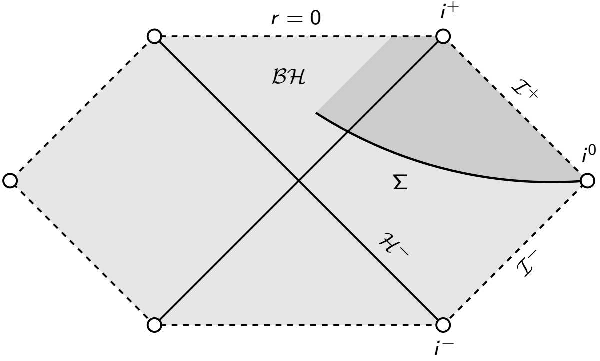
If an asymptotically flat spacetime contains a trapped surface and satisfies the null energy condition, it is geodesically incomplete.

Refresher on gravitational collapse

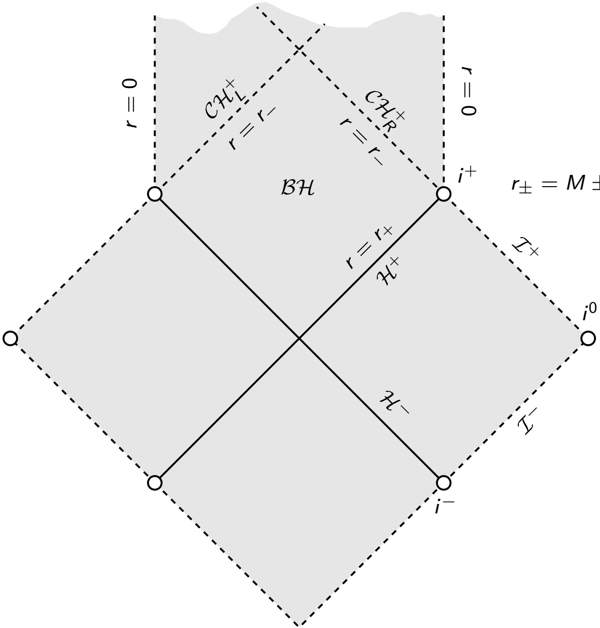


Penrose diagram of gravitational collapse. One-ended Cauchy data!

Refresher on gravitational collapse



Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



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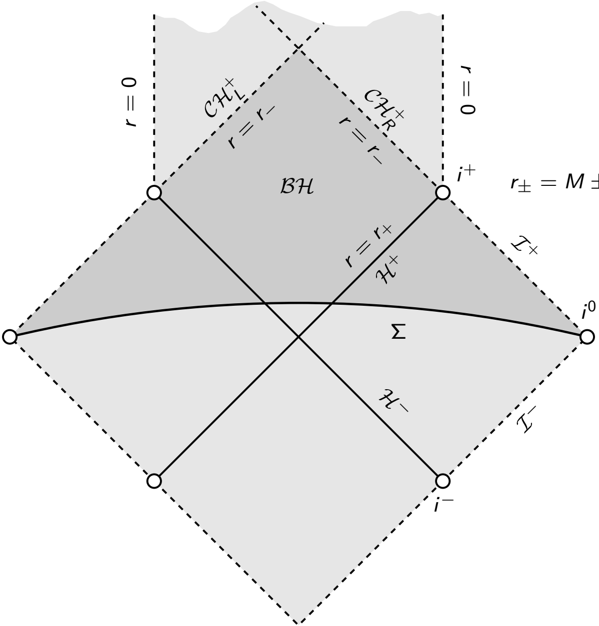

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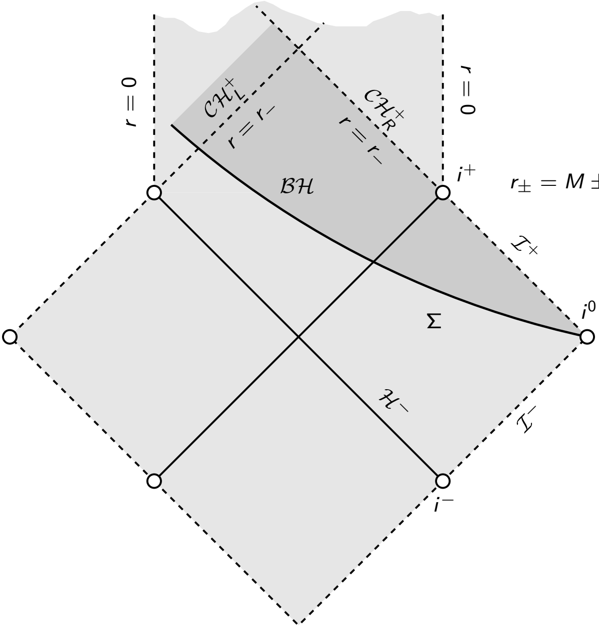

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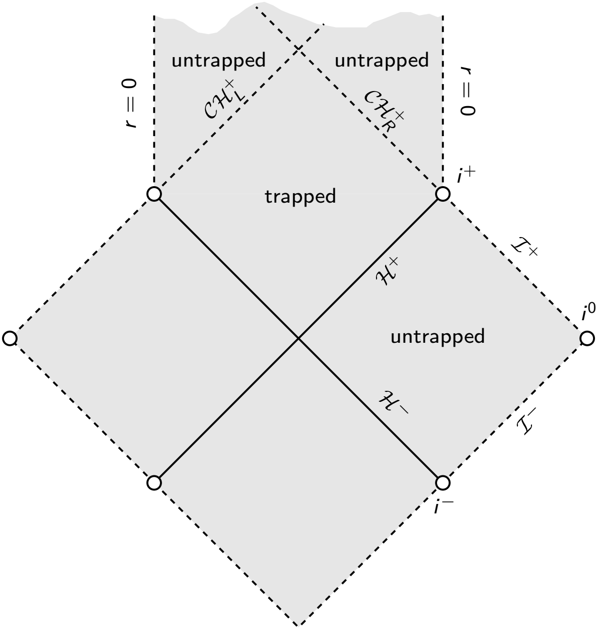

