Retiring the third law of black hole thermodynamics

Ryan Unger

Department of Mathematics, Princeton University

Edinburgh analysis seminar
February 2023

joint work with Christoph Kehle (ETH Zürich)

Classical thermodynamics

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Zeroth	T constant throughout body in thermal equilibrium

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First	$dE = TdS +$ work terms (PdV)
Second	$dS \geq 0$ in any process
Third	Impossible to achieve $T = 0$ in finite time

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
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- What is a black hole?
- What is the analogue of temperature T?
- What is the analogue of entropy S?

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \tag{EFE}$$

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Example: Minkowski space. $\mathcal{M}^{3+1} = \mathbb{R}^{3+1}_{t,x,y,z}$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

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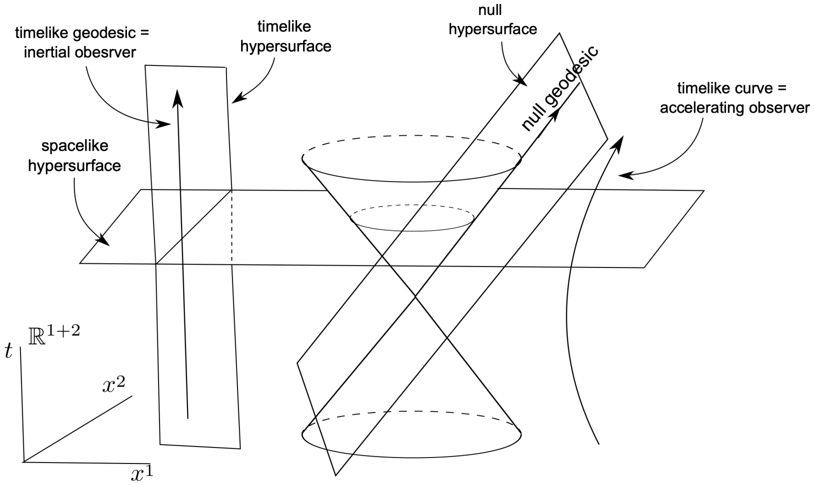
Given a vector $v \in T_p\mathcal{M}$:

- $g(v, v) < 0$, v is *timelike*
- $g(v, v) = 0$, v is *null*
- $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

Crash course on black holes: Geometry of Minkowski space



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Examples to come shortly!

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The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be known to exist, much less be located, *without knowing the entire future of the spacetime*.

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The formation, dynamics, and stability of black holes can be rigorously studied using the tools of Lorentzian geometry and hyperbolic PDE.

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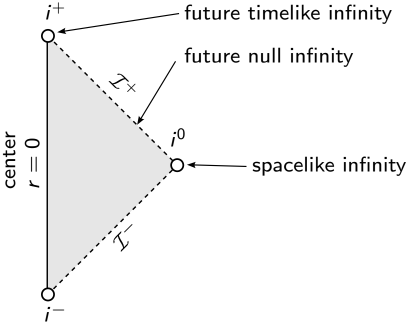
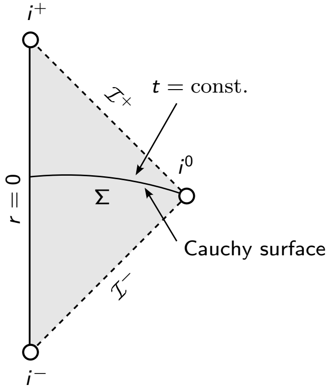
- 45 degree lines correspond to null hypersurfaces in $(3+1)$ dimensions
- steeper than 45: timelike
- shallower than 45: spacelike



Penrose diagram of Minkowski space

Crash course on black holes: Minkowski space



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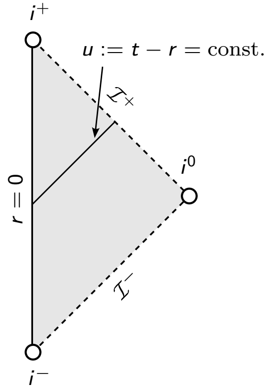

