

Crete summer school spherical symmetry examples sheet

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1 The spherically symmetric equations

Let g be a spherically symmetric Lorentzian metric

$$g = -\Omega^2 dudv + r^2 g_{S^2}, \quad (1.1)$$

on \mathcal{M}^{3+1} where $g_{S^2} \doteq d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the round metric on the unit sphere. We use the notation $\not{g} \doteq r^2 g_{S^2}$.

a) Show that the Christoffel symbols involving null coordinates are given by

$$\Gamma_{uu}^u = \partial_u \log \Omega^2, \quad (1.2)$$

$$\Gamma_{vv}^v = \partial_v \log \Omega^2, \quad (1.3)$$

$$\Gamma_{AB}^u = \frac{2\partial_v r}{\Omega^2 r} \not{g}_{AB}, \quad (1.4)$$

$$\Gamma_{AB}^v = \frac{2\partial_u r}{\Omega^2 r} \not{g}_{AB}, \quad (1.5)$$

$$\Gamma_{Bu}^A = \frac{\partial_u r}{r} \delta_B^A, \quad (1.6)$$

$$\Gamma_{Bv}^A = \frac{\partial_v r}{r} \delta_B^A, \quad (1.7)$$

and the totally spatial Christoffel symbols Γ_{BC}^A are the same as for g_{S^2} in the coordinates (ϑ, φ) .

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b) Show that the Ricci tensor is given by

$$R_{uu} = -\frac{2\Omega^2}{r}\partial_u\left(\frac{\partial_u r}{\Omega^2}\right), \quad (1.8)$$

$$R_{vv} = -\frac{2\Omega^2}{r}\partial_v\left(\frac{\partial_v r}{\Omega^2}\right), \quad (1.9)$$

$$R_{uv} = -\partial_u\partial_v\log\Omega^2 - \frac{2}{r}\partial_u\partial_v r, \quad (1.10)$$

$$R_{\vartheta\vartheta} = 1 + \frac{4\partial_u r\partial_v r}{\Omega^2} + \frac{4r}{\Omega^2}\partial_u\partial_v r, \quad (1.11)$$

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta}. \quad (1.12)$$

c) Let ϕ be a *free massless scalar field* on (\mathcal{M}, g) , i.e., a smooth function $\phi : \mathcal{M} \rightarrow \mathbb{R}$ which solves the *linear wave equation*

$$\square_g \phi \doteq \nabla^\mu \nabla_\mu \phi = 0. \quad (1.13)$$

If ϕ is also spherically symmetric, i.e., $\phi = \phi(u, v)$, show that

$$\square_g \phi = -\frac{4}{\Omega^2} \left(\frac{\partial_v r \partial_u \phi}{r} + \frac{\partial_u r \partial_v \phi}{r} + \partial_u \partial_v \phi \right). \quad (1.14)$$

d) Define the *energy-momentum tensor* of ϕ by

$$T_{\mu\nu} \doteq \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi. \quad (1.15)$$

Prove that

$$\nabla^\mu T_{\mu\nu} = \square_g \phi \partial_\nu \phi. \quad (1.16)$$

(This part of the exercise does not rely on spherical symmetry.)

e) For a spherically symmetric scalar field, show that

$$T_{uu} = (\partial_u \phi)^2, \quad (1.17)$$

$$T_{vv} = (\partial_v \phi)^2, \quad (1.18)$$

$$T_{uv} = 0, \quad (1.19)$$

$$T_{AB} = \frac{2}{\Omega^2} \partial_u \phi \partial_v \phi g_{AB}. \quad (1.20)$$

f) The *Einstein field equations* for a *self-gravitating massless scalar field* (with zero cosmological constant) read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2T_{\mu\nu} \quad (1.21)$$

with $T_{\mu\nu}$ as in (1.15). Prove that

$$R_{\mu\nu} = 2\partial_u \phi \partial_v \phi, \quad (1.22)$$

$$\square_g \phi = 0. \quad (1.23)$$

(This part of the exercise does not rely on spherical symmetry.)

g) For a spherically symmetric scalar field, derive the *spherically symmetric Einstein equations*