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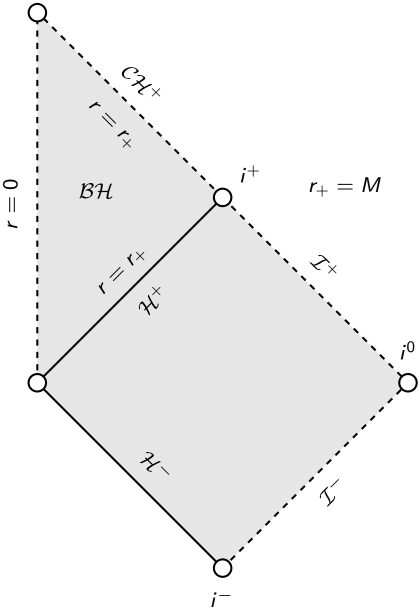

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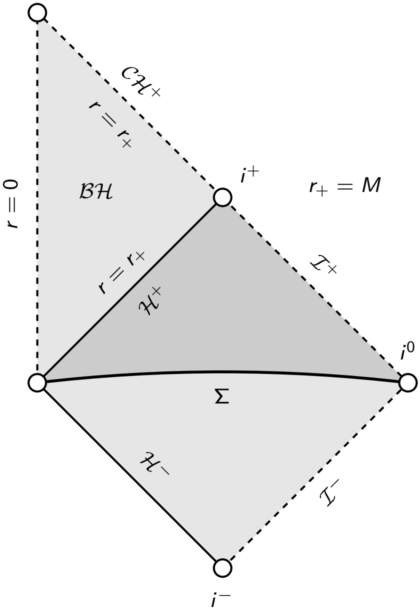

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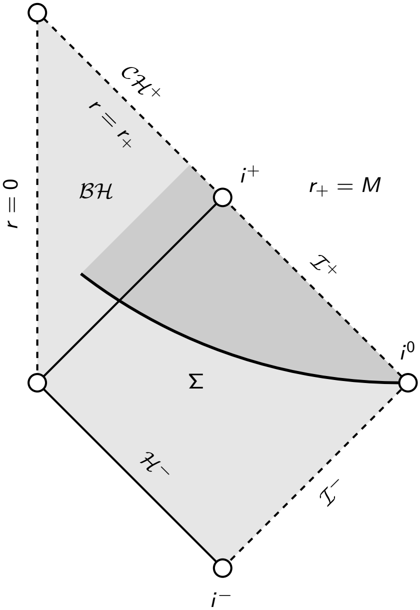

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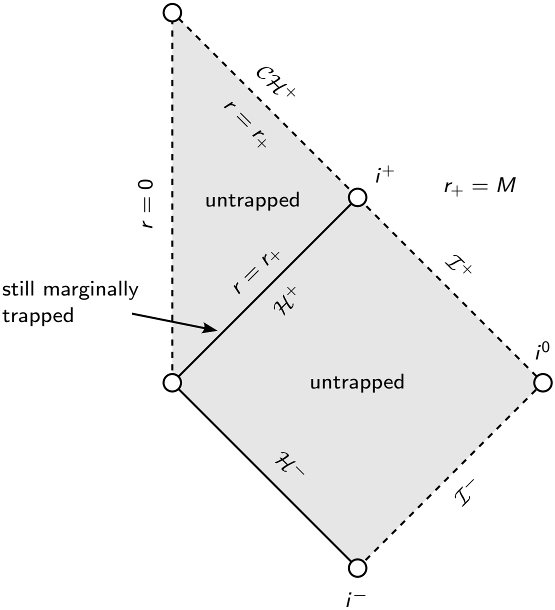

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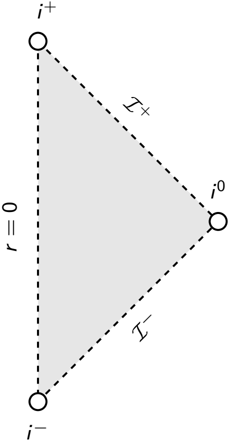

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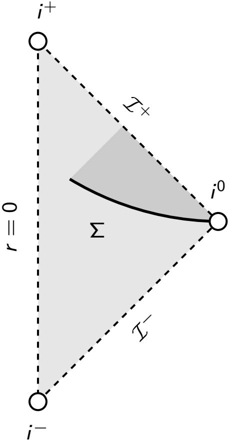

Surface gravity κ of Reissner–Nordström

- RN with mass M and charge e, $|e| \leq M$, has

$$\kappa = 2\pi T = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Subextremal:** $\kappa > 0$
- **Extremal:** $\kappa = 0$

The third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Difficult to interpret κ dynamically!

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VOLUME 57, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JULY 1986

Third Law of Black-Hole Dynamics: A Formulation

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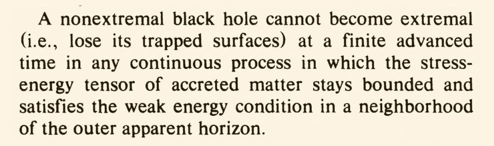
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4. Extremizing associated with "losing trapped surfaces."
   - This parenthetical remark is related to the "mistake" in Israel's paper and we will discuss it later.

