

Crete summer school GR basics examples sheet

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1 Orthogonality in Lorentzian vector spaces

Let (V, m) be an $(n + 1)$ -dimensional Lorentzian vector space. Show that:

- Two timelike vectors are never orthogonal.
- A timelike vector is never orthogonal to a null vector.
- Two null vectors are orthogonal if and only if they are collinear.
- The orthogonal complement of a null vector is a codimension-one subspace with a degenerate scalar product. The kernel of the scalar product restricted to this subspace is one-dimensional and equal to the span of the null vector.
- Continuing the previous point, show that if v is null, then $(v^\perp)^\perp = \mathbb{R}v$. (In fact, for any subspace it holds that $(W^\perp)^\perp = W$, just as in an inner product space.)

Hint: While one can try to prove all of these in a “coordinate free” manner, it is much simpler to work in a standard orthonormal basis.

2 Null hypersurfaces

Let (\mathcal{M}^{n+1}, g) be a Lorentzian manifold and $\mathcal{H}^n \subset \mathcal{M}^{n+1}$ a smooth embedded hypersurface. We say that \mathcal{H} is a *null hypersurface* if $T_p\mathcal{H} \subset T_p\mathcal{M}$ is null (as a codimension-one subspace of the Lorentzian vector space $(T_p\mathcal{M}, g_p)$) for every $p \in \mathcal{H}$. Show that:

- There exists a null vector field L defined along \mathcal{H} such that $L_p \in T_p\mathcal{H}$ for every $p \in \mathcal{H}$. This L is both tangent and normal to \mathcal{H} !

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- b) Show that there exists a function $f : \mathcal{H} \rightarrow \mathbb{R}$ such that

$$\nabla_L L = fL. \quad (2.1)$$

Hint: The goal is show that for any section X of $T\mathcal{H}$, $g(X, \nabla_L L) = 0$. Let $p \in \mathcal{H}$, $X_p \in T_p\mathcal{H}$, and extend X_p to a vector field $X \in \Gamma(T\mathcal{H})$ (at least locally near p) such that $[L, X] = 0$. (Why can this be done?) Now use the symmetry of the connection to show that $g(\nabla_L L, X) = 0$.

- c) Show that there exists a null vector field L' , pointwise proportional to L , such that

$$\nabla_{L'} L' = 0. \quad (2.2)$$

- d) Interpret and prove the statement that “any null hypersurface is ruled by null geodesics.”

3 The energy-momentum tensor of a scalar field

A *scalar field* is a function $\phi \in C^\infty(\mathcal{M})$ (which often solves a wave equation). We define the *energy-momentum tensor* of ϕ by

$$T_{\mu\nu} \doteq \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi, \quad (3.1)$$

where $\partial^\alpha \phi = g^{\alpha\beta} \partial_\beta \phi$.

- a) Show that if X is future-directed causal, then $-T^\mu{}_\nu X^\nu$ is as well. *Hint:* Choose an orthonormal basis of the tangent space so that X is “standard,” i.e., is of the form e_0 or $e_0 + e_1$. Use the fact that $T_{\mu\nu} X^\mu X^\nu$ is coordinate invariant.
- b) Let X and Y be future-directed timelike vectors at p and $\{V_\mu\}$ a basis of $T_p\mathcal{M}$. Show that there exists a constant $c > 0$ such that

$$T(X, Y) \geq c \sum_\mu |V_\mu \phi|^2. \quad (3.2)$$

In this sense $T(X, Y)$ quantitatively “controls all derivatives of ϕ ” pointwise.

- c) Let X be future-directed timelike and Y be future-directed null. Let $\{e_1, \dots, e_{n-1}\}$ be spacelike vectors so that $\{Y, e_1, \dots, e_{n-1}\}$ spans Y^\perp . Show that there exists a constant $c > 0$ such that

$$T(X, Y) \geq c \left(|Y\phi|^2 + \sum_{i=1}^{n-1} |e_i \phi|^2 \right). \quad (3.3)$$

This combination misses one direction!

- d) Let Y be future-directed null and let \underline{Y} be a future-directed null vector so that $g(Y, \underline{Y}) = -2$. Show that there exists a constant $c > 0$ such that

$$T(Y, \underline{Y}) \geq c \sum_{i=1}^{n-2} |e_i \phi|^2, \quad (3.4)$$

where $\{e_1, \dots, e_{n-1}\}$ is a basis of spacelike vectors for $Y^\perp \cap \underline{Y}^\perp$. This combination misses two directions!