$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} - \frac{\partial_u \partial_v r}{r}, \quad (1.24)$$

$$\partial_u \partial_v \log \Omega^2 = \frac{\Omega^2}{2r^2} + \frac{2\partial_u \partial_v r}{r^2} - 2\partial_u \phi \partial_v \phi, \quad (1.25)$$

$$\partial_u \partial_v \phi = -\frac{\partial_u r \partial_v \phi}{r} - \frac{\partial_v r \partial_u \phi}{r}, \quad (1.26)$$

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2, \quad (1.27)$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_v \phi)^2. \quad (1.28)$$

2 The Hawking mass and monotonicities in spherical symmetry

In this problem we again consider a spherically symmetric self-gravitating scalar field.

a) The *Hawking mass* of a spherically symmetric spacetime is defined implicitly by the relation

$$1 - \frac{2m}{r} = g(\nabla r, \nabla r). \quad (2.1)$$

Show that in double null coordinates,

$$m = \frac{r}{2} \left(1 + \frac{4\partial_u r \partial_v r}{\Omega^2} \right). \quad (2.2)$$

b) Show that in the Schwarzschild spacetime of mass M , the Hawking mass m is globally constant and equals M . *Hint:* Use the definition (2.1).

c) Derive the equations

$$\partial_u m = -\frac{2r^2 \partial_v r}{\Omega^2} (\partial_u \phi)^2, \quad (2.3)$$

$$\partial_v m = -\frac{2r^2 \partial_u r}{\Omega^2} (\partial_v \phi)^2. \quad (2.4)$$

d) Show that if $\partial_u r(u, v) \leq 0$ (resp., < 0), then $\partial_u r(u', v) \leq 0$ (resp., < 0) for all $u' \geq u$.

e) Show that if $\partial_v r(u, v) \leq 0$ (resp., < 0), then $\partial_v r(u, v') \leq 0$ (resp., < 0) for all $v' \geq v$.

f) If $\partial_u r(u, v) \leq 0$, show that $\partial_v m(u, v) \geq 0$.

g) If $\partial_v r(u, v) \geq 0$, show that $\partial_u m(u, v) \leq 0$.

h) If $\partial_u r < 0$ at a point (u, v) , show the following equivalences at (u, v) :

$$\partial_v r > 0 \iff \frac{2m}{r} < 1, \quad (2.5)$$

$$\partial_v r = 0 \iff \frac{2m}{r} = 1, \quad (2.6)$$

$$\partial_v r < 0 \iff \frac{2m}{r} > 1. \quad (2.7)$$

i) Show that if $\partial_u r < 0$, then

$$\partial_u \left(\frac{\Omega^2}{-\partial_u r} \right) \leq 0. \quad (2.8)$$

3 The characteristic initial value problem

In this problem, we set up and solve the characteristic initial value problem for the spherically symmetric Einstein-scalar field system (away from the center).

3.1 Definitions and the initial data

Given $U_0 < U_1$ and $V_0 < V_1$, set

$$\mathcal{C}(U_0, U_1, V_0, V_1) \doteq (\{U_0\} \times [V_0, V_1]) \cup ([U_0, U_1] \times \{V_0\}), \quad (3.1)$$

$$\mathcal{R}(U_0, U_1, V_0, V_1) \doteq [U_0, U_1] \times [V_0, V_1], \quad (3.2)$$

so that \mathcal{C} is the past boundary of \mathcal{R} when viewed as subsets of $\mathbb{R}_{u,v}^2$ equipped with the standard Minkowski metric $-dudv$ and time orientation. A C^k *characteristic data set* for the spherically symmetric Einstein-scalar field system on \mathcal{C} consists of continuous functions $\mathring{r}, \mathring{\Omega}^2, \mathring{\phi} : \mathcal{C} \rightarrow \mathbb{R}$ such that \mathring{r} and $\mathring{\Omega}^2$ are strictly positive, \mathring{r} is C^{k+1} when restricted to the two intervals in \mathcal{C} , and $\mathring{\Omega}^2$ and $\mathring{\phi}$ are C^k when restricted to the two intervals. Furthermore, we require that (1.27) and (1.28) hold on \mathcal{C} for $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi})$.

- a) Show that $\mathring{\Omega}^2$ and $\mathring{\phi}$, together with $\mathring{r}(U_0, V_0)$, $\partial_u \mathring{r}(U_0, V_0)$, and $\partial_v \mathring{r}(U_0, V_0)$, determine a unique characteristic data set on $(\{U_0\} \times [V_0, V_1']) \cup ([U_0, U_1'] \times \{V_0\})$ if $V_1' - V_0$ and $U_1' - U_0$ are sufficiently small.