Conjecture (The third law, BCH '73, Israel '86).
A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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Theorem (Kehle–U. '22).
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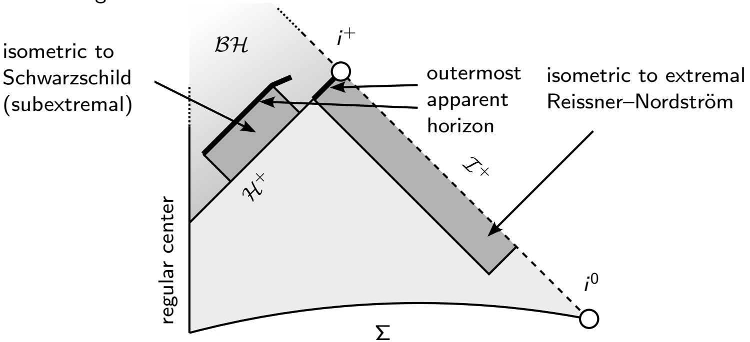
Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field (self-gravitating charged massless scalar field) system with the following behavior:

Counterexample to the third law: detailed statement of our theorem

There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



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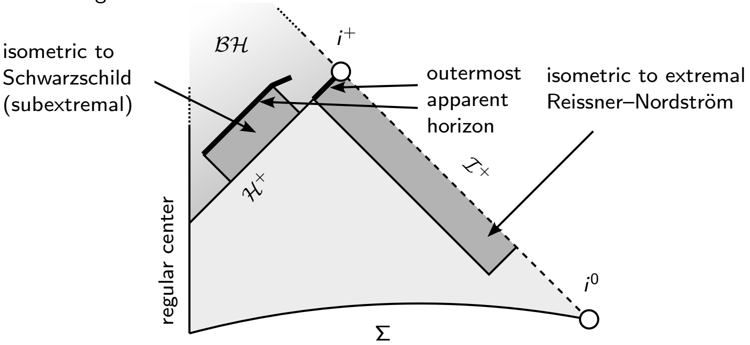

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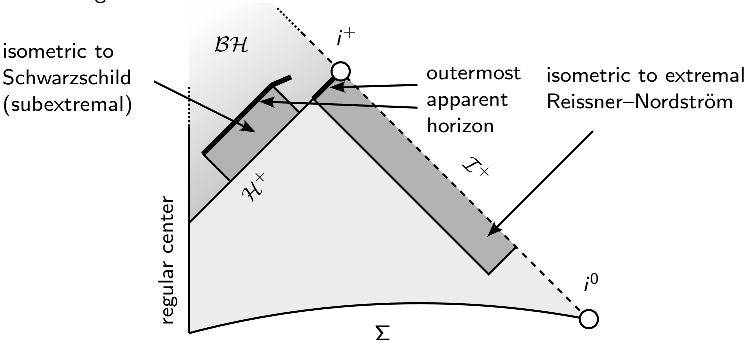

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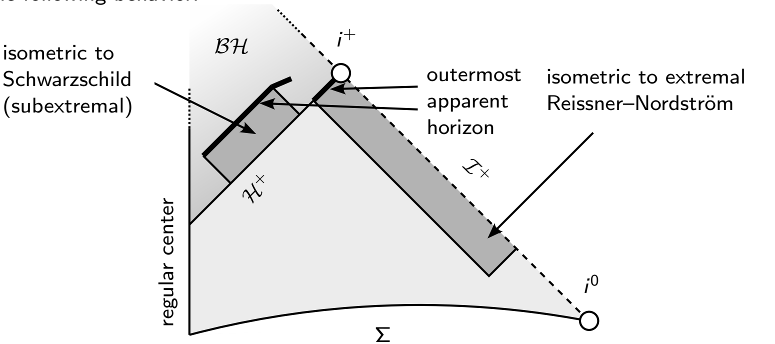

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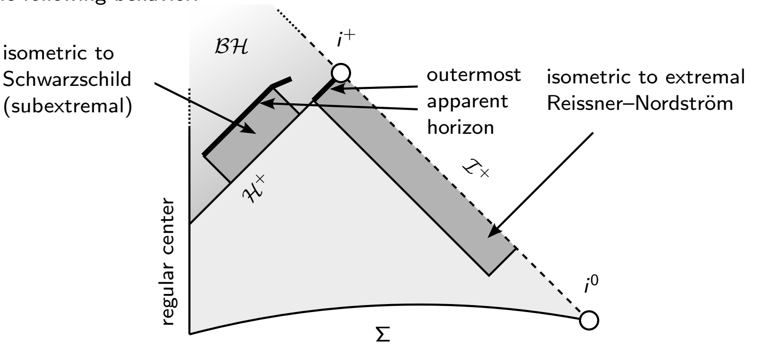

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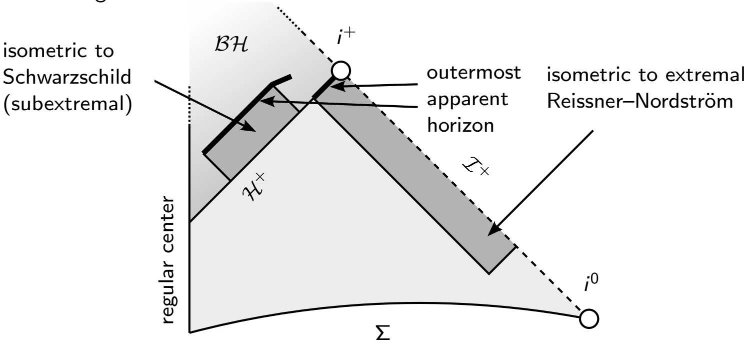

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- Model satisfies the dominant energy condition.

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- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ).

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$$\begin{aligned} R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) && (1)\\ \nabla^{\mu}F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu\phi}) && (2)\\ g^{\mu\nu}D_\mu D_\nu\phi &= 0 && (3)\\ T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} && (4)\\ T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} && (5) \end{aligned}$$

- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ).
- Spherical symmetry, ϕ degree of freedom breaks rigidity from Birkhoff.

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- Under this implicit assumption, Israel concluded the third law to be true on the basis of Raychaudhuri's equation.

