Crete summer school spherical symmetry examples sheet

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Contents

1 The spherically symmetric equations

Let g be a spherically symmetric Lorentzian metric

$$g = -\Omega^2\, dudv + r^2 g_{S^2}, \tag{1.1}$$

on $\mathcal{M}^{3+1}$ where $g_{S^2} \doteq d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the round metric on the unit sphere. We use the notation $\not{g} \doteq r^2 g_{S^2}$.

a) Show that the Christoffel symbols involving null coordinates are given by

$$\Gamma^u_{uu} = \partial_u \log \Omega^2, \qquad \Gamma^v_{vv} = \partial_v \log \Omega^2, \tag{1.2}$$

$$\Gamma^u_{AB} = \frac{2\partial_v r}{\Omega^2 r}\not{g}_{AB}, \qquad \Gamma^v_{AB} = \frac{2\partial_u r}{\Omega^2 r}\not{g}_{AB}, \tag{1.3}$$

$$\Gamma^A_{Bu} = \frac{\partial_u r}{r}\delta^A_B, \qquad \Gamma^A_{Bv} = \frac{\partial_v r}{r}\delta^A_B, \tag{1.4}$$

and the totally spatial Christoffel symbols Γ^A_{BC} are the same as for g_{S^2} in the coordinates (ϑ, φ).

b) Show that the Ricci tensor is given by

$$R_{uu} = -\frac{2\Omega^2}{r}\partial_u\left(\frac{\partial_u r}{\Omega^2}\right), \qquad R_{uv} = -\partial_u\partial_v \log \Omega^2 - \frac{2}{r}\partial_u\partial_v r, \tag{1.5}$$

$$R_{vv} = -\frac{2\Omega^2}{r}\partial_v\left(\frac{\partial_v r}{\Omega^2}\right), \qquad R_{\vartheta\vartheta} = 1 + \frac{4\partial_u r\partial_v r}{\Omega^2} + \frac{4r}{\Omega^2}\partial_u\partial_v r, \tag{1.6}$$

$$R_{\varphi\varphi} = \sin^2\vartheta\, R_{\vartheta\vartheta}. \tag{1.7}$$

c) Let ϕ be a *free massless scalar field* on $(\mathcal{M}, g)$, i.e., a smooth function $\phi : \mathcal{M} \to \mathbb{R}$ which solves the *linear wave equation*

$$\Box_g \phi \doteq \nabla^\mu \nabla_\mu \phi = 0. \tag{1.8}$$

If ϕ is also spherically symmetric, i.e., $\phi = \phi(u, v)$, show that

$$\Box_g \phi = -\frac{4}{\Omega^2}\left(\frac{\partial_v r\partial_u \phi}{r} + \frac{\partial_u r\partial_v \phi}{r} + \partial_u\partial_v \phi\right). \tag{1.9}$$

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d) Define the *energy-momentum tensor* of ϕ by

$$T_{\mu\nu} \doteq \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial^\alpha\phi\partial_\alpha\phi. \tag{1.10}$$

Prove that

$$\nabla^\mu T_{\mu\nu} = \Box_g\phi\partial_\nu\phi. \tag{1.11}$$

(This part of the exercise does not rely on spherical symmetry.)

e) For a spherically symmetric scalar field, show that

$$T_{uu} = (\partial_u\phi)^2, \qquad T_{uv} = 0, \tag{1.12}$$

$$T_{vv} = (\partial_v\phi)^2, \qquad T_{AB} = \frac{2}{\Omega^2}\partial_u\phi\partial_v\phi\,\not{g}_{AB}. \tag{1.13}$$

f) The *Einstein field equations* for a *self-gravitating massless scalar field* (with zero cosmological constant) read

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu} \tag{1.14}$$

with $T_{\mu\nu}$ as in (1.10). Prove that

$$R_{\mu\nu} = 2\partial_\mu\phi\partial_\nu\phi, \tag{1.15}$$

$$\Box_g\phi = 0. \tag{1.16}$$

(This part of the exercise does not rely on spherical symmetry.)

g) For a spherically symmetric scalar field, derive the *spherically symmetric Einstein equations*