Remark 1. In general, the constants c in (3.2), (3.3), and (3.4) depend on the metric g , the point p , the vector fields X and Y , and the choice of basis vectors on the right-hand side. In practice one needs some quantitative control this constant so that these estimates are “effective.”

4 Topology of Lorentzian manifolds

In this exercise, we will see some basic topological properties of Lorentzian manifolds. We wish to prove the following:

Theorem 1. *Let \mathcal{M} be a smooth manifold. The following are equivalent:*

- i) \mathcal{M} admits a Lorentzian metric.
- ii) \mathcal{M} admits a time-oriented Lorentzian metric.
- iii) \mathcal{M} admits a nonvanishing vector field (i.e., $X(p) \neq 0$ for every $p \in \mathcal{M}$).
- iv) \mathcal{M} is noncompact or is compact with vanishing Euler characteristic.

Proof. The logic of the proof is as follows: ii) \Rightarrow i) is trivial, iii) \Leftrightarrow iv), iii) \Leftrightarrow ii), and i) \Rightarrow iv). Here is an outline of some of the parts. We will need the following deep theorem in topology:

Theorem 2 (Poincaré–Hopf). *A closed manifold \mathcal{M} carries a nonzero vector field if and only if the Euler characteristic $\chi(\mathcal{M}) = 0$.*

Here are now some hints to prove Theorem 1.

- a) Let \mathcal{M} be a smooth manifold carrying a nonzero vector field X . Let g_0 be a Riemannian metric on \mathcal{M} . (Does such a g_0 always exist?) Show that there exists a positive function $f \in C^\infty(\mathcal{M})$ so that

$$g = -f^2 g_0(\cdot, X) \otimes g_0(\cdot, X) + g_0 \quad (4.1)$$

is a Lorentzian metric on \mathcal{M} .

- b) (*) Show that every noncompact manifold carries a nonvanishing vector field. *Hint:* Construct a vector field with discrete zeros and isotope the zeros to infinity.
- c) Show that a time-orientable Lorentzian manifold carries an everywhere nonvanishing *timelike vector field*.
- d) Show that a Lorentzian manifold admits a time-orientable double cover.
- e) Show that a closed Lorentzian manifold has $\chi(\mathcal{M}) = 0$. *Hint:* How does $\chi(\mathcal{M})$ behave under finite covering maps?

□

Corollary 1. *Any odd-dimensional smooth manifold admits a Lorentzian metric.*

- f) (*) Prove this. *Hint:* Use Poincaré duality.

Corollary 2. *A nonempty closed Lorentzian manifold is not simply connected.*

- g) (*) Prove this. *Hint:* Use Poincaré duality again and aim to show that $b_1(\mathcal{M}) \neq 0$.

Proposition 2. *Let (\mathcal{M}, g) be a closed Lorentzian manifold. Then \mathcal{M} contains a closed timelike curve.*

- h) Prove this. *Hint:* Cover \mathcal{M} by finitely many sets of the form $I^+(p_i)$, $i = 1, \dots, N$. Show that for some i , $p \in I^+(p_i)$.

5 Uniformization of Lorentzian surfaces

The goal of this exercise is to prove the following important theorem in Lorentzian geometry:

Theorem 3. *Let (\mathcal{M}^2, g) be a Lorentzian surface. For any $p \in \mathcal{M}$ there exist coordinates (t, x) defined in a neighborhood $U \subset \mathcal{M}$ of p and a smooth, positive function Ω on U such that*

$$g = \Omega^2(-dt^2 + dx^2) \quad (5.1)$$

in U .

The proof of this fact for Lorentzian metrics is much simpler than for Riemannian metrics. Here is a suggested solution:

- Show there exist two null vectors fields X and Y defined in a neighborhood U of p that are linearly independent at every point of U .
- Show that there exist functions $\alpha, \beta \in C^\infty(U)$ such that $[X, Y] = \alpha X + \beta Y$.
- Show that there exist nowhere vanishing functions $\theta, \zeta \in C^\infty(U')$, where U' is a possibly smaller neighborhood of U , such that $[\theta X, \zeta Y] = 0$. *Hint:* Compute out $[\theta X, \zeta Y]$ and derive a first order PDE system for θ and ζ that makes this vanish. Why does a solution exist?
- Show that there exists a coordinate chart (u, v) defined near p such that $\theta X = \partial_u$ and $\zeta Y = \partial_v$. *Hint:* Recall (or prove that) commuting vector fields induce coordinate charts via their flows.
- Show that $t = u + v$, $x = v - u$ has the desired properties.

