

Retiring the third law of black hole thermodynamics

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joint work with Christoph Kehle (ETH Zürich)
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Outline

1. What is the third law?
2. Main result: the third law is false
3. Geometry and physics of third law violating spacetimes
4. Extremal black hole formation as a critical phenomenon

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)
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The Four Laws of Black Hole Mechanics

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Fundamentally, these are statements about Einstein gravity coupled to classical matter fields (and are already interesting in vacuum!).

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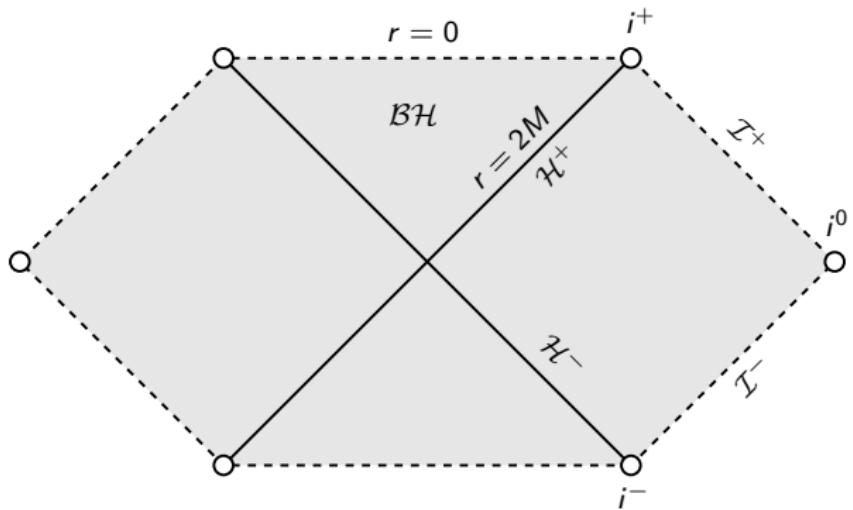
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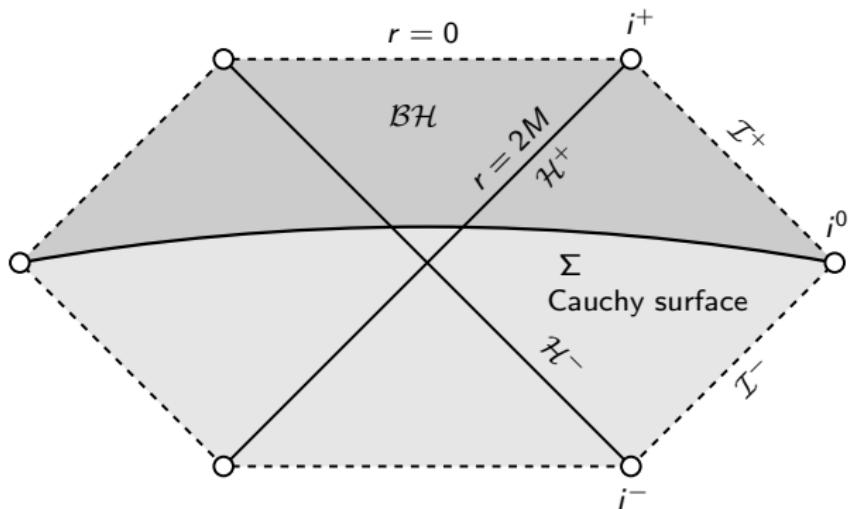
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- ▶ These identifications are further justified physically using QFT, but are **formal analogies** in classical GR.
- ▶ Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Refresher on Schwarzschild

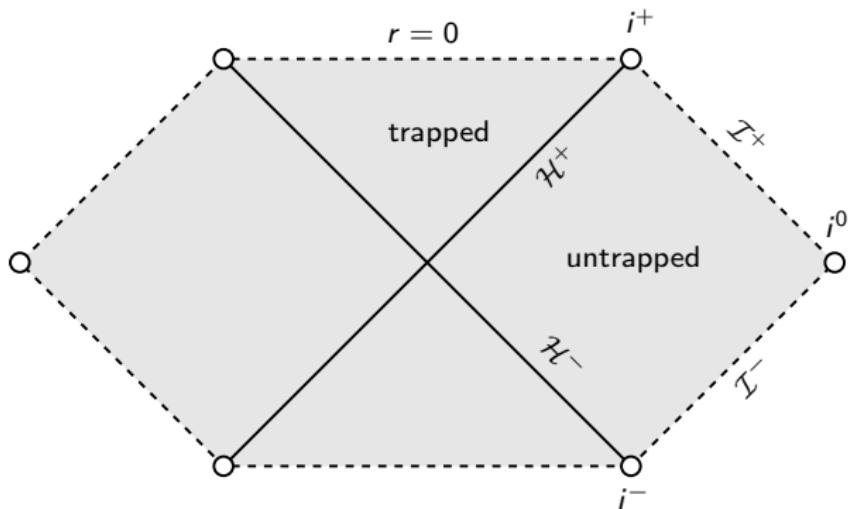


Refresher on Schwarzschild



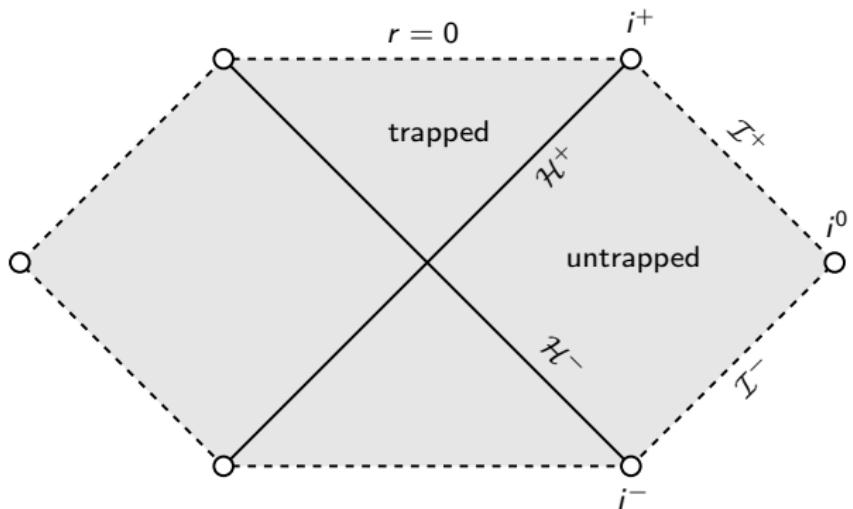
Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface Σ as depicted here.

Refresher on Schwarzschild



The black hole interior is foliated by **trapped surfaces** (both future null expansions negative).

Refresher on Schwarzschild

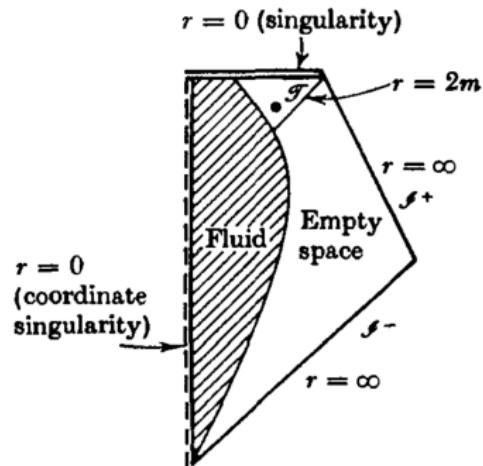


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Theorem (Penrose's incompleteness theorem).

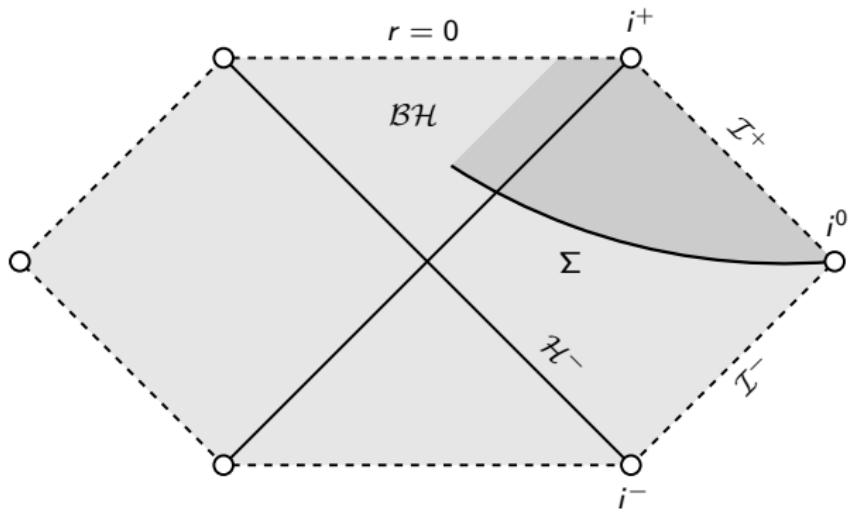
If an asymptotically flat spacetime contains a trapped surface and satisfies the null energy condition, it is geodesically incomplete.

Refresher on gravitational collapse

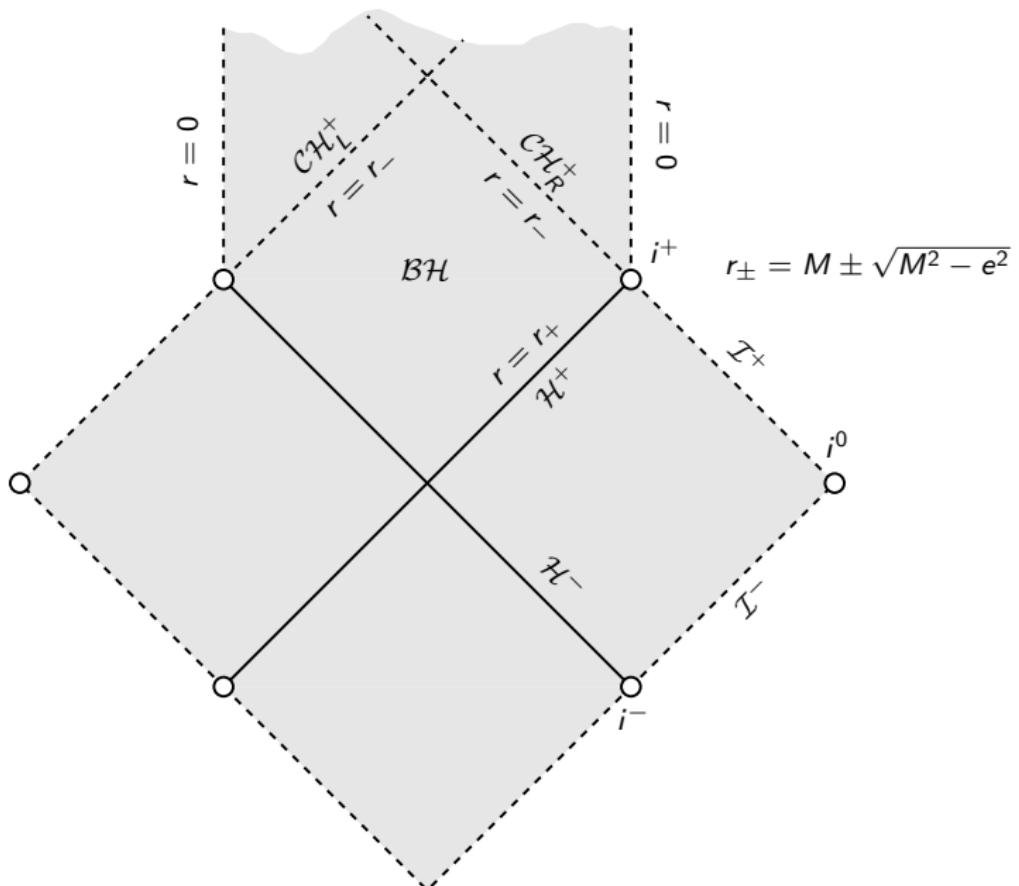


Penrose diagram of gravitational collapse. One-ended Cauchy data!