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u r\partial_v r}{r}, \tag{1.17}$$

$$\partial_u\partial_v\log\Omega^2 = \frac{\Omega^2}{2r^2} + \frac{2\partial_u r\partial_v r}{r^2} - 2\partial_u\phi\partial_v\phi, \tag{1.18}$$

$$\partial_u\partial_v\phi = -\frac{\partial_u r\partial_v\phi}{r} - \frac{\partial_v r\partial_u\phi}{r}, \tag{1.19}$$

$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_u\phi)^2, \tag{1.20}$$

$$\partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -\frac{r}{\Omega^2}(\partial_v\phi)^2. \tag{1.21}$$

2 The Hawking mass and monotonicities in spherical symmetry

In this problem we again consider a spherically symmetric self-gravitating scalar field. Recall the *Hawking mass* m, defined by

$$m = \frac{r}{2}(1 - g(\nabla r, \nabla r)) = \frac{r}{2}\left(1 + \frac{4\partial_u r\partial_v r}{\Omega^2}\right). \tag{2.1}$$

a) Derive the equations

$$\partial_u m = -\frac{2r^2\partial_v r}{\Omega^2}(\partial_u\phi)^2, \quad \partial_v m = -\frac{2r^2\partial_u r}{\Omega^2}(\partial_v\phi)^2. \tag{2.2}$$

b) Show that if $\partial_u r(u,v) \le 0$ (resp., < 0), then $\partial_u r(u',v) \le 0$ (resp., < 0) for all $u' \ge u$.

c) Show that if $\partial_v r(u,v) \le 0$ (resp., < 0), then $\partial_v r(u,v') \le 0$ (resp., < 0) for all $v' \ge v$.

d) If $\partial_u r(u,v) \le 0$, show that $\partial_v m(u,v) \ge 0$.

e) If $\partial_v r(u,v) \ge 0$, show that $\partial_u m(u,v) \le 0$.

f) If $\partial_u r < 0$ at a point (u, v), show the following equivalences at (u, v):

$$\partial_v r > 0 \iff \frac{2m}{r} < 1, \tag{2.3}$$

$$\partial_v r = 0 \iff \frac{2m}{r} = 1, \tag{2.4}$$

$$\partial_v r < 0 \iff \frac{2m}{r} > 1. \tag{2.5}$$

g) Show that if $\partial_u r < 0$, then

$$\partial_u \left(\frac{\Omega^2}{-\partial_u r} \right) \leq 0. \tag{2.6}$$

3 The characteristic initial value problem

In this problem, we set up and solve the characteristic initial value problem for the spherically symmetric Einstein-scalar field system (away from the center).

3.1 Definitions and the initial data

Given $U_0 < U_1$ and $V_0 < V_1$, set

$$\mathcal{C}(U_0, U_1, V_0, V_1) \doteq (\{U_0\} \times [V_0, V_1]) \cup ([U_0, U_1] \times \{V_0\}), \tag{3.1}$$

$$\mathcal{R}(U_0, U_1, V_0, V_1) \doteq [U_0, U_1] \times [V_0, V_1], \tag{3.2}$$

so that $\mathcal{C}$ is the past boundary of $\mathcal{R}$ when viewed as subsets of $\mathbb{R}^2_{u,v}$ equipped with the standard Minkowski metric $-dudv$ and time orientation. A C^k *characteristic data set* for the spherically symmetric Einstein-scalar field system on $\mathcal{C}$ consists of continuous functions $\mathring{r}, \mathring{\Omega}^2, \mathring{\phi} : \mathcal{C} \to \mathbb{R}$ such that $\mathring{r}$ and $\mathring{\Omega}^2$ are strictly positive, $\mathring{r}$ is C^{k+1} when restricted to the two intervals in $\mathcal{C}$, and $\mathring{\Omega}^2$ and $\mathring{\phi}$ are C^k when restricted to the two intervals. Furthermore, we require that (1.20) and (1.21) hold on $\mathcal{C}$ for $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi})$.

a) Show that $\mathring{\Omega}^2$ and $\mathring{\phi}$, together with $\mathring{r}(U_0, V_0)$, $\partial_u \mathring{r}(U_0, V_0)$, and $\partial_v \mathring{r}(U_0, V_0)$, determine a unique characteristic data set on $(\{U_0\} \times [V_0, V_1']) \cup ([U_0, U_1'] \times \{V_0\})$ if $V_1' - V_0$ and $U_1' - U_0$ are sufficiently small.