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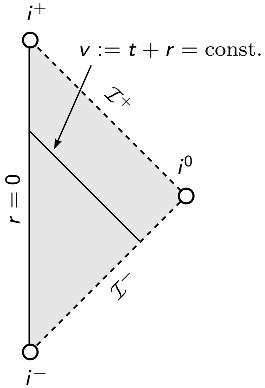

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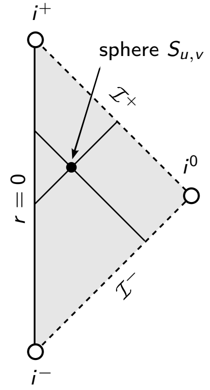

Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- Solves the **Einstein vacuum** equations
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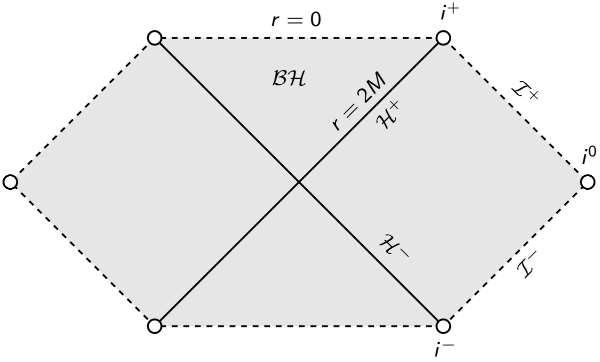

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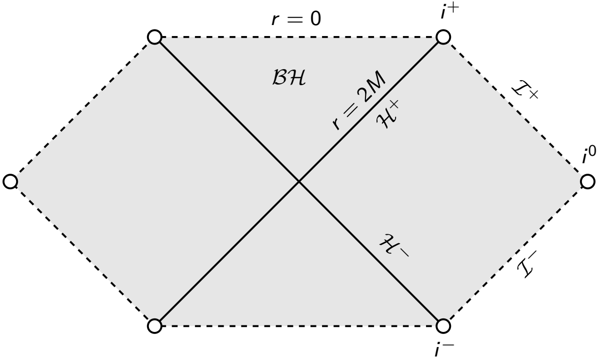

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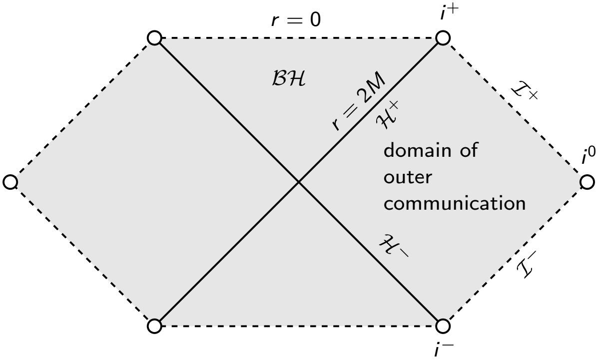

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$$g_{M,e} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

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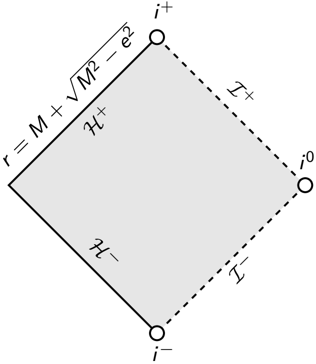

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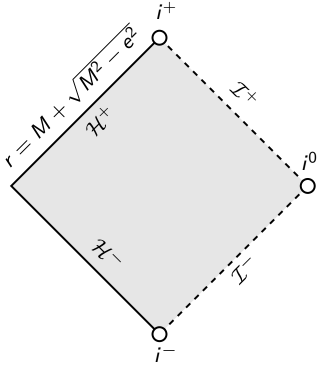

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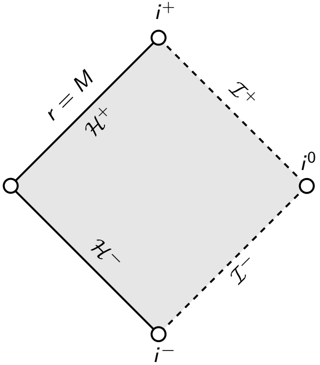

