Some Problems in Scalar Curvature Geometry

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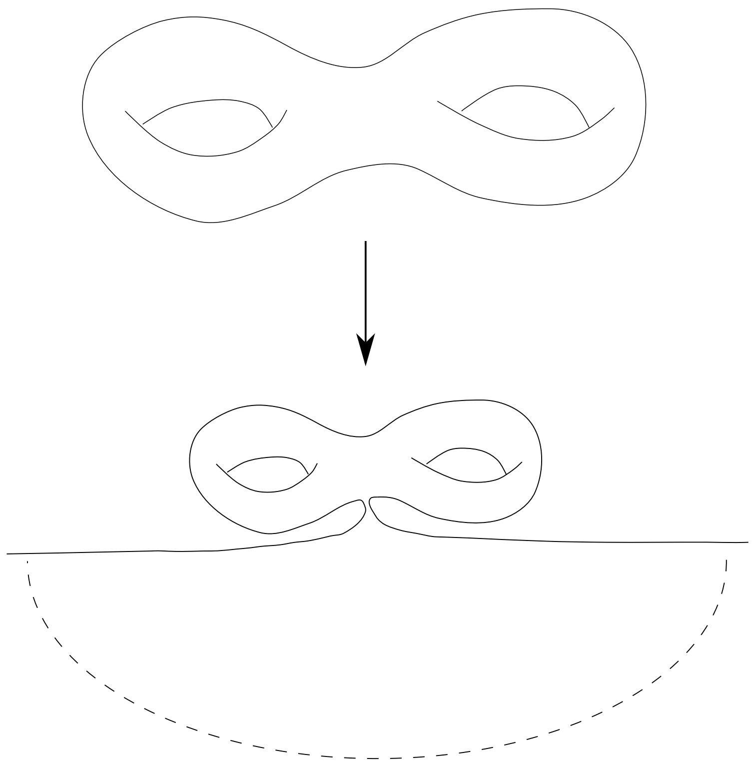

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Mathematicians are like theologians: we regard existence as the prime attribute of what we study. But unlike theologians, we need not always rely upon faith alone.

—L.C. Evans

Preface

The goal of this thesis is to give a detailed account of four deep problems in scalar curvature geometry: the Yamabe problem, Kazdan and Warner's prescribed scalar curvature problem, the Liouville theorem for locally conformally flat manifolds, and the positive mass theorem.

In 1960, H. Yamabe conjectured that given a compact Riemannian manifold, there exists a conformal deformation of the metric to one of constant scalar curvature. Yamabe's original proof of his theorem had an error in the "positive case." The combined works of Yamabe, N. Trudinger, T. Aubin, and R. Schoen gave an affirmative resolution of the conjecture in 1984. Schoen's contribution makes use of the positive mass theorem of general relativity in a very surprising manner.

The Yamabe Problem. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there is a smooth, positive function u on M such that the metric $u^{4/(n-2)}g$ has constant scalar curvature. The sign of the constant is determined uniquely by g.*

In 1975, J.L. Kazdan and F.W. Warner answered the following question: given a smooth function f on a compact manifold M, when is there a smooth Riemannian metric on M with f as its scalar curvature?

Kazdan–Warner Trichotomy. *Let M be a compact surface with Euler characteristic $\chi(M)$. Let $f \in C^\infty(M)$.*

(i) If $\chi(M) > 0$, then f is a scalar curvature if and only if $f > 0$ somewhere.
(ii) If $\chi(M) = 0$, then f is a scalar curvature if and only if f changes sign or vanishes identically.
(iii) If $\chi(M) < 0$, then f is a scalar curvature if and only if $f < 0$ somewhere.

For $n \geq 3$, the class of compact n-manifolds falls into three disjoint groups:

(i) M admits a positive scalar curvature metric.
(ii) M does not admit a positive scalar curvature metric, but does admit a scalar-flat metric.
(iii) M does not admit a metric with $R \geq 0$.

In case (i), any smooth function is a scalar curvature on M. In cases (ii) and (iii), any function that is negative somewhere is a scalar curvature on M.

We say that a Riemannian manifold is *locally conformally flat* (LCF) if it admits local conformal maps into $\mathbb{R}^n$. In 1949, N. Kuiper showed that any simply connected LCF manifold admits a global conformal immersion into the standard sphere. In 1988, R. Schoen and S.-T. Yau gave conditions under which this map is actually injective, and hence an embedding.

Liouville Theorem. *Let (M, g) be a complete LCF manifold of dimension $n \geq 7$ with nonnegative scalar curvature. Then any conformal map from the universal cover of M to the sphere is injective. When $n = 5, 6$, this also holds under the assumption that the scalar curvature is bounded below by a positive constant. When $n = 4$, it holds under the additional assumptions that the scalar curvature is bounded, ∇R is bounded with respect to g, and that the Ricci curvature is bounded below.*

Finally, we say that a Riemannian manifold is *asymptotically flat* if it has one end which is diffeomorphic to the exterior of a ball in $\mathbb{R}^n$ and the metric decays sufficiently rapidly to the Euclidean metric on the end.

Positive Mass Theorem. *Let (M, g) be an asymptotically flat manifold with decay rate $\tau > (n-2)/2$ and such that $R \in L^1(M)$. If $R \geq 0$ and (i) $3 \leq n \leq 7$ or (ii) M is a spin manifold, then the ADM mass is nonnegative. The mass is zero if and only if (M, g) is isometric to $\mathbb{R}^n$ with the standard metric.*

The main geometric content of the positive mass theorem is contained in the following scalar curvature rigidity theorem:

Theorem. *Let M be a compact manifold such that (i) $n \leq 7$ or (ii) M is spin. Then the manifold $M \# T^n$ does not admit a metric of positive scalar curvature, where T^n is the n-dimensional torus.*

In this thesis we give a complete proof of all of the theorems written above, as stated here. The Yamabe and Kazdan–Warner results are optimal, but the Liouville and positive mass theorems have some restrictions. The restrictions in the positive mass theorem are thought to be technical and can be removed, see the discussion in Chapter 7. Although the proof of the Liouville theorem does not use the positive mass theorem, if the positive mass theorem were known for complete manifolds with possibly non-asymptotically flat ends, then the Liouville theorem would hold for all dimensions $n \geq 3$ and $R \geq 0$ without extra hypotheses. To the author's knowledge, this general form of the positive mass theorem is not in the literature.

Here is a rough outline of the contents:

In Chapter 1 we fix our notational conventions for Riemannian geometry. We discuss the conformal transformation properties of curvature, which gives rise to the Yamabe equation. We also derive a dual approach to the Yamabe problem by considering the variational theory of the total scalar curvature.

In Chapter 2 we derive the best constant in the Sobolev inequality $W^{1,p} \hookrightarrow L^{p^\star}$ in $\mathbb{R}^n$ and any compact Riemannian manifold. Manifolds with boundary are also considered in anticipation of Chapter 5.

In Chapter 3 we prove the basic existence and regularity theorem for elliptic equations of the form $-\Delta u + Vu = \lambda u^{p-1}$, where $p \in [2, 2n/(n-2)]$. When p takes the value $2n/(n-2)$ (as in the Yamabe equation), we can only guarantee the existence of a nontrivial solution under an additional requirement.

In Chapter 4 we attack the Yamabe problem. Using results from the previous chapter, the Yamabe problem is immediately solved in the "nonpositive cases." In the positive case, there is a condition that has to be checked, called Aubin's criterion. We check the Yamabe conjecture in the case of S^n by hand. For all other manifolds, we show that Aubin's criterion holds. To do this, we give an account of J.M. Lee and T.H. Parker's conformal normal coordinates. Assuming the positive mass theorem, this allows us to solve the Yamabe conjecture. Finally, we give Z. Jin's example of a complete manifold that does not admit a complete constant scalar curvature metric in its conformal class. This shows that the Yamabe problem for noncompact manifolds is more subtle, but we do not pursue this topic any further.

In Chapter 5 we prove the trichotomy using the negative case of the Yamabe problem and the linearized scalar curvature functional.

In Chapter 6 we discuss Schoen and Yau's conformal development theory, which leads to the positive energy theorem for Yamabe positive locally conformally flat manifolds in dimensions $n \geq 4$. We streamline the original argument because the primary concern here is the positive energy. The main step is to prove the Liouville theorem stated above.

In Chapter 7 we prove the positive mass theorem when $n \leq 7$ or when M is a spin manifold. Using R. Bartnik's elliptic theory on and a method of J. Lohkamp, we show precisely how to compactify the positive mass theorem. In the spin case, the result now follows immediately from results of M. Gromov and H.B. Lawson. When $n \leq 7$, classical methods of minimal surface theory give the result. We also consider the case of multiple ends by showing how to close nonnegative mass ends into almost-spherical bubbles.

Finally, the appendix contains a proof of the Cheng–Yau gradient inequality for Schrödinger operators. This result is needed to extend the Liouville theorem to $n = 4$.

Large parts of this thesis were presented in talks at the University of Tennessee at Knoxville during the 2017–2018 school year in the geometric analysis and general relativity seminars.

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Ryan Unger
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Knoxville, TN

Contents

1

Preliminaries

1.1 Riemannian Geometry

Geometry is a magic that works...
—R. Thom

We begin by fixing some notation and recalling some standard facts from Riemannian geometry. Unless otherwise stated, everything in this section can be found in [15], [38], or [100]. In particular [15] and [38] have very good reviews of Riemannian geometry in their introductory chapters. We follow the conventions of [38].

M will always denote a smooth, n-dimensional, paracompact, Hausdorff manifold. It will be boundaryless unless otherwise stated. We equip M with a Riemannian metric g, and in terms of local coordinates $\{x^i\}$ we write

$$g = g_{ij}\, dx^i\, dx^j.$$

The cone of all Riemannian metrics on M is written as $\mathfrak{M}$. We will also write $\langle X, Y\rangle$ for $g(X, Y)$. The Levi-Civita connection is denoted by ∇, and our convention for the Riemann tensor is

$$\operatorname{Riem}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By taking the trace in an orthonormal basis $\{e_i\}$, we obtain the Ricci tensor

$$\operatorname{Ric}(X, Y) = \sum_{i=1}^{n} \langle \operatorname{Riem}(X, e_i)e_i, Y\rangle$$

and then the scalar curvature

$$R = \sum_{i=1}^{n} \operatorname{Ric}(e_i, e_i).$$

In terms of the Christoffel symbols $\Gamma^i{}_{jk} = dx^i(\nabla_j \partial_k)$ in a chart, the components of the curvature tensors are

$$R^l{}_{kij} = \partial_i \Gamma^l{}_{jk} - \partial_j \Gamma^l{}_{ik} + \Gamma^l{}_{im}\Gamma^m{}_{jk} - \Gamma^l{}_{jm}\Gamma^m{}_{ik},$$

$$R_{kj} = R^i{}_{kij}, \quad R = R^j{}_j.$$

Riemannian measure and integration is defined in the usual way, see [100]. In local coordinates, the integration measure may be represented by a volume form

$$d\mu(g) = \sqrt{\det(g_{ij})}\, dx,$$

modulo orientation. If we take geodesic polar coordinates (r, θ) at p, then we have

$$d\mu = J(r, \theta)\, dr \wedge d\theta,$$

where $d\theta$ is the volume form of S^{n-1} and

$$J(r, \theta) \sim r^{n-1}$$

as $r \downarrow 0$. This measure gives L^p spaces with the usual properties. The total measure (volume) of (M, g) is finite if M is compact. In this case we will often scale g to make (M, g) a probability space. An important consequence is that for $p < q$ and $f \in L^q(M)$, then $f \in L^p(M)$ and

$$\|f\|_{L^p} \leq \|f\|_{L^q}$$

if M has unit volume.

Recall that two metrics g and g' are *conformal* if there is a positive C^∞ function u such that $g' = ug$. This is clearly an equivalence relation on $\mathfrak{M}$, and we denote the *conformal class* of g by $[g]$. Since u is positive, we will often write it as e^{2f} or similar. For two symmetric (0,2) tensors H, K define the *Kulkarni-Nomizu* product as the (0,4) tensor

$$\begin{aligned} H \owedge K(v_1, v_2, v_3, v_4) = {} & H(v_1, v_3)K(v_2, v_4) + H(v_2, v_4)K(v_1, v_3) \\ & - H(v_1, v_4)K(v_2, v_3) - H(v_2, v_3)K(v_1, v_4). \end{aligned}$$

Note that $H \owedge K = K \owedge H$. When $n \geq 3$, the *Weyl tensor* is the (0,4) tensor defined by

$$\mathrm{Weyl} = \mathrm{Riem} + \frac{R}{2(n-1)(n-2)} g \owedge g - \frac{1}{n-2} \mathrm{Ric} \owedge g.$$

In components, this reads

$$W_{ijkl} = R_{ijkl} + \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{jk}g_{il}) - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}).$$

The important properties of the Weyl tensor are that all of its traces are zero, and under conformal deformations we have

$$\mathrm{Weyl}(ug) = u\mathrm{Weyl}(g).$$

Furthermore, when $n \geq 4$, the Weyl tensor vanishes in an open set if and only if the metric is locally conformally flat (LCF) there. When $n = 3$, the obstruction is the *Cotton tensor*, defined in coordinates by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R\, g_{jk} - \nabla_j R g_{ik}).$$

The transformation formula for the Riemann tensor is

$$e^{-2f}\mathrm{Riem}(e^{2f}g) = \mathrm{Riem}(g) - \left(\mathrm{Hess}\, f - \nabla f \otimes \nabla f + \frac{1}{2}|\nabla f|^2 g\right) \owedge g.$$

Here $\mathrm{Hess}\, f = \nabla\nabla f$ is the *Hessian* of f. The *Laplace-Beltrami operator* or simply Laplacian is defined as the trace of the Hessian, or

$$\Delta f = \mathrm{tr}\,\mathrm{Hess}\, f = \nabla^i \nabla_i f.$$

The transformation for the Ricci tensor is

$$\mathrm{Ric}(e^{2f}g) = \mathrm{Ric}(g) - (n-2)\,\mathrm{Hess}\, f - \Delta f g + (n-2)(\nabla f \otimes \nabla f - |\nabla f|^2 g)$$

and for the scalar curvature

$$e^{2f}R(e^{2f}g) = R(g) - 2(n-1)\Delta f - (n-2)(n-1)|\nabla f|^2.$$

A Riemannian metric g induces a topological metric d (distance function) on M by setting

$$d(x,y) = \inf\left\{\int |\gamma'| : \gamma \text{ is a piecewise } C^\infty \text{ curve from } x \text{ to } y\right\}.$$

If (M,d) is a complete metric space, then geodesics are defined for all time, and vice-versa. In particular, this is true if M is compact. We say that (M,g) is a *complete Riemannian manifold* if it is complete as a metric space.

If $P \subset T_pM$ is a 2-plane, then the *sectional curvature* of P is defined by

$$K(P) = \langle \mathrm{Riem}(e_1,e_2)e_2, e_1\rangle$$

where $\{e_1, e_2\}$ is an orthonormal basis of T_pM. When K is the same for all points in M and 2-planes, we say that (M,g) has *constant curvature* and is a *space form* if it is complete. If $\tilde{M}$ is the universal cover of a complete manifold M, we may give it a complete pullback metric $\tilde{g}$. If M is a space form and $K=-1$, then the cover is isometric to H^n, if $K=0$, the cover is isometric to $\mathbb{R}^n$, and if $K=1$, the cover is isometric to S^n. In particular, this holds for M itself when it is simply connected.

An *Einstein metric* is a Riemannian metric for which there exists a constant k such that

$$\mathrm{Ric}(g) = kg.$$

The pair (M,g) is then an *Einstein manifold*. When $n=2$, we have for any metric

$$\mathrm{Ric} = \frac{1}{2}Rg,$$

so g is Einstein if and only if M has constant curvature (since all curvatures are proportional in 2 dimensions). When $n=3$, g is Einstein if and only if it has constant sectional curvature. This is less trivial, but follows from the algebraic fact that the Ricci tensor determines the entire Riemann tensor in 3 dimensions.

1.2 The Yamabe Conjecture

In this section we will motivate the Yamabe problem by examining the *total scalar curvature functional*. This is also called the *Einstein–Hilbert functional* or the *Einstein–Hilbert action*. For more information on Riemannian functionals, see [15] Chapter 4 or [118]. Fix a compact manifold M.

The Einstein–Hilbert functional is defined on $\mathfrak{M}$ by

$$S(g) = \int_M R(g)\, d\mu(g).$$

Since $S(cg) = c^{n/2-1}S(g)$ we will often normalize by a factor of $(\operatorname{vol} g)^{1-2/n}$, which is homogenous of degree $n/2-1$, and the *normalized Einstein-Hilbert functional* is

$$\overline{S}(g) = \frac{1}{(\operatorname{vol} g)^{1-2/n}} \int_M R(g)\, d\mu(g).$$

It obviously satisfies $\overline{S}(cg) = \overline{S}(g)$ for any $c > 0$. If $\mathfrak{M}_1$ is the set of metrics with unit volume, then S and $\overline{S}$ are the same on $\mathfrak{M}_1$. Given two $(0,2)$ tensors H and K on (M,g) we define the inner product

$$(H,K)_g := \int_M \langle H, K\rangle\, d\mu(g), \quad \langle H,K\rangle = H^{ij}K_{ij} = g^{ij}g^{kl}H_{ik}K_{jl}.$$

Proposition 1.1 (Hilbert)**.** *The Einstein-Hilbert functional has the Gâteaux derivative*

$$dS(g).h = (R(g)g/2 - \operatorname{Ric}(g), h)_g$$

and therefore

$$\operatorname{grad} S(g) = \frac{1}{2}Rg - \operatorname{Ric}.$$

Here $h \in \Gamma(S_2M)$, the module of sections of the bundle of $(0,2)$ symmetric tensors on M.

Consult [15] or any textbook on general relativity for the proof. For the rest of the section assume $n \geq 3$ unless stated otherwise.

Theorem 1.2 (Hilbert)**.** *For a compact Riemannian manifold (M,g) of unit volume the following are equivalent:*

(a) *(M,g) is Einstein.*

(b) *g is a critical point of $\overline{S}$.*

(c) *g is a critical point of S in $\mathfrak{M}_1$.*

Proof. That (b) and (c) are equivalent follows from the scaling properties.

(c) $\Rightarrow$ (a) We let $g = g(s)$ be a smooth family of metrics in $\mathfrak{M}_1$ such that

$$\left.\frac{d}{ds}S(g(s))\right|_{s=0} = 0$$

and $g(0) = g_0$. We consider the variation tensors

$$v = \frac{\partial}{\partial s} g, \quad V = \operatorname{tr} v.$$

As shown on p. 104 of [38], we have

$$\frac{\partial}{\partial s}\, d\mu = \frac{1}{2} V\, d\mu.$$

We then differentiate the equation $1 = \int_M d\mu(g(s))$ with respect to s to obtain

$$0 = \int_M V\, d\mu = (v, g)_g.$$

Since v was arbitrary besides being orthogonal to g, we have from Hilbert's derivative formula that

$$(Rg_0/2 - \mathrm{Ric}, v)_{g_0} = 0$$

for all $v \in T_{g_0}\mathfrak{M}_1 = \{v \in \Gamma(S_2 M) : (v, g_0)_{g_0} = 0\}$. This means that

$$\frac{1}{2} R g_0 - \mathrm{Ric} = c g_0$$

for some constant c, hence g_0 is Einstein by Schur's lemma.

(a) $\Rightarrow$ (c) We show that the orthogonal projection of $S(g)$ onto $T_g\mathfrak{M}_1$ is zero. This is easy because R is a constant and $\mathrm{Ric} = Rg/n$, so $dS(g).v = 0$ for any $v \in T_g\mathfrak{M}_1$. □

Using this result, Yamabe hoped to prove the famous Poincaré conjecture.

Conjecture 1.3 (Poincaré). *If M is a compact, simply connected 3-manifold, then M is diffeomorphic to S^3.*

Namely, by the observation at the end of the first section, any Einstein metric on M would have constant sectional curvature. Since M is compact and simply connected, it must then be S^3 by the space form theorem. By Hilbert's theorem, finding an Einstein metric on M amounts to finding a critical point of $\overline{S}$, which seems like a reasonable problem.

Yamabe asked whether we can minimize $\overline{S}$ in a given *conformal class*, that is, given a metric g, can we find a metric g' pointwise conformal to g that minimizes $\overline{S}$ over $[g]$. The following result gives this question a more obviously geometric flavor.

Proposition 1.4. *For a compact Riemannian manifold (M, g), R is constant if and only if g is a critical point of $\overline{S}$ restricted to $[g]$.*

Proof. ($\Leftarrow$) We must characterize variations in $[g] \cap \mathfrak{M}_1$. We let the original metric be g_0 and denote the variation by $g(s) = e^{u(s)} g_0$ since it is through conformal transformations. We then have

$$v = \frac{\partial}{\partial s} g = \frac{\partial u}{\partial s} g.$$

But since $v \in T_{g_0}\mathfrak{M}_1$ at $s = 0$, we find that

$$\int_M \left.\frac{\partial u}{\partial s}\right|_{s=0} d\mu(g_0) = 0.$$

This implies that

$$dS(g_0).(fg_0) = 0$$

for every $f \in C^\infty(M)$ whose integral is zero. Using the formula for dS, this gives

$$\int_M Rf\, d\mu(g_0) = 0.$$

We now choose

$$f = R - \fint_M R\, d\mu(g_0),$$

which is orthogonal to both R and the constants, hence to itself, implying R is constant.

($\Rightarrow$) Simply repeat the argument in reverse. $\square$

So to find a critical point of $\overline{S}$ in a conformal class, one should find a conformal deformation to constant scalar curvature. This leads to the *Yamabe problem*, which is to prove the following conjecture:

Conjecture 1.5 (Yamabe 1960 [123])**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a metric g' conformal to g such that g' has constant scalar curvature.*

This question was answered affirmatively by Yamabe, Trudinger, Aubin, and Schoen in the papers [123], [117], [6], and [101], respectively. This thesis contains the full solution of the conjecture.

This problem was asked in the 1880s for algebraic curves by Klein and Poincaré.

Theorem 1.6 (Uniformization)**.** *Every Riemannian 2-manifold is conformal to a complete manifold of constant Gauss curvature.*

This is more ambitious because it makes no compactness assumption. The classical proof of this fact rests on the existence of isothermal coordinates and a covering argument, which reduces the proof to classifying simply connected Riemann surfaces. However, this was done by Koebe using potential theory. See Wolf [122] for details. When M is compact and $\chi(M) < 0$, Berger found a proof using the calculus of variations [14]. For closed manifolds, a proof using the Ricci flow was found by Hamilton [56], Chow [37], and Chen–Lu–Tian [32]. It turns out that the Ricci flow in two dimensions is conformal, and is exactly the *Yamabe flow*. See the end of this section for more information.

In the higher-dimensional cases we do not get a nice PDE. Indeed, if we set $\tilde{g} = e^{2f}g$ with scalar curvature R' constant, then we have

$$e^{2f}R' = R - 2(n-1)\Delta f - (n-2)(n-1)|\nabla f|^2.$$

Note that R does not depend on f. To get rid of the exponential nonlinearity, we write $e^{2f} = u^{2b}$ for u a positive function and b a constant to be determined. Then $f = b \log u$ and the equation becomes

$$u^{2b} R' = R - 2b(n-1)\frac{\Delta u}{u} + \left[2b(n-1) - b^2(n-2)(n-1)\right] \frac{|\nabla u|^2}{u^2}.$$

We wish to remove the gradient term, so we set $b = 2/(n-2)$. Simplifying, the equation becomes

$$-4\frac{n-1}{n-2}\Delta u + Ru = \tilde{R} u^{(n+2)/(n-2)}.$$

If we define

$$a = 4\frac{n-1}{n-2}$$

and the *conformal Laplacian*

$$L = -a\Delta + R,$$

the Yamabe problem can be reformulated as

Conjecture 1.7 (Yamabe, PDE version)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists a constant μ and a smooth positive solution u to the* Yamabe equation

$$Lu = \mu u^{(n+2)/(n-2)}.$$

The metric $u^{4/(n-2)} g$ then has constant scalar curvature μ.

The natural approach to finding a solution is to construct a functional for which the Yamabe equation is the Euler-Lagrange equation. But we got to this point from the normalized Einstein-Hilbert functional, so we just use it again. From the conformal transformation rules,

$$\overline{S}(u^{4/(n-2)} g) = \frac{4\frac{n-1}{n-2}\int_M |\nabla u|^2 \, d\mu(g) + \int_M Ru^2 \, d\mu(g)}{\left(\int_M |u|^{2n/(n-2)} \, d\mu(g)\right)^{(n-2)/n}}.$$

From now on, we will omit the integration measures and regions when there is no possibility for confusion. This motivates the following definition: we define the *Yamabe functional*

$$J(u) = \frac{E(u)}{\|u\|^2_{L^{2^\star}}}, \quad E(u) = a\int |\nabla u|^2 + \int Ru^2, \quad 2^\star := \frac{2n}{n-2}$$

for $u \in C^\infty(M)$ not identically zero.

Conjecture 1.8 (Yamabe, variational version)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then the infimum of J over $C^\infty(M) \setminus \{0\}$ is attained by a postive smooth function which solves the Yamabe equation.*

We will show that this is true in every case. Thus the Yamabe problem is solved.

Remark 1.9. We prove the Yamabe problem using a variational method together with a quite difficult test function estimate. There are alternative approaches:

Yamabe flow. There is a lot to say about the Yamabe flow, but we will only make a few comments. The Yamabe flow was introduced by R. Hamilton to solve the Yamabe problem. The idea is to flow the metric according to the equation

$$\frac{\partial}{\partial t} g = -(R_g - r_g) g,$$

where r_g is the integral-averaged scalar curvature, and obtain a constant scalar curvature metric as $t \to \infty$. This flow turns out to be conformal, i.e. is just a scalar PDE for the conformal factor. Indeed, if $g(t) = u(t)^{4/(n-2)} g_0$, then

$$\frac{\partial}{\partial t} u^{(n+2)/(n-2)} = \frac{n+2}{4} \left(r_g u^{(n+2)/(n-2)} - L_{g_0} u \right).$$

The first general result was obtained by Ye, who showed the convergence for all LCF manifolds [125]. For a short summary of Ye's proof, see Appendix B of Chow, Lu, and Ni's book [38]. Later, Brendle showed the convergence for all manifolds in dimensions $3 \leq n \leq 5$, and under some conditions in higher dimensions [19] [20]. Yamabe flow is currently an active research area.

The topological approach to the Yamabe problem. For our test function estimate, the positive mass theorem is essential. However, the positive mass theorem is actually not actually needed for every proof of the Yamabe conjecture. Bahri and Brezis have shown how to prove the Yamabe conjecture in the hard cases by using techniques of algebraic topology and calculus of variations [10] [11].

Remark 1.10. Finally, we mention the *compactness of the Yamabe equation.* The compactness theorem for the Yamabe equation says roughly that the solution set of the Yamabe equation is bounded in $C^{2,\alpha}$. By the work of Khuri, Marques, and Schoen, this is known to be true in dimensions $3 \leq n \leq 24$ when the manifold has positive Yamabe invariant and is not conformally diffeomorphic to the standard sphere [70]. By the work of Brendle [21] and Brendle and Marques [22], this is known to be false in dimensions $n \geq 25$. See the survey by Brendle and Marques [23] and also Hebey's book on compactness and stability for nonlinear elliptic equations [60].

2

The Best Constant in the Sobolev Inequality

In this chapter we prove Aubin's result for the best constant in the Sobolev inequality on a compact Riemannian manifold. The proof uses the best constant for the Sobolev inequality in $\mathbb{R}^n$, so we give that proof at the beginning. We finish the technical part with a discussion of the optimal inequality on a manifold with boundary, useful for solving a Dirichlet problem in Chapter 6. Finally, we show some graphs of spherical rearrangements generated in Mathematica.

2.1 The Case of $\mathbb{R}^n$

The usual "Sobolev inequality" in $\mathbb{R}^n$ says that if $1 \leq p < n$, there exists a constant $C > 0$ such that

$$\|u\|_{L^{p^\star}} \leq C\|\nabla u\|_{L^p} \tag{2.1}$$

for every $u \in C_c^\infty(\mathbb{R}^n)$, where $p^\star = np/(n-p)$. Consequently, the inclusion $W^{1,p}(\mathbb{R}^n) \subset L^{p^\star}(\mathbb{R}^n)$ is continuous, and the same estimate holds. A typical proof of this inequality is via the Gagliardo–Nirenberg–Sobolev inequality

$$\|u\|_{L^{n/(n-1)}} \leq \prod_{i=1}^{n} \|\partial_i u\|_{L^1}^{1/n},$$

which holds for all $\varphi \in C_c^\infty(\mathbb{R}^n)$, [24] page 280. This method gives the constant

$$C = \frac{(n-1)p}{n-p}$$

in (2.1). This constant is not optimal, but since (2.1) holds for *some* $C > 0$, we know that

$$K(n,p) = \inf\{C > 0 : (2.1) \text{ holds with constant } C\} = \sup_{u \in W^{1,p}\setminus\{0\}} \frac{\|u\|_{L^{p^\star}}}{\|\nabla u\|_{L^p}}$$

exists and is some positive real number. The goal of the present section is to compute this number and show that the supremum is actually attained when $p > 1$. This result was obtained independently by Aubin [5] and Talenti [116] in 1976 with essentially the same proof. Proposition 2.9 seems to be due to Sperner [111] in 1974, but he is rarely given credit for it. His paper seems to have the only correct proof of this result, which is needed for Aubin and Talenti's proofs. Sperner actually came very close to obtaining the best constant – the only missing ingredient was Bliss' Theorem 2.14. Lieb [82] found a different proof in 1983 using the optimal Hardy–Littlewood–Sobolev inequality. See Lieb and Loss [83] for a textbook treatment of that approach. Here we follow Aubin's approach, which simplifies things a bit by working with Morse functions.

Theorem 2.1. *Let $1 < p < n$ and $p^\star = np/(n-p)$. Then the best constant in the Sobolev inequality is given by*

$$K(n,p) = \frac{p-1}{n-p^\star}\left(\frac{n-p}{n(p-1)}\right)^{1/p}\left(\frac{\Gamma(n+1)}{\Gamma(n/p^\star)\Gamma(n+1-n/p^\star)\omega_{n-1}}\right)^{1/n}$$

The equality in the optimal Sobolev inequality is attained by the functions

$$u_\lambda(x) = \left(\frac{1}{\lambda + |x|^{p/(p-1)}}\right)^{n/p-1}, \quad \lambda \in (0,\infty).$$

Here ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$. The $p = 1$ case is due to Federer and Fleming [48].

Theorem 2.2. *If $u \in W^{1,1}(\mathbb{R}^n)$, then*

$$\|u\|_{L^{n/(n-1)}} \leq n^{-1}\alpha_n^{-1/n}\|\nabla u\|_{L^1}.$$

This inequality is optimal.

Here α_n is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Although we will not need this result, the proof when $p = 1$ illustrates some of the geometric measure theoretic techniques required for the $p > 1$ case.

Proof. It suffices to prove the result for $u \in C_c^\infty(\mathbb{R}^n)$. We define the sets

$$\Omega_t = \{x \in \mathbb{R}^n : |u(x)| > t\} \quad \text{and} \quad \Sigma_t = \{x \in \mathbb{R}^n : |u(x)| = t\},$$

and the truncation of u at level $t \geq 0$ by

$$u_t(x) = \begin{cases} t & \text{if } u(x) > t \\ -t & \text{if } u(x) < -t \\ u(x) & \text{otherwise} \end{cases}.$$

Finally, let $f(t) = \|u_t\|_{L^{n/(n-1)}}$. Note that $f(0) = 0$. We have

$$|u_{t+h}| \leq |u_t| + h\chi_{\Omega_t}$$

for any $h \geq 0$, hence

$$f(t+h) \leq f(t) + h\mathcal{L}^n(\Omega_t)^{(n-1)/n}.$$

Since $\mathcal{L}^n(\Omega_t) \leq \mathcal{L}^n(\operatorname{supp} u) < \infty$, f is in fact Lipschitz. By Rademacher's theorem, it is differentiable almost everywhere, and

$$f'(t) \leq \mathcal{L}^n(\Omega_t)^{(n-1)/n}$$

for almost every t. By Sard's theorem, Σ_t is a C^∞ hypersurface for almost every t, and equal to the boundary of Ω_t when this happens. The isoperimetric inequality gives

$$\mathcal{L}^n(\Omega_t)^{(n-1)/n} \leq n^{-1}\alpha_n^{-1/n}\mathcal{H}^{n-1}(\Sigma_t).$$

We now combine everything to estimate $\|u\|_{L^{n/(n-1)}}$:

$$\begin{aligned}
\|u\|_{L^{n/(n-1)}} &= f(\infty) - f(0) \\
&= \int_0^\infty f'(t)\,dt \\
&\leq n^{-1}\alpha_n^{-1/n}\int_0^\infty \mathcal{H}^{n-1}(\Sigma_t)\,dt \\
&= n^{-1}\alpha_n^{-1/n}\int_{\mathbb{R}^n}|\nabla u|,
\end{aligned}$$

where in the last step we used Federer's coarea formula.

To show that this result is optimal, we exhibit a sequence of Lipschitz functions (u_k) with compact support such that

$$\lim_{k\to\infty}\frac{\|u_k\|_{L^{n/(n-1)}}}{\|\nabla u_k\|_{L^1}} = n^{-1}\alpha_n^{-1/n}.$$

We define the u_k's such that they approximate the characteristic function χ of the closed unit ball $\overline{B_1}(0)$. In that case $\|\chi\|_{L^{n/(n-1)}} = \alpha_n^{(n-1)/n}$, and $\|\nabla\chi\|_{L^1} = \omega_{n-1}$ is just the perimeter, so equality holds. However, χ is not in $W^{1,1}$, so we need to approximate. To be precise, we define the u_k's by

$$u_k(x)\begin{cases} 1 & \text{if } |x| \leq 1 \\ 1 + k(1-|x|) & \text{if } 1 \leq |x| \leq 1 + k^{-1} \\ 0 & \text{if } |x| \geq 1 + k^{-1} \end{cases}.$$

An elementary calculation shows

$$\|u_k\|_{L^{n/(n-1)}}\|\nabla u_k\|_{L^1}^{-1} = 1 + O(k^{-1}),$$

which completes the proof. □

We now begin the proof of the harder result, Theorem 2.1. The proof follows from a sophisticated approximation scheme. We show that it is sufficient to reduce attention to rotationally symmetric decreasing Lipschitz functions.

Lemma 2.3. *Let f be a smooth function on a Riemannian manifold. The set S of points $x \in M$ satisfying $f(x) = 0$ and $\nabla f(x) \neq 0$, has measure zero in M.*

Proof. If $x \in S$ then ∇f is nonzero in a neighborhood U_x of x and $U_x \cap f^{-1}(0)$ is a properly embedded $(n-1)$-dimensional submanifold by standard rank arguments. Consequently, $U_x \cap f^{-1}(0)$ has measure zero. Since countably many such U_x's cover S, we conclude that S has measure zero. □

We recall that a *Morse function* on a Riemannian manifold is a smooth function whose Hessian at every critical point is a nondegenerate bilinear form. The Morse lemma says that these critical points are isolated.

Theorem 2.4 (Morse)**.** *Let M be a Riemannian manifold and f a bounded smooth function on M. Then f can be uniformly approximated by Morse functions on M. Further, given $k \in \mathbb{N}$ and $\Omega \subset M$ with compact closure, the approximating Morse functions can be chosen to approximate f in $C^k(\overline{\Omega})$ as well.*

For the proof, see Section 6 in Milnor [89].

Proposition 2.5. *Let f be a smooth function on a Riemannian manifold M with compact support and not identically zero. Then f can be approximated in $W^{1,p}(M)$ by a sequence of functions (f_k) such that for each k:*
(i) $f_k \in C_c^0(M) \cap \mathrm{Lip}(M)$,
(ii) $\operatorname{supp} f_k = K_k \subset K$,
(iii) ∂K_k *is of class* C^∞,
(iv) f_k *is a* C^∞ *Morse function on* $\{x : f_k(x) \neq 0\}$.
(v) f_k *has finitely many critical points in* $\{x : f_k(x) \neq 0\}$.

Proof. By Morse's theorem, for each $k \in \mathbb{N}$ there is a Morse function $g_k \in C^\infty$ such that $|f - g_k| < k^{-1}$ on M and $|\nabla(f - g_k)| < k^{-1}$ on K. Choose a real number a_k satisfying $k^{-1} < a_k < 2k^{-1}$ such that neither a_k nor $-a_k$ are critical values of g_k. (The existence of such a number follows from Sard's theorem.) By the rank theorem, $g_k^{-1}(a_k)$ and $g_k^{-1}(-a_k)$ are codimension 1 properly embedded submanifolds of M, or they are empty. Define $A_k = [g_k \geq a_k]$ and $A_{-k} = [g_k \leq -a_k]$. We now construct the f_k's by

$$f_k(x) = [g_k(x) - a_k]\chi_{A_k}(x) + [g_k(x) + a_k]\chi_{A_{-k}}(x).$$

By inspection $K_k := \operatorname{supp} f_k = A_k \cup A_{-k}$ and $K_k \subset K$. Further, $f \in C^\infty(K_k)$, which implies $f \in W^{1,p}(M)$. The regularity of ∂K_k follows from the regularity of $g_k^{-1}(a_k)$ and $g_k^{-1}(-a_k)$.

Since $|f - f_k| \leq 3k^{-1}\chi_K$, $\|f - f_k\|_p \to 0$ as $k \to \infty$. If $f(x) \neq 0$ for some $x \in M$, then $|\nabla(f(x) - f_k(x))| \to 0$ because x is in K_k for k sufficiently large. If $f(x) = 0$, then x is never in K_k. However, by Lemma 2.3, the set of points where $f(x) = 0$ and $\nabla f(x) \neq 0$ is a null set. If $f(x) = 0$ and $\nabla f(x) = 0$, then in fact $\nabla(f(x) - f_k(x)) = 0$. Thus $|\nabla(f(x) - f_k(x))| \to 0$ for a.e. $x \in M$. Finally, we need to show that $|\nabla(f - f_k)|$ is bounded from above on M. Note that it vanishes outside of K. On K_k, we have $|\nabla(f - f_k)| < k^{-1} \leq 1$ by the definition of g_k. But on $K \setminus K_k$, f_k vanishes, and the best we can do is bound above by $\sup |\nabla f|$. We combine these to obtain

$$|\nabla(f - f_k)| \leq \max\{\sup |\nabla f|, 1\}\chi_K \quad \text{on } M.$$

This is enough to apply the dominated convergence theorem to obtain $\|\nabla(f - f_k)\|_p \to 0$ as $k \to \infty$. □

Corollary 2.6. *To prove Theorem 2.1, it suffices to consider functions satisfying (i)-(v) of Proposition 2.5, that are in addition nonnegative.*

Proof. Since the functions satisfying (i)–(v) of Proposition 2.5 are dense in $C_c^\infty(\mathbb{R}^n)$, it clearly suffices to prove Theorem 2.1 for them. But the norms involved in (2.1) are unaffected by replacing u with $|u|$, so we may consider only those functions satisfying (i)–(v) that are also nonnegative. □

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a Borel measureable function that *vanishes at infinity* in the sense that $\mathcal{L}^n\{|f| > t\}$ is finite for all $t > 0$. If u is nonnegative, we define its *spherical rearrangement* $u^\star : \mathbb{R}^n \to [0, \infty)$ by

$$u^\star(x) = \sup\{t \in [0, \infty) : \mu(t) > \omega_n |x|^n\},$$

where μ is the *distribution function*

$$\mu(t) = \mathcal{L}^n\{x \in \mathbb{R}^n : u(x) > t\}.$$

If A is a Borel set in $\mathbb{R}^n$ with finite measure, then its *spherical rearrangement* $A^\star$ is defined by

$$A^\star = \{x \in \mathbb{R}^n : \omega_n |x|^n < \mathcal{L}^n(A)\}.$$

In other words, $A^\star$ is an open ball with the same measure as A. In what follows, we will assume u is always measurable. See Section 2.4 for some pictures of spherically rearranged functions.

Proposition 2.7. *Let u be a nonnegative function vanishing at infinity. Then for every $x \in \mathbb{R}^n$*

$$u^\star(x) = \int_0^\infty \chi_{\{u>t\}^\star}(x)\, dt.$$

Proof. Fix $x \in \mathbb{R}^n$, then the function $t \mapsto \chi_{\{u>t\}^\star}(x)$ is monotone decreasing. By inspection, this function is 1 for $0 \le t \le u^\star(x)$ and vanishing otherwise. We conclude that

$$\int_0^\infty \chi_{\{u>t\}^\star}(x)\, dt = \int_0^{u^\star(x)} dt = u^\star(x).$$ □

Compare this with the *layer cake representation*

$$u(x) = \int_0^\infty \chi_{\{u>t\}}(x)\, dt,$$

valid whenever u is a nonnegative measurable function. In fact, for any Borel measure ν on $[0, \infty)$ satisfying $\nu[0, t) = \nu[0, t]$ for all $t > 0$,

$$\int_{\mathbb{R}^n} \nu([0, u(x)])\, dx = \int_0^\infty \mu(t)\, d\nu(t).$$

Proposition 2.8. *Let u be a nonnegative function vanishing at infinity. Then:*

(i) The super-level sets of $u^\star$ are the spherical rearrangements of those of u, i.e.

$$\{u^\star > t\} = \{u > t\}^\star.$$

(ii) u *and* $u^\star$ *are Lebesgue equimeasurable in the sense that* $\mu_u = \mu_{u^\star}$.
(iii) u *and* $u^\star$ *have the same* L^p*-norm for any* $p \geq 1$.

Proof. (i) follows directly from the definition of $u^\star$ and (ii) from the defining fact that $\mathcal{L}^n(A) = \mathcal{L}^n(A^\star)$. We obtain (iii) by applying the layer cake formula with $\nu([0,t]) = t^p$ and using (ii). □

We now study spherical rearrangements when the function has some regularity.

Proposition 2.9. *Let* $u \in C_c^0(\mathbb{R}^n)$ *be nonnegative, and Lipschitz with constant* L. *Then* $u^\star$ *is also Lipschitz, with constant no greater than* L.

Proof. Let g be the function on $[0,\infty)$ defined by $u^\star(x) = g(|x|)$. Let $0 \leq r_0 \leq r_1$, $\sigma = g(r_0)$, $\tau = g(r_1)$ with $\sigma \geq \tau \geq 0$, $\sigma \geq \eta = \sigma - \tau \geq 0$. Suppose that η and σ are positive. For $x \in u^{-1}[\sigma,\infty)$, $y \in B_{\eta/L}(x)$ we have, because u is L-Lipschitz,

$$\sigma \leq u(x) \leq u(y) + L|x-y| < u(y) + \eta.$$

Thus $u(y) > \sigma - \eta = \tau$, i.e. $y \in u^{-1}(\tau,\infty)$, which means

$$u^{-1}[\sigma,\infty) + B_{\eta/L}(0) \subset u^{-1}(\tau,\infty).$$

The Brunn–Minkowski inequality ([47] 3.2.41) implies

$$\begin{aligned}\mathcal{L}^n(u^{-1}(\tau,\infty))^{1/n} &\geq \mathcal{L}^n(u^{-1}[\sigma,\infty) + B_{\eta/L}(0))^{1/n} \\ &\geq \mathcal{L}^n(u^{-1}[\sigma,\infty))^{1/n} + \omega_n^{1/n}\frac{\eta}{L}.\end{aligned}$$

This is equivalent to

$$\left[\frac{\mathcal{L}^n(u^{-1}(\tau,\infty))}{\omega_n}\right]^{1/n} - \left[\frac{\mathcal{L}^n(u^{-1}[\sigma,\infty))}{\omega_n}\right]^{1/n} \geq \frac{\eta}{L}. \tag{2.2}$$

Now

$$\left[\frac{\mathcal{L}^n(u^{-1}(g(r),\infty))}{\omega_n}\right] = r$$

for any $r \geq 0$, so (2.2) implies

$$r_1 - r_0 \geq \eta L = \frac{g(r_0) - g(r_1)}{L}.$$

Thus g is Lipschitz, and

$$\begin{aligned}|u^\star(x) - u^\star(y)| &= |g(|x|) - g(|y|)| \\ &\leq L||x| - |y|| \\ &\leq L|x-y|\end{aligned}$$

for any $x, y \in \mathbb{R}^n$. □

Proposition 2.10. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function as in Corollary 2.6 and μ its distribution function. Then μ is absolutely continuous and where differentiable, we have*

$$\mu'(t) = -\int_{\Sigma_t} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}, \tag{2.3}$$

where $\Sigma_t = f^{-1}(t)$.

Proof. Let $\Omega_t = \{x : u(x) > t\}$. We apply the coarea formula to

$$\mu(t) = \int_{\Omega_t} dx = \int_{\Omega_t} |\nabla u|^{-1}|\nabla u|\, dx,$$

where $|\nabla u|^{-1}$ is defined to be ∞ when $\nabla u = 0$. Since u is a Morse function this situation only occurs on a set of measure zero, so this manipulation is legitimate. The coarea formula gives

$$\mu(t) = \int_0^\infty \left(\int_{\Sigma_s \cap \Omega_t} |\nabla u|^{-1}\, d\mathcal{H}^{n-1} \right) ds.$$

Note that $\Sigma_s \cap \Omega_t = \emptyset$ when $t < s$, so the integral becomes

$$\mu(t) = \int_t^\infty \left(\int_{\Sigma_s} |\nabla u|^{-1}\, d\mathcal{H}^{n-1} \right) ds.$$

That μ is absolutely continuous, hence differentiable a.e., and its derivative is given by (2.3), follows from this formula by standard measure theory. □

Theorem 2.11 (Pólya–Szegö Inequality)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function as in Corollary 2.6 and $u^\star$ be its spherical rearrangement. Then*

$$\|\nabla u^\star\|_{L^p} \le \|\nabla u\|_{L^p}$$

for any $p \ge 1$.

Pólya and Szegö proved the result in the $p = 2$ case [99]. This generalization is due to Sperner, but the proof here is due to Aubin. The restriction to Morse functions is not essential, but it suffices for our purposes. Note that in this case, we showed that $u^\star$ is Lipschitz, hence $\nabla u^\star$ is defined a.e. To lift this condition, see Sperner's paper, or the article by Fusco [50]. There is a further generalization of this result due to Almgren and Lieb [2] for integrals of the form $\int \Psi(|\nabla u|)$, where Ψ is a suitable convex function.

Proof. We again consider the level sets $\Sigma_t = u^{-1}(t)$. By the coarea formula,

$$\|\nabla u\|_{L^p}^p = \int_{\mathbb{R}^n} |\nabla u|^p = \int_0^\infty \left(\int_{\Sigma_t} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1} \right) dt. \tag{2.4}$$

We apply Hölder's inequality to the relationship

$$\mathcal{H}^{n-1}(\Sigma_t) = \int_{\Sigma_t} d\mathcal{H}^{n-1} = \int_{\Sigma_t} |\nabla u|^{-(p-1)/p} |\nabla u|^{(p-1)/p}\, d\mathcal{H}^{n-1},$$

which gives

$$\mathcal{H}^{n-1}(\Sigma_t) \le |\mu'(t)|^{(p-1)/p} \left(\int_{\Sigma_t} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} \right)^{1/p}, \tag{2.5}$$

where we have used (2.3). Since Σ_t is the boundary of Ω_t for almost every t, the Euclidean isoperimetric inequality implies that $\mathcal{H}^{n-1}(\Sigma_t)$ is bounded below by the surface area of a ball with volume $\mu(t)$, equivalently,

$$\mu(t)^{(n-1)/n} \le n^{-1} \alpha_n^{-1/n} \mathcal{H}^{n-1}(\Sigma_t). \tag{2.6}$$

Combining (2.4), (2.5), and (2.6) gives

$$(n\alpha_n^{1/n})^p \int_0^\infty \mu(t)^{p(n-1)/n} |\mu'(t)|^{1-p} \, dt \le \int_{\mathbb{R}^n} |\nabla u|^p. \tag{2.7}$$

This inequality becomes an equality when u is spherically symmetric: in this case Σ_t is a coordinate sphere and we have equality in (2.6). We also have equality in (2.5) when u is constant on coordinate spheres, as is the case for spherically symmetric functions. So if we consider $u^\star$, then we have

$$(n\alpha_n^{1/n})^p \int_0^\infty \mu(t)^{p(n-1)/n} |\mu'(t)|^{1-p} \, dt = \int_{\mathbb{R}^n} |\nabla u^\star|^p. \tag{2.8}$$

Here we used Proposition 2.9 to justify the use of the coarea formula. Comparing (2.7) and (2.8) completes the proof. □

We remark that the obstruction to generalizing this to the sphere and hyperbolic space is the isoperimetric inequality. Since these are space forms, the results in Dinghas [43] may be employed to reach the same conclusion in those cases.

By combining Proposition 2.8 (iii) and Theorem 2.11, we have

Corollary 2.12. *Let u be a function as in Corollary 2.6. If $\|u^\star\|_{L^{p^\star}} \le C\|\nabla u^\star\|_{L^p}$, then $\|u\|_{L^{p^\star}} \le C\|\nabla u\|_{L^p}$.*

So now we just have to prove Theorem 2.1 for functions of the form $u^\star$.

Proposition 2.13. *Let g be a monotone decreasing, compactly supported, and absolutely continuous function on $[0,\infty)$. Then:*

$$\omega_{n-1}^{-1/n} \left(\int_0^\infty |g(r)|^{p^\star} r^{n-1} \, dr \right)^{1/p} \le K(n,p) \left(\int_0^\infty |g'(r)|^p r^{n-1} \, dr \right)^{1/p}.$$

This follows from the following result of Bliss [17].

Theorem 2.14. *Let m and n be constants such that $n > m > 1$, and let $f : [0,\infty) \to [0,\infty)$ be such that*

$$J = \int_0^\infty f^m < \infty.$$

Then the integral

$$y(x) = \int_0^x f$$

is finite for every x and

$$\int_0^\infty \frac{y^n}{x^{n-r}}\, dx \le kJ^{n/m}, \tag{2.9}$$

where

$$l = \frac{n}{m} - 1, \quad k = \frac{1}{n-l-1}\left(\frac{r\Gamma(n/l)}{\Gamma(1/l)\Gamma((n-1)/l)}\right).$$

The inequality (2.9) holds with the equality sign for every function of the form

$$f(x) = \frac{c}{(dx^r+1)^{(l+1)/l}},$$

and for every function differing from one of these only on a set of measure zero, but for no others.

Proof. Step 1. From Hölder's inequality,

$$\int_0^x f \le \left(\int_0^x f^m\right)^{1/m} x^{(m-1)/m},$$

and so

$$\lim_{x\to 0} x^{-(m-1)/m} y(x) = 0. \tag{2.10}$$

Next, consider the function

$$\psi(p,w) = (m-1)p^m - mp^{m-1}w + w^m,$$

for p, w nonnegative real numbers. This is positive except at its roots $w = p$. Indeed, $\psi_w(p,p) = 0$ and $\psi_{ww}(p,w) > 0$ for $w > 0$.