3.2 The proxy system

We will prove local well-posedness for systems of nonlinear wave equations on $\mathbb{R}^2_{u,v}$ of the form

$$\partial_u \partial_v \Psi = F(\Psi, \partial \Psi), \tag{3.3}$$

where $\Psi : \mathcal{D} \to \mathbb{R}^N$, $F : \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}^N$ is smooth, and $\mathcal{D} \subset \mathbb{R}^2_{u,v}$.

a) Show that the spherically symmetric Einstein-sclar field system can be brought into this form if $r > 0$, with the variables $\Psi_1 = \log r$, $\Psi_2 = \log \Omega^2$, and $\Psi_3 = \phi$.

b) We say that the nonlinearity F satisfies the *null condition* if there exist functions $F_0, F_{ij} : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$F(\Psi, \partial \Psi) = F_0(\Psi) + \sum_{i,j} F_{ij}(\Psi) \partial_u \Psi_i \partial_v \Psi_j. \tag{3.4}$$

Show that the spherically symmetric Einstein-scalar field model satisfies the null condition.

3.3 Uniqueness

A C^1 function $\Psi : \mathcal{D} \to \mathbb{R}^N$ is said to be a C^1 *solution* of (3.3) if for any $\mathcal{R} = \mathcal{R}(U_0, U_1, V_0, V_1) \subset \mathcal{D}$, the integrated form of (3.3) holds on $\mathcal{R}$:

$$\Psi(u,v) = \int_{U_0}^{u} \int_{V_0}^{v} F(\Psi, \partial\Psi)\, dv' du' + \Psi(u, V_0) + \Psi(U_0, v) - \Psi(U_0, V_0) \tag{3.5}$$

for every $(u, v) \in \mathcal{R}$. We wish to show:

Theorem 1. *Let Ψ_1 and Ψ_2 be two C^1 solutions of* (3.3) *on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which agree along $\mathcal{C}(U_0, U_1, V_0, V_1)$. Then $\Psi_1 = \Psi_2$ identically in $\mathcal{R}$.*

a) Show that any classical (C^2) solution of (3.3) is a C^1 solution.

b) Prove the following lemma:

   Lemma 3.1. *For any constant $C_\dagger > 0$ there exists a constant $\delta = \delta(C_\dagger) > 0$ such that if Ψ is a C^1 $\mathbb{R}^N$-valued function on $\mathcal{R}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \delta$, $0 < V_1 - V_0 < \delta$, satisfying*

   $$\Psi(u,v) = \int_{U_0}^{u} \int_{V_0}^{v} f_1 \cdot \Psi + f_2 \cdot \partial\Psi\, dv' du' \tag{3.6}$$

   for every $(u, v) \in \mathcal{R}$, where f_1 and f_2 are continuous $N \times N$-matrix valued functions satisfying

   $$\sup_{\mathcal{R}}(|f_1| + |f_2|) \le C_\dagger, \tag{3.7}$$

   then Ψ vanishes identically in $\mathcal{R}$.

   Hint: Use (3.6) to directly estimate $\|\Psi\|_{C^1(\mathcal{R})}$ in terms of itself.

c) Use Lemma 3.1 to prove Theorem 1. *Hint*: Let $\Psi \doteq \Psi_2 - \Psi_1$. Show that Ψ satisfies (3.6) for an appropriate choice of f_1 and f_2. Then cover the domain by small rectangles.

3.4 Existence in small rectangles

The goal of this section is to prove the following:

Theorem 2. *For any $C_* > 0$ there exists a constant $\varepsilon_{\mathrm{loc}} > 0$ depending on C_* and F with the following property. Let Ψ_0 be a C^1 characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_2 < \varepsilon_{\mathrm{loc}}$, $0 < V_1 - V_0 < \varepsilon_{\mathrm{loc}}$, and*

$$\|\mathring{\Psi}\|_{C^1(\mathcal{C})} \le C_*. \tag{3.8}$$

Then there exists a unique C^1 solution Ψ of (3.3) *on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\mathring{\Psi}$. Moreover, it holds that*

$$\|\Psi\|_{C^1(\mathcal{R})} \le 10C_*. \tag{3.9}$$

The theorem is proved by constructing the solution Ψ as the limit of an iteration scheme. Set $\Psi_1 = 0$ on $\mathcal{R}$ and, for $n \ge 2$, let Ψ_n solve the linear inhomogeneous wave equation

$$\partial_u \partial_v \Psi_n = F(\Psi_{n-1}, \partial\Psi_{n-1}), \tag{3.10}$$

$$\Psi_n|_{\mathcal{C}} = \mathring{\Psi}. \tag{3.11}$$

a) Find an explicit recursive formula for $\Psi_n(u, v)$ using the method of characteristics.