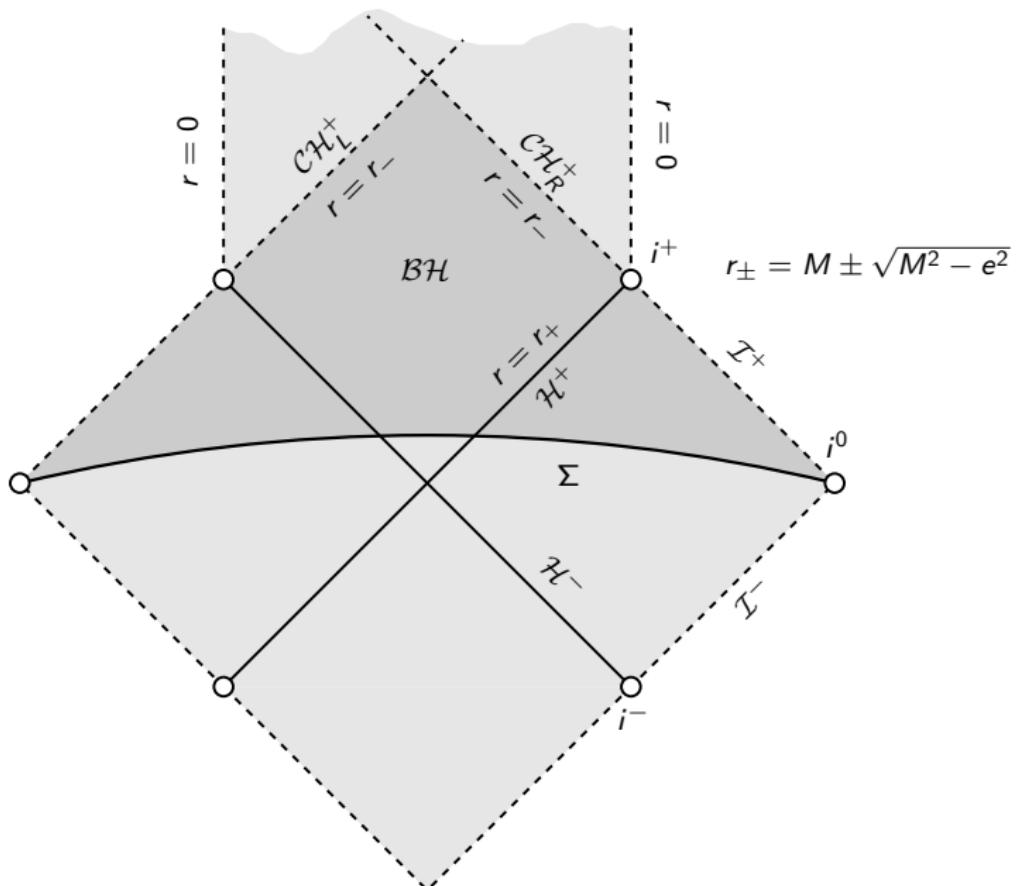
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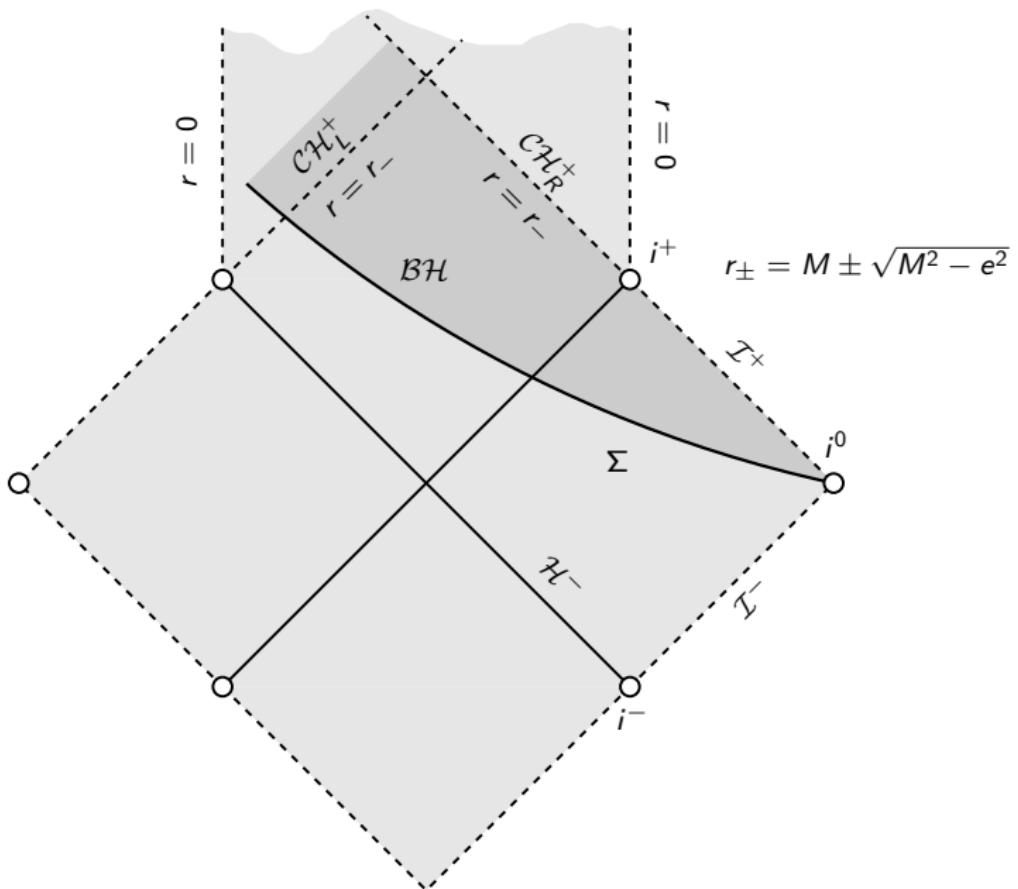
Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



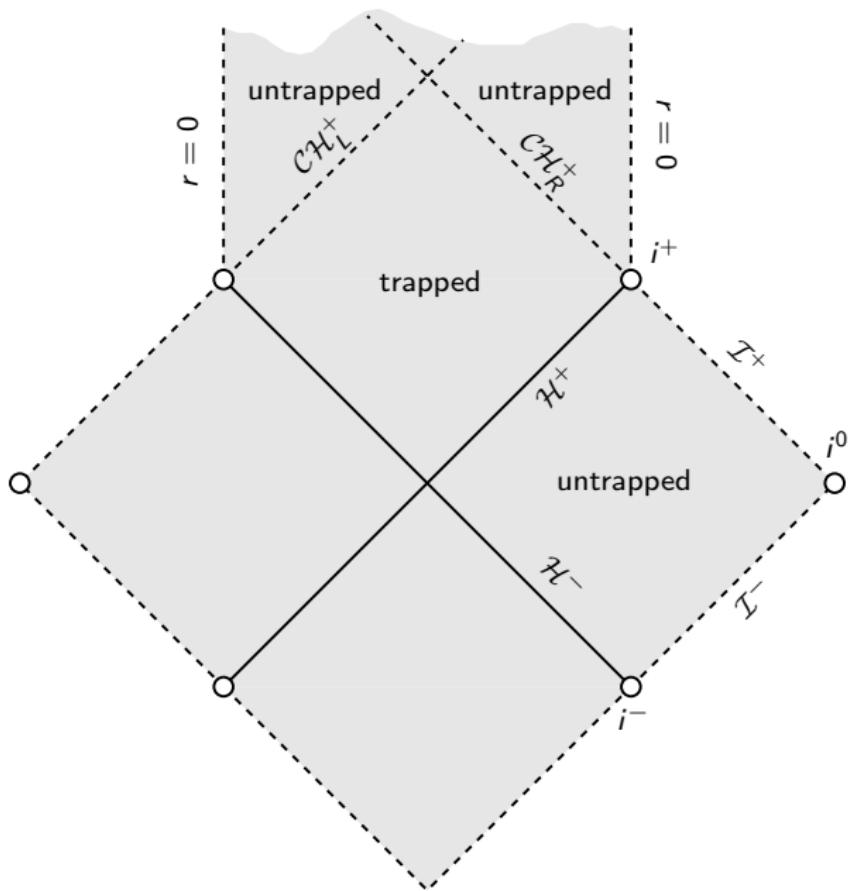
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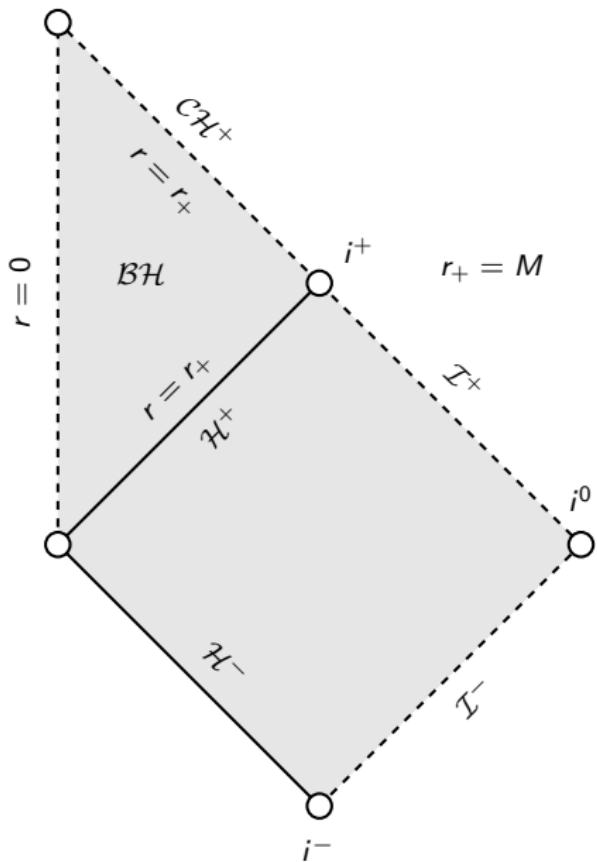
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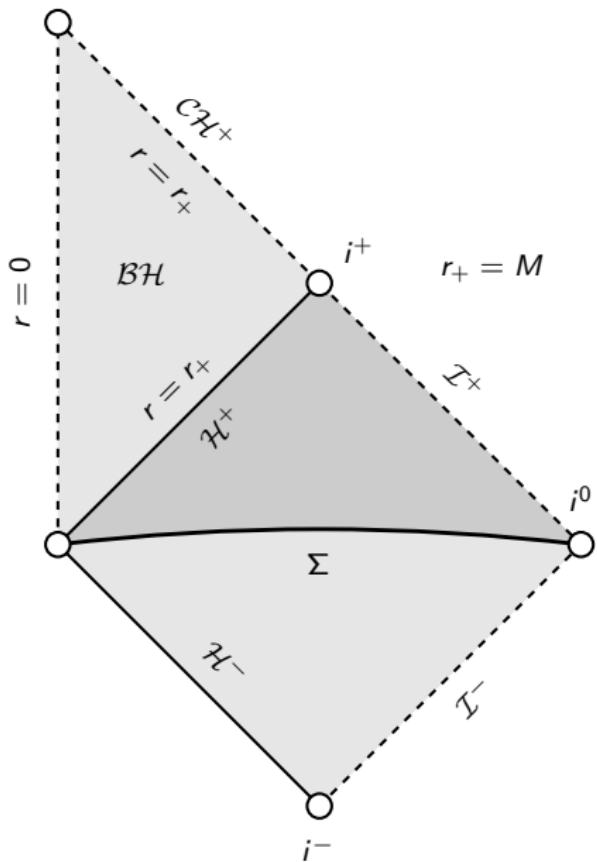
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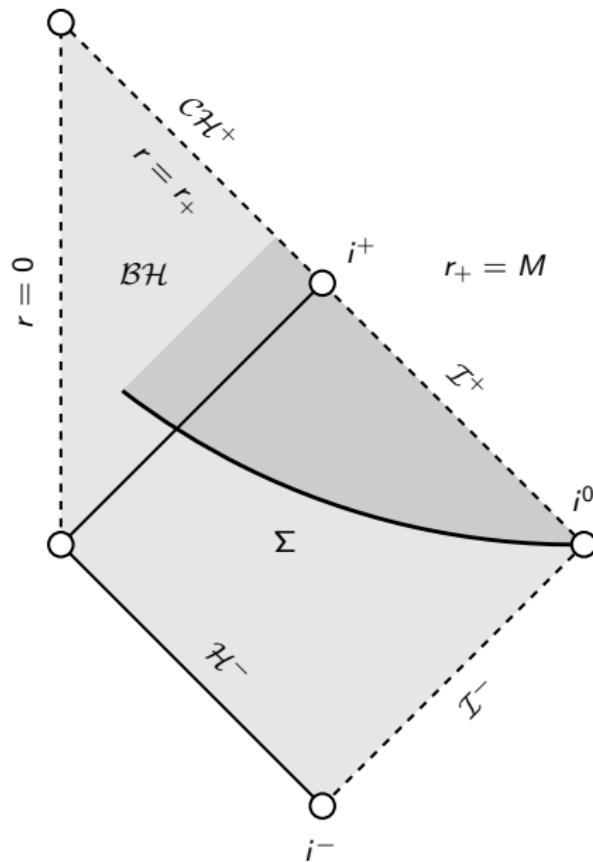
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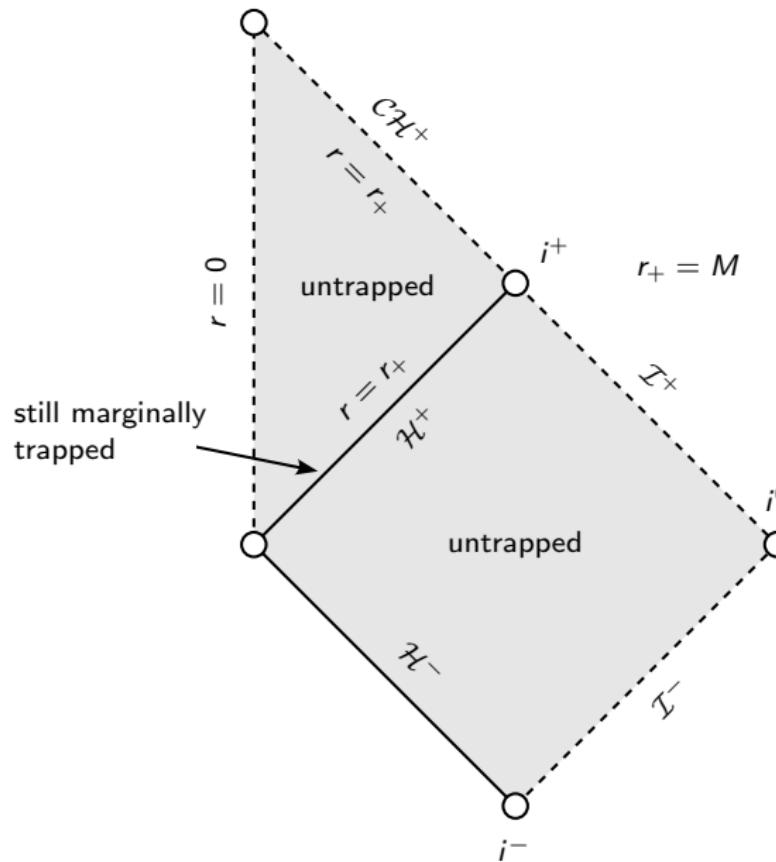
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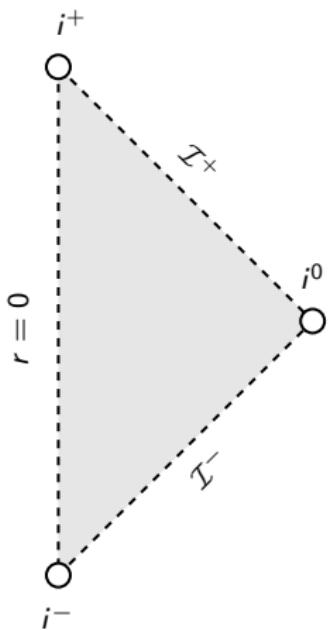
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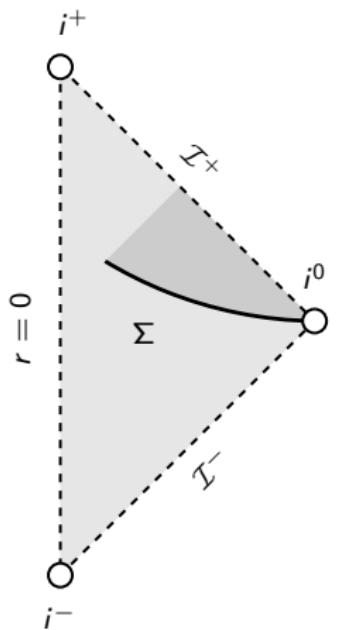
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Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



Surface gravity κ of Reissner–Nordström

- ▶ $\nabla_T T = \kappa T$ on \mathcal{H}^+
- ▶ RN with mass M and charge e , $|e| \leq M$, has

$$\kappa = 2\pi T = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- ▶ **Subextremal:** $\kappa > 0$
- ▶ **Extremal:** $\kappa = 0$

The third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Difficult to interpret κ dynamically!