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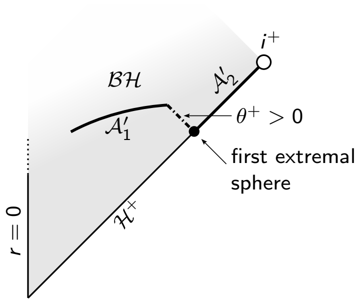

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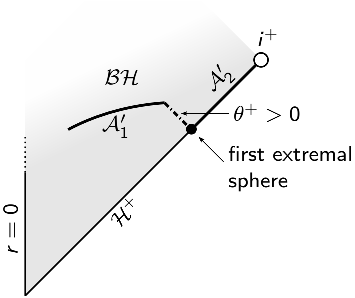

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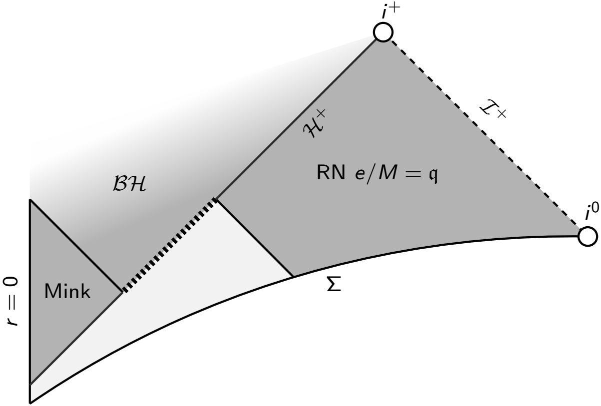
However, this is a feature, not a glitch!

Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:



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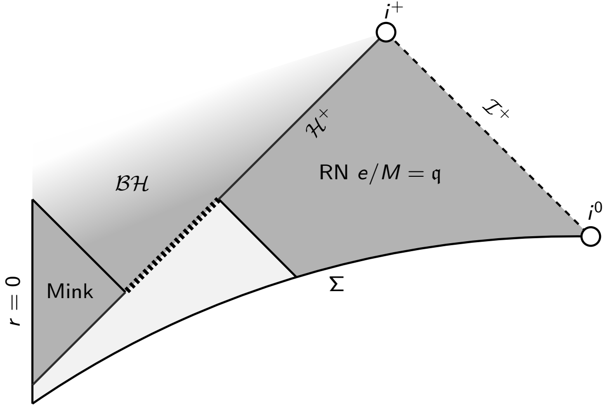

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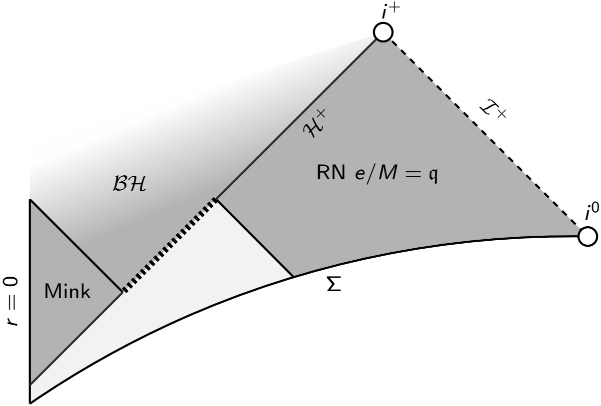

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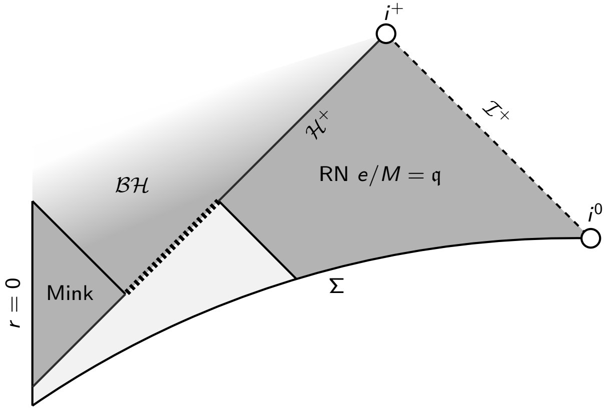

- From the point of view of our construction, the extremal case is exactly the same as the subextremal case!
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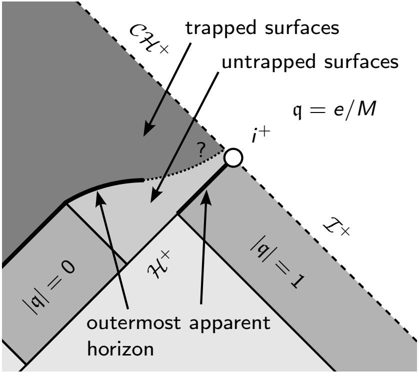
Interior structure of third law violating solutions



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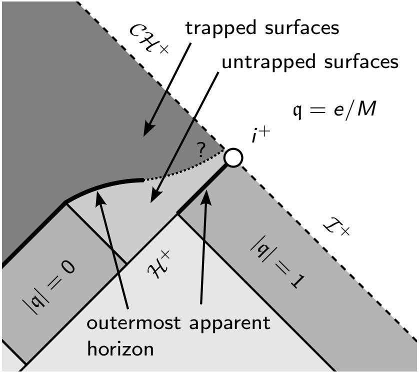

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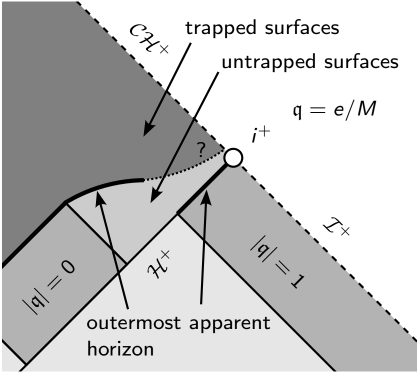
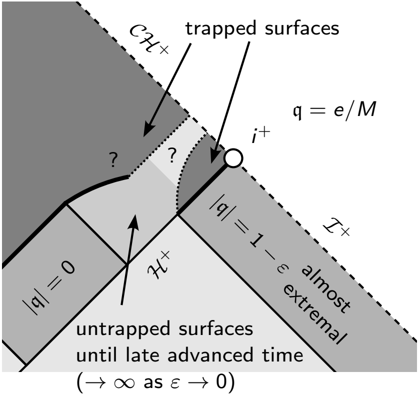


- The outermost apparent horizon becomes disconnected, yet the spacetime is highly regular.
- Trapped surfaces persist for all time and are found at any retarded time in the black hole interior.