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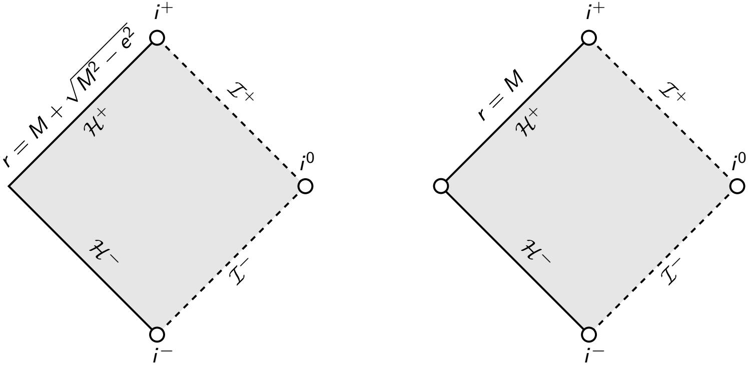

Subextremal $|e| < M$ **Extremal** $|e| = M$

Superextremal $|e| > M$ does not contain a black hole!

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- **Subextremal horizon:** $\kappa > 0$
- **Extremal horizon:** $\kappa \equiv 0$

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- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Retiring the third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the “third law of black hole thermodynamics” is false.

More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field system (which satsifies the weak energy condition) with the following behavior:

Counterexample to the third law: detailed statement of our theorem

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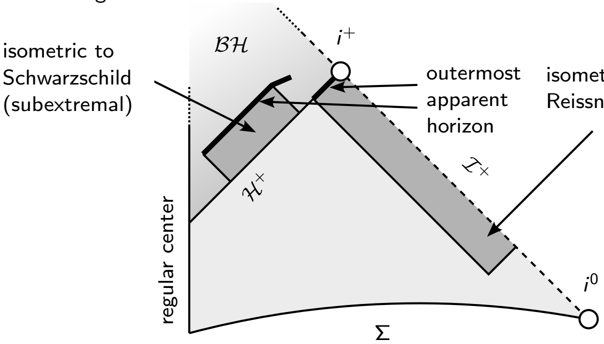

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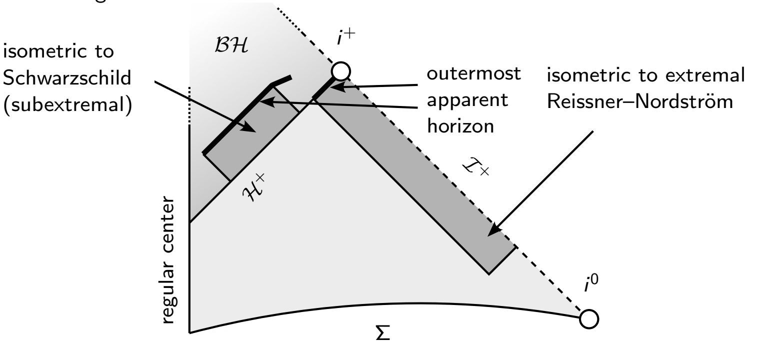

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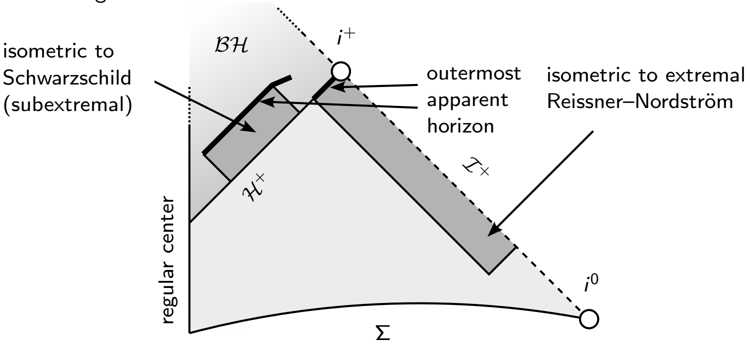