Step 2. Let $0 \le u < 1$ and define

$$\phi(u) = u^m \int_0^1 \frac{\eta^{n-2}}{(1-u+u\eta^r)^{n/r}}\, d\eta = \frac{1}{(1-u)^{1/r}u^{(m-1)/r}} \int_0^U \frac{\zeta^{n-2}}{(1+\zeta^r)^{n/r}}\, d\zeta,$$

where $U = (u/(1-u))^{1/r}$ and $\zeta = U\eta$. Consider $(x,y,z) \in \mathcal{O} := (0,\infty)^3$. Then the equation

$$\phi(u) = x^{m-1}y^{-m}z$$

has a unique solution $(x,y,z) \mapsto u \in [0,1)$ since

$$\phi(0) = 0, \quad \phi(1) = \infty, \quad \phi'(u) > 0 \text{ for } 0 < u < 1.$$

To see that $\phi(u) \to \infty$ as $u \to 1$, consider the second form of ϕ given. Then $U = \infty$ and the integrand is $O(\zeta^{-2})$, so the integral diverges. By the inverse function theorem, u is smooth on $\mathcal{O}$ (even real analytic).

If we approach $(x_1, 0, 1), x_1 > 0$, along a curve in $\mathcal{O}$, it follows from the definition of u that $u \to 1$ and also that

$$\lim_{x \to x_1} \frac{x^{m-1}}{y^m}(1-u)^{1/r} = \int_0^\infty \frac{\zeta^{n-2}}{(1+\zeta^r)^{n/r}}\, d\zeta = \frac{1}{r}\frac{\Gamma(1/r)\Gamma((n-1)/r)}{\Gamma(n/r)}, \tag{2.11}$$

where the second equality follows from standard beta function formulas. Using (2.10), this is still seen to be true in case $(x, y, z) \to (0, 0, 1)$ along a curve of the form

$$y = \int_0^x f, \quad z = \int_x^\infty f^m. \tag{2.12}$$

If $x_2, y_2 > 0$ and $(x, y, z) \to (x_2, y_2, 0)$ along a curve in $\mathcal{O}$, then $u \to 0$ and

$$\lim_{x \to x_2} \frac{u^m}{z} = (n-1)x_2^{m-1}y_2^{-m}. \tag{2.13}$$

Step 3. Define a function $W : \mathcal{O} \to \mathbb{R}$ by

$$W(x, y, z) = -\frac{1}{n-r-1}\left(\frac{1}{1-u}\frac{y^n}{x^{n-r-1}} + \frac{z}{(1-u)u^{m-1}}\frac{y^{n-m}}{x^{(m-1)r}}\right),$$

where $u = u(x, y, z)$ is as defined in step 2. If $p, \lambda : \mathcal{O} \to \mathbb{R}$ are given by

$$p(x, y, z) = \frac{y}{x}u, \quad \lambda(x, y, z) = -\frac{1}{m-1}\frac{y^{mr}}{x^{r(m-1)}}\frac{1}{(1-u)u^{m-1}},$$

then

$$\begin{aligned} W_x &= x^{-n+r}y^n - (m-1)\lambda p^m \\ W_y &= m\lambda p^{m-1} \\ W_z &= \lambda. \end{aligned}$$

When $(x, y, z) \to (x_1, 0, 1)$ or $(x_2, y_2, 0)$ in $\mathcal{O}$, the equations (2.11) and (2.13) show that

$$\begin{aligned} &\lim_{x \to x_1} W = -k \\ &\lim_{x \to x_2} W = -\frac{1}{n-r-1}\frac{y_2^n}{x_2^{n-r-1}}. \end{aligned}$$

Step 4. By scaling, we may assume without loss of generality that

$$\int_0^\infty f^m = 1.$$

Consider the curve C in $\mathcal{O}$ defined by (2.12). Note that $z(0) = 1$. For this curve there exists a smallest interval $(x_1, x_2) \subset [0, \infty]$ such that

$$y(x) = \begin{cases} 0 & \text{if } 0 \le x \le x_1 \\ \text{const.} & \text{if } x_2 \le x < \infty \end{cases}, \quad z(x) = \begin{cases} 1 & \text{if } 0 \le x \le x_1 \\ 0 & \text{if } x_2 \le x < \infty \end{cases}.$$

(We allow the values $x_1 = 0$ and $x_2 = \infty$ in case no such interval exists.) Define $\tilde{W}(x) = W(x, y(x), z(x))$. A calculation shows

$$\frac{d\tilde{W}}{dx} = x^{-n+r}y^n - \lambda\left[(m-1)p^m - mp^{m-1}f + f^m\right].$$

We define functions I, I^* on intervals by

$$I(a,b) = \int_a^b x^{-n+r}y^n\,dx, \quad I^*(a,b) = \int_a^b \frac{d\tilde{W}}{dx}\,dx = \tilde{W}(b) - \tilde{W}(a).$$

If $x_2 < \infty$, then on the the intervals $(0, x_1), (x_1, x_2), (x_2, \infty)$,

$$\begin{aligned} I(0,x_1) &= 0 \\ I^*(x_1,x_2) &= \tilde{W}(x_2) - \tilde{W}(x_1) = k - \frac{1}{n-r-1}\frac{y_2^n}{x_2^{n-r-1}} \\ I(x_2,\infty) &= \frac{1}{n-r-1}\frac{y_2^n}{x_2^{n-r-1}}, \end{aligned}$$

where $y_2 = y(x_2)$. Furthermore,

$$I(x_1,x_2) = I^*(x_1,x_2) + \int_{x_1}^{x_2} \lambda\psi(p,f)\,dx.$$

As shown in step 1, $\psi \geq 0$, and $\lambda < 0$, hence

$$I(0,\infty) = I(x_1,x_2) + I(x_2,\infty) \leq I^*(x_1,x_2) + I(x_2,\infty) = k.$$

When $x_2 = \infty$ the value of $I(0,\infty)$ can be calculated as the limit of $I(x_1,x_3)$ as $x_3 \to \infty$. In this case,

$$I(x_1,x_3) = I^*(x_1,x_3) + \int_{x_1}^{x_3} \lambda\psi(p,f)\,dx \leq I^*(x_1,x_3).$$

But

$$I^*(x_1,x_3) = W(x_3) - W(x_1) = k + W(x_3) \leq k.$$

This proves the inequality for all functions $f \in L^m$. It remains to be shown that equality is attained for only that f which is written in the statement. However this is not needed for our purposes, so we omit the proof. □

To prove Proposition 2.13, one applies Theorem 2.14 with $m = p$, $n = p^\star$, $l = p^\star/n$, $x = r^{(p-n)/(p-1)}$, and $f = g'$. This completes the proof of Theorem 2.1.

2.2 Aubin's Result for Compact Manifolds

Aubin noticed that one can localize optimal inequalities on $\mathbb{R}^n$ to obtain optimal inequalities on certain complete Riemannian manifolds. We will only deal with the compact case here, but Aubin's original proof worked for any complete manifold with positive injectivity radius and bounded sectional curvature [5]. Later Hebey showed it is good enough to replace the curvature bound with Ricci curvature bounded *below* [58]. In the noncompact case, one has to get a good cover with uniform properties. To deal with this, Aubin used an unpublished lemma of Calabi and Hebey used the Bishop–Gromov volume comparison theorem. In the compact case, there are of course no such difficulties, so the proof here is quite simple.

Theorem 2.15. *Let (M, g) be a smooth, compact Riemannian n-manifold. For any $\varepsilon > 0$ and $1 \le p < n$, there exists a constant B such that for any $u \in W^{1,p}$,*

$$\left(\int_M |u|^{2^\star}\right)^{p/p^\star} \le (K(n,p)^p + \varepsilon) \int_M |\nabla u|^p + B \int_M |u|^p. \tag{2.14}$$

This is a bit stronger than the form originally stated by Aubin, but the proof is the same. The stated result in [5] is

Corollary 2.16. *For any $\varepsilon > 0$ there exists a constant B such that for every $u \in W^{1,p}$,*

$$\|u\|_{L^{p^\star}} \le (K(n,p) + \varepsilon)\|\nabla u\|_{L^p} + B\|u\|_{L^p}. \tag{2.15}$$

Consequently, the embedding $W^{1,p} \hookrightarrow L^{p^\star}$ is continuous.

Proof. Since $p \ge 1$, $(x+y)^{1/p} \le x^{1/p} + y^{1/p}$ for $x, y \ge 0$, so one can take the $1/p$-th root of (2.14) to conclude (2.15). $\square$

Proof of Theorem 2.15. By an approximation argument with the Euclidean Sobolev inequality, each $x \in M$ is contained a chart (U, ϕ) such that

$$\left(\int_M |u|^{2^\star}\right)^{p/p^\star} \le \left(K(n,p)^p + \frac{\varepsilon}{2}\right) \int_M |\nabla u|^p \tag{2.16}$$

for every $u \in C_c^\infty(U)$. Let $(U_i, \phi_i)_{i=1}^N$ be a finite covering of M by such charts. Choose a partition of unity (α_i) subordinate to the covering, and set

$$\theta_i = \frac{\alpha_i^p}{\sum_{k=1}^N \alpha_k^p}.$$

This defines a new partition of unity such that each $\theta_i^{1/p}$ is smooth and compactly supported in U_i for any i. Note that

$$\begin{aligned}
\|u\|_{L^{p^\star}}^p &= \|u^p\|_{p^\star/p} \\
&= \left\|\sum_{i=1}^N \theta_i u^p\right\|_{L^{p^\star/p}} \\
&\le \sum_{i=1}^N \|\theta_i u^p\|_{L^{p^\star/p}} \\
&= \sum_{i=1}^N \|\theta_i^{1/p} u\|_{L^{p^\star}}^p
\end{aligned}$$

Let $H = \max_{i=1,\ldots,N} \|\nabla\theta_i^{1/p}\|_{L^\infty}$ and μ, ν be such that $(1+t)^p \le 1 + \mu t + \nu t^p$ for all $t \ge 0$ (see Lemma 2.18 below), then by inserting the previous inequality into (2.16), we find

$$\begin{aligned}\left(K(n,p)^p + \frac{\varepsilon}{2}\right)^{-1}\left(\int_M |u|^{2^\star}\right)^{p/p^\star} &\le \sum_{i=1}^N \int_M \left(\theta_i^{1/p}|\nabla u| + |u||\nabla\theta_i^{1/p}|\right)^p \\ &\le \int_M \sum_{i=1}^N (|\nabla u|^p\theta_i + \mu|\nabla u|^{p-1}|\nabla\theta_i^{1/p}|\theta_i^{(p-1)/p}|u| \\ &\quad + \nu|u|^p|\nabla\theta_i^{1/p}|^p) \\ &\le \|u\|_{L^p}^p + \mu N H\|\nabla u\|_{L^p}^{p-1}\|u\|_{L^p}^p + \nu N H^p\|u\|_{L^p}^p.\end{aligned}$$

Now let $\varepsilon' > 0$ be such that

$$\left(K(n,p)^p + \frac{\varepsilon}{2}\right)(1+\varepsilon') \le K(n,p)^p + \varepsilon.$$

By Young's inequality with ε,

$$px^{p-1}y \le \lambda(p-1)x^p + \lambda^{1-p}y^p$$

for all $x, y, \lambda > 0$. By taking $x = \|\nabla u\|_{L^p}$, $y = \|u\|_{L^p}$, and $\lambda = p\varepsilon'/(\mu(p-1)NH)$, one gets that for any $u \in C^\infty(M)$,

$$\mu N H\|\nabla u\|_{L^p}^{p-1} \le \varepsilon'\|\nabla u\|_{L^p}^p + C\|u\|_{L^p}^p, \quad C = \frac{\mu N H}{q}\left(\frac{p\varepsilon'}{\mu(p-1)NH}\right)^{1-p}.$$

Hence, for any $u \in C^\infty(M)$,

$$\begin{aligned}\left(\int_M |u|^{p^\star}\right)^{p/p^\star} &\le \left(K(n,p)^p + \frac{\varepsilon}{2}\right)(1+\varepsilon')\int_M |\nabla u|^p + B\int_M |u|^p \\ &\le (K(n,p)^p + \varepsilon)\int_M |\nabla u|^p + B\int_M |u|^p,\end{aligned}$$

where

$$B = \left(K(n,p)^p + \frac{\varepsilon}{2}\right)(C + \nu N H^q). \qquad \square$$

Remark 2.17. The B constructed here blows up as ε, hence ε', tends to zero. Hebey and Vaugon have shown that when $p = 2$, the Sobolev inequality holds for $\varepsilon = 0$ and some constant B finite [61]. See also Chapter 7 of Hebey's book [59].

Lemma 2.18. *If $p \ge 1$, there exist constants μ, ν (depending on p) such that*

$$(1+t)^p \le 1 + \mu t + \nu t^p$$

for any $t \ge 0$.

Proof. When $q = 1$, take $\mu = 0$ and $\nu = 1$. For $p > 1$, the function $x \mapsto x^p$ is convex, so the derivative test for convexity implies

$$x^p \geq y^p + py^{p-1}(x - y).$$

Taking $x = 1$ and $y = 1 + t$, we find

$$(1 + t)^p \leq 1 + p(1 + t)^{p-1}t.$$

$1 < p \leq 2$: Now the function $x \mapsto x^{p-1}$ is concave, so $(1 + t)^{p-1} \leq 1 + t^{p-1}$, and

$$(1 + t)^p \leq 1 + pt + pt^p.$$

$2 < p$: The function $x \mapsto x^{p-1}$ is still convex, so midpoint convexity gives

$$\left(\frac{1+t}{2}\right)^{p-1} \leq \frac{1}{2}1 + \frac{1}{2}t^{p-1}.$$

We thus obtain

$$(1 + t)^p \leq 1 + p2^{p-2}t + p2^{p-2}t^p. \qquad \square$$

We now characterize the sense in which $K(n, p)$ is the "best" constant.

Proposition 2.19. *Let (M, g) be a Riemannian n-manifold (not necessarily compact), and $1 \leq p < n$. Suppose there exist constants A and B such that*

$$\|u\|_{L^{p^\star}} \leq A\|\nabla u\|_{L^p} + B\|u\|_{L^p}$$

for every $u \in C_c^\infty(M)$. Then $A \geq K(n, p)$.

Proof. Suppose the inequality holds on some manifold for $A < K(n, p)$. Let $\varepsilon > 0$. For any $x \in M$ there is a chart (U, ϕ) around x, with $\phi(U) = B_\delta(0)$ a Euclidean ball, and such that the components of g satisfy

$$(1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)g_{ij}$$

in the sense of bilinear forms. Choosing ε small enough we get that by the supposed Sobolev inequality,

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leq A'\|\nabla u\|_{L^p(\mathbb{R}^n)} + B'\|u\|_{L^p(\mathbb{R}^n)}, \quad A' < K(n, p),$$

for any $u \in C_c^\infty(B_{\delta'}(0))$ and $\delta' \leq \delta$. But by Hölder's inequality,

$$\|u\|_{L^{p^\star}(B_\delta(0))} \leq \mathrm{vol}(B_\delta(0))^{1/n}\|u\|_{L^p(B_\delta(0))}.$$

Hence, by choosing δ small enough, we get that there exists $A'' < K(n, p)$ such that for any $u \in C_c^\infty(B_\delta(0))$,

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leq A''\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

But now we use the scale-invariance of the Euclidean Sobolev inequality to conclude that

$$\|u\|_{L^{p^\star}(\mathbb{R}^n)} \leq A''\|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for every $u \in C_c^\infty(\mathbb{R}^n)$. This contradicts $K(n, p)$ being the optimal constant in the Euclidean case. $\square$

2.3 Manifolds with Boundary

We consider a smooth Riemannian manifold $(\overline{M}, g)$ with a smooth boundary. For our purposes, manifolds with boundary occur as subsets of proper Riemannian manifolds. We define the Sobolev spaces $W_0^{k,p}(M)$ by the completion of $C_c^\infty(M)$ in the $W^{k,p}$-norm on M. These spaces encode "zero boundary data" in the following sense:

Theorem 2.20 (Trace Theorem)**.** *Let $(\overline{M}, g)$ be a compact Riemannian manifold with smooth boundary. Then there exists a bounded linear operator $T : W^{1,p}(M) \to L^p(\partial M)$ such that*

(i) $Tu = u|_{\partial M}$ *if* $u \in W^{1,p}(M) \cap C(\overline{M})$,

(ii) $\|Tu\|_{L^p(\partial M)} \leq C\|u\|_{W^{1,p}(M)}$ *for each* $u \in W^{1,p}(M)$, *with the constant C depending only on p and M.*

(iii) If $u \in W^{1,p}(M)$, *then* $u \in W_0^{1,p}(M)$ *if and only if* $Tu = 0$.

Compare Section 5.5 in [46]. We refer to the results there as the "Euclidean trace theorem."

Proof. Let $\{\theta_i\}_{i=1}^N$ be a finite partition of unity on $\overline{M}$ such that each θ_i is nonzero on a coordinate ball if $\Theta_i := \operatorname{supp}\theta_i \subset M$ or nonzero on a coordinate half-ball if $\Theta_i \cap \partial M \neq \emptyset$. It is easy to check that $\theta_i u \in W^{1,p}(\Theta_i)$ whenever $u \in W^{1,p}(M)$. By standard arguments this implies the coordinate representative of $\theta_i u$ is in the Euclidean Sobolev space on a ball or half-ball. Thus we have natural trace operators T_i on $W^{1,p}(\Theta_i)$ defined by this coordinate representation. We thus define the trace operator $T : W^{1,p}(M) \to L^p(\partial M)$ on the manifold by

$$Tu = \sum_{i=1}^N T_i(\theta_i u). \tag{2.17}$$

This is a bit of an abuse of notation: the codomain of T_i is $L^p(\Theta_i \cap \partial M)$ when $\Theta \cap \partial M \neq \emptyset$, but by exending by zero we may assume the codomain is $L^p(\partial M)$. If $\Theta_i \cap \partial M = \emptyset$, then $\theta_i u \in W_0^{1,p}(\Theta_i)$ and $T_i(\theta_i u) = 0$ by the Euclidean trace theorem.

We now show that T defined by (2.17) has properties (i) and (ii). If $u \in W^{1,p}(M) \cap C(\overline{M})$, the same is true for $\theta_i u$ in Θ_i. Hence by the Euclidean trace theorem, $T_i(\theta_i u) = (\theta_i u)|_{\partial\Theta_i}$. Since $\sum \theta_i = 1$, we see that $Tu = u|_{\partial M}$. We also have the estimate

$$\begin{aligned}
\|Tu\|_{L^p(\partial M)} &\leq \sum_{i=1}^N \|T_i(\theta_i u)\|_{L^p(\partial\Theta_i)} \\
&\leq C \sum_{i=1}^N \|\theta_i u\|_{W^{1,p}(\Theta_i)} \\
&\leq C \sum_{i=1}^N \|u\|_{W^{1,p}(\Theta_i)} \\
&\leq C\|u\|_{W^{1,p}(M)}.
\end{aligned}$$

By continuity and density of $C^\infty(\overline{M})$ in $W^{1,p}(M)$, T is the only bounded linear operator with properties (i) and (ii), hence it is well defined, i.e. the values of T on $W^{1,p}(M) \setminus C(\overline{M})$ do not depend on the partition of unity $\{\theta_i\}$.

Property (iii) is now obvious from the definition (2.17). □

With small amount of work, we can transfer the best constant result to the spaces $W_0^{1,p}$.

Theorem 2.21. *Let $(\overline{M}, g)$ be a compact Riemannian manifold with boundary. For $1 \le p < n$ and any $\varepsilon > 0$, there exists a constant $B \ge 0$ such that*

$$\left(\int_M |u|^{p^\star}\right)^{p/p^\star} \le (K(n,p)^p + \varepsilon) \int_M |\nabla u|^p + B \int_M |u|^p$$

for every $u \in W_0^{1,p}$.

For $u \in W^{1,p}$ one must use $2^{p/n} K(n,p)^p$ instead [34]. The proof of Theorem 2.21 is based off of the result for closed manifolds. The trick is to embed $\overline{M}$ in a closed manifold.

Lemma 2.22. *If $(\overline{M}, g)$ is a compact Riemannian manifold, then it is isometrically embedded in a closed Riemannian manifold.*

Proof. Let D be the double of $\overline{M}$. Specifically, we take another copy of M, M', and glue the boundaries along the identity map. The result is a smooth manifold and we identify $\overline{M}$ with its image in the double. We claim that D admits a Riemannian metric $h \in C^\infty$ such that $h|_{\overline{M}} = g$.

Since D is a smooth manifold, it has a C^∞ Riemannian metric g'. The goal is to blend this with g but not to touch g on $\overline{M}$. Since g is smooth up to the boundary, we may extend g to some smooth symmetric tensor s in a neighborhood Ω of $\overline{M}$ in D. Since "positive definite" is an open condition, there is a neighborhood $\Omega' \subset \Omega$ of $\overline{M}$ such that s is a metric tensor on Ω'. It is well known that $\overline{M}'$ contains a collar, i.e. an open neighborhood C of $B \subset \overline{M}'$ such that C is diffeomorphic to $B \times [0,1)$. It is attached to the common boundary in the obvious way when we apply the embedding $\overline{M}' \hookrightarrow D$. Now C might be larger than $\Omega' \cap \bar{M}'$, but by reducing the diffeomorphism to $B \times [0, \delta)$ for some $\delta > 0$, calling the result C_δ, we can arrange for $\overline{M} \cup C_\delta \subset \Omega'$. Note that $\overline{M} \cap C_\delta = B$. We define $C_{\delta/2}$ in the obvious way. Using a partition of unity there exists a $\theta \in C^\infty(D; [0,1])$ such that $\theta|_{\overline{C_{\delta/2}}} = 1$ and $\theta|_{(D\setminus\Omega')} = 0$. (This complicated construction is to guarantee $\overline{C_{\delta/2}} \cap (D \setminus \Omega') = \emptyset$.)

Now θs is, in the natural way, a C^∞ symmetric tensor field on all of D. Let $\eta = 1 - \theta \in C^\infty(D; [0,1])$ so that $\eta|_{(D\setminus\Omega')} = 1$ and $\eta|_{\overline{C_{\delta/2}}} = 0$. We now consider the symmetric C^∞ tensor field

$$g := \theta s + \eta g'$$

on D. It is clear that $h|_{\overline{M}} = g$. Since h is a convex linear combination of positive definite forms at each point, it must be a positive definite form. Hence h is the desired metric on D. □

Proof of Theorem 2.21. Using Lemma 2.22, we isometrically embed $(\overline{M}, g)$ into a compact Riemannian manifold (N, h). From Theorem 2.15, we have that

$$\left(\int_N |u|^{2^\star}\right)^{p/p^\star} \leq (K(n,p)^p + \varepsilon) \int_N |\nabla u|^p + B \int_N |u|^p$$

for every $u \in C^\infty(N)$. But since $C_c^\infty(M) \subset C^\infty(N)$ in the natural way, this integral holds for $u \in C_c^\infty(M)$ and with the integrals over N replaced by integrals over M. By the density of $C_c^\infty(M)$ in $W_0^{1,p}(M)$, this inequality then holds for every $u \in W_0^{1,p}(M)$. □

2.4 Some Examples of Spherically Rearranged Functions

It is quite difficult to find spherical rearrangements in practice. Here is an example where the rearrangement is exactly calculable:

Proposition 2.23. *Let $a, b > 0$. Define $u : \mathbb{R} \to [0,1]$ by*

$$u(x) = \begin{cases} ax + 1 & \text{if } -a^{-1} \leq x \leq 0 \\ -bx + 1 & \text{if } 0 \leq x \leq b^{-1} \\ 0 & \text{otherwise} \end{cases}.$$

The graph of u is a triangle with base length $a^{-1} + b^{-1}$, height 1, and side slopes a and $-b$. Then the spherical rearrangement $u^\star$ is given by

$$u(x) = \begin{cases} -2|x|h + 1 & \text{if } |x| \leq 2h^{-1} \\ 0 & \text{otherwise} \end{cases},$$

where h is the harmonic mean of the slopes,

$$h = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}.$$

The graph of $u^\star$ is a triangle with side slopes $\pm h/2$ and base length $4h^{-1}$.

Proof. We first compute the distribution function μ of u. This is easily done with a bit of geometry: the line $y = t$ intersects the graph at the points x_1 and x_2 satisfying

$$ax_1 + 1 = t, \quad -bx_2 + 1 = t.$$

The measure of the super-level set is then just $|x_1| + |x_2|$, which gives

$$\mu(t) = \left|\frac{t-1}{a}\right| + \left|\frac{1-t}{b}\right| = (1-t)\left(\frac{1}{a} + \frac{1}{b}\right).$$

We then obtain $u^\star(x)$ by solving $\mu(t) = 2|x|$ for t, which gives

$$u^\star(x) = 1 - 2h|x|,$$

whenever $|x| \leq 2h^{-1}$. □

Figure 2.1: Example of Proposition 2.23

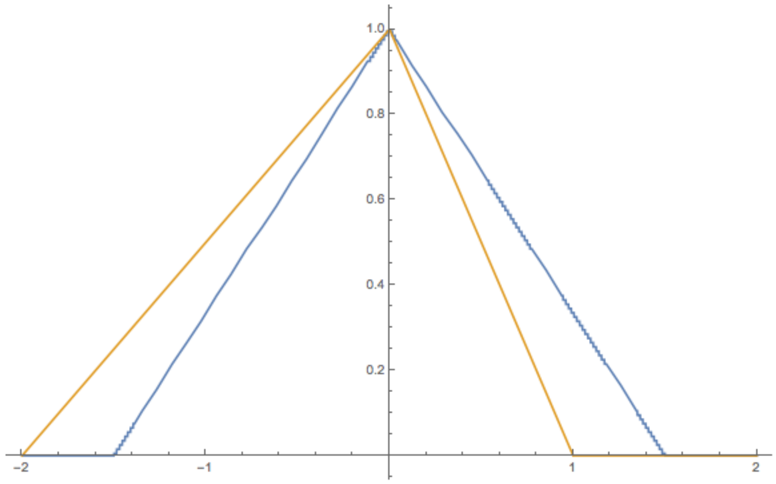


Figure 2.2: Another example of spherical rearrangement

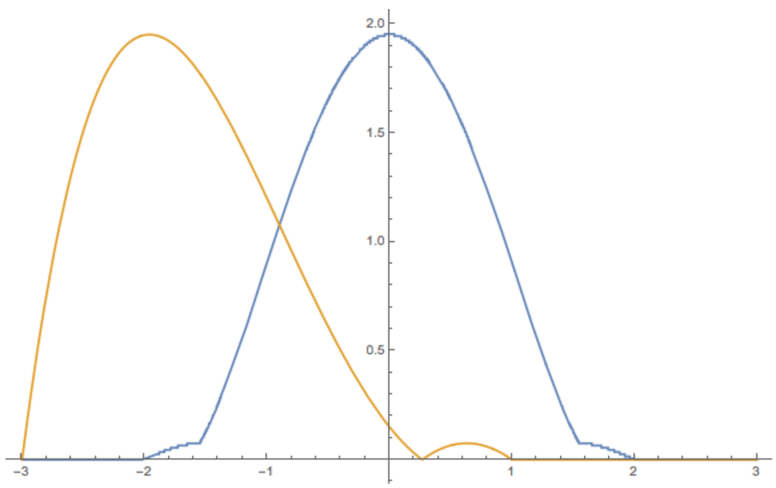
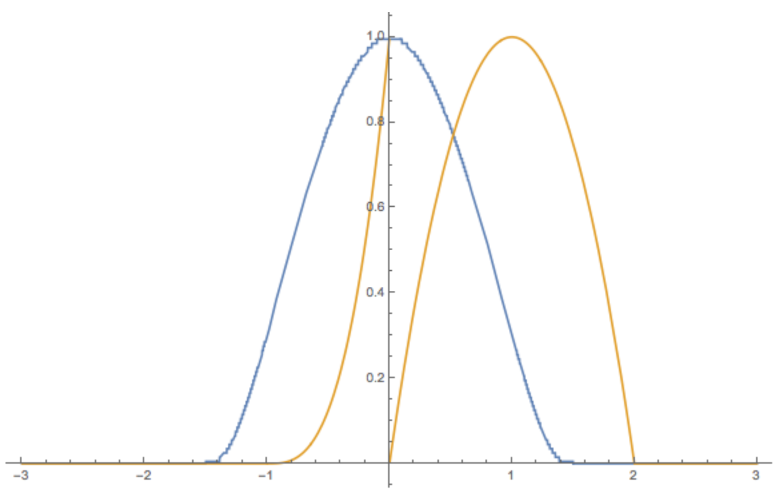


Figure 2.3: Another example of spherical rearrangement



3

Semilinear Schrödinger Equations

The setting of this chapter is a compact Riemannian manifold (M, g) of dimension $n \geq 3$. Yamabe's equation

$$-a\Delta u + Ru = \mu u^{(n+2)/(n-2)}$$

is a special case of the general *semilinear Schrödinger equation*

$$Lu := -\Delta u + Vu = \lambda u^{p-1},$$

where $V \in L^\infty(M)$, $\lambda \in \mathbb{R}$, and $p \in [2, 2^\star]$. When $p < 2^\star$, the existence and regularity for these equations is almost immediate, but when $p = 2^\star$, the best constant in the Sobolev inequality is used in a crucial manner and the regularity is also nontrivial. This chapter draws heavily from [60].

3.1 Existence Theorem

As explained in Chapter 1, the Yamabe equation may be obtained as the Euler–Lagrange equation for a constrained functional. More generally, any semilinear Schrödinger equation may be obtained from the functional

$$J_{V,p}(u) = \frac{E(u)}{\|u\|_{L^p}^2}, \quad E(u) = \int_M |\nabla u|^2 + \int_M Vu^2.$$

We set

$$\lambda = \inf\{J_{V,p}(u) : u \in H^1 \setminus \{0\}\}$$

and say that $u \in H^1$ solves the *variational equation* $Lu = \lambda u^{p-1}$ *weakly* if

$$\int_M \langle \nabla u, \nabla \varphi \rangle + \int_M Vu\varphi = \lambda \int_M u^{p-1}\varphi \quad \forall \varphi \in H^1.$$

Since $V \in L^\infty$, λ is finite. Indeed, if $V \geq -C$ a.e., then

$$E(u) \geq -C\|u\|_{L^2}^2 \geq -C\|u\|_{L^p}^2,$$

so λ is finite.

Note that $u \in L^{2^\star}$ by the Sobolev inequality, so that $u^{p-1} \in L^q$, where $q = 2^\star/(p-1)$. The Hölder conjugate of q is

$$q' = \frac{2^\star}{2^\star - p + 1} < 2^\star.$$

Therefore $q > (2^\star)'$ and $u^{p-1} \in (H^1)'$.

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and suppose $V \in L^\infty$. Then:*

(i) If $p \in [2, 2^\star)$, there exists a nontrivial weak solution of the variational equation.

(ii) If $p = 2^\star$ and $\lambda < K_n^{-2}$, where K_n is the optimal Sobolev constant of $H^1 \hookrightarrow L^{2^\star}$, there exists a nontrivial weak solution of the variational equation.

In both cases, the solution can be taken nonnegative (almost everywhere).

Remark 3.2. When $p = 2$, λ is just the principal eigenvalue of L (Rayleigh quotient) and this gives the existence of the principal eigenfunction.

Proof of (i). We write J for $J_{p,V}$ for simplicity. Since $J(u) = J(|u|)$ and $J(cu) = J(u)$ for every $u \in H^1 \setminus \{0\}$ and $c \in \mathbb{R} \setminus \{0\}$, we restrict attention to the set

$$\mathcal{H} = \{u \in H^1 : u \not\equiv 0, u \geq 0 \text{ a.e.}, \|u\|_{L^p} = 1\}.$$

So we now minimize the energy $E(u)$ over $u \in \mathcal{H}$. Let $(u_i) \subset \mathcal{H}$ be a minimizing sequence for E. The sequence is bounded in L^2 by Hölder's inequality and the gradients are bounded in L^2 because E is bounded on the sequence and

$$\left| \int_M V u_i^2 \right| \leq \|V\|_{L^\infty} \int_M u_i^2 \leq C\|V\|_{L^\infty},$$

again using Hölder's inequality. Now, since $p < 2^\star$, we may apply the Rellich compactness theorem [59] to select a subsequence (u_j) of (u_i) such that

(i) $u_j \rightharpoonup u$ in H^1,

(ii) $u_j \to u$ in L^p, and

(iii) $u_j \to u$ a.e.

for some $u \in H^1$. From (i) we conclude

$$\|u\|_{H^1} \leq \liminf_{j\to\infty} \|u_j\|_{H^1},$$

from (ii) we conclude $\|u\|_{L^p} = 1$, and from (iii) we conclude $u \geq 0$ a.e. We also have

$$\begin{aligned} \left| \int_M V u_j^2 - \int_M V u^2 \right| &= \left| \int_M V(u_j + u)(u_j - u) \right| \\ &\leq \|V\|_{L^\infty} \|u_j + u\|_{L^2} \|u_j - u\|_{L^2} \to 0 \end{aligned}$$

as $j \to \infty$. Thus

$$E(u) \leq \liminf_{j\to\infty} E(u_j) \implies E(u) = \lambda.$$

Since u is a minimum, it solves the Euler–Lagrange equations for J weakly. □

The proof here depends critically on the embedding of H^1 into L^p being compact. The fact that this is *not* true for $H^1 \hookrightarrow L^{2^\star}$ is what makes the Yamabe problem so difficult. We now consider the "critical" case.

Proposition 3.3. *The embedding $H^1 \hookrightarrow L^{2^\star}$ is never compact on a compact Riemannian manifold.*

Proof. We exhibit a bounded sequence of functions in H^1 with disjoint supports, but bounded away from zero in $L^{2^\star}$. Thus it will have no convergent subsequences in $L^{2^\star}$, and compactness fails.

Since the topologies of H^1 and $L^{2^\star}$ are independent of the particular metric, we choose to work with a metric which is flat in the coordinate domain $\Omega \subset M$. We identify Ω with its isometric image in $\mathbb{R}^n$. Let ϕ be a C_c^∞ function on $B_1(0)$ in $\mathbb{R}^n$. Let (a_i) be a sequence of points in Ω and (r_i) a bounded set of numbers such that the balls $B_i = B_{r_i}(a_i)$ are pairwise disjoint and contained in Ω. We define the functions

$$\phi_i(x) = r_i^{1-n/2}\phi(y), \quad x = a_i + r_i y.$$

Clearly $\phi_i \in C_c^\infty(B_i)$. Since $D_x\phi_i(x) = r_i^{-n/2} D_y\phi(y)$ and $dx = r_i^n\, dy$, we have

$$\int_\Omega |\phi_i(x)|^2\, dx = r_i^2 \int_{B_1(0)} |\phi(y)|^2\, dy$$
$$\int_\Omega |D_x\phi_i(x)|^2\, dx = \int_{B_1(0)} |D_y\phi(y)|^2\, dy$$

for all i, hence (ϕ_i) is bounded in H^1. On the other hand, we compute

$$\int_\Omega |\phi_i(x)|^{2^\star}\, dx = \int_{B_1(0)} |\phi(y)|^{2^\star}\, dy > 0$$

uniformly for all i. This completes the proof. □

A similar proof works for all "critical" Sobolev embeddings. See Example 6.12 in [1].

The proof of (ii) in Theorem 3.1 we give here is via a direct minimization of the functional and is due to Brezis and Lieb. We require a simple measure theoretic lemma due to these authors [25]:

Lemma 3.4 (Missing Term in Fatou's Lemma AKA Brezis–Lieb Lemma)**.** *Let (f_i) be a bounded sequence in $L^p(X,\Sigma,\mu)$ with $1 < p < \infty$. Suppose there exists a measurable function f such that $f_i \to f$ a.e. Then $f \in L^p$, $f_i \rightharpoonup f$ weakly in L^p and*

$$\int_X \big||f_i|^p - |f_i - f|^p - |f|^p\big| = o(1) \quad \text{as } i \to \infty.$$

Proof. That $f \in L^p$ follows immediately from the standard Fatou's lemma. We next show that $f_i \rightharpoonup f$. Let $g \in (L^p)' \cong L^{p'}$. If X has finite measure we may use Egorov's theorem and the absolute continuity of the integral to find a set $A \subset X$ such that $f_i \to f$ uniformly on A and $\|g\|_{L^{p'}(X \setminus A)} < \varepsilon$. Then

$$\begin{aligned}\left|\int_X f_i g - \int_X f g\right| &\le \int_X |(f_i - f)g| \\ &= \int_A |(f_i - f)g| + \int_{X\setminus A} |(f_i - f)g| \\ &\le C_1 \sup_A |f_i - f| + C_2\varepsilon \to 0\end{aligned}$$

as $\varepsilon \to 0$ and $i \to \infty$. If X does not have finite measure we may restrict attention to $\{|g| \ge n^{-1}\}$, which has finite measure, because

$$\int_{\{|g|<n^{-1}\}} |g|^{p'} \to 0$$

as $n \to \infty$.

The missing term in Fatou's lemma follows essentially from the following elementary fact: for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that

$$||a+b|^p - |b|^p| \le \varepsilon|b|^p + C|a|^p \quad \forall a, b \in \mathbb{R}. \tag{3.1}$$

This follows from the convexity of $t \mapsto |t|^p$. Indeed, for any $0 < \lambda < 1$,

$$|a+b|^p \le (|a| + |b|)^p \le (1-\lambda)^{1-p}|a|^p + \lambda^{1-p}|b|^p.$$

The claim now follows by choosing $\lambda = (1+\varepsilon)^{-1/(p-1)}$.

To apply this, write $g_i = f_i - f$ so that $g_i \to 0$ a.e. We claim that

$$G_{i,\varepsilon} := \max\{0, ||f + g_i|^p - |g_i|^p - |f|^p| - \varepsilon|g_i|^p\}$$

tends to zero in L^1 as $i \to \infty$. By (3.1),

$$\begin{aligned}|f + g_i|^p - |g_i|^p - |f|^p| &\le ||f + g_i|^p - |g_i|^p| + |f|^p \\ &\le \varepsilon|g_i|^p + (1 + C)|f|^p,\end{aligned}$$

which implies $G_{i,\varepsilon} \le (1 + C)|f|^p$. Since $G_{i,\varepsilon} \to 0$ a.e., the claim follows from the dominated convergence theorem. From the definition,

$$\int_X ||f + g_i|^p - |g_i|^p - |f|^p| \le \varepsilon \int_X |g_i|^p + \int_X G_{i,\varepsilon}.$$

Using convexity,

$$\int_X |g_i|^p \le 2^p \int_X (|f|^p + |f_i|^p) \le C.$$

Therefore,

$$\lim_{i\to\infty} \int_X ||f + g_i|^p - |g_i|^p - |f|^p| \le \varepsilon C$$

for any ε, which completes the proof. □

Proof of Theorem 3.1 (ii). Let (u_i) be a minimizing sequence for J in $\mathcal{H}$, which was defined before. The sequence is bounded, so we again take a subsequence (u_j) such that

(i) $u_j \rightharpoonup u$ in H^1,

(ii) $u_j \to u$ in L^2, and

(iii) $u_j \to u$ a.e.

Note that we do not claim the subsquence converges in $L^{2^\star}$, but just in L^2. From these properties we have $u \geq 0$ and

$$\lim_{j\to\infty} \int_M \langle \nabla u_j, \nabla u \rangle = \int_M |\nabla u|^2.$$

This is easily seen to imply

$$\int_M |\nabla u_j|^2 = \int_M |\nabla(u_j - u)|^2 + \int_M |\nabla u|^2 + o(1). \tag{3.2}$$

The Brezis–Lieb lemma gives

$$1 = \int_M |u_j|^{2^\star} = \int_M |u_j - u|^{2^\star} + \int_M |u|^{2^\star} + o(1). \tag{3.3}$$

We apply the Sobolev inequality Theorem 2.21, which says that for any $\varepsilon > 0$ there exists a $B \geq 0$ such that $\|v\|^2_{L^{2^\star}} \leq (K_n^2 + \varepsilon)\|\nabla v\|^2_{L^2} + B\|v\|^2_{L^2}$ for every $v \in H^1$. So by combining (3.2) and (3.3) we find

$$\left(1 - \int_M |u|^{2^\star}\right)^{2/2^\star} \leq (K_n^2 + \varepsilon)\left(\int_M |\nabla u_j|^2 - \int_M |\nabla u|^2\right) + o(1). \tag{3.4}$$

We claim this inequality shows u is not identically zero. Indeed, if $u \equiv 0$ then also $\nabla u \equiv 0$, and we have

$$1 \leq (K_n^2 + \varepsilon)\int_M |\nabla u_j|^2 + o(1)$$

and also

$$\lambda = \lim_{i\to\infty} E(u_i) = \lim_{i\to\infty} \int_M |\nabla u_i|^2.$$

We thus take $i \to \infty$ and $\varepsilon \downarrow 0$ to conclude $1 \leq K_n^2\lambda$, which contradicts the assumption $\lambda < K_n^{-2}$.

Next, we use the fact that $u_j \to u$ in L^2 to estimate

$$\begin{aligned}
\int_M |\nabla u_i|^2 - \int_M |\nabla u|^2 &= E(u_i) - E(u) + o(1) \\
&= \lambda - E(u) + o(1) \\
&= \lambda - \frac{E(u)}{\|u\|^2_{L^{2^\star}}}\|u\|^2_{L^{2^\star}} + o(1) \\
&\leq \lambda\bigl(1 - \|u\|^2_{L^{2^\star}}\bigr) + o(1),
\end{aligned}$$

because $\lambda \leq E(u)/\|u\|_{L^{2^\star}}^2 = J(u)$. We insert this into the Sobolev inequality (3.4) to obtain

$$\left(1 - \int_M |u|^{2^\star}\right)^{2/2^\star} \leq K_n^2 \lambda \big(1 - \|u\|_{L^{2^\star}}^2\big)$$

as $i \to \infty$ and $\varepsilon \downarrow 0$. Since $u_i \rightharpoonup u$ in $L^{2^\star}$, $\|u\|_{L^{2^\star}} \leq 1$. From the concavity of the function $b \mapsto |b|^{2/2^\star}$ [1],

$$\begin{aligned} 1 &= \left(1 - \int_M |u|^{2^\star} + \int_M |u|^{2^\star}\right)^{2/2^\star} \\ &\leq \left(1 - \int_M |u|^{2^\star}\right)^{2/2^\star} + \left(\int_M |u|^{2^\star}\right)^{2/2^\star}, \end{aligned}$$

hence

$$\left(1 - \int_M |u|^{2^\star}\right)^{2/2^\star} \leq K_n^2 \lambda \left(1 - \int_M |u|^{2^\star}\right)^{2/2^\star}.$$

Since $K_n^2\lambda < 1$, $\|u\|_{L^{2^\star}}^2 = 1$ and $u \in \mathcal{H}$. Finally,

$$E(u) \leq \liminf_{i\to\infty} E(u_i),$$

whence $E(u) = \lambda$, so u is in fact a minimizer and a weak solution of the variational equation by standard arguments. □

3.2 Regularity

In this section we derive the regularity theory for the equations considered in the previous section. When $p > 2$, the nonlinearity causes the source term to lie in a worse space than u. After a little work, the L^p theory of elliptic equations works in the subcritical cases. In the critical case, it will turn out that "standard" arguments give nothing – one has to use a trick of Trudinger's [117] to get anything. We first present proofs based on the L^p theory of elliptic equations as found in Gilbarg and Trudinger [53] and Morrey [91].

Theorem 3.5 (Subcritical Regularity)**.** *Let (M, g) be a closed Riemannian manifold, $n \geq 3$, $p \in [2, 2^\star)$, and let $u \in H^1$, $u \geq 0$, be a weak solution to $Lu = \lambda u^{p-1}$. If $V \in L^\infty$, then $u \in W^{2,q}$ for all $q \geq 1$.*

Proof. We will prove that $u \in \bigcap_{q\geq 1} L^q$ via a bootstrapping procedure. Choosing $q = 2(p-1)$ then gives $u \in W^{2,2}$ by Theorem 8.8 in [53], then the L^p theory, Theorem 9.19 in [53], gives $u \in W^{2,q}$ for every $q \geq 1$.

We begin by writing $p_1 = 2^\star$. By the Sobolev embedding theorem, $u \in L^{p_1}$. Clearly $u^{p-1} \in L^{p_1/(p-1)}$ and also $u \in L^{p_1/(p-1)}$ since $L^{p_1/(p-1)} \subset L^{p_1}$ (note that $p - 1 > 1$). If we write $Lu = \lambda u^{p-1}$ as

$$-\Delta u = -Vu + \lambda u^{p-1},$$

[1] In general, if f is a concave function with $f(0) \geq 0$, then $f(a + b) \leq f(a) + f(b)$ for $a, b \geq 0$.

the right-hand side is in $L^{p_1/(p-1)}$, so from Theorem 6.2.5 in [91] we have that $u \in W^{2,p_1/(p-1)}$. If $p_1/(n-1) > n/2$, then $u \in C^{0,\alpha}$ by Morrey's inequality and we are done. If $p_1/(n-1) = n/2$, then $u \in \bigcap_{q \geq 1} L^q$, so we would be done again. So we focus on the remaining case $p_1/(p-1) < n/2$.

In this case the Sobolev inequality gives $u \in L^{p_2}$, where

$$p_2 = \frac{np_1}{n(p-1) - 2p_1}.$$

Now $u, u^{p-1} \in L^{p_2/(p-1)}$, and we may repeat the argument, assuming at each step that u is not in every L^q space. Namely, we obtain a sequence (p_j) such that $p_1 = 2^\star$, $p_j \leq n(p-1)/2$ for all j,

$$p_{j+1} = \frac{np_j}{n(p-1) - 2p_j},$$

and $u \in L^{p_j}$ for all j. By direct computation one checks that this sequence is increasing. Therefore $p_j > n(p-2)/2$ since $p_1 > n(p-2)/2$, which follows easily from $p_1 < 2^\star = 2n/(n-2)$. We now assume the sequence is bounded, for otherwise u is in every L^q space. But then $p_\infty = \lim p_j < \infty$ must satisfy

$$p_\infty = \frac{np_\infty}{n(p-1) - 2p_\infty}.$$

This can be solved for $p_\infty = n(p-2)/2$, which is a contradiction because (p_j) is increasing and lies above $n(p-2)/2$. Therefore $u \in \bigcap_{q \geq 1} L^q$. □

Regularity in the critical case $p = 2^\star$ is tricky. Namely, if $u \in L^{2^\star}$, then $u^{2^\star - 1} \in L^{2^\star/(2^\star - 1)}$, so the regularity theory gives $u \in W^{2,2^\star/(2^\star-1)}$. But then the Sobolev inequality just gives $u \in L^{2^\star}$, so nothing new. However, if we somehow get a little more integrability for u, then the bootstrap procedure works. We employ a neat cutoff argument due to Trudinger and Struwe.

Theorem 3.6 (Critical Regularity). *Let (M, g) be a closed Riemannian manifold, $n \geq 3$, and let $u \in H^1$, $u \geq 0$, be a weak solution to $Lu = \lambda u^{2^\star - 1}$. If $h \in L^\infty$, then $u \in W^{2,q}$ for every $q \geq 1$.*

Proof. We show that $u \in L^r$ for some $r > 2^\star$. Then the previous remark implies the bootstrap procedure of the theorem applies and the corollary gives the result. If $p = 2/(n-2)$, we aim to show that $u \in L^{2^\star(p+1)}$.

For $k \in \mathbb{N}$ we let $f_k : [0, \infty) \to [0, \infty)$ be defined by

$$f_k(x) = \begin{cases} x^{p+1} & x \leq k \\ kx & x > k \end{cases}.$$

This function is Lipschitz, so by standard Sobolev space theory we have $v_k := f_k \circ u \in H^1$. Note that $v_k = u \min\{u^p, k\}$. Similarly, the function $w_k := u \min\{u^{2p}, k^2\}$ is in H^1. Computations yield:

$$\int_M \langle \nabla u, \nabla w_k \rangle = \int_M |\nabla u|^2 \min\{u^{2p}, k^2\} + 2p \int_{\{u^p \leq k\}} u^{2p} |\nabla u|^2,$$

$$\int_M |\nabla v_k|^2 = \int_M |\nabla u|^2 \min\{u^{2p}, k^2\} + p^2 \int_{\{u^p \le k\}} u^{2p} |\nabla u|^2.$$

So we clearly have

$$\int_M |\nabla v_k|^2 \le C \int_M \langle \nabla u, \nabla w_k \rangle \tag{3.5}$$

for each k and C depending only on p. Note that $w_k \le u^{2p+1}$ and $2p+2 = 2^\star$, hence

$$\left| \int_M V u w_k \right| \le \|V\|_{L^\infty} \left| \int_M u^{2^\star} \right| < \infty. \tag{3.6}$$

The plan is to use w_k as a test function in the weak equation, namely we have

$$\int_M \langle \nabla u, \nabla w_k \rangle + \int_M V u w_k = \lambda \int_M u^{2^\star - 1} w_k. \tag{3.7}$$

Using (3.5), (3.6), and (3.7), we estimate

$$\int_M |\nabla v_k|^2 \le C \left(1 + \int_M u^{2^\star} \min\{u^{2p}, k^2\} \right). \tag{3.8}$$

We now choose another cutoff b for u, which we use estimate the last term in (3.8)

$$\begin{aligned} \int_M u^{2^\star} \min\{u^{2p}, k^2\} &\le \int_{\{u \le b\}} u^{2^\star} \min\{u^{2p}, k^2\} + \int_{\{u \ge b\}} u^{2^\star} \min\{u^{2p}, k^2\} \\ &= b^{2^\star + 2p} \operatorname{vol}(M) + \int_{\{u \ge b\}} u^{2^\star} \min\{u^{2p}, k^2\}. \end{aligned} \tag{3.9}$$

Now we use Hölder's inequality on the last term in (3.9):

$$\int_{\{u \ge b\}} u^{2^\star} \min\{u^{2p}, k^2\} \le \left(\int_{\{u \ge b\}} u^{2^\star} \right)^{(2^\star - 2)/2^\star} \left(\int_{\{u \ge b\}} v_k^{2^\star} \right)^{2/2^\star}. \tag{3.10}$$

By convexity,

$$v_k^{2^\star} \le 2^{2^\star} (|v_k - \overline{v_k}|^{2^\star} + \overline{v_k}^{2^\star}) \le C(1 + |v_k - \overline{v_k}|^{2^\star}), \tag{3.11}$$

where the second inequality follows from the fact that $u \in L^{2^\star}$ and $\overline{v_k}$ is thus bounded. Combining the Sobolev and Poincaré inequalities gives

$$\begin{aligned} \left(\int_M |v_k - \overline{v_k}|^{2^\star} \right)^{2/2^\star} &\le \|v_k - \overline{v_k}\|_{H^1}^2 \\ &= \int_M |v_k - \overline{v_k}|^2 + \int_M |\nabla v_k|^2 \\ &\le \left(1 + \frac{1}{\lambda_1} \right) \int_M |\nabla v_k|^2, \end{aligned} \tag{3.12}$$

where λ_1 is the first nonzero eigenvalue of the Laplacian on M. Combining (3.10), (3.11), and (3.12) gives

$$\int_{\{u \ge b\}} u^{2^\star} \min\{u^{2p}, k^2\} \le C I(b)^{(2^\star - 2)/2^\star} \left(\operatorname{vol}(\{u \ge b\}) + \int_M |\nabla v_k|^2 \right), \tag{3.13}$$

where

$$I(b) = \int_{\{u \geq b\}} u^{2^\star}.$$

Now take (3.8), estimate the second term using (3.9), and the second term in (3.9) using (3.13) to obtain

$$\begin{aligned}\int_M |\nabla v_k|^2 &\leq C\left[1 + b^{2^\star+2p}\,\mathrm{vol}(M) + CI(b)^{(2^\star-2)/2^\star}\left(\mathrm{vol}(\{u \geq b\}) + \int_M |\nabla v_k|^2\right)\right] \\ &= CI(b)^{(2^\star-2)/2^\star}\int_M |\nabla v_k|^2 + R(b), \qquad (3.14)\end{aligned}$$

where $R(b)$ is a number depending on b, but not k. Since $u \in L^{2^\star}$, $I(b) \to 0$ as $b \to \infty$. Choose b sufficiently large that $CI(b)^{(2^\star-2)/2} < 1$, so that (3.14) gives

$$\int_M |\nabla v_k|^2 \leq C$$

for all k. Since

$$\int_M v_k^2 \leq C$$

trivially, we get from the Sobolev inequality that

$$\int_M v_k^{2^\star} \leq C$$

for some $C > 0$ and all k. Now the monotone convergence theorem gives

$$\int_M u^{2^\star(p+1)} \leq C.$$

This completes the proof of the theorem. □

Corollary 3.7. *Let u solve $Lu = \lambda u^{p-1}$ weakly, $p \in (2, 2^\star]$, but suppose that $h \in C^{0,\alpha}$. Then $u \in C^{2,\alpha}$ and is either strictly positive or identically zero.*

If $h \in C^\infty$, then $u \in C^\infty$.