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(Received 19 May 1986)

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5. “In any continuous process” included because we are dealing with fundamentally nongeneric behavior.
 - ▶ Adding any kind of genericity assumption to the third law renders it a completely uninteresting and utterly trivial statement, which does not even use the Einstein equations!

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the “third law of black hole thermodynamics” is false.

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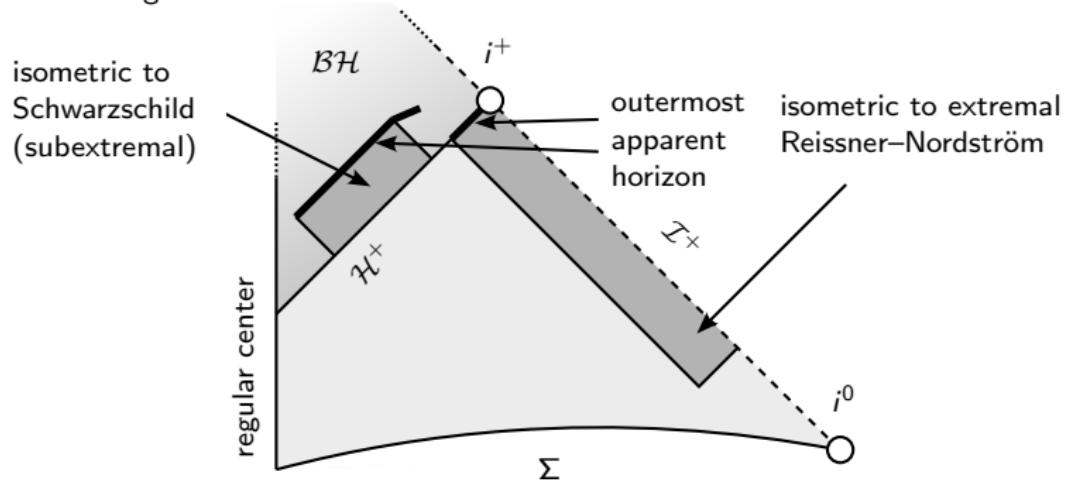
Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the “third law of black hole thermodynamics” is false.

More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field system (which satisfies the weak energy condition) with the following behavior:

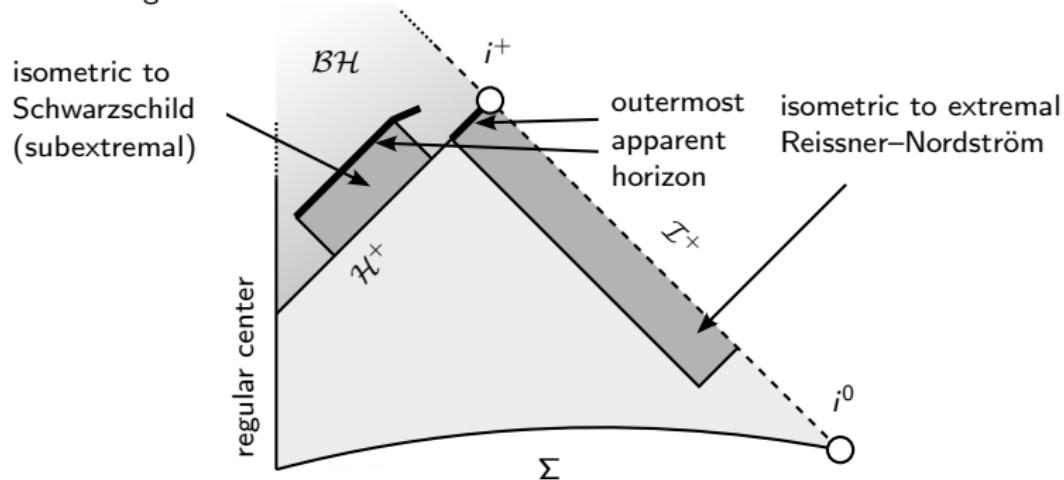
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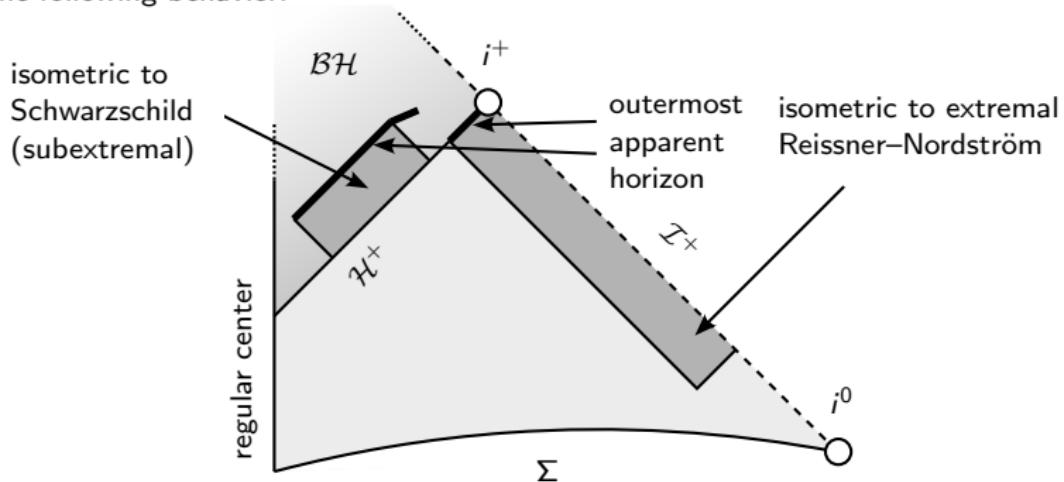
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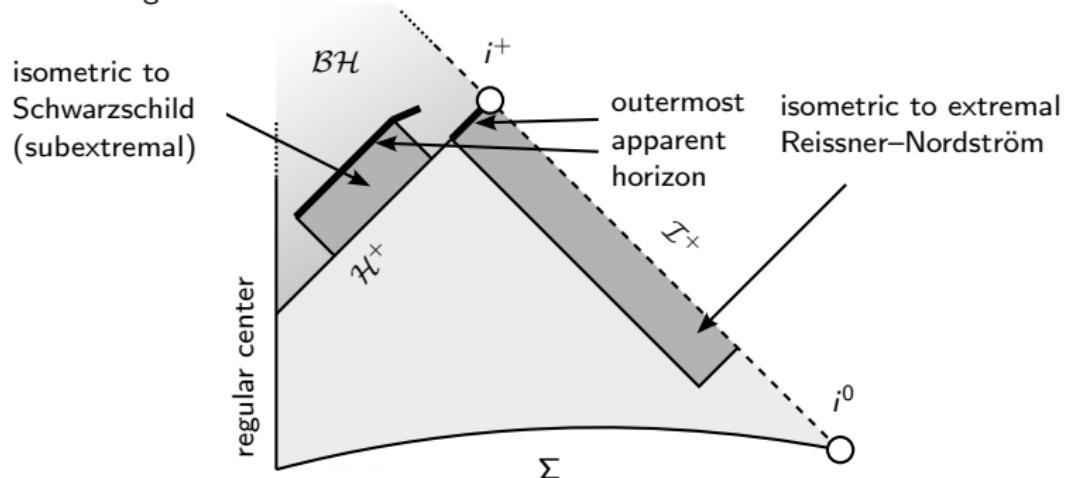
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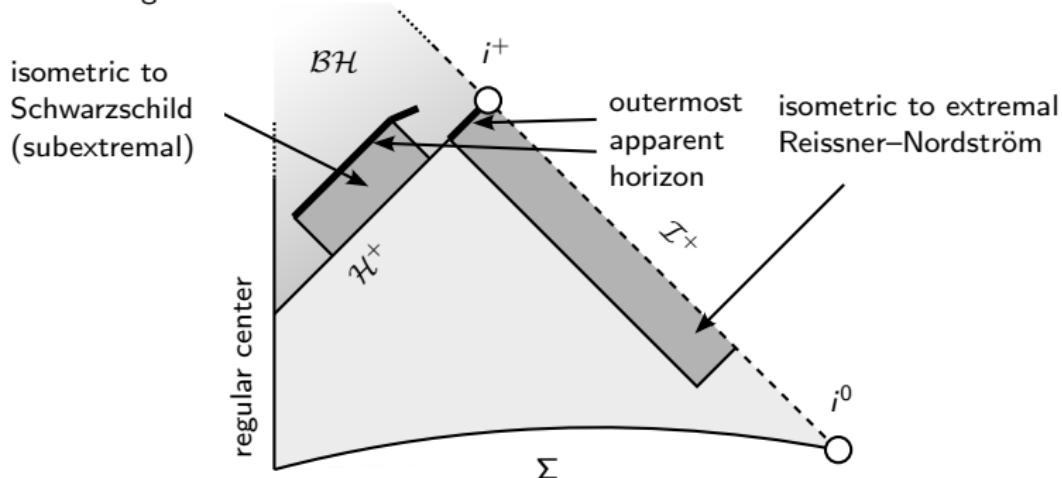
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- ▶ Forms an exactly Schwarzschild “apparent horizon.”
- ▶ Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.
- ▶ These solutions are arbitrarily **regular**: for any $k \in \mathbb{N}$, there exists a C^k example.

Einstein–Maxwell-charged scalar field system

- ▶ Lorentzian manifold (\mathcal{M}^{3+1}, g)
- ▶ 2-form $F = dA$ (electromagnetism)
- ▶ Charged (complex) massless scalar field ϕ

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$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = 2 \left(T_{\mu\nu}^{\text{EM}} + T_{\mu\nu}^{\text{CSF}} \right) \quad (1)$$

$$\nabla^\mu F_{\mu\nu} = 2\epsilon \operatorname{Im}(\phi \overline{D_\nu \phi}) \quad (2)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0 \quad (3)$$

$$T_{\mu\nu}^{\text{EM}} = g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \quad (4)$$

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- ▶ Spherical symmetry, ϕ degree of freedom breaks rigidity from Birkhoff.

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- ▶ 2-form $F = dA$ (electromagnetism)
- ▶ Charged (complex) massless scalar field ϕ
- ▶ Equations:

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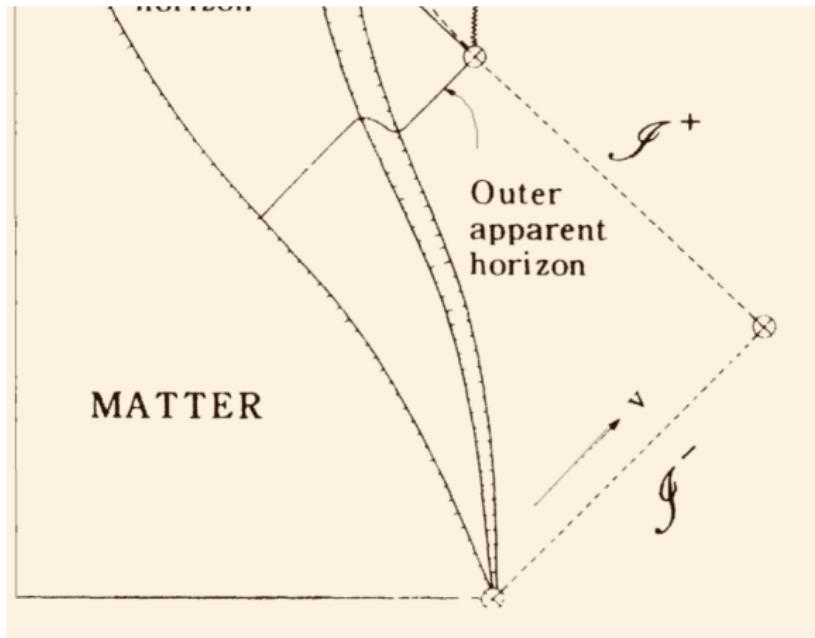
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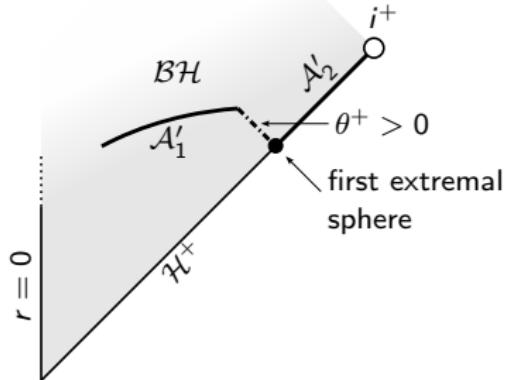
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- ▶ Under this implicit assumption, Israel concluded the third law to be true on the basis of Raychaudhuri's equation.