So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



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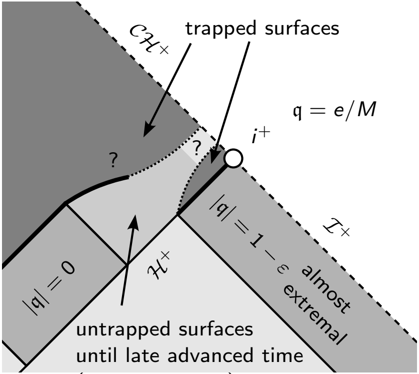

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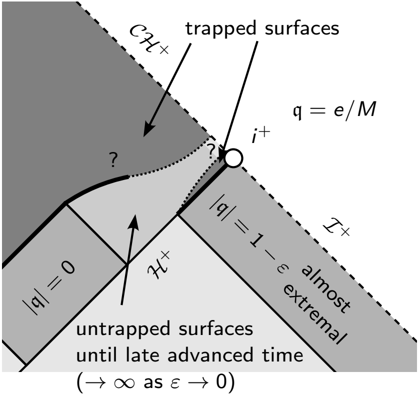

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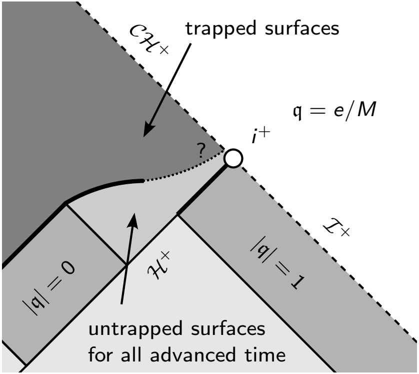

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

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- However, such a reformulation would be completely trivial (and has nothing to do with $T = 0$).

The third law and supercharging

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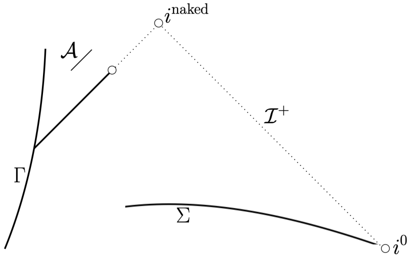

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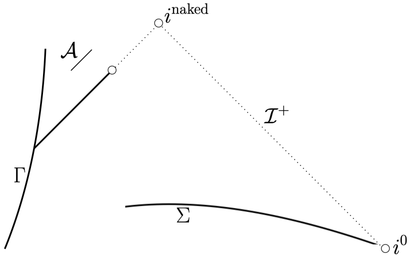

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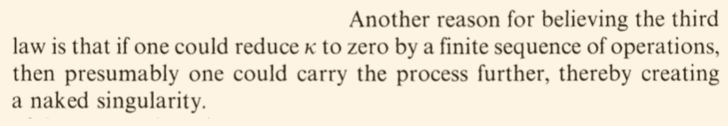
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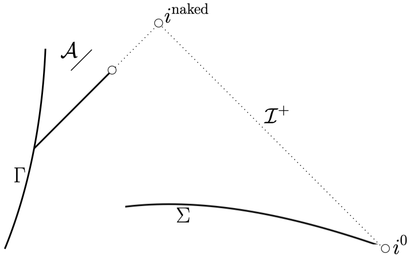

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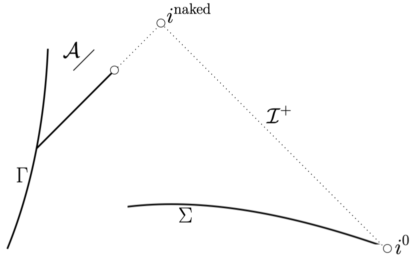

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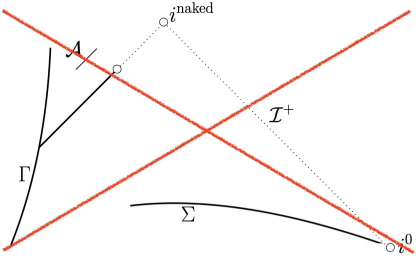

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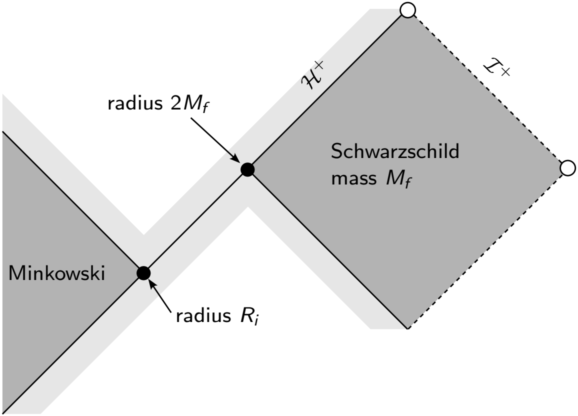
The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be located *without knowing the entire future of the spacetime*.

Prototype: Minkowski to Schwarzschild gluing



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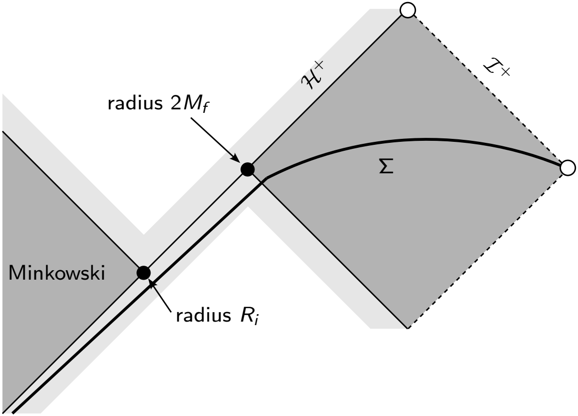

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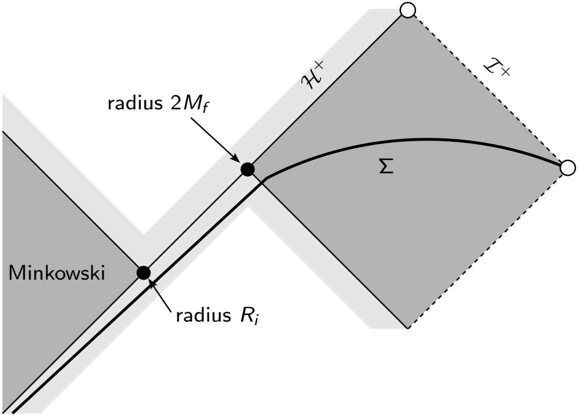