3.2 The proxy system

We will prove local well-posedness for systems of nonlinear wave equations on $\mathbb{R}_{u,v}^2$ of the form

$$\partial_u \partial_v \Psi = F(\Psi, \partial \Psi), \quad (3.3)$$

where $\Psi : \mathcal{D} \rightarrow \mathbb{R}^N$, $F : \mathbb{R}^N \times \mathbb{R}^{2N} \rightarrow \mathbb{R}^N$ is smooth, and $\mathcal{D} \subset \mathbb{R}_{u,v}^2$.

- a) Show that the spherically symmetric Einstein-scalar field system can be brought into this form if $r > 0$, with the variables $\Psi_1 = \log r$, $\Psi_2 = \log \Omega^2$, and $\Psi_3 = \phi$.
- b) We say that the nonlinearity F satisfies the *null condition* if there exist functions $F_0, F_{ij} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that

$$F(\Psi, \partial \Psi) = F_0(\Psi) + \sum_{i,j} F_{ij}(\Psi) \partial_u \Psi_i \partial_v \Psi_j. \quad (3.4)$$

Show that the spherically symmetric Einstein-scalar field model satisfies the null condition.

3.3 Uniqueness

A C^1 function $\Psi : \mathcal{D} \rightarrow \mathbb{R}^N$ is said to be a C^1 *solution* of (3.3) if for any $\mathcal{R} = \mathcal{R}(U_0, U_1, V_0, V_1) \subset \mathcal{D}$, the integrated form of (3.3) holds on \mathcal{R} :

$$\Psi(u, v) = \int_{U_0}^u \int_{V_0}^v F(\Psi, \partial \Psi) dv' du' + \Psi(u, V_0) + \Psi(U_0, v) - \Psi(0, 0) \quad (3.5)$$

for every $(u, v) \in \mathcal{R}$. We wish to show:

Theorem 1. *Let Ψ_1 and Ψ_2 be two C^1 solutions of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which agree along $\mathcal{C}(U_0, U_1, V_0, V_1)$. Then $\Psi_1 = \Psi_2$ identically in \mathcal{R} .*

- a) Show that any classical (C^2) solution of (3.3) is a C^1 solution.
- b) Prove the following lemma:

Lemma 3.1. *For any constant $C_{\dagger} > 0$ there exists a constant $\delta = \delta(C_{\dagger}) > 0$ such that if Ψ is a C^1 \mathbb{R}^N -valued function on $\mathcal{R}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \delta$, $0 < V_1 - V_0 < \delta$, satisfying*

$$\Psi(u, v) = \int_{U_0}^u \int_{V_0}^v f_1 \cdot \Psi + f_2 \cdot \partial \Psi \, dv' du' \quad (3.6)$$

for every $(u, v) \in \mathcal{R}$, where f_1 and f_2 are continuous $N \times N$ -matrix valued functions satisfying

$$\sup_{\mathcal{R}} (|f_1| + |f_2|) \leq C_{\dagger}, \quad (3.7)$$

then Ψ vanishes identically in \mathcal{R} .

Hint: Use (3.6) to directly estimate $\|\Psi\|_{C^1(\mathcal{R})}$ in terms of itself.

- c) Use Lemma 3.1 to prove Theorem 1. *Hint:* Let $\Psi \doteq \Psi_2 - \Psi_1$. Show that Ψ satisfies (3.6) for an appropriate choice of f_1 and f_2 . Then cover the domain by small rectangles.

3.4 Existence in small rectangles

The goal of this section is to prove the following:

Theorem 2. *For any $C_* > 0$ there exists a constant $\varepsilon_{\text{loc}} > 0$ depending on C_* and F with the following property. Let Ψ_0 be a C^1 characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \varepsilon_{\text{loc}}$, $0 < V_1 - V_0 < \varepsilon_{\text{loc}}$, and*

$$\|\mathring{\Psi}\|_{C^1(\mathcal{C})} \leq C_*. \quad (3.8)$$

Then there exists a unique C^1 solution Ψ of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\mathring{\Psi}$. Moreover, it holds that