b) Use this formula to show that $\|\Psi_n\|_{C^1(\mathcal{R})} \le 10C_*$ if $\varepsilon_{\mathrm{loc}}$ is chosen sufficiently small.

c) Show that

   $$\|\Psi_n - \Psi_{n-1}\|_{C^1(\mathcal{R})} \le \tfrac{1}{2}\|\Psi_{n-1} - \Psi_{n-2}\|_{C^1(\mathcal{R})} \tag{3.12}$$

   for $\varepsilon_{\mathrm{loc}}$ sufficiently small.

d) Conclude that Ψ_n converges to the desired unique C^1 solution Ψ. *Hint*: Show that Ψ_n is a Cauchy sequence in C^1.

3.5 Higher regularity

In fact, Theorem 2 can be upgraded to the following:

Theorem 3. *Let $k \geq 2$. For any $C_* > 0$ there exists constants $C_1, C_2, \ldots, C_k > 0$ depending on C_* and F with the following property. Let $\varepsilon_{\text{loc}}(C_*, F) > 0$ be as in Theorem 2. Let $\mathring{\Psi}$ be a C^k characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_2 < \varepsilon_{\text{loc}}$, $0 < V_1 - V_0 < \varepsilon_{\text{loc}}$, and*

$$\|\Psi_0\|_{C^1(\mathcal{C})} \leq C_*. \tag{3.13}$$

Then there exists a unique classical C^k solution Ψ of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\mathring{\Psi}$. Moreover, it holds that

$$\|\Psi\|_{C^k(\mathcal{R})} \leq C_k \|\mathring{\Psi}\|_{C^k(\mathcal{C})} \tag{3.14}$$

for all k.

Note that the size of the region on which Ψ exists depends only on the C^1 norm of the initial data. The easiest way to prove this result is to directly argue that the iterates Ψ_n in the proof of Theorem 2 are bounded and Cauchy in C^k.

Proof of boundedness for $k = 2$. We claim that there exists constants $\hat{C}_2, \tilde{C}_2$ such that

$$|\partial_u^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 v}, \tag{3.15}$$

$$|\partial_v^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 u} \tag{3.16}$$

on $\mathcal{R}$ for every n. Indeed, differentiating (3.10) in u, we find

$$\partial_v(\partial_u^2 \Psi_n) = f_1 + f_2 \partial_u^2 \Psi_{n-1}, \tag{3.17}$$

where f_1 and f_2 are uniformly bounded functions by the C^1 estimate for Ψ_{n-1}. By integrating this and choosing $\hat{C}_2, \tilde{C}_2$ sufficiently large, (3.15) is easily established by induction. (Note that we used (3.10) again to eliminate the mixed term $\partial_u \partial_v \Psi_{n-1}$ that could have appeared.) The estimate (3.16) is obtained similarly by differentiating (3.10) in v. By commuting the equation further, one can show (3.9) for $k = 2$. $\square$

a) Generalize this argument to arbitrary k.

b) Is it true that Ψ_n is Cauchy in C^k?

3.6 Existence in thin slabs

The region of existence in Theorems 2 and 3 is a small rectangle. If the nonlinearity F satisfies the null condition (3.4), then this local existence result can be upgraded to include a full neighborhood of the bifurcate characteristic hypersurface $\mathcal{C}$.

Theorem 4. *For any $A > 0$, $L > 0$, and nonlinearity F satisfying the null condition (3.4), there exists a constant $\varepsilon_{\text{slab}} = \varepsilon_{\text{slab}}(A, L, F) > 0$ with the following property. Let $\mathring{\Psi}$ be a C^k characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \varepsilon_{\text{slab}}$, $0 < V_0 - V_1 < L$, and*

$$\|\mathring{\Psi}\|_{C^1(\mathcal{C})} \leq A. \tag{3.18}$$

Then there exists a unique smooth solution of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\mathring{\Psi}$.