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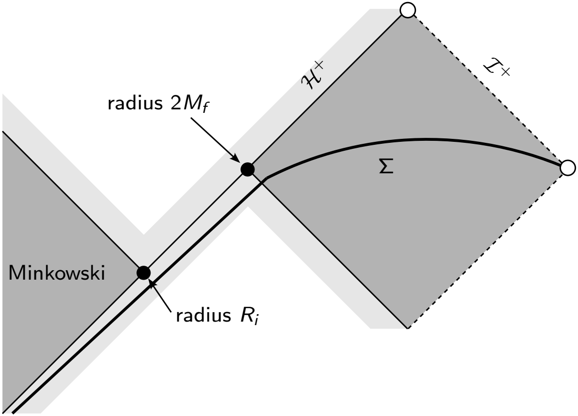

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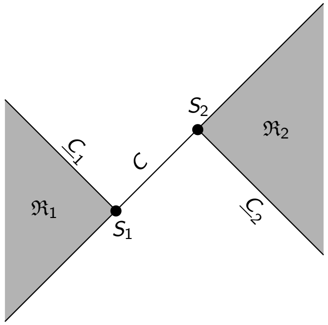
The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Characteristic/null gluing for the linear wave equation



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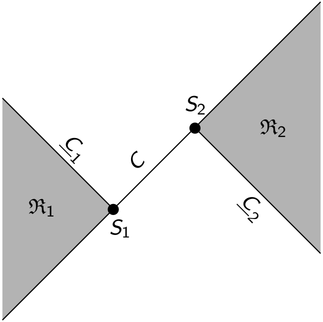

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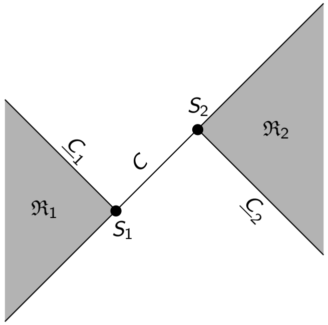

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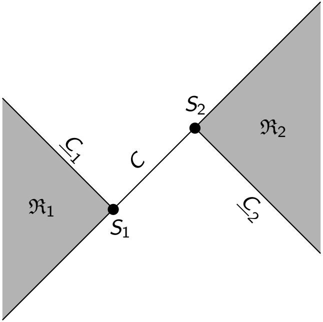

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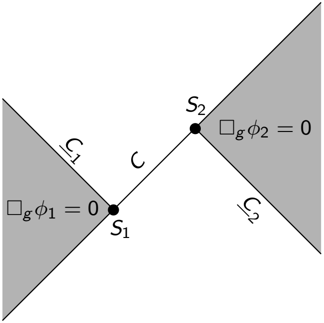

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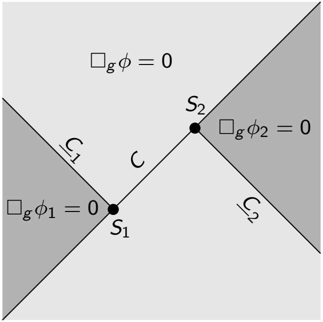

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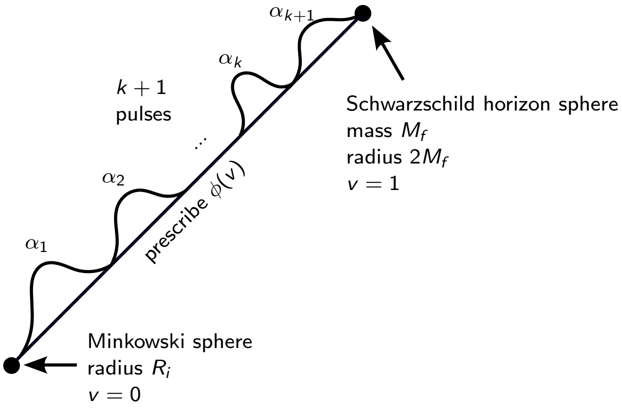

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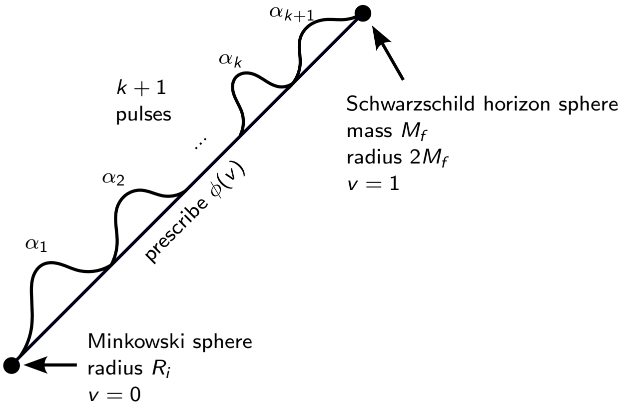

Pulse amplitudes α_j are chosen using the Borsuk–Ulam theorem, making use of the symmetry $\phi \mapsto -\phi$ of the equations.