$$\|\Psi\|_{C^1(\mathcal{R})} \leq 10C_*. \quad (3.9)$$

The theorem is proved by constructing the solution Ψ as the limit of an iteration scheme. Set $\Psi_1 = 0$ on \mathcal{R} and, for $n \geq 2$, let Ψ_n solve the linear inhomogeneous wave equation

$$\partial_u \partial_v \Psi_n = F(\Psi_{n-1}, \partial \Psi_{n-1}), \quad (3.10)$$

$$\Psi_n|_{\mathcal{C}} = \mathring{\Psi}. \quad (3.11)$$

- a) Find an explicit recursive formula for $\Psi_n(u, v)$ using the method of characteristics.
- b) Use this formula to show that $\|\Psi_n\|_{C^1(\mathcal{R})} \leq 10C_*$ if ε_{loc} is chosen sufficiently small.
- c) Show that

$$\|\Psi_n - \Psi_{n-1}\|_{C^1(\mathcal{R})} \leq \frac{1}{2} \|\Psi_{n-1} - \Psi_{n-2}\|_{C^1(\mathcal{R})} \quad (3.12)$$

for ε_{loc} sufficiently small.

- d) Conclude that Ψ_n converges to the desired unique C^1 solution Ψ . *Hint:* Show that Ψ_n is a Cauchy sequence in C^1 .

3.5 Higher regularity

In fact, Theorem 2 can be upgraded to the following:

Theorem 3. *Let $k \geq 2$. For any $C_* > 0$ there exists constants $C_1, C_2, \dots, C_k > 0$ depending on C_* and F with the following property. Let $\varepsilon_{\text{loc}}(C_*, F) > 0$ be as in Theorem 2. Let $\mathring{\Psi}$ be a C^k characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \varepsilon_{\text{loc}}$, $0 < V_1 - V_0 < \varepsilon_{\text{loc}}$, and*

$$\|\mathring{\Psi}_0\|_{C^1(\mathcal{C})} \leq C_*. \quad (3.13)$$

Then there exists a unique classical C^k solution Ψ of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\mathring{\Psi}$. Moreover, it holds that

$$\|\Psi\|_{C^k(\mathcal{R})} \leq C_k \|\mathring{\Psi}\|_{C^k(\mathcal{C})} \quad (3.14)$$

for all k .

Note that the size of the region on which Ψ exists depends only on the C^1 norm of the initial data. The easiest way to prove this result is to directly argue that the iterates Ψ_n in the proof of Theorem 2 are bounded in Cauchy in C^k .

Proof of boundedness for $k = 2$. We claim that there exists constants \hat{C}_2, \tilde{C}_2 such that

$$|\partial_u^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 v}, \quad (3.15)$$

$$|\partial_v^2 \Psi_n| \leq \tilde{C}_2 e^{\hat{C}_2 u} \quad (3.16)$$

on \mathcal{R} for every n . Indeed, differentiating (3.10) in u , we find

$$\partial_v(\partial_u^2 \Psi_n) = f_1 + f_2 \partial_u^2 \Psi_{n-1}, \quad (3.17)$$

where f_1 and f_2 are uniformly bounded functions by the C^1 estimate for Ψ_{n-1} . By integrating this and choosing \hat{C}_2, \tilde{C}_2 sufficiently large, (3.15) is easily established by induction. (Note that we used (3.10) again to eliminate the mixed term $\partial_u \partial_v \Psi_{n-1}$ that could have appeared.) The estimate (3.16) is obtained similarly by differentiating (3.10) in v . By commuting the equation further, one can show (3.9) for $k = 2$. \square

a) Generalize this argument to arbitrary k .

b) Is it true that Ψ_n is Cauchy in C^k ?

3.6 Existence in thin slabs

The region of existence in Theorems 2 and 3 is a small rectangle. If the nonlinearity F satisfies the null condition (3.4), then this local existence result can be upgraded to include a full neighborhood of the bifurcate characteristic hypersurface \mathcal{C} .

Theorem 4. *For any $A > 0$, $L > 0$, and nonlinearity F satisfying the null condition (3.4), there exists a constant $\varepsilon_{\text{slab}} = \varepsilon_{\text{slab}}(A, L, F) > 0$ with the following property. Let $\dot{\Psi}$ be a C^k characteristic data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < \varepsilon_{\text{slab}}$, $0 < V_0 - V_1 < L$, and*

$$\|\dot{\Psi}\|_{C^1(\mathcal{C})} \leq A. \quad (3.18)$$

Then there exists a unique smooth solution of (3.3) on $\mathcal{R}(U_0, U_1, V_0, V_1)$ which extends the initial data $\dot{\Psi}$.