The same statement holds for data on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < L$ and $0 < V_0 - V_0 < \varepsilon_{\text{slab}}$.

a) Prove the following "matrix Grönwall" lemma:

Lemma 3.2. *Let $X, Y : [0, T] \to \mathbb{R}^N$ be C^1 and satisfy $X' = Y + MX$, where $M : [0, T] \to \mathbb{R}^{N \times N}$. Then*

$$|X(T)| \leq \left(|X(0)| + \int_0^T |Y(t)|\, dt \right) \exp\left(\int_0^T |M(t)|\, dt \right). \tag{3.19}$$

Hint: Consider the equation satisfied by $x(t) = \sqrt{|X(t)|^2 + \varepsilon^2}$.

b) I will outline a proof utilizing a bootstrap argument based on the pointwise bounds

$$|\Psi| \le 10A, \tag{3.20}$$
$$|\partial_u \Psi| \le 10B, \tag{3.21}$$
$$|\partial_v \Psi| \le 10A \tag{3.22}$$

and the local existence statement Theorem 2. (Here B is a large constant to be determined in the course of the proof.) Define the bootstrap set $\mathcal{A}_{A,B}$ to be the component of

$$\{\tilde{V} \in [V_0, V_1] : \Psi \text{ exists, is } C^\infty\text{, and satisfies (3.20)–(3.22) on } \mathcal{R}(U_0, U_1, V_0, \tilde{V})\}. \tag{3.23}$$

containing V_0. The goal is to show that $\mathcal{A}_{A,B}$ is nonempty, open, and closed for B sufficiently large and $\varepsilon_{\text{slab}}$ sufficiently small.

Using Theorem 2, show that if $B \ge A$ and $\varepsilon_{\text{slab}}$ is sufficiently small depending on A, then $\mathcal{A}_{A,B} \neq \emptyset$.

c) Show that $\mathcal{A}_{A,B}$ is closed.

d) We separate the proof that $\mathcal{A}_{A,B}$ is open into two parts. Let $\tilde{V} \in \mathcal{A}_{A,B}$. First, show that if $\varepsilon_{\text{slab}}$ is chosen to be sufficiently small and B sufficiently large, then the bounds (3.20)–(3.22) hold on $\mathcal{R}(U_0, U_1, V_0, \tilde{V})$ with "better constants," i.e.,

$$|\Psi| \le 2A, \tag{3.24}$$
$$|\partial_u \Psi| \le 2B, \tag{3.25}$$
$$|\partial_v \Psi| \le 2A. \tag{3.26}$$

Hint: To estimate Ψ and $\partial_v \Psi$, use thinness of the slab in the u-direction. To estimate $\partial_u \Psi$, use the fact that the null condition implies that $\partial_u \Psi$ satisfies a *linear ODE* in v. Use Lemma 3.2 to estimate $|\partial_u \Psi|$.

e) Using these "improved" estimates, carry out a continuity argument to show that $\mathcal{A}_{A,B}$ is open.

3.7 Propagation of constraints

We now return to the spherically symmetric Einstein-scalar field system. Using Theorems 2 and 3, we can solve the characteristic initial value problem for the wave equations (1.17)–(1.19). But how do we obtain Raychauduri's equations (1.20) and (1.21)?

a) Using only (1.17), (1.18), and (1.19), prove the pair of identities

$$\partial_u \left(r\Omega^2 \partial_v \left(\frac{\partial_v r}{\Omega^2} \right) + r^2 (\partial_v \phi)^2 \right) = 0, \quad \partial_v \left(r\Omega^2 \partial_u \left(\frac{\partial_u r}{\Omega^2} \right) + r^2 (\partial_u \phi)^2 \right) = 0. \tag{3.27}$$

b) Conclude that (1.20) and (1.21) hold on $\mathcal{R}$ if they do on $\mathcal{C}$.

4 The extension principle away from the center*

The goal of this exercise is to prove the following:

Theorem 5. *Let $(\mathcal{Q}, r, \Omega^2, \phi)$ be a solution of the spherically symmetric-Einstein scalar field system, where $\mathcal{Q} \subset \mathbb{R}^2_{u,v}$ is an open set. Suppose that the following hold:*

i) $\mathcal{R}' \subset \mathcal{Q}$, where $\mathcal{R}' \doteq ([0, U] \times [0, V]) \setminus \{(U, V)\}$ and U, V are finite,

ii) $\partial_u r < 0$ on $\mathcal{R}'$, and *iii)* $\partial_v r \geq 0$ *on* $\mathcal{R}'$.