Israel's argument

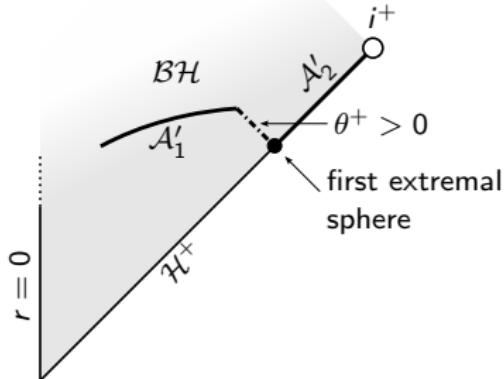


Israel's paper reinterpreted



Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

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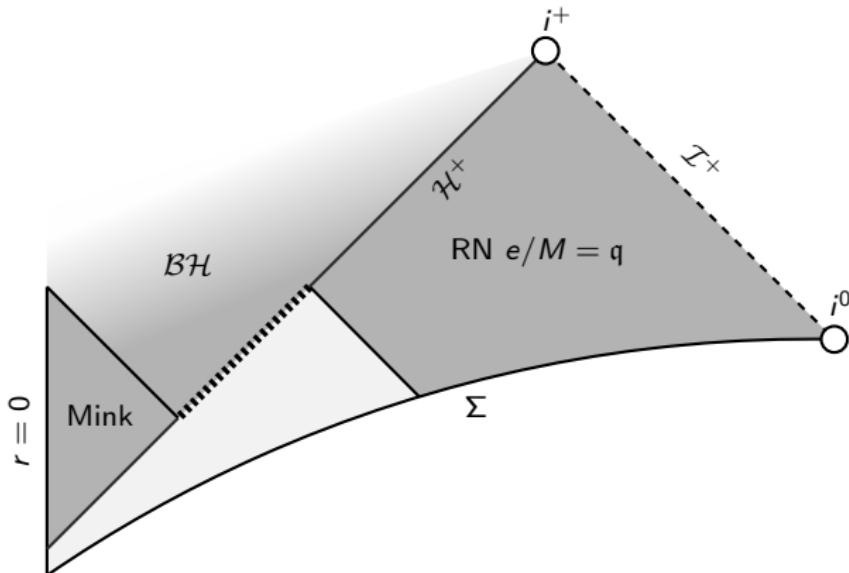
Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

However, this is a feature, not a glitch!

Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

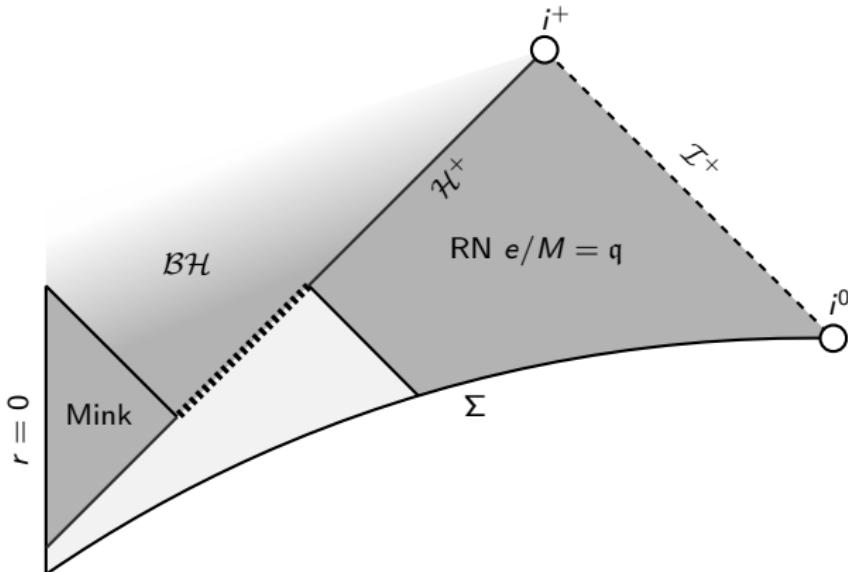
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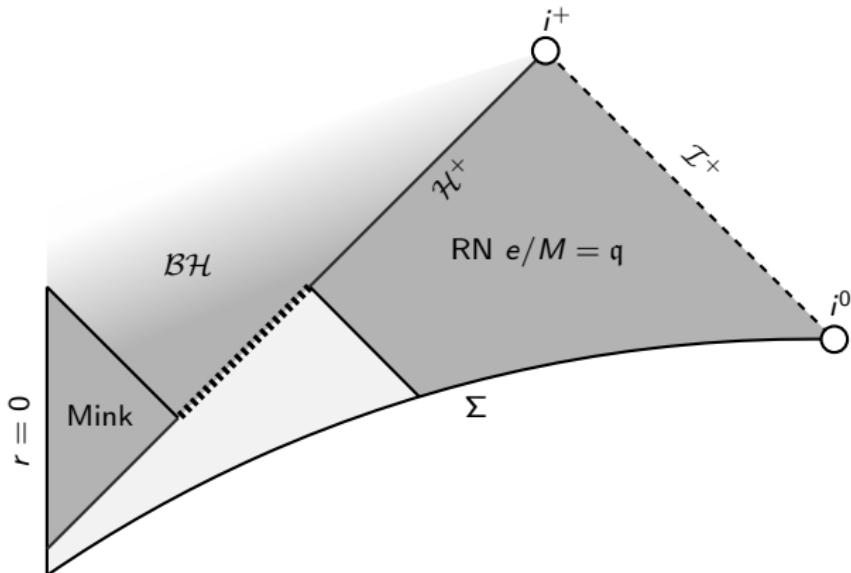


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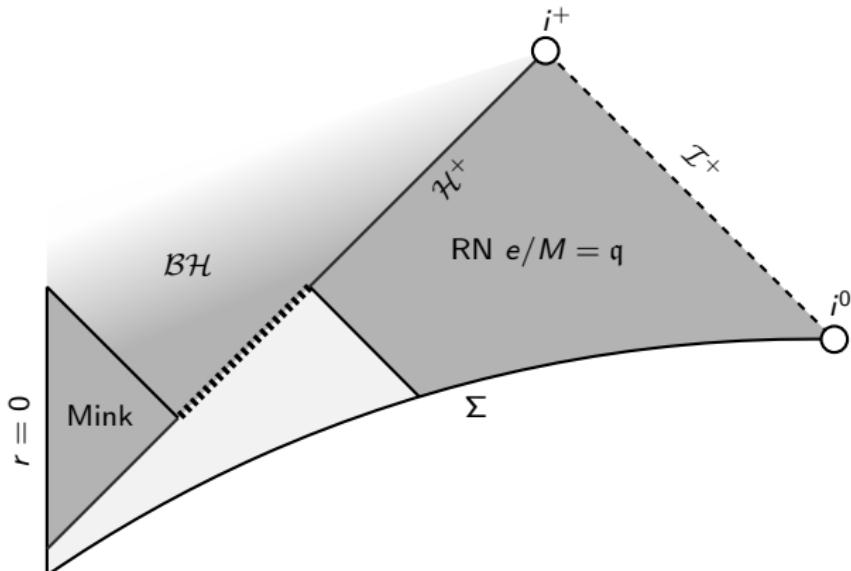


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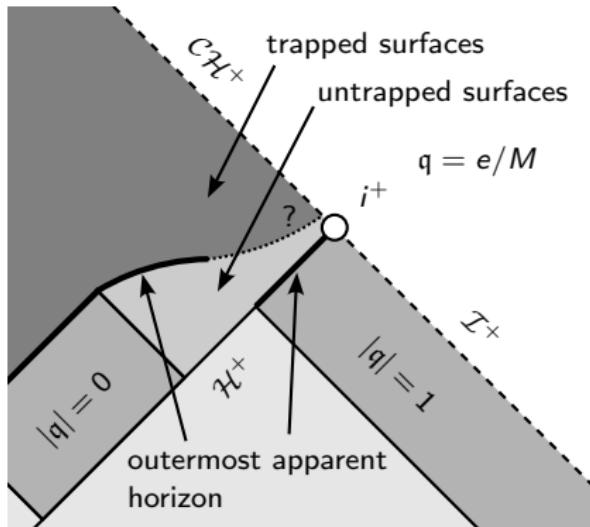
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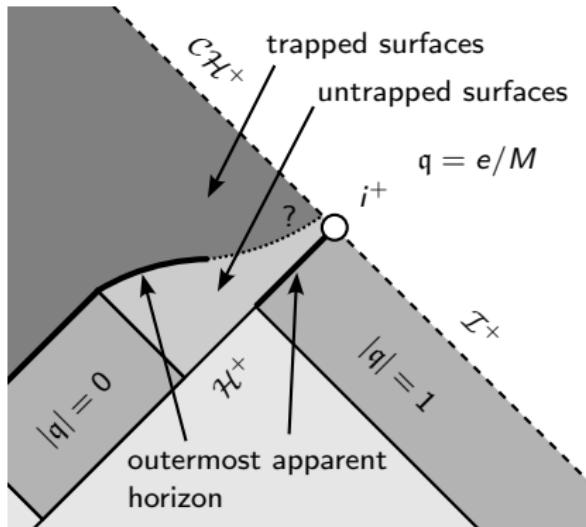


- ▶ From the point of view of our construction, the extremal case is exactly the same as the subextremal case!
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- ▶ $T = 0$ is not fundamentally different than $T > 0$ in these examples.

Interior structure of third law violating solutions

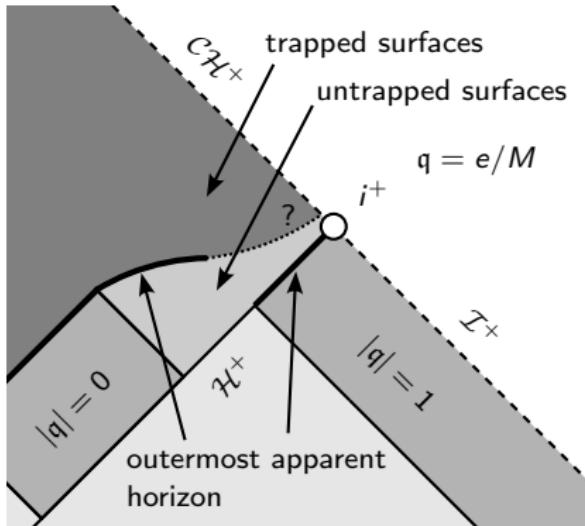


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- The outermost apparent horizon becomes disconnected, yet the spacetime is highly regular.

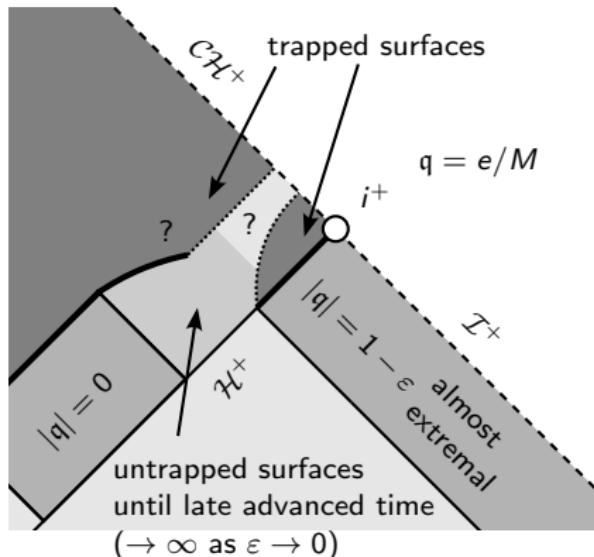
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- Trapped surfaces persist for all time and are found at any retarded time in the black hole interior.

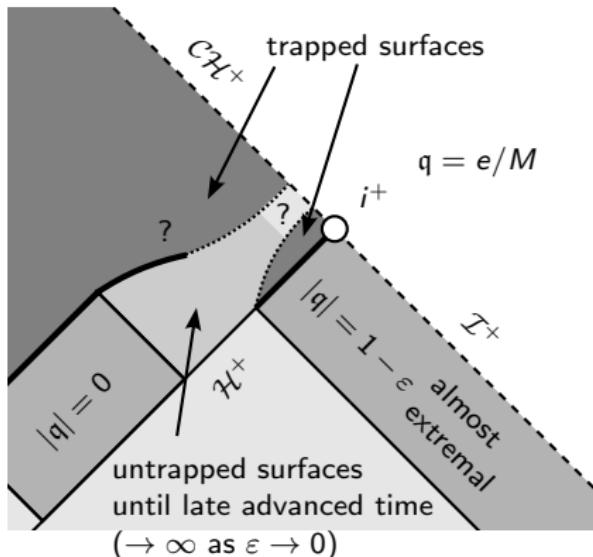
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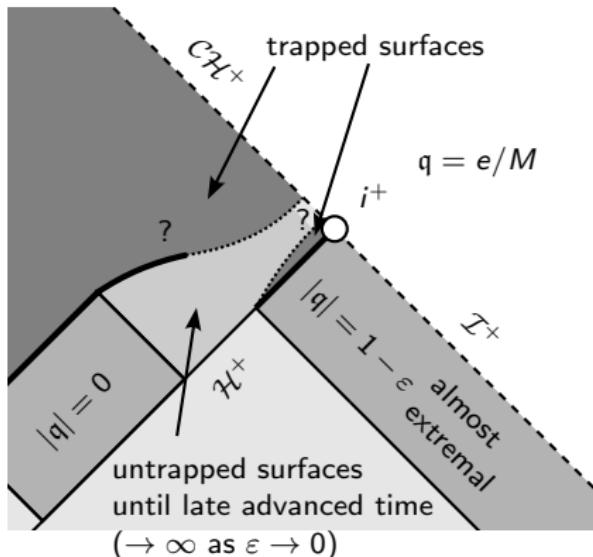
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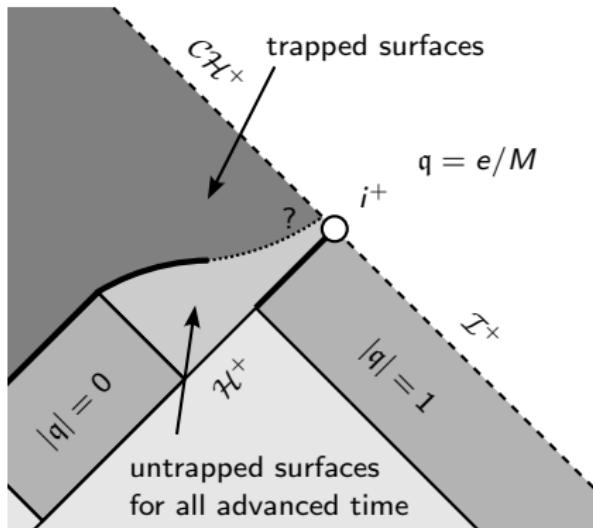
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The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

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- ▶ However, such a reformulation would be utterly trivial (and has nothing to do with $T = 0$).