The same statement holds for data on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $0 < U_1 - U_0 < L$ and $0 < V_0 - V_1 < \varepsilon_{\text{slab}}$.

a) Prove the following “matrix Gronwall” lemma:

Lemma 3.2. *Let $X, Y : [0, T] \rightarrow \mathbb{R}^N$ be C^1 and satisfy $X' = Y + MX$, where $M : [0, T] \rightarrow \mathbb{R}^{N \times N}$. Then*

$$|X(T)| \leq \left(|X(0)| + \int_0^T |Y(t)| dt \right) \exp \left(\int_0^T |M(t)| dt \right). \quad (3.19)$$

Hint: Consider the equation satisfied by $x(t) = \sqrt{|X(t)|^2 + \varepsilon^2}$.

b) I will outline a proof utilizing a bootstrap argument based on the pointwise bounds

$$|\Psi| \leq 10A, \quad (3.20)$$

$$|\partial_u \Psi| \leq 10B, \quad (3.21)$$

$$|\partial_v \Psi| \leq 10A \quad (3.22)$$

and the local existence statement Theorem 2. (Here B is a large constant to be determined in the course of the proof.) Define the bootstrap set $\mathcal{A}_{A,B}$ to be the component of

$$\{\tilde{V} \in [V_0, V_1] : \Psi \text{ exists, is } C^\infty, \text{ and satisfies (3.20)–(3.22) on } \mathcal{R}(U_0, U_1, V_0, \tilde{V})\}. \quad (3.23)$$

containing V_0 . The goal is to show that $\mathcal{A}_{A,B}$ is nonempty, open, and closed for B sufficiently large and $\varepsilon_{\text{slab}}$ sufficiently small.

Using Theorem 2, show that if $B \geq A$ and $\varepsilon_{\text{slab}}$ is sufficiently small depending on A , then $\mathcal{A}_{A,B} \neq \emptyset$.

- c) Show that $\mathcal{A}_{A,B}$ is closed.
- d) We separate the proof that $\mathcal{A}_{A,B}$ is open into two parts. Let $\tilde{V} \in \mathcal{A}_{A,B}$. First, show that if $\varepsilon_{\text{slab}}$ is chosen to be sufficiently small and B sufficiently large, then the bounds (3.20)–(3.22) hold on $\mathcal{R}(U_0, U_1, V_0, \tilde{V})$ with “better constants,” i.e.,

$$|\Psi| \leq 2A, \quad (3.24)$$

$$|\partial_u \Psi| \leq 2B, \quad (3.25)$$

$$|\partial_v \Psi| \leq 2A. \quad (3.26)$$

Hint: To estimate Ψ and $\partial_v \Psi$, use thinness of the slab in the u -direction. To estimate $\partial_u \Psi$, use the fact that the null condition implies that $\partial_u \Psi$ satisfies a *linear ODE* in v . Use Lemma 3.2 to estimate $|\partial_u \Psi|$.

- e) Using these “improved” estimates, carry out a continuity argument to show that $\mathcal{A}_{A,B}$ is open.

3.7 Propagation of constraints

We now return to the spherically symmetric Einstein-scalar field system. Using Theorems 2 and 3, we can solve the characteristic initial value problem for the wave equations (1.24)–(1.26). But how do we obtain Raychaudhuri’s equations (1.27) and (1.28)?

- a) Using only (1.24), (1.25), and (1.26), prove the pair of identities

$$\partial_u \left(r \Omega^2 \partial_v \left(\frac{\partial_v r}{\Omega^2} \right) + r^2 (\partial_v \phi)^2 \right) = 0, \quad (3.27)$$

$$\partial_v \left(r \Omega^2 \partial_u \left(\frac{\partial_u r}{\Omega^2} \right) + r^2 (\partial_u \phi)^2 \right) = 0. \quad (3.28)$$

- b) Conclude that (1.27) and (1.28) hold on \mathcal{R} if they do on \mathcal{C} .

