Crete summer school GR basics examples sheet

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Contents

1 Orthogonality in Lorentzian vector spaces

Let (V, m) be an $(n+1)$-dimensional Lorentzian vector space. Show that:

a) Two timelike vectors are never orthogonal.

b) A timelike vector is never orthogonal to a null vector.

c) Two null vectors are orthogonal if and only if they are collinear.

d) The orthogonal complement of a null vector is a codimension-one subspace with a degenerate scalar product. The kernel of the scalar product restricted to this subspace is one-dimensional and equal to the span of the null vector.

e) Continuing the previous point, show that if v is null, then $(v^\perp)^\perp = \mathbb{R}v$. (In fact, for any subspace it holds that $(W^\perp)^\perp = W$, just as in an inner product space.)

Hint: While one can try to prove all of these in a "coordinate free" manner, it is much simpler to work in a standard orthonormal basis.

2 Null hypersurfaces

Let $(\mathcal{M}^{n+1}, g)$ be a Lorentzian manifold and $\mathcal{H}^n \subset \mathcal{M}^{n+1}$ a smooth embedded hypersurface. We say that $\mathcal{H}$ is a *null hypersurface* if $T_p\mathcal{H} \subset T_p\mathcal{M}$ is null (as a codimension-one subspace of the Lorentzian vector space $(T_p\mathcal{M}, g_p)$) for every $p \in \mathcal{H}$. Show that:

a) There exists a null vector field L defined along $\mathcal{H}$ such that $L_p \in T_p\mathcal{H}$ for every $p \in \mathcal{H}$. This L is both tangent and normal to $\mathcal{H}$!

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b) Show that there exists a function $f : \mathcal{H} \to \mathbb{R}$ such that

$$\nabla_L L = fL. \tag{2.1}$$

Hint: The goal is show that for any section X of $T\mathcal{H}$, $g(X, \nabla_L L) = 0$. Let $p \in \mathcal{H}$, $X_p \in T_p\mathcal{H}$, and extend X_p to a vector field $X \in \Gamma(T\mathcal{H})$ (at least locally near p) such that $[L, X] = 0$. (Why can this be done?) Now use the symmetry of the connection to show that $g(\nabla_L L, X) = 0$.

c) Show that there exists a null vector field L', pointwise proportional to L, such that

$$\nabla_{L'} L' = 0. \tag{2.2}$$

d) Interpret and prove the statement that "any null hypersurface is ruled by null geodesics."

3 The energy-momentum tensor of a scalar field

A *scalar field* is a function $\phi \in C^\infty(\mathcal{M})$ (which often solves a wave equation). We define the *energy-momentum tensor* of ϕ by

$$T_{\mu\nu} \doteq \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi, \tag{3.1}$$

where $\partial^\alpha\phi = g^{\alpha\beta}\partial_\beta\phi$.

a) Show that if X is future-directed causal, then $-T^\mu{}_\nu X^\nu$ is as well. *Hint*: Choose an orthonormal basis of the tangent space so that X is "standard," i.e., is of the form e_0 or $e_0 + e_1$. Use the fact that $T_{\mu\nu}X^\mu X^\nu$ is coordinate invariant.

b) Let X and Y be future-directed timelike vectors at p and $\{V_\mu\}$ a basis of $T_p\mathcal{M}$. Show that there exists a constant $c > 0$ such that

$$T(X, Y) \geq c\sum_\mu |V_\mu\phi|^2. \tag{3.2}$$

In this sense $T(X, Y)$ quantitatively "controls all derivatives of ϕ" pointwise.

c) Let X be future-directed timelike and Y be future-directed null. Let $\{e_1, \ldots, e_{n-1}\}$ be spacelike vectors so that $\{Y, e_1, \ldots, e_{n-1}\}$ spans $Y^\perp$. Show that there exists a constant $c > 0$ such that

$$T(X, Y) \geq c\left(|Y\phi|^2 + \sum_{i=1}^{n-1}|e_i\phi|^2\right). \tag{3.3}$$

This combination misses one direction!

d) Let Y be future-directed null and let $\underline{Y}$ be a future-directed null vector so that $g(Y, \underline{Y}) = -2$. Show that there exists a constant $c > 0$ such that

$$T(Y, \underline{Y}) \geq c\sum_{i=1}^{n-2}|e_i\phi|^2, \tag{3.4}$$

where $\{e_1, \ldots, e_{n-1}\}$ is a basis of spacelike vectors for $Y^\perp \cap \underline{Y}^\perp$. This combination misses two directions!