The third law and supercharging

Bardeen–Carter–Hawking:

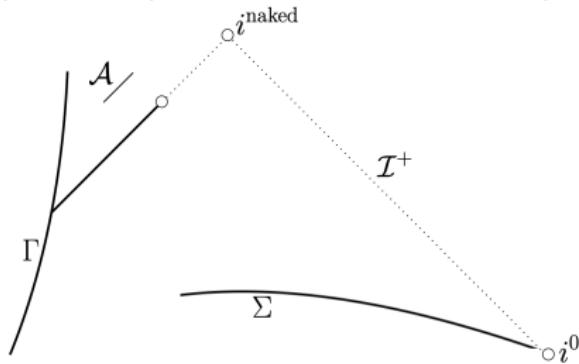
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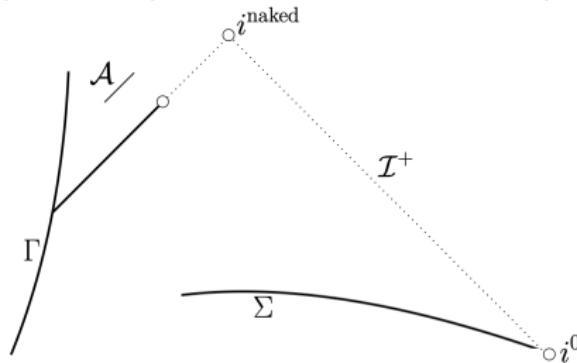


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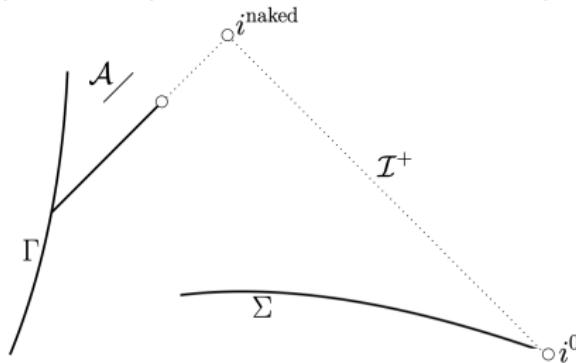
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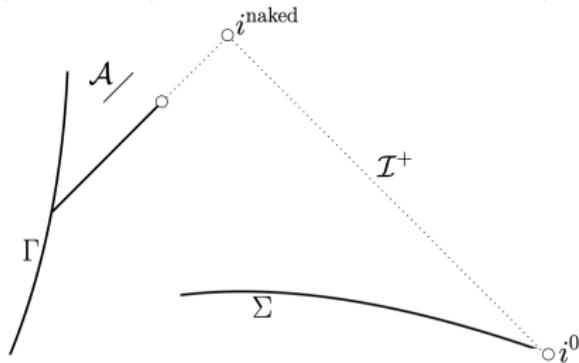
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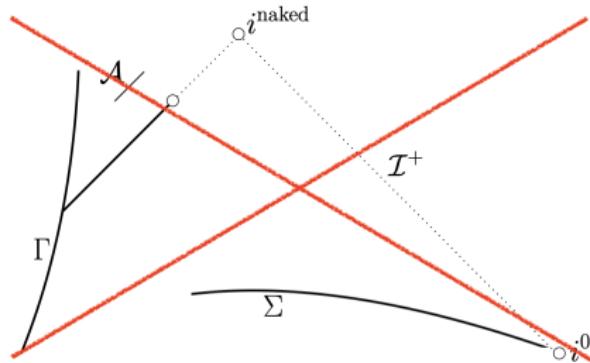
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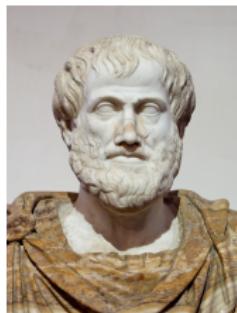
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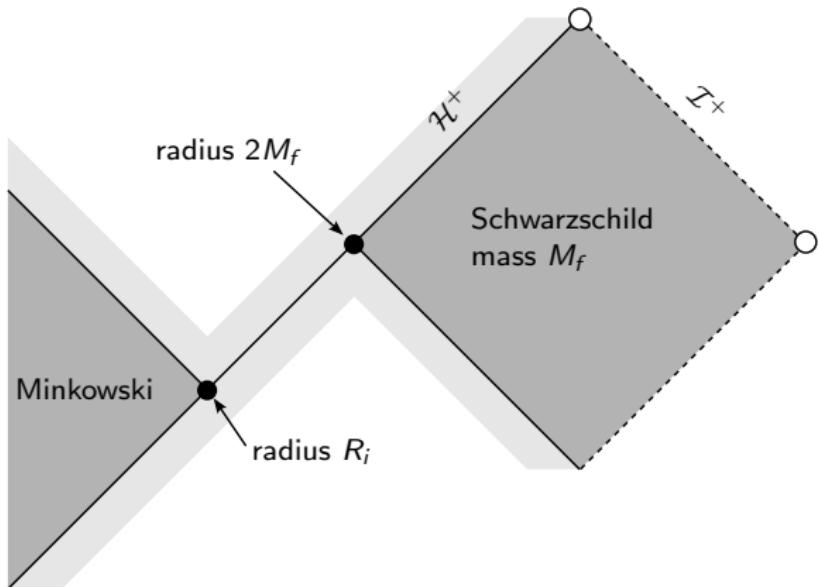
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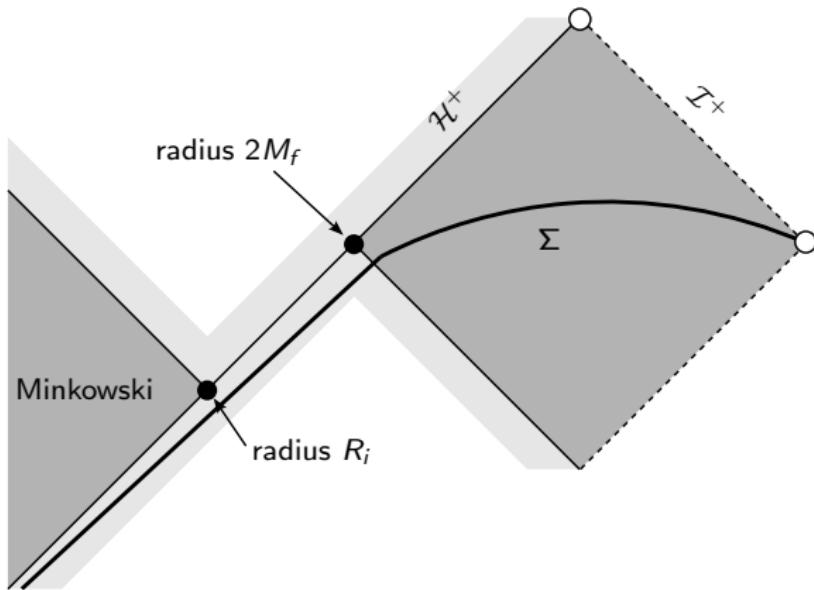


The black hole region \mathcal{BH} and event horizon \mathcal{H}^+ of a spacetime are **teleological** notions—they can't be located *without knowing the entire future of the spacetime*.

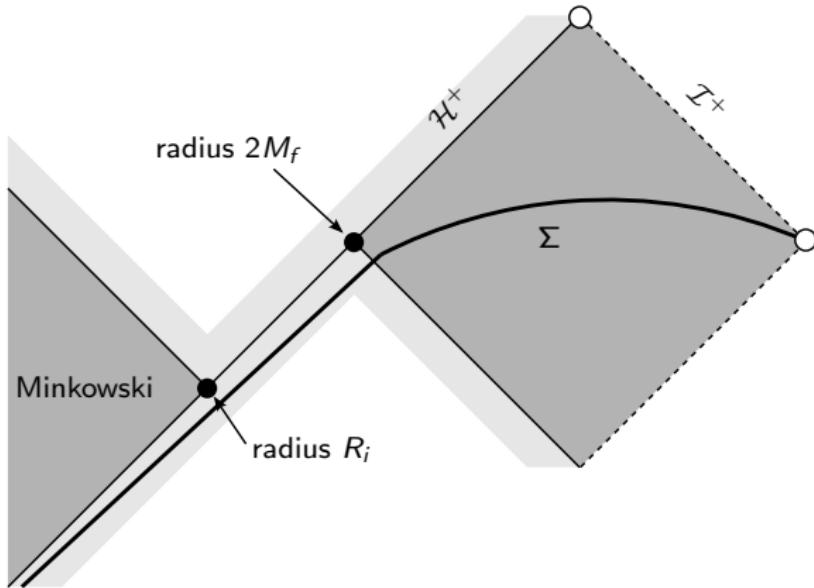
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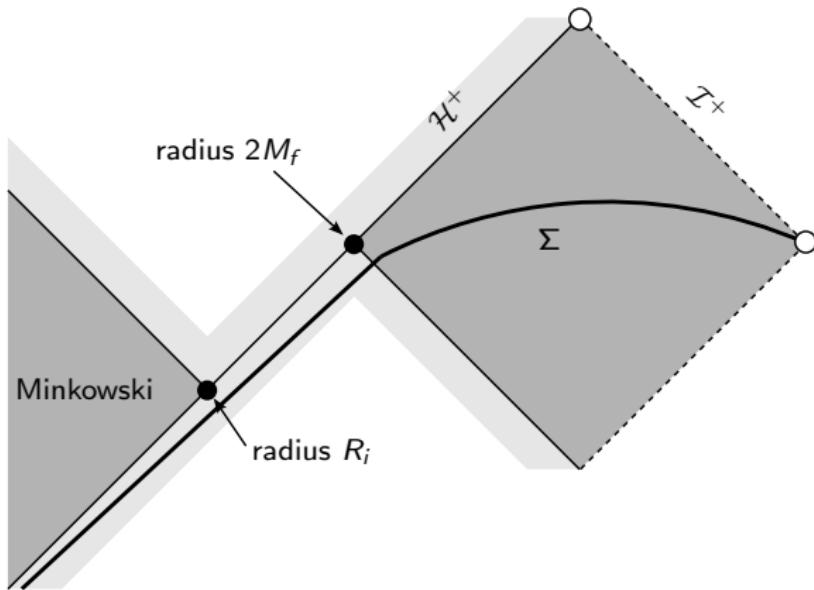


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Characteristic gluing for the vacuum Einstein equations recently initiated in fundamental works of Aretakis–Czimek–Rodnianski and refined by Czimek–Rodnianski, but in a **perturbative** regime around Minkowski space which is inapplicable here.

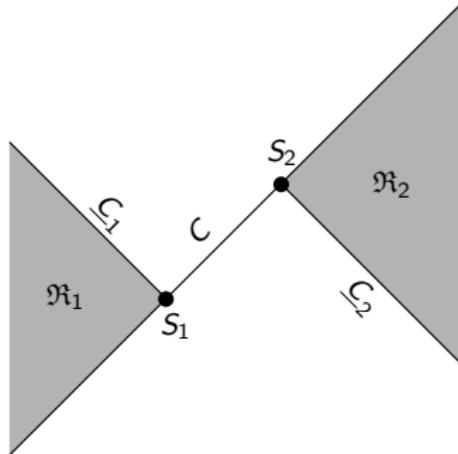
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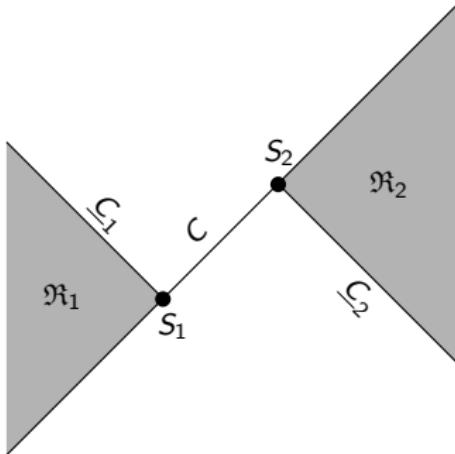
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The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Characteristic/null gluing for the linear wave equation



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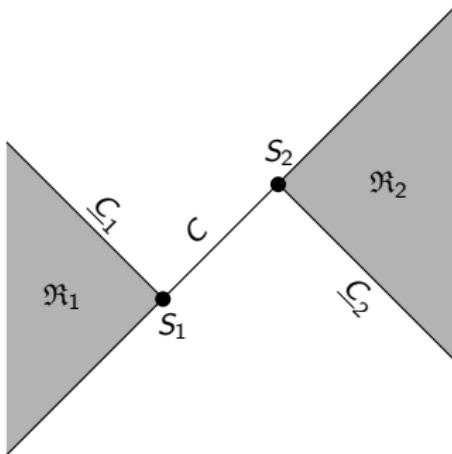