Then the solution extends smoothly to a neighborhood of $(U, V) \in \overline{\mathcal{Q}}$.

This theorem says that a "first singularity" in the spherically symmetric Einstein-scalar field model either occurs along the axis Γ (so that no such $\mathcal{R}'$ exists) or where $\partial_v r < 0$. We now sketch the proof as a series of exercises:

a) Argue using the well-posedness statement from Problem 3 that it suffices to show that (r, Ω^2, ϕ) are bounded in C^1 on $\mathcal{R}'$ and (r, Ω^2) are bounded below away from zero.

b) Show that $r \sim 1$ on $\mathcal{R}'$.

c) Show that $0 \leq -\Omega^2/\partial_u r \lesssim 1$ on $\mathcal{R}'$. *Hint*: Use Raychaudhuri's equation.

d) Show that $|m| \lesssim 1$ on $\mathcal{R}'$ and hence that

$$\sup_{u\in[0,U]} \int_{\{u\}\times[0,V]} \frac{-\partial_u r}{\Omega^2} r^2 (\partial_v \phi)^2 \, dv \lesssim 1, \tag{4.1}$$

$$\sup_{v\in[0,V]} \int_{[0,U]\times\{v\}} \frac{\partial_v r}{\Omega^2} r^2 (\partial_u \phi)^2 \, du \lesssim 1. \tag{4.2}$$

These are fundamental *energy estimates* for the spherically symmetric Einstein-scalar field system.

e) Show that $|\phi| \lesssim 1$ on $\mathcal{R}'$. *Hint*: Use the fundamental theorem in calculus in v and parts c) and d).

f) Show that the r wave equation can be written as

$$\partial_v \partial_u r = -\frac{\Omega^2}{2r^2} m. \tag{4.3}$$

g) Multiply and divide (4.3) by $\partial_u r$ and use the method of integrating factors to estimate $\partial_u r \sim -1$ on $\mathcal{R}'$.

h) Show that $\partial_v r \lesssim 1$ on $\mathcal{R}'$.

i) Show that $\Omega^2 \lesssim 1$ on $\mathcal{R}'$.

j) Derive the equation

$$\partial_u \partial_v (r\phi) = -\frac{\Omega^2 m}{2r^2} \phi \tag{4.4}$$

and complete the argument.

5 Formation of trapped surfaces in spherical symmetry*

The goal of this exercise is to prove the following:

Theorem 6 (Christodoulou)**.** *Black holes can form dynamically in the spherically symmetric Einstein-scalar field model, starting from data at "past null infinity": There exists a solution* (r, Ω^2, ϕ) *on* $\mathcal{D} \doteq (-\infty, -1] \times [0, \delta]$ *(where* $\delta > 0$ *is a small parameter) with the following properties:*

1. *The initial ingoing cone is Minkowskian:* $\partial_v r(u, 0) = -\partial_u r(u, 0) = \frac{1}{2}$, $\Omega^2(u, 0) = 1$, *and* $\phi(u, 0) = 0$ *for* $u \in (-\infty, -1]$.

2. *The initial outgoing cone (formally "*$u = -\infty$*") is a portion of null infinity in the sense that* $r(-\infty, v) = \infty$ *and* $\partial_v r(\infty, v) > 0$ *for* $v \in [0, \delta]$. *(These are to be understood as limiting statements.)*

3. *The solution has no antitrapped surfaces:* $\partial_u r \sim -1$ *in* $\mathcal{D}$.

4. The sphere $(-1,\delta)$ is trapped: $\partial_v r(-1,\delta)<0$.

Remark 5.1. In fact, this theorem holds true for the Einstein vacuum equations, where it necessarily requires a departure from spherical symmetry. The proof strategy given below is essentially an interpretation of Christodoulou's proof for the Einstein vacuum equations for the spherically symmetric Einstein-scalar field system.