4 The extension principle away from the center*

The goal of this exercise is to prove the following:

Theorem 5. *Let $(\mathcal{Q}, r, \Omega^2, \phi)$ be a solution of the spherically symmetric-Einstein scalar field system, where $\mathcal{Q} \subset \mathbb{R}_{u,v}^2$ is an open set. Suppose that the following hold:*

- i) $\mathcal{R}' \subset \mathcal{Q}$, where $\mathcal{R}' \doteq ([0, U] \times [0, V]) \setminus \{(U, V)\}$ and U, V are finite,
- ii) $\partial_u r < 0$ on \mathcal{R}' , and
- iii) $\partial_v r \geq 0$ on \mathcal{R}' .

Then the solution extends smoothly to a neighborhood of $(U, V) \in \overline{\mathcal{Q}}$.

This theorem says that a “first singularity” in the spherically symmetric Einstein-scalar field model either occurs along the axis Γ (so that no such \mathcal{R}' exists) or where $\partial_v r < 0$. We now sketch the proof as a series of exercises:

- a) Argue using the well-posedness statement from Problem 3 that it suffices to show that (r, Ω^2, ϕ) are bounded in C^1 on \mathcal{R}' and (r, Ω^2) are bounded below away from zero.
- b) Show that $r \sim 1$ on \mathcal{R}' .
- c) Show that $0 \leq -\Omega^2 / \partial_u r \lesssim 1$ on \mathcal{R}' . *Hint:* Use Raychaudhuri’s equation.

d) Show that $|m| \lesssim 1$ on \mathcal{R}' and hence that

$$\int_{[0,U] \times \{v\}} \frac{-\partial_u r}{\Omega^2} r^2 (\partial_v \phi)^2 dv \lesssim 1 \quad (4.1)$$

for any $v \in [0, V]$ and

$$\int_{\{u\} \times [0,V]} \frac{\partial_v r}{\Omega^2} r^2 (\partial_u \phi)^2 du \lesssim 1 \quad (4.2)$$

for any $u \in [0, U]$. These are fundamental *energy estimates* for the spherically symmetric Einstein-scalar field system.

e) Show that $|\phi| \lesssim 1$ on \mathcal{R}' . *Hint*: Use the fundamental theorem in calculus in v and parts c) and d).

f) Show that the r wave equation can be written as

$$\partial_v \partial_u r = -\frac{\Omega^2}{2r^2} m. \quad (4.3)$$

g) Multiply and divide (4.3) by $\partial_u r$ and use the method of integrating factors to estimate $\partial_u r \sim -1$ on \mathcal{R}' .

h) Show that $\partial_v r \lesssim 1$ on \mathcal{R}' .

i) Show that $\Omega^2 \lesssim 1$ on \mathcal{R}' . *Hint*: Multiply c) and g).

j) Derive the equation

$$\partial_u \partial_v (r\phi) = -\frac{\Omega^2 m}{2r^2} \phi \quad (4.4)$$

and complete the argument.

5 Formation of trapped surfaces in spherical symmetry*

The goal of this exercise is to prove the following:

Theorem 6 (Christodoulou). *Black holes can form dynamically in the spherically symmetric Einstein-scalar field model, starting from data at “past null infinity”: There exists a solution (r, Ω^2, ϕ) on $\mathcal{D} \doteq (-\infty, -1] \times [0, \delta]$ (where $\delta > 0$ is a small parameter) with the following properties:*

1. *The initial ingoing cone is Minkowskian: $\partial_v r(u, 0) = -\partial_u r(u, 0) = \frac{1}{2}$, $\Omega^2(u, 0) = 1$, and $\phi(u, 0) = 0$ for $u \in (-\infty, -1]$.*
2. *The initial outgoing cone (formally “ $u = -\infty$ ”) is a portion of null infinity in the sense that $r(-\infty, v) = \infty$ and $\partial_v r(\infty, v) > 0$ for $v \in [0, \delta]$. (These are to be understood as limiting statements.)*
3. *The solution has no antitrapped surfaces: $\partial_u r \sim -1$ in \mathcal{D} .*
4. *The sphere $(-1, \delta)$ is trapped: $\partial_v r(-1, \delta) < 0$.*

Remark 5.1. In fact, this theorem holds true for the Einstein vacuum equations, where it necessarily requires a departure from spherical symmetry. The proof strategy given below is essentially an interpretation of Christodoulou’s proof for the Einstein vacuum equations for the spherically symmetric Einstein-scalar field system.