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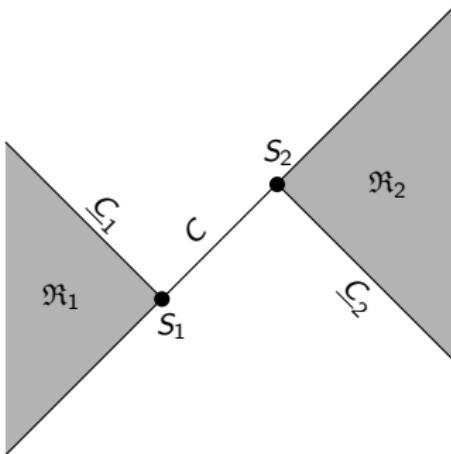
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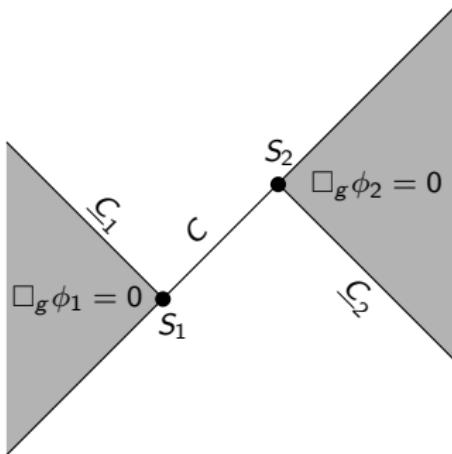
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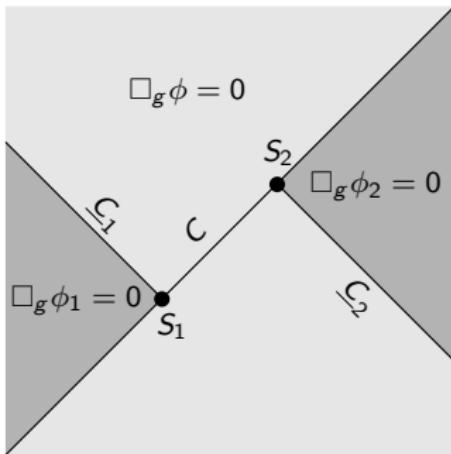
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- ▶ C will be the event horizon \mathcal{H}^+

Minkowski to Schwarzschild gluing

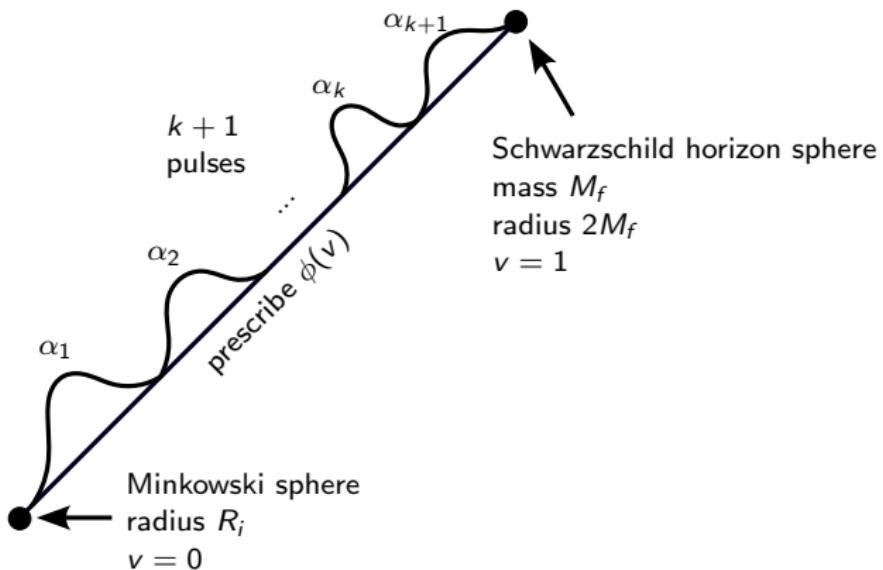
Theorem (Kehle–U. '22).

For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.

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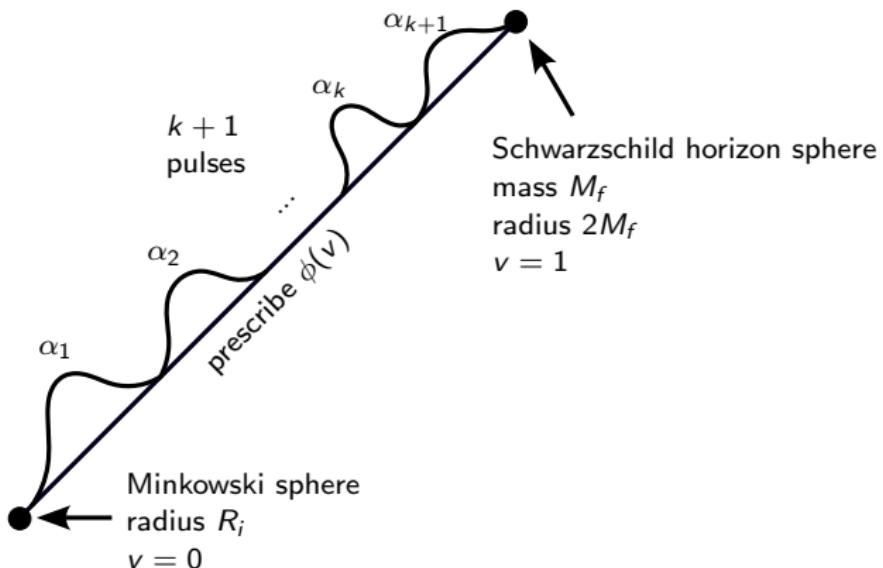
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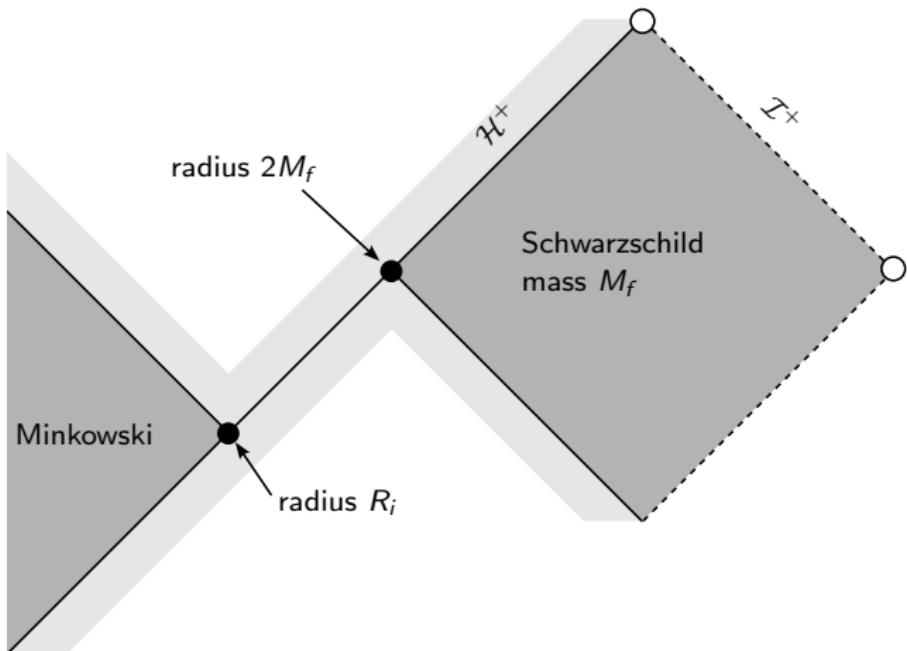
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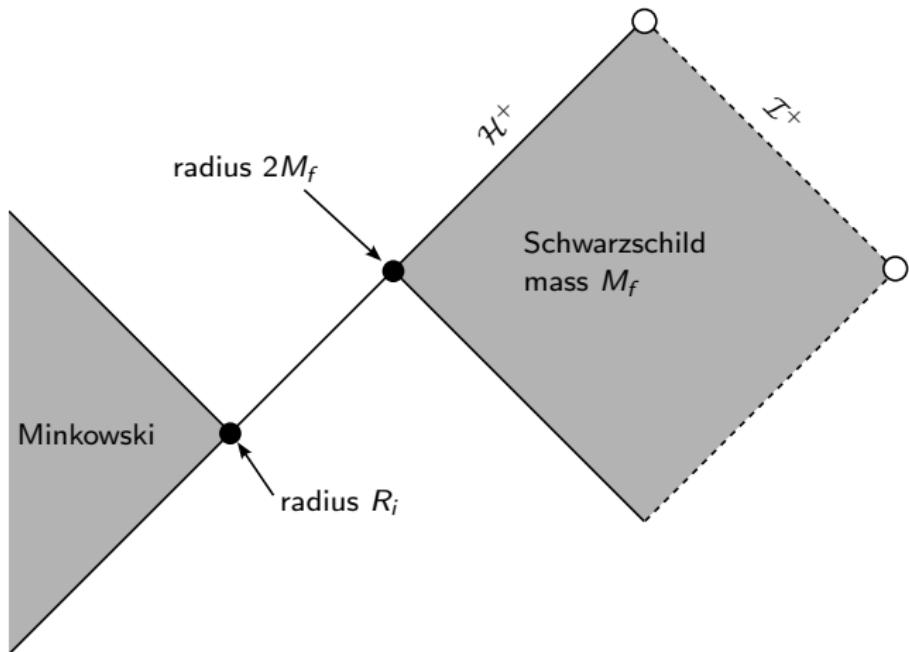


Pulse amplitudes α_j are chosen using the Borsuk–Ulam theorem, making use of the symmetry $\phi \mapsto -\phi$ of the equations.

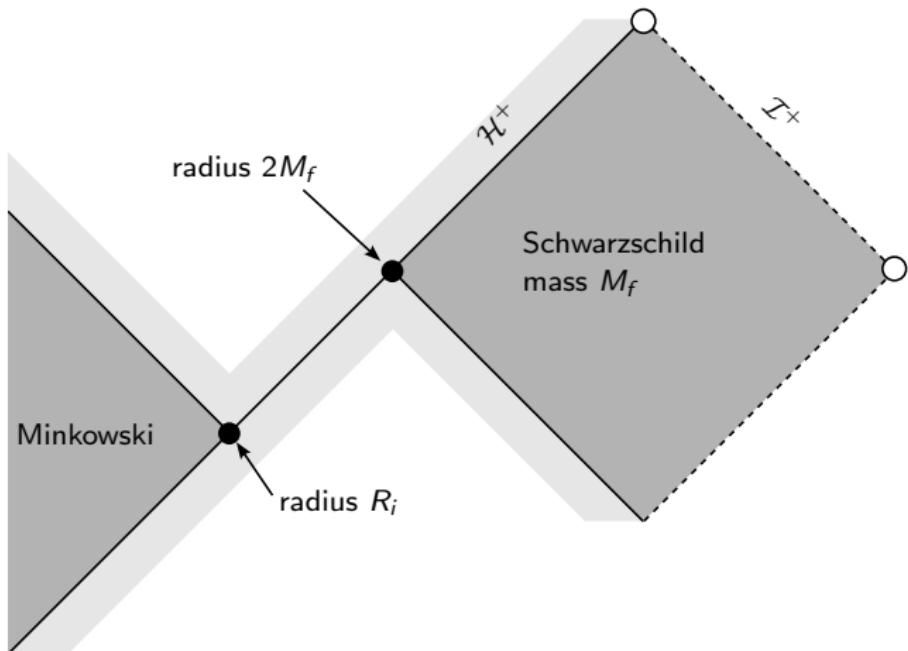
Minkowski to Schwarzschild gluing



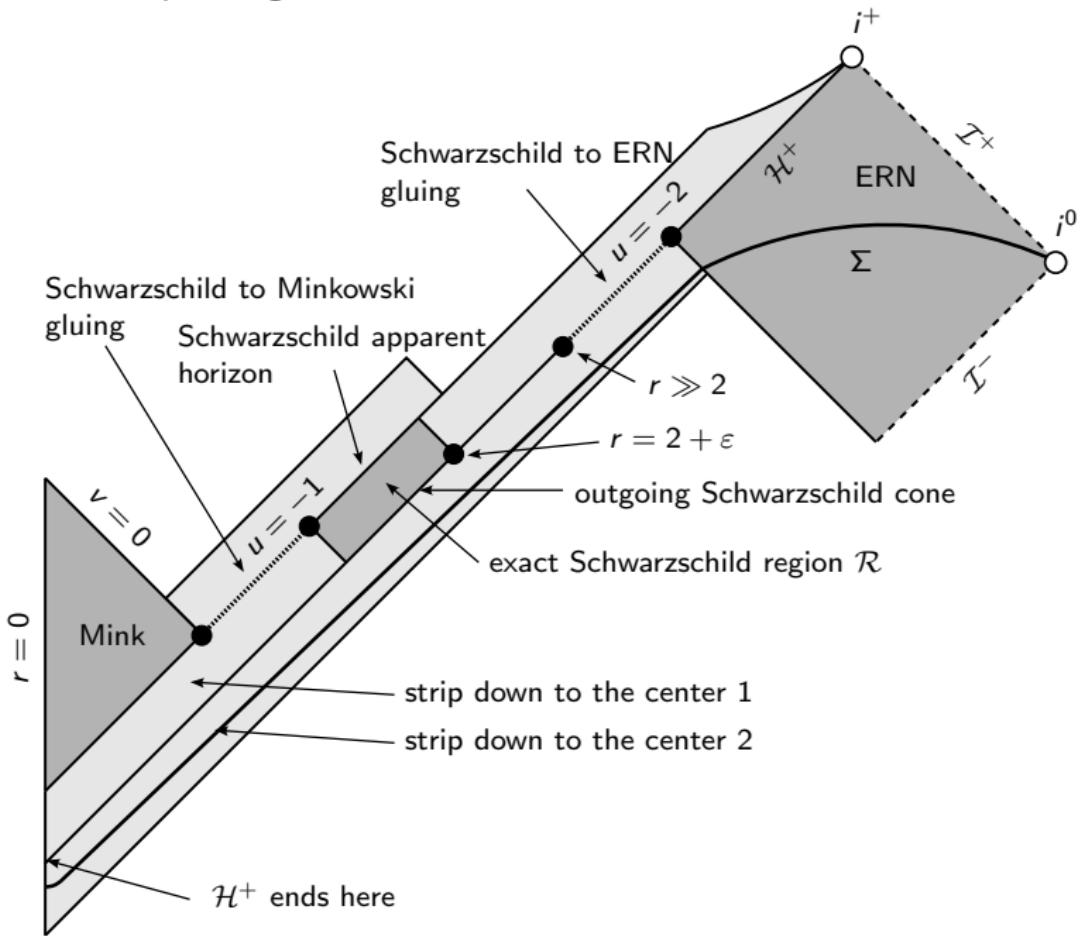
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Blueprint to disproving the third law



The vacuum case

We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

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We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

Conjecture.

There exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, already in vacuum, the “third law of black hole thermodynamics” is false.

The very slowly rotating case

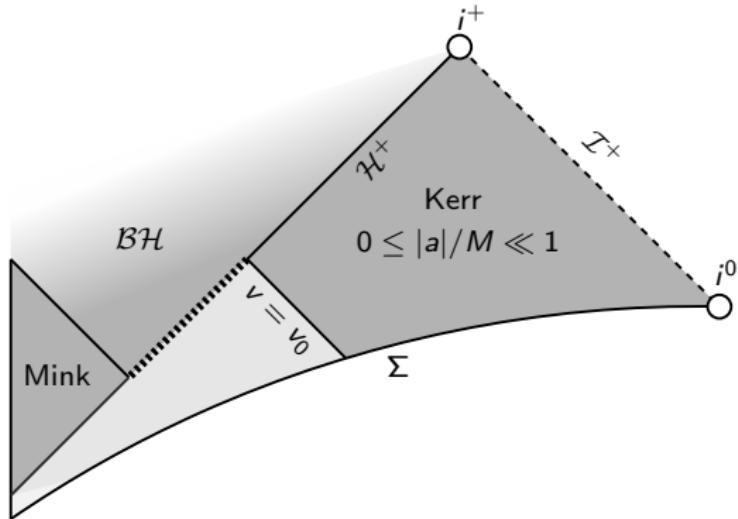
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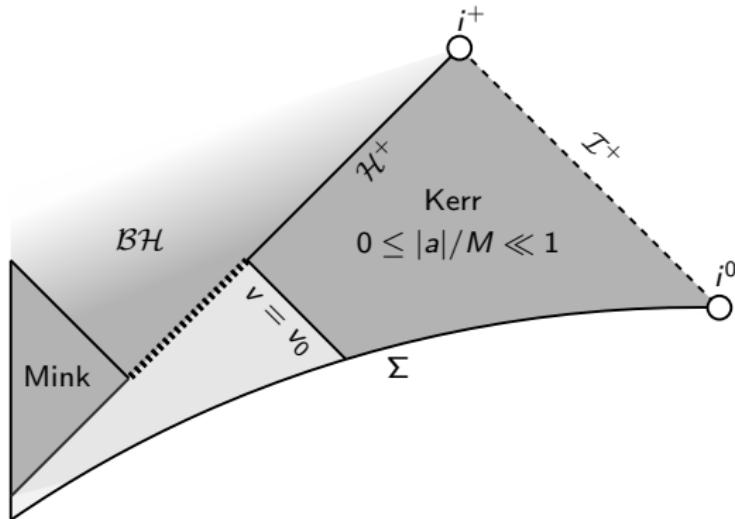
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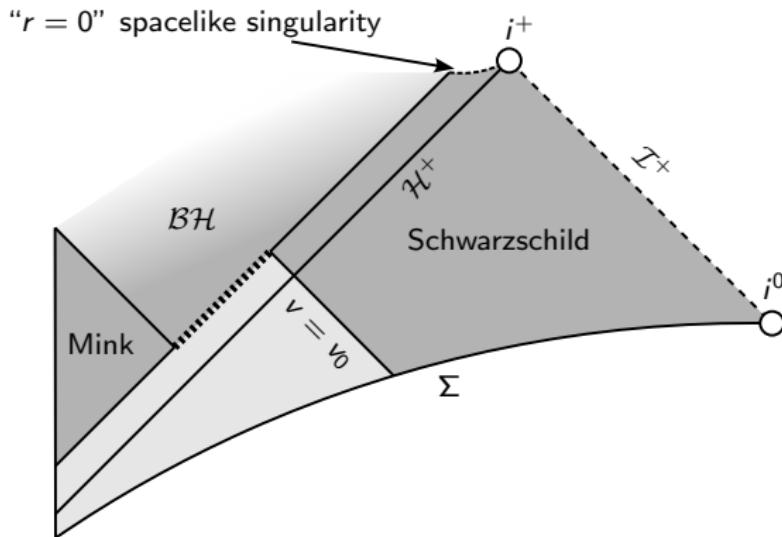


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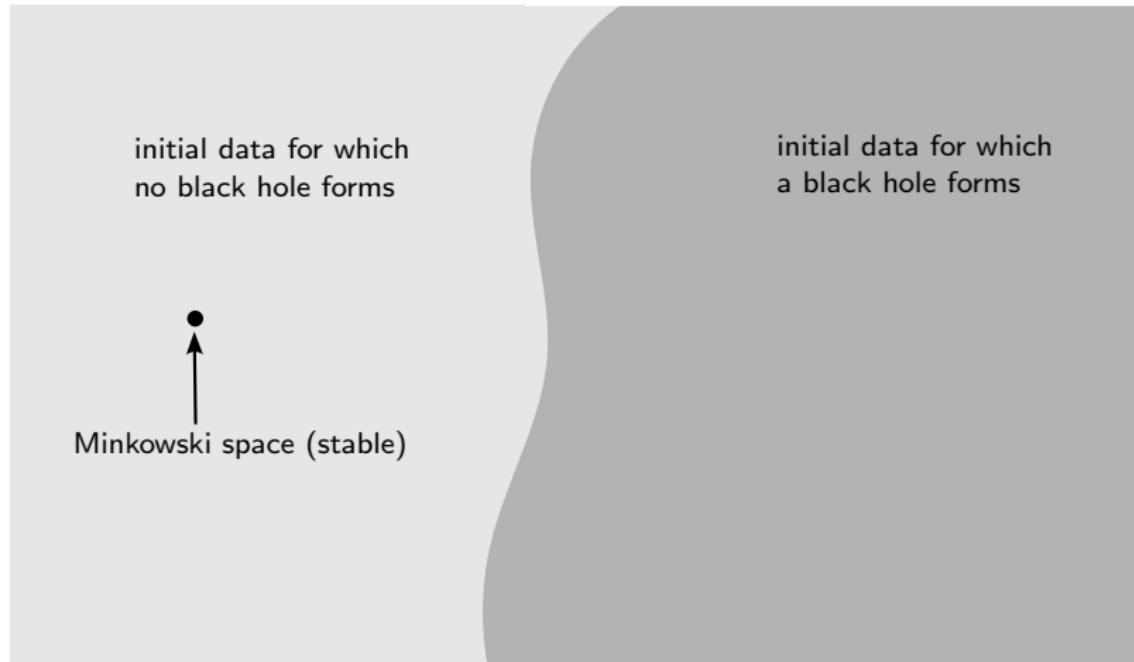


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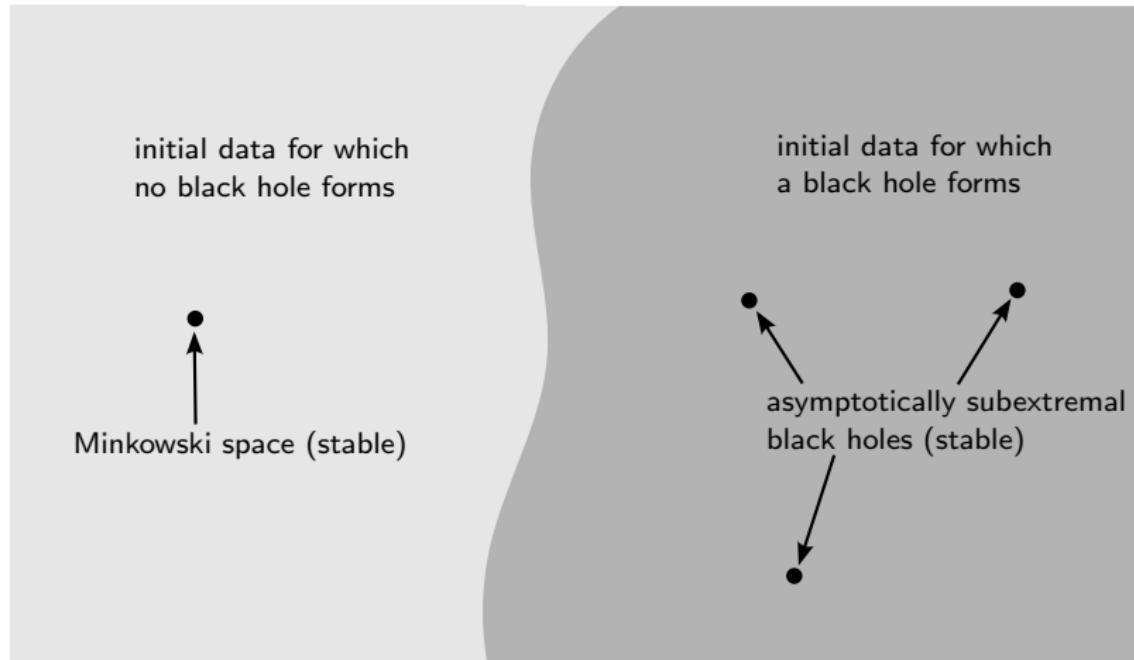
Outlook: the “moduli space” of gravitational (non-)collapse



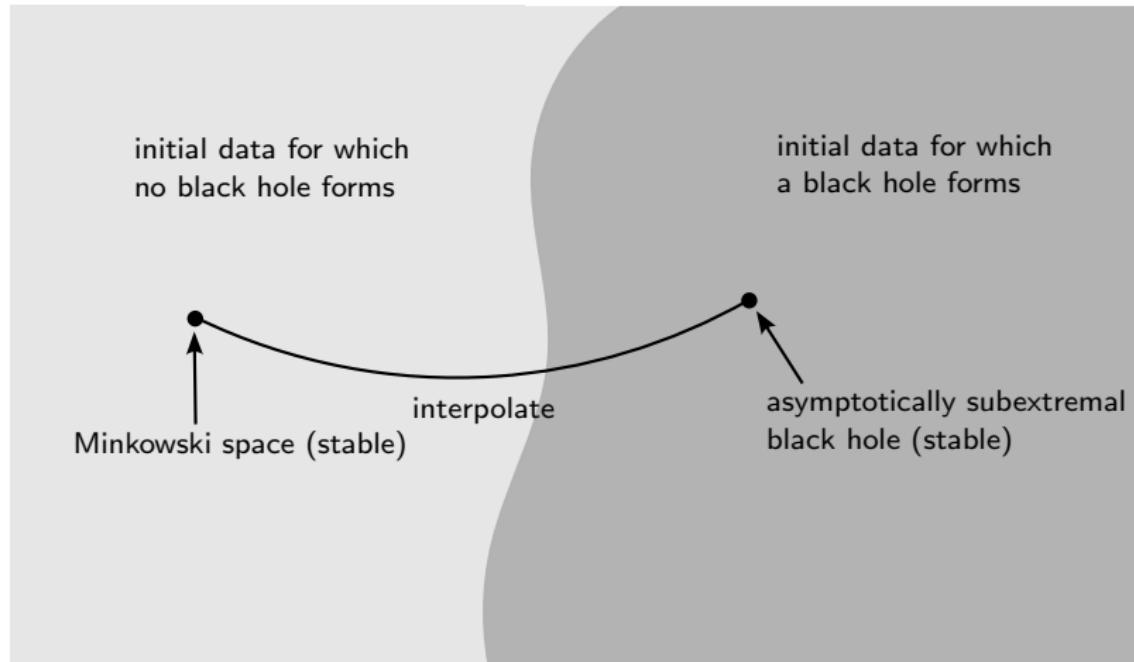
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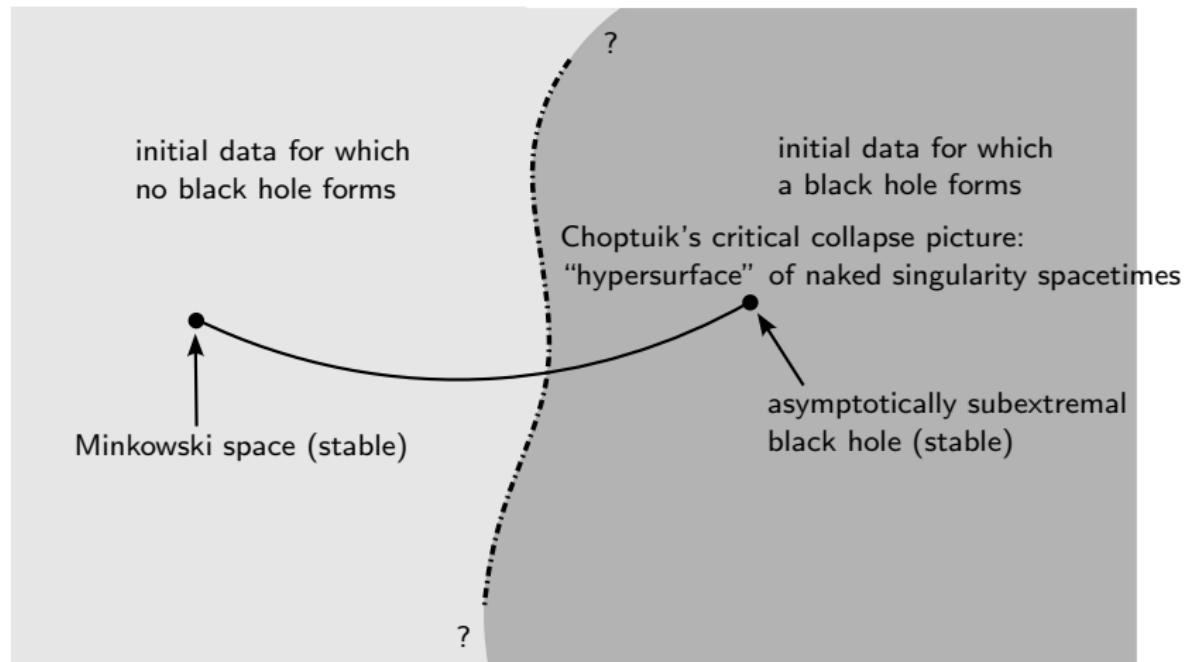
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Critical behavior: naked singularities expected when a “low energy” black hole forms.