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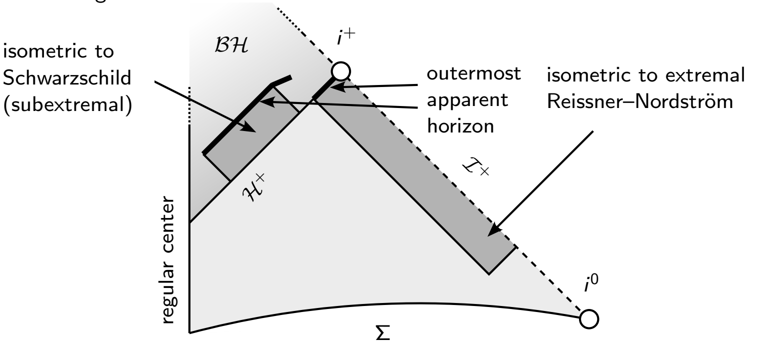

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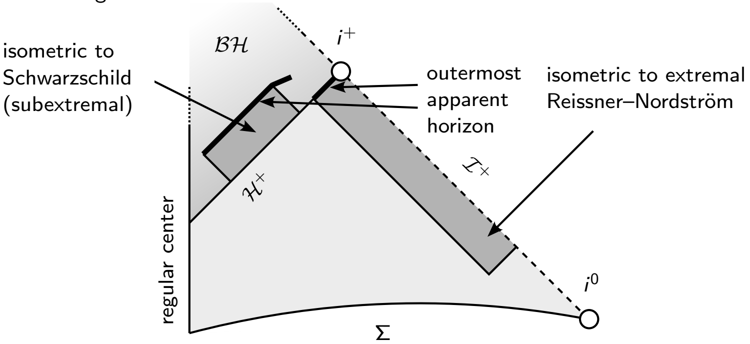

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- Forms an exactly Schwarzschild "apparent horizon."
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- These solutions are arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example.

Einstein–Maxwell-charged scalar field system

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- 2-form $F = dA$ (electromagnetism)
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- Equations:

$$\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) &(1)\\
\nabla^{\mu}F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_{\nu}\phi}) &(2)\\
g^{\mu\nu}D_{\mu}D_{\nu}\phi &= 0 &(3)\\
T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} &(4)\\
T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_{\mu}\phi\overline{D_{\nu}\phi}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_{\alpha}\phi\overline{D_{\beta}\phi} &(5)
\end{aligned}$$

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- 2-form $F = dA$ (electromagnetism)
- Charged (complex) scalar field ϕ
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\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2} R(g) g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) && (1)\\
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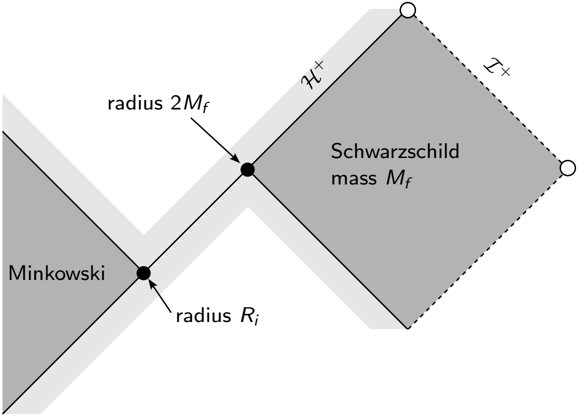
- Model satisfies the dominant energy condition ($\Longrightarrow$ weak energy condition).

Prototype: Minkowski to Schwarzschild gluing



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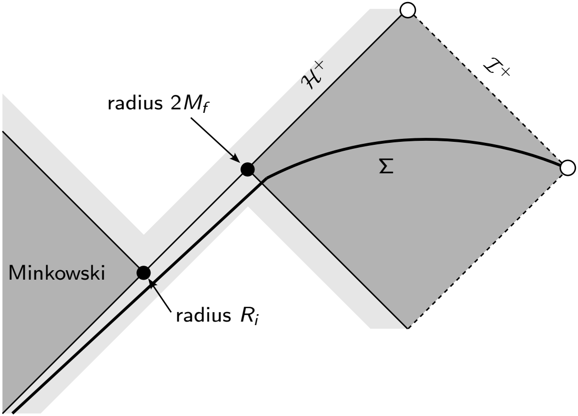