Proof. We will construct the desired solution by a limiting procedure (i.e., sending the initial outgoing cone to $u = -\infty$). Consider a double null rectangle $\mathcal{R} \doteq [u_0, -1] \times [0, \delta] \subset \mathbb{R}_{u,v}^2$, where $u_0 < -1$ and $\delta > 0$. We consider a characteristic data set $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi})$ on \mathcal{C} (the past boundary of \mathcal{R}) chosen as follows: Fix a function $f \in C_c^\infty(0, 1)$ with $\|f'\|_{L^2} = 1$ and set

$$\mathring{\phi}(u_0, v) = \frac{\delta^{1/2}}{|u_0|} f\left(\frac{v}{\delta}\right) \quad (5.1)$$

on the initial outgoing cone C_{u_0} . On the initial ingoing cone \underline{C}_0 , set $\mathring{\phi}(u, 0) = 0$. On \mathcal{C} , set $\Omega^2 = 1$. At the bifurcation sphere $(u_0, 0)$, set

$$\mathring{r}(u_0, 0) = 1 + \frac{1}{2}|u_0|, \quad \partial_v \mathring{r}(u_0, 0) = \frac{1}{2}, \quad \partial_u \mathring{r}(u_0, 0) = -\frac{1}{2}. \quad (5.2)$$

- a) Show that there exists $\delta_0 > 0$ such that if $|u_0|$ is sufficiently large, and $0 < \delta < \delta_0$, then the above seed data defines a regular characteristic data set on \mathcal{C} satisfying the following estimates:

$$|r\phi| \lesssim \delta^{1/2}, \quad (5.3)$$

$$|r\partial_u \phi| \lesssim \delta^{1/2}, \quad (5.4)$$

$$|r\partial_v \phi| \lesssim \delta^{-1/2}, \quad (5.5)$$

$$\partial_u r \sim -1, \quad (5.6)$$

$$\frac{1}{4} \leq \partial_v r \leq \frac{3}{4}, \quad (5.7)$$

$$m(u_0, \delta) = 1 + O(\delta) \quad (5.8)$$

Here, the notation $x \lesssim y$ means that there exists a constant C , independent of δ and u_0 , but depending possibly on f , such that $x \leq Cy$.

Hint: This is easily proved by a bootstrap argument in v on C_{u_0} (for example one can try to improve the assumptions $\frac{1}{2}|u_0| \leq r \leq 2 + \frac{1}{2}|u_0|$ and $0 \leq \partial_v r \leq 1$ on C_{u_0}).

- b) Let $u_* \in [u_0, -1]$. Suppose there exists a number $B > 0$ such that the following bounds hold in $[u_0, u_*] \times [0, \delta]$:

$$-2B \leq \partial_u r \leq -\frac{1}{2B}, \quad (5.9)$$

$$|\partial_v r| \leq 2B, \quad (5.10)$$

$$\frac{1}{2B} \leq \Omega^2 \leq 2B. \quad (5.11)$$

We will refer to these estimates as the “bootstrap assumptions.” Use the bootstrap assumptions to infer the following estimates in $[u_0, u_*] \times [0, \delta]$:

$$|r - 1 + \frac{1}{2}u| \leq 2B\delta, \quad (5.12)$$

$$|r\phi| \lesssim_B \delta^{1/2}, \quad (5.13)$$

$$|r^2 \partial_u \phi| \lesssim_B \delta^{1/2}, \quad (5.14)$$

$$|r\partial_v \phi| \lesssim_B \delta^{-1/2} \quad (5.15)$$

if δ is sufficiently small (independent of u_0). Here, the notation $x \lesssim_B y$ means that there exists a constant C , independent of δ and u_0 , but depending possibly on f and B , such that $x \leq Cy$.

Hint: Write the wave equation in the form $\partial_u \partial_v (r\phi) = \dots$ and estimate the right-hand side using the bootstrap assumptions. Then integrate in u and v , and use the fact that the integral in v gives a good power of δ . Don’t forget to include the initial data (estimated in the previous step) when integrating!

- c) Use the above estimates for the scalar field to show that for B sufficiently large and δ sufficiently small (depending on B), the estimates (5.9)–(5.11) hold in $[u_0, u_*] \times [0, \delta]$ with $2B$ replaced by B .