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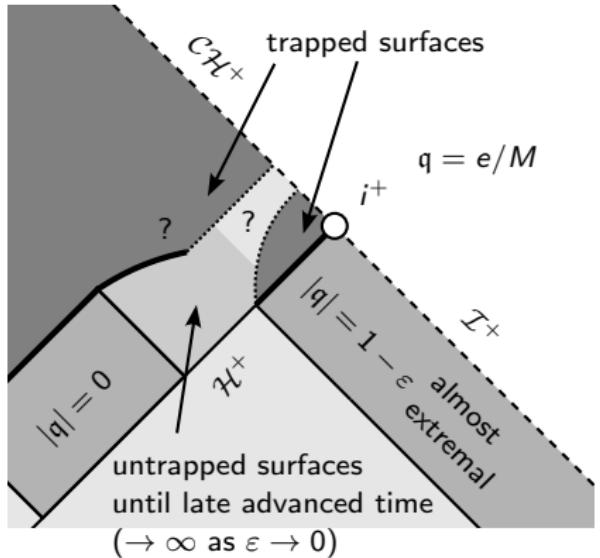
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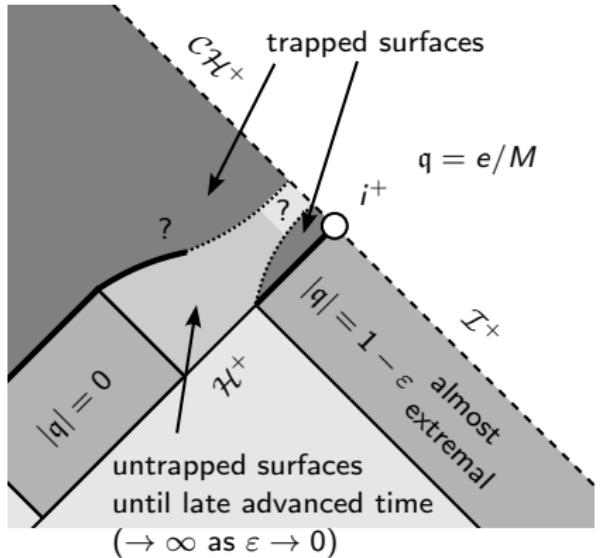
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Critical behavior: extremal black holes can be perturbed so black hole formation is delayed and the event horizon jumps just as the extremal threshold is crossed.

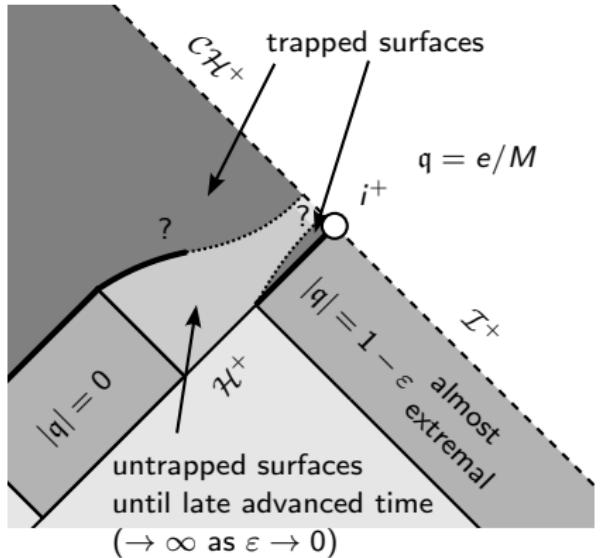
Event horizon jumping



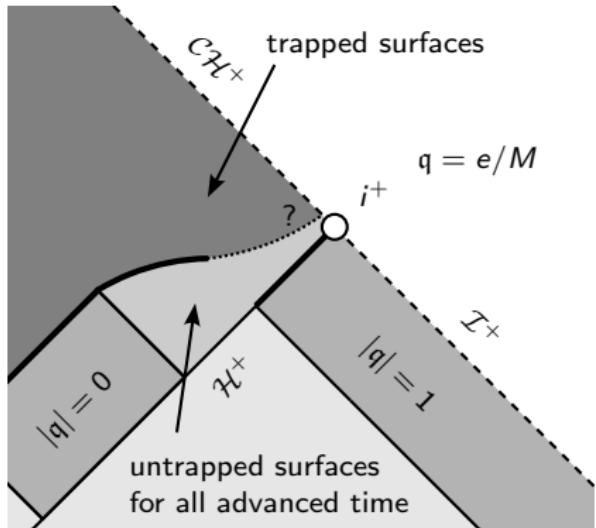
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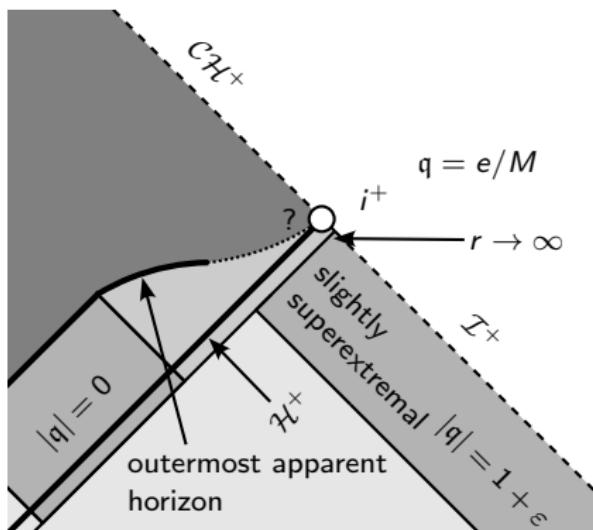
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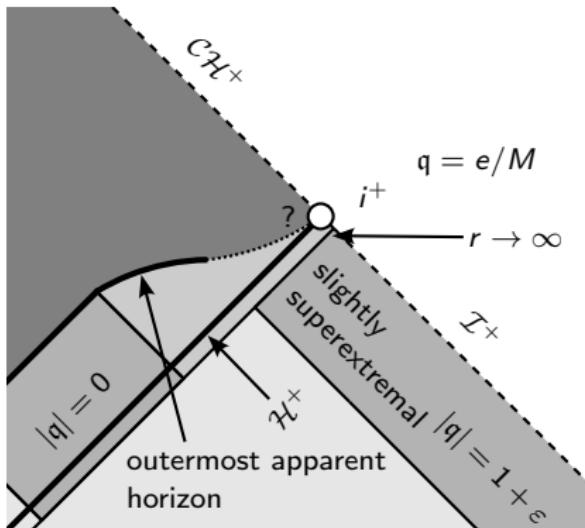
Event horizon jumping



So what does happen if you “go past extremality”?



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Critical behavior: We expect that there are smooth 1-parameter families of solutions passing through extremality where the event horizon “jumps” as soon as the exterior becomes superextremal. There is no naked singularity.

Instability of black hole formation due to electromagnetic repulsion

Conjecture.

There exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system, leading to the formation of an extremal Reissner–Nordström black hole, that can be perturbed to disperse, i.e. no black hole forms.

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- ▶ (g, F, f)

$$G_{\mu\nu}[g] = 2 \left(T_{\mu\nu}^{\text{EM}}[F] + T_{\mu\nu}^{\text{Vl}}[f] \right) \quad (6)$$

$$\nabla^\nu F_{\mu\nu} = \epsilon N_\mu[f] \quad (7)$$

$$\left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu{}_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\mu} + \epsilon F^\mu{}_\nu p^\nu \frac{\partial}{\partial p^\mu} \right) f = 0 \quad (8)$$

$$\text{spt } f \subset \{p^2 = 0\} \quad (9)$$

Injection of charged massless Vlasov beam into Minkowski space

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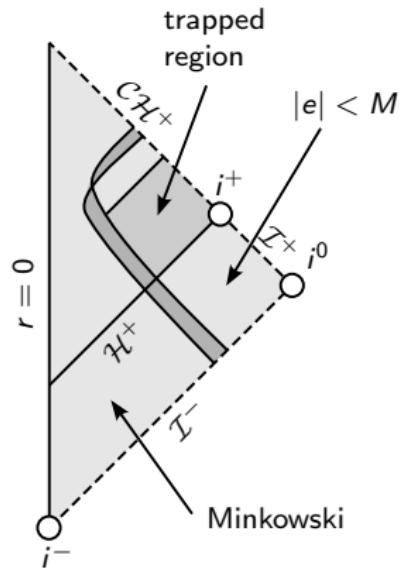
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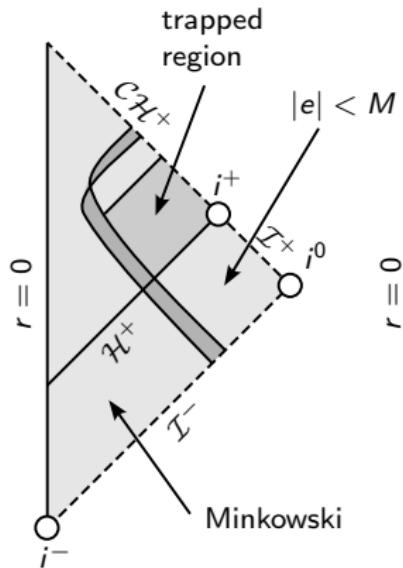


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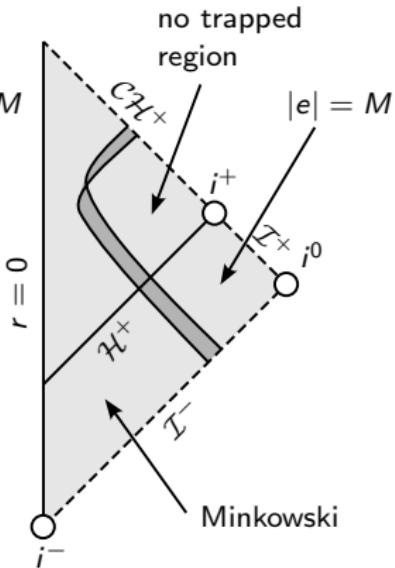
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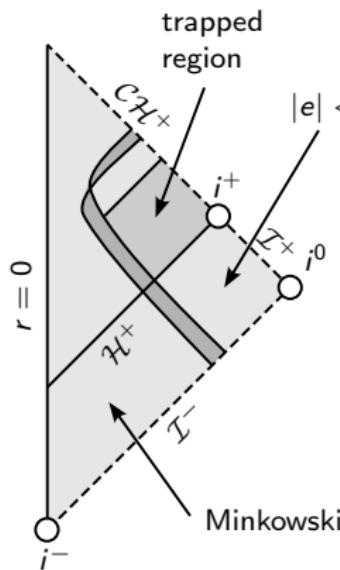


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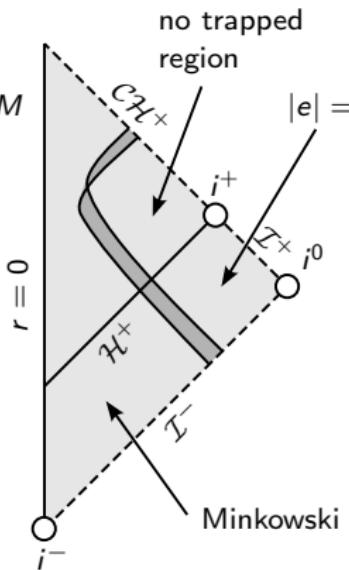
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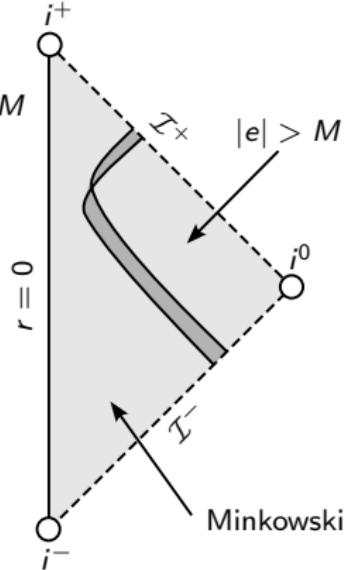
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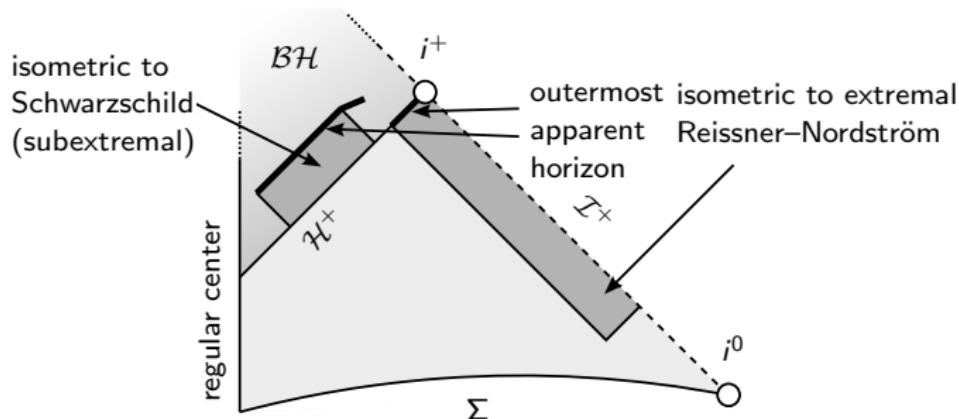
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Thank you!