Proof. Since $u \in W^{2,q}$ for $q \geq 1$ arbitrarily large, we in fact have $u \in C^{0,\alpha}$ by Morrey's inequality and $u \in L^\infty$. Since $[0,\infty) \ni a \mapsto a^{p-1}$ is locally Hölder continuous, $u^{p-1} \in C^{0,\alpha}$. We thus have $\Delta u \in C^{0,\alpha}$ and the result follows from [53] Theorem 9.19 and the maximum principle.

Suppose h is now smooth. If u vanishes identically, we are done. But if not, then the nonlinearity is bounded away from zero and is as smooth as u. Thus the result follows from bootstrapping. □

Theorem 3.8 (Maximum Principle)**.** *Let (M, g) be a compact Riemannian manifold. If $\psi \in C^2(M)$ is nonnegative, and satisfies $\Delta\psi \leq \psi f(\cdot, \psi)$ where $f : M \times \mathbb{R} \to \mathbb{R}$ is continuous, then ψ is either strictly positive or identically zero.*

If $\psi \in C^2(\overline{\Omega})$ for some $\Omega \subset M$ satisfies $\Delta\psi \leq \psi f(\cdot, \psi)$, then the infimum of ψ over Ω is attained on $\partial\Omega$, and ψ is either zero everywhere or $\psi > 0$ in Ω.

Proof. Since ψ is continuous and M is compact, we have $\Delta\psi + a\psi \leq 0$ everywhere for some constant $a < 0$. If we define $L = \Delta + a$, then $L\psi \leq 0$ implies via the strong maximum principle [53] Theorem 3.5 that ψ cannot achieve a nonpositive minimum unless it is identically zero.

To prove the second statement, proceed as above. Then the strong maximum principle says ψ cannot attain its infimum in the interior unless it is constant. □

Remark 3.9. We have chosen to state this as a separate theorem because the maximum principle in [53] has a sign requirement on the reaction term which is kind of a red herring.

In Chapter 6 we will need to solve the Yamabe equation on a manifold with boundary. Here is the regularity theorem for that situation:

Theorem 3.10. *Let u solve $Lu = \lambda u^{2^\star - 1}$ weakly in $H_0^1(M)$, where $(\overline{M}, g)$ is a compact Riemannian manifold with smooth boundary. If $V \in C^\infty(\overline{M})$, then $u \in C^\infty(\overline{M})$.*

The proof is done by breaking up u in Lu using a cutoff function, and then treating the nonlinearity as a source term, exactly as in the proof of Theorem 3.5. Only this time, we use Theorem 6.4.8 in [91] to increase regularity at each step. The method of Theorem 3.6 then gives the result, together with Poincaré's inequality for H_0^1 (which is equivalent to the principal eigenvalue of the Laplacian being positive on a compact manifold with boundary).

The following result is useful in controlling possibly singular functions and will be used in the proof of the positive mass theorem.

Lemma 3.11 (Removable Singularities Theorem)**.** *Let Ω be an open set in (M, g) and $p \in \Omega$. Suppose u is a distributional solution of $Lu = 0$ in $\Omega \setminus \{p\}$, with $V \in L^{n/2}(\Omega)$ and $u \in L^q(\Omega)$ for some $q > 2^\star/2$. Then u satisfies $Lu = 0$ distributionally in all of Ω. If $V \in C^\infty(\Omega)$, then u is in fact smooth in all of Ω.*

Proof. We need to show that

$$\int_\Omega u\, L\phi = 0$$

for any $\phi \in C_c^\infty(\Omega)$. Let η be the usual cutoff function for the unit ball, and define $\eta_\varepsilon(x) = \eta(r/\varepsilon)$. Then η_ε is supported in $B_{2\varepsilon}(p)$, which is contained in Ω for $\varepsilon > 0$ small enough. Since $(1 - \eta_\varepsilon)\phi$ is compactly supported in $\Omega \setminus \{p\}$ and $Lu = 0$ there,

$$\int_\Omega u\, L\phi = \int_\Omega u\, L(\eta_\varepsilon \phi).$$

We will show that the RHS goes to zero as ε does.

Since $Vu \in L^1$ by Hölder's inequality, the potential term vanishes as the support of η_ε shrinks. As for the kinetic term,

$$\Delta(\eta_\varepsilon \phi) = \phi \Delta \eta_\varepsilon + \eta_\varepsilon \Delta \phi + 2\langle \nabla \eta_\varepsilon, \nabla \phi \rangle.$$

We have $|\nabla\eta_\varepsilon| \le C/\varepsilon$ and $|\Delta\eta_\varepsilon| \le C/\varepsilon^2$. Therefore, if q' denotes the Hölder conjugate of q,

$$\begin{aligned}\left|\int_\Omega u\Delta(\eta\phi)\right| &\le \frac{C}{\varepsilon^2}\int_{B_{2\varepsilon}}|u| \\ &\le \frac{C}{\varepsilon^2}\|u\|_{L^q}\left(\mathrm{vol}(B_{2\varepsilon})\right)^{1/q'} \\ &\le C\varepsilon^{-2+n/q'}\|u\|_{L^q}.\end{aligned}$$

Since $q > 2^\star/2$ implies $n/q' > 2$, this goes to zero with ε.

When $V \in C^\infty$, then $u \in C^\infty$ by a standard regularity theorem for distributional equations, see Corollary 7.20 in [55]. □

Note that the value of q was chosen so that u does not blow up quicker than the Green's function, which is $\sim r^{2-n}$ at the pole.

3.3 A Historical Remark

In the original proof, Yamabe couldn't directly obtain a solution to the Yamabe equation, but he did know how to solve the "subcritical" equations

$$-a\Delta u + Ru = \mu_q u^{q-1}, \quad q \in [2, 2^\star).$$

His idea was to take a limit $q \to 2^\star$ and obtain a solution of the critical equation. This works when $\mu < 0$ (the $\mu = 0$ case is more or less trivial), but he made a mistake when $\mu > 0$. This mistake was pointed out by Trudinger [117].

The idea is to use the Green's function for the Laplacian to write

$$u_q(x) = \fint_M u_q + \int_M G(x,y)a^{-1}[\mu_q u_q^{q-1}(y) - R(y)u_q(y)]\,d\mu(y).$$

Then, if the μ_q's and u_q's are uniformly bounded, the Arzela–Ascoli theorem together with basic estimates on the Green's function gives a uniformly convergent limit of the u_q's (up to a subsequence). The boundedness of the u_q's follows from the Hardy–Littlewood–Sobolev inequality. Aubin then showed how to guarantee the limit is nontrivial as long as $\mu < \Lambda$ (see the following chapter for the notation).

4

The Yamabe Problem

The goal of this chapter is to show that we can always make a conformal change of g to obtain a metric with constant scalar curvature.

$\boxed{\text{Assume } n = \dim M \geq 3 \text{ in this chapter.}}$

4.1 The Yamabe Equation

Recall from Chapter 1:

Theorem 4.1 (Yamabe's Equation). *If $\varphi \in C^\infty(M)$ is strictly positive and satisfies*

$$L\varphi = \mu\varphi^{(n+2)/(n-2)}$$

for a constant μ, then $\varphi^{4/(n-2)}g$ has constant scalar curvature μ.

This is exactly the kind of equation we showed how to solve in the previous chapter. Note that the normalization here is a little different, with

$$L = -a\Delta + R, \quad a = 4\frac{n-1}{n-2}.$$

Accordingly, we define

$$J(u) = \frac{E(u)}{\|u\|_{L^{2^\star}}^2}, \quad E(u) = a\int_M |\nabla u|^2 + \int_M Ru^2.$$

We define the *Yamabe energy* of (M, g) by

$$\mu(M, g) = \inf\{J(u) : u \in H^1(M) \setminus \{0\}\}.$$

In the notation of the previous chapter, we have $\mu = a\lambda$, where λ is the infimum of $J_{2^\star, R/a}$. We also set

$$\Lambda = \frac{a}{K_n^2} = n(n-1)\omega_n^{n/2},$$

where
$$K_n = K(n,2) = \frac{2}{n(n-2)\omega_n^{1/n}}$$
is the best constant in the Sobolev inequality and ω_n is the volume of the unit sphere in $\mathbb{R}^{n+1}$. From Theorem 3.1, we obtain immediately:

Theorem 4.2. *If $\mu(M) < \Lambda$, there exists a strictly positive C^∞ solution u of the Yamabe equation with $\|u\|_{L^{2^\star}} = 1$ and*
$$R(u^{4/(n-2)}g) = \mu(M).$$

Yamabe showed this when $\mu \leq 0$ [123] and Aubin showed it when $0 < \mu < \Lambda$ [6]. As mentioned earlier, the proof given here is due to Brezis–Lieb [25].

Corollary 4.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and with Yamabe energy $\mu \leq 0$. Then g is conformal to a metric with scalar curvature of constant sign equal to the sign of μ.*

The difficulty in completing the proof of Yamabe's conjecture is the $\mu > 0$ case. Before attacking that, we list some simple but important observations.

Proposition 4.4. *The following hold for any compact Riemannian manifold:*

(i) The sign of μ agrees with the sign of the principal eigenvalue λ_1 of L.
(ii) The Yamabe energy μ is a conformal invariant.

Proof. (i) If $\lambda_1 \geq 0$, then
$$E(u) \geq \lambda_1 \|u\|_{L^2}^2 \geq 0$$
for any $u \in H^1$, by the Rayleigh eigenvalue formula. Thus $\mu \geq 0$. If $\lambda_1 = 0$, take $u \in \ker L$. Then $J(u) = 0$, so $\mu = 0$. If $\lambda_1 < 0$, let u be the associated eigenfunction. Then
$$E(u) = \lambda_1 \|u\|_{L^2}^2 < 0,$$
so $\mu < 0$. If $\mu = 0$, Theorem 3.1 shows that L has a kernel. Thus $\lambda_1 > 0$ cannot imply $\mu = 0$, so in that case $\mu > 0$.

(ii) This follows from the construction of the Yamabe functional in Chapter 1. It can also be verified by a direct computation. □

Theorem 4.5. *For any compact Riemannian manifold of dimension $n \geq 3$, if $\mu > 0$, then the metric is conformal to a metric with strictly positive scalar curvature.*

Proof. By Theorem 3.1 and the regularity theory, there exists a smooth positive solution of the eigenvalue problem $Lu = \lambda_1 u$. By Proposition 4.4, $\lambda_1 > 0$. By the conformal curvature formula,
$$\lambda_1 u = R(u^{4/(n-2)}g)u^{(n+2)/(n-2)}.$$
Therefore, the conformal metric has strictly positive scalar curvature. □

In applications this turns out to be a very useful result. For example:

Corollary 4.6. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with scalar curvature $R \geq 0$ but $R(p) > 0$ for some $p \in M$. Then g is conformal to a metric with strictly positive scalar curvature.*

Proof. We clearly have $\mu \geq 0$ because $R \geq 0$. We claim that $\mu > 0$, and then the result follows from the theorem. If $\mu = 0$, then the principal eigenvalue of L, λ_1, must vanish. By elliptic theory there is a positive $u \in C^\infty$ such that $Lu = \lambda_1 u = 0$. We now integrate $uLu = 0$ over M and apply the divergence theorem to obtain

$$0 = a \int_M |\nabla u|^2 + \int_M Ru^2.$$

This contradicts the fact that R is positive in some open set. Hence $\mu > 0$. □

This result is critical to Lohkamp's compactification argument for the positive mass theorem, see Section 7.3.

Theorem 4.7 (Weak Yamabe Theorem)**.** *Let (M, g) be a compact Riemannian manifold with $n \geq 3$. Then g is conformal to a metric $\tilde{g}$ such that $R(\tilde{g})$ has constant sign if and only if $R(\tilde{g})$ has the same sign as $\mu(M)$.*

Proof. We have seen that if μ has a certain sign, there is a conformal metric with the same sign. We thus have to show that if the conformal metric g's scalar curvature R has constant sign, it must be the same sign as μ. For this, we use Proposition 4.4: If R is identically zero, then $\lambda_1 = 0$ and $\mu = 0$. If $R > 0$, we compute as in Corollary 4.6 to conclude $\lambda_1 > 0$, hence $\mu > 0$. If $R < 0$, then we estimate

$$\mu \leq J(1) = \int_M R < 0,$$

which concludes the proof. □

This is one of the main workhorse theorems used in the positive case because then we can assume R is strictly positive. It is easily shown that the conformal Laplacian is strictly coercive in the sense of Gårding when $R > 0$.

4.2 Models for the Positive Case

In this chapter we begin the difficult task of solving the Yamabe problem when the Yamabe energy $\mu(M)$ is positive. We begin with the model cases: S^n and $\mathbb{R}^n$. Of course $\mathbb{R}^n$ is not a compact manifold, but we can define its Yamabe energy by using functions in $C_c^\infty(\mathbb{R}^n)$. The resulting expression will be obviously related to the best constant in the Sobolev inequality. We show how the best constant is related to solving the Yamabe equation when $\mu(M) < \mu(S^n)$. The rest of the chapter is devoted to showing that for manifolds not conformal to the sphere, $\mu(M) < \mu(S^n)$.

Let δ be the Euclidean metric $(dx^1)^2 + \cdots + (dx^n)^2$ on $\mathbb{R}^n$. We define the Yamabe energy of $(\mathbb{R}^n, \delta)$ by

$$\inf \frac{a \displaystyle\int_M |\nabla u|_\delta^2 \, dx}{\left(\displaystyle\int_M |u|^{2n/(n-2)} \, dx \right)^{(n-2)/n}}, \tag{4.1}$$

where the infimum is taken over those $u \in C_c^\infty(\mathbb{R}^n)$ that are not identically zero. Equivalently, the infimum is over those $u \in H^1(\mathbb{R}^n)$ that are not identically zero. By comparison with the optimal Sobolev inequality (Theorem 2.1), we see at once that the Yamabe energy of $\mathbb{R}^n$ is

$$\Lambda = \frac{a}{K_n^2}.$$

The Euler-Lagrange equation corresponding to (4.1) is of course just the Yamabe equation in $\mathbb{R}^n$:

$$-a\Delta u = \Lambda u^{2^\star - 1}.$$

A remarkable result of Caffarelli, Gidas, and Spruck [26] says that the sole nontrivial, non-negative solutions of this equation in $C^2(\mathbb{R}^n)$ are

$$u_{\lambda, x_0}(x) = \left(\sqrt{\frac{2}{\omega_n^{1/n}}} \frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{(n-2)/2} \qquad \text{for any } \lambda > 0, x_0 \in \mathbb{R}^n.$$

This result could have been anticipated from the proof of the best constant theorem.

Now consider the energies $\mu(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^n$, defined by taking the infimum in the Yamabe quotient over $C_c^\infty(\Omega) \setminus \{0\}$. We will later associate a Yamabe energy to any Riemannian manifold in this manner.

Proposition 4.8. *Let Ω be any bounded open set in $\mathbb{R}^n$. Then $\mu(\Omega) = \Lambda$.*

Proof. It is easy to see that $\mu(\Omega)$ is translation invariant, i.e. $\mu(\Omega + x) = \mu(\Omega)$ for any $x \in \mathbb{R}^n$. Also, if $\Omega \subset \Omega'$, then $\mu(\Omega) \geq \mu(\Omega')$ since $C_c^\infty(\Omega) \subset C_c^\infty(\Omega')$. So we may assume that $B_{r_1}(0) \subset \Omega \subset B_{r_2}(0)$, for $0 < r_1 < r_2$, which implies $\mu(B_{r_1}(0)) \geq \mu(\Omega) \geq \mu(B_{r_2}(0))$. However, since the Yamabe energy is scale invariant, $\mu(B_r(0))$ is a constant independent of r. But since any $u \in C_c^\infty(\mathbb{R}^n)$ has support in some $B_r(0)$, we have that $\mu(B_r(0)) = \Lambda$ for any r. Thus $\mu(\Omega) = \Lambda$. $\square$

We now consider the sphere S^n with the standard round metric $\mathcal{S}$. The solution of the Yamabe problem for the sphere is:

Theorem 4.9. *The Yamabe functional on $(S^n, \mathcal{S})$ is minimized by constant multiples of the standard metric and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on S^n that have constant scalar curvature.*

Let $N = (0, \ldots, 0, 1)$ denote the north pole on $S^n \subset \mathbb{R}^{n+1}$. The *stereographic projection* $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$ is defined by $\sigma(\zeta^1, \ldots, \zeta^n, \xi) = (x^1, \ldots, x^n)$, where $x^i = \zeta^i/(1 - \xi)$. A computation shows that

$$(\sigma^{-1})^* \mathcal{S} = \frac{4}{(1 + |x|^2)^2} \delta,$$

i.e. σ is a conformal diffeomorphism [113]. Note the similarity between the term in front of δ and the CGS solution above.

To begin the proof of Theorem 4.9, we prove a uniqueness result due to Obata [95].

Theorem 4.10. *If g is a metric on S^n that is conformal to the standard metric $\mathcal{S}$ and has constant scalar curvature, then up to a constant, g is obtained from $\mathcal{S}$ by a conformal diffeomorphism of the sphere.*

Proof. We begin by showing that g is Einstein. Note that $\mathcal{S}$ is Einstein since the round sphere has constant sectional curvature. Since $g \in [\mathcal{S}]$, there is a strictly positive smooth function φ on S^n such that $\mathcal{S} = \varphi^{-2} g$. The conformal transformation law for the Ricci tensor gives

$$R_{ij}(\mathcal{S}) = R_{ij}(g) + \varphi^{-1}\left((n-2)\nabla^g_{ij}\varphi - (n-1)\frac{|\nabla\varphi|^2_g}{\varphi} g_{ij} + \Delta_g \varphi g_{ij}\right).$$

If $T_{ij} = R_{ij} - R g_{ij}/n$ denotes the traceless Ricci tensor, then

$$0 = T_{ij}(\mathcal{S}) = T_{ij}(g) + (n-2)\varphi^{-1}(\nabla^g_{ij}\varphi + n^{-1}\Delta_g\varphi g_{ij})$$

since $\mathcal{S}$ is Einstein. Since the scalar curvature of g is constant, the contracted Bianchi identity $dR = 2\,\mathrm{div}\,\mathrm{Ric}$ implies $\mathrm{div}\,\mathrm{Ric} = 0$, and thus $\mathrm{div}\,T = 0$ as well. We now show that T is identically zero. Indeed, because T is traceless,

$$\begin{aligned}
\int_{S^n} \varphi |T|^2 &= \int_{S^n} \varphi T_{ij} T^{ij} \\
&= -(n-2)\int_{S^n} T^{ij}(\nabla^g_{ij}\varphi + n^{-1}\Delta_g\varphi g_{ij}) \\
&= -(n-2)\int_{S^n} T^{ij}\nabla^g_{ij}\varphi \\
&= (n-2)\int_{S^n} \nabla\varphi \cdot \mathrm{div}\,T = 0,
\end{aligned}$$

where all integrals are with respect to $d\mu(g)$. This implies $T=0$ identically, hence $\mathrm{Ric}(g) = (R/n)g$, which implies by Schur's lemma that g is Einstein.

Since g is conformal to the standard metric $\mathcal{S}$ on the sphere, which is locally conformally flat, we additionally have $\mathrm{Weyl}(g) = 0$. From the formula of the Weyl tensor, we can now show that $\mathrm{Riem}(g)$ is proportional to $g \owedge g$, which implies (S^n, g) is a space form. From the space form theorem, (S^n, g) is isometric to a scaled version of $(S^n, \mathcal{S})$. □

The next result was proved by Aubin in [6].

Theorem 4.11. *The Yamabe energy of the round sphere is $\Lambda = n(n-1)\omega_n^{n/2}$ and the infimum of the Yamabe functional is attained.*

Proof. We will show below independently that $\mu(M) \leq \Lambda$ for any compact Riemannian manifold M. We thus have to show $\mu(S^n) \geq \Lambda$. Let u be a smooth, nonnegative function on

S^n. If we can show that $J_{S^n}(u) \geq \Lambda$, we have proved the first part of the theorem. Define the function ρ on $\mathbb{R}^n$ by

$$\rho^{2^\star-2}(x) = \frac{4}{(1+|x|^2)^2}.$$

Then the conformal invariance of the Yamabe functional implies $J_{S^n}(u) = J_{\mathbb{R}^n}(\overline{u})$, where $\overline{u} = \rho\, u \circ \sigma^{-1}$. We claim that there exists a sequence $(\overline{u}_i) \subset C_c^\infty$ such that $J_{\mathbb{R}^n}(\overline{u}_i) \to J_{\mathbb{R}^n}(u)$ as $i \to \infty$. Since $J_{\mathbb{R}^n}(\overline{u}_i) \geq \Lambda$ by (4.1), this sequence proves $\mu(S^n) \geq \Lambda$. To construct the sequence, let $\eta_i \in C_c^\infty(\mathbb{R}^n)$ be a cutoff function identically one for $|x| \leq i$, and identically zero for $|x| \geq 2i$, with $\|D\eta_i\|_{L^\infty} \leq C/i$. If we define $\overline{u}_i = \eta_i \overline{u}$, then it is clear that $\|\overline{u}_i\|_{L^{2^\star}} \to \|\overline{u}\|_{L^{2^\star}}$ by the monotone convergence theorem, so we just have to take care of the gradient term. We estimate:

$$\int_{\mathbb{R}^n} |D\overline{u}_i|^2 = \int_{\mathbb{R}^n} \overline{u}^2 |D\eta_i|^2 + \int_{\mathbb{R}^n} \eta_i^2 |D\overline{u}|^2 + 2\int_{\mathbb{R}^n} \eta_i \overline{u} \langle D\eta_i, D\overline{u}\rangle. \tag{4.2}$$

The first term tends to zero:

$$\int_{\mathbb{R}^n} \overline{u}^2 |D\eta_i|^2 \leq \frac{C}{i} \int_{\mathbb{R}^n} \overline{u}^2 \to 0$$

because $\overline{u} \in L^2(\mathbb{R}^n)$. To see this, note that $u \circ \sigma^{-1}$ is bounded,

$$\rho(x) = \frac{C}{(1+|x|^2)^{(n-2)/2}},$$

hence $\rho \in L^2(\mathbb{R}^n)$. The second term in (4.2) converges to $\int |D\overline{u}|^2$ by the monotone convergence theorem. For the third term in (4.2), we estimate:

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} \eta_i \overline{u} \langle D\eta_i, D\overline{u}\rangle \right| &\leq \int_{\mathbb{R}^n} \overline{u} |\langle D\eta_i, D\overline{u}\rangle| \\
&\leq \int_{\mathbb{R}^n} \overline{u} |D\eta_i||D\overline{u}| \\
&\leq \frac{C}{i} \|\overline{u}\|_{L^2} \|D\overline{u}\|_{L^2} \to 0.
\end{aligned}$$

This completes the proof that $\mu(S^n) = \Lambda$.

Recall that the scalar curvature of $(S^n, \mathcal{S})$ is $n(n-1)$. This follows from direct computation or any number of tricks for constant curvature spaces. We exhibit a solution u of the geometric Yamabe equation

$$-\Delta u + \frac{n(n-2)}{4} u = u^{2^\star-1},$$

which is just the conformal scalar curvature equation for the round sphere to itself, and Δ is with respect to the round metric. Aubin found that the functions

$$u_{x_0,\beta} = \left(\frac{n(n-2)}{4} (\beta^2 - 1) \right)^{(n-2)/4} (\beta - \cos r)^{(2-n)/2}$$

are solutions of the Yamabe equation, where $x_0 \in S^n$ and $\beta > 1$ are arbitrary, and $r = d_{\mathcal{S}}(x, x_0)$. He also showed that these, together with the constant limit as $\beta \to \infty$, are the sole positive solutions. Furthermore,

$$J_{\mathcal{S}}(u_{x_0,\beta}) = \Lambda$$

for all x_0 and β. These computations can be carried out in polar coordinate about x_0, and the uniqueness result follows in a straightforward manner from the CGS result. See [59] Theorem 5.1 for details on the calculation and [60] Theorem 4.1 for details on the uniqueness proof. $\square$

Theorem 4.9 now follows immediately from Theorems 4.10 and 4.11. The Yamabe energy Λ is actually maximal:

Theorem 4.12. *For any compact Riemannian manifold (M, g), $\mu(M) \leq \Lambda$.*

Proof. The inequality $\mu(M) \leq \Lambda$ is equivalent to the fact that any constant A in the Sobolev inequality has to be such that $A \geq K_n$. $\square$

In his paper [6], Aubin showed that this conditions holds whenever $n \geq 6$ and (M, g) is not locally conformally flat. We will give a simple version of his proof of this result in the next section. Since "most" manifolds are of this type, Aubin's result was quite successful. Choosing the test function $u = 1$, we have a basic result:

Proposition 4.13. *If $\int_M R \leq \Lambda$, there exists a conformal metric with constant scalar curvature.*

When equality holds either, $\mu < \Lambda$ and Aubin's theorem applies, or $\mu = \Lambda$ and 1 is a minimizer. In that case it follows from Aubin's conjecture that (M, g) is conformal to $(S^n, \mathcal{S})$.

Conjecture 4.14 (Aubin)**.** *For any compact manifold (M, g) of dimension $n \geq 3$ not conformally diffeomorphic to $(S^n, \mathcal{S})$, $\mu(M) < \Lambda$.*

Schoen [101] completed the proof of this conjecture in the remaining cases. We prove this using Lee and Parker's method [78].

4.3 Conformal Normal Coordinates

In this section we show how to get better decay of metrical quantities in normal coordinates by passing to a conformal metric. These estimates are essential for understanding the expansions in the next section. Here is the classical expansion for the metric determinant in normal coordinates:

Proposition 4.15. *Let (M,g) be a Riemannian manifold. In normal coordinates $(x^1,\dots,x^n)$ about $p \in M$ with radial coordinate $r = ((x^1)^2 + \cdots + (x^n)^2)^{1/2}$, we have*

$$\begin{aligned}\det g = 1 &- \frac{1}{3}R_{ij}x^ix^j - \frac{1}{6}R_{ij;k}x^ix^jx^k \\ &- \left(\frac{1}{20}R_{ij;kl} + \frac{1}{90}R_{hijm}R_{hklm} - \frac{1}{18}R_{ij}R_{kl}\right)x^ix^jx^kx^l + O(r^5),\end{aligned}$$

where all of the coefficients are evaluated at p.

Proof. We use a standard Jacobi field expansion technique. Let J be a Jacobi field with initial data $J(0) = 0$ and $J'(0) = \xi \in T_pM$ along the geodesic γ with $\gamma'(0) = \tau \in T_pM$. So, for $T = \gamma'(t)$, the Jacobi equation $\nabla_T^2 J = R(T,J)T$ can be used to compute terms in the Taylor expansion of $f(t) = |J(t)|^2$:

$$\begin{aligned}&\nabla_T f(0) = 0 \\ &\nabla_T^2 f(0) = 2\langle\xi,\xi\rangle \\ &\nabla_T^3 f(0) = 0 \\ &\nabla_T^4 f(0) = 8\langle R(\tau,\xi)\tau,\xi\rangle \\ &\nabla_T^5 f(0) = 20\langle(\nabla_\tau R(\tau,\cdot)\tau)\xi,\xi\rangle \\ &\nabla_T^6 f(0) = 36\langle(\nabla_\tau^2 R(\tau,\cdot)\tau)\xi,\xi\rangle + 32\langle R(\tau,\xi)\tau,\xi\rangle,\end{aligned}$$

where everything is evaluated at $t = 0$. Then, using the fact that $J(t) = \exp_*(t\xi)$, we find

$$\begin{aligned}\langle\xi,\xi\rangle_{t\tau} &= t^{-2}f(t) \\ &= \langle\xi,\xi\rangle + \frac{t^2}{3}\langle R(\tau,\xi)\tau,\xi\rangle + \frac{t^3}{6}\langle(\nabla_\tau R(\tau,\cdot)\tau)\xi,\xi\rangle \\ &\quad + t^4\left(\frac{1}{20}\langle(\nabla_\tau^2 R(\tau,\cdot)\tau)\xi,\xi\rangle + \frac{2}{45}\langle R(\tau,\xi)\tau,\xi\rangle\right) + O(t^5).\end{aligned}$$

By polarization,

$$\begin{aligned}g_{ij}(x) = \delta_{ij} &+ \frac{1}{3}R_{ipqj}x^px^q + \frac{1}{6}R_{ipqj;k}x^px^qx^k \\ &+ \left(\frac{1}{20}R_{ipqj;kl} + \frac{2}{45}R_{ipqm}R_{jklm}\right)x^px^qx^kx^l + O(t^5).\end{aligned} \tag{4.3}$$

One can write g_{ij} as the exponential of another matrix a_{ij}, and then use the relation $\det g = \exp \operatorname{tr} a$ to obtain the result. □

By an inductive process, Lee and Parker showed how to kill off the curvature terms via a conformal change of the metric. This significantly simplifies later calculations and gives a nice general form for the expansion of the conformal Green's function.

Theorem 4.16 (Lee and Parker)**.** *Let (M,g) be a Riemannian manifold of dimension $n \geq 3$ and let $p \in M$. For any integer $N \geq 2$, there exists a conformal metric g on M such that in a normal coordinate system for g at p,*

$$\det g = 1 + O(r^N).$$

Furthermore, if $N \geq 5$ then we may require

$$R = O(r^2) \quad \textit{and} \quad \Delta R(p) = \frac{1}{6}|\mathrm{Weyl}(p)|^2.$$

We call such a coordinate sytem *conformal normal coordinates*. Here we used g for both the original metric and the conformally related metric. We will often do this in the sequel when it is not an issue – since the Yamabe problem is conformally invariant we are usually free to make such conformal deformations. As an immediate application we have

Theorem 4.17 (Aubin's $n \geq 6$ Theorem)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$, and suppose there exists a point $p \in M$ where the Weyl tensor is nonvanishing. Then $\mu(M) < \Lambda$.*

We prove this by constructing an appropriate test function φ for which $J(\varphi) < \Lambda$. We make use of the functions

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(n-2)/2}, \quad \varepsilon > 0$$

which are scaled versions of the functions in Section 4.2. Direct computation verifies that

$$-\Delta u_\varepsilon = n(n-2)u_\varepsilon^{2^\star - 1}. \tag{4.4}$$

Multiplying this equation by u_ε and integrating, we find

$$\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 = n(n-2)\int_{\mathbb{R}^n} u_\varepsilon^{2^\star},$$

so we have

$$\Lambda = \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2}{\left(\displaystyle\int_{\mathbb{R}^n} u_\varepsilon^{2^\star}\right)^{2/2^\star}} = n(n-2)\left(\int_{\mathbb{R}^n} u_\varepsilon^{2^\star}\right)^{2/n}.$$

Proof of Theorem 4.17. Let p be a point where the Weyl tensor is nonvanishing. Let (x^i) be a conformal normal coordinate system at p, and use these coordinates to define a cutoff function $\eta \in C^\infty(M)$ such that $\eta = \eta(r)$, $0 \leq \eta \leq 1$, $\eta = 1$ on $B_\rho = B_\rho(p)$, and $\eta = 0$ on $M \setminus B_{2\rho}$. For example, take the usual cutoff function in $\mathbb{R}^n$ and replace the argument by r/ρ, then extend by zero to all of M. Here ρ is a sufficiently small positive number. With this choice of cutoff function we have $|\nabla \eta| \leq C/\rho$ for a constant C independent of ρ. We define

$$\varphi(r) = \begin{cases} \eta(r)u_\varepsilon(r) & \text{if } r \leq 2\rho \\ 0 & \text{if } r \geq 2\rho \end{cases}$$

We will prove that for ε small enough, $J(\varphi) < \Lambda$.

Since φ depends only on r, we have

$$\begin{aligned}\int_M |\nabla\varphi|^2\,d\mu &= \int_{B_{2\rho}} |\partial_r\varphi|^2\sqrt{\det g}\,dx\\ &\le \int_{B_{2\rho}} |\partial_r\varphi|^2(1+Cr^N)\,dx\\ &= \int_{B_\rho} |\nabla u_\varepsilon|^2\,dx + C\int_{B_\rho} r^N|\nabla u_\varepsilon|^2\,dx + \int_{B_{2\rho}\setminus B_\rho} |\nabla(\eta u_\varepsilon)|^2(1+Cr^N)\,dx,\end{aligned}$$

where we will choose N later. By the definition of u_ε, we can show that

$$|\partial_r u_\varepsilon| \le (n-2)\varepsilon^{(n-2)/2}r^{1-n}, \tag{4.5}$$

so the last two integrals on the RHS are $O(\varepsilon^{n-2})$ for ρ fixed. We integrate by parts in the first integral and use (4.4)

$$\int_{B_\rho} |\nabla u_\varepsilon|^2\,dx = n(n-2)\int_{B_\rho} u_\varepsilon^{2^\star}\,dx + \int_{\partial B_\rho} u_\varepsilon\frac{\partial u_\varepsilon}{\partial r}.$$

Since u_ε is strictly radially decreasing, we have

$$\begin{aligned}\int_{B_\rho} |\nabla u_\varepsilon|^2\,dx &< n(n-2)\left(\int_{B_\rho} u_\varepsilon^{2^\star}\,dx\right)^{2/n}\left(\int_{B_\rho} u_\varepsilon^{2^\star}\,dx\right)^{2/2^\star}\\ &< \Lambda\left(\int_{B_\rho} u_\varepsilon^{2^\star}\,dx\right)^{2/2^\star}.\end{aligned}$$

Therefore,

$$\int_M |\nabla\varphi|^2\,d\mu < \Lambda\left(\int_{B_\rho} u_\varepsilon^{2^\star}\,dx\right)^{2/2^\star} + C\varepsilon^{n-2}.$$

On the other hand,

$$\begin{aligned}\int_M \varphi^{2^\star}\,d\mu &= \int_{B_\rho} u_\varepsilon^{2^\star}\sqrt{\det g}\,dx + \int_{B_{2\rho}\setminus B_\rho} (\eta u_\varepsilon)^{2^\star}\sqrt{\det g}\,dx\\ &\ge \int_{B_\rho} u_\varepsilon^{2^\star}\,dx - C\int_{B_\rho} r^N u_\varepsilon^{2^\star}\,dx - \int_{B_{2\rho}\setminus B_\rho} u_\varepsilon^{2^\star}(1+Cr^N)\,dx\\ &\ge \int_{B_\rho} u_\varepsilon^{2^\star}\,dx - C\varepsilon^n.\end{aligned}$$

Note that this estimate is actually quite crude since we are basically just flipping the sign on positive terms to make them negative. Choosing N large enough we have $R = O(r^2)$ and

$\Delta R(p) = -|W(p)|^2/6$. Therefore,

$$\begin{aligned}
\int_M R\varphi^2\, d\mu &= \int_{B_{2\rho}} \left(\frac{1}{2}\partial_{ij}R(p)x^ix^j + O(r^3)\right)\eta^2 u_\varepsilon^2\, dx \\
&\le \frac{1}{2}\int_{B_{2\rho}} \partial_{ij}R(p)x^ix^j\eta^2u_\varepsilon^2\, dx + C\int_{B_{2\rho}} \eta^2u_\varepsilon^2 r^3\, dx \\
&= \frac{1}{2}\int_0^{2\rho} \eta^2u_\varepsilon^2\, dr \int_{|x|=r} \partial_{ij}R(p)x^ix^j\, d\mathcal{H}^{n-1} + C\int_{B_{2\rho}} \eta^2u_\varepsilon^2r^3\, dx \\
&= \frac{\omega_{n-1}}{n}\Delta R(p)\int_0^{2\rho} \eta^2u_\varepsilon^2 r^{n+1}\, dr + C\omega_{n-1}\int_0^{2\rho}\eta^2u_\varepsilon^2r^{n+2}\, dr,
\end{aligned}$$

where in the last line we used the well-known integral

$$\int_{|x|=r} x^ix^j\, d\mathcal{H}^{n-1} = \frac{\omega_{n-1}}{n}r^{n+1}\delta_{ij}.$$

By Lemma 4.18 below, we may estimate the first term as

$$\frac{\omega_{n-1}}{n}\Delta R(p)\int_0^{2\rho}\eta^2u_\varepsilon^2r^{n+1}\, dr \le -C|W(p)|^2\cdot\begin{cases}\varepsilon^4|\log\varepsilon| & \text{if } n=6\\ \varepsilon^4 & \text{if } n\ge 7\end{cases}$$

and the second term by $O(\varepsilon^5)$. Putting everything together, we find that

$$E(\varphi) \le \begin{cases}\Lambda\|\varphi\|_{L^{2^\star}}^2 - C|W(p)|^2\varepsilon^4|\log\varepsilon| + O(\varepsilon^5) & \text{if } n=6\\ \Lambda\|\varphi\|_{L^{2^\star}}^2 - C|W(p)|^2\varepsilon^4 + O(\varepsilon^5) & \text{if } n\ge 7\end{cases}.$$

Since $|W(p)|>0$ by hypothesis, it is clear that $J(\varphi) = E(\varphi)/\|\varphi\|_{L^{2^\star}}^2$ if ε is small enough. $\square$

Lemma 4.18. *Suppose $k > -n$. Then as $\varepsilon \downarrow 0$,*

$$I(\varepsilon) = \int_0^\rho r^k u_\varepsilon^2 r^{n-1}\, dr$$

satisfies

$$I(\varepsilon) = \begin{cases} O(\varepsilon^{k+2}) & \textit{if } n > k+4\\ O(\varepsilon^{k+2}|\log\varepsilon|) & \textit{if } n = k+4\\ O(\varepsilon^{n-2}) & \textit{if } n < k+4\end{cases}.$$

Proof. The substitution $y = r/\varepsilon$ gives

$$I(\varepsilon) = \varepsilon^{k+2}\int_0^{\rho/\varepsilon} y^{k+n-1}(y^2+1)^{2-n}\, dy.$$

Observe that $y^2 \le y^2+1 \le 2y^2$ for $y \ge 1$, so $I(\varepsilon)$ is bounded above and below by positive multiples of

$$\varepsilon^{k+2}\left(C + \int_1^{\rho/\varepsilon} y^{k+3-n}\, dy\right).$$

When $n > k+4$, the term in parentheses is bounded, when $n < k+4$ it is comparable to ε^{n-k-4}, and to $|\log\varepsilon|$ if $n = k+4$. $\square$

Let us now move towards the proof of Theorem 4.16. The first lemma is due to C.R. Graham, but it seems the paper was never published. Let $\mathcal{P}_k$ denote the space of homogenous polynomials n variables. The variables $x^1, \dots, x^n$ will typically the Cartesian coordinates in a normal coordinate system. These are of course not defined for arbitrarily large values, but we can formally extend polynomial quantities near 0 to be defined for all values of the x^i's.

Lemma 4.19. *Let $p \in M$, and let T be a symmetric $(k+2)$-tensor on T_pM, $k \geq 0$. There exists a unique polynomial $f \in \mathcal{P}_{k-2}$ such that in normal coordinates for g, the metric $\tilde{g} = e^{2f} g$ satisfies*

$$\mathrm{Sym}(\tilde{\nabla}^k \mathrm{Ric}(\tilde{g})) = T.$$

Here Sym is the projection operator from the tensor algebra to the symmetric tensor algebra, normalized so that $\mathrm{Sym}(S) = S$ if S is symmetric.

Proof. Let (x^i) be normal coordinates and set $F(x) = R_{ij}(x)x^i x^j$. If we define

$$F^{(m)}(x) = \frac{1}{(m-2)!} \sum_{|\alpha|=m-2} D^\alpha R_{ij}(p) x^i x^j x^\alpha \in \mathcal{P}_m,$$

then Taylor's expansion can be written as

$$F(x) = \sum_{m=2}^{k+2} F^{(m)}(x) + O(r^{k+3}).$$

Note that

$$\nabla^\alpha \mathrm{Ric}(p) = D^\alpha \mathrm{Ric}(p) + S^\alpha,$$

where S^α is a polynomial in the curvature and its derivatives of order $< |\alpha|$ at p. If $\tilde{g} = e^{2f} g$ with $f \in \mathcal{P}_{k+2}$, then $\tilde{S}^\alpha = S^\alpha$ when $|\alpha| = k$, because $\tilde{S}^\alpha$ differs from S^α by terms which vanish because f vanishes to order $k+2$. Thus the lemma is equivalent to finding $f \in \mathcal{P}_{k+2}$ such that

$$0 = \sum_{|\alpha|=k} (\tilde{\nabla}^\alpha \tilde{R}_{ij} - T_{ij\alpha}) x^i x^j x^\alpha = k! \tilde{F}^{(k+2)}(x) + \sum_{|\alpha|=k} (S_{ij\alpha} - T_{ij\alpha}) x^i x^j x^\alpha,$$

where all of the coefficients are evaluated at p. By Euler's homogenous function theorem

$$x^i x^j \partial_{ij} f = (x^i \partial_i)(x^j \partial_j) f - x^i \partial_i f = (k+1)(k+2) f.$$

We also have

$$\Delta f = \Delta_0 f + O(r^{k+1}),$$

where Δ_0 is the Euclidean Laplacian in the normal coordinates. The transformation rules give

$$\tilde{F}^{(k+2)}(x) = F^{(k+2)}(x) + x^i x^j [(n-2)\partial_{ij} f - (\Delta_0 f)\delta_{ij}],$$

where we are calculating modulo higher order terms. So $\tilde{F}^{(k+2)}(x) = F^{(k+2)}(x)$ to leading order if we can uniquely solve

$$[r^2 \Delta_0 + (n-2)(k+2)(k+1)] f = F^{(k+2)}(x) - \tilde{F}^{(k+2)}(x) \in \mathcal{P}_{k+2}$$

on $\mathcal{P}_{k+2}$. This is indeed possible by Lemma 4.20 below and the rank-nullity theorem. $\square$

Lemma 4.20. *The eigenvalues of $r^2\Delta_0$ on $\mathcal{P}_k$ are*

$$\{\lambda_j = 2j(n-2+2m-2j) : j = 0, \dots, \lfloor m/2 \rfloor\}.$$

Further, the eigenfunctions corresponding to λ_j are the functions of the form $r^{2j}\psi$, where $\psi \in \mathcal{P}_{m-2j}$ is a harmonic polynomial.

Proof. This holds for $m = 0$ or 1 because $r^2\Delta_0 = 0$ on $\mathcal{P}_m$ in those cases. Assume now $m \geq 2$ and let $f \in \mathcal{P}_m$ satisfy $r^2\Delta_0 f = \lambda f$. We have $\Delta_0 \in \mathcal{P}_{m-2}$ and using Euler's theorem we calculate

$$\begin{aligned}\lambda\Delta_0 f &= \Delta_0(r^2\Delta_0 f)\\ &= \Delta_0(r^2)\Delta_0 f + 4x^i\partial_i\Delta_0 f + r^2\Delta_0^2 f\\ &= 2n\Delta_0 f + 4(m-2)\Delta_0 f + r^2\Delta_0^2 f.\end{aligned}$$

So we have

$$r^2\Delta_0^2 f = (\lambda - 2n - 4m + 8)\Delta_0 f.$$

This implies that either $\Delta_0 f = 0$, hence $\lambda = 0$ and f is harmonic, or $\lambda - 2n - 4m + 8$ is an eigenvalue of $r^2\Delta_0$ on $\mathcal{P}_{m-2}$. In the latter case f has the expression $f = \lambda^{-1}r^2\Delta_0 f$. The result now follows from induction. □

Corollary 4.21. *Given $p \in M$ and $N \geq 0$, there exists a metric conformal to g such that all symmetrized covariant derivatives of the Ricci tensor of order $\leq N$ vanish at p.*

Proof. Induction on N with $T = 0$ above. Note that $f \in \mathcal{P}_{N+2}$ implies $\tilde{\nabla}^k\widetilde{\mathrm{Ric}}(p) = \nabla^k \mathrm{Ric}(p)$ for all $k < N$. □

Proof of Theorem 4.16. We proceed again by induction on N. Assume that $\det g = 1 + O(r^N)$ for $N \geq 2$. The $N = 2$ case was shown in Proposition 4.15. A computation using Jacobi fields as in Proposition 4.15 shows that

$$\det g = 1 + \sum_{|\alpha|=N-2} c_N(\nabla^\alpha R_{ij}(p) - T_{ij\alpha})x^i x^j x^\alpha + O(r^{N+1}),$$

where T is a symmetric tensor constructed from derivatives of the curvature of order less than $N-2$. By Lemma 4.19, we can kill this term with a conformal transformation, but without causing lower order terms to come back. We thus obtain a conformal metric $\tilde{g}$ with

$$\det \tilde{g} = 1 + O(r^{N+1})$$

in normal coordinates for g, which are also normal for $\tilde{g}$.

Now let $N \geq 5$. Then it holds at p that:

$$\begin{aligned}0 &= R_{ij}\\ 0 &= \mathrm{Sym}(R_{ij;k})\\ 0 &= \mathrm{Sym}(R_{ij;kl} + \tfrac{2}{9}R_{pijm}R_{pklm}).\end{aligned}$$

Then $R_{ijkl} = W_{ijkl}$ at p from the first equation and by the Ricci identity $R_{ij;kl} = R_{ij;lk}$ at p. From the third equation we obtain

$$\begin{aligned}0 &= (R_{ij;kl} + R_{kl;ij} + 2R_{ik;jl} + 2R_{jl;ik})x^i x^j \\ &+ \frac{2}{9}(W_{pijm}W_{pklm} + W_{pikm}W_{pjlm} + W_{pkim}W_{pjlm} + W_{pjkm}W_{plim} \\ &+ W_{pkjm}W_{plim} + W_{plkm}W_{pjim})x^i x^j,\end{aligned}$$

where the coefficients are evaluated at p. Now contract on k and l, note that by the symmetries of the Weyl tensor,

$$W_{pikm}W_{pjkm} = \tfrac{1}{2}W_{pikm}(W_{pkjm} - W_{pmjk}) = \tfrac{1}{2}W_{pikm}W_{pjkm},$$

so the contracted Bianchi identity

$$R^i{}_{m;i} = \tfrac{1}{2}R_{,m}$$

implies

$$(3R_{;ij} + R_{ij;kk} + \tfrac{2}{3}W_{ipkm}W_{jpkm})x^i x^j = 0.$$

Finally, contracting i and j, we have

$$\Delta R(p) = R_{,ii}(p) = -\tfrac{1}{6}|W(p)|^2. \qquad \square$$

4.4 Asymptotic Expansion of the Conformal Green's Function

This section is concerned with exploring the asymptotic behavior of the conformal Green's function near its pole. The growth estimates will be essential for constructing appropriate test functions for the Yamabe functional in dimensions $3, 4$, and 5, and for $n \geq 6$ when the Weyl tensor is identically zero.

Schoen's big idea [101] was to (i) construct test functions and conformal factors using the Green's function of the conformal Laplacian L (*conformal Green's function*), (ii) apply the positive mass theorem of general relativity to get a key positivity estimate in the computation of the Yamabe functional evaluated on the test functions. However, his proof is pretty difficult because it treats the low dimensional cases ($n = 3, 4, 5$) separately from the others. The Lee and Parker estimates considerably simplify the proof [78].

By definition, the Green's function G_p for L at p satisfies

$$\varphi(p) = \int_M G_p L\varphi$$

for any $\varphi \in C^\infty(M)$. Note that M is compact so we don't have to stipulate that the support of φ is compact. We say that

$$LG_p = \delta_p$$

in the *distributional sense*. Note that G_p defines both a function on $M\setminus\{p\}$ and a distribution (element of the dual of $C^\infty(M)$). By taking $\varphi \in C_c^\infty(M \setminus \{p\})$, we see that

$$LG_p = 0 \quad \text{in } M \setminus \{p\}.$$

Schoen's big idea involves a generalized stereographic projection. To motivate his definition, let us first prove

Proposition 4.22. *Under the stereographic projection $\sigma : S^n \setminus \{N\} \to \mathbb{R}^n$, the Euclidean metric δ is pulled back to a metric conformal to the round metric $\mathcal{S}$. The conformal factor is proportional to $H^{4/(n-2)}$, where H is the conformal Green's function of $(S^n, \mathcal{S})$ at N.*

The existence of the conformal Green's function on the sphere follows immediately from Appendix A of [45] since the conformal Laplacian is coercive on the sphere.

Proof. We already know that

$$(\sigma^{-1})^*\mathcal{S} = \frac{4}{(1+|x|^2)^2}\delta.$$

Rearranging, we have

$$\sigma^*\delta = \frac{1}{4}\left(1 + \frac{|\zeta|^2}{(1-\xi)^2}\right)^2 \mathcal{S} =: \frac{1}{4}F^{4/(n-2)}\mathcal{S}.$$

Since this must be scalar flat, the usual conformal transformation rule shows that $L_{\mathcal{S}}F = 0$ away from N. To show that F is proportional to the Green's function, we show it has the same growth behavior. Then it must be proportional to the Green's function by the maximum principle. Indeed, F must lie either completely above or below the Green's function, else they are equal. Suppose it lies below. Then εF must lie completely above for any $\varepsilon > 1$, again by the strong maximum principle and the fact that the growths are equal. But then taking $\varepsilon \downarrow 1$, we see that F lies above, which is a contradiction. A similar argument works if F lies above.

The singularity is caused by $\xi \to 1$ near the north pole. If θ denotes the polar angle from the north pole, then $\xi = \cos\theta$ and each ζ^i contains a factor of $\sin\theta$ together with other angular terms that are all close to 1 near the north pole. Thus $|\zeta|/(1-\xi) \sim 4/\theta$ from the Taylor expansion. We thus see that F has leading term $\propto 1/\theta^{n-2}$. But on the sphere the great circles are the geodesics, so θ is also the geodesic distance to N. Thus F has the same growth as the Green's function. $\square$

Let us now prove the existence of conformal Green's functions on Yamabe positive manifolds.