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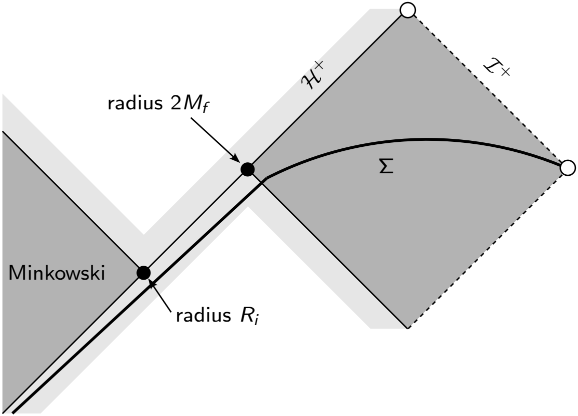

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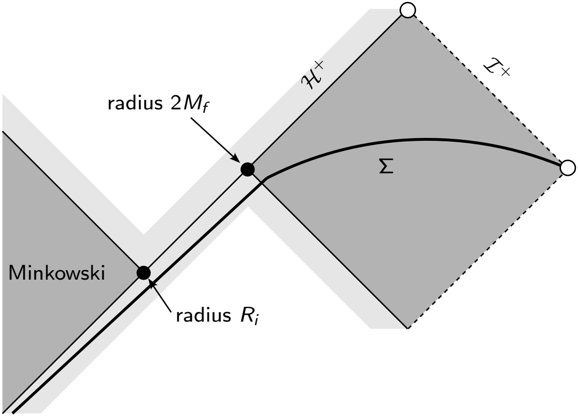

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The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

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Given two domains $\mathfrak{R}_1, \mathfrak{R}_2 \subset \mathcal{M}$ and functions

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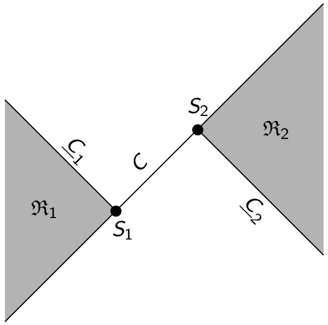

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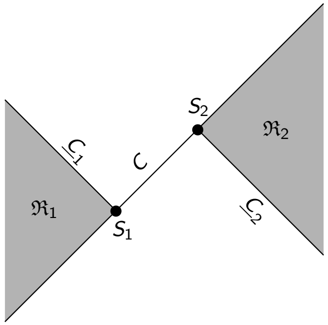

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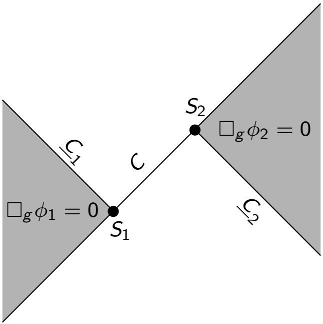

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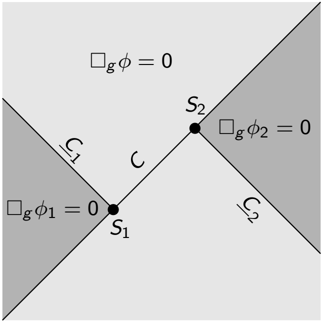

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The characteristic initial value problem

Bifurcate null hypersurface: $C \cup \underline{C}$ where C and $\underline{C}$ meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \to \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

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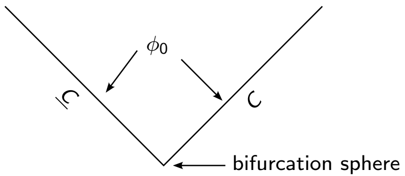

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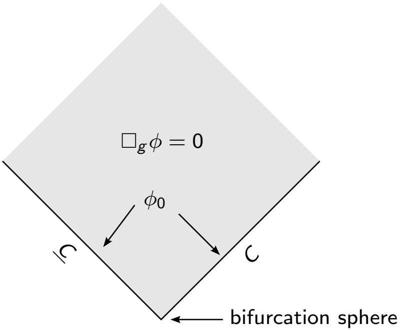