Hint: To estimate $\partial_v r$, either use the v -Raychaudhuri equation or first bound the Hawking mass m to get an improved estimate on $|\partial_u \partial_v r|$.

d) Show that the solution exists in the full rectangle $[u_0, -1] \times [0, \delta]$ and satisfies

$$|r - 1 + \frac{1}{2}u| \lesssim \delta, \quad (5.16)$$

$$\partial_u r \sim -1, \quad (5.17)$$

$$|\partial_v r| \lesssim 1, \quad (5.18)$$

$$|r\phi| \lesssim \delta^{1/2}, \quad (5.19)$$

$$|r^2 \partial_u \phi| \lesssim \delta^{1/2}, \quad (5.20)$$

$$|r \partial_v \phi| \lesssim \delta^{-1/2}. \quad (5.21)$$

Hint: Use a continuity argument: Consider the set \mathcal{A} consisting of $u_* \in [u_0, -1]$ such that the solution exists on the rectangle $[u_0, u_*] \times [0, \delta]$ and satisfies the bootstrap assumptions (5.9)–(5.11). Show that if B is sufficiently large and δ is sufficiently small, then \mathcal{A} is nonempty, closed, and open.

e) Conclude trapped surface formation as follows: Using the above heirarchy of estimates, compute $r(-1, \delta)$ and $m(-1, \delta)$ and show that $\frac{2m}{r}(-1, \delta) > 1$ for δ sufficiently small.

Hint: Estimate $\partial_u m$.

□

Extended hints:

b) (5.12) is proved by integrating (5.10). To estimate the scalar field, we write the wave equation as

$$\partial_u \partial_v (r\phi) = \left(-\frac{\Omega^2}{4r^2} - \frac{\partial_u r \partial_v r}{r^2} \right) r\phi, \quad (5.22)$$

which using the bootstrap assumptions implies

$$|\partial_u \partial_v (r\phi)| \lesssim_B \frac{\|r\phi\|_{L^\infty}}{r^2}. \quad (5.23)$$

Integrating in u , using the estimate for $\partial_v(r\phi) = \phi \partial_v r + r \partial_v \phi$ obtained from part a) on C_{u_0} , and the fact that r^{-2} is integrable in u on $[u_0, u_1]$, we obtain

$$\|\partial_v(r\phi)\|_{L^\infty} \lesssim_B \delta^{-1/2} + \|r\phi\|_{L^\infty}. \quad (5.24)$$

Integrating in v , we find $\|r\phi\|_{L^\infty} \lesssim_B \delta^{1/2} + \delta \|r\phi\|_{L^\infty}$. The second term can be absorbed and we conclude (5.13). Inserting this into (5.24), we conclude (5.15). Integrating (5.22) in v , we obtain $|\partial_u(r\phi)| \lesssim_B r^{-2} \delta^{3/2}$. We then have

$$|r^2 \partial_u \phi| \leq |\partial_u r| |r\phi| + r |\partial_u(r\phi)| \lesssim_B \delta^{1/2}, \quad (5.25)$$

which is (5.14).

c) The bootstrap assumption on Ω^2 is immediately improved by integrating the wave equation and using smallness in the v direction. This works similarly for $\partial_u r$ by integrating the wave equation for r in v . To estimate $\partial_v r$, we integrate Raychaudhuri's equation¹ in v :

$$|\Omega^{-2} \partial_v r - \frac{1}{2}| \lesssim \int_0^\delta r \Omega^{-2} (\partial_v \phi)^2 dv' \lesssim 1. \quad (5.26)$$

¹The use of Raychaudhuri's equation here can be avoided if one is willing to take $|u_0| \lesssim 1$. In that case, the estimate can be retrieved by using an integrating factor on the wave equation for r , but I think this generates terms diverging in $|u_0|$.

e) It follows from the first estimate in part d) that $r(u_1, \delta) = 1 + O(\delta)$ by the definition of u_1 . We then estimate

$$|m(u_1, \delta) - m(u_0, \delta)| \leq \int_{u_0}^{-1} 2\Omega^{-2} r^2 |\partial_v r| (\partial_u \phi)^2 du' \lesssim \delta. \quad (5.27)$$

Combined with $m(u_0, \delta) = 1 + O(\delta)$, this implies $m(u_1, \delta) = 1 + O(\delta)$ and consequently,

$$\frac{2m}{r}(u_1, \delta) = 2 + O(\delta) > 1 \quad (5.28)$$

for δ sufficiently small.