Lemma 4.23 (Invariance of the Conformal Laplacian)**.** *If g is a Riemannian metric on any manifold M and $\tilde{g} = \varphi^{4/(n-2)}g$ is a conformal metric, then*

$$\tilde{L}v = \varphi^{-(n+2)/(n-2)}L(\varphi v)$$

for every $v \in C^\infty(M)$.

Proof. We first check the Laplacian term

$$-a\Delta(\varphi v) = -a\varphi\Delta v - av\Delta\varphi - 2a\langle\nabla\varphi, \nabla v\rangle_g.$$

Thus, by using the rule for conformal tranformation of scalar curvature

$$-a\Delta\varphi + R\varphi = \tilde{R}\varphi^{(n+2)/(n-2)},$$

we see that we must only prove

$$\varphi^{4/(n-2)}\tilde{\Delta} v = \Delta v - \frac{2}{\varphi}\langle \nabla\varphi, \nabla v\rangle_g.$$

If ψ is any compactly supported smooth function, then

$$\begin{aligned}\int_M \tilde{\Delta} v\psi \, d\mu(\tilde{g}) &= \int_M \langle \nabla v, \nabla\psi\rangle \, d\mu(\tilde{g})_{\tilde{g}} \\ &= \int_M \langle \nabla v, \nabla\psi\rangle_g \varphi^2 \, d\mu(g) \\ &= \int_M (\varphi^2 \Delta v - 2\varphi\langle\nabla v, \nabla\varphi\rangle_g)\psi \, d\mu(g) \\ &= \int_M \varphi^{-4/(n-2)}\left(\Delta v - \frac{2}{\varphi}\langle\nabla\varphi, \nabla v\rangle_g\right) d\mu(\tilde{g}),\end{aligned}$$

where we have used twice that $d\mu(g) = \varphi^{-2n/(n-2)} \, d\mu(\tilde{g})$. Since ψ was arbitrary, this gives the result. □

Theorem 4.24. *Suppose (M, g) is a compact Riemannian manifold with $n \geq 3$ and $\mu(M) > 0$. Then at each $p \in M$ the Green's function G_p for L exists, is strictly positive, and C^∞ away from p. It has the property that*

$$LG_p = \delta_p$$

in the distributional sense on M. Furthermore, we have the asymptotic expansion

$$G_p(x) = \frac{1}{(n-2)\omega_{n-1}a}\frac{1}{r^{n-2}}(1 + o(1)),$$

where $r = d(p, x)$.

Proof. Choose a conformal metric $g' = \varphi^{4/(n-2)}g$ with strictly positive scalar curvature (Theorem 4.5). Then L' is coercive, so the statement for G'_p follows from Appendix A of [45]. Now define $G_p(x) = \varphi(p)\varphi(x)G'_p(x)$. This is obviously positive and smooth, but we have to show that $LG_p = \delta_p$. Indeed, for any $f \in C^\infty(M)$:

$$\begin{aligned}\varphi^{-1}(p)f(p) &= \int_M G'_p L'(\varphi^{-1}f) \, d\mu(g') \\ &= \varphi^{-1}(p)\int_M \varphi^{-1}G_p\varphi^{1-2^\star} Lf\varphi^{2^\star} \, d\mu(g) \\ &= \varphi^{-1}(p)\int_M G_p \, Lf \, d\mu(g).\end{aligned}$$

The asymptotic expansion is somewhat subtle because in the corresponding expansion for G'_p, the distance is computed in the g' metric. From standard arguments, we see that

$$d_{g'}(p, x) \sim \varphi^{2/(n-2)}(x)d_g(p, x)$$

as $x \to p$. Thus

$$(d_{g'}(p, x))^{2-n} \sim \varphi^{-2}(x)(d_g(p, x))^{2-n}.$$

Finally, since $\varphi(p)\varphi(x)\cdot\varphi^{-2}(x) \sim 1$ as $x \to p$, we obtain the desired expansion. □

From the proof, we obtain the important result

Proposition 4.25. *If G and G' are the conformal Green's functions for g and g', respectively, and $g' = \varphi^{4/(n-2)}g$, then*

$$G(x) = \varphi(p)\varphi(x)G'(x).$$

Suppose (M, g) is a compact Riemannian manifold with $\mu > 0$ (this is the only case we consider at this point). For $p \in M$ define the metric $\hat{g} = G^{4/(n-2)}g$ on $\hat{M} = M \setminus \{p\}$, where

$$G = (n-2)\omega_{n-1}aG_p.$$

(We have just renormalized the Green's function to get rid of the numerical factor in the expansion. Such a normalization will be assumed for the rest of the thesis.) The pair $(\hat{M}, \hat{g})$ is the *stereographic projection* of (M, g). We showed above that in the case of the standard sphere this is homothetic to the standard stereographic projection. To utilize this construction, we must determine the $o(1)$ terms in the expansion of G. We will then show that $(\hat{M}, \hat{g})$ is asymptotically flat, and then apply the positive mass theorem of general relativity to obtain more information about G.

To make writing down the expansion easier, let $f = O_k(r^j)$ denote $\partial^\alpha f \in O(r^{j-|\alpha|})$ for any multiindex α with length $\leq k$.

Theorem 4.26 (Schoen, Lee and Parker)**.** *In a conformal normal coordinate system, the conformal Green's function has the generic asymptotic expansion:*

$$G(x) = r^{2-n}\left(1 + \sum_{k=4}^{n} \psi_k(x)\right) + C\log r + O_2(1),$$

where $r = |x|$, $\psi_k \in \mathcal{P}_k$, and $C = 0$ when n is odd. The leading terms are:

(i) For $n = 3, 4, 5$, or (M, g) is LCF around p,

$$G(x) = r^{2-n} + A + O_2(r),$$

(ii) For $n = 6$,

$$G(x) = r^{-4} - \frac{1}{288a}|W(p)|^2\log r + O_2(1).$$

(iii) For $n \geq 7$,

$$G(x) = r^{2-n}\left[1 + \frac{1}{12a(n-4)}\left(\frac{r^4}{12(n-6)}|W(p)|^2 - R_{,ij}(p)x^ix^jr^2\right)\right] + O_2(r^{7-n}).$$

Before we give the proof, we need a lemma. Let $\mathcal{C}_k$ denote the set of smooth functions defined near p whose k-jets vanish at p and let B be a small geodesic ball around p.

Lemma 4.27. *Consider conformal normal coordinates of order $N \geq 5$. There exists a function*

$$\overline{\psi} \in C^0(B) \cap C^\infty(B \setminus \{0\})$$

such that

$$L(r^{2-n}\overline{\psi}) + r^{2-n}R \in \mathcal{C}_1 \oplus \mathcal{C}_1 \log r \quad \textit{and} \quad \overline{\psi} = o(1). \tag{4.6}$$

In odd dimensions, the logarithmic term is absent and $\overline{\psi}$ *can be chosen to lie in* $\bigoplus_{k=4}^{n} \mathcal{P}_k$*. For* n *even, there can be logarithms multiplied by homogenous polynomials of degrees* $n-2$*,* $n-1$*, and* n*, in addition to the polynomials.*

More specifically: In dimension 3*, we may take* $\overline{\psi} \equiv 0$*. In dimension* 4*, we may take* $\overline{\psi} = Q_4 + P_4 \log r$*, where* $Q_4, P_4 \in \mathcal{P}_4$*. In dimension* 5*,* $\overline{\psi}$ *can be a sum of a polynomial in* $\mathcal{P}_4$ *and one in* $\mathcal{P}_5$*. When* $n = 6$*, the leading terms are*

$$-\frac{1}{24a}\left(R_{,kl}(p)x^k x^l r^2 + \frac{r^4}{12}|W(p)|^2 \log r\right).$$

Finally, when $n \geq 7$*, the leading terms are*

$$\frac{1}{12(n-4)a}\left(\frac{r^4}{12(n-6)}|W(p)|^2 - R_{,kl}(p)x^k x^l r^2\right).$$

Proof. Let

$$L_0 = -r^2\Delta_0 + 2(n-2)r\partial_r$$

and

$$K = r^2(\Delta - \Delta_0) + 2(n-2)(r\partial_r - g^{ij}x^i\partial_j).$$

Suppose $\overline{\psi} \in C^\infty(B \setminus \{0\})$ satisfies $\overline{\psi} = o(1)$ and

$$L_0\overline{\psi} - K\overline{\psi} - a^{-1}Rr^2(1+\overline{\psi}) \in \mathcal{C}_{n+1} \oplus \mathcal{C}_{n+1}\log r. \tag{4.7}$$

A short calculation shows this implies (4.6). We construct $\overline{\psi}$ as a formal series solution to (4.7) of the form $\overline{\psi} = \psi_1 + \cdots + \psi_n$, where $\psi_k \in \mathcal{C}_k$. We also introduce the notational device $\overline{\psi}_k = \psi_1 + \cdots + \psi_k$, so that $\overline{\psi}_n = \overline{\psi}$. Consider first the case when n is odd. Set $\psi_1 = \psi_2 = \psi_3 = 0$. Assume inductively that we have $\overline{\psi}_{k-1} = \psi_1 + \cdots + \psi_{k-1}$ such that

$$L_0\overline{\psi}_{k-1} - K\overline{\psi}_{k-1} - a^{-1}Rr^2(1+\overline{\psi}_{k-1}) \in \mathcal{C}_k. \tag{4.8}$$

Since $R = O(r^2)$, $a^{-1}Rr^2 \in \mathcal{C}_4$, and one may take $\overline{\psi}_k = 0$ for $k \leq 3$. Now write the RHS of (4.7) as $b_k + \mathcal{C}_{k+1}$, where $b_k \in \mathcal{P}_k$. Since $L_0 = -r^2\Delta_0 + 2k(n-2)$ on $\mathcal{P}_k$, Lemma 4.20 asserts that L_0 is invertible on $\mathcal{P}_k$ for n odd. Let $\psi_k = L_0^{-1}(-b_k)$. Then $\overline{\psi}_k = \psi_1 + \cdots + \psi_k$ satisfies (4.7) with $\mathcal{C}_k$ replaced by $\mathcal{C}_{k+1}$ (see below). Thus $\overline{\psi}_n$ satisfies (4.7) with $\mathcal{C}_{n+1}$ on the RHS, hence also satisfies (4.7).

To justify the claim that $\overline{\psi}_k$ satisfies (4.7) with $\mathcal{C}_{k+1}$ on the right, we have to analyze $K\psi_k$. We claim that K maps $\mathcal{C}_k$ into $\mathcal{C}_{k+1}$. Written out, we have

$$Ku = \frac{1}{\sqrt{\det g}}\partial_i(g^{ij}\sqrt{\det g}\,\partial_j u) - \delta^{ij}\partial_i\partial_j u + 2(n-2)(\delta^{ij} - g^{ij})x^i\partial_j u.$$

Assume $u \in \mathcal{C}_k$. The last term is in $\mathcal{C}_{k+2}$ by (4.3). Now in the first two terms everything with two derivatives on u cancels and the remainder has one derivative on g or its determinant. Thus by (4.3) again, these terms are in $\mathcal{C}_{k+1}$, so $u \in \mathcal{C}_{k+1}$.

When n is even, the previous construction works for $k < n-2$. But for $k \geq n-2$, L_0 is not longer invertible on $\mathcal{P}_k$. However, L_0 is self-adjoint with respect to the natural inner product

$$\left\langle \sum_\alpha a_\alpha x^\alpha, \sum_\alpha b_\alpha x^\alpha \right\rangle = \sum_\alpha a_\alpha b_\alpha$$

on $\mathcal{P}_k$. So by the spectral theorem, $\mathcal{P}_k = \operatorname{im} L_0 \oplus \ker L_0$. If $\ker L_0 \neq \{0\}$, we take $\psi_k = p_k + q_k \log r$, where $p_k, q_k \in \mathcal{P}_k$. Computations show

$$L_0(p_k + q_k \log r) = L_0 p_k + (n-2-2k)q_k + (L_0 q_k)\log r. \tag{4.9}$$

Since any $b_k \in \mathcal{P}_k$ can be written as $b_k = L_0 p_k + q_k$, where $L_0 q_k = 0$, (4.9) shows that $L_0 \psi_k = -b_k$ has the solution

$$\psi_k = p_k + (n-2-2k)^{-1} q_k \log r.$$

When $k = n-2$, Lemma 4.20 shows that $\ker L_0$ is spanned by r^{n-2}. Therefore,

$$\psi_{n-2} = p_{n-2} + C r^{n-2} \log r.$$

From definition of K and the expansion (4.3), if $f = f(r)$ then

$$Kf = r^2 \partial_i \left[\left(\frac{1}{3} R_{iklj} x^k x^l + \mathcal{C}_3 \right) r^{-1} x^j f'(r) \right].$$

By by the antisymmetry of the curvature, $R_{iklj} x^k x^j = 0$. Hence, if $f(r) = C r^{n-2} \log r$, then a calculation reveals $Kf \in \mathcal{C}_{n+1} \oplus \mathcal{C}_{n+1} \log r$. We thus have

$$L_0 \overline{\psi}_{n-2} - K \overline{\psi}_{n-2} - a^{-1} R r^2 (1 + \overline{\psi}_{n-2}) \in \mathcal{C}_{n-1} \oplus \mathcal{C}_{n+1} \log r.$$

Writing the RHS as $b_{n-2} + \mathcal{C}_{n-1} + \mathcal{C}_{n+1} \log r$, we can solve for $\psi_{n-1} \in \mathcal{P}_{n-1} + \mathcal{P}_{n-1} \log r$, and then for $\psi_n \in \mathcal{P}_n + \mathcal{P}_n \log r$. The result is a solution to (4.7), but with possibly three logarithmic terms.

When $n = 3$, we showed above that $\overline{\psi}$ identically zero is a valid solution. When $n = 4$, we only get ψ_4, which is in $\mathcal{P}_4 \oplus \mathcal{P}_4 \log r$. And when $n = 5$, we get $\overline{\psi} = \psi_4 + \psi_5 \in \mathcal{P}_4 \oplus \mathcal{P}_5$. For $n \geq 6$, we compute ψ_4. Expanding R as a Taylor series, the above proof implies ψ_4 solves

$$L_0 \psi_4 = -b_4 = -\frac{1}{2a} r^2 R_{,kl}(p) x^k x^l.$$

It can be verified directly that the leading terms written in the statement of the lemma satisfy this equation. $\square$

Proof of Theorem 4.26. We use geodesic polar coordinates constructed from the radial variable $r = |x|$ in standard normal coordinates, and as an angular variable ξ the point on the unit $(n-1)$-sphere that specifies the initial velocity of the geodesic. By the Gauss lemma, this induces a splitting of the metric

$$g = dr^2 + h_{ij}(r, \theta)\, d\theta^i d\theta^j, \quad h \sim r^2 d\Omega^2 \text{ as } r \downarrow 0,$$

where $d\Omega^2$ is the metric of S^{n-1}. In particular, there are no r-θ cross-terms. This splitting is valid in some ball B containing the point p, hereafter referred to as the origin 0. From the usual coordinate transformation rules,

$$\sqrt{\det h} = r^{n-1}\sqrt{\det g}.$$

Thus, if f is a C^2 function of r only, then

$$\Delta f = \frac{1}{r^{n-1}\sqrt{\det g}}\frac{\partial}{\partial r}\left(r^{n-1}\sqrt{\det g}\frac{\partial f}{\partial r}\right).$$

Using this formula, let us compute $\Delta - \Delta_0$ on radial functions. Since in Euclidean space we have $\det g = 1$, the above formula shows that

$$(\Delta - \Delta_0)f = \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial r}\left(\sqrt{\det g}\right)\frac{\partial f}{\partial r} = \frac{1}{2}\frac{\partial}{\partial r}(\log\det g)\frac{\partial f}{\partial r}.$$

If $f(r) = r^{2-n}$, then we find

$$(\Delta - \Delta_0)r^{2-n} = \frac{2-n}{2}\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\log\det g$$

By choosing an appropriate conformal metric, we may assume that the non-Euclidean part $\mathcal{N} = \det g - 1$ is in $\mathcal{C}_N$ for whatever N we want. The function $\log\det g$ is likewise in $\mathcal{C}_N$. Indeed, $\log\det g$ is certainly $o(1)$ since $\det g \to 1$, and for any first derivative ∂, we have

$$\partial \log\det g = \frac{\partial\mathcal{N}}{1+\mathcal{N}}.$$

At the origin, the denominator goes to 1 and the numerator vanishes. Repeating this process, we conclude that

$$\frac{\partial}{\partial r}\log\det g \in \mathcal{C}_{N-1}.$$

We thus have

$$(\Delta - \Delta_0)r^{2-n} =: \theta \in \mathcal{C}_{N-n}.$$

We are free to choose $N = n+1$, for then $\theta \in \mathcal{C}_1$. It is important that $N \geq 5$ also, so that $R = O(r^2)$ and $\Delta R(p) = |W(p)|^2/6$.

From potential theory, we have

$$\Delta_0 r^{2-n} = (n-2)\omega_{n-1}\delta_p \implies \Delta r^{2-n} = (n-2)\omega_{n-1}\delta_p + \theta.$$

If we write $G = r^{n-2}(1+\psi)$, where $\psi = o(1)$, then the equation $LG = (n-2)\omega_{n-1}a\delta_p$ becomes

$$L(r^{2-n}\psi) + Rr^{2-n} = -\theta. \tag{4.10}$$

By the lemma, there exists a there exists a function $\overline{\psi} \in C^0(B) \cap C^\infty(B \setminus \{0\})$ such that

$$L(r^{2-n}\overline{\psi}) + r^{2-n}R \in \mathcal{C}_1 \oplus \mathcal{C}_1 \log r.$$

The exact form of $\overline{\psi}$ was described in the lemma. If we define $\phi = \psi - \overline{\psi}$, then

$$L(r^{2-n}\phi) \in \mathcal{C}_1 + \mathcal{C}_1 \oplus \mathcal{C}_1 \log r = \mathcal{C}_1 \oplus \mathcal{C}_1 \log r.$$

For any function f in $\mathcal{C}_1$ or $\mathcal{C}_1 \log r$, ∇f is bounded, hence $f \in C^{0,\theta}(B)$ for any $\theta \in (0,1)$. Thus $L(r^{2-n}\phi) \in C^{0,\theta}$. The goal here is to show $r^{2-n}\phi$ has no singularity at 0. We can't directly apply Schauder theory to $L(r^{2-n}\phi)$ because we don't know *a priori* what the regularity of $r^{2-n}\phi$ is. One option is to use Lemma 3.11, but we opt for a more geometric proof here. We consider the Schauder-type Dirichlet problem

$$\begin{cases} Lv = L(r^{2-n}\phi) & \text{in } B \\ \quad v = r^{2-n}\phi & \text{on } \partial B. \end{cases}$$

To apply the usual Schauder theory (Section 6.3 in [53]) one must have $R \geq 0$ in B. As before, conformally deform the metric so that this is true, then apply the reverse transformation, and the equation can be solved with $v \in C^{2,\theta}$. Now $w = r^{2-n}\phi - v$ satisfies

$$\begin{cases} Lw = 0 & \text{in } B \\ \quad w = 0 & \text{on } \partial B. \end{cases}$$

Since $\phi = o(1)$ and $v = O(1)$, $w = o(r^{2-n})$. Hence for any $\varepsilon > 0$ we see that $\varepsilon G > w$ at $r = 1$ and for r sufficiently small. (Heuristically, the pole of G always beats out w since its growth is never as bad as r^{2-n}.) Also recall that $LG = 0$ away from 0. It follows from the maximum principle (again, making the appropriate conformal deformation if necessary, or using a trick as in Appendix D), that $\varepsilon G > w$ on $B \setminus \{0\}$. Letting $\varepsilon \downarrow 0$ gives $w \leq 0$. Repeating the argument with $\varepsilon < 0$ and letting $\varepsilon \uparrow 0$ gives $w \equiv 0$. Thus $r^{2-n}\phi = v \in C^{2,\theta}$ and

$$G = r^{2-n}(1 + \overline{\psi}) + v. \tag{4.11}$$

From the lemma,

$$\overline{\psi} \in \bigoplus_{k=4}^{n} \mathcal{P}_k \oplus (\mathcal{P}_{n-2} \oplus \mathcal{P}_{n-1} \oplus \mathcal{P}_n) \log r.$$

This, together with (4.11), gives the desired general expansion for G. We now consider more specific cases.

(i) In the LCF case, we may choose a conformal metric that is flat in B. Now (4.10) says that $r^{2-n}\psi$ is harmonic in the Euclidean sense, and it follows that $\overline{\psi} \equiv 0$ and $G = r^{2-n} + v$, where $v \in C^\infty(B)$ is harmonic. If we let $A = v(0)$ and $\alpha = v - A$, then $\alpha = O_2(r)$ and so we are done in that case. When $n = 3, 4$, and 5, we have an explicit description of $\overline{\psi}$. Indeed, in these cases $r^{2-n}\overline{\psi} = O_2(r)$, so we are done there was well.

(ii) When $n = 6$, the most singular term in $\overline{\psi}$ is

$$-\frac{r^4}{288a}|W(p)|^2 \log r.$$

So we multiply this by $r^{2-6} = r^4$ and all other terms are subleading $O_2(1)$.

(iii) When $n \geq 7$, we use the entire ψ_4 of the lemma, and all other terms are subleading $O_2(r^{7-n})$. $\square$

4.5 Test Functions for the Positive Case

To complete the proof of the Yamabe conjecture, we need the correct test functions. Recall the family of functions

$$u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{(n-2)/2}, \quad \varepsilon > 0,$$

whose properties were listed in Section 4.3. Here we are only concerned with the cases $n < 6$, and when (M, g) is LCF. We showed in Theorem 4.26 (i) that near the pole,

$$G(x) = r^{2-n} + A + \alpha(x),$$

where $\alpha = O(r^2)$ and $\alpha \in C^{2,\theta}$. Following Schoen [101], we use the Green's function to construct the test functions. We choose the test function

$$\phi(x) = \begin{cases} u_\varepsilon(x) & \text{if } r \le \rho \\ \delta(G(x) - \eta(x)\alpha(x)) & \text{if } \rho \le r \le 2\rho \\ \delta G(x) & \text{if } r \ge 2\rho. \end{cases},$$

where $\eta \in C^\infty(M)$ is identically 1 on $B_\rho(p)$, zero outside of $B_{2\rho}(p)$, and such that $|\nabla\eta| \le C/\rho$. Here ρ, ε, and δ are positive and will be chosen small, and such that

$$\delta(\rho^{2-n} + A) = \left(\frac{\varepsilon}{\varepsilon^2 + \rho^2}\right)^{(n-2)/2}. \tag{4.12}$$

In that case, we can see that ϕ will be continuous, hence Lipschitz. We will explain later how exactly to choose these parameters.

Theorem 4.28 (Schoen)**.** *Let (M, g) be a compact manifold that is locally conformally flat, or of dimension $n < 6$. Also assume (M, g) is not conformally diffeomorphic to the sphere. Then for suitable values of the parameters ρ, ε, and δ, the test function ϕ satisfies $J(\phi) < \Lambda$.*

Corollary 4.29 (Yamabe's Theorem)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \ge 3$. Then there exists a C^∞ metric in the conformal class of g with constant scalar curvature.*

Proof. When (M, g) is conformally diffeomorphic to the standard sphere, Theorem 4.9 gives the result. When (M, g) is not locally conformally flat and $n \ge 6$, then the result follows from Theorem 4.17 and Theorem 4.2. In all other cases, Schoen's theorem together with Theorem 4.2 gives the result. □

So once we prove Theorem 4.28, we have completed the proof of the Yamabe conjecture. We will be very careful in proving the test function estimate because there are multiple parameters involved that have to be chosen in the right order.

Lemma 4.30. *Let (M, g) be LCF, $p \in M$, and choose $\rho > 0$ such that it is possible to make the metric flat in $B_{2\rho}(p)$ by a conformal transformation. Assume δ is small enough such that ε may be chosen to satisfy (4.12). Then there exists a constant $C > 0$, independent of these parameters, such that*

$$E(\phi) \le \Lambda\|\phi\|^2_{L^{2^\star}} - \delta^2[(n-2)\omega_{n-1}A - C\rho] + o(\delta^2)\rho^{-n}.$$

Proof. We estimate the energy on B_ρ and $M \setminus B_\rho$, then combine the result. Since $R = 0$ in $B_{2\rho}$, it holds that

$$E_{M\setminus B_\rho}(\phi) = \int_{M\setminus B_\rho} \delta^2(a|\nabla G|^2 + RG^2) + \int_{B_{2\rho}\setminus B_\rho} \delta^2 a \left(|\nabla(\eta\alpha)|^2 - 2\langle \nabla G, \nabla(\eta\alpha)\rangle\right).$$

Since $\alpha = O(r)$, $\nabla\alpha = O(1)$, and $|\nabla(\eta\alpha)| \leq C_1(1+\rho^{-1})$. Further, we have that $|\nabla G| \leq C_2 r^{n-1}$, so together

$$\int_{B_{2\rho}\setminus B_\rho} \delta^2 a \left(|\nabla(\eta\alpha)|^2 - 2\langle \nabla G, \nabla(\eta\alpha)\rangle\right) \leq C_3\delta^2\rho.$$

Using the equation $a\Delta G = -RG$, we find using integrating by parts

$$\int_{M\setminus B_\rho} a|\nabla G|^2 = -\int_{M\setminus B_\rho} RG^2 - \int_{\partial B_\rho} G\frac{\partial G}{\partial r},$$

where $\partial/\partial r$ is the geodesic radial vector field. The first term here cancels out the same term in $E_{M\setminus B_\rho}(\phi)$, so we find

$$E_{M\setminus B_\rho}(\phi) \leq C_3\delta^2\rho - \int_{\partial B_\rho} \delta^2 G\frac{\partial G}{\partial r}.$$

Next,

$$\begin{aligned} E_{B_\rho}(\phi) &= \int_{B_\rho} a|\nabla u_\varepsilon|^2 \\ &= -\int_{B_\rho} a u_\varepsilon \Delta u_\varepsilon + \int_{\partial B_\rho} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \\ &= 4n(n-1)\int_{B_\rho} u_\varepsilon^{2^\star} + \int_{\partial B_\rho} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r} \\ &\leq \Lambda\left(\int_{B_\rho} u_\varepsilon^{2^\star}\right)^{2/2^\star} + \int_{\partial B_\rho} u_\varepsilon \frac{\partial u_\varepsilon}{\partial r}. \end{aligned}$$

We also have

$$\int_M \phi^{2^\star} \geq \int_{B_\rho} \phi^{2^\star} = \int_{B_\rho} u_\varepsilon^{2^\star}.$$

Consequently, the total energy is estimated by

$$E(\phi) \leq \Lambda\|\phi\|_{L^{2^\star}}^2 + C_3\delta^2\rho + \int_{\partial B_\rho}\left(u_\varepsilon\frac{\partial u_\varepsilon}{\partial r} - \delta^2 G\frac{\partial G}{\partial r}\right).$$

Note that in deriving C we only used that ρ is small enough for $B_{2\rho}$ to be flat. If we want to decrease ρ, the same estimate holds, though δ and ε will have to be adjusted to account for (4.12). Using the expansion $G = r^{2-n} + A + \alpha$, one computes

$$G\frac{\partial G}{\partial r}\bigg|_{r=\rho} = -(n-2)(\rho^{3-2n} + A\rho^{1-n} + O(\rho^{2-n})).$$

Estimating the other term is a little more involved. Direct computation shows

$$\left.\frac{\partial u_\varepsilon}{\partial r}\right|_{r=\rho} = -(n-2)\frac{\rho}{\varepsilon^2+\rho^2}\left(\frac{\varepsilon}{\varepsilon^2+\rho^2}\right)^{(n-2)/2}.$$

Now multiply by $u_\varepsilon(\rho)$ and use (4.12) to obtain

$$u_\varepsilon \left.\frac{\partial u_\varepsilon}{\partial r}\right|_{r=\rho} = -(n-2)\frac{\delta^2}{\rho}(\rho^{2-n}+A)^2\left[\left(\frac{\varepsilon}{\rho}\right)^2+1\right]^{-1}.$$

Now use the inequality $(t^2+1)^{-1} \geq 1-t^2$, and note the RHS is manifestly nonpositive to obtain

$$\begin{aligned} u_\varepsilon \left.\frac{\partial u_\varepsilon}{\partial r}\right|_{r=\rho} &\leq -(n-2)\delta^2(\rho^{3-2n}+2A\rho^{1-n}+A^2\rho^{-1})\left[1-\left(\frac{\varepsilon}{\rho}\right)^2\right] \\ &\leq -(n-2)\delta^2(\rho^{3-2n}+2A\rho^{1-n}+A^2\rho^{-1})+C_4\varepsilon^2\delta^2\rho^{1-2n}. \end{aligned}$$

Inserting these estimates into the energy inequality and integrating, we obtain

$$\begin{aligned} E(\phi) &\leq \Lambda\|\phi\|^2_{L^{2^\star}} + \delta^2[(n-2)\omega_{n-1}A - C_5\rho] - (n-2)A^2\rho^{n-2} + C_6\varepsilon^2\delta^2\rho^{-n} \\ &\leq \Lambda\|\phi\|^2_{L^{2^\star}} + \delta^2[(n-2)\omega_{n-1}A - C_5\rho] + C_6\varepsilon^2\delta^2\rho^{-n} \end{aligned}$$

Since ε and δ are not independent, we estimate ε by δ. Indeed, by (4.12),

$$\varepsilon = (\varepsilon^2+\rho^2)\left((\rho^{2-n}+A)\delta\right)^{2/(n-2)} \leq C_7\delta^{2/(n-2)}. \tag{4.13}$$

Thus we obtain a term like $\delta^{2+4/(n-2)}$, which is $o(\delta^2)$. This completes the proof. □

Lemma 4.31. *Let (M,g) have dimension $n<6$. Choose ρ smaller than the injectivity radius at p. Assume δ is small enough such that ε may be chosen to satisfy (4.12). Then there exists a constant $C>0$, independent of these parameters, such that*

$$E(\phi) \leq \Lambda\|\phi\|^2_{L^{2^\star}} - \delta^2[(n-2)\omega_{n-1}A - C\rho] + o(\delta^2)\rho^{-n}.$$

Proof. We still have $G = r^{2-n}+A+\alpha$, where $\alpha = O(r)$ and $\nabla\alpha = O(1)$, and use the same test function. But now, we cannot set $R=0$ in $B_{2\rho}$. Take conformal normal coordinates so that

$$g_{ij} = \delta_{ij} + O(r^2), \quad \det g = 1 + O(r^N), \quad R = O(r^2).$$

Here is how the estimate on $M\setminus B_\rho$ is modified:

$$\begin{aligned} E_{M\setminus B_\rho} &= \int_{M\setminus B_\rho} \delta^2(a|\nabla G|^2 + RG^2) \\ &\quad + \delta^2\int_{B_{2\rho}\setminus B_\rho}\left(a|\nabla(\eta\alpha)|^2 - 2a\langle\nabla G, \nabla(\eta\alpha)\rangle + R(\eta^2\alpha^2 - 2\eta\alpha G)\right) \\ &\leq C_1\delta^2\rho + \int_{M\setminus B_\rho}\delta^2(a|\nabla G|^2 + RG^2). \end{aligned}$$

Since the metric is not flat in $B_{2\delta}$, we have to compute the surface element (volume form) of ∂B_ρ. By the Gauss lemma, the normal to ∂B_ρ is

$$\operatorname{grad} r = \partial_r = \frac{x^i}{r}\,\partial_i$$

So the surface element $d\sigma$ on ∂B_ρ is given by

$$d\sigma = \partial_r \lrcorner\, d\mu = \sqrt{\det g}\frac{x^i}{r}\partial_i \lrcorner\, dx = \sqrt{\det g}\, d\theta,$$

where $d\theta$ is the Euclidean volume element on ∂B_ρ. Then $-a\Delta G + RG = 0$ gives

$$\int_{M\setminus B_\rho} (a|\nabla G|^2 + RG^2) = -\int_{\partial B_\rho} G\frac{\partial G}{\partial r}\sqrt{\det g}\, d\theta$$

Since $\det g$ differs from 1 by an arbitrarily large power of r, we obtain

$$E_{M\setminus B_\rho}(\phi) \le C_2\delta^2\rho - \delta^2\int_{\partial B_\rho} G\frac{\partial G}{\partial r}\, d\theta.$$

Next, since u_ε depends only on r, we may estimate

$$\begin{aligned} E_{B_\rho}(\phi) &= \int_{B_\rho} (a|\nabla u_\varepsilon|^2 + Ru_\varepsilon^2)\sqrt{\det g}\, dx \\ &\le C_3\rho^{6-n}\delta^2 + \int_{B_\rho} |\nabla u_\varepsilon|^2\, dx, \end{aligned}$$

where we used (4.5), (4.13), (4.5), and the gradient and norm in the integral are now that of the flat ball in $\mathbb{R}^n$. The rest of the proof is as in Lemma 4.30. □

We will use the Liouville theorem and the positive mass theorem to show that $A > 0$ unless (M, g) is conformally equivalent to the sphere – see Theorem 6.42 and Corollary 7.12.

Proof of Theorem 4.28. We first explain how to choose ρ. The estimates in Lemmas 4.30 and 4.31 both contain the term

$$-\delta^2[(n-2)\omega_{n-1}A - C\rho],$$

where C is independent of ρ, δ, and ε. Since (M, g) is not equivalent to the sphere, $A > 0$. So for $\rho > 0$ small enough, we have

$$-\delta^2[(n-2)\omega_{n-1}A - C\rho] < 0$$

and ρ is fixed. Now we explain how to choose δ and ε using (4.12). Consider the function

$$t \mapsto (\rho^{2-n} + A)^{-1}\left(\frac{t}{t^2+\rho^2}\right)^{(n-2)/2}.$$

By calculating the first derivative of this function, we see it is strictly increasing for $t \in [0, \eta)$, where $\eta = \eta(\rho) > 0$. Also, it clearly maps 0 to 0. So we see that for δ sufficiently small, there is a unique assignment $\delta \mapsto \varepsilon(\delta)$, and that $\varepsilon \downarrow 0$ as $\delta \downarrow 0$. Now, since $o(\delta^2)$ decays faster than δ^2 (by definition), there is a $\delta > 0$ such that

$$-\delta^2[(n-2)\omega_{n-1}A - C\rho] + o(\delta^2)\rho^{-n} < 0.$$

Choosing the corresponding ε, we use Lemmas 4.30 and 4.31 to find a function $\phi \in H^1$ such that

$$E(\phi) < \Lambda \|\phi\|^2_{L^{2^\star}}.$$

Since this implies $J(\phi) < \Lambda$, we are done. □

4.6 A Word on the Noncompact Case

As mentioned in Chapter 1, the uniformization theorem is true even for noncompact 2-manifolds. That is, given a Riemannian 2-manifold, we can always find a conformal deformation to a complete metric with constant curvature. Thus, one might ask the folowing:

Question 4.32. *Given a noncompact Riemannian manifold, can one always make a conformal deformation to a complete manifold with constant scalar curvature?*

In this section we present Jin's answer: in general, this is not possible [65].

Theorem 4.33 (Jin)**.** *Let M be a compact n-manifold, $n \geq 3$. Take points $p_1, \ldots, p_k \in M$ and set*

$$X = M \setminus \{p_1, \ldots, p_k\}.$$

Then there is a complete Riemannian metric g on X such that g is not conformal to any complete metric with constant scalar curvature.

Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let X be as in the theorem. By a classical theorem of Nomizu and Ozeki, there is a conformal factor $\phi \in C^\infty(X)$ such that $g = \phi^{4/(n-2)}g_0$ is a complete metric on X [94]. We now consider deformations of g, $\tilde{g} = u^{4/(n-1)}g$. By the conformal curvature formula, the problem is to solve

$$\begin{cases} L_g u = \mu u^{(n+2)/(n-2)} \\ u > 0, u \in C^\infty(X) \\ u^{4/(n-2)}g \text{ is a complete metric on } X \end{cases}$$

for some constant μ. On the other hand setting $v = u\phi$, this is equivalent to

$$\begin{cases} L_{g_0} v = \mu v^{(n+2)/(n-2)} \\ v > 0, v \in C^\infty(X) \\ v^{4/(n-2)}g_0 \text{ is a complete metric on } X \end{cases}. \tag{4.14}$$

We set $\lambda_1(g_0)$ to be the principal eigenvalue of L_{g_0} on the compact manifold M. The sign of this number will be important even on the noncompact X.

Proposition 4.34. *The equation* (4.14) *has no solution for* $\mu < 0$.

Proof. By an (admittedly nontrivial) theorem of Aviles, the hypothesis on μ implies that a smooth solution v of (4.14) in ball $\setminus \{p_i\}$ can be extended to a C^1 function on the ball ([9] Theorem 2.2). In particular, v must be bounded at p_i. However, since g_0 is an incomplete metric, v must blow up at p_i if $v^{4/(n-2)}g_0$ is to be complete. $\square$

Let $\overline{g}$ be a metric on M, $B_\rho(p_i)$ geodesic balls in M. Set

$$\Omega_\rho = M \setminus \bigcup_{i=1}^{k} B_\rho(p_i).$$

Further, we define $\lambda_1(\overline{g}, \rho)$ to be the principal Dirichlet eigenvalue of $L_{\overline{g}}$ on Ω_ρ.

Lemma 4.35. *As* $\rho \downarrow 0$, $\lambda_1(\overline{g}, \rho) \to \lambda_1(\overline{g})$. *In particular, if* $\lambda_1(\overline{g}) < 0$, *then there is a* $\rho_0 > 0$ *such that* $\lambda_1(\overline{g}, \rho) < 0$ *for* $\rho \in (0, \rho_0)$.

Proof. Let u be the eigenfunction for $\lambda_1(\overline{g})$ so that $\lambda_1(\overline{g})$ is given by the Rayleigh quotient of u. We define a smooth cutoff function

$$\psi = \begin{cases} 0 & \text{on } \bigcup_{i=1}^{k} B_\rho(p_i) \\ 1 & \text{on } M \setminus \bigcup_{i=1}^{k} B_{2\rho}(p_i) \end{cases}$$

and such that $|\nabla \psi| \le C/\rho$. Setting $v = \psi u$, we have the inequalities

$$\begin{aligned} \left| \int_M |\nabla u|^2 - \int_M |\nabla v|^2 \right| &\le C\rho^{n-2} \\ \left| \int_M R(\overline{g}) u^2 - \int_M R(\overline{g}) v^2 \right| &\le C\rho^{n} \\ \left| \int_M u^2 - \int_M v^2 \right| &\le C\rho^{n}. \end{aligned}$$

Therefore, for any $\varepsilon > 0$ there is a $\delta > 0$ such that for $0 < \rho < \delta$,

$$\frac{\displaystyle\int_{\Omega_\rho} vLv}{\displaystyle\int_{\Omega_\rho} v^2} = \frac{\displaystyle\int_M vLv}{\displaystyle\int_M v^2} < \lambda_1(\overline{g}) + \varepsilon.$$

But this means $\lambda_1(\overline{g}, \rho) \le \lambda_1(\overline{g}) + \varepsilon$ for $0 < \rho < \delta$. Since clearly $\lambda_1(\overline{g}, \rho) \ge \lambda_1(\overline{g})$, the result follows. $\square$

Proof of Theorem 4.33. We will show in Theorem 5.1 that any compact manifold admits a metric with scalar curvature -1. Using a constant test function, this implies $\lambda_1 < 0$ for such a metric. So given a compact manifold M as in the statement of the theorem, let g_0 be a metric on M with scalar curvature -1.

Suppose v is a solution to (4.14). We take $\rho_0 > 0$ smaller than the injectivity radius of (M, g_0) and also small enough so that $\lambda_1(g_0, \rho) < 0$. Let u be the principal eigenfunction of L_{g_0} on Ω_ρ, chosen to be positive. By using the equations and integrating by parts, we have (all measures are with respect to g_0)

$$\begin{aligned}\mu \int_{\Omega_\rho} uv^{(n+2)/(n-2)} &= \int_{\Omega_\rho} uL_{g_0}v \\ &= \int_{\Omega_\rho} vL_{g_0}u - a\sum_{i=1}^{k}\int_{\partial B_\rho(p_i)} v\frac{\partial u}{\partial n} \\ &= \lambda_1(g_0,\rho)\int_{\Omega_\rho} uv - a\sum_{i=1}^{k}\int_{\partial B_\rho(p_i)} v\frac{\partial u}{\partial n},\end{aligned}$$

where u is the outer normal to $\partial B_\rho(p_i)$, equivalently the inner normal to $\partial\Omega_\rho$. Since $u > 0$ in Ω_ρ and $u = 0$ on $\partial\Omega_\rho$, $\partial u/\partial n \geq 0$. Then, since $\lambda_1(g_0, \rho) < 0$, we conclude that $\mu < 0$. But by Proposition 4.34, (4.14) cannot have a solution with $\mu < 0$. □

Actually, Jin considers more general prescribed conformal curvature equations. We partially proved the following theorem here, but will not give the rest of the proof.

Theorem 4.36. *Let M and X be as above. Fix a metric g_0 on M. Assume $\lambda_1(g_0) \leq 0$. Under either of the following conditions, (4.14) has no solution:*

(i) $R(\tilde{g})$ is bounded above and below by negative constants, or
(ii) $R(\tilde{g}) \geq 0$.

Now assume $\lambda_1(g_0) > 0$. Then:

(i) If $R(\tilde{g}) = 0$, then (4.14) has a solution,
(ii) $R(\tilde{g})$ is bounded above and below by negative constants, (4.14) has no solution, and
(iii) If $R(\tilde{g}) > 0$, sometimes there are no solutions, other times there are multiple solutions.

Some more references for the noncompact case are Kim [71] and Bettiol–Piccione [16].

5

The Kazdan–Warner Trichotomy

The basic *prescribed scalar curvature problem* is to ask whether given a smooth function f on M, there is a metric g on M with f as its scalar curvature. The following is an early result that settles the question for the function $f = -1$. The following theorem appears to be due to Avez [8]. Aubin gave a proof using the Yamabe theorem in [4]. We follow Bérard Bergery's proof [13].

Theorem 5.1. *A compact manifold M, $\dim M \geq 3$, carries a metric with constant negative scalar curvature.*

Proof. We construct a metric g with negative total scalar curvature. The result then follows from the basic estimate

$$\mu \leq J(1) = \int_M R(g)$$

and the negative case of the Yamabe problem.

Consider an open ball in M. This contains a submanifold diffeomorphic to a solid donut $S^p \times B^q$, where $p + q = n$, $p \geq 1$, and $q \geq 2$. Take a positive radial function f on B^q which is identically 1 in a neighborhood of the boundary. If g_2 and g_3 denote the canonical metrics on S^p and B^q, appropriately pushed forward onto the manifold, the formula

$$g_1 = \frac{f^2 g_2 \oplus g_3}{f^{2p/n-1}}$$

defines a metric on $S^p \times B^q$ that extends to a metric g on all of M. The total scalar curvature is

$$\int_M R = \int_{M \setminus S^p \times B^q} R + \int_{S^p \times B^q} R.$$

The first term does not depend on f, so we show that f can be chosen to make the second term arbitrarily large and negative. A calculation using the warped product formula in [44] gives

$$\int_{S^p \times B^q} R(g_1)\, d\mu(g_1) = \omega_p \int_{B^q} f^{p/(n-1)-2} \left(R(g_2) - \frac{p(n-1-p)}{n-1} |\nabla f|^2 \right) d\mu(g_3).$$

The dimensional constraints make the $|\nabla f|^2$ term negative, so by choosing f to oscillate wildly near the core of the donut while keeping it bounded gives the result. $\square$

The main Kazdan–Warner theorem is as follows [68] [69]:

Theorem 5.2. *Let M be a compact manifold of dimension $n \geq 3$. If f is a smooth function on M, then f is the scalar curvature of some metric on M if either*

(i) f is negative somewhere on M,

(ii) $f \geq 0$ and M admits a metric with constant positive scalar curvature.

Part (i) is significantly easier and will be attacked first. Using Avez's theorem, we select a metric with scalar curvature -1. It is then easy to show that the conformal curvature equation is solvable in an L^p-neighborhood of -1. Using a diffeomorphism and a scaling, we can "smear out" any function to be close to -1 in L^p. So we solve the equation for the smeared function and then pull back by the diffeomorphism to get a solution for any f that is negative somewhere.

Proof of Part (i). Using Avez's theorem, we select a metric g on M with scalar curvature identically -1. The scalar curvature of a conformal metric $g' = u^{4/(n-2)}$ satisfies

$$-a\Delta u - u = R' u^{2^\star - 1}.$$

Let $\Omega = \{u \in W^{2,p} : u > 0 \text{ a.e.}\}$ for $p > n$ (recall that then $W^{2,p} \hookrightarrow C^1$) and define $\Gamma : \Omega \times L^p \to L^p$ by

$$\Gamma(u, R) = -a\Delta u - u - R u^{2^\star - 1}.$$

It is easy to see that Γ is C^1 in the Fréchet sense and that its partial derivative in the u direction is given by

$$D_1\Gamma(u, R)v = -a\Delta v - v - (2^\star - 1)R u^{2^\star - 2} v, \quad v \in W^{2,p}.$$

At $(1, -1) \in \Omega \times L^p$, we have

$$D_1\Gamma(1, -1)v = -a\Delta v + (2^\star - 2)v.$$

Since $2^\star - 2 = 4/(n-2) > 0$, the operator $D_1\Gamma(1,-1)$ is a coercive differential operator, so by standard elliptic theory, $D_1\Gamma(1,-1) \in \mathcal{B}(W^{2,p}, L^p)$ is invertible. By the implcit function theorem, there exists $\varepsilon > 0$ such that if R satisfies $\|R + 1\|_{L^p} < \varepsilon$, the equation $\Gamma(u, K) = \Gamma(1, -1) = 0$ has a solution in a neighborhood of 1 in $W^{2,p}$. By shrinking ε, we may make this neighborhood as small as desired. Using the Sobolev embedding $W^{2,p} \hookrightarrow C^1 \hookrightarrow C^0$, we may take this neighborhood to contain only positive functions u.

We now show that by deforming M, we may "smear out" f to be almost -1 in L^p. Indeed, let $\eta > 0$ and use the continuity of f to find a $\delta > 0$ such that $f(y) - \inf_M f < \eta$ for $y \in B_\delta(x)$, where $f(x) = \inf_M f$. Using Lemma 5.3 below, take a diffeomorphism ϕ of M such that $\phi(B_\delta(x))$ has comeasure h. Then $f \circ \phi$ will be η-close to $\inf f$ in a set of comeasure h, and we have

$$\|f \circ \phi - \inf f\|_{L^p} \leq \eta \operatorname{vol} M + 2 \sup |f| h.$$

So by taking η and h sufficiently small, we see there exists an $\alpha > 0$ such that

$$\|\alpha f \circ \phi + 1\|_{L^p} < \varepsilon.$$

By the existence theorem above, $R = \alpha f \circ \phi$ is the scalar curvature of some conformal metric $u^{4/(n-2)}g$. Then the diffeomorphism invariance of curvature implies αf is the scalar curvature of $(\phi^{-1})^*(u^{4/(n-2)}g)$. So f is the scalar curvature of a metric homothetic to this. That $u \in C^\infty$ follows results in Section 3.2. □

From now on (contrary to the rest of the thesis!) we only suppose $n = \dim M \geq 2$, unless explicitly stated otherwise.

Lemma 5.3. *Let (M, g) be a compact Riemannian manifold and let $x \in M$, $\delta > 0$ be smaller than the injectivity radius of M, and $\varepsilon > 0$. Then there exists a C^∞ diffeomorphism ϕ of M such that*

$$\mathrm{vol}(M \setminus \phi(B_\delta(x))) < \varepsilon.$$

In other words, ϕ maps the small ball $B_\delta(x)$ into "ε-almost all" of M in the measure theoretic sense.

Proof. Recall that there exists a continuous function $\mu : \{v \in T_xM : |v| = 1\} \to (0, \infty)$ such that $\mu(v) = r$ whenever $\exp_x(rv)$ is in the cut locus $C(x)$ of x. Further, the open cell $E = \{tv : 0 \leq t < \mu(v), |v| = 1\}$ is mapped diffeomorphically by $\exp_x$ onto an open set of M, and $M \setminus \exp_x E$ has measure zero. Let $k \in \mathbb{N}$ and define the sets

$$E_k = \{w \in E : \mathrm{dist}(w, \partial E) \geq k^{-1}\}.$$

These are all star-shaped open sets, so there exist diffeomorphisms $f_k : B \to E_k$, where B is the open unit ball in $\mathbb{R}^n$. Let V be a radial vector field on $\mathbb{R}^n$ with support $\overline{B}$ and source at 0 [1], then consider the vector fields $df_k(V)$ on $E_k \subset E$. Each of these may be pushed forward by the exponential map to a vector field X_k on M with compact support in $\exp_x E_k$. Since $E = \bigcup E_k$, for k large enough,

$$\mathrm{vol}(M \setminus \exp_x E_k) < \frac{\varepsilon}{2}.$$

Then the flow $\Phi_t^{X_k}$ of X_k will expand $B_\delta(x)$ such that

$$\mathrm{vol}(M \setminus \Phi_t^{X_k}(B_\delta(x))) < \varepsilon$$

for large enough t. Indeed, the flow of V expands any smaller ball in B centered at the origin to be arbitrarily close to the unit sphere. Therefore, $\Phi_t^{X_k}$ expands $B_\delta(x)$ to be arbitrarily close to the boundary of $\exp_x E_k$. □

Proof of Theorem 5.2 (ii) First Part. We show that if M admits a metric with positive scalar curvature, then M admits a metric with zero scalar curvature. This corresponds to the function $f \equiv 0$. Let g_0 be the positive scalar curvature metric, and use Avez's theorem to select a metric g_1 with scalar curvature -1 everywhere. We consider the linear homotopy of metrics $g_t = (1-t)g_0 + tg_1$, and the associated function $t \mapsto \mu(g_t)$. By Lemma 5.5 below, this is continuous, and we clearly have $\mu(g_0) > 0$ and $\mu(g_1) < 0$. So by the intermediate value theorem, there is some $s \in (0, 1)$ for which $\mu(g_s) = 0$, and by the zero case of the Yamabe conjecture, g_s is conformal to a metric with zero scalar curvature. □

[1] For example, let $V = \eta\vec{x}$, where η is a cutoff function with support equal to $\overline{B}$.

Lemma 5.4. *Let (X, d) be a metric space and $f_\alpha : X \to \mathbb{R}$ a family of functions, each upper semicontinuous at $x \in X$. Then the lower envelope function $f = \inf_\alpha f_\alpha$ is upper semicontinuous at x.*

Proof. Let $c > f(x)$. As $f(x) = \inf_\alpha f_\alpha(x)$, there exists a β with $c > f_\beta(x)$. Since β is upper semicontinuous at x, for y in a neighborhood U of x we have $c > f_\beta(y)$. As $f_\beta \geq f$, it follows that $c > f$ on U as well. Therefore f is upper semicontinuous at x. □

Lemma 5.5. *The function $\mu : \mathfrak{M} \to \mathbb{R}$ is continuous in the C^2 topology of metrics.*

Proof. With the C^2 topology, $\mathfrak{M}$ becomes a metric space. So given $g \in \mathfrak{M}$, we aim to show that μ is upper and lower semicontinuous at g.