6 Cauchy stability for ODEs

Cauchy stability, or continuous dependence on initial data, is a very useful tool when studying wave equations. Here we will see the simplest possible example, which will also introduce us to the notion of *bootstrap arguments*.

Theorem 4. *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function and suppose $\bar{y} : I \rightarrow \mathbb{R}$ is a smooth solution of the ODE*

$$\frac{d\bar{y}}{dt}(t) = F(t, \bar{y}(t)) \quad (6.1)$$

with initial condition $\bar{y}(0) = \bar{y}_0$, where $I = [0, T_0]$ is a closed and bounded interval. For any $\varepsilon > 0$ there exists a $\delta > 0$ (depending on F , T_0 , and \bar{y}_0) such that the solution $y(t)$ of the initial value problem

$$\frac{dy}{dt} = F(t, y(t)), \quad (6.2)$$

$$y(0) = y_0 \quad (6.3)$$

with $|y_0 - \bar{y}_0| \leq \delta$, exists for $t \in I$ and satisfies the estimate

$$\sup_{t \in I} |y(t) - \bar{y}(t)| \leq \varepsilon. \quad (6.4)$$

We emphasize that this theorem has two parts: the solution $y : I \rightarrow \mathbb{R}$ exists and moreover satisfies the estimate (6.4).

- First, prove the following *continuation criterion*.

Proposition 3. *Let $y(t)$ solve the ODE (6.2) on an interval of the form $[0, T_*)$ with $T_* < \infty$. If*

$$\limsup_{t \nearrow T_*} |y(t)| < \infty, \quad (6.5)$$

then y can be uniquely smoothly extended to an interval $[0, T')$ with $T' > T_$.*

Hint: Use the ODE to prove that $y(t)$ and all of its derivatives are bounded on $[0, T_*)$.

b) For $M \geq 1$, define the set

$$\mathcal{A}_{\delta, M} \doteq \left\{ T_* \in [0, T_0] : y(t) \text{ exists on } [0, T_*] \text{ and } \sup_{t \in [0, T_*)} |y(t) - \bar{y}(t)| \leq 2\delta e^{MT_*} \right\}. \quad (6.6)$$

Show that $\mathcal{A}_{\delta, M}$ is nonempty and closed.

c) Show that $\mathcal{A}_{\delta, M}$ is open for M sufficiently large and δ sufficiently small (depending on M and T_0).

Hint: Let $\tilde{y} \doteq y - \bar{y}$ and use the calculation (mean value theorem)

$$|\tilde{y}(t) - \tilde{y}(0)| \leq \int_0^t |F(t', y(t')) - F(t', \bar{y}(t'))| dt' \quad (6.7)$$

$$\leq \left(\max_{t' \in [0, t], z \in [\bar{y}(t') - 2\delta e^{Mt'}, \bar{y}(t') + 2\delta e^{Mt'}]} D_z F(t', z) \right) \int_0^t \delta e^{Mt'} dt'. \quad (6.8)$$

Show that for M sufficiently large, δ sufficiently small, and $T_* \in \mathcal{A}$, this estimate proves that

$$\sup_{t \in [0, T_*)} |\tilde{y}(t)| \leq \delta e^{MT_*}. \quad (6.9)$$

d) Conclude Theorem 4 by performing a continuity argument and using Proposition 3.

e) Generalize Theorem 4 as follows:

Theorem 5. Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function and suppose $\bar{y} : I \rightarrow \mathbb{R}$ is a smooth solution of the ODE

$$\frac{d\bar{y}}{dt} = F(t, \bar{y}(t)) \quad (6.10)$$

with $\bar{y}(0) = \bar{y}_0$, where $I = (T_{-1}, T_1) \ni 0$ is the maximal domain of definition.¹ For any $\varepsilon > 0$ and compact subinterval $K \subset I$ containing t_0 , there exists a $\delta > 0$ (depending on F , K , t_0 , and \bar{y}_0) such that the solution $y(t)$ of

$$\frac{dy}{dt} = F(t, y(t)), \quad (6.11)$$

$$y(0) = y_0 \quad (6.12)$$

with $|y_0 - \bar{y}(t_0)| \leq \delta$, exists for $t \in K$ and satisfies the estimate

$$\sup_{t \in K} |y(t) - \bar{y}(t)| \leq \varepsilon. \quad (6.13)$$

¹That is, $T_1 = \infty$ or $T_1 < \infty$ and $|y(t)|$ blows up as $t \nearrow T_1$, and similarly for T_{-1} .