Proof. We will construct the desired solution by a limiting procedure (i.e., sending the initial outgoing cone to $u=-\infty$). Consider a double null rectangle $\mathcal{R}\doteq[u_0,-1]\times[0,\delta]\subset\mathbb{R}^2_{u,v}$, where $u_0<-1$ and $\delta>0$. We consider a characteristic data set $(\mathring{r},\mathring{\Omega}^2,\mathring{\phi})$ on $\mathcal{C}$ (the past boundary of $\mathcal{R}$) chosen as follows: Fix a function $f\in C_c^\infty(0,1)$ with $\|f'\|_{L^2}=1$ and set

$$\mathring{\phi}(u_0,v)=\frac{\delta^{1/2}}{|u_0|}f\left(\frac{v}{\delta}\right) \tag{5.1}$$

on the initial outgoing cone C_{u_0}. On the initial ingoing cone $\underline{C}_0$, set $\mathring{\phi}(u,0)=0$. On $\mathcal{C}$, set $\Omega^2=1$. At the bifurcation sphere $(u_0,0)$, set

$$\mathring{r}(u_0,0)=\tfrac{1}{2}+\tfrac{1}{2}|u_0|,\quad \partial_v\mathring{r}(u_0,0)=\tfrac{1}{2},\quad \partial_u\mathring{r}(u_0,0)=-\tfrac{1}{2}. \tag{5.2}$$

a) Show that there exists $\delta_0>0$ such that if $|u_0|$ is sufficiently large, and $0<\delta<\delta_0$, then the above seed data defines a regular characteristic data set on $\mathcal{C}$ satisfying the following estimates:

$$|r\phi|\lesssim\delta^{1/2}, \tag{5.3}$$

$$|r^2\partial_u\phi|\lesssim\delta^{1/2}, \tag{5.4}$$

$$|r\partial_v\phi|\lesssim\delta^{-1/2}, \tag{5.5}$$

$$\partial_u r\sim-1, \tag{5.6}$$

$$\tfrac{1}{4}\le\partial_v r\le\tfrac{3}{4}, \tag{5.7}$$

$$m(u_0,\delta)=1+O(\delta) \tag{5.8}$$

Here, the notation $x\lesssim y$ means that there exists a constant C, independent of δ and u_0, but depending possibly on f, such that $x\le Cy$.

Hint: This is easily proved by a bootstrap argument in v on C_{u_0} (for example one can try to improve the assumptions $\frac{1}{2}|u_0|\le r\le 2+\frac{1}{2}|u_0|$ and $0\le\partial_v r\le 1$ on C_{u_0}).

b) Let $u_*\in[u_0,-1]$. Suppose there exists a number $B>0$ such that the following bounds hold in $[u_0,u_*]\times[0,\delta]$:

$$-2B\le\partial_u r\le-\frac{1}{2B}, \tag{5.9}$$

$$|\partial_v r|\le 2B, \tag{5.10}$$

$$\frac{1}{2B}\le\Omega^2\le 2B. \tag{5.11}$$

We will refer to these estimates as the "bootstrap assumptions." Use the bootstrap assumptions to infer the following estimates in $[u_0,u_*]\times[0,\delta]$:

$$|r-\tfrac{1}{2}+\tfrac{1}{2}u|\le 2B\delta, \tag{5.12}$$

$$|r\phi|\lesssim_B\delta^{1/2}, \tag{5.13}$$

$$|r^2\partial_u\phi|\lesssim_B\delta^{1/2}, \tag{5.14}$$

$$|r\partial_v\phi|\lesssim_B\delta^{-1/2} \tag{5.15}$$

if δ is sufficiently small (independent of u_0). Here, the notation $x\lesssim_B y$ means that there exists a constant C, independent of δ and u_0, but depending possibly on f and B, such that $x\le Cy$.

Hint: Write the wave equation in the form $\partial_u\partial_v(r\phi)=\cdots$ and estimate the right-hand side using the bootstrap assumptions. Then integrate in u and v, and use the fact that the integral in v gives a good power of δ. Don't forget to include the initial data (estimated in the previous step) when integrating!

c) Use the above estimates for the scalar field to show that for B sufficiently large and δ sufficiently small (depending on B), the estimates (5.9)–(5.11) hold in $[u_0, u_*] \times [0, \delta]$ with $2B$ replaced by B.