To prove upper semicontinuity, we use the Lemma 5.4 above. Recall from [59] that H^1 is defined independently of the metric. Let (u_α) be a list (of course uncountable) of the elements of $H^1 \setminus \{0\}$. If J_g denotes the Yamabe functional of the metric g, we define

$$f_\alpha : \mathfrak{M} \to \mathbb{R}, \quad g \mapsto J_g(u_\alpha).$$

If f denotes the lower envelope as in Lemma 5.4, we see that

$$f(g) = \inf_\alpha f_\alpha(g) = \inf_\alpha J_g(u_\alpha) = \inf_{H^1\setminus\{0\}} J_g(u) = \mu(g).$$

Thus μ is upper semicontinuous at g since each f_α is.

To prove lower semicontinuity, we argue by contradiction. Let $(g_j) \subset \mathfrak{M}$ be a sequence of metrics such that $\|g_j - g\|_{C^2} \to 0$ but $\mu(g_j) < \mu(g) - \varepsilon$ for some ε and all j. From the convergence, we obtain uniform estimates on the convergence of all metrical quantities up to order 2. In particular, there exist constants $C_j > 1$ such that

$$\frac{1}{C_j} J_{g_j}(u) \leq J_g(u) \leq C_j J_{g_j}(u)$$

for all $u \in H^1 \setminus \{0\}$ and $C_j \to 1$ as $j \to \infty$. But this means we can find a function $u \in H^1 \setminus \{0\}$ such that $J_{g_j}(u) < \mu(g) - \varepsilon$ and then choosing j large enough, such that $J_g(u) < \mu(g) - \varepsilon$, since $J_{g_j}(u)$ can be made arbitrarily close to $J_g(u)$. But this contradicts the definition of $\mu(g)$. □

This is due to Bérard-Bergery [13], but since we were unable to find a copy of this reference, the proof there might be different. In the original paper [68], they used the fact that the assignment $g \mapsto$ princpal eigenvalue of $\lambda(g)$ is analytic. This follows from Kato's perturbation theory [66].

From Lemma 5.5 we obtain a result that should be compared to Corollary 4.6. Note here that we do not claim the metric is conformal to g.

Lemma 5.6. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with scalar curvature $R \geq 0$ but $R(p) > 0$ for some $p \in M$. Then there exists a metric on M with constant positive scalar curvature.*

Proof. We showed in the proof of Corollary 4.6 that $\mu(g) > 0$ under these hypotheses on the scalar curvature. So by the method above, we can find a metric g_1 with zero scalar curvature. If $\mu(g) < \Lambda$, then using Theorem 4.2, we can find a metric with constant positive scalar curvature. If not, then Lemma 5.5 implies a suitable convex combination of g and g_1 will have Yamabe energy less than Λ, but still positive. So Theorem 4.2 may be applied to this metric. This completes the proof. $\square$

This result also follows from the full Yamabe conjecture and the proof of Corollary 4.6. In that case, one of course obtains a conformal transformation to constant positive scalar curvature.

We now begin the proof of the significantly more difficult $f \geq 0$ case. The main lemma is the following:

Lemma 5.7. *Let $f_0 \in C^\infty(M)$ be the scalar curvature of some metric. If $f_1 \in C^\infty(M)$ is such that, for a positive real number λ, the inclusion $\lambda(\operatorname{im} f_0) \subset \operatorname{im} f_1$ holds, then f_1 is also a scalar curvature.*

Proof of Theorem 5.2 (ii) Second Part. Let f be a nonnegative function on M which is not identically zero. We show that there is a metric on M with scalar curvature f. By the hypothesis, there is a metric g_0 on M with scalar curvature R_0 constant. There exists a $\lambda > 0$ such that $\lambda R_0 \in \operatorname{im}(f)$. We apply Lemma 5.7 to conclude that f is a scalar curvature. $\square$

Corollary 5.8 (Kazdan–Warner Trichotomy, $n \geq 3$)**.** *For $n \geq 3$, the class of compact n-manifolds falls into three disjoint groups:*

(i) M admits a positive scalar curvature metric.

(ii) M does not admit a positive scalar curvature metric, but admits a scalar-flat metric.

(iii) M does not admit a metric with $R \geq 0$.

In case (i), any function is a scalar curvature on M. In case (ii), any scalar flat metric is Ricci flat. In cases (ii) and (iii), any function that is negative somewhere is a scalar curvature on M.

Proof. It follows from Lemma 5.6 that if f is a scalar curvature that is nonnegative and positive at some point, then M admits a metric with constant positive scalar curvature. In that case, part (ii) of Theorem 5.2 implies every function is a scalar curvature of M. The rest of the proof follows immediately from part (i) of Theorem 5.2. $\square$

We now move towards the proof of Lemma 5.7. To prove it, we need to understand the surjectivity of the scalar curvature map $g \mapsto R_g \equiv \operatorname{scal}(g)$. Unfortunately, this is quite a bit more technical than part (i) because we need to deal with elliptic systems on vector bundles now. Standard references for the terminology and theorems we use are Gilkey [54] and Lawson and Michselson [76]. For $p > n$ we consider the Sobolev space of Riemannian metrics $W^{2,p}(\mathfrak{M})$ defined using a finite atlas in the obvious way. Since M is compact, all such definitions will be equivalent. It follows from the Sobolev embedding theorem that $W^{2,p}(\mathfrak{M})$ is a Banach algebra, see Theorem 4.39 in [1] for the analogous result on $\mathbb{R}^n$. In particular, we may define the scalar curvature of a metric in $W^{2,p}(\mathfrak{M})$. Further, note that $W^{2,p}(\mathfrak{M}) \hookrightarrow C^1(\mathfrak{M})$ and the scalar curvature is quasilinear.

Proposition 5.9. *The scalar curvature map* scal : $W^{2,p}(\mathfrak{M}) \to L^p(M)$, $g \mapsto \operatorname{scal}(g)$, *is* C^1 *in the Fréchet sense and for* $h \in W^{2,p}(S_2M)$, *we have the* linearization formula

$$\operatorname{scal}'(g)(h) = -\Delta_g \operatorname{tr}_g h + \operatorname{div}_g \operatorname{div}_g h - g(\operatorname{Ric}_g, h).$$

Proof. The linearization formula is well known in multiple fields, see for instance Theorem 1.174 in [15] or Section I.11 in [35]. It remains to be shown that the scalar curvature map is continuously differentiable, namely, we must show that the map

$$\operatorname{scal}' : W^{2,p}(\mathfrak{M}) \to \mathcal{B}(W^{2,p}(S_2M), L^p(M))$$

is continuous. Let $g \in W^{2,p}(\mathfrak{M})$ and (g_j) be a sequence in $W^{2,p}(\mathfrak{M})$ converging to g. By the Sobolev inequality, $g_j \to g$ in $C^1(\mathfrak{M})$ as well. Using this, and the fact that $\partial^2 g_i \to \partial g$ in L^p, we conclude from the linearization formula that for any $\varepsilon > 0$,

$$\|\operatorname{scal}'(g_j)h - \operatorname{scal}'(g)h\|_{L^p(M)} < C\varepsilon \|h\|_{W^{2,p}(S_2M)}$$

for i sufficiently large, and for all $h \in W^{2,p}(S_2M)$. Thus $\operatorname{scal}'(g_j) \to \operatorname{scal}'(g)$ in operator norm, so the scalar curvature is C^1. □

For convenience, we denote $\operatorname{scal}'(g)$ by A when the fixed metric g is clear from context.

Proposition 5.10. *Fix a smooth metric g. Then the linearized scalar curvature at g is a surjectively elliptic operator from $C^\infty(S_2M)$ to $C^\infty(M)$. Its principal symbol is given by*

$$\sigma_\xi(A)(h) = |\xi|^2 \operatorname{tr}_g h - h(\xi^\#, \xi^\#)$$

for $\xi \in T_x^*M$, $h \in S_{2x}M$. The formal adjoint A^* of A is an injectively elliptic operator from $C^\infty(M)$ to $C^\infty(S_2M)$ given by

$$A^* f = \operatorname{Hess}_g f - (\Delta_g f) g - f \operatorname{Ric}_g .$$

with principal symbol

$$\sigma_\xi(A)(f) = (|\xi|_g^2 g - \xi \otimes \xi) f.$$

Proof. Note first that since g is smooth, A is a smooth differential operator. Its symbol may be read off immediately from the linearization formula. Since the symbol is not identically zero for $\xi \neq 0$, it is surjective. To compute the adjoint, we integrate by parts (the fixed metric g is implied):

$$\begin{aligned}
\langle Ah, f\rangle_{L^2(M)} &= \int_M (-\Delta \operatorname{tr} h f + \operatorname{div}\operatorname{div} f - g(\operatorname{Ric}, h)) \\
&= \int_M h_{ik}(-\Delta f g_{jl} + \nabla_j \nabla_l f - f R_{jl}) g^{ij} g^{kl} \\
&= \langle h, \operatorname{Hess} f - (\Delta f) g - f \operatorname{Ric}\rangle_{L^2(S_2M)} \\
&= \langle h, A^* f\rangle_{L^2(S_2M)}.
\end{aligned}$$

The principal symbol of A^* is the Hermitian conjugate of A's symbol. Thus the former will be injective since the latter is surjective. Alternatively, since $\sigma_\xi(A)$ is an operator on $\mathbb{R}$, if it has a kernel it is identically zero. But by taking the trace, we find $(n-1)|\xi|_g^2$, which is not identically zero. □

We can now characterize the local surjectivity of the scalar curvature map. This result is due to Fischer and Marsden [49] in connection with their work on the stability of the Einstein constraint equations.

Theorem 5.11. *Let g be a smooth metric and suppose that either*

(i) $R_g \equiv 0$ but $\mathrm{Ric}_g \not\equiv 0$, or

(ii) $R_g/(n-1)$ is not a positive constant which is an eigenvalue of Δ_g.

Then the $W^{2,p}(\mathfrak{M})$ scalar curvature map is locally surjective at g, i.e. there exists an $\varepsilon > 0$ such that if $f \in L^p(M)$ and $\|f - R_g\|_{L^p} < \varepsilon$, there is a $g' \in W^{2,p}(\mathfrak{M})$ such that $f = \mathrm{scal}(g')$. Furthermore, if f is a smooth function, then g' can be chosen to be smooth.

Proof. One can show that if X and Y are Banach spaces and $F \in C^1(X,Y)$ has surjective differential at $x \in X$, then F is an open map near x, [14] Theorem 3.1.19. In particular, it is locally surjective at x. In our situation, this reduces the local surjectivity of the scalar curvature map to the surjectivity of A.

We now show that this is equivalent to the injectivity of A^*. The idea is to use elliptic theory to conclude a Fredholm-type splitting

$$L^p(M) = A(W^{2,p}(S_2M)) \oplus \ker A^*, \tag{5.1}$$

which of course gives the desired criterion. We consider the differential operator

$$P = AA^* : C^\infty(M) \to C^\infty(M).$$

By the symbol calculus and linear algebra, the principal symbol of this will be an isomorphism, hence P is elliptic. Furthermore, if $f \in C^\infty(M)$, then

$$\langle Pf, f\rangle = \langle AA^*f, f\rangle = \|A^*f\|^2,$$

which shows that $\ker P = \ker A^*$. We now invoke the main theorems of elliptic pseudodifferential operators to conclude that P is a Fredholm operator $H^s(M) \to H^{s-4}(M)$, $\ker P$ and $\ker P^*$ are finite-dimensional, contained in $C^\infty(M)$, and we have the orthogonal splitting

$$L^2(M) = P(H^4(M)) \oplus \ker P^*. \tag{5.2}$$

We easily compute $\langle Pf_1, f_2\rangle = \langle f_1, P^*f_2\rangle$, i.e. P is self-adjoint, so $\ker P^* = \ker P = \ker A^* \subset C^\infty(M)$. Since M is compact and $p > n \geq 2$, $L^p(M) \hookrightarrow L^2(M)$. From (5.2), $f = Ph + \alpha$, where $\alpha \in \ker A^* \subset C^\infty(M)$ and $h \in H^4(S_2M)$. But then $Ph = f - \alpha \in L^p(M)$, so the classical L^p regularity theory for elliptic systems implies $h \in W^{4,p}(S_2M)$. Letting $\tilde{h} = A^*h \in W^{2,p}(S_2M)$, we find that $f = A\tilde{h} + \alpha$, i.e.

$$L^p(M) = A(W^{2,p}(S_2M)) + \ker A^*,$$

but we don't know yet that $A(W^{2,p}(S_2M)) \cap \ker A^* = \{0\}$. To show the intersection is trivial, take $k \in A(W^{2,p}(S_2M)) \cap \ker A^*$. Then $k = Ah \in C^\infty(M)$ and $A^*Ah = 0$, so

$$0 = \langle A^*Ah, h\rangle = \|Ah\|^2.$$

This completes the proof of (5.1).

We have established that A is surjective if and only if A^* is injective. We now show that this is implied by (i) and (ii) in the statement of the theorem. Since $\ker A^* \subset C^\infty(M)$, we may freely use the standard methods of Riemannian geometry. Let $f \in C^\infty(M)$ satisfy

$$A^* f = \operatorname{Hess} f - \Delta f g - f \operatorname{Ric} = 0.$$

The trace of this formula yields

$$(1-n)\Delta f - fR = 0$$

Thus $R = 0$ implies f is constant, and so $A^* f = -f \operatorname{Ric} = 0$. Since $\operatorname{Ric} \not\equiv 0$, f vanishes identically. This takes care of case (i). Consider now (ii). Taking the divergence of $A^* f = 0$ gives

$$\frac{1}{2} f \, dR = 0.$$

Indeed, from the Ricci identity $(\nabla_i \nabla_j - \nabla_j \nabla_i) A^k = R^k{}_{lij} A^l$ we obtain

$$\operatorname{div} \operatorname{Hess} f - d\Delta f = \operatorname{Ric}(df, \cdot).$$

Recall also the contracted Bianchi identity

$$\operatorname{div} \operatorname{Ric} = \frac{1}{2}\, dR.$$

We now compute

$$\begin{aligned}
\operatorname{div} A^* f &= \operatorname{div}(\operatorname{Hess} f - \Delta f g - f \operatorname{Ric}) \\
&= \operatorname{div} \operatorname{Hess} f - d\Delta f - \operatorname{Ric}(df, \cdot) - f \operatorname{div} \operatorname{Ric} \\
&= -\frac{1}{2} f \, dR.
\end{aligned}$$

If f is never zero, dR vanishes everywhere and R is constant, which together with $(1-n)\Delta f = Rf$, contradicts (ii). Suppose there is an $x \in M$ such that $f(x) = 0$. We must have $df(x) \neq 0$; indeed, if $df(x) = 0$, let γ be a geodesic starting at x and define $\tilde{f} = f \circ \gamma$. From $A^* f = 0$ and $(1-n)\Delta f = Rf$ we find

$$\operatorname{Hess} f = \left(\operatorname{Ric} - \frac{R}{n-1} g \right) f$$

and hence $\tilde{f}$ satisfies the second order linear ODE

$$\begin{aligned}
\tilde{f}''(t) &= (\operatorname{Hess} f)(\gamma'(t), \gamma'(t)) \\
&= \left(\left(\operatorname{Ric} - \frac{R}{n-1} g \right) (\gamma'(t), \gamma'(t)) \right) \tilde{f}(t),
\end{aligned}$$

with $\tilde{f}(0) = 0$ and $\tilde{f}'(0) = 0$. By by the theorem of Picard and Lindelöf, we must have $\tilde{f} \equiv 0$ and hence $f = 0$ along γ. Since γ was arbitrary, f vanishes everywhere on M. Thus if f is to be a nontrivial element of $\ker A^*$, 0 must be a regular value and $f^{-1}(0)$ is a set of measure zero in M. We conclude that dR vanishes identically, contradicting (ii). Thus $f = 0$ and A^* is injective.

We conclude the proof by showing that if $f \in C^\infty(M)$ is sufficiently close to $\mathrm{scal}(g)$, we may find a *smooth* metric g' that solves $\mathrm{scal}(g') = f$. Let $U \subset W^{4,p}(M)$ be a neighborhood of 0 and let $V = A^*(U) \subset W^{2,p}(S_2M)$. Pick U so small that $g + V$ contains only proper metrics. This is possible due to the embedding $W^{2,p}(S_2M) \hookrightarrow C^1(S_2M)$. More precisely, these tensors would be metrics if they were smoother, but they can be taken to define positive definite bilinear forms in each tangent space. Now define the quasilinear operator

$$Q : U \to L^p(M), \quad Q(u) = \mathrm{scal}(g + A^*u).$$

The differential of Q at g is AA^*, i.e. P above, which is an isomorphism $H^4(M) \to L^2(M)$ since $\ker P = \ker A^* = \{0\}$ and (5.2). It follows that $P : W^{4,p}(M) \to L^p(M)$ is an isomorphism by the L^p regularity theory. Thus $Q : U \to L^p(M)$ is a diffeomorphism of a neighborhood of 0. We work with Q because it is properly elliptic in some possibly smaller neighborhood of 0, still denoted by U. Indeed, the principal symbol of the linearization of Q varies continuously with the metric. Since the symbol is invertible at 0, it is invertible in a neighborhood U of 0. We now use the L^p theory to show that Q is hypoelliptic in U. This is standard. Locally, the equation $Q(u) = f$ can be written as

$$\sum_{|\alpha| \leq 4} a_\alpha(x, \partial^\beta u)\partial^\alpha u = w(x, \partial^\beta u) + f(x),$$

where $|\beta| \leq 3$, a_α, w are C^∞ in their variables, and $f \in C^\infty(M)$. Thus $a_\alpha(x, \partial^\beta u)$ and $w(x, \partial^\beta u)$ are in $W^{1,p}$ locally, so $u \in W^{5,p}$ locally by Theorem 6.2.5 in [91]. Then $u \in C^{4,\alpha}$ and we may increase the regularity arbitrarily. □

Proof of Lemma 5.7. The λ condition implies we may find $\phi \in \mathrm{Diff}(M)$ and $\alpha > 0$, as in the proof of Theorem 5.2, such that

$$\|\alpha f_1 \circ \phi - f_0\|_{L^p} < \varepsilon.$$

If $\ker \mathrm{scal}'(g_0)^* = 0$, apply the Fischer–Marsden theorem to find a metric with scalar curvature $\alpha f_1 \circ \phi$. Then undo the diffeomorphism and the scaling to find the metric for f_1 itself. But we know that $\ker \mathrm{scal}'(g_0)^* \neq 0$ only when $\mathrm{scal}(g_0)$ is constant. But then, by a very small perturbation of g_0, we can make it have nonconstant scalar curvature, while still preserving the λ condition. Then apply the Fischer–Marsden theorem to the perturbed metric. □

By the same method and Theorem 1.6 together with the Gauss–Bonnet formula instead of the negative case of the Yamabe conjecture, we also have

Theorem 5.12 (Kazdan–Warner Trichotomy, $n = 2$)**.** *Let M be a compact 2-manifold with Euler characteristic $\chi(M)$. Let $f \in C^\infty(M)$.*

(i) If $\chi(M) > 0$, then f is a scalar curvature if and only if $f > 0$ somewhere.

(ii) If $\chi(M) = 0$, then f is a scalar curvature if and only if f changes sign or vanishes identically.

(iii) If $\chi(M) < 0$, then f is a scalar curvature if and only if $f < 0$ somewhere.

The approach here is based on the note [67] instead of the original papers [68] [69]. The original approach is based on highly nontrivial holomorphic perturbation theory.

6

Locally Conformally Flat Manifolds

The goal of this chapter is to prove $A > 0$ for LCF manifolds of dimension at least 4. This completes the solution of the Yamabe problem. It is based on Chapter VI of [106] and the paper [105].

More precisely, the goal is:

Theorem 6.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. If the conformal class of g has positive Yamabe energy and (M, g) locally conformally flat, but not conformally diffeomorphic to the sphere, then for any point $p \in M$, $A > 0$, where A is the number in Theorem 4.26 (i) for the conformal Green's function at p in a flat coordinate system at p.*

6.1 The Developing Map

Let (M, g) and (N, h) be two Riemannian manifolds of the same dimension. A smooth map $F : M \to N$ is said to be *conformal* if there exists a smooth positive function u on M such that $F^*h = ug$. If F is in addition a diffeomorphism, we say that it is a *conformal diffeomorphism.* If F is a conformal map, dF can never have a kernel, so F is an immersion.

An n-dimensional manifold M is said to be *locally conformally flat (LCF)* if it admits an atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$, $\phi_\alpha : U_\alpha \subset M \to S^n$, such that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the transition function

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a conformal diffeomorphism of open subsets of S^n with the standard round metric. We then call $\mathcal{A}$ a *locally conformally flat structure* on M. On the surface, this appears to be different than our previous definition. We say that a Riemannian manifold (M, g) is *locally conformally flat in the Riemannian sense (RLCF)* if at each point in M, the Weyl tensor of g is zero when $n \geq 4$, and the Cotton tensor is zero when $n = 3$. Using the Frobenius theorem, one can show that this is equivalent to each point having a neighborhood in which there exists a pointwise conformal transformation to a ball in $\mathbb{R}^n$. We will show that the conditions LCF and RLCF are effectively equivalent.

Given an LCF manifold $(M, \mathcal{A})$ and a Riemannian metric g on M, we say that g is *compatible* with $\mathcal{A}$ if for each α, $\phi_\alpha : (U_\alpha, g) \to S^n$ is a conformal map.

Proposition 6.2. *Given an LCF manifold $(M, \mathcal{A})$, there is a Riemannian metric g on M compatible with $\mathcal{A}$. Conversely, given an RLCF manifold (M, g), we may equip M with an LCF structure $\mathcal{A}$ such that g is compatible with $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ be the LCF structure. By passing to a refinement, we may assume there is a partition of unity $\{\theta_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Set

$$g = \sum_\alpha \theta_\alpha \phi_\alpha^* \mathcal{S},$$

where $\mathcal{S}$ is the round metric on S^n. This is obviously a compatible metric.

Conversely, let $\{U_\alpha\}$ be a covering of M by open sets such that for each α, there exists a map $\phi_\alpha : U_\alpha \to S^n$, diffeomorphic onto its image, and a function $f_\alpha \in C^\infty(\phi_\alpha(U_\alpha))$, such that

$$(\phi_\alpha^{-1})^* g = e^{f_\alpha} \mathcal{S}.$$

Then the chart overlaps will be conformal maps of the sphere, so this is indeed an LCF structure, and is compatible with g from the definition. □

Our focus is Riemannian geometry, so given an RLCF manifold (M, g), we associate an LCF structure $\mathcal{A}$ to it. We also stop using the terminology "RLCF" now.

Proposition 6.3. *Let M be a smooth manifold and suppose $\Phi : M \to S^n$ is an immersion. Then Φ induces an LCF structure on M.*

The proof is clear. When M already has an LCF structure, we can ask when there exists an immersion $\Phi : M \to S^n$ such that the induced structure agrees with the preexisting one. Clearly we could compose such a Φ with a conformal diffeomorphism of S^n and obtain another compatible immersion. Thankfully, we have a nice classification of these conformal diffeomorphisms:

Theorem 6.4 (Liouville)**.** *Let $n \geq 3$. The group of conformal diffeomorphisms of $\mathbb{R}^n$ is generated by translations, rotations, scalings, and inversions. By extending these maps to the point at infinty and using stereographic projection, this also describes the conformal group of S^n, which we denote by* $\mathrm{Conf}(n)$*. These transformations of the sphere are called* Möbius transformations*. Any conformal map of a connected open set of S^n is the restriction of a unique Möbius transformation on all of S^n.*

For the proof, see Spivak [112] ($n = 3$) and [113] ($n \geq 3$). We now come to the main theorem of this section, which was proved by Kuiper in a slightly broader pseudo-Riemannian context [74].

Theorem 6.5. *Let (M, g) be a simply connected LCF Riemannian manifold of dimension $n \geq 3$. Then there exists an immersion $\Phi : M \to S^n$ such that the LCF structure of M is induced by Φ. Any two such maps are related by a Möbius transformation in S^n.*

We say that Φ is a *developing map* for (M, g) and that M *develops over* S^n. By the preceding results, Φ is a conformal map. We give a modernized rendition of Kuiper's very geometric proof.

Lemma 6.6. *Let $\gamma : [0,1] \to M$ be continuous curve in an LCF manifold (M, g). Then there exists a neighborhood of the image of γ that is conformally immersed in S^n. This map is unique up to Möbius transformations.*

Proof. We give M an LCF structure $\mathcal{A}$ compatible with the metric g. There exists a finite number of open sets $U_1, \ldots, U_k$ in $\mathcal{A}$ and partition

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$$

such that U_i covers $\gamma([t_{i-1}, t_i])$ for each $i = 1, \ldots, k$. Let ϕ_i be the conformal chart associated to U_i. The diffeomorphism $\phi_1 : U_1 \to \phi_1(U_1)$ maps a neighborhood of $\gamma([t_0, t_1])$ conformally into S^n. We denote this map by Φ_1. The open set $U_1 \cap U_2$ contains a connected neighborhood V_1 of $\gamma(t_1)$. We have a coordinate transition map

$$\psi_{1,2} = \phi_1 \circ \phi_2^{-1} : \phi_2(V_1) \to \phi_1(V_2).$$

By Liouville's theorem, $\psi_{1,2}$ extends to a unique global Möbius transformation of S^n. We define Φ_2 on U_2 by $\psi_{1,2} \circ \phi_2$. Then Φ_2 maps a neighborhood of $\gamma([t_1, t_2])$ conformally into S^n, and agrees with Φ_1 near $\gamma(t_1)$. In this way we construct Φ_{i+1}, covering a neighborhood of $\gamma([t_i, t_{i+1}])$, and agreeing with Φ_i near $\gamma(t_i)$, $i = 2, \ldots, k-1$. This gives an immersion Φ of a neighborhood of $\gamma([0,1])$ in S^n.

To prove the uniqueness, we consider anther pair $(U', \phi') \in \mathcal{A}$ which covers $\gamma([t', t''])$ for some times $t' < t''$. We suppose that ϕ' agrees with Φ_j near some point of $\gamma([t_{j-1}, t_j])$. Then by Liouville's theorem it agrees with Φ_j in a neighborhood of $\gamma([t', t'']) \cap \gamma([t_{j-1}, t_j])$. Hence it agrees near $\gamma([t', t'']) \cap \gamma(t_{j-1})$ with Φ_j and Φ_{j-1} and near $\gamma([t', t'']) \cap \gamma(t_j)$ with Φ_j and Φ_{j+1}. Hence it agrees near $\gamma([t', t'']) \cap \gamma([t_{j-2}, t_{j-1}])$ with Φ_{j-1} and near $\gamma([t', t'']) \cap \gamma([t_j, t_{j+1}])$ with Φ_{j+1}, etc. It follows that ϕ' agrees with Φ in a neighborhood of $\gamma([t', t''])$.

Now suppose there are two conformal immersions $\Phi, \Phi' : U \supset \gamma \to S^n$. We have that locally $\Phi \circ \Phi^{-1}$ is a conformal diffeomorphism. So by the previous argument we may find a global Möbius transformation f of S^n such that $\Phi = f \circ \Phi'$. ☐

Proof of Theorem 6.5. We now have a way for immersing a neighborhood of a curve into S^n. So the goal is to fix a base point $p \in M$, and define $\Phi(q)$, $q \in M \setminus \{p\}$ by connecting p and q with a curve, and then immersing a neighborhood of the curve. However, the issue is that this might depend on the curve chosen and there will compatibility issues. So the goal here is to show that no such issues actually arise when M is simply connected.

So fix $p \in M$ and let $q \in M \setminus \{p\}$ be arbitrary. Let γ_0 be a curve from p to q. Let γ_1 be any other curve from p to q, and $H : [0,1] \times [0,1] \to M$ the associated path homotopy such that $H(\cdot, 0) = \gamma_0$ and $H(\cdot, 1) = \gamma_1$. Such an H exists because of the simple connectedness of M. Fix a conformal diffeomorphism from a neighborhood of p to S^n. By Lemma 6.6, this determines a conformal diffeomorphism into S^n of a neighborhood of q. We now show that as γ_0 is homotoped to γ_1, this diffeomorphism remains the same.

Let the square $[0,1] \times [0,1]$ be covered by (relatively) open subsquares such that the image of each subsquare under H is contained in some element of $\mathcal{A}$. We now push the line $[0,1] \times \{0\}$ up, one square at a time, until it reaches $[0,1] \times \{1\}$. This moves γ_0 to γ_1 in a finite number of steps such that in each step, the curve is modified in a conformal coordinate neighborhood. Then each step leaves the conformal diffeomorphism near q invariant. We conclude that γ_0 and γ_1 determine the same diffeomorphism.

Thus Φ is uniquely determined up to the original diffeomorphism. But any two such diffeomorphisms are related by a Möbius transformation, so the proof is complete. $\square$

Corollary 6.7. *If M is a simply connected compact LCF manifold, the developing map $\Phi : M \to S^n$ is a conformal diffeomorphism.*

Proof. Since Φ is an immersion, it is a local diffeomorphism. It is also proper since M is compact. Therefore, it follows from standard covering space theory ([77] Proposition 4.46) that Φ is a covering map. But M is simply connected, so Φ is just a diffeomorphism. $\square$

In other words, the only compact, simply connected LCF manifold is the standard sphere (up to a conformal diffeomorphism).

If M is not simply connected we may take its universal cover $\pi : \tilde{M} \to M$ and then find a developing map $\Phi : \tilde{M} \to S^n$. Since we may identify $\pi_1(M)$ with the group of deck transformations of π, each $\gamma \in \pi_1(M)$ induces a developing map via $\Phi \circ \gamma : \tilde{M} \to S^n$. By Kuiper's theorem there is a $\rho(\gamma) \in \mathrm{Conf}(n)$ such that $\Phi \circ \gamma = \rho(\gamma) \circ \Phi$. It can be checked that $\gamma \mapsto \rho(\gamma)$ is a homomorphism $\pi_1(M) \to \mathrm{Conf}(n)$, called the *holonomy representation* of $\pi_1(M)$. Since $\ker \rho$ is normal in $\pi_1(M)$, we may consider the manifold $\hat{M} = \tilde{M}/\ker \rho$, and the developing map descends to it. We thus have $\Phi : \hat{M} \to S^n$ and the group $\hat{\Gamma} = \pi_1(M)/\ker \rho$ acts by deck transformations on $\hat{M}$ with $M = \hat{M}/\hat{\Gamma}$. Since we took out the kernel, the induced homomorphism $\hat{\rho} : \hat{\Gamma} \to \mathrm{Conf}(n)$ is injective. We give $\hat{M}$ the natural covering metric $\hat{g}$, and then $(\hat{M}, \hat{g})$ is called the *holonomy covering* of M.

Using universal covers, we also have

Corollary 6.8. *Let (M, g) be a compact LCF manifold with finite fundamental group. Then M has universal cover S^n.*

Proof. If $\pi_1(M)$ is finite, then the universal covering map $\tilde{M} \to M$ is finite-sheeted. Hence $\tilde{M}$ is compact and we may apply Corollary 6.7. $\square$

In the rest of this chapter we will develop tools that give a convenient condition for Φ to be injective.

6.2 Yamabe Energy of Conformal Immersions in the Sphere

Recall first the *Yamabe energy*, which we define for a Riemannian manifold (M, g) by

$$\mu(M) = \inf\{J(u) : u \in C_c^\infty(M) \setminus \{0\}\},$$

where $J(u)$ is the *Yamabe quotient*

$$J(u) = \frac{E(u)}{\|u\|_{L^{2^\star}}^2}, \quad E(u) = a \int_M |\nabla u|^2 + \int_M R u^2.$$

Note that we are now not requiring M to be compact. If $(\overline{M}, g)$ is a Riemannian manifold with boundary, we associate to it the Yamabe energy of the interior. When M is complete (in particular compact), this agrees with our previous definition by virtue of the density theorem [59] Theorem 2.4. It is still true that $\mu(M)$ is a conformal invariant. We begin with a simple lemma that will let us estimate $\mu(M)$ using subsets.

For us a *compact exhaustion* is a sequence of open sets $\Omega_1, \Omega_2, \ldots$ of M with smooth boundaries and compact closures such that $\overline{\Omega_i} \subset \Omega_{i+1}$ for every i and

$$\bigcup_{i \geq 0} \Omega_i = M.$$

Such an exhaustion always exists, see for instance Proposition 2.28 in [77]. An important fact is that if K is a compact set in M, then $K \subset \Omega_i$ for some i.

Lemma 6.9. *Let (M, g) be any Riemannian manifold. If $\{\Omega_i\}$ is a compact exhaustion of M, then*

$$\lim_{i \to \infty} \mu(\Omega_i) = \mu(M).$$

Proof. Since $C_c^\infty(\Omega_i) \subset C_c^\infty(M)$ for each i, $\liminf_{i \to \infty} \mu(\Omega_i) \geq \mu(M)$. Choose $\varepsilon > 0$ and $u \in C_c^\infty(M)$ such that $\mu(M) > J(u) + \varepsilon$. But $\operatorname{supp} u \subset \Omega_j$ for some j, so we have $\mu(M) > \mu(\Omega_j) + \varepsilon$. This shows that $\limsup_{i \to \infty} \mu(\Omega_i) \leq \mu(M)$, which completes the proof. $\square$

We reduced the Yamabe problem to showing that when M is closed, $\mu(M) \leq \mu(S^n)$, and $\mu(M) < \mu(S^n)$ unless M is conformally equivalent to S^n. We now consider what happens when M is not necessarily compact. Since we are going to embed manifolds in the sphere, understanding domains of the sphere is important.

Proposition 6.10. *If Ω is an open subset of S^n, then $\mu(\Omega) = \Lambda$.*

Proof. If Ω is not all of S^n, then we may consider the equivalent problem on $\mathbb{R}^n$ by stereographic projection and conformal invariance. Further, by Lemma 6.9 we may restrict attention to $\Omega \subset \mathbb{R}^n$ bounded. But we computed the Yamabe energy of such a set in Proposition 4.8, and found it was Λ. $\square$

Theorem 6.11. *Assume there is a conformal immersion $\Phi : M \to S^n$. Then $\mu(M) = \Lambda$.*

For the proof of this, note that existence and regularity theory of Chapter 3 for the Yamabe equation generalizes to a compact manifold with boundary if we prescribe a Dirichlet boundary condition.

Proof. Since Φ is an immersion, there is an open set U in M such that $\Phi|_U$ is a diffeomorphism onto its image. Then we see $\mu(U) = \mu(\Phi(U)) = \Lambda$ by Proposition 6.10. It follows that $\mu(M) \leq \mu(U) = \Lambda$. The reverse inequality is harder.

Let $\{\Omega_i\}$ be a smooth compact exhaustion of M. We showed above that $\lim \mu(\Omega_i) = \mu(M)$, so it suffices to show that $\mu(\Omega) \geq \Lambda$ for any subdomain Ω with smooth boundary and compact closure. Assume instead that $\mu(\Omega) < \Lambda$. By the existence theorem, we may solve

$$\begin{cases} Lu = \mu(\Omega) u^{2^\star - 1} & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

for $u \in C^\infty(\overline{\Omega})$, with $u > 0$ in M and $\|u\|_{L^{2^\star}} = 1$. Extend u by zero to all of M. Since $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, $\partial u/\partial n \leq 0$ on the boundary, where n is the outer-facing normal field. It follows that for any $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_M u\, L\varphi &= \int_\Omega u\, L\varphi \\ &= \int_\Omega \varphi\, Lu + \int_{\partial\Omega} \varphi \frac{\partial u}{\partial n} \\ &\leq \int_\Omega \varphi\, Lu \\ &= \mu(\Omega) \int_M u^{2^\star-1}\varphi,\end{aligned}$$

i.e. the extension satisfies $Lu \leq \mu(\Omega) u^{2^\star-1}$ in the sense of distributions. Now we define a function v on S^n as follows: Set $v = 0$ on $S^n \setminus \Phi(\overline{\Omega})$ and for $y \in \Phi(\overline{\Omega})$ define

$$v(y) = \max\{|\Phi'(x)|^{-(n-2)/2} u(x) : x \in \Phi^{-1}(y) \cap \overline{\Omega}\}.$$

Here $|\Phi'|$ is defined by $\Phi^*\mathcal{S} = |\Phi'|^2 g$, where $\mathcal{S}$ is the round metric on S^n. Since Φ is an immersion and Ω is relatively compact, the set $\Phi^{-1}(y) \cap \overline{\Omega}$ is finite, and for each x in this set there is a neighborhood V_x of x such that $\Phi|_{V_x}$ is a diffeomorphism onto its image, with associated local inverse Φ_x^{-1}. Notice that the function v_x on $\Phi(V_x)$ defined by

$$v_x(y) = |(\Phi_x^{-1})'(y)|^{(n-2)/2} u(\Phi_x^{-1}(y))$$

Satisfies $L_{\mathcal{S}} v_x \leq \mu(\Omega) v_x^{2^\star-1}$ on $\Phi(V_x)$. This follows from a calculation as in Lemma 4.23. Thus we see that v is a nonnegative Lipschitz function on S^n satisfying $L_{\mathcal{S}} v \leq \mu(\Omega) v^{2^\star-1}$ in the sense of distributions. By conformal invariance we have

$$\int_{\Phi(V_x)} v_x^{2^\star} = \int_{V_x} u^{2^\star},$$

and hence $\|v\|_{L^{2^\star}} \leq 1$. Integrating the differential inequality satisfied by v we get

$$E_{\mathcal{S}}(v) = \int_{S^n} v\, Lv \leq \mu(\Omega) \int_{S^n} v^{2^\star}.$$

This implies $\Lambda = \mu(S^n) \leq \mu(\Omega)$, a contradiction. ☐

Corollary 6.12. *Assume there is a conformal immersion $\Phi : M \to S^n$. Then $\mu(M) = \mu(S^n)$. Then $\mu(\Omega) = \Lambda$ for every open subset Ω of M.*

Proposition 6.13. *Let Ω be a relatively compact open set in a Riemannian manifold (M, g). Then $\mu(\Omega)$ has the same sign as the principal eigenvalue $\lambda_1(\Omega)$ of L on Ω with the Dirichlet boundary condition.*

Proof. The proof is the same as Proposition 4.4 (i), but we use Rayleigh's formula over the Sobolev space $H_0^1(\Omega)$ in the $\mu \geq 0$ cases, and in the $\mu < 0$ case solve $Lu = \lambda_1 u$ in $H_0^1(\Omega)$. ☐

6.3 Existence of the Conformal Green's Function

In this section we prove the existence of a unique minimal Greens's function for the conformal Laplacian on certain open manifolds.

Theorem 6.14. *Let (M, g) be a locally conformally flat Riemannian manifold of dimension $n \geq 3$. Suppose there exists a conformal map $\Phi : M \to S^n$. For any $p \in M$, there is unique minimal and positive Green's function for the conformal Laplacian with pole p. It is C^∞ away from p and satisfies, for any $f \in C_c^\infty(M)$:*

$$\int_M G\, Lf = f(p).$$

Here *minimal* means that if G' is any other positive Green's function for the conformal Laplacian with pole p, $G \leq G'$ on $M \setminus \{p\}$. The proof of Theorem 6.14 is based on the fact that the conformal Green's function on S^n can be pulled back along Φ and provides a convenient barrier for the construction of G. In the case of standard Laplacian, such a barrier is provided by a positive superharmonic function that is decreasing at infinity [80].

Let $\{\Omega_i\}$ be a compact exhaustion of M with associated Dirichlet conformal Green's functions (G_i) such that $p \in \Omega_1$. The existence of the Green's functions on the exhaustion follows from Corollary 6.12, Proposition 6.13, methods from Section 4.2.4 of [7], and Appendix A of [45]. The idea in this section is that the G_i's converge to G uniformly on compact sets, and G inherits its properties from the rigidity of conformally harmonic functions. We now see that the sequence (G_i) is increasing.

Lemma 6.15. *For any $x \in \Omega_i \setminus \{p\}$, $G_i(x) \leq G_{i+1}(x)$.*

Proof. Let $\varepsilon > 0$. We want to show that $(1+\varepsilon)G_{i+1} \geq G_i$ on Ω_i. Since $G_i = 0$ on $\partial\Omega_i$ and $G_{i+1} > 0$ in $\Omega_{i+1} \supset \Omega_i$, $(1+\varepsilon)G_{i+1} \geq G_i$ on $\partial\Omega_i$. Near p we have the expansions

$$G_i(x) = \frac{c_n}{r^{n-2}}(1 + \psi_i(x)), \quad G_{i+1} = \frac{c_n}{r^{n-2}}(1 + \psi_{i+1}(x)),$$

where ψ_i and ψ_{i+1} are $o(1)$ as $r = d(x,p) \downarrow 0$. We have

$$(1+\varepsilon)G_{i+1}(x) - G_i(x) = \varepsilon\frac{c_n}{r^{n-2}} + \frac{o(1)}{r^{n-2}},$$

so for $\eta > 0$ small enough, $(1+\varepsilon)G_{i+1} \geq G_i$ on the sphere $\partial B_{\eta'}(p)$ for any $\eta' \leq \eta$. From the fact that R is bounded on Ω_i, we find that

$$a\Delta G \leq CG_i, \quad a\Delta G \leq CG_{i+1}$$

in $\Omega_i \setminus B_{\eta'}(p)$. Thus by the maximum principle, $(1+\varepsilon)G_{i+1} \geq G_i$ on $\Omega_i \setminus B_{\eta'}(p)$. Now let $\eta' \downarrow 0$ and $\varepsilon \downarrow 0$ to conclude $G_i(x) \leq G_{i+1}(x)$ for any $x \neq p$. ◻

This trick of replacing the curvature by its supremum is interesting because we can circumvent the sign requirement of the maximum principle. We may define the entire Green's function on M to be

$$G(x) = \lim_{i\to\infty} G_i(x)$$

in the pointwise sense, because this limit exists everywhere.

Lemma 6.16. *G is positive everywhere. It is smaller than any entire Green's function. Further, G is defined independently of of the exhaustion sequence.*

Proof. By the maximum principle, each G_i is positive in Ω_i. So for every x, the sequence $(G_i(x))$ is eventually positive and is increasing, so the limit is positive.

Suppose G' is a positive Green's function. By a similar argument to the proof of Lemma 6.15, $G' \geq G_i$ on Ω_i, for every i. So now take $i \to \infty$ to conclude that $G \leq G'$. The uniqueness of G follows from a similar argument, using the fact that if $\{\Omega_i\}$ and $\{\Omega_i'\}$ are compact exhaustions, then any Ω_i is contained in some Ω_j'. □

Remark 6.17. At this point we do not know that G is actually an entire Green's function, so the second statement does not imply the third. Our proof that G is an entire Green's function is independent of the third statement here, however.

The utility of G being unique is that we may use any compact exhaustion to construct it, and the limit will be the same in each case. We still have to show that G is a Green's function. We now construct the barrier for the sequence of Dirichlet Green's functions.

Since $\Phi : M \to S^n$ is a conformal map, there is a smooth function $|\Phi'|$ on M such that

$$\Phi^*\mathcal{S} = |\Phi'|^2 g,$$

where $\mathcal{S}$ is the round metric on S^n. Note that $|\Phi'|$ is the natural norm of $d\Phi : TM \to TS^n$.

Proposition 6.18. *Let $p \in M$ and $\Phi(p) = y \in S^n$. If H denotes the conformal Green's function of $(S^n, \mathcal{S})$ at y and we define $\overline{H} = |\Phi'|^{(n-2)/2} H \circ \Phi$, then*

$$L\overline{H} = \sum_{q \in \Phi^{-1}(y)} |\Phi'(q)|^{-(n-2)/2} \delta_q.$$

In their paper [105], Schoen and Yau have an incorrect formula for $L\overline{H}$. This follows from an erroneous application of the transformation formula for L. In a sense, this is due to the delta functions having different "normalizations" depending on which metric is being used to compute the Laplacian. Indeed, it is well known in the physics literature that the delta function of "curved space" differs from that of "flat space" by a factor of $\sqrt{\det g}$.

Proof. Let $g_0 = \Phi^*\mathcal{S}$ and u be the smooth positive function on M such that $g = u^{4/(n-2)} g_0$, or, equivalently, $u = |\Phi'|^{-(n-2)/2}$. We first show that

$$L_0(H \circ \Phi) = \sum_{q \in \Phi^{-1}(y)} \delta_q. \tag{6.1}$$

Note that Φ is an immersion, hence a diffeomorphism of a neighborhood U_q of each $q \in \Phi^{-1}(y)$, and also an isometry in the g_0 metric. Further, the support of the distribution $L_0(H \circ \Phi)$ must be contained in $\Phi^{-1}(y)$ because everywhere else it is smooth and L_0-harmonic. Thus to prove (6.1), it suffices to consider test functions supported in U_q, for any test function may be written as a sum of test functions supported in each U_q (or possibly zero), and a test function whose support avoids $\Phi^{-1}(y)$. For $\varphi \in C_c^\infty(U_q)$, we have

$$\int_{U_q} H \circ \Phi \, L_0 \varphi \, d\mu(g_0) = \int_{\Phi(U_q)} H \, L_{\mathcal{S}}(\varphi \circ \Phi|_{U_q}^{-1}) \, d\mu(\mathcal{S}) = \varphi(q),$$

which verifies (6.1), and we have

$$\int_{U_p} u^{-1} H \circ \Phi \, L\varphi \, d\mu(g) = \int_{U_p} H \circ \Phi \, L_0(u\varphi) \, \mathrm{d}\mu(g_0) = u(q)\varphi(q).$$

Thus for a general $\varphi \in C_c^\infty(M)$,

$$\int_M \overline{H} \, L\varphi = \sum_{q \in \Phi^{-1}(y)} u(q)\varphi(q). \qquad \square$$

Each of the numbers $|\Phi(q)|^{-(n-2)/2}$ are positive, so by rescaling $\overline{H}$ by $|\Phi'(p)|^{(n-2)/2}$ we obtain a new distribution $\overline{G}$, called the *pullback Green's function*, satisfying

$$L\overline{G} = \sum_{q \in \Phi^{-1}(y)} a_q \delta_q,$$

where each a_q is positive and $a_p = 1$.

Proposition 6.19. *$G \leq \overline{G}$ on $M \setminus \{p\}$.*

It is in this sense that $\overline{G}$ is a barrier for the construction.

Proof. We have to characterize the growth of $\overline{G}$ near p before we apply the maximum principle. From the estimates on H in Appendix A of [45] and the fact that Φ is a local isometry $(M, g_0) \to (S^n, \mathcal{S})$,

$$H \circ \Phi(x) = \frac{c_n}{d_{g_0}(x,p)^{n-2}}(1 + \psi(x))$$

near p, where $\psi(x) = o(1)$. It is easily seen that

$$d_{g_0}(x,p) \sim |\Phi'(x)| d_g(x,p)$$

as $x \to p$. Thus, there is a $\psi_1(x) = o(1)$ such that

$$\overline{H}(x) = |\Phi'(p)|^{-(n-2)/2} \frac{c_n}{d_g(x,p)^{n-2}}(1 + \psi_1(x)).$$

So when we normalize $\overline{H}$ to $\overline{G}$, we obtain the correct growth near p. Then the maximum principle applies as shown in Lemma 6.15. $\square$

Proof of Theorem 6.14. The first task is to show that the only pole G has is p. Indeed, $\overline{G}$ has as many poles as the cardinality of $\Phi^{-1}(y)$, so it's not clear *a priori* that G can't blow up at some of these points. However, each G_i is conformally harmonic, so we may use the Harnack inequality to get a bound. Let $q \in \Phi^{-1}(y) \setminus \{p\}$. If there is no such point, this step can be skipped. Let i be large enough such that $q \in \Omega_i$. Then each G_i solves the same elliptic equation in a small ball B around q. If $x \in B \setminus \{q\}$, the sequence $(G_i(x))$ is increasing and bounded by $\overline{G}(x) < \infty$, so it converges. Hence it is Cauchy, and for $\varepsilon > 0$ there is a number N such that $0 \leq G_m(x) - G_n(x) < \varepsilon$ for all $m \geq n > N$. But $G_m - G_n$ is a nonnegative conformally harmonic function, so the Harnack inequality ([53] Theorem 8.20) gives a constant C, independent of m and n, such that

$$\sup_B |G_m - G_n| \le C \inf_B |G_m - G_n| \le C\varepsilon.$$

Thus (G_i) is a Cauchy sequence in $L^\infty(B)$, so it converges uniformly to G in B.

Now consider a decreasing sequence (δ_i) of small numbers such that $\delta_i \to 0$. We may repeat the previous argument on the sets

$$D_i = \Omega_i \setminus B_{\delta_i}(p)$$

to conclude that G_i converges uniformly to G in each set D_i. Let us show that G is distributionally conformally harmonic in D_i, hence smooth ([55] Corollary 7.20). Indeed, for any $\varphi \in C_c^\infty(D_i)$, we have

$$0 = \int_{D_i} LG_j\, \varphi = \int_{D_i} G_j\, L\varphi \underset{j\to\infty}{\to} \int_{D_i} G\, L\varphi$$

since the convergence is uniform and D_i is compact. Thus $G \in C^\infty(M \setminus \{p\})$.

The only thing left is to show that G is actually a Green's function. For $\varepsilon > 0$, we use uniform convergence again to see that for i large enough,

$$0 \le G(x) - G_i(x) \le \varepsilon \quad \text{on } \partial B_1(p).$$

The maximum principle asserts that

$$G_i(x) \le G(x) \le G_i(x) + \varepsilon \quad \text{on } B_1(p) \setminus \{p\}.$$

Now, if $\varphi \in C_c^\infty(M)$, then

$$\begin{aligned} \left| \int_M G\, L\varphi - \int_M G_i\, L\varphi \right| &\le \int_M |G - G_i||L\varphi| \\ &\le \varepsilon \int_{B_1(p)} |L\varphi| + \int_{M\setminus B_1(p)} |G - G_i||L\varphi|. \end{aligned}$$

The first term vanishes as $\varepsilon \downarrow 0$, and the second vanishes as $i \to \infty$ because of the uniform convergence of G_i on the compact set $\operatorname{supp} L\varphi \setminus B_1(p)$. Then, using the fact that

$$\int_M G_i\, L\varphi = \int_{\Omega_i} G_i\, L\varphi = \varphi(p),$$

we conclude that

$$\int_M G\, L\varphi = \varphi(p)$$

for any $\varphi \in C_c^\infty(M)$. $\square$

Let us prove some auxiliary results involving G that will be important later in the chapter.