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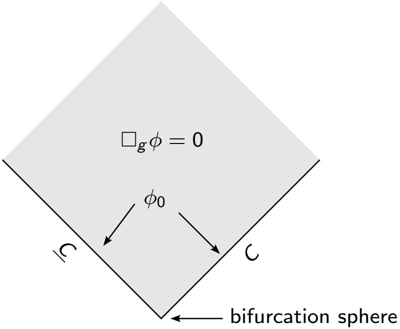

Characteristic data for the wave equation does not involve derivatives.

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- $(*)$ lets us compute $\partial_u^i\phi$ on C to all orders

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These equations have become known as the *null "constraints"*.

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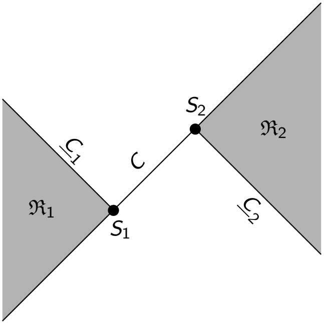

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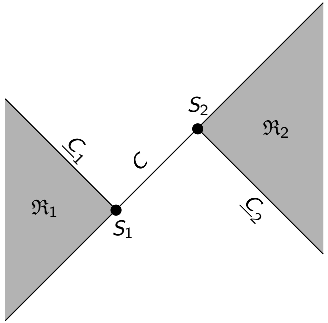

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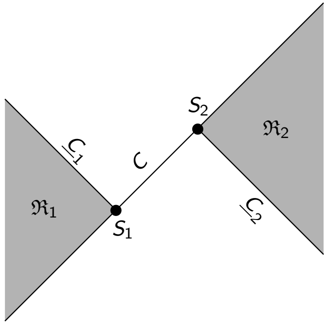

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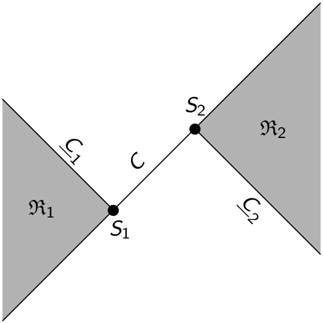

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- Initial/boundary value problem for a system of coupled ODEs where you are allowed to (almost) freely prescribe the potential

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- The linearized Einstein equations around Minkowski space and extremal Reissner–Nordström have conservation laws of the above type.

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- For the linear wave equation, not all solutions can be characteristically glued.
- Example: Minkowski space

$$\partial_v \int_{S^2} \partial_u(r\phi)\, d\vartheta = \int_{S^2} \not{\Delta}(r\phi)\, d\vartheta = 0$$

$$\int_{S^2} \partial_u(r\phi)\, d\vartheta \text{ is conserved for any solution of the wave equation}$$

- Therefore, in general S_1 and S_2 have to satisfy compatibility conditions.
- A complete geometric characterization of first order conservation laws for $\Box_g \phi = 0$ was given by Aretakis '17.
- These linear obstructions are a fundamental difficulty if one wishes to approach the nonlinear problem via linearization (as in Aretakis–Czimek–Rodnianski '21)
- The linearized Einstein equations around Minkowski space and extremal Reissner–Nordström have conservation laws of the above type.
- By exploiting "high frequency" perturbations, Czimek–Rodnianski '22 have shown how to avoid some of the conservation laws in the perturbative regime around Minkowski space.
- We completely avoid these conservation laws by having a fully nonperturbative mechanism.