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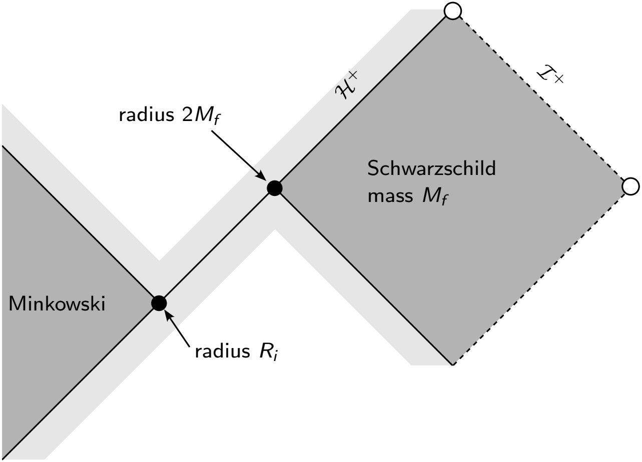

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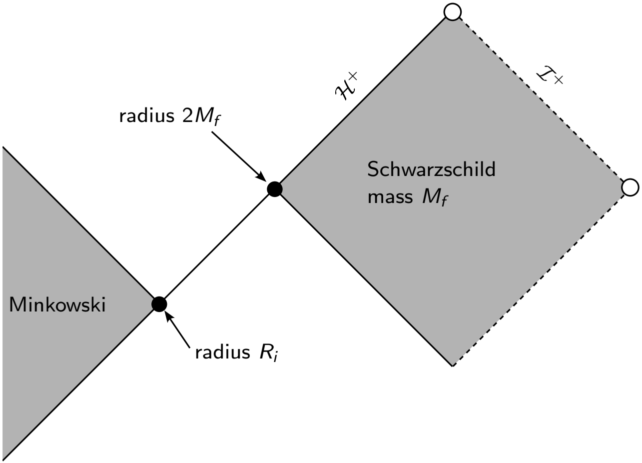

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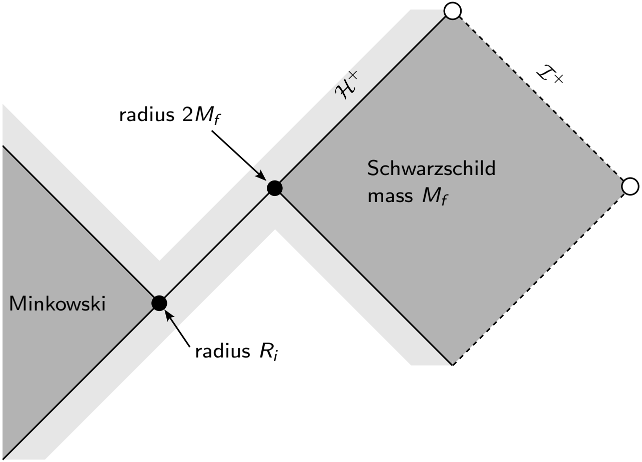
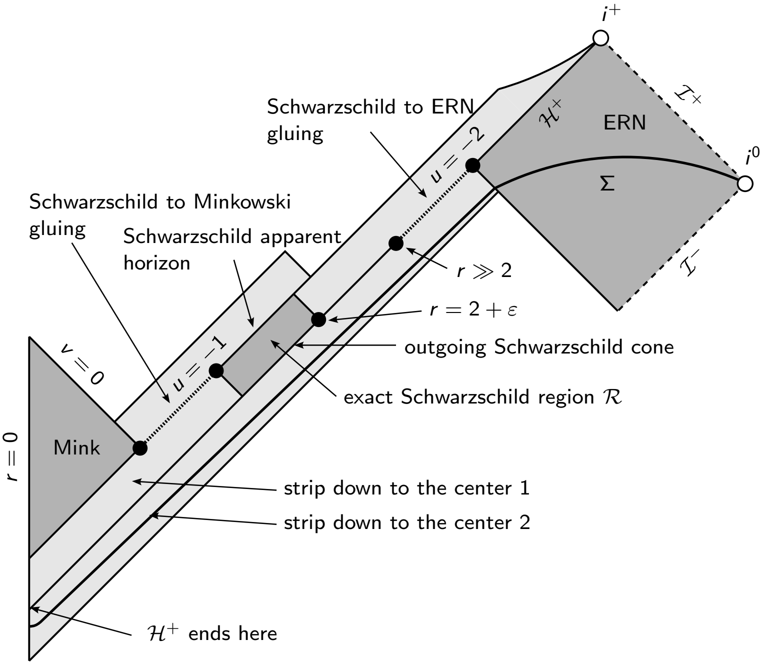


Blueprint to disproving the third law



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This would follow from characteristic gluing, in vacuum, of a Minkowski sphere to a Kerr exterior sphere.

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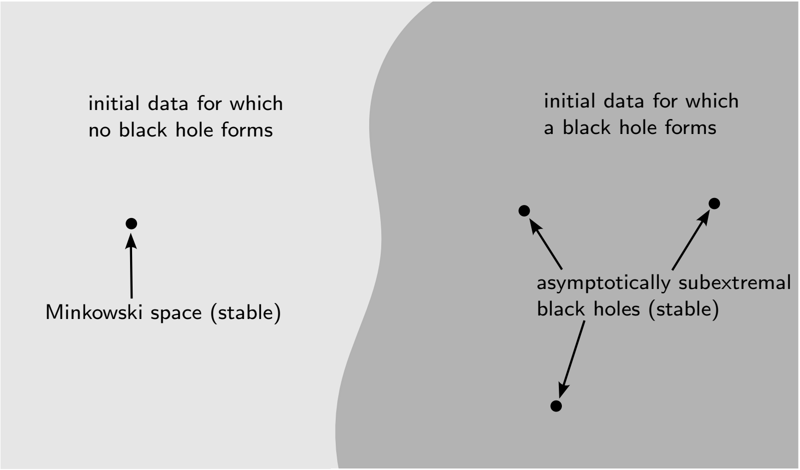

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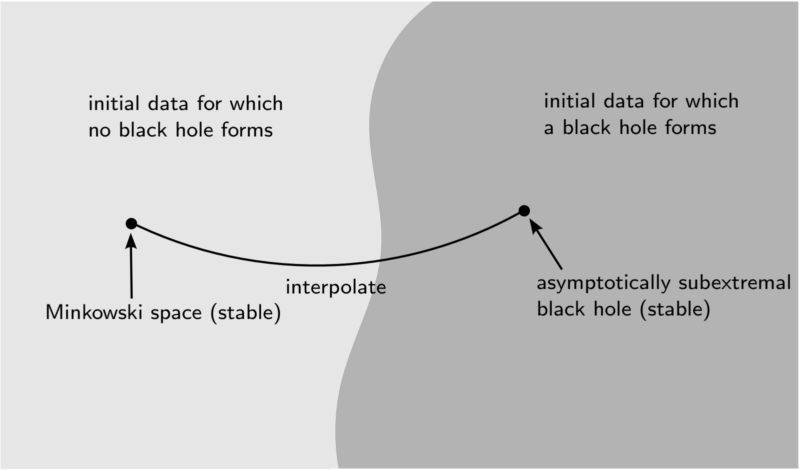