Proposition 6.20. *$G < \overline{G}$ on $M \setminus \{p\}$ if $\Phi^{-1}(y)$ contains more than one point.*

Proof. Let $q \in \Phi^{-1}(y)$ be a point other than y. Then $\overline{G}$ blows up at q, but we have seen that G is finite near q. Consider a compact exhaustion $\{\Omega_i\}$ of M with $q \in \Omega_1$, and let $\delta > 0$ be small enough so that $G < \overline{G}$ on $\partial B_{2\delta}(q)$. If $G(x) = \overline{G}(x)$ for some $x \in M$, then $x \in \Omega_i$ for some i. Then, applying the trick of the proof of Lemma 6.15, we use the strong maximum principle to conclude that $G = \overline{G}$ on all of $\Omega_i \setminus B_\delta(q)$. This contradicts $G < \overline{G}$ on $\partial B_{2\delta}(q) \subset \Omega_i \setminus B_\delta(q)$. □

Proposition 6.21. *Suppose there exists a Riemannian covering map $\pi : \tilde{M} \to M$. Then for any $\tilde{p} \in \tilde{M}$ there exists a minimal positive Green's function $\tilde{G}$ for the conformal Laplacian on $\tilde{M}$. If the covering is nontrivial, $\tilde{G} < G \circ \pi$ on $\tilde{M} \setminus \{\tilde{p}\}$, where G is the minimal positive Green's function at $\pi(\tilde{p}) \in M$.*

Proof. The existence of $\tilde{G}$ is clear from everything above. The only difference is that π is a local isometry so the business with $|\Phi'|$ can be avoided. The inequality has the exact same proof as Proposition 6.20. □

It turns out that the injectivity of the map $\Phi : M \to S^n$ is dependent on the growth of G "at infinity". This will be made precise in the following section, but for now we derive a fundamental estimate.

Theorem 6.22. *Assume there is a conformal immersion $\Phi : M \to S^n$. Let g be a compatible metric on M with minimal conformal Green's function G with pole $p \in M$. For any relatively compact open neighborhood $\mathcal{O}$ of p we have*

$$\int_{M\setminus\mathcal{O}} G^{2^\star} < \infty.$$

Proof. Let $\{\Omega_i\}$ be a compact exhaustion with Dirichlet Green's functions $\{G_i\}$. Without loss of generality assume $\mathcal{O} \subset\subset \Omega_1$ and L is coercive on $\Omega_1 \setminus \mathcal{O}$. Let $\zeta \in C_c^\infty(M)$ be unity on $\mathcal{O}$ and with support in Ω_1. Since G_i is conformally harmonic in Ω_i, it minimizes energy in $\Omega_i \setminus \mathcal{O}$ for its boundary values by the variational formulation of the Lax–Milgram theorem ([24] Corollary 5.8). Consequently,

$$E_{\Omega_i\setminus\mathcal{O}}(G_i) \leq E_{\Omega_i\setminus\mathcal{O}}(\zeta G_i) \tag{6.2}$$

for every i. We claim the quantity on the right is bounded independently of i. Note that $G_i(x) \uparrow G(x)$ for every $x \neq p$, and G integrable on compact sets away from p. Thus the curvature term satisfies

$$\begin{aligned} \left| \int_{\Omega_i\setminus\mathcal{O}} R\zeta^2 G_i^2 \right| &= \left| \int_{\Omega_1\setminus\mathcal{O}} R\zeta^2 G_i^2 \right| \\ &\leq \sup_{\Omega_1\setminus\mathcal{O}} |R| \int_{\Omega_1\setminus\mathcal{O}} G_i^2 \\ &\to \sup_{\Omega_1\setminus\mathcal{O}} |R| \int_{\Omega_1\setminus\mathcal{O}} G^2 < \infty, \end{aligned}$$

where the convergence is due to the monotone convergence theorem. It is thus bounded independently of i. The kinetic term is a little more delicate. The goal is to use Schauder theory to estimate ∇G_i since it satisfies the elliptic equation $LG_i = 0$ in $\Omega_i \setminus \{p\}$. First, note that ζG_i satisfies

$$|\nabla(\zeta G_i)|^2 = \zeta^2 |\nabla G_i|^2 + G_i^2 |\nabla \zeta|^2 + 2\langle \nabla \zeta, \nabla G_i \rangle.$$

The integral of the second term is of course bounded by a monotone convergence argument, and the integral of the third term can be estimated above by the integral of the first term. If $\mathcal{W}$ is a neighborhood of $\Omega_1 \setminus \mathcal{O}$ such that $\Omega_1 \setminus \mathcal{O} \subset\subset \mathcal{W} \subset\subset M$, the Schauder interior estimates imply that

$$\sup_{\Omega_1 \setminus \mathcal{O}} |\nabla G_i| \leq C \sup_{\mathcal{W}} G_i,$$

for C a constant independent of i. Squaring and integrating this inequality completes the proof that $E_{\Omega_i \setminus \mathcal{O}}(G_i)$ is bounded above.

In particular, this suggests that the long-range behavior of the G_i's does not increase energy much. We now claim that $E_{\Omega_i \setminus \mathcal{O}}((1-\zeta)G_i)$ is bounded above. To see this, split the integrals into the regions where $1 - \zeta \equiv 1$ and where $1 - \zeta$ is variable. Where it is unity, the bound on $E_{\Omega_i \setminus \mathcal{O}}(G_i)$ shows their contribution must be bounded. The other integral is over a relatively compact domain and a similar argument as above establishes the bound.

The Sobolev inequality Proposition 6.10 implies

$$\int_{\Omega_i \setminus \mathcal{O}} [(1-\zeta)G_i]^{2^\star} \leq C$$

for $C > 0$ independent of i, because $\mu(\Omega_i) = \Lambda$ for all i. Note that $(1-\zeta)G_i$ is a smooth function with zero boundary values, so it is in the closure of C_c^∞ by Theorem 2.20 and we may apply the Sobolev inequality. The monotone convergence theorem gives

$$\int_{M \setminus \mathcal{O}} [(1-\zeta)G]^{2^\star} < \infty.$$

Since the $1 - \zeta$ factor modifies the integral on a region where G is bounded, this completes the proof. □

6.4 Conformal Invariants

It turns out the growth of the Green's function is closely tied to the injectivity of the developing map Φ. For an LCF manifold M with holonomy cover $\hat{M}$, we define a number $p(M)$ called the *conformal critical exponent* by

$$p(M) := \inf \left\{ p \geq 0 : \int_{\hat{M} \setminus \mathcal{O}} G^p < \infty \quad \forall \text{ open } \mathcal{O} \text{ containing the pole of } G \right\}.$$

By Theorem 6.22, we have that $p(M)$ is finite. Also note that we do not require a uniform bound on the integrals over $\hat{M} \setminus \mathcal{O}$, so this doesn't actually measure the growth of the pole. To show that this number is well defined, we need the following:

Proposition 6.23. *The value of $p(M)$ is independent of the position of the pole of the Green's function used in the definition.*

Proof. Suppose we have minimial positive conformal Green's functions G and G' with poles at q and q', respectively. Let $\{\Omega_i\}$ be a compact exhaustion of $\hat{M}$ such that $q, q' \in \Omega_1$ and let $\mathcal{O}$ be any bounded open set containing q and q'. If (G_i) and (G_i') are sequences of Dirichlet Green's functions with poles at q and q' respectively, then the uniqueness result Lemma 6.16 says that $G = \lim G_i$ and $G' = \lim G_i'$, with the convergence uniform on compact sets avoiding the poles. There exists a number $c \geq 1$ such that

$$c^{-1}G' \leq G \leq cG' \quad \text{on } \partial\mathcal{O}.$$

So by the uniform convergence,

$$c^{-1}G_i' \leq G_i \leq cG_i' \quad \text{on } \partial\mathcal{O}$$

for any i large enough. Since $G_i = G_i' = 0$ on $\partial\Omega_i$, we have by the maximum principle (see the trick in the proof of Lemma 6.15),

$$c^{-1}G_i' \leq G_i \leq cG_i' \quad \text{on } \Omega_i \setminus \mathcal{O}.$$

Letting $i \to \infty$, we see that

$$c^{-1}G' \leq G \leq cG' \quad \text{on } \hat{M} \setminus \mathcal{O}.$$

This means the growth rate is independent of the pole. $\square$

We also consider a scaled version of $p(M)$, namely the *conformal dimension*

$$d(M) := \frac{n-2}{2}p(M).$$

By Theorem 6.22, we always have

$$d(M) \leq n.$$

Proposition 6.24. *For the standard sphere S^n, we have $d(S^n) = 0$.*

Proof. S^n is its own holonomy cover, so the result follows from compactness. $\square$

In a sense, $d(M)$ measures how far away a manifold is from being conformally equivalent to the standard sphere. The following result justifies our use of this invariant in the context of the Yamabe problem.

Proposition 6.25. *If (M, g) is a compact LCF manifold, then $d(M)$ depends only on the conformally flat structure of M and is independent of the choice of conformal metric.*

Proof. Let g' be a pointwise conformal metric on M. Then the conformal factor lifts to $\hat{M}$ and is a bounded smooth function u. The conformal Green's functions are related by $G(x) = u(p)u(x)G'(x)$ (Proposition 4.25). Since u is bounded, $G \in L^p$ if and only if $G' \in L^p$. $\square$

Here are the fundamental estimates of $d(M)$:

Theorem 6.26. *Let (M, g) be an LCF manifold.*

(i) *Suppose R is strictly positive and bounded away from zero, i.e. $R \geq R_0 > 0$ with R_0 constant. Then $d(M) \leq (n-2)/2$.*

(ii) *Suppose $R \geq 0$ and let $\lambda_1(\hat{M})$ denote the first eigenvalue of the Laplacian of $\hat{M}$ in $H^1_0(\hat{M})$.* (a) *If $\lambda_1(\hat{M}) > 0$, then $d(M) \leq (n-2)/2$.* (b) *If $\lambda_1(\hat{M}) = 0$, then $d(M) \leq n/2$.*

(iii) *Suppose $R \geq R_0 > 0$. Additionally, assume that $|R| + |\nabla R| \leq C$ and $\mathrm{Ric} \geq -bg$ for some constants $C, b > 0$. Then $d(M) < (n-2)/2$.*

Proof. (i) Let $\{\Omega_i\}$ be an exhaustion of $\hat{M}$ with Green's functions (G_i). Using coercivity, we may find functions $v_1, v_2, \ldots$ solving

$$\begin{cases} Lv_i = 1 & \text{in } \Omega_i \\ v_i = 0 & \text{on } \partial\Omega_i \end{cases}$$

with $v_i \in C^\infty(\overline{\Omega_i})$ and positive in the interiors. Then we have

$$v_i(p) = \int_{\Omega_i} G_i \, Lv_i = \int_{\Omega_i} G_i$$

At an interior maximum x, $\Delta v_i(x) \leq 0$, which implies

$$\max_{\Omega_i} v_i \leq \frac{1}{R_0}.$$

Thus $v_i(p)$ is bounded for all i, and

$$\int_{\hat{M}} G = \lim_{i\to\infty} \int_{\Omega_i} G_i < \infty.$$

(ii) Since $R \geq 0$ we have $\Delta G_i \geq 0$ on $\Omega_i \setminus \mathcal{O}$. Letting $\varepsilon > 0$ and integrating by parts gives

$$\begin{aligned} \int_{\Omega_i\setminus\mathcal{O}} |\nabla G_i^{(1+\varepsilon)/2}|^2 &= -\int_{\Omega_i\setminus\mathcal{O}} G_i^{(1+\varepsilon)/2} \Delta G_i^{(1+\varepsilon)/2} + \int_{\partial\mathcal{O}} G_i^{(1+\varepsilon)/2} \frac{\partial}{\partial n} G_i^{(1+\varepsilon)/2} \\ &= -\frac{1+\varepsilon}{2} \int_{\Omega_i\setminus\mathcal{O}} G_i^{\varepsilon} \Delta G_i + \frac{1-\varepsilon}{1+\varepsilon} \int_{\Omega_i\setminus\mathcal{O}} |\nabla G_i^{(1+\varepsilon)/2}|^2 \\ &\quad + \int_{\partial\mathcal{O}} G_i^{(1+\varepsilon)/2} \frac{\partial}{\partial n} G_i^{(1+\varepsilon)/2}. \end{aligned}$$

Using the Schauder estimates, one can check that the last term on the right is bounded independently of i. So we have a constant C independent of i such that

$$\frac{2\varepsilon}{1+\varepsilon} \int_{\Omega_i\setminus\mathcal{O}} |\nabla G_i^{(1+\varepsilon)/2}|^2 + \frac{1+\varepsilon}{2} \int_{\Omega_i\setminus\mathcal{O}} G_i^{\varepsilon} \Delta G_i \leq C. \tag{6.3}$$

Both terms are nonnegative, so both are bounded by C.

When $\lambda_1 > 0$ and $R \geq 0$, we obtain from the Rayleigh principle the Poincaré inequality

$$\int_{\hat{M}} u^2 \leq \frac{1}{\lambda_1} \int_{\hat{M}} |\nabla u|^2 \quad \forall u \in H^1_0.$$

We apply the Poincaré inequality to the first term in (6.3) after rounding off G_i in a controlled way with a cutoff function near $\partial\mathcal{O}$ to obtain

$$\int_{\Omega_i\setminus\mathcal{O}} G_i^{1+\varepsilon} \le C(\varepsilon).$$

Letting $i\to\infty$ and then $\varepsilon\downarrow 0$ gives $d(M)\le (n-2)/2$.

In the other case, we don't have the Poincaré inequality available. However, we can still estimate

$$\left(\int_{\Omega_i\setminus\mathcal{O}} \zeta^{2^\star} G_i^{(1+\varepsilon)2^\star/2}\right)^{2/2^\star} \le \mu(\Omega_i\setminus\mathcal{O})^{-1} E_{\Omega_i\setminus\mathcal{O}}\left(\zeta G_i^{(1+\varepsilon)/2}\right), \tag{6.4}$$

where $\mu(\Omega_i\setminus\mathcal{O})=\Lambda$ and ζ is a cutoff function near $\partial\mathcal{O}$. On the other hand,

$$E_{\Omega_i\setminus\mathcal{O}} = a\int_{\Omega_i\setminus\mathcal{O}} |\nabla(\zeta G^{(1+\varepsilon)/2})|^2 + \int_{\Omega_i\setminus\mathcal{O}} R\zeta^2 G_i^{1+\varepsilon}.$$

By (6.3) and the Schauder estimates, the first integral on the right is bounded by $C(\varepsilon)$. For the second integral, using $LG_i=0$ on $\Omega_i\setminus\mathcal{O}$ we have

$$\int_{\Omega_i\setminus\mathcal{O}} R\zeta^2 G_i^{1+\varepsilon} = \int_{\Omega_i\setminus\mathcal{O}} \zeta^2 G_i^{\varepsilon}\Delta G_i.$$

This is also bounded by (6.3) and the Schauder estimates. So we see from (6.4) that $p(M)\le (1+\varepsilon)2^\star/2$, so by taking $\varepsilon\downarrow 0$ we conclude $d(M)\le n/2$.

(iii) We use the stronger curvature estimates to improve the result (i). Let $B_r=B_r(p)$ and $B_b=B_b(p)$, where $1\le r<b$. Define two cutoff functions $\phi,\eta\in C_c^\infty(\hat{M})$ such that

$$\phi=\begin{cases}0 & \text{on } \hat{M}\setminus B_{2b}\\ 1 & \text{on } B_b\end{cases},\quad |\nabla\phi|\le 2/b$$

and

$$\eta=\begin{cases}0 & \text{on } B_r\\ 1 & \text{on } M\setminus B_{r+1}\end{cases},\quad |\nabla\eta|\le 2.$$

Then we have

$$\int_{B_{2b}\setminus B_{r+1}} G \le \int_{\hat{M}\setminus B_r}\phi\eta G.$$

Since $-a\Delta G+RG=0$ on $\hat{M}\setminus B_r$ implies $R_0G\le a\Delta G$ there, we have

$$\begin{aligned} R_0\int_{\hat{M}\setminus B_r}\phi\eta G &\le a\int_{\hat{M}\setminus B_r}\phi\eta\Delta G\\ &= a\int_{\hat{M}\setminus B_r}\langle\phi\nabla\eta+\eta\nabla\phi,\nabla G\rangle\\ &\le C\left(\int_{B_{r+1}\setminus B_r} G+\frac{1}{b}\int_{B_{2b}\setminus B_b} G\right),\end{aligned}$$

where in the last line we used the Cheng–Yau gradient estimate Theorem A.6. Now, letting $b \to \infty$ in this inequality, we get

$$\int_{\hat{M}\setminus B_r} G \le C_1 \int_{B_{r+1}\setminus B_r} G,$$

where C_1 is a constant independent of r. Then, by writing

$$\int_{\hat{M}\setminus B_r} G = \int_{\hat{M}\setminus B_{r+1}} G + \int_{B_{r+1}\setminus B_r} G,$$

we obtain

$$\int_{\hat{M}\setminus B_{r+1}} G \le \lambda \int_{\hat{M}\setminus B_r} G,$$

where $\lambda = C_1/(1+C_1) < 1$. This inequality can be easily iterated. We find for $N = 1, 2, \ldots,$

$$\int_{\hat{M}\setminus B_N} G \le \lambda^{N-1} \int_{\hat{M}\setminus B_1} G.$$

Under the assumption $\mathrm{Ric} \ge -bg$, the Bishop–Gromov volume comparison theorem can be used to show that

$$\mathrm{vol}(B_N) \le C_2 e^{C_3 N},$$

where C_2 and C_3 are constants independent of N. Let $\varepsilon > 0$. By the Hölder inequality,

$$\begin{aligned}\int_{\hat{M}\setminus B_1} G^{1-\varepsilon} &= \sum_{N=2}^{\infty} \int_{B_N\setminus B_{N-1}} G^{1-\varepsilon} \\ &\le \sum_{N=2}^{\infty} \left(\int_{\hat{M}\setminus B_{N-1}} G \right)^{1-\varepsilon} \mathrm{vol}(B_N \setminus B_{N-1})^{\varepsilon} \\ &\le C_3 \sum_{N=2}^{\infty} \lambda^{(N-2)(1-\varepsilon)} e^{\varepsilon C_3 N}.\end{aligned}$$

By choosing $\varepsilon < 1/C_3$, we can ensure the series converges. Thus $G^{1-\varepsilon}$ is integrable away from the pole, so $p(M) < 1$. □

6.5 Injectivity of the Developing Map

The goal of this section is to show that many LCF manifolds are in fact conformally equivalent to subsets of S^n. The main hypothesis is smallness of $d(M)$, which in turns controls the growth of the Green's function at infinity. The main embedding theorem is

Theorem 6.27. *Let (M, g) be a complete Riemannian manifold and $\Phi : M \to S^n$ a conformal map. Assume that R is bounded below on M. For $n = 3$ or 4 also assume that R is bounded. If*

$$d(M) < \frac{(n-2)^2}{n},$$

then Φ is injective and gives a conformal diffeomorphism of M onto $\Phi(M) \subset S^n$.

The condition on $d(M)$ is convenient for the proof, but inconvenient for applications. We postpone the proof of this theorem and now show how to apply the fundamental estimates of Theorem 6.26 to obtain a conformal embedding.

Corollary 6.28. *Let (M, g) be a complete Riemannian manifold with $R \geq 0$ and let $\Phi : M \to S^n$ be a conformal map. Suppose one of the following holds:*

(i) $n \geq 4$, $R \geq R_0 > 0$, and if $n = 4$, $|R| + |\nabla R| \leq C$ and $\mathrm{Ric} \geq -bg$.
(ii) $n \geq 5$, and $\lambda_1(\hat{M}) > 0$.
(iii) $n \geq 7$.

Then Φ is injective.

Proof. By Theorem 6.26, under the hypotheses (i) or (ii) we have $d(M) \leq (n-2)/2$. Since $(n-2)/2 \leq (n-2)^2/n$ for $n \geq 5$ we may apply Theorem 6.27 to establish the conclusion. When $n = 4$, then $(n-2)/2 = (n-2)^2/n = 1$, so part (iii) of Theorem 6.26 gives the result. Under hypothesis (iii) we use the second statement of part (ii) of Theorem 6.26, namely that $d(M) \leq n/2$, and then note that $n/2 \leq (n-2)^2/n$ when $n \geq 7$. □

Corollary 6.29. *Let (M, g) be a complete LCF manifold satisfying the hypotheses of Corollary 6.28. Then $\hat{M}$ is just $\tilde{M}$, the universal cover of M (with the covering metric) and the developing map $\Phi : \tilde{M} \to S^n$ is injective.*

If (M, g) is also compact with positive Yamabe invariant and $n \geq 4$, then after a conformal deformation of g, the conclusion holds.

Proof. Consider Corollary 6.28 applied to $\tilde{M}$, which has a conformal map Φ into S^n by Kuiper's theorem. So $\Phi : \tilde{M} \to S^n$ is injective because the curvature conditions lift to the covering metric of M. This implies that the holonomy representation $\rho : \pi_1(M) \to \mathrm{Conf}(n)$ is injective. Indeed, suppose there is a deck transformation $\gamma : \tilde{M} \to \tilde{M}$ that is mapped to the identity map of S^n, i.e. $\Phi \circ \gamma = \Phi$. But since Φ is injective, γ must just be the identity.

If M is compact and has positive Yamabe invariant, we can make a conformal deformation to positive scalar curvature. When M is compact, the curvature conditions in (i) hold automatically. □

We now begin the proof of Theorem 6.27. Recall that by scaling $|\Phi'|^{(n-2)/2} H \circ \Phi$, we obtain a "almost Green's function" $\overline{G}$ satisfying

$$L\overline{G} = \sum_{q \in \Phi^{-1}(\Phi(p))} a_q \delta_q,$$

with $a_q > 0$ for all q and $a_p = 1$. Therefore, to prove Φ is injective it suffices to prove $G = \overline{G}$, because then $\Phi^{-1}(\Phi(p)) = \{p\}$. To do this, we consider the function $v = G/\overline{G}$ and want to show $v \equiv 1$. It will be convenient to denote $\Phi^{-1}(\Phi(p))$ by W.

Lemma 6.30. *The function v is a positive harmonic function on $M \setminus W$ with respect to the metric $\overline{g} = \overline{G}^{4/(n-2)} g$, and we have in normal coordinates centered at p:*

$$v(x) = 1 + h(x), \quad h \in C^\infty, \quad h = O(|x|^{n-2}).$$

Moreover, $\overline{g}$ is a flat metric on $M \setminus W$.

Proof. We first show that $\overline{g}$ is flat. Note that $H^{4/(n-2)}\mathcal{S}$ is a flat metric on $S^n \setminus \{y\}$ by Proposition 4.22. If $\sigma : S^n \setminus \{y\} \to \mathbb{R}^n$ is an appropriately scaled stereographic projection, then

$$\begin{aligned}(\Phi^* \circ \sigma^*)\delta &= \Phi^*(H^{4/(n-2)}\mathcal{S}) \\ &= (H \circ \Phi)^{4/(n-2)}\Phi^*\mathcal{S} \\ &= (H \circ \Phi)^{4/(n-2)}|\Phi'|^2 g \\ &= \overline{G}^{4/(n-2)} g \\ &= \overline{g}.\end{aligned}$$

Thus $\overline{g}$ is locally isometric to the Euclidean metric via $\sigma \circ \Phi$.

Next we show that v is $\overline{g}$-harmonic. By definition, $\overline{g} = (G/v)^{4/(n-2)}g$. Hence by the formula of Lemma 4.23,

$$L_{\overline{g}}v = L_g G = 0 \quad \text{on } M \setminus W.$$

On the other hand, Proposition 6.19 states that $0 \leq v \leq 1$. It follows from the proof of 6.19 that

$$v(p) = \lim_{x \to p} \frac{G(x)}{\overline{G}(x)} = 1.$$

Now if $v = 1 + h$, then $\overline{G} - G = hG$ and $L(hG) = 0$ on $B_\delta(p) \setminus \{p\}$. But $hG = O(|x|)G = O(|x|^{3-n})$, so by an application of Lemma 3.11, we see that p is a removable singularity. Thus hG is smooth near p, so $h = O(|x|^{n-2})$. $\square$

This next result gives pointwise decay of the Green's function at infinity.

Lemma 6.31. *Suppose (M, g) is complete and $\Phi : M \to S^n$ is a conformal map. Assume that $R \geq -C$ for a constant C. Then*

$$\lim_{d(x,p) \to \infty} G(x) = 0.$$

Proof. By Theorem 6.22,

$$\int_{M \setminus \mathcal{O}} G^{2^\star} < \infty.$$

So by standard measure theory,

$$\lim_{\sigma \to \infty} \int_{\rho(x,p) \geq \sigma} G^{2^\star} = 0.$$

Therefore, the proof will be done if we can establish the estimate

$$G(x) \leq C \left(\int_{B_1(x)} G^{2^\star} \right)^{1/2^\star}$$

for some constant C and all $x \in M \setminus B_2(p)$. Since $\mu(M) = \Lambda$, the Sobolev inequality implies

$$\left(\int_M |\phi|^{2^\star} \right)^{2/2^\star} \leq \Lambda^{-1} \int_M \left(a|\nabla \phi|^2 + R^+ \phi^2 \right) \tag{6.5}$$

for any $\phi \in C_c^\infty(M \setminus B_1(p))$, where $R^+ = \max\{R, 0\}$. The equation $LG = 0$ combined with the lower bound on R implies

$$a\Delta G - R^+G \geq -CG,$$

for some constant C. Choose some $q \geq 2^\star$. Multiplying this inequality by $G^{q-1}\phi^2$ and integrating by parts gives

$$(q-1)\int_M \phi^2 G^{q-2}|\nabla G|^2 + a^{-1}\int_M \phi^2 G^q R^+ \leq 2\int_M \phi G^{q-1}|\nabla\phi||\nabla G| + C\int_M G^q\phi^2. \quad (6.6)$$

For any $\alpha > 0$, the Cauchy inequality gives

$$2\int_M \phi G^{q-1}|\nabla\phi||\nabla G| \leq \alpha\int_M \phi^2 G^{q-2}|\nabla G|^2 + \frac{1}{4\alpha}\int_M |\nabla\phi|^2 G^q. \quad (6.7)$$

Now combine (6.6) and (6.7) with $\alpha = q-2$ to get

$$\int_M \left(\phi^2 G^{q-2}|\nabla G|^2 + a^{-1}R^+\phi^2 G^q\right) \leq \frac{1}{q-2}\int_M |\nabla\phi|^2 G^q + C\int_M \phi^2 G^q. \quad (6.8)$$

Letting $\alpha = 1$ in (6.7) and using (6.8) gives

$$2\int_M \phi G^{q-1}|\nabla\phi||\nabla G| \leq \frac{q-1}{q-2}\int_M |\nabla\phi|^2 G^q + C\int_M \phi^2 G^q.$$

It then follows that

$$\begin{aligned}\int_M \left(|\nabla(\phi G^{q/2})|^2 + a^{-1}R^+\phi^2 G^q\right) &\leq \int_M \big(|\nabla\phi|^2 G^q + q|\nabla\phi||\nabla G|\phi G^{q-1} \\ &\qquad + \frac{q^2}{4}\phi^2 G^{q-2}|\nabla G|^2 + a^{-1}R^+\phi^2 G^q\big) \\ &\leq a^{-1}q^2\int_M \left(|\nabla\phi|^2 + \phi^2\right)G^q.\end{aligned}$$

Applying (6.5), we have

$$\left(\int_M (\phi G^{q/2})^{2^\star}\right)^{2/2^\star} \leq Cq^2\int_M G^q\left(|\nabla\phi|^2 + \phi^2\right), \quad (6.9)$$

where C is independent of q. We perform Moser iteration using (6.9).

Define $q_k = q_0 r^k$, where $q_0 = 2^\star = 2r$. Define a sequence of cutoff functions as follows: Set $a_0 = 1$,

$$a_k = 1 - \sum_{i=1}^k 3^{-i}$$

for $k \geq 1$, and require for each k that the function $\phi_k \in C_c^\infty(M)$ satisfies

$$\phi_k(y) = \begin{cases} 1 & \text{if } y \in B_{a_k}(x) \\ 0 & \text{if } y \notin B_{a_{k-1}}(x) \end{cases},$$

$0 \le \phi_k \le 1$, and $|\nabla \phi_k| \le 2 \cdot 3^k$. We we get iteratively from (6.9) that

$$\begin{aligned}\left(\int_{B_{a_{k+1}}} G^{q_{k+1}}\right)^{1/q_{k+1}} &\le (Cq_k^2)^{1/q_k}(4\cdot 3^{2k+2}+1)^{1/q_k}\left(\int_{B_k} G^{q_k}\right)^{1/q_k}\\ &\le \prod_{j=1}^{k}(Cr^{2j})^{1/(2^\star r^j)}\left(\int_{B_1} G^{2^\star}\right)^{1/2^\star}.\end{aligned}$$

To see that the product converges, we simply take the logarithm

$$\log\left(\prod_{j=1}^{k}(Cr^{2j})^{1/(2^\star r^j)}\right) = \sum_{j=1}^{k}\frac{1}{2^\star r^j}\left(\log C + 2j\log r\right),$$

which converges as $k \to \infty$. Then, since

$$\sup_{y\in B_{1/2}(x)} G(y) = \lim_{k\to\infty}\left(\int_{B_{a_{k+1}}} G^{q_{k+1}}\right)^{1/q_{k+1}},$$

we conclude that

$$\sup_{y\in B_{1/2}(x)} G(y) \le C\left(\int_{B_1(x)} G^{2^\star}\right)^{1/2^\star},$$

which concludes the proof. □

Lemma 6.32. *Suppose there is a conformal map $\Phi : M \to S^n$. Then there exists a constant $C > 0$ such that for any $\phi \in C_c^\infty(M)$,*

$$\int_M \phi^2|\nabla \log \overline{G}|^2 \le C\int_M \left(\phi^2|\nabla\log G|^2 + |\nabla\phi|^2\right). \tag{6.10}$$

Proof. First choose $\phi \in C_c^\infty(M\setminus W)$, i.e. the support avoids all of the poles of $\overline{G}$ and G. Away from the poles we have

$$\begin{aligned}\Delta\log\overline{G} &= \overline{G}^{-1}\Delta\overline{G} - |\nabla\log\overline{G}|^2\\ &= a^{-1}R - |\nabla\log\overline{G}|^2\\ &= G^{-1}\Delta G - |\nabla\log\overline{G}|^2\\ &= \Delta\log G + |\nabla\log G|^2 - |\nabla\log\overline{G}|^2.\end{aligned}$$

Multiplying by ϕ^2 on both sides and integrating by parts gives

$$\int_M \phi^2|\nabla\log\overline{G}|^2 \le \int_M \phi^2|\nabla\log G|^2 + 2\int_M \phi\langle\nabla\phi, \nabla(\log\overline{G} - \log G)\rangle. \tag{6.11}$$

Now use Cauchy's inequality with ε to estimate

$$\begin{aligned}\phi\langle\nabla\phi,\nabla(\log\overline{G}-\log G)\rangle &\le \phi|\nabla\phi||\nabla(\log\overline{G}-\log G)|\\ &\le \frac{1}{4\varepsilon}|\nabla\phi^2| + \varepsilon\phi^2|\nabla(\log\overline{G}-\log G)|^2\\ &\le \frac{1}{4\varepsilon}|\nabla\phi|^2 + 3\varepsilon\phi^2\left(|\nabla\log\overline{G}|^2 + |\nabla\log G|^2\right).\end{aligned}$$

Inserting this into (6.11) and choosing ε suitably small gives (6.10) for a constant $C > 0$ independent of ϕ. For a general $\phi \in C_c^\infty(M)$, use a cutoff technique to create a sequence of functions $\phi_j \in C_c^\infty(M \setminus W)$ that approximate it in C^1_{loc}. Then the limit of (6.10) with ϕ_j is (6.10) with ϕ, so it is valid in that case. $\square$

To continue we need a refinement of Kato's inequality $|\nabla|\nabla u||^2 \le |\operatorname{Hess} u|^2$ due to Stein, [115] Section VII.3.1.2.

Lemma 6.33. *Let A be a traceless symmetric $n \times n$ matrix with real entries. If $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm and $\|\cdot\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm, then*

$$\|A\|_{\mathrm{op}}^2 \le \frac{n-1}{n}\|A\|_{\mathrm{HS}}^2.$$

Proof. It is clear that both norms are invariant under the action of $\mathrm{SO}(n)$, so we may work in a basis where A is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$. Since A is traceless we get $\sum_{j=1}^n \lambda_j = 0$, so $\lambda_{j_0} = -\sum_{j \ne j_0} \lambda_j$ and the Cauchy-Schwarz inequality for sums gives

$$\lambda_{j_0}^2 \le (n-1)\sum_{j \ne j_0} \lambda_j^2.$$

Now let $\lambda_{j_0}^2 = \max \lambda_j^2$ and add it $n-1$ times to both sides of this inequality. This gives the result after evaluating the two norms for the diagonal matrix A. $\square$

Proposition 6.34 (Harmonic Kato inequality)**.** *Let u be a harmonic function in an open set of $\mathbb{R}^n$. Then*

$$|\nabla|\nabla u||^2 \le \frac{n-1}{n}|\operatorname{Hess} u|^2,$$

where we are interpreting $|\nabla u|$ as an H^1 function [53, Lemma 7.6].

Proof. This is trivially true where $|\nabla u| = 0$, so we assume that $|\nabla u| > 0$. Then, as in the proof of the usual Kato inequality we have

$$\nabla|\nabla u| = \frac{\nabla\langle\nabla u, \nabla u\rangle}{2|\nabla u|} = \frac{\langle \operatorname{Hess} u, \nabla u\rangle}{|\nabla u|}.$$

We use Lemma 6.33 to estimate the right-hand side. Letting $\operatorname{Hess} u = A$ and $\nabla u = X$, we find

$$\frac{|\langle \operatorname{Hess} u, \nabla u\rangle|^2}{|\nabla u|^2} = \frac{|A(X)|^2}{|X|^2} \le \|A\|_{\mathrm{op}}^2 \le \frac{n-1}{n}\|A\|_{\mathrm{HS}}^2 = \frac{n-1}{n}|\operatorname{Hess} u|^2. \qquad \square$$

Whenever we put a bar on a quantity or operator, we mean these are with respect to the flat metric $\overline{g} = \overline{G}^{4/(n-2)} g$. Since $\overline{\Delta} v = 0$ and $\overline{g}$ is flat, Bochner's formula gives immediately:

Lemma 6.35. *The function $v = G/\overline{G}$ satisfies*

$$\overline{\Delta}|\overline{\nabla} v|^2 = 2|\overline{\operatorname{Hess}}\, v|^2.$$

Lemma 6.36. *Suppose there is a conformal map $\Phi : M \to S^n$. Then there exists a constant $C > 0$ such that for any $\phi \in C_c^\infty(M)$,*

$$\int_M \phi^2 |\overline{\nabla}|\overline{\nabla} v||^2 V \, d\overline{\mu} \le C \int_M |\nabla \phi|^2 \overline{G}^q |\nabla v|^q \, d\mu, \tag{6.12}$$

where $q = 2(n-2)/n$ and $V = |\overline{\nabla} v|^{q-2}$ when $\overline{\nabla} v \neq 0$, and 0 otherwise.

Proof. We first prove (6.12) for $\phi \in C_c^\infty(M \setminus W)$. The original argument of Schoen and Yau proceeds by estimating $\overline{\Delta}|\overline{\nabla} v|^q$ and then integrating it against ϕ^2. This works when $q \ge 2$, but the case we are interested in has $q < 2$, so one has to supplement the argument because then $|\overline{\nabla} v|^q$ is not necessarily C^2.

To remedy this, let $\varepsilon > 0$ and define $e = |\overline{\nabla} v|^2$ (the energy density). Then, since $e + \varepsilon > 0$, the following calculation holds everwhere on $M \setminus W$:

$$\begin{aligned}
\overline{\Delta}(e+\varepsilon)^{q/2} &= \frac{q}{2} \overline{\nabla}_i \left[(e+\varepsilon)^{(q-2)/2} \overline{\nabla}_i e \right] \\
&= \frac{q(q-2)}{4} (e+\varepsilon)^{(q-4)/2} |\overline{\nabla} e|^2 + \frac{q}{2} (e+\varepsilon)^{(q-2)/2} \overline{\Delta} e.
\end{aligned}$$

Now, according to Proposition 6.34 and Lemma 6.35, we have

$$\overline{\Delta} e = 2 |\overline{\mathrm{Hess}}\, v|^2 \ge \frac{2n}{n-1} |\overline{\nabla}|\overline{\nabla} v||^2$$

almost everywhere. Furthermore,

$$|\overline{\nabla} e|^2 = |2 |\overline{\nabla} v| \overline{\nabla} |\overline{\nabla} v||^2 = 4e |\overline{\nabla}|\overline{\nabla} v||^2$$

almost everywhere. Putting everything together, we have

$$\overline{\Delta}(e+\varepsilon)^{q/2} \ge q |\overline{\nabla}|\overline{\nabla} v||^2 \left\{ \frac{n}{n-1} (e+\varepsilon)^{(q-2)/2} + (q-2)(e+\varepsilon)^{(q-4)/2} e \right\} \tag{6.13}$$

almost everywhere. We multiply this inequality by ϕ^2, $\phi \in C_c^\infty(M \setminus W)$, and integrate over $M \setminus W$. Since $(e+\varepsilon)^{q/2} \in C^\infty$, we have

$$\begin{aligned}
\int_{M \setminus W} \phi^2 \overline{\Delta}(e+\varepsilon)^{q/2} \, d\overline{\mu} &= - \int_{M \setminus W} \langle \overline{\nabla} \phi^2, \overline{\nabla}(e+\varepsilon)^{q/2} \rangle \, d\overline{\mu}. \\
&= -q \int_{M \setminus W} \phi (e+\varepsilon)^{(q-2)/2} \langle \overline{\nabla} \phi, \overline{\nabla} e \rangle \, d\overline{\mu} \\
&\le q \int_{M \setminus W} |\phi| (e+\varepsilon)^{(q-2)/2} |\overline{\nabla} \phi| |\overline{\nabla} e| \, d\overline{\mu} \\
&= 2q \int_{M \setminus W} |\phi| |\overline{\nabla} \phi| (e+\varepsilon)^{(q-2)/2} e |\overline{\nabla}|\overline{\nabla} v|| \, d\overline{\mu}.
\end{aligned}$$

Now let $\eta > 0$ and estimate

$$|\phi| |\overline{\nabla}|\overline{\nabla} v|| |\overline{\nabla} \phi| e \le \eta \phi^2 |\overline{\nabla}|\overline{\nabla} v||^2 + \frac{1}{4\eta} |\overline{\nabla} \phi|^2 e^2.$$

So, by using (6.13), we have

$$q \int_{M\setminus W} \phi^2 |\overline{\nabla}|\overline{\nabla} v||^2 \left\{ \frac{n}{n-1}(e+\varepsilon)^{(q-2)/2} + (q-2)(e+\varepsilon)^{(q-4)/2} e \right\} d\overline{\mu}$$

$$\leq 2q\eta \int_{M\setminus W} \phi^2 |\overline{\nabla}|\overline{\nabla} v|^2 (e+\varepsilon)^{(q-2)/2}\, d\overline{\mu} + \frac{q}{2\eta} \int_{M\setminus W} |\overline{\nabla}\phi|^2 (e+\varepsilon)^{(q-2)/2} e^2\, d\overline{\mu}.$$

Now set $q = 2(n-2)/n$. This is clearly strictly less than 2, so $q-2$ and $q-4$ are negative. Consequently, for $x \in M \setminus W$, the function

$$\varepsilon \mapsto (e(x)+\varepsilon)^{(q-k)/2}, \quad k = 2, 4$$

is increasing as $\varepsilon \downarrow 0$. So we may apply the monotone convergence theorem to the inequality. Define the function

$$V(x) = \begin{cases} |\overline{\nabla} v|^{q-2}(x) & \text{if } \overline{\nabla} v(x) \neq 0 \\ 0 & \text{if } \overline{\nabla} v(x) = 0. \end{cases}$$

Notice also that if $\overline{\nabla}|\overline{\nabla} v|$ is defined at a point where $\overline{\nabla} v = 0$, then it is zero. We then have the following pointwise limits (almost everywhere):

$$\begin{aligned} \lim_{\varepsilon\to 0} |\overline{\nabla}|\overline{\nabla} v||^2 (e+\varepsilon)^{(q-2)/2} &= |\overline{\nabla}|\overline{\nabla} v||^2 V \\ \lim_{\varepsilon\to 0} |\overline{\nabla}|\overline{\nabla} v||^2 (e+\varepsilon)^{(q-4)/2} e &= |\overline{\nabla}|\overline{\nabla} v||^2 V. \\ \lim_{\varepsilon\to 0} (e+\varepsilon)^{(q-4)/2} e^2 &= |\overline{\nabla} v|^q. \end{aligned}$$

So, by using the monotone convergence theorem, we find

$$\left(\frac{n}{n-1} + q - 2 - 2\eta \right) \int_{M\setminus W} \phi^2 |\overline{\nabla}|\overline{\nabla} v||^2 V\, d\overline{\mu} \leq \frac{1}{2\eta} \int_{M\setminus W} |\overline{\nabla}\phi|^2 |\overline{\nabla} v|^q\, d\overline{\mu}.$$

For $n/(n-1) + q - 2$ to be positive, we need $q > (n-2)/(n-1)$. It is easy to see that this indeed the case for our choice of q. So for η small enough, there is a $C > 0$ such that

$$\int_{M\setminus W} \phi^2 |\overline{\nabla}|\overline{\nabla} v||^2 V\, d\overline{\mu} \leq C \int_{M\setminus W} |\overline{\nabla}\phi|^2 |\overline{\nabla} v|^q\, d\overline{\mu}.$$

Changing the measure and norms the right to those of g gives a factor of $\overline{G}^q$, where $q = 2(n-2)/n$. So in this case we have verified (6.12).

Next observe that near a pole of $\overline{G}$ we have

$$|\nabla v| = O(|x|^{n-2}), \quad \overline{G} = O(|x|^{2-n}).$$

Enumerate W as $\{y_i\}$ and choose $\psi \in C_c^\infty(M \setminus W)$ to be a function satisfying

$$\psi(x) = \begin{cases} 0 & \text{on } B_{r_i}(y_i) \\ 1 & \text{on } M \setminus \bigcup B_{2r_i}(y_i) \end{cases}$$

and

$$|\nabla\psi| \le r_i^{-1} \quad \text{on } B_{2r_i}(y_i).$$

For any $\phi \in C_c^\infty(M)$, the special case we have already proved implies

$$\int_M \psi^2\phi^2|\overline{\nabla} v|^{q-2}|\overline{\nabla}|\overline{\nabla} v||^2\, d\overline{\mu} \le C\int_M |\nabla(\psi\phi)|^2\overline{G}^q|\nabla v|^q\, d\mu.$$

Cauchy's inequality gives

$$|\nabla(\psi\phi)|^2 \le C\big(\psi^2|\nabla\phi|^2 + \phi^2|\nabla\psi|^2\big),$$

so we have

$$\begin{aligned}\int_M |\nabla(\psi\phi)|^2\overline{G}^q|\nabla v|^q\, d\mu \le{}& C\int_M \psi^2|\nabla\phi|^2\overline{G}^q|\nabla v|^q\, d\mu \\ &+ C\sum_{y_i\in W}\int_M \phi^2|\nabla\psi|^2 d(x,y_i)^{-q}\, d\mu.\end{aligned}$$

By the definition of ψ, this last term is bounded by

$$\sum_{y_i\in W} C_i r_i^{n-2-q}.$$

Since $n-2-q>0$ we may let $r_i \to 0$ for each i and conclude (6.12) holds for every ϕ. □

Lemma 6.37. *Suppose there is a conformal map $\Phi : M \to S^n$. Then there exists a constant $C > 0$ such that for any $\rho > 0$ sufficiently large,*

$$\int_{B_\rho(p)} |\overline{\nabla}|\overline{\nabla} v||^2 V\, d\overline{\mu} \le \frac{C}{\rho^2}\int_{B_{4\rho(p)}\setminus B_{\rho/2}(p)} G^q\big(1+|\nabla\log G|^2\big)\, d\mu, \tag{6.14}$$

where $q = 2(n-2)/n$.

Proof. We want to make use of (6.12). Away from poles, note that

$$\begin{aligned}\overline{G}^q|\nabla v|^q &= |\nabla G - G\overline{G}^{-1}\nabla\overline{G}|^q \\ &\le C(|\nabla G|^q + G^q|\nabla\log\overline{G}|^q)\end{aligned}$$

by virtue of

$$\begin{aligned}|a+b|^p &\le (2\max\{a,b\})^p \\ &= 2^p\max\{a^p,b^p\} \\ &\le 2^p(a^p+b^p),\end{aligned}$$

valid for any $a,b,p \ge 0$. Thus, if we take $\phi \in C_c^\infty(M)$ such that

$$\phi = \begin{cases} 1 & \text{on } B_\rho(p) \\ 0 & \text{on } M\setminus B_{2\rho}(p)\end{cases}, \quad |\nabla\phi| \le 2/\rho,$$

we see that the RHS of (6.12) is controlled by

$$\frac{C}{\rho^2}\int_{B_{2\rho}\setminus B_\rho}\big(|\nabla G|^q + G^q|\nabla\log\overline{G}|^q\big)\,d\mu. \tag{6.15}$$

Note that for any function $f : U \subset M \to [0,\infty)$,

$$f^q \leq 1 + f^2.$$

Indeed, when $f \leq 1$, $f^q \leq 1$, and when $f \geq 1$, $f^q \leq f^2$ because $q < 2$. So we have

$$|\nabla\log G|^q \leq 1 + |\nabla\log G|^2,$$

or,

$$|\nabla G|^q \leq G^q(1 + |\nabla\log G|^2).$$

So we derive from (6.15) that

$$\int_{B_\rho}|\overline{\nabla}|\overline{\nabla}v||^2 V\,d\overline{\mu} \leq \frac{C}{\rho^2}\int_{B_{2\rho}\setminus B_\rho} G^q\big(1 + |\nabla\log\overline{G}|^q + |\nabla\log G|^2\big).$$

Now estimate

$$|\nabla\log\overline{G}|^q \leq 1 + |\nabla\log\overline{G}|^2,$$

which gives

$$\int_{B_\rho}|\overline{\nabla}|\overline{\nabla}v||^2 V\,d\overline{\mu} \leq \frac{C}{\rho^2}\int_{B_{2\rho}\setminus B_\rho} G^q\big(1 + |\nabla\log\overline{G}|^2 + |\nabla\log G|^2\big). \tag{6.16}$$

Now let $\phi = \psi G^{q/2}$ in (6.10), where $\psi \in C_c^\infty(M \setminus \{p\})$ to get

$$\begin{aligned}\int_M \psi^2 G^q|\nabla\log\overline{G}|^2 &\leq C\int_M\big(\psi^2G^q|\nabla\log G|^2 + |\nabla(G^{q/2}\psi)|^2\big)\\ &\leq C\int_M\big(\psi^2|\nabla\log G|^2 + |\nabla\psi|^2\big)G^q.\end{aligned}$$

We choose $\psi = 0$ on $B_{\rho/2}$ and on $M \setminus B_{4\rho}$ and $\psi = 1$ on $B_{2\rho}\setminus B_{\rho/2}$ with $|\nabla\psi| \leq 1$. We then conclude (6.14) from (6.16). □

We are now ready to prove the final estimate.

Lemma 6.38. *Under the hypotheses of Theorem 6.27,*

$$\int_{B_\rho}|\overline{\nabla}|\overline{\nabla}v||^2 V\,d\overline{\mu} \leq \frac{C}{\rho^2}\int_{M\setminus B_{\rho/4}} G^s\,d\mu, \tag{6.17}$$

where $s = q$ if $n \neq 4$ and $s \in (0,1)$ when $n = 4$.

Proof. Consider first the case $n \geq 5$. Then $q > 1$ and using $R \geq -C$ we have

$$a\Delta G \leq CG.$$

Multiplying by $\psi^2 G^{q-1}$ with $\psi \in C_c^\infty(M \setminus \{p\})$ from the previous proof and integrating by parts gives

$$a(q-1)\int_M \psi^2 G^{q-2}|\nabla G|^2 + 2a\int_M G^{q-1}\psi\langle\nabla\psi,\nabla G\rangle \leq C\int_M \psi^2 G^q.$$

Now move the second term on the left to the right, use the Cauchy–Schwarz inequality, then Cauchy's inequality with ε to obtain

$$\int_M \psi^2 G^{q-2}|\nabla G|^2 \leq \frac{C}{q-1}\int_M G^q\big(\psi^2 + |\nabla\psi|^2\big). \tag{6.18}$$

Note that the LHS may be written as

$$\int_M G^q|\nabla \log G|^2\psi^2.$$

So by using the properties of ψ, we reach the conclusion.

Consider next $n = 3$. Here $q = 2/3$ and by hypothesis, $|R| \leq C$. We now estimate

$$\Delta G \leq a^{-1}CG$$

and multiply by $\psi^2 G^{q-1}$, which leads to (6.18), but now with $1-q$ instead of $q-1$. This also gives the conclusion.

When $n = 4$, $q = 1$, and we cannot use this trick. Recall Lemma 6.31, which assures that G is bounded on $M\backslash B_1$. Thus estimate (6.14) holds for $s \in (0,1)$ in place of q. Consequently, we obtain (6.18) with s in place of q. This completes the proof. □

We are finally ready to prove that $v \equiv 1$.

Proof of Theorem 6.27. Since by assumption

$$p(M) = \frac{2}{n-2}d(M) < \frac{2(n-2)}{n} = q,$$

there exist $q_i > p(M)$ and $q_i \to p(M)$ such that

$$\int_{M\backslash B_1} G^{q_i} < \infty.$$

Let i be such that $q_i < q$. Then applying Lemma 6.31 we have

$$\int_{M\backslash B_1} G^q \leq \sup_{M\backslash B_1} G^{q-q_i}\int_{M\backslash B_1} G^{q_i} < \infty. \tag{6.19}$$

Now use (6.17) (choose s very close to 1 when $n = 4$), combined with (6.14) to conclude

$$\int_{B_\rho} |\overline{\nabla}|\overline{\nabla}v||^2 V\, d\overline{\mu} \leq \frac{C}{\rho^2},$$

where $C > 0$ is independent of ρ. Then let $\rho \to \infty$ to see

$$|\overline{\nabla} v| = \text{const}.$$

Since

$$|\overline{\nabla} v|(p) = \overline{G}^{-1}|\nabla G - v\nabla\overline{G}|(p) = 0$$

(take limits to be rigorous here), we have $|\nabla v| \equiv 0$. But since $v(p) = 1$, we conclude that $v = 1$ everywhere. This completes the proof. ☐

6.6 A Positive Mass Theorem

We now discuss the case $R \geq 0$ and relate it to the positive mass theorem. Suppose (M, g) is complete LCF and that for every $p \in M$ the minimal conformal Green's function G for L at p exists. As in Chapter 4, we consider the manifold $\hat{M} = M \setminus \{p\}$ with the metric $\hat{g} = G^{4/(n-2)}g$. Consider a conformally flat chart near p,

$$\Psi : \mathcal{O} \to \mathbb{R}^n \cup \{\infty\},$$

where $\mathcal{O}$ is a neighborhood of p and such that $\Psi(p) = \infty$. To construct such a chart, choose a standard conformally flat chart that maps p to zero, and then perform an inversion in $\mathbb{R}^n$. Let $\delta(x) = (dx^1)^2 + \cdots + (dx^n)^2$ denote a Euclidean metric. Writing

$$\hat{g} = h^{4/(n-2)}\delta(x),$$

we see that h is an asymptotically constant harmonic function defined outside of a ball in $\mathbb{R}^n$. Indeed, the metric $\hat{g}$ has zero scalar curvature since $LG = 0$. Then, by the formula in Proposition 4.23,

$$0 = L_{\hat{g}}1 = h^{-(n+2)/(n-2)}L_\delta h,$$

which implies

$$\Delta_\delta h = L_\delta h = 0.$$

Letting a denote the asymptotic value of h, we have $h = a + f$, for f a harmonic function tending to zero at infinity. By Lemma 7.14 below, we have the expansion

$$h = a + b|x|^{2-n} + O(|x|^{1-n})$$

as $x \to \infty$, with $a > 0$.