Remark 1. In general, the constants c in (3.2), (3.3), and (3.4) depend on the metric g, the point p, the vector fields X and Y, and the choice of basis vectors on the right-hand side. In practice one needs some quantitative control this constant so that these estimates are "effective."

4 Topology of Lorentzian manifolds

In this exercise, we will see some basic topological properties of Lorentzian manifolds. We wish to prove the following:

Theorem 1. *Let $\mathcal{M}$ be a smooth manifold. The following are equivalent:*

i) $\mathcal{M}$ admits a Lorentzian metric.

ii) $\mathcal{M}$ admits a time-oriented Lorentzian metric.

iii) $\mathcal{M}$ admits a nonvanishing vector field (i.e., $X(p) \neq 0$ for every $p \in \mathcal{M}$).

iv) $\mathcal{M}$ is noncompact or is compact with vanishing Euler characteristic.

Proof. The logic of the proof is as follows: ii) $\Rightarrow$ i) is trivial, iii) $\Leftrightarrow$ iv), iii) $\Leftrightarrow$ ii), and i) $\Rightarrow$ iv). Here is an outline of some of the parts. We will need the following deep theorem in topology:

Theorem 2 (Poincaré–Hopf)**.** *A closed manifold $\mathcal{M}$ carries a nonzero vector field if and only if the Euler characteristic $\chi(\mathcal{M}) = 0$.*

Here are now some hints to prove Theorem 1.

a) Let $\mathcal{M}$ be a smooth manifold carrying a nonzero vector field X. Let g_0 be a Riemannian metric on $\mathcal{M}$. (Does such a g_0 always exist?) Show that there exists a positive function $f \in C^\infty(\mathcal{M})$ so that

$$g = -f^2 g_0(\cdot, X) \otimes g_0(\cdot, X) + g_0 \tag{4.1}$$

is a Lorentzian metric on $\mathcal{M}$.

b) (*) Show that every noncompact manifold carries a nonvanishing vector field. *Hint*: Construct a vector field with discrete zeros and isotope the zeros to infinity.

c) Show that a time-orientable Lorentzian manifold carries an everywhere nonvanishing *timelike vector field*.

d) Show that a Lorentzian manifold admits a time-orientable double cover.

e) Show that a closed Lorentzian manifold has $\chi(\mathcal{M}) = 0$. *Hint*: How does $\chi(\mathcal{M})$ behave under finite covering maps?

$\square$

Corollary 1. *Any odd-dimensional smooth manifold admits a Lorentzian metric.*

f) (*) Prove this. *Hint:* Use Poincaré duality.

Corollary 2. *A nonempty closed Lorentzian manifold is not simply connected.*

g) (*) Prove this. *Hint*: Use Poincaré duality again and aim to show that $b_1(\mathcal{M}) \neq 0$.

Proposition 2. *Let $(\mathcal{M}, g)$ be a closed Lorentzian manifold. Then $\mathcal{M}$ contains a closed timelike curve.*

h) Prove this. *Hint*: Cover $\mathcal{M}$ by finitely many sets of the form $I^+(p_i)$, $i = 1, \dots, N$. Show that for some i, $p \in I^+(p_i)$.

5 Uniformization of Lorentzian surfaces

The goal of this exercise is to prove the following important theorem in Lorentzian geometry:

Theorem 3. *Let $(\mathcal{M}^2, g)$ be a Lorentzian surface. For any $p \in \mathcal{M}$ there exist coordinates (t, x) defined in a neighborhood $U \subset \mathcal{M}$ of p and a smooth, positive function Ω on U such that*

$$g = \Omega^2(-dt^2 + dx^2) \tag{5.1}$$

in U.

The proof of this fact for Lorentzian metrics is much simpler than for Riemannian metrics. Here is a suggested solution:

a) Show there exist two null vectors fields X and Y defined in a neighborhood U of p that are linearly independent at every point of U.

b) Show that there exist functions $\alpha, \beta \in C^\infty(U)$ such that $[X, Y] = \alpha X + \beta Y$.

c) Show that there exist nowhere vanishing functions $\theta, \zeta \in C^\infty(U')$, where U' is a possibly smaller neighborhood of U, such that $[\theta X, \zeta Y] = 0$. *Hint*: Compute out $[\theta X, \zeta Y]$ and derive a first order PDE system for θ and ζ that makes this vanish. Why does a solution exist?

d) Show that there exists a coordinate chart (u, v) defined near p such that $\theta X = \partial_u$ and $\zeta Y = \partial_v$. *Hint*: Recall (or prove that) commuting vector fields induce coordinate charts via their flows.

e) Show that $t = u + v$, $x = v - u$ has the desired properties.