Hint: To estimate $\partial_v r$, either use the v-Raychaudhuri equation or first bound the Hawking mass m to get an improved estimate on $|\partial_u \partial_v r|$.

d) Show that the solution exists in the full rectangle $[u_0, -1] \times [0, \delta]$ and satisfies

$$|r - \tfrac{1}{2} + \tfrac{1}{2}u| \lesssim \delta, \tag{5.16}$$
$$\partial_u r \sim -1, \tag{5.17}$$
$$|\partial_v r| \lesssim 1, \tag{5.18}$$
$$|r\phi| \lesssim \delta^{1/2}, \tag{5.19}$$
$$|r^2 \partial_u \phi| \lesssim \delta^{1/2}, \tag{5.20}$$
$$|r \partial_v \phi| \lesssim \delta^{-1/2}. \tag{5.21}$$

Hint: Use a continuity argument: Consider the set $\mathcal{A}$ consisting of $u_* \in [u_0, -1]$ such that the solution exists on the rectangle $[u_0, u_*] \times [0, \delta]$ and satisfies the bootstrap assumptions (5.9)–(5.11). Show that if B is sufficiently large and δ is sufficiently small, then $\mathcal{A}$ is nonempty, closed, and open.

e) Conclude trapped surface formation as follows: Using the above heirarchy of estimates, compute $r(-1, \delta)$ and $m(-1, \delta)$ and show that $\frac{2m}{r}(-1, \delta) > 1$ for δ sufficiently small.

Hint: Estimate $\partial_u m$.

□

Extended hints:

b) (5.12) is proved by integrating (5.10). To estimate the scalar field, we write the wave equation as

$$\partial_u \partial_v (r\phi) = \left(-\frac{\Omega^2}{4r^2} - \frac{\partial_u r \partial_v r}{r^2} \right) r\phi, \tag{5.22}$$

which using the bootstrap assumptions implies

$$|\partial_u \partial_v (r\phi)| \lesssim_B \frac{\|r\phi\|_{L^\infty}}{r^2}. \tag{5.23}$$

Integrating in u, using the estimate for $\partial_v (r\phi) = \phi \partial_v r + r \partial_v \phi$ obtained from part a) on C_{u_0}, and the fact that r^{-2} is integrable in u on $[u_0, u_1]$, we obtain

$$\|\partial_v (r\phi)\|_{L^\infty} \lesssim_B \delta^{-1/2} + \|r\phi\|_{L^\infty}. \tag{5.24}$$

Integrating in v, we find $\|r\phi\|_{L^\infty} \lesssim_B \delta^{1/2} + \delta \|r\phi\|_{L^\infty}$. The second term can be absorbed and we conclude (5.13). Inserting this into (5.24), we conclude (5.15). Integrating (5.22) in v, we obtain $|\partial_u (r\phi)| \lesssim_B r^{-2} \delta^{3/2}$. We then have

$$|r^2 \partial_u \phi| \leq |\partial_u r| |r\phi| + r |\partial_u (r\phi)| \lesssim_B \delta^{1/2}, \tag{5.25}$$

which is (5.14).

c) The bootstrap assumption on Ω^2 is immediately improved by integrating the wave equation and using smallness in the v direction. This works similarly for $\partial_u r$ by integrating the wave equation for r in v. To estimate $\partial_v r$, we integrate Raychaudhuri's equation[1] in v:

$$|\Omega^{-2} \partial_v r - \tfrac{1}{2}| \lesssim \int_0^\delta r \Omega^{-2} (\partial_v \phi)^2 \, dv' \lesssim 1. \tag{5.26}$$

[1] The use of Raychaudhuri's equation here can be avoided if one is willing to take $|u_0| \lesssim 1$. In that case, the estimate can be retrieved by using an integrating factor on the wave equation for r, but I think this generates terms diverging in $|u_0|$.

e) It follows from the first estimate in part d) that $r(u_1, \delta) = 1 + O(\delta)$ by the definition of u_1. We then estimate

$$|m(u_1, \delta) - m(u_0, \delta)| \leq \int_{u_0}^{-1} 2\Omega^{-2} r^2 |\partial_v r| (\partial_u \phi)^2 \, du' \lesssim \delta. \tag{5.27}$$

Combined with $m(u_0, \delta) = 1 + O(\delta)$, this implies $m(u_1, \delta) = 1 + O(\delta)$ and consequently,

$$\frac{2m}{r}(u_1, \delta) = 2 + O(\delta) > 1 \tag{5.28}$$

for δ sufficiently small.