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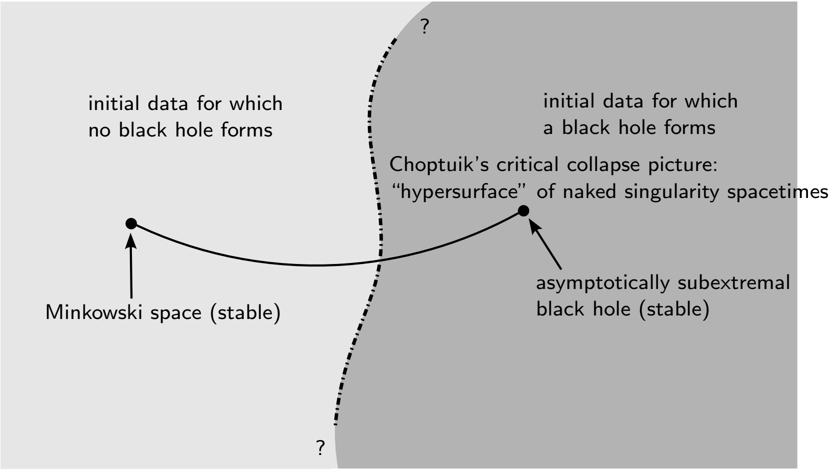

Critical behavior: naked singularities expected when a "low energy" black hole forms.

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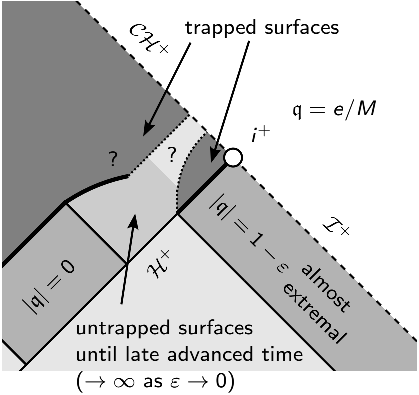
Critical behavior: the very black-holeness of certain spacetimes is unstable.

Event horizon jumping



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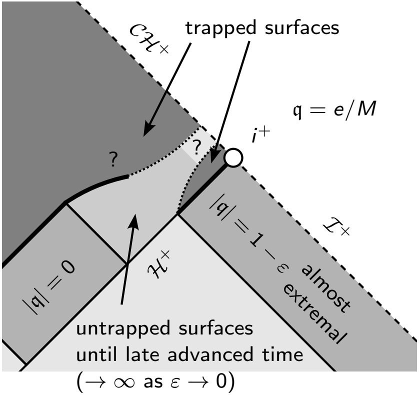

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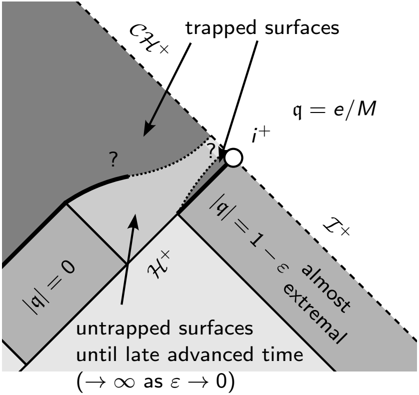

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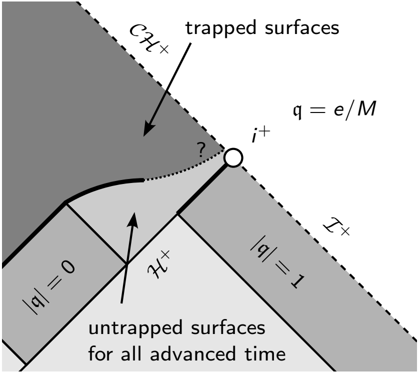
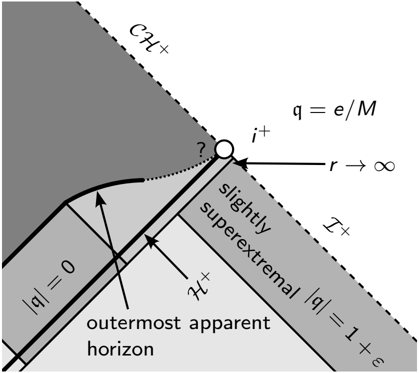


So what does happen if you “go past extremality”?



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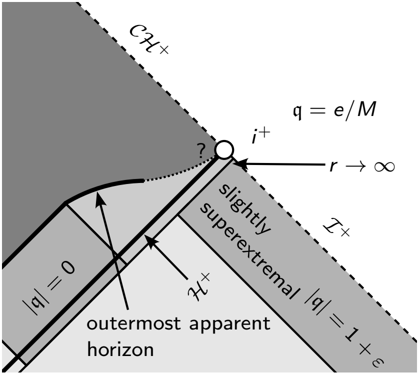

Critical behavior: We expect that there are smooth 1-parameter families of solutions passing through extremality where the event horizon "jumps" as soon as the exterior becomes superextremal. There is no naked singularity.

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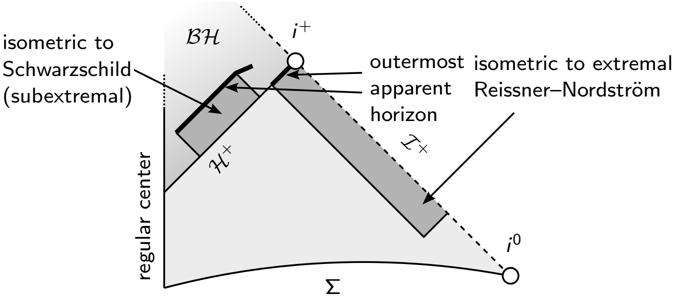

Thank you!