Einstein-scalar field in spherical symmetry

- $\mathcal{M}^{3+1} = \mathcal{Q}^{1+1} \times S^2$
$$g = -\Omega^2 du\, dv + r^2 g_{S^2}$$
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$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r} \tag{6}$$

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- Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2 \tag{9}$$

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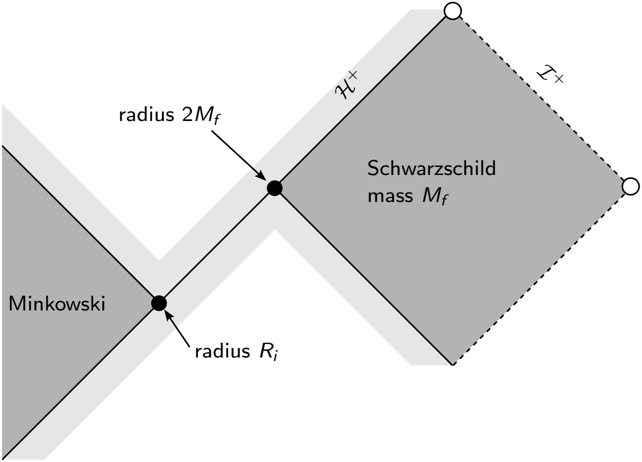
Minkowski to Schwarzschild gluing



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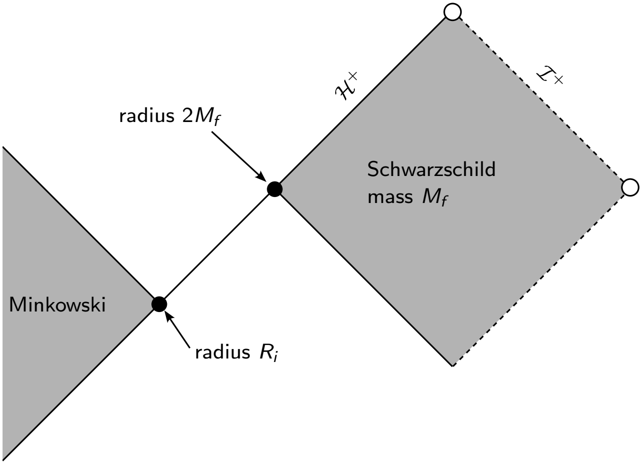

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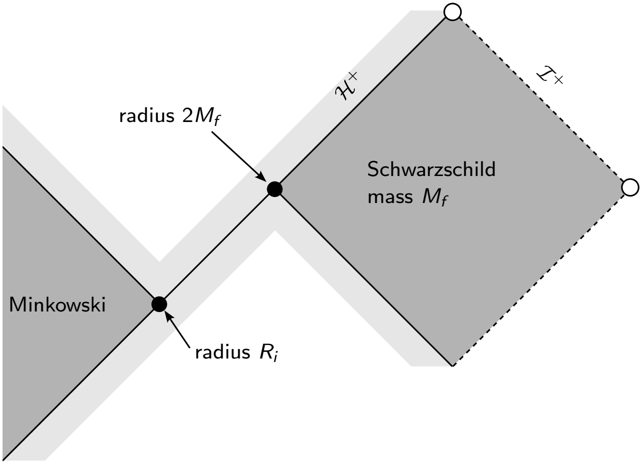
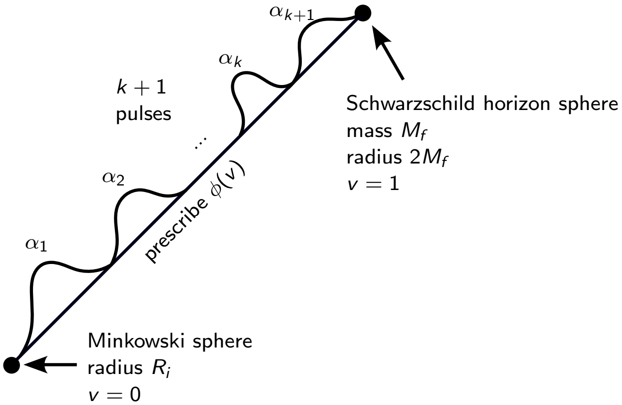


Theorem (Kehle–U. '22).