If we had chosen another chart Ψ' with $\Psi'(p) = \infty$ and denote its coordinates by y, then we see that $\Psi \circ \Psi'^{-1} : y \mapsto x$ is a conformal transformation fixing ∞, hence by Liouville's theorem is a composition of a rotation, reflection, scaling, and translation:

$$x = lSy + t, \quad l > 0, S \in \mathrm{O}(n), t \in \mathbb{R}^n.$$

Therefore we have

$$\begin{aligned}\hat{g} &= h(x)^{4/(n-2)}\delta(x) \\ &= h(lSy + t)^{4/(n-2)}l^2\delta(y) \\ &=: h'(y)^{4/(n-2)}\delta(y).\end{aligned}$$

Then $h'(y)$ has the expansion

$$h'(y) = a' + b'|y|^{2-n} + O(|y|^{1-n}),$$

where $a' = l^{(n-2)/2}a$ and $b' = l^{(2-n)/2}b$. It follows that $a'b' = ab$ so we define $E(p) = ab$ and observe that $E(p)$ depends only on p and the metric g. If we replace g by a conformal metric $\tilde{g} = u^{4/(n-2)}g$, then from Proposition 4.25 we obtain $\tilde{E}(p) = u(p)^{(n+2)/(n-2)}E(p)$, i.e. the sign of $E(p)$ depends only on the conformal class of g. We call E the *energy function* of (M, g).

Proposition 6.39. *Suppose (M, g) is an LCF manifold and $\Phi : M \to S^n$ is a conformal map. The function E is nonnegative if and only if Φ is injective.*

Proof. We have seen that Φ is injective if and only if for every $p \in M$, $G \equiv \overline{G}$. We compare the energy terms of G and $\overline{G}$ for $p \in M$ fixed. Let $\Psi : \mathcal{O} \to \mathbb{R}^n \cup \{\infty\}$ be as above with $\Psi(p) = \infty$, and let $h, \overline{h}$ be the corresponding harmonic functions, with Ψ is chosen so that $h \to 1$ at infinity. Since the metric $\overline{h}^{4/(n-2)}\delta$ is a Euclidean metric we clearly have $\overline{h}(x) = 1$, i.e. $\overline{E}(p) = 0$. Note that $\overline{h} \to 1$ because $\overline{G}$ has a pole of the same strength as G at p and $h \to 1$. We also have $h \le \overline{h}$ near infinity since $G \le \overline{G}$. Thus by the maximum principle we either have $h \equiv \overline{h}$ or $h < \overline{h}$ near infinity. We write the expansion for h

$$h(x) = 1 + E(p)|x|^{2-n} + O(|x|^{1-n}).$$

If $h < \overline{h}$, then $\overline{h} - h$ is a positive harmonic function on $\mathbb{R}^n \setminus B_\sigma$ for some $\sigma > 0$. If we choose a number δ with

$$0 < \delta < \min\{\overline{h}(x) - h(x) : x \in \partial B_\sigma(0)\},$$

then we must have $\overline{h}(x) - h(x) > \delta|x|^{n-2}$ for $|x| \ge \sigma$ since $\delta|x|^{2-n}$ is harmonic and by the maximum principle. Therefore we have $E(p) \le -\delta$. Thus we either have $E(p) < 0$ or $G \equiv \overline{G}$ on M. □

Lemma 6.40. *Suppose $\pi : M' \to M$ is a nontrivial covering of M (i.e. not a diffeomorphism), where (M, g) is LCF and for each $y \in M$ the minimal Green's function with pole at y exists. Then energy functions satisfy $E'(x) < E(\pi(x))$ for every $x \in M'$.*

Proof. This is immediate from Proposition 6.21. □

We now have a type of positive mass theorem.

Theorem 6.41. *Let (M, g) be a complete LCF manifold with $R \ge 0$ and suppose the holonomy cover $\hat{M}$ satisfies one of the hypotheses of Corollary 6.28. Suppose M has a minimal positive conformal Green's function at every point. Then the energy function E on M is nonnegative at every point. If M is not simply connected then E is strictly positive at each point, while if M is simply connected then E vanishes identically.*

Proof. By Corollary 6.28, we obtain an injective conformal map $\Phi : \tilde{M} \to S^n$. Since this is injective, Proposition 6.39 implies $\tilde{E} \ge 0$. If M is not simply connected, then $E > \tilde{E}$ by Lemma 6.40. If M is simply connected, then there is an injective conformal map $\Phi : M \to S^n$. By Lemma 6.30, the metric $\hat{g}$ is flat, since $G \equiv \overline{G}$. But then $h \equiv 1$, so $E(p) = 0$. □

Theorem 6.42. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. If the conformal class of g has positive Yamabe energy and (M, g) locally conformally flat, but not conformally diffeomorphic to the sphere, then for any point $p \in M$, $A > 0$, where A is the number in Theorem 4.26 (i) for the conformal Green's function at p in a flat coordinate system at p.*

Proof. By the results in this section, the energy at p is nonnegative. Note that by Corollary 6.7, M is not simply connected, so in fact $E(p) > 0$. What's left is to relate $E(p)$ to A. Note first that $E(p)$ is positive even when we make the metric flat in a neighborhood of p. So that E is positive if and only if A is follows from Equation (7.2) and the remark after it. □

7

The Positive Mass Theorem in Low Dimensions

7.1 Asymptotically Flat Manifolds and the ADM Mass

When $n < 6$ or (M, g) is locally conformally flat, the expansion of the conformal Green's function in conformal normal coordinates was found to be

$$G(x) = r^{2-n} + A + O_2(r),$$

where A is a constant. It turns out that every test function estimate for the Yamabe functional in these cases requires A to be positive. This is true for both Schoen's original test functions that use the Green's functions directly (Section 4.5) and Lee and Parker's test functions which are more natural and do not use the Green's function on the surface (which we did not use). In this chapter we will show that $A > 0$ is implied by the positive mass theorem (PMT) of general relativity. We justified the claim $A > 0$ for LCF manifolds in the previous chapter when $n \geq 4$.

We say that a complete Riemannian manifold (M, g) with $n \geq 3$ is *asymptotically flat* (AF) of *order* τ if there exists a decomposition $M = M_0 \sqcup M_\infty$, with M_0 compact and M_∞ such that (M_∞, g) is isometric to $(\mathbb{R}^n \setminus B, g^*)$, where B is a closed ball and g^* is a metric such that

$$g^*_{ij} = \delta_{ij} + O(|x|^{-\tau}), \quad \partial_k g^*_{ij} = O(|x|^{-\tau-1}), \quad \partial_j \partial_k g^*_{ij} = O(|x|^{-\tau-2})$$

in some rectangular coordinates on $\mathbb{R}^n$. We call M_0 the *core* and M_∞ the *end* of the manifold. When there is no risk of confusion, we write g^*_{ij} as just g_{ij}. The physical interpretation of this is a constant-time slice of a spacetime containing an isolated graviational source. Physicists naturally wondered about the total mass or energy of such a system. Einstein and Weyl developed a definition by eliminating the second order part of the Einstein–Hilbert action by adding a divergence to the action, and then applying Noether's theorem [120] [39]. It was later rediscovered by Arnowitt, Deser, and Misner, and it now bears the name *ADM mass* in their honor [3]. The ADM mass $m(g)$ is defined by the formula

$$m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii}) n^j \, d\mathcal{H}^{n-1},$$

where S_r is the coordinate sphere of radius r and n is its normal vector. If this definition is to be physically reasonable, then it should be nonnegative and a geometric invariant under physically reasonable conditions on (M, g). Further, if $m(g) = 0$, there should be no sources at all, i.e. (M, g) should be flat. Bartnik showed that under sufficiently strong decay conditions, the mass is indeed a geometric invariant [12].

Theorem 7.1. *If (M, g) is AF of order $\tau > (n-2)/2$ and its scalar curvature R is in $L^1(M)$, then the numerical value of $m(g)$ is independent of the particular coordinate system used, as long as one remains in the class $\tau > (n-2)/2$.*

This result is optimal: Denisov and Solov'ev constructed a 3-dimensional AF manifold with $\tau = 1/2$ that has has any real number as its mass depending on how the coordinates are chosen [42]. Bartnik actually worked in the category of weighted Sobolev spaces, which has good analytic properties for elliptic PDE. We will explain this approach in Section 6.2.

The celebrated *positive mass theorem* says that the ADM mass is indeed positive when one expects so physically:

Theorem 7.2 (Positive Mass Theorem)**.** *Let (M, g) be asymptotically flat of order $\tau > (n-2)/2$ and with $R \in L^1(M)$. If $R \geq 0$, then $m(g) \geq 0$, and $m(g) = 0$ if and only if (M, g) is isometric to $\mathbb{R}^n$ with the standard metric.*

The condition $R \geq 0$ is implied by the Hamiltonian constraint equation for a maximal (zero mean curvature) slice in a spacetime obeying the dominant energy condition. See [35] or [57] for more physical background.

The history of the proof of this theorem is quite colorful. The first big result was by Schoen and Yau in 1979 for 3-manifolds that were close to the $t = 0$ slice of Schwarzschild spacetime [103]. In [104] they extended the result to 4-dimensional spacetimes (not discussed here). In 1981, Witten gave a heuristic proof of the spacetime positive mass theorem for 4-dimensional spacetimes that admit spin structures [121]. The key was to use a Dirac spinor to write a formula for the mass which is obviously nonnegative when $R \geq 0$. His work was formalized by Parker and Taubes in [96]. In 1984, Schoen announced in his Yamabe problem paper that he and Yau had the proof of the general PMT in all dimensions [101]. It wasn't until 1986 that Bartnik proved the mass is a geometric invariant, however [12]. Bartnik also adopted Witten's method and proved the Riemannian positive mass theorem on for spin manifolds in all dimensions. Lee and Parker sketched a method for extending the Riemannian PMT to all manifolds in dimensions ≤ 7, but were unable to go higher because of technical issues with the singularity set of minimal surfaces in dimensions 8 and above [78]. However, they did manage to prove the rigidity part of the PMT in all dimensions, though their proof assumed the implication of the inequality part. Finally in 1989 the ≤ 7 cases were definitely resolved by Schoen [108]. In 1990 Kuwert wrote a review on the situation up to that point [75]. His account is very complete and readable.

The next breakthrough was in 1999 when Lohkamp essentially compactified the positive mass theorem [84]. He showed that the validity of the positive mass theorem is implied by the following positive scalar curvature theorem:

Theorem 7.3. *Let M be a compact n-manifold, $n \geq 3$. Then the connected sum $T^n \# M$, where T^n is the torus $S^1 \times \cdots \times S^1$, does not admit a metric with positive scalar curvature.*

The connection to the positive mass theorem is when M is the one-point compactification of the asymptotically flat manifold. We will explain this reduction in Section 6.3. This method immediately gives a proof of the positive mass theorem for spin manifolds by appealing to Gromov and Lawson's theorems on enlargeable spin manifolds. In Section 6.4 we will prove Theorem 7.3 using minimal surface techniques when $n \leq 7$. In [85], Lohkamp proved this theorem in complete generality using his method of "skin structures."

It should be noted that [85] was published on the arXiv in 2006, but reuploaded in 2016 with major changes. It was then that Lohkamp also put a proof of the fully general spacetime positive mass theorem on the arXiv [86].

In 2017, Schoen and Yau finally gave their promised proof of the positive mass theorem in the form of a new proof of Theorem 7.3 [107]. As far as I know, the implication Theorem 7.3 $\Longrightarrow$ Theorem 7.2 was not known when Schoen announced their proof in 1984. See also the lecture notes by Li of a course Schoen gave in 2017 on the PMT [102]. We do not attempt to reproduce the proof of the positive mass theorem in complete generality here, because for our purposes it suffices to know the result in low dimensions. We also feel that the approach taken here is easier than the one in [108] and [75] for reasons that will be explained at the end of Section 6.4.

7.2 Asymptotics in Weighted Sobolev Spaces

Before we come to the geometric content of the positive mass theorem, we have to simply the asymptotic behavior of the metric. This involves solving several Poisson-type elliptic equations on the asymptotically flat manifold and obtaining precise control on the asymptotic decay of the solutions. The basic tool for elliptic PDE on asymptotically flat manifolds are the weighted Sobolev spaces. These spaces were introduced by Nirenberg and Walker in [93] and Cantor in [27] [28] [29] [30]. They were subsequently studied by many authors, in particular Choquet-Bruhat and Christodoulou [36] and Bartnik [12]. We will mainly utilize the theory developed by Bartnik. However, we will not attempt to repeat the purely analytical proofs in that paper, so this treatment is not entirely self-contained.

There is another side to this story, namely the weighted Hölder category. These spaces were pioneered by Chaljub-Simon and Choquet-Bruhat in [31]. The main advantage of these spaces is that the existence theory for elliptic equations becomes easier. For a treatment of the positive mass theorem in this category, see Lee and Parker [78]. When we cite [78], we will always refer to results (or proofs) that can be carried over to the weighted Sobolev category in a natural way.

To define the weighted Sobolev spaces spaces on a manifold we first consider the case of $\mathbb{R}^n$. We work with $n \geq 3$, set $r = |x|$, and $\sigma(x) = \sqrt{1 + r^2}$. Relative to this weight, we define the *weighted Lebesgue spaces* $L^p_\delta(\mathbb{R}^n)$ $(1 \leq p < \infty)$ with weight $\delta \in \mathbb{R}$ by taking the completion of the space of those C^∞ functions for which the norm

$$\|u\|_{p,\delta} = \left(\int_M |u|^p \, \sigma^{-\delta p - n} \, dx \right)^{1/p}$$

is finite. The *weighted Sobolev spaces* $W^{k,p}_\delta(\mathbb{R}^n)$ are now defined via the norms

$$\|u\|_{k,p,\delta} = \sum_{j=0}^{k} \|D^j u\|_{p,\delta-j}.$$

The weighted spaces on open sets in $\mathbb{R}^n$ are defined in the obvious way.

A smooth n-manifold M with complete Riemannian metric g is said to be *asymptotically flat* if there is a compact $K \subset M$ such that $M \setminus K$ has a *structure of infinity*: There is $R \geq 1$ and a C^∞ diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_R(0)}$ which satisfies the conditions

- $d\Phi(g)$ is uniformly equivalent to the flat metric δ on $\mathbb{R}^n \setminus \overline{B_R(0)}$, i.e. there is some constant $\lambda \geq 1$ such that

$$\lambda^{-1}|\xi|^2 \leq d\Phi(g)_{ij}(x)\xi^i\xi^j \leq \lambda|\xi|^2 \quad \forall x \in \mathbb{R}^n \setminus \overline{B_R(0)}, \xi \in \mathbb{R}^n$$

- $d\Phi(g) - \delta \in W^{2,q}_{-\tau}(\mathbb{R}^n \setminus \overline{B_R(0)})$ for some $q > n$ and *decay rate/order* $\tau > 0$.

Alternatively, we can view Φ as giving coordinates via $x^i(p) = \Phi^i(p)$ for $p \in M$. We then set $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) = d\Phi(g)_{ij}$ and omit the Φ from the notation when there is no risk for confusion.

To define the weighted spaces on an AF manifold, let $\sigma \in C^\infty(M)$ be a strictly positive function satisfying

$$\sigma(p) = |\Phi(p)| \quad \forall p \in M \setminus K.$$

The Lebesgue spaces are defined in the natural way, but there is a question of what derivatives to use for the Sobolev spaces. On the end, we use the derivatives determined by Φ. In the core, any choices will be equivalent. Thus the Sobolev spaces do depend on the particular structure of infinity, but it turns out that this is not really an issue.

Note that since

$$|f(x)| \leq C|x|^\delta \implies f \in L^q_\delta \quad \forall q > n,$$

the definition of asymptotic flatness in Section 6.1 implies the weaker definition used here.

We now define the natural setting for the ADM mass. Consider the set

$$\mathcal{M}_\tau = \{\text{Riemannian metrics } g : g - \delta \in W^{2,q}_{-\tau}, \operatorname{scal}(g) \in L^1\}$$

topologized by

$$|||g_1 - g_2|||_\tau = \|g_1 - g_2\|_{2,q,-\tau} + \|\operatorname{scal}(g_1) - \operatorname{scal}(g_2)\|_{L^1}.$$

We say that a metric $g \in \mathcal{M}_\tau$ satisfies the *mass condition* if $\tau > (n-2)/2$.

Lemma 7.4. *For any g satisfying the mass condition, $m(g)$ is finite. Let (g_k) be a sequence of metrics satisfying the mass condition such that $g_k \to g$ in $\mathcal{M}_\tau$. Then $m(g_k) \to m(g)$ as $k \to \infty$. Therefore, the mass is a continuous affine functional on $\mathcal{M}_\tau$.*

This is Lemma 3.4 in [75] and follows from results in [12]. We have the following fundamental theorem, [12] Theorem 4.2.

Theorem 7.5 (Bartnik)**.** *Let (M, g) be an asymptotically flat manifold satisfying the mass condition. Then the mass is finite and does not depend on the coordinates used to compute the integral. By this we mean that for any structure of infinity relative to which $g - \delta \in W^{2,q}_{-\tau}$, the numerical value of the mass is the same.*

As mentioned in Section 6.1, the mass condition is optimal. We now state the positive mass theorem in the form which we will prove it.

Theorem 7.6. *Let (M, g) be an asymptotically flat manifold satisfying the mass conditions. If $\mathrm{scal}(g) \geq 0$ and $3 \leq n \leq 7$ or if M is a spin manifold, then $m(g) \geq 0$. The mass is zero if and only if (M, g) is isometric to $(\mathbb{R}^n, \delta)$.*

The proof of the positive mass theorem will be completed in Section 6.5 and extended to a more general class of AF manifolds with multiple ends in Section 6.6.

It follows from Lemma 7.4 that the mass is a smooth function on $\mathcal{M}_\tau$. We have the following useful formula for its derivative.

Theorem 7.7. *Let $g_t, t \in (-\varepsilon, \varepsilon)$, be a differentiable family of metrics satisfying the mass condition. If S denotes the Einstein–Hilbert action, G the Einstein tensor of g_0, and $h = g_t'$, then*

$$2(n-1)\omega_{n-1}\frac{dm}{dt} = \frac{dS}{dt} + \int_M \langle G, h\rangle.$$

This follows from the well-known first variation formula for the Einstein-Hilbert action, see Section 8 in [78] and pages 468-469 in [119] for a physical interpretation.

Remark 7.8. Suppose $\{g(t)\}_{t\in[0,T)}$ is a smooth family of metrics in $\mathcal{M}_\tau$ satisfying the mass condition and evolving by Ricci flow $\partial_t g = -2\,\mathrm{Ric}$. Then by the evolution equations for Ricci flow and the variational formula for the mass,

$$2(n-1)\omega_{n-1}\frac{d}{dt}m(g(t)) = \int_M \Delta R = \int_{S_\infty} \partial_i R\, n^i\, d\mathcal{H}^{n-1}.$$

It was first noticed by Dai–Ma [41] that under sufficiently strong decay conditions for the initial metric, this integral is zero because Shi's estimate gives gradient decay not present at the initial time. Later, McFeron–Székelyhidi [88] and Li [81] showed that the mass is constant under the weaker decay conditions we assume for the positive mass theorem. The integrand in the mass integral is to leading order the scalar curvature, but the lower order terms do not in general decay quickly enough to integrate to zero. Li showed that if the flow exists for all time, then it converges to a metric g_∞, for which the lower order terms in the mass integral *do* integrate to zero. Then one concludes

$$m(g_0) = m(g_\infty) = \frac{1}{2(n-1)\omega_{n-1}}\int_M R(g_\infty),$$

which is manifestly nonnegative by the maximum principle. He also showed that in 3 dimensions, Ricci flow with surgery may be used to conclude the positive mass theorem for any metric.

The following formula is useful to prove the positive mass theorem, but is also what relates the Yamabe problem to the positive mass theorem.

Proposition 7.9. *Let g be a metric satisfying the mass condition such that $g = \varphi^{4/(n-2)}\delta$ outside of some large ball. Suppose*

$$\varphi = 1 + \frac{A}{|x|^{n-2}} + W^{2,q}_{2-n-\varepsilon}$$

for some $\varepsilon > 0$ and $A \in \mathbb{R}$. Then the ADM mass is given by

$$m(g) = 2A.$$

Proof. We first show that for $u \in W^{k,q}_{-\delta}$, $u > -1$, and $s \in \mathbb{R}$, $(1+u)^s - 1 \in W^{k,q}_{-\delta}$. By the weighted Sobolev inequality ([12] Theorem 1.2 (iv)), $\sup u \leq C\|u\|_{k,q,-\tau}$ is finite. Let $-1 \leq l_1 \leq l_2 \leq \sup u$. Then

$$\begin{aligned} |(1+l_2)^s - (1+l_1)^s| &\leq |s| \int_{l_1}^{l_2} (1+x)^{s-1}\,dx \\ &\leq C(s, l_1, l_2)|l_1 - l_2|. \end{aligned}$$

Choosing $l_1 = 0$, it follows that $|(1+u)^s - 1| \leq C|u|$, therefore, $(1+u)^s - 1 \in L^q_{-\delta}$. The integrability of the derivatives follows by induction. We note for later that this argument also implies if $u_i \to u$ in $W^{k,q}_{-\delta}$, then $(1+u_i)^s - 1 \to (1+u)^s - 1$ in $W^{k,q}_{-\delta}$.

We can write this suggestively as $(1 + W^{k,q}_{-\delta})^s = 1 + W^{k,q}_{-\delta}$. It follows that

$$\begin{aligned} \left(1 + \frac{A}{|x|^{n-2}} + W^{2,q}_{2-n-\varepsilon}\right)^{4/(n-2)} &= \left(1 + \frac{A}{|x|^{n-2}}\right)^{4/(n-2)} \left(1 + W^{2,q}_{2-n-\varepsilon}\right) \\ &= 1 + \frac{4}{n-2}\frac{A}{|x|^{n-2}} + W^{2,q}_{2-n-\varepsilon}. \end{aligned}$$

So, near infinity, a direct calculation shows

$$\partial_i g_{ij} - \partial_j g_{ii} = \frac{4(n-1)A}{|x|^{n-1}} n^j + W^{1,q}_{1-n-\varepsilon}.$$

By the Sobolev inequality, $W^{1,q}_{1-n-\varepsilon} = o(r^{1-n-\varepsilon})$, so this term does not contribute to the integral and

$$m(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{R\to\infty} \int_{|x|=R} \frac{4(n-1)A}{R^{n-1}} (n^j)^2\,d\mathcal{H}^{n-1} = 2A. \qquad \square$$

Corollary 7.10. *The ADM mass of the Schwarzschild metric*

$$g = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{4/n-2} \delta,$$

defined on $\mathbb{R}^n \setminus \{0\}$, is m.

Corollary 7.11. *Let (M, g) be a compact Riemannian manifold with positive Yamabe energy and such that $n = 3, 4, 5$, or (M, g) is LCF. In an inverted conformal normal coordinate system, the stereographic projection $(\hat{M}, \hat{g})$ is asymptotically flat of order*

$$\begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n = 4, 5 \\ n-2 & \text{if } (M, g) \text{ is LCF} \end{cases} . \tag{7.1}$$

The ADM mass of the projection is given by

$$m(\hat{g}) = 2A,$$

where A is the expansion coefficient in Theorem 4.26 (i).

Proof. Let $\{x^i\}$ be normal coordinates in a neighborhood $\mathcal{O}$ of p. Define the "inverted" coordinates by

$$z^i = \frac{x^i}{r^2}, \quad \rho = |z| = \frac{1}{r}.$$

Now 0 in the x-coordinates corresponds to ∞ in the z-coordinates. Using the chain rule, we easily obtain

$$\frac{\partial}{\partial z^i} = \rho^{-2} \left(\delta_{ij} - 2 \frac{z^i z^j}{\rho^2} \right) \frac{\partial}{\partial x^j}$$

and, letting $\gamma = r^{n-2} G$,

$$\begin{aligned} \hat{g}_{ij}(z) &= \gamma^{4/(n-2)} \rho^4 g \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right) \\ &= \gamma^{4/(n-2)} \left(\delta_{ik} - 2\rho^{-2} z^i z^k \right) \left(\delta_{jl} - 2\rho^{-2} z^j z^l \right) g_{kl}(\rho^{-2} z) \\ &= \gamma^{4/(n-2)} \left(\delta_{ij} + O_2(\rho^{-2}) \right). \end{aligned} \tag{7.2}$$

In the LCF case, one can arrange for the $O_2(\rho^{-2})$ to vanish identically in $\mathcal{O}$.

We have

$$\gamma(z) = 1 + A\rho^{2-n} + O_2(\rho^{1-n}).$$

So by expanding $\gamma^{4/(n-2)}$ at infinity, we see from (7.2) and the remark following it that $(\hat{M}, \hat{g})$ is asymptotically flat with the order given by (7.1). The formula for the mass in the LCF case follows immediately from Proposition 7.9 and in the other cases after noticing that the derivative of the $O_2(\rho^{-2})$ term cannot contribute to the integral. □

Corollary 7.12. *Let (M, g) satisfy the hypotheses of Corollary 7.11. Then the constant A is nonnegative. It is zero if and only if (M, g) is conformally equivalent to the standard sphere.*

Proof. Note that in every case, the order of the projection is greater than $(n-2)/2$. Furthermore, since $LG = 0$ we have $R(\hat{g}) = 0$ by the curvature formula. By the positive mass theorem, we have $A \geq 0$, with $A = 0$ if and only if $(\hat{M}, \hat{g})$ is isometric to $(\mathbb{R}^n, \delta)$. But $(\mathbb{R}^n, \delta)$ is conformal to $(S^n \setminus \{N\}, \mathcal{S})$, so if $A = 0$, (M, g) is conformally equivalent to $(S^n, \mathcal{S})$. □

We can now state and prove the main simplification result.

Theorem 7.13. *Let (M, g) satisfy the mass condition and suppose additionally that $R(g) \geq 0$. Then for any $\varepsilon > 0$ there is a metric $\overline{g}$ such that $(M, \overline{g})$ is asymptotically flat, scalar flat everywhere, conformally flat near infinity, and $m(\overline{g}) \leq m(g) + \varepsilon$.*

Proof. We first construct a metric $\tilde{g}$ that is asymptotically flat, has nonnegative scalar curvature, conformally flat near infinity, and satisfies $m(\tilde{g}) \leq m(g) + \varepsilon$. The second step is to make a conformal change to zero scalar curvature.

Let $\eta \in C^\infty(\mathbb{R}; [0, 1])$ be a cutoff function with the properties

$$\begin{cases} \eta'(t) \leq 0 & \text{for all } t \\ \eta(t) = 1 & \text{for } t \leq 1 \\ \eta(t) = 0 & \text{for } t \geq 2 \end{cases}.$$

For sufficiently large R, let $\eta_R \in C^\infty(R)$ be defined by $\eta(|x|/R)$ for large x and extended by 1 in the core. The C^k-norm of η_R is bounded independently of R and the support of $d\eta_R$ is contained in the annulus $\{x : R < |x| < 2R\}$. We now interpolate between g and the Euclidean metric:

$$g_R = \eta_R g + (1 - \eta_R)\delta.$$

The conformal Laplacian of g_R will be written as

$$L_R = -a\Delta_R + \operatorname{scal}_R .$$

We want to solve the PDE

$$L_R u_R = \eta_R \operatorname{scal}(g) u_R \quad \text{with } u_R > 0,$$

for then the metric $\tilde{g}_R = u_R^{4/(n-2)} g_R$ has scalar curvature

$$\operatorname{scal}(\tilde{g}_R) = u_R^{(n+2)/(n-2)} \eta_R \operatorname{scal}(g) \geq 0$$

and is conformally scalar flat near infinity. We make the substitution $u_R = 1 + v_R$, so that we obtain

$$P_R v_R := (-a\Delta_R + \gamma_R) v_R = -\gamma_R, \tag{7.3}$$

where $\gamma_R = \operatorname{scal}_R - \eta_R \operatorname{scal}(g)$.

Choose μ between $(n-2)/2$ and τ. By Proposition 2.2 and Theorem 1.10 of [12], $-a\Delta : W^{2,q}_{-\mu} \to W^{0,q}_{-\mu-2}$ is boundedly invertible for any asymptotically flat metric. To use the method of continuity to solve (7.3), we have to show that $\gamma_R \in W^{0,q}_{-\mu-2}$ and $P_R \to -a\Delta$ in operator norm as $R \to \infty$. Then P_R will also be boundedly invertible for sufficiently large R. Firstly, we have

$$\begin{aligned} \|g - g_R\|_{2,q,-\tau} &= \|(1 - \eta_R)(g - \delta)\|_{2,q,-\tau} \\ &\leq C \|1 - \eta_R\|_{C^2} \|g - \delta\|_{2,q,-\tau} \\ &\leq C \end{aligned}$$

uniformly in R. This shows that since the support of $g - g_R$ is contained in $\{x : |x| > R\}$ and $\mu < \tau$, $g_R \to g$ in $W^{2,q}_{-\mu}$ as $R \to \infty$. Using the coordinate expression for the scalar curvature, one can show that this implies $\operatorname{scal}(g_R) \to \operatorname{scal}(g)$ in $W^{0,q}_{-\mu-2}$. For details see Lemma 2.4 in [75]. On the other hand we have

$$\|(1-\eta_R)\operatorname{scal}(g)\|_{q,-\tau-2} \leq \|1-\eta_R\|_{C^0}\|\operatorname{scal}(g)\|_{q,-\tau-2}.$$

Together, we have

$$\|\gamma_R\|_{q,-\mu-2} = \|[\operatorname{scal}(g_R) - \operatorname{scal}(g)] - [\eta_R - 1]\operatorname{scal}(g)\|_{q,-\mu-2} \to 0.$$

Next is to show that $P_R \to -a\Delta$. Let $u \in W^{2,q}_{-\mu}$ be arbitrary. Then

$$\begin{aligned}\|\gamma_R u\|_{q,-\mu-2} &\leq \|\gamma_R\|_{q,-2}\sup(u\sigma^\mu)\\ &\leq C\|\gamma_R\|_{q,-2}\|u\|_{2,q,-\mu},\end{aligned}$$

where the second inequality follows from the weighted Sobolev inequality. Therefore the operator norm of γ_R goes to zero. On the other hand, using coordinates one can show that

$$\|\Delta u - \Delta_R u\|_{q,-\mu-2} \leq C\|g-g_R\|_{2,q,-\mu}\|u\|_{2,q,-\mu}.$$

Thus, by the method of continuity, $P_R : W^{2,q}_{-\mu} \to W^{0,q}_{-\mu-2}$ is a bounded isomorphism for sufficiently large R. Let v_R be the corresponding solution of (7.3). Since the data $-\gamma_R \to 0$ as $R \to 0$, we have from the uniform boundedness of P_R^{-1} that $v_R \to 0$ in $W^{2,q}_{-\mu}$. Applying Morrey's inequality again, we find that $\sup|v_R| \to 0$, hence $u_R = 1 + v_R$ is strictly positive for all sufficiently large R.

Note that $\gamma_R \equiv 0$ and $\Delta_R = \Delta_\delta$ for $|x| > 2R$, so v_R is a Euclidean harmonic function near infinity. It follows that

$$v_R(x) = \frac{A_R}{|x|^{n-2}} + O_\infty(|x|^{1-n})$$

for some constant A_R and sufficiently large $|x| > 2R$. This is proved in Lemma 7.14 below and is a special case of Theorem 1.17 in [12]. Therefore $\tilde{g}_R = u_R^{4/(n-2)} g_R$ is asymptotically flat, has nonnegative scalar curvature, and is conformally flat near infinity. We now show that for sufficiently large R, $m(\tilde{g}_R)$ is within ε of $m(g)$.

By Lemma 7.4 it suffices to show $\tilde{g}_R \to g$ in $W^{2,q}_{-\mu}$ and $\operatorname{scal}(\tilde{g}_R) \to \operatorname{scal}(g)$ in L^1 to conclude $m(\tilde{g}_R) \to m(g)$. Firstly,

$$g - \tilde{g}_R = g - g_R + \left[1 - (1+v_R)^{4/(n-2)}\right] g_R.$$

Since $v_R \to 0$ in $W^{2,q}_{-\mu}$, $1-(1+v_R)^{4/(n-2)} \to 0$ in the same space (see Lemma 2.2 in [75] for details). We now observe

$$\begin{aligned}|\operatorname{scal}(g) - \operatorname{scal}(\tilde{g}_R)| &= |\operatorname{scal}(g) - u_R^{-4/(n-2)}\eta_R\operatorname{scal}(g)|\\ &\leq |(1-\eta_R)\operatorname{scal}(g)| + |1-u_R^{-4/(n-2)}||\operatorname{scal}(g)|,\end{aligned}$$

which implies

$$\int_M |\operatorname{scal}(g) - \operatorname{scal}(\tilde{g}_R)| \leq \left(1 + \sup|1-u_R^{-4/(n-2)}|\right)\int_M |\operatorname{scal}(g)|.$$

Since by assumption $\operatorname{scal}(g) \in L^1$, Lemma 7.4 applies. We now choose R sufficiently large that $|m(\tilde{g}_R) - m(g)| < \varepsilon$. Write $\tilde{g}$ for this $\tilde{g}_R$.

Now to the second step, where we make the scalar curvature identically zero. We use the Fredholm alternative for the conformal Laplacian and the strong maximum principle (Theorem 3.5 in [53]). Note that $\operatorname{scal}(\tilde{g})$ vanishes for large $|x|$, so is in every weighted Sobolev space. We wish to solve the equation $L_{\tilde{g}}\varphi = 0$ with $\varphi \to 1$ at infinity so that $\overline{g} = \varphi^{4/(n-2)}\tilde{g}$ is scalar flat. Writing $\varphi = 1 + \psi$ gives the equation $L_{\tilde{g}}\psi = -\operatorname{scal}(\tilde{g})$. By combining [12] Corollary 2.3 with the discussion on the bottom of page 672 in [12], we see that $L_{\tilde{g}} : W^{2,q}_{\frac{5}{2}-n} \to W^{0,q}_{\frac{1}{2}-n}$ is surjective as long as $L_{\tilde{g}}$ has no kernel in $W^{2,q}_{-\frac{1}{2}}$. By Theorem 1.2 (iv) in [12], $q > n$ implies any $f \in W^{2,q}_{-\frac{1}{2}}$ has decay $|f(x)| = o(r^{-1/2})$. Suppose that $f \in W^{2,q}_{-\frac{1}{2}} \cap \ker L_{\tilde{g}}$. Since $\operatorname{scal}(\tilde{g}) \geq 0$, the strong maximum principle implies f cannot have a non-negative maximum or a non-positive minimum unless it is constant. This implies f vanishes identically in view of its decay. Therefore $L_{\tilde{g}}$ is a surjection and there exists a $\psi \in W^{2,q}_{\frac{5}{2}-n}$ satisfying $L_{\tilde{g}}\psi = -\operatorname{scal}(\tilde{g})$. Again by Theorem 1.2 (iv) in [12], $\psi \to 0$ at infinity. The function $\varphi = 1 + \psi$ satisfies $L_{\tilde{g}}\varphi = 0$, so cannot have a non-positive minimum unless it is constant. Since it tends to 1 at infinity, we conclude that φ is strictly positive. Then, since $\operatorname{scal}(\tilde{g})$ vanishes for large $|x|$, φ is a Euclidean harmonic function for large $|x|$, hence has an expansion as in Lemma 7.14. This shows that $\overline{g} = \varphi^{4/(n-2)}\tilde{g}$ is asymptotically flat. By the strong maximum principle applied to ψ, we have $\psi \leq 0$, since it cannot have a non-negative maximum unless it is constant.

Finally, we have to compute the mass of $\overline{g}$. We arrived at $\overline{g}$ after performing two conformal transformations, so $\overline{g} = (\varphi u_R)^{4/(n-2)}\delta$ for large x, where u_R was constructed in step one. Because φ lies below 1, we have $A_\varphi \leq 0$ in the expansion

$$\varphi(x) = 1 + \frac{A_\varphi}{|x|^{n-2}} + \text{l.o.t.}$$

Therefore,

$$\varphi u_R = 1 + \frac{A_\varphi + A_R}{|x|^{n-2}} + \text{l.o.t.},$$

so by Proposition 7.9,

$$m(\overline{g}) = 2(A_\varphi + A_R).$$

Since R was chosen such that $|m(g) - 2A_R| < \varepsilon$, the mass of $\overline{g}$ satisfies

$$m(\overline{g}) \leq m(g) + \varepsilon,$$

as desired. □

Lemma 7.14. *Let u be a harmonic function defined outside of a ball in $\mathbb{R}^n$, $n \geq 3$, with the standard metric. If $u \to 0$ at infinity, then there exists a constant C and homogeneous polynomials $P_1, P_2, \ldots$ such that*

$$u(x) = \frac{C}{|x|^{n-2}} + \sum_{j=1}^{\infty} \frac{P_j(x)}{|x|^{n-2-2j}},$$

with the convergence in the C^∞ sense.

Proof. Consider the Kelvin transform

$$v(x) = |x|^{2-n} u \left(\frac{x}{|x|^2} \right).$$

It is well known that v is harmonic if u is ([62] Theorem 2.8.1). The singularity of v at 0 is removable by Lemma 3.11 since $v(x)$ grows slower than the Green's function as $x \to 0$. Therefore, v is harmonic in a neighborhood of zero, hence is analytic and has a C^∞ expansion in homogeneous polynomials. The claimed expansion follows. □

7.3 Lohkamp's Compactification Argument

We now show how to use the simplified asymptotics derived in the previous section to reduce the proof of the positive mass theorem to a problem in scalar curvature geometry. We argue by contradiction, so assume the mass is strictly negative.

Proposition 7.15. *Let (M, g) satisfy the hypotheses of Theorem 7.6. If $m(g) < 0$, we can find another complete metric $\overline{g}$ on M with $\operatorname{scal}(\overline{g}) \geq 0$, $\operatorname{scal}(\overline{g}) > 0$ at some point, and such that the end of $(M, \overline{g})$ is isometric to the exterior of a ball in $\mathbb{R}^n$ with the standard metric.*

This was first proved by Lohkamp in [84] using a finicky approximation procedure and scaling argument on annuli. The proof we give here is a precise version of the sketch in Section 5 of [107]. We first recall some standard material that will be useful again later.

Let $\Omega \subset \mathbb{R}^n$. A $C^0(\Omega)$ function u is said to be *superharmonic* in Ω if for every ball $B \subset\subset \Omega$ and every harmonic function $h \in C^2(B) \cap C^0(\overline{B})$ satisfying $u \geq h$ on ∂B, we also have $u \geq h$ in B. It is immediate that the minimum of two superharmonic functions is superharmonic.

Lemma 7.16. *A $C^0(\Omega)$ function u is superharmonic in Ω if and only if it satisfies the mean value inequality locally; that is, for every $x \in \Omega$ there exists $\delta = \delta(x) > 0$ such that*

$$u(x) \geq \fint_{\partial B_r(x)} u \, d\mathcal{H}^{n-1} \quad \forall r \leq \delta.$$

Proof. ($\Rightarrow$) Let $B \subset\subset \Omega$ be any ball with center x and h be the harmonic function in $\overline{B}$ satisfying $h|_{\partial B} = u|_{\partial B}$. By the mean value property and the fact that u is superharmonic,

$$u(x) \geq h(x) = \fint_{\partial B} u.$$

($\Leftarrow$) Exactly the same as Theorem 2.2 in [53]. □

Lemma 7.17. *A C^2 subharmonic function u satisfies $\Delta u \leq 0$.*

Proof. Suppose by contradiction that $\Delta u > 0$ for some $x \in \Omega$. Since Δu is continuous, this happens in a whole ball about x. Consider the function

$$\phi(r) = \fint_{\partial B_r(x)} u.$$

A simple calculation shows

$$\phi'(r) = \frac{r}{n} \fint_{B_r(x)} \Delta u,$$

so $\phi(r)$ is strictly increasing for sufficiently small r. On the other hand, by Lebesgue's theorem, $\phi(r) \to u(x)$ as $r \downarrow 0$. Therefore $\phi(r) > u(x)$ for small r, which is a contradiction. □

Proposition 7.18. *The mollification of a harmonic function is the harmonic function itself. The mollification of a superharmonic function is superharmonic.*

Proof. Let η be the standard mollifier, $\eta_\varepsilon = \varepsilon^{-n}\eta(\cdot/\varepsilon)$, and $u_\varepsilon = \eta_\varepsilon * u$. First take u to be harmonic, so that it satisfies the mean value property. Then

$$\begin{aligned} u_\varepsilon(x) &= \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x)} \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\, dy \\ &= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\Big(\frac{r}{\varepsilon}\Big) \int_{\partial B_r(x)} u \, d\mathcal{H}^{n-1}\, dr \\ &= \frac{1}{\varepsilon^n} u(x) \int_0^\varepsilon \eta\Big(\frac{r}{\varepsilon}\Big) \mathcal{H}^{n-1}(\partial B_r(x))\, dr \\ &= u(x) \int_{B_\varepsilon(0)} \eta_\varepsilon\, dy = u(x). \end{aligned}$$

If instead u is superharmonic, then Lemma 7.16 and Fubini's theorem imply

$$\begin{aligned} \fint_{B_R(x)} u_\varepsilon(y)\, d\mathcal{H}^{n-1}(y) &= \fint_{B_R(x)} \int u(y-z)\eta_\varepsilon(z)\, dz\, d\mathcal{H}^{n-1}(y) \\ &= \int \eta_\varepsilon(z) \fint_{B_R(x)} u(y-z)\, d\mathcal{H}^{n-1}(y)\, dz \\ &\le \int \eta_\varepsilon(z) u(z-x)\, dz = u_\varepsilon(x). \end{aligned}$$

Therefore u_ε is superharmonic. □

Proof of Proposition 7.15. By Theorem 7.13 we may assume that the metric g at large x is equal to $\varphi^{4/(n-2)}\delta$, where φ is a Euclidean harmonic function tending to 1 at infinity. We also have that φ is not constant, since in the expansion

$$\varphi(x) = 1 + \frac{A}{|x|^{n-2}} + O(|x|^{1-n}),$$

A has the same sign as $m(g)$, namely strictly negative by hypothesis. Choose $\varepsilon > 0$ and $R > 0$ such that

$$\max_{|x|=R} \varphi < 1 - \varepsilon.$$

By the maximum principle, φ has to be larger on spheres of larger radius. In particular, the function $\min\{\varphi, 1 - \frac{1}{2}\varepsilon\}$ is equal to φ for $|x| \le R$ and equal to $1 - \frac{1}{2}\varepsilon$ for all x sufficiently large. It is also weakly superharmonic since φ and $1-\frac{1}{2}\varepsilon$ are harmonic. We seek a smooth superharmonic approximation to this function.

The function $\min\{\cdot, 1-\frac{1}{2}\varepsilon\} : \mathbb{R} \to \mathbb{R}$ is of course not smooth, but by mollifying it we obtain a smooth function $f(x)$ which equals $\min\{x, 1-\frac{1}{2}\varepsilon\}$ unless $|x| \leq \frac{1}{10}\varepsilon$. Set $u = f(\varphi)$. Then, since u is harmonic and by Proposition 7.18,

$$\Delta u = f''(\varphi)|\nabla\varphi|^2 \leq 0.$$

Since f is not linear, there is some point where $f'' < 0$. Also, since φ is not constant, this will happen at some point where $\nabla\varphi \neq 0$. Therefore, as φ approaches $1-\frac{1}{2}\varepsilon$ from below, there is some point where $\Delta u < 0$. So we define a new metric

$$\overline{g} = \begin{cases} g & \text{for } |x| \leq R \\ u^{4/(n-2)}\delta & \text{for } |x| \geq R \end{cases}$$

on M. By the curvature formula, $\mathrm{scal}(\overline{g}) \geq 0$ since $\Delta u \leq 0$, but $\mathrm{scal}(\overline{g}) > 0$ somewhere because $\Delta u < 0$ somewhere. Finally note that since u is constant for all sufficiently large x, $\overline{g}$ is totally flat near infinity. □

This allows us to "compactify" the positive mass theorem in the following sense:

Theorem 7.19. *Let (M, g) satisfy the hypotheses of Theorem 7.6. Let $\overline{M}$ be the one-point compactification of M. If the compact manifold $\overline{M} \# T^n$ does not admit a metric of positive scalar curvature, then the ADM mass of (M, g) is nonnegative.*

Proof. We suppose that $m(g) < 0$ and apply Proposition 7.15 to give M a metric $\overline{g}$ with nonnegative scalar curvature, positive in some nonempty open set, and such that the end of M is totally flat. We may think of the curved region as being contained in some large cube whose sides are in the flat region. By identifying opposing sides of this cube, we obtain a manifold diffeomorphic to $\overline{M} \# T^n$ carrying a metric with nonnegative, but not incidentally zero, scalar curvature. By Corollary 4.6, $\overline{M} \# T^n$ carries a metric of positive scalar curvature. This contradicts the hypothesis, so $m(g) \leq 0$. □

We can now easily prove the positive mass theorem in the case when M is a spin manifold.

Theorem 7.20. *Let M be a compact spin manifold of dimension $n \geq 3$. Then $M \# T^n$ does not carry a metric of positive scalar curvature.*

Proof. Since T^n is parallelizable, it is a spin manifold. By Lemma 7.21 below, $T^n \# M$ is a spin manifold. That $T^n \# M$ admits no positive scalar curvature metrics follows from a theorem of Gromov–Lawson. Indeed, by Theorems IV.5.3 (D) and IV.5.4 (B) of [76], $T^n \# M$ is enlargeable, so the fundamental Theorem IV.5.5 in [76] completes the proof. □

Lemma 7.21. *Let M and N be compact spin manifolds of dimension $n \geq 3$. Then $M \# N$ is a spin manifold.*

In this proof, $H^j(X)$ denotes the cohomology group $H^j(X; \mathbb{Z}_2)$.

Proof. Let B be a ball in M and $i_M : M \setminus B \hookrightarrow M$ the inclusion. We claim that i_M induces an isomorphism

$$i_M^* : H^2(M) \xrightarrow{\sim} H^2(M \setminus B). \tag{7.4}$$

Let U be a chart around B, and $V \subset M$ open such that $U \cap V$ is homeomorphic to an annulus and $U \cup V = M$. Then U is contractible and $U \cap V$ is homotopy equivalent to S^{n-1}. The Mayer–Vietoris sequence collapses to

$$0 \to H^2(M) \to H^2(M \setminus B) \to 0 \tag{7.5}$$

when $n \geq 4$, for then $H^2(S^{n-1}) \cong 0$. Since the sequence is exact, $H^2(M) \xrightarrow{\sim} H^2(M \setminus B)$. Now suppose $n = 3$. Since M is compact, $H^3(M) \cong \mathbb{Z}_2$ and $H^3(M \setminus B) \cong 0$. So the sequence

$$0 \to H^2(M) \to H^2(M \setminus B) \to H^2(S^2) \to H^3(M) \to H^3(M \setminus B) \to 0$$

becomes

$$0 \to H^2(M) \to H^2(M \setminus B) \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0.$$

Obviously $\mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$ is surjective, so $H^2(M \setminus B) \to \mathbb{Z}_2$ must be the zero map by exactness, hence (7.5) holds here too. This verifies (7.4).

Letting j denote the inclusion $M \setminus B \amalg N \setminus B \hookrightarrow M \# N$, a similar argument shows

$$j^* : H^2(M \# N) \xrightarrow{\sim} H^2(M \setminus B) \oplus H^2(N \setminus B),$$

so we have isomorphisms

$$H^2(M \# N) \xrightarrow{\sim} H^2(M \setminus B) \oplus H^2(N \setminus B) \xrightarrow{\sim} H^2(M) \oplus H^2(N).$$

If $w_2(X)$ denotes the second Stiefel–Whitney class of TX, the naturality axiom implies

$$\begin{aligned} w_2(M \# N) &= j^*(w_2(M \setminus B) + w_2(N \setminus B)) \\ &= j^*(i_M^* w_2(M) + i_N^* w_2(N)) \\ &= 0, \end{aligned}$$

since $w_2(M) = w_2(N) = 0$. This completes the proof. □

From the proof, we also have

Lemma 7.22. *Let M be a compact manifold of dimension $n \geq 3$ that is not spin. Then $M \setminus \{\mathrm{pt}\}$ is not spin either. If an asymptotically flat manifold (with finitely many ends) is spin, then so is its endwise compactification.*

This essentially repeats the Witten–Bartnik proof of the positive mass theorem for spin manifolds. However, since the spinorial mass formula is of independent interest, both proofs have their merits. For example, the Witten–Bartnik mass formula was used by Masood-ul Alam to prove the fluid ball conjecture in [87]. The advantage to Lohkamp's method is that we can prove Theorem 7.20 for non-spin manifolds as well.

Remark 7.23. In three dimensions, the spinorial proof is completely general. This is due to the following classical theorem of Wu. (Stiefel gave an incomplete proof in 1936.)

Theorem 7.24. *Let M be a closed, orientable 3-manifold. Then M is spin.*

Proof. Recall the Wu class $v = 1 + v_1 + v_2 + \cdots \in H^*(M)$ defined by the duality relation

$$\langle v_j \smile x, [M]\rangle = \langle \mathrm{Sq}^j(x), [M]\rangle$$

for all $x \in H^{n-j}(M)$, where $\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots$ is the Steenrod squaring homomorphism. Then we have Wu's formula

$$w = \mathrm{Sq}(v) \quad \text{or} \quad w^i = \sum_{k+l=i} \mathrm{Sq}^k(v_l).$$

See, for instance, Milnor and Stasheff [90]. At the first level,

$$0 = w_1 = v_1 + \mathrm{Sq}^1(1) = v_1,$$

where the first equality is due to the orientability of M and the third equality is by the axioms of Steenrod squares. By a similar argument, Wu's formula gives

$$w_2 = v_2 + \mathrm{Sq}^1(v_1) + \mathrm{Sq}^2(1) = v_2.$$

However, since $\mathrm{Sq}^2(x) = 0$ for every $x \in H^1(M)$ by the axioms, $v_2 = 0$. Therefore M is spin. Similarly, one can show that $w_3 = 0$, so the total Stiefel–Whitney class is trivial. □

We use this result to prove an interesting theorem which is hard to localize in the literature.[1] Recall the Stiefel manifold $V_k(\mathbb{R}^n)$, the manifold of all orthonormal k-frames in $\mathbb{R}^n$. Given a vector bundle ξ, we can consider the fiber bundle of k-frames $V_k(\xi)$ which has fiber $V_k(\mathbb{R}^n)$.