6 Cauchy stability for ODEs

Cauchy stability, or continuous dependence on initial data, is a very useful tool when studying wave equations. Here we will see the simplest possible example, which will also introduce us to the notion of *bootstrap arguments*.

Theorem 4. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and suppose $\bar{y} : I \to \mathbb{R}$ is a smooth solution of the ODE*

$$\frac{d\bar{y}}{dt}(t) = F(t, y(t)) \tag{6.1}$$

with initial condition $\bar{y}(0) = \bar{y}_0$, where $I = [0, T_0]$ is a closed and bounded interval. For any $\varepsilon > 0$ there exists a $\delta > 0$ (depending on F, T_0, and $\bar{y}_0$) such that the solution $y(t)$ of the initial value problem

$$\frac{dy}{dt} = F(t, y(t)), \tag{6.2}$$

$$y(0) = y_0 \tag{6.3}$$

with $|y_0 - \bar{y}_0| \leq \delta$, exists for $t \in I$ and satisfies the estimate

$$\sup_{t \in I} |y(t) - \bar{y}(t)| \leq \varepsilon. \tag{6.4}$$

We emphasize that this theorem has two parts: the solution $y : I \to \mathbb{R}$ *exists* and moreover satisfies the estimate (6.4).

a) First, prove the following *continuation criterion.*

Proposition 3. *Let $y(t)$ solve the ODE* (6.11) *on an interval of the form $[0, T_*)$ with $T_* < \infty$. If*

$$\limsup_{t \nearrow T_*} |y(t)| < \infty, \tag{6.5}$$

then y can be uniquely smoothly extended to an interval $[0, T')$ with $T' > T_*$.

Hint: Use the ODE to prove that $y(t)$ and all of its derivatives are bounded on $[0, T_*)$.

b) For $M \geq 1$, define the set

$$\mathcal{A}_{\delta,M} \doteq \left\{ T_* \in [0, T_0] : y(t) \text{ exists on } [0, T_*] \text{ and } \sup_{t\in[0,T_*)} |y(t) - \bar{y}(t)| \leq 2\delta e^{MT_*} \right\}. \tag{6.6}$$

Show that $\mathcal{A}_{\delta,M}$ is nonempty and closed.

c) Show that $\mathcal{A}_{\delta,M}$ is open for M sufficently large and δ sufficiently small (depending on M and T_0). *Hint*: Let $\tilde{y} \doteq y - \bar{y}$ and use the calculation (mean value theorem)

$$|\tilde{y}(t) - \tilde{y}(0)| \leq \int_0^t \left| F(t', y(t')) - F(t', \bar{y}(t')) \right| dt' \tag{6.7}$$

$$\leq \left(\max_{t'\in[0,t], z\in[\bar{y}(t')-2\delta e^{Mt'}, \bar{y}(t')+2\delta e^{Mt'}]} D_z F(t', z) \right) \int_0^t \delta e^{Mt'}\, dt'. \tag{6.8}$$

Show that for M sufficiently large, δ sufficiently small, and $T_* \in \mathcal{A}$, this estimate proves that

$$\sup_{t\in[0,T_*)} |\tilde{y}(t)| \leq \delta e^{MT_*}. \tag{6.9}$$

d) Conclude Theorem 4 by performing a continuity argument and using Proposition 3.

e) Generalize Theorem 4 as follows:

Theorem 5. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and suppose $\bar{y} : I \to \mathbb{R}$ is a smooth solution of the ODE*

$$\frac{d\bar{y}}{dt} = F(t, y(t)) \tag{6.10}$$

with $\bar{y}(0) = \bar{y}_0$, where $I = (T_{-1}, T_1) \ni 0$ is the maximal domain of definition.[1] *For any $\varepsilon > 0$ and compact subinterval $K \subset I$ containing t_0, there exists a $\delta > 0$ (depending on F, K, t_0, and $\bar{y}_0$) such that the solution $y(t)$ of*

$$\frac{dy}{dt} = F(t, y(t)), \tag{6.11}$$

$$y(0) = y_0 \tag{6.12}$$

with $|y_0 - \bar{y}(t_0)| \leq \delta$, exists for $t \in K$ and satisfies the estimate

$$\sup_{t\in K} |y(t) - \bar{y}(t)| \leq \varepsilon. \tag{6.13}$$

[1] That is, $T_1 = \infty$ or $T_1 < \infty$ and $|y(t)|$ blows up as $t \nearrow T_1$, and similarly for T_{-1}.