For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



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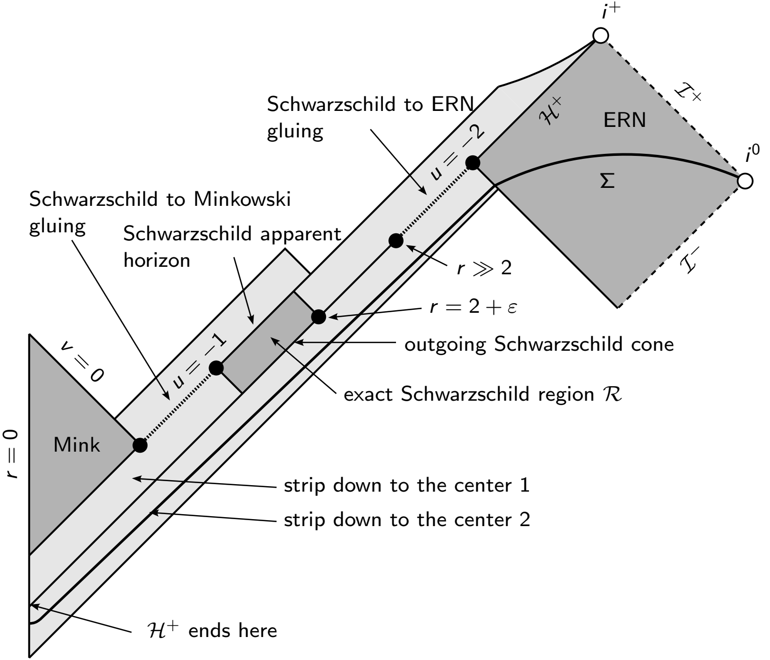
- **Borsuk–Ulam theorem:** there exists α_* such that

$$\left(\partial_u \phi_{\alpha_*}(1), \ldots, \partial_u^k \phi_{\alpha_*}(1)\right) = 0.$$

Blueprint to disproving the third law



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Theorem (Kehle–U. forthcoming).
For any $a/M \ll 1$, Minkowski space can be characteristically glued to the event horizon of a Kerr black hole with parameters M and a.

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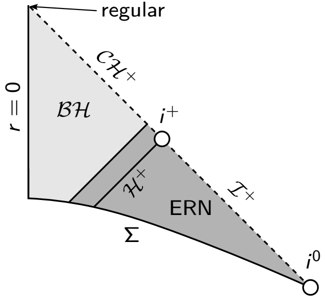

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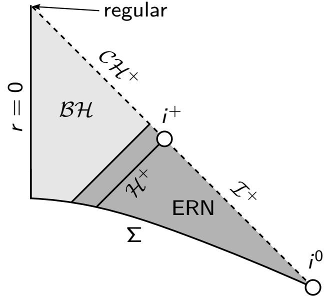

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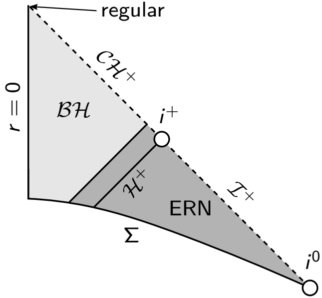

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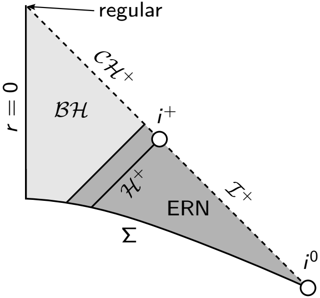

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Critical behavior: the very black-holeness of certain spacetimes is unstable.

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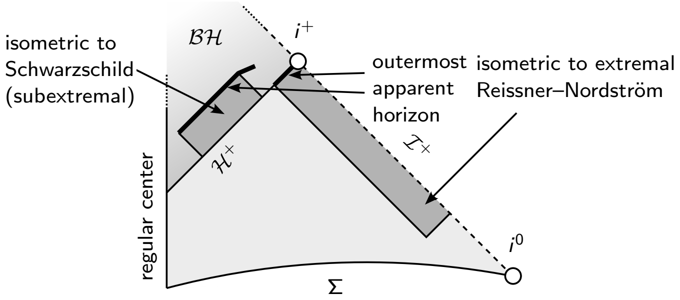

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Σ

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