Corollary 7.25 (Stiefel)**.** *Every smooth orientable 3-manifold with boundary is parallelizable, i.e. TM is smoothly isomorphic to the trivial bundle $M \times \mathbb{R}^3$.*

Proof. By approximation theory it suffices to find a *continuous* 3-frame on M ([114] Theorem 6.7). We first show that every closed 3-manifold is trivializable. Since M is smooth, it can be triangulated ([92]). By basic obstruction theory, the vanishing of $w_i(TM)$ for $i \geq 1$ shows that $V_2(TM)$ has a section over the 2-skeleton of M ([90] page 143). Let C be the boundary of a 3-simplex S in the triangulation of M. The section of $V_2(TM)$ over $C \approx S^2$ gives rise to a map $f : S^2 \to V_2(\mathbb{R}^3)$. Since $V_2(\mathbb{R}^3) \approx \mathrm{SO}(3)$, it has $\pi_2 = 0$ and the section over C extends to S (just foliate S by 2-spheres and define the section by the nullhomotopy of f). This gives a global 2-frame, so the tangent bundle splits as

$$TM \cong (M \times \mathbb{R}^2) \oplus \eta,$$

where η is a line bundle. But since TM and $M \times \mathbb{R}^2$ are orientable, η is orientable and hence trivial. This proves the claim when M is closed.

Suppose now M is compact but has a boundary. If M is not parallelizable, then neither is its double DM. But we just showed that DM is parallelizable, so M must be as well.

[1] The compact case is treated in Steenrod, pages 203-204 [114].

If M is open, we consider an exhaustion of M by compact sets K_j with smooth boundaries. By the previous step we know that $TM|K_j$ is trivial for each j. However, this is not enough to show that TM itself is trivial.[2] If $i_j : K_j \hookrightarrow M$ is the inclusion, the naturality axiom implies $i_j^* w_2(TM) = w_2(TK_j)$. Now we know that $w_2(TK_j) = 0$ for every j. If $c \in H_2(M; \mathbb{Z}_2)$, then it is represented by some cycle $c \in C_2(M) \otimes \mathbb{Z}_2$. The cycle is compact, hence is in some K_j. Since $\mathbb{Z}_2$ is a field, the universal coefficient theorem shows

$$H^2(M; \mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(H_2(M; \mathbb{Z}_2), \mathbb{Z}_2).$$

Therefore,

$$\langle w_2(TM), c\rangle = \langle w_2(TK_i), c\rangle = 0$$

implies $w_2(TM) = 0$. Therefore M is spin and we can run the obstruction argument again to conclude that M is parallelizable.

Finally, if M is noncompact and has a boundary, we double it and apply the previous step. □

7.4 Minimal Surfaces and the Heredity Principle

In this section we prove the positive mass theorem when $3 \leq n \leq 7$ without the spin assumption. The proof is based on the *positive scalar curvature heredity principle*, discovered by Schoen and Yau in the 70s, and applied to the positive mass theorem in [103] and [108], but in a slighly different way that we will sketch later. The theorem to prove is

Theorem 7.26. *Let M be a compact manifold of dimension $n \leq 7$. Then $M \# T^n$ does not carry a metric of positive scalar curvature.*

To prove this, we take the idea from [107] and apply it to minimal surfaces instead of minimal slicings. This works when $n \leq 7$ and greatly simplifies the proof in this case. The method in [107] works without dimensional restrictions but is very hard.

We first recall some basic terminology. Consider a Riemannian manifold (M, g) (not necessarily compact), and $\Sigma \subset M$ a smooth immersed hypersurface. Let $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a variation of Σ with compact support that fixes $\partial\Sigma$. Then the *first variation formula* reads

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, \vec{H}\rangle,$$

where $\vec{H} = \mathrm{tr}\, A$ is the mean curvature vector of Σ in M. We say that Σ is a *minimal surface* if $\vec{H} \equiv 0$, equivalently, if Σ is a critical point of the area functional. It is often of interest to

[2] The issue is that we can construct a global frame on $TM|K_i$ but it is not clear that it extends to a frame on $TM|K_{i+1}$, even though both bundles are trivial. In the CW setting, one can use the fact that $[\Sigma\mathbb{C}P^\infty, \mathrm{BO}(3)] \cong [\mathbb{C}P^\infty, S^3] \neq 0$ to construct a 3-plane bundle on $\Sigma\mathbb{C}P^\infty$ which is trivial on each finite subcomplex $\Sigma\mathbb{C}P^n$ but is globally nontrivial. This situation can be homotoped to a locally finite cell complex. In the setting of smooth manifolds, one can construct a smooth 3-manifold $M \hookrightarrow \mathbb{R}^4$ with $[M, \mathbb{C}P^\infty] \neq 0$, so there exists a nontrivial complex line bundle over M. The base is constructed in such a way that any compact set has a neighborhood which is homotopy equivalent to S^1, and complex line bundles over S^1 are trivial. These constructions can be found in the MathOverflow thread [63]. It is not clear if such pathological situations can happen for tangent bundles.

know that Σ is actually locally minimizing instead of just being a critical point. The *second variation formula* is

$$\frac{d^2}{dt^2}\operatorname{vol}(F(\Sigma,t))\bigg|_{t=0} = -\int_\Sigma \langle F_t, JF_t\rangle,$$

where J is the *stability operator* and Σ is assumed to be minimal. For proofs and precise definitions, see [40]. When the normal bundle of Σ is trivial and we can write a generic normal section as $X = f\nu$, where ν is the normal to Σ, then

$$\langle X, JX\rangle = f\left(\Delta_\Sigma f + |A|^2 f + \operatorname{Ric}_M(\nu,\nu) f\right).$$

It is therefore convenient to just write

$$Jf = \Delta_\Sigma f + |A|^2 f + \operatorname{Ric}_M(\nu,\nu) f.$$

We also define the *stability form* of Σ by

$$S(\varphi,\varphi) = -\int_\Sigma \varphi J\varphi$$

for $\varphi \in C^1_c(\Sigma)$. We say that Σ is a *stable minimal surface* if the second variation is nonnegative. When Σ is two-sided, this is equivalent to

$$S(\varphi,\varphi) \geq 0 \quad \forall \varphi \in C^1_c(\Sigma).$$

Written out, this reads

$$\int_\Sigma |\nabla\varphi|^2 - \left(|A|^2 + \operatorname{Ric}_M(\nu,\nu)\right)\varphi^2 \geq 0$$

With this terminology out of the way, we can now state the heredity principle.

Theorem 7.27 (Heredity Principle)**.** *Let M be a Riemannian manifold with positive scalar curvature and $\Sigma \subset M$ a closed stable minimal surface with trivial normal bundle. If $n = \dim M \geq 4$, then Σ is Yamabe positive in the induced metric. When $n = 3$, each connected component of Σ is either S^2 or $\mathbb{R}P^2$.*

Proof. We show that the first eigenvalue of the conformal Laplacian of (Σ, g_Σ) is positive when $n \geq 4$. First, note that by the Gauss equation,

$$2\operatorname{Ric}_M(\nu,\nu) = R_M - R_\Sigma - |A|^2.$$

Then, for any $\varphi \in C^1_c(\Sigma)$ not identically zero,

$$\begin{aligned}
0 &\leq S(\varphi,\varphi)\\
&< \int_\Sigma \left(|\nabla\varphi|^2 + \frac{1}{2}R_\Sigma\varphi^2\right)\\
&= \frac{1}{2}\int_\Sigma \left(2|\nabla\varphi|^2 + R_\Sigma\varphi^2\right)\\
&\leq \frac{1}{2}\int_\Sigma \left(a|\nabla\varphi|^2 + R_\Sigma\varphi^2\right),
\end{aligned}$$

where $a = 4(n-2)/(n-3)$, which is the right constant for the $(n-1)$-dimensional Σ, and we used $2 < a$.

When Σ is 2-dimensional, the stability inequality with $\varphi = 1$ implies

$$0 < \frac{1}{2}\int_\Sigma \left(R_M + |A|^2\right) \le \frac{1}{2}\int_\Sigma R_\Sigma = 2\pi\chi(\Sigma),$$

so $\Sigma = S^2$ or $\mathbb{R}P^2$. □

There is a version of this result for stable MOTS (marginally trapped outer surfaces) due to Galloway and Schoen [51]. They show that stable black hole horizons have positive Yamabe type or are spheres.

So the goal is to create a tower of stable minimal surfaces, eventually ending up at an S^2, which contradicts some part of the construction. The existence and regularity of these surfaces follows from standard geometric measure theory when $n \le 7$. For the terminology we use in the following, see [47] [110] [52].

Theorem 7.28. *Let M be a closed orientable n-manifold with dimension $2 \le n \le 7$. Suppose there exists a smooth map $F : M \to T^n$ such that $\deg F = s \neq 0$. Then M admits no metric of positive scalar curvature.*

Proof. First consider the case $n = 2$. Let t^1 and t^2 be the coordinates on T^2 with periods equal to 1. Letting $\omega^1 = F^* dt^1$ and $\omega^2 = F^* dt^2$, we find from the degree formula

$$\int_M \omega^1 \wedge \omega^2 = \deg F \int_{T^2} dt^1 \wedge dt^2 = s \neq 0.$$

This shows that M has at least two distinct H^1_{dR} classes. However, by the Gauss–Bonnet formula, the only closed orientable 2-manifold admitting a positive scalar curvature metric is S^2, which has $H^1_{\mathrm{dR}} = 0$.

Now let $3 \le n \le 7$ and suppose $M =: \Sigma_n$ has a positive scalar curvature metric g_n. For convenience we use the Nash embedding theorem to view (Σ_n, g_n) as isometrically embedded in some $\mathbb{R}^N$. Let the coordinates on T^n be $t^1, \dots, t^n$, each with period 1. We define $\omega^j = F^* dt^j$ and $\Omega^j = \omega^1 \wedge \cdots \wedge \omega^j$. Consider the set

$$\mathcal{C}_{n-1} = \{T \in \mathcal{R}_{n-1}(\mathbb{R}^N) : \operatorname{supp} T \subset \Sigma_n, \partial T = 0, T(\Omega^{n-1}) = s\},$$

where $\mathcal{R}_{n-1}(\mathbb{R}^N)$ denotes the space of integer multiplicity rectifiable $(n-1)$-currents in $\mathbb{R}^N$. We seek a smooth minimal surface minimizing volume in $\mathcal{C}_{n-1}$. We first show that $\mathcal{C}_{n-1}$ is nonempty. Letting $u^n = t^n \circ F : \Sigma_n \to S^1$, let v be a regular value of u^n, and consider the embedded hypersurface $S_{n-1} = \{u^n = v\}$ in Σ_n. By the degree formula,

$$\int_{S_{n-1}} \Omega^{n-1} = \deg F \int_{T^{n-1}\times\{v\}} dt^1 \wedge \cdots \wedge dt^{n-1} = s.$$

Therefore, the current $T = [\![S_{n-1}]\!]$ lies in $\mathcal{C}_{n-1}$. We now consider a sequence (T_i) in $\mathcal{C}_{n-1}$ such that

$$\lim_{i\to\infty} \mathbb{M}(T_i) = \inf_{T \in \mathcal{C}_{n-1}} \mathbb{M}(T).$$

By the Federer–Fleming compactness theorem, (T_i) converges weakly to some $\Sigma_{n-1} \in \mathcal{R}_{n-1}(\mathbb{R}^N)$ after passing to a subsequence. By the definition of this convergence, $\partial\Sigma_{n-1} = 0$ and $\Sigma_{n-1}(\Omega^{n-1}) = s$. Furthermore, one can easily check from the properties of weak convergence of currents that $\operatorname{supp}\Sigma_{n-1} \subset \operatorname{supp}\Sigma_n$ since $\operatorname{supp}\Sigma_n$ is a closed subset of $\mathbb{R}^N$ and

$$\mathbb{M}(\Sigma_{n-1}) \leq \lim_{i\to\infty} \mathbb{M}(T_i).$$

Because Ω^{n-1} is closed, Σ_{n-1} is actually a *homologically* mass minimizing current. Indeed, if Σ'_{n-1} were such that $\Sigma_{n-1} - \Sigma'_{n-1} = \partial R$, then

$$(\Sigma_{n-1} - \Sigma'_{n-1})(\Omega^{n-1}) = \partial R(\Omega^{n-1}) = R(d\Omega^{n-1}) = 0.$$

Therefore $\Sigma'_{n-1} \in \mathcal{C}_{n-1}$ and $\mathbb{M}(\Sigma_{n-1}) \leq \mathbb{M}(\Sigma'_{n-1})$. It now follows from the strong regularity theorem of mass minimizing currents in Riemannian manifolds that Σ_{n-1} is a smooth, closed, oriented submanifold of Σ_n [98]. Since Σ_{n-1} is homologically minimizing it is stable (small variations do not change the homology class).

By the heredity principle we have either $n-1 = 2$ and Σ_{n-1} is a sphere or $n-1 \geq 3$ and it is Yamabe positive in the induced metric. If it is Yamabe positive, choose a conformal metric with positive scalar curvature. Assume inductively that we have constructed Σ_j, $j \geq 3$, with a positive scalar curvature metric. We minimize mass in the set

$$\mathcal{C}_{j-1} = \{T \in \mathcal{R}_{j-1}(\mathbb{R}^N) : \operatorname{supp} T \subset \Sigma_j, \partial T = 0, T(\Omega^{j-1}) = s\}.$$

Consider the function $u^j : \Sigma_j \to S^1$ defined by projecting $F : \Sigma_j \to T^n$ onto the j-th factor. Let v be a regular value and define $S_{j-1} = \{u^j = v\}$. By the degree formula,

$$\int_{S_{j-1}} \Omega^{j-1} = s,$$

so $\mathcal{C}_{j-1}$ is nonempty. Arguing as before, we may find a smooth, stable, mass minimizing hypersurface Σ_{j-1} in $\mathcal{C}_{j-1}$.

We proceed in this manner until we arrive at the closed Σ_2 which is diffeomorphic to S^2 by the heredity principle. But since

$$\int_{\Sigma_2} \omega^1 \wedge \omega^2 = s \neq 0,$$

we obtain a contradiction. □

Proof of Theorem 7.26. We first assume M is orientable and construct a degree 1 smooth map $M \# T^n \to T^n$. We first construct an approximate continuous map $F : M \# T^n \to T^n$ by letting F be the identity on a neighborhood of the connecting tube in T^n, and collapsing M and the tube to a point in T^n. By a standard approximation theorem ([73] III.2.5), we may find a smooth $\tilde{F} : M \# T^n \to T^n$ that equals F where F is smooth, and satsifies

$$\operatorname{dist}(F(x), \tilde{F}(x)) < \varepsilon \quad \forall x \in M \# T^n.$$

But this means $\tilde{F}$ is still the identity on some part of the torus, hence has degree 1.

In the nonorientable case there exists a double cover $\pi : \tilde{M} \to M$ with $\tilde{M}$ orientable. Using this, we can construct a double covering $p : \tilde{M} \# T^n \# T^n \to M \# T^n$. If $M \# T^n$ had a PSC metric g, then p^*g would be a PSC metric on $\tilde{M} \# T^n \# T^n$. But $\tilde{M} \# T^n \# T^n$ is the connected sum of the orientable manifold $\tilde{M} \# T^n$ with T^n, so this contradicts the result in the orientable case. □

We also give a proof of the more general nonexistence theorem of [107] in low dimensions. In the following, let $D : H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$, $\alpha \mapsto [M] \frown \alpha$ be the Poincaré duality map. Recall also the rule $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$, for α a chain and φ, ψ cochains.

Theorem 7.29. *Suppose M is a compact oriented n-manifold with $2 \leq n \leq 7$ and carrying a metric of positive scalar curvature. If $\alpha_1, \ldots, \alpha_{n-2}$ are cohomology classes in $H^1(M;\mathbb{Z})$ with the property that the homology class $\sigma_2 \in H_2(M;\mathbb{Z})$ given by*

$$\sigma_2 = D(\alpha_1 \smile \cdots \smile \alpha_{n-2})$$

is nonzero, then the class σ_2 can be represented by a disjoint union of two-spheres. If α_{n-1} is any class in $H^1(M;\mathbb{Z})$, then we must have $\sigma_2 \frown \alpha_{n-1} = 0$.

In particular, if M has classes $\alpha_1, \ldots, \alpha_{n-1}$ with

$$D(\alpha_1 \smile \cdots \smile \alpha_{n-1}) \neq 0,$$

then M cannot carry a metric of positive scalar curvature.

We first need a basic lemma.

Lemma 7.30. *Let M be a manifold with Σ_0 and Σ_1 submanifolds of M whose inclusion maps are homotopic. Then the fundamental classes of Σ_0 and Σ_1 are homologous.*

Proof. By definition, Σ_0 and Σ_1 are copies of some manifold Σ and there is a continuous map $f : \Sigma \times [0,1] \to M$ such that $f(\cdot, 0)$ is the inclusion of Σ_0 and $f(\cdot, 1)$ is the inclusion of Σ_1. Clearly the fundamental classes of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ represent the same homology class in $\Sigma \times [0,1]$, up to a sign. Therefore, by considering the induced homomorphism $f_* : H_\bullet(\Sigma \times [0,1];\mathbb{Z}) \to H_\bullet(M;\mathbb{Z})$, we see that Σ_0 and Σ_1 represent the same homology class, up to a sign. □

Proof of Theorem 7.29. We write M as Σ_n and let g_n be a metric on Σ_n with positive scalar curvature. We construct a tower of stable minimal surfaces Σ_{n-j}, each having dimension $n-j$ and representing the integral homology class $D(\alpha_1 \smile \cdots \smile \alpha_j)$.

To construct Σ_{n-1}, let S_{n-1} be an embedded, orientable hypersurface which is Poincaré dual to α_1. Such a hypersurface exists by a theorem of Thom ([18] Theorem VI.11.16). We now choose Σ_{n-1} to be a smooth, homologically g_n-mass minimizing current in the class of S_{n-1}. Clearly Σ_{n-1} is stable. Using the heredity principle, we may choose a metric g_{n-1} on Σ_{n-1} with positive scalar curvature unless $n-1=2$, in which case we are done.

Assuming $n-1 \geq 3$, we now construct Σ_{n-2}. We want to minimize mass inside of (Σ_{n-1}, g_{n-1}) within a specific homology class. Represent the Poincaré dual of α_2 in Σ_n by a smooth submanifold S_{n-2}, again using Thom's theorem. By the transversality homotopy theorem, we may perturb S_{n-2} to obtain a homotopic submanifold $\tilde{S}_{n-2}$ that is transversal to

Σ_{n-1}. By Lemma 7.30, $\tilde{S}_{n-2}$ is also Poincaré dual to α_2. The intersection $\Sigma_{n-1} \cap \tilde{S}_{n-2} = T_{n-2}$ is a nonempty codimension 2 submanifold of Σ_n if $\alpha_1 \smile \alpha_2 \neq 0$ and is Poincaré dual to $\alpha_1 \smile \alpha_2$ ([18] Theorem IV.11.9). We may also regard T_{n-2} as a hypersurface in Σ_{n-1}. We let Σ_{n-2} be the g_{n-1}-mass minimizing smooth hypersurface in Σ_{n-1} which is homologous to T_{n-2}. The hypothesis on σ_2 implies Σ_{n-2} is not the zero current.

We repeat this until we reach a 2-dimensional minimization problem. The resulting minimal surface will be a union of 2-spheres by the heredity principle and is Poincaré dual to $\alpha_1 \smile \cdots \smile \alpha_{n-2}$. If α_{n-1} is any class in $H^1(M;\mathbb{Z})$, then $\sigma_2 \frown \alpha_{n-1}$ is a class in $H_1(\sqcup S^2;\mathbb{Z})$, and is therefore zero. In particular, if a manifold admits a metric of positive scalar curvature, the Poincaré dual of the cup product of $n-1$ $H^1(M;\mathbb{Z})$ classes must vanish. □

Remark 7.31. Theorem 7.29 implies Theorem 7.28. Indeed, let $\beta_1, \ldots, \beta_n$ be a basis of $H^1(T^n;\mathbb{Z}) \cong \mathbb{Z}^n$. We pull back along the degree s map F to obtain classes $\alpha_1, \ldots, \alpha_n \in H^1(M;\mathbb{Z})$. The class $\gamma := \beta_1 \smile \cdots \smile \beta^n$ satisfies $[T^n] \frown \gamma = 1$. By the homological definition of degree, $F_*[M] = s[T^n]$. By the naturality of the cap product,

$$F_*[M] \frown \gamma = F_*([M] \frown F^*\gamma),$$

we find that

$$F_*([M] \frown F^*\gamma) = s \implies [M] \frown F^*\gamma \neq 0.$$

By the naturality of the cup product,

$$F^*\gamma = F^*\beta_1 \smile \cdots \smile F^*\beta_n = \alpha_1 \smile \cdots \smile \alpha_n.$$

Therefore,

$$\big([M] \frown (\alpha_1 \smile \cdots \smile \alpha_{n-1})\big) \frown \alpha_n \neq 0,$$

which implies M does not admit a positive scalar curvature metric by Theorem 7.29.

Remark 7.32. To show that $M = T^3$ does not have a PSC metric, there is the following slick argument. Suppose it does admit a PSC metric. Take a homologically nontrivial $T^2 \hookrightarrow T^3$ and minimize area in $[T^2]$. By the heredity principle, the minimizer Σ is a disjoint union of spheres. It follows from the Alexander–Schoenflies theorem and a covering argument that each of these spheres bound 3-cells in T^3 ([109]). Therefore Σ is nullhomologous, but the original T^2 was not.

7.5 Rigidity and Completing the Proof

In this section we use the positivity of the mass to deduce the rigidity statement. We first show that zero mass implies the manifold is Ricci flat. This first step is originally due to Schoen and Yau [103], but was cast by Lee and Parker [78] and Kuwert [75] in the categories of weighted Hölder and Sobolev spaces, respectively. We follow Kuwert's approach. In 3 dimensions, one has that a Ricci flat manifold is totally flat. In higher dimensions Bartnik adopted an argument of Witten to show $\mathrm{Ric} = 0 \implies \mathrm{Riem} = 0$. He used harmonic coordinates $y^1, \ldots, y^n$ to construct harmonic 1-forms $\omega^i = dy^i$ and then used Bochner's formula to show

$$m(g) = \frac{1}{\omega_{n-1}} \sum_i \int_M \left(|\nabla \omega^i|^2 + \mathrm{Ric}(\omega^i, \omega^i)\right).$$

Since Ric $= 0$, the ω^i's are parallel. Then the map $y : M \to \mathbb{R}^n$ is a local isometry, even a covering map, hence a global isometry. However, this step has since been significantly simplified using the Bishop–Gromov volume comparison theorem (I'm not sure by whom).

We prove these two steps as two separate lemmas, then put everything together at the end of this section.

Lemma 7.33. *Let (M,g) satisfy the mass condition, $R(g) \geq 0$, and $m(g) = 0$. Then g is Ricci flat.*

Proof. Let h be a compactly supported symmetric tensor field on M, and consider the metrics $g_t = g + th$. Suppose we can solve the equations

$$L_t\varphi_t = \mathrm{scal}(g)\varphi_t, \quad \varphi_t \to 1 \text{ as } x \to \infty \tag{7.6}$$

such that $t \mapsto \varphi_t$ is C^1 in some topology. Suppose further that the φ_t's decay quickly enough that the metrics $\tilde{g}_t = \varphi_t^{4/(n-2)} g$ satisfy the energy condition. From the curvature formula,

$$\mathrm{scal}(\tilde{g}_t) = \varphi_t^{(n+2)/(n-2)}\,\mathrm{scal}(g) \geq 0,$$

and by the positive mass theorem,

$$m(\tilde{g}_t) \geq 0 = m(g) \implies \left.\frac{d}{dt}\right|_{t=0} m(g_t) = 0.$$

From the variation formula Theorem 7.7,

$$\begin{aligned} 0 = 2(n-1)\omega_{n-1}\left.\frac{d}{dt}\right|_{t=0} m(g_t) &= \left.\frac{d}{dt}\right|_{t=0}\int_M \mathrm{scal}(\tilde{g}_t)\,d\mu(\tilde{g}_t) \\ &\quad + \int_M \left\langle \mathrm{Ric}(g) - \frac{1}{2}\mathrm{scal}(g)g, \left.\frac{d}{dt}\right|_{t=0}\tilde{g}_t \right\rangle_g d\mu(g). \end{aligned}$$

Let the variation of φ_t be ψ, so that

$$\left.\frac{d}{dt}\right|_{t=0}\tilde{g}_t = h + \frac{4}{n-2}\psi g$$

and

$$\left.\frac{d}{dt}\right|_{t=0} d\mu(\tilde{g}_t) = \frac{1}{2}\left\langle h + \frac{4}{n-2}\psi g, g\right\rangle_g d\mu(g)$$

by the standard formula. Therefore,

$$\left.\frac{d}{dt}\right|_{t=0}(\mathrm{scal}(\tilde{g}_t)\,d\mu(\tilde{g}_t)) = 2\psi\,\mathrm{scal}(g)\,d\mu(g) + \frac{1}{2}\mathrm{scal}(g)\langle g, h\rangle\,d\mu(g),$$

so the variational formula reads

$$0 = 2(n-1)\omega_{n-1}\left.\frac{d}{dt}\right|_{t=0} m(\tilde{g}_t) = \int_M \langle \mathrm{Ric}(g), h\rangle.$$

Since h was arbitrary, $\mathrm{Ric}(g)$ vanishes identically.

We now solve (7.6) for $|t|$ small. We do the usual trick of writing $\varphi_t = 1 + \psi_t$ and solving

$$P_t\psi_t = -\gamma_t, \quad \psi_t \to 0 \text{ as } x \to \infty, \tag{7.7}$$

where $P_t = -a\Delta_t + \gamma_t$ and $\gamma_t = \mathrm{scal}(g_t) - \mathrm{scal}(g)$. Since $g_t \to g$ as $t \to 0$ in $W^{2,q}_{-\tau}$, the method of continuity gives unique solutions of (7.7) as in Theorem 7.13 for small $|t|$. Now P_t and γ_t depend on t in the C^1 sense, so since the inversion map on Banach spaces is C^∞, we have that $\psi_t = P_t^{-1}(-\gamma_t)$ is C^1 in t. The technical point is that this shows $t \mapsto \psi_t$ is C^1 as a map $(-\varepsilon, \varepsilon) \to W^{2,q}_{-\tau}$. We have to show the function $(x,t) \mapsto \psi_t(x)$ is regular enough that the formal calculations above hold. However, by the Sobolev embedding $W^{2,q}_{-\tau} \hookrightarrow C^{1,\alpha}_{-\tau}$ ([78] Lemma 9.1), we have $\psi_t \in C^1((-\varepsilon,\varepsilon); C^{1,\alpha}_{-\tau})$. Therefore we may differentiate under the integral sign and the formal calculation is correct. $\square$

Lemma 7.34. *Let (M,g) be an asymptotically flat manifold satisfying $\mathrm{Ric}(g) \geq 0$. Then (M,g) is isometric to $(\mathbb{R}^n, \delta)$.*

Proof. A simple argument using the Hopf–Rinow theorem shows any asymptotically flat manifold is complete. Choose some $p \in M_0$. The volume comparison theorem says the function

$$\mathcal{V}(r) = \frac{\mathrm{vol}(B_r(p))}{\omega_n r^n}$$

is nonincreasing. Note that $\mathcal{V}(r) \to 1$ as $r \to 0$ for any Riemannian manifold. Using the asymptotically flat structure of M, there is some R_0 such that $M_0 \subset B_{R_0}(p)$ and

$$\mathrm{vol}(B_r(p)) = \mathrm{vol}(B_{R_0}(p)) + \mathrm{vol}(B_r(p) \setminus B_{R_0}(p)).$$

Now we divide by $\omega_n r^n$ and send $r \to \infty$. The constant term gets killed, but using asymptotic flatness, $\mathrm{vol}(B_r(p) \setminus B_{R_0}(p))$ becomes comparable to the volume of the annulus $B_r(0) \setminus B_{R_0}(0)$ in $\mathbb{R}^n$. Therefore, $\mathcal{V}(r) \to 1$ as $r \to \infty$. So by the comparison theorem, $\mathcal{V} \equiv 1$. By the rigidity statment of the comparison theorem, each ball $B_r(p)$ in M is isometric to $B_r(0)$ in $\mathbb{R}^n$. Therefore M is flat and simply connected, hence must be isometric to $(\mathbb{R}^n, \delta)$ because it is complete. $\square$

We can now prove the positive mass theorem (in the cases we considered here).

Proof of Theorem 7.6. We first show that $m(g) \geq 0$. By Theorem 7.19 it suffices to show that $\overline{M} \# T^n$, where $\overline{M}$ is the one-point compactification of M, does not carry a metric of positive scalar curvature. When $3 \leq n \leq 7$, this follows from Theorem 7.26. When M is spin, this follows from Theorem 7.20 after noting that Lemma 7.22 implies that $\overline{M}$ is spin as well.

When $m(g) = 0$, Lemmas 7.33 and 7.34 imply immediately that (M,g) is isometric to $(\mathbb{R}^n, \delta)$. $\square$

It is instructive to compare the present proof to the original one, which is far more challenging. Here are the main steps, cf. pages 55-56 of [75]:

1. Using similar methods as in the proof of Theorem 7.13, one reduces consideration to the asymptotically conformally flat case but instead makes the scalar curvature strictly positive. Therefore one seeks to prove that if $g = \varphi^{4/(n-2)}\delta$ for large x, where
$$\varphi(x) = 1 + \frac{A}{|x|^{n-2}} + \text{l.o.t.}$$
and $\operatorname{scal}(g) > 0$, then $A \geq 0$.

2. Assume $A < 0$.

3. Let $S(\sigma, h) = \{x : (x^1)^2 + \cdots + (x^{n-1})^2 = \sigma, x^n = h\}$. Using minimal surface theory one constructs solutions $\Sigma(\sigma, h)$ of the Plateau problem for $S(\sigma, h)$, namely embedded volume-minimizing hypersurfaces $\Sigma(\sigma, h)$ with boundary equal to $S(\sigma, h)$.

4. Using the assumption $A < 0$ one shows that for each σ there exists an h_σ so that $\Sigma(\sigma, h_\sigma)$ has minimal area for all h. Further, one shows that for each σ, $\Sigma(\sigma, h_\sigma)$ is contained in some cylinder $\Sigma(\sigma, h_\sigma) = \{x : |x|^n \leq h_0\}$. To do this one has to compare the growth of the Riemannian Hausdorff measure to the Euclidean one.

5. The cylinder acts as a barrier and can be used to extract a limit $\Sigma(\sigma, h_\sigma) \to \Sigma$ as $\sigma \to \infty$. This results from detailed uniform area estimates.

6. Using Allard's regularity theorem, the height and area estimates show that Σ can be viewed as a graph over the set $\{x : (x^1)^2 + \cdots + (x^{n-1})^2 \geq R_0, x^n = 0\}$ for some $R_0 \gg 1$.

7. Using the minimal surface equation for the function whose graph represents Σ, one obtains improved estimates on the function.

8. Using these estimates, one obtains a stability-like inequality for Σ.

9. In $n = 3$, this stability inequality violates the Gauss–Bonnet theorem, using the fact that $\chi(\Sigma) \leq 1$ for an open surface. When $4 \leq n \leq 7$, a version of the heredity principle shows that Σ has a conformal metric that is AF, has zero scalar curvature, and negative mass. The theorem then follows by induction on n.

7.6 Asymptotically Flat Manifolds with Multiple Ends

It is traditional to say some words about the generalization of the positive mass theorem to the case of multiple ends (finitely many). This is because the original proof requires the AF manifold to be orientable, but taking the orientation cover of a nonorientable AF manifold produces an AF manifold with two ends. Further, there are many interesting manifolds with multiple ends, such as the $t = 0$ slice of Schwarzschild spacetime. The following proof seems to be mostly folklore.

Firstly, one has to extend the elliptic theory of AF manifolds to the case of multiple ends. This was done in some cases in [30], but as far as I know, no one has written down the details of generalizing [12] to the case of multiple ends. So, assuming this, the entire discussion of Section 6.2 carries over. The mass is defined *per end* by taking the integral to be over spheres on that particular end. The more general version of the positive mass theorem becomes:

Theorem 7.35. *Let (M, g) be an asymptotically flat manifold with finitely many ends satisfying the mass condition on each end. Suppose* $\operatorname{scal}(g) \geq 0$ *and one of the following conditions holds:*

- $3 \leq n \leq 7$ *or*
- *M is finitely covered by a spin manifold.*

Then the mass of each end is nonnegative, and if one mass is zero, then (M, g) is isometric to $\mathbb{R}^n$ with the standard metric.

Of course, if one accepts the validity of Theorem 7.3, then this should be true without any hypothesis on the dimension or topology of M.

We first note that the rigidity proof carries over nicely. If there are k ends, it is easy to see that the asymptotic volume ratio is k. This contradicts the volume comparison theorem unless $k = 1$, so there is one end and the rigidity theorem for one end applies immediately.

We proceed to prove Theorem 7.35 by contradiction. So we have to deal with at least one end having negative mass, but multiple ends could possibly have positive or zero mass. If there are two ends with negative mass, proceed as in Proposition 7.15 to compactify one of them to a torus. Then we have an AF manifold with $k-1$ ends and at least one negative mass. By induction on k we easily arrive at a contradiction. However, this does not preclude the existence of just one end with negative mass.

So we have only one end with negative mass and the rest have mass ≥ 0. If we can somehow close off one of those ends while maintaining $\operatorname{scal} \geq 0$ and not changing the other masses, we arrive at a contradiction again via induction on k.

Fix an end E with mass ≥ 0. We can compactify M along E by adding a point at the infinity of E and giving this the natural smooth structure, denoted by CM. Note that this does not introduce any new topology. By Theorem 7.13 we may assume that the metric on E looks like $\varphi^{4/(n-2)}\delta$, where φ is δ-harmonic and has the usual expansion

$$\varphi(x) = 1 + \frac{m(E)}{4(n-1)|x|^{n-2}} + O(|x|^{1-n}).$$

If $m(E) < 0$, we showed in Proposition 7.15 that one can compactify to a torus component. When $m(E) \geq 0$, this fails because φ lies *above* 1 due to the maximum principle.

We cannot really repair the proof because, say, $\max(\varphi, 1 + \frac{1}{2}\varepsilon)$ is *subharmonic*, which leads to a *negative scalar curvature metric*. So some other compactification is needed. To see what is needed, consider the inverse stereographic projection $F : \mathbb{R}^n \to S^n \setminus \{N\}$. If $\mathcal{S}$ denotes the round metric on S^n, then it is well known that

$$F^*\mathcal{S} = \frac{4}{(1+|x|^2)^2}\delta.$$

Because $\mathcal{S}_R$ has positive scalar curvature and δ has zero scalar curvature, the conformal curvature formula implies that

$$\xi(x) := \left(\frac{4}{(1+|x|^2)^2}\right)^{(n-2)/4} \tag{7.8}$$

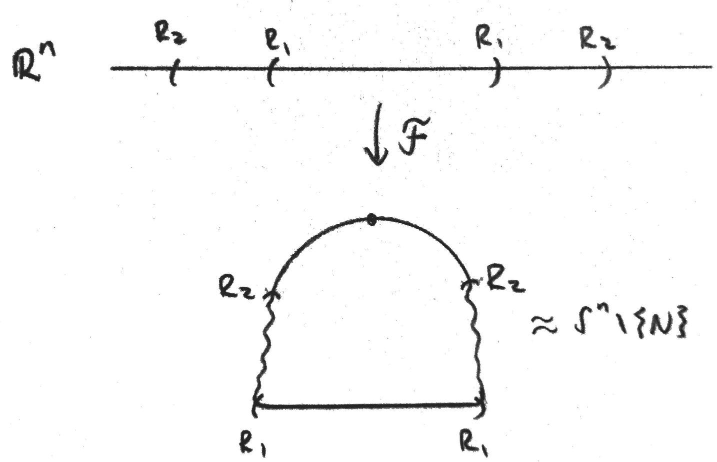
is strictly superharmonic.

So the idea is to replace φ by something of the form $u = \min(\varphi, \xi)$. This is superharmonic and equals ξ near infinity since $\xi \to 0$ while $\varphi \to 1$. However, it is not smooth and it is not possible to smooth it while keeping the exact ξ asymptotics. We show that by using a mollifier, we can arrange $u_\varepsilon(x) = u(x)$ when $|x|$ is "small" and u^ε is very close to ξ in the C^2 sense for x large. Therefore, under some modified version of inverse stereographic projection, we can expect to get something "close" to a sphere. One might call this a scal > 0 "bubble". Now on to the details of this construction.

We require the following obvious geometric result.

Lemma 7.36. *Consider two balls $B_{R_1} \subset B_{R_2}$ about the origin in $\mathbb{R}^n$. There exists a $\rho > 0$ and a diffeomorphism $\mathcal{F} : \mathbb{R}^n \to S^n \setminus \{N\}$ such that $\mathcal{F}|_{B_{R_1}} = \mathrm{id}$ and $\mathcal{F}|_{\mathbb{R}^n \setminus B_{R_2}}$ is inverse stereographic projection onto S^n with radius ρ.*

Figure 7.1: Modified stereographic projection



Proof of Theorem 7.35. As remarked above, it suffices to provide a compactification algorithm for an end E with nonnegative mass. We also assume that the metric on E is $\varphi^{4/(n-2)}\delta$, where φ is a harmonic function.

First consider the case when the mass is positive. Let $\beta > 0$. Since $\varphi \to 1$ as $x \to \infty$, by the maximum principle we may assume that for some $R_1 > 0$, $|x| > R_1$ implies $1 < \varphi < 1+\beta$. Let ξ be the function on E defined by (7.8). Choose α so large that $\alpha\xi(R_1) > 1 + 2\beta$. Then the function

$$u = \begin{cases} \varphi & \text{for } |x| \le R_1 \\ \min(\varphi, \alpha\xi) & \text{for } |x| \ge R_1 \end{cases}$$

is continuous, equals u for $|x| < R_1$ and equals $\alpha\xi$ when $|x| > R_2$ for some $R_2 \gg 1$. It is also superharmonic since φ is harmonic and ξ is superharmonic. By Proposition 7.18, the mollification u^ε equals φ for $|x| < R_1$ and is superharmonic everywhere. Note as well that u_ε is strictly positive and since u is uniformly continuous, can be made arbitrarily close to u in $L^\infty(E)$.

Using Lemma 7.36, we construct a diffeomorphism $\mathcal{F} : M \to CM \backslash \{\infty\}$ with the following properties: $F(x) = x$ whenever $|x| < R_1$ and F is inverse stereographic projection whenever $|x| > R_2$. We claim that $\overline{g} = d\mathcal{F}(u_\varepsilon^{4/(n-2)}\delta)$ extends smoothly to CM and has nonnegative scalar curvature. Since u_ε is superharmonic, the curvature formula implies $u_\varepsilon^{4/(n-2)}\delta$ has nonnegative scalar curvature.

In a punctured neighborhood of ∞ in CM, we have

$$\overline{g} = \left\{ \left(\frac{u_\varepsilon}{\xi}\right)^{4/(n-2)} \circ \mathcal{F}^{-1} \right\} \mathcal{S},$$

where $\mathcal{S}$ is the round metric on S^n. To show that $\overline{g}$ extends to ∞, we study the ratio u_ε/ξ. For $|x| > R_2$,

$$u_\varepsilon(x) = \alpha \int_{B_\varepsilon(0)} \eta_\varepsilon(y)\xi(x-y)\, dy.$$

Therefore, since η and ξ are rotationally symmetric, so is u_ε. We immediately have that

$$\frac{u_\varepsilon(x)}{\xi(x)} \to \alpha \quad \text{as } x \to \infty.$$

Similarly, by examining the growth rates of ξ and its derivatives,

$$D^\gamma\left(\frac{u_\varepsilon(x)}{\xi(x)}\right) \to 0 \quad \text{as } x \to \infty$$

for any multiindex γ with $|\gamma| \geq 1$. Therefore, $\overline{g}$ extends to all of CM in a C^∞ fashion and $\mathrm{scal}(\overline{g}) \geq 0$.

Suppose now we want to compactify an end with zero mass. The problem now is that there is no information on the leading term of φ, so we could have $\varphi > 1$, $\varphi < 1$, or $\varphi \equiv 1$. In the first case, however, the preceeding proof applies. In the latter two cases, we can compactify to a torus as before.

Finally, the arguments at the beginning of this section show how this compactification algorithm proves the theorem inductively. $\square$

Appendix A

Cheng–Yau Gradient Estimate

The Cheng–Yau gradient estimate is an important global gradient estimate for supersolutions of certain elliptic equations [33]. Originally Yau proved it for eigenfunctions of the Laplacian [124] (see also Chapter 6 of [79] for the sharp estimate).

Theorem A.1. *Let (M, g) be an $n(\geq 2)$-dimensional complete Riemannian manifold with*

$$\mathrm{Ric} \geq -(n-1)K, \quad K \geq 0.$$

Suppose that u is a positive function satisfying

$$\Delta u = -\lambda u$$

on M and B_ρ is a metric ball in M. Then

$$\frac{|\nabla u|}{u} \leq C \quad \text{on } B_{\rho/2},$$

for $C = C(n, K, \lambda, \rho) > 0$ depending only on these parameters. When $\lambda = 0$, we can choose

$$C = C_n \left(\frac{1 + \rho\sqrt{K}}{\rho} \right).$$

We also have the sharp estimate

$$\frac{|\nabla u|^2}{u^2} \leq \frac{(n-1)^2 K}{2} - \lambda + \sqrt{\frac{(n-1)^4 K^2}{4} - (n-1)^2 \lambda K} \tag{A.1}$$

on all of M.

Note that the constant does not depend on the center of the ball. The proof is based on the second derivative test and the Laplacian comparison theorem.

Theorem A.2 (Laplacian Comparison Theorem)**.** *Let (M, g) be an n-dimensional complete Riemannian manifold with* Ric $\geq (n-1)K$*, $K \in \mathbb{R}$. Then for fixed $p \in M$, the function $r(x) = d(x, p)$ satisfies*

$$\Delta r \leq \begin{cases} (n-1)\sqrt{K} \cot\left(\sqrt{K} r\right) & \text{if } K > 0 \\ (n-1) r^{-1} & \text{if } K = 0 \\ (n-1)\sqrt{-K} \coth\left(\sqrt{-K} r\right) & \text{if } K < 0 \end{cases}$$

pointwise where it is smooth (inside of the cut locus), and globally in the sense of distributions.

Corollary A.3. *Let (M, g) be complete with $\mathrm{Ric} \geq -(n-1)K$, $K \geq 0$. Then*

$$\Delta r \leq \frac{n-1}{r}(1 + \sqrt{K} r).$$

Proof. This follows from the Laplacian comparison theorem once we show the inequality

$$x \cosh x \leq 1 + x$$

for all $x \geq 0$. Indeed, we have

$$x\left(\coth x - 1\right) = x\left(\frac{2e^{-x}}{e^{x} - e^{-x}}\right). \tag{A.2}$$

Since $1 + 2x \leq e^{2x}$, we have

$$\frac{2}{e^{2x}-1} \leq \frac{1}{x}.$$

So from (A.2) it follows that

$$x(\coth x - 1) \leq 1.$$

□

For the proof, see again Section 1.12 in [38]. A typical application of the gradient estimate is the following Harnack inequality.

Corollary A.4. *Let (M, g) and u be as in Theorem A.1. Fix $p \in M$ and let $r(x) = d(x, p)$. Then there exists a constant $C > 0$ such that*

$$u(y) \leq u(x) e^{Cd(x,y)}$$

for all $x, y \in M$.

Proof. Fix $x, y \in M$. Let γ be a geodesic from x to y such that $L(\gamma) = d(x, y)$ (use completeness). Then Yau's gradient estimate and the fundamental theorem of calculus give

$$\begin{aligned}
\log \frac{u(y)}{u(x)} &= \log u(y) - \log u(x) \\
&\leq \int_\gamma |\nabla \log u| \, ds \\
&\leq C \int_\gamma ds \\
&= C d(x, y).
\end{aligned}$$

Now exponentiate this inequality and rearrange to conclude. □

We also have a Liouville theorem.

Corollary A.5. *On a complete Riemannian manifold with nonnegative Ricci curvature there does not exist a nonconstant positive harmonic function.*

Proof. This is immediate from the sharp estimate (A.1) when $\lambda = K = 0$. $\square$

In Chapter 6, we require a slightly more general form of the Yau gradient inequality, but much less general than the Cheng–Yau inequality.

Theorem A.6. *Let (M, g) be a complete n-dimensional Riemannian manifold with*

$$\mathrm{Ric} \geq -(n-1)K, \quad K \geq 0.$$

Let h be a bounded smooth Lipschitz function on M and suppose $u > 0$ satisfies

$$\Delta u = hu.$$

Then

$$\frac{|\nabla u|}{u} \leq C \quad \text{on } B_\rho,$$

where the constant $C > 0$ depends on $n, K, \rho, \|h\|_{L^\infty}$, and the Lipschitz constant of h.

Remark A.7. Here Lipschitz means ∇h is bounded globally with respect to the metric g. This condition obviously depends on the metric.

The proof, while simple in idea, is quite long and is complicated by the fact that r is not a smooth function everywhere. We begin with some lemmas.

Lemma A.8. *Let γ be a segment (length minimizing geodesic) in M from p to q. Suppose that q is in the cut locus of p. Let x be a point on γ. Then x has no cut point on γ.*

Proof. Suppose that x has a cut point y between itself and q along γ. First suppose that x and y are conjugate. Then by using Proposition VIII.3.5 in [72], we could show that p and y are conjugate, which is impossible since y is not in the cut locus of p. The other option is that there is a second segment σ from x to y that is not γ. But then we could start at p, go to x along γ, and then take σ to y to get a segment from p to y that is not γ, which again contradicts y not being a cut point of p. A dual argument works when y is between p and x using the reflexivity of the cut locus. $\square$

Lemma A.9. *There exists a smooth function $\phi : [0, \infty) \to [0, 1]$ such that*

$$\phi = \begin{cases} 1 & \text{on } [0, \rho] \\ 0 & \text{on } [2\rho, \infty) \end{cases},$$

$$-C\rho^{-1}\phi^{1/2} \leq \phi' \leq 0 \quad \text{on } [\rho, 2\rho],$$

and

$$|\phi''| \leq C\rho^{-2},$$

where $C > 0$ is independent of ρ.

Proof. We construct the function when $\rho = 1$. It is easy to see that $\phi(r/\rho)$ then works for any $\rho > 0$ with the same constant C. So for $\rho = 1$, let ϕ be any standard plateau function that is decreasing on $[\rho, 2\rho]$. Clearly there exists a C_1 such that $|\phi''| \leq C_2$, so we just have to find a C_2 for which $-C_2\phi^{1/2} \leq \phi'$, and then set $C = \max\{C_1, C_2\}$. It suffices to set

$$C_2 = \sup_{[\rho,2\rho]} \frac{|\phi'|^2}{\phi}.$$

That this is finite follows easily from an ε-regularization argument ([64] Lemma 6.6). $\square$

Lemma A.10. *Assume the hypotheses of Theorem A.6. Set $v = \log u$ and $Q = |\nabla \log u|^2$. Then*

$$\Delta Q - \frac{n}{2(n-1)}\frac{|\nabla Q|^2}{Q} + \frac{\langle \nabla v, \nabla Q\rangle}{Q}\left(\frac{2h}{n-1} - \frac{2(n-2)}{n-1}Q\right)$$

$$\geq \frac{2}{n-1}Q^2 - \left(\frac{4}{n-1}h + 2(n-1)K\right)Q + \frac{2}{n-1}h^2 + 2\langle \nabla v, \nabla h\rangle \tag{A.3}$$

at every point where $Q \neq 0$.

Proof. By a direct computation,

$$\Delta v = -|\nabla v|^2 + h.$$

Letting $v_i = \nabla v_i$ and $v_{ij} = (\operatorname{Hess} v)_{ij}$ in an orthonormal frame, Bochner's formula implies

$$\begin{aligned}\Delta Q &= 2\sum_{i,j=1}^n v_{ij}^2 + 2\operatorname{Ric}(\nabla v, \nabla v) + 2\langle \nabla v, \nabla \Delta v\rangle \\ &\geq 2\sum_{i,j=1}^n v_{ij}^2 - 2(n-1)KQ - 2\langle \nabla v, \nabla Q\rangle + 2\langle \nabla v, \nabla h\rangle. \end{aligned}\tag{A.4}$$

Choosing the frame so that $|\nabla v|e_1 = \nabla v$ at p, we can write at p

$$\begin{aligned}|\nabla|\nabla v|^2|^2 &= 4\sum_{j=1}^n\left(\sum_{i=1}^n v_i v_{ij}\right)^2 \\ &= 4v_i^2\sum_{j=1}^n v_{1j}^2 \\ &= 4|\nabla v|^2\sum_{j=1}^n v_{1j}^2 \end{aligned}\tag{A.5}$$

On the other hand,

$$\begin{aligned}\sum_{i,j=1}^n v_{ij}^2 &\geq v_{11}^2 + 2\sum_{j=2}^n v_{1j}^2 + \sum_{j=2}^n v_{jj}^2 \\ &\geq v_{11}^2 + 2\sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1}\left(\sum_{j=2}^n v_{jj}^2\right)^2 \\ &= v_{11}^2 + 2\sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1}(\Delta v - v_{11})^2 \\ &= v_{11}^2 + 2\sum_{j=2}^n v_{1j}^2 + \frac{1}{n-1}(|\nabla v|^2 - h + v_{11})^2 \\ &\geq \frac{n}{n-1}\sum_{j=1}^n v_{1j}^2 + \frac{1}{n-1}(|\nabla v|^2 - h)^2 + \frac{2v_{11}}{n-1}(|\nabla v|^2 - h).\end{aligned}$$

However, using the identity

$$\begin{aligned}2v_1 v_{11} &= e_1(|\nabla v|^2) \\ &= \frac{\langle \nabla|\nabla v|^2, \nabla v\rangle}{|\nabla v|},\end{aligned}$$

we conclude that

$$2v_{11} = \frac{\langle \nabla|\nabla v|^2, \nabla v\rangle}{|\nabla v|^2}.$$

This gives

$$\sum_{i,j=1}^n v_{ij}^2 \geq \frac{n}{n-1}\sum_{j=1}^n v_{1j}^2 + \frac{1}{n-1}(|\nabla v|^2 - h)^2 + \frac{1}{n-1}\frac{\langle \nabla|\nabla v|^2, \nabla v\rangle}{|\nabla v|^2}(|\nabla v|^2 - h).$$

Combining this with (A.4) and (A.5), we obtain

$$\begin{aligned}\Delta Q \geq{}& \frac{n}{2(n-1)}\frac{|\nabla Q|^2}{Q} - 2(n-1)KQ - \frac{\langle \nabla v, \nabla Q\rangle}{Q}\left(\frac{2h}{n-1} + \frac{2(n-2)}{n-1}Q\right) \\ &+ \frac{2}{n-1}Q^2 - \frac{4}{n-1}hQ + \frac{2}{n-1}h^2 + 2\langle \nabla v, \nabla h\rangle.\end{aligned}$$

□

We now give an improved and generalized version of the proof in [79].

Proof of Theorem A.6. Choose $\rho > 0$ and $p \in M$. We show the estimate in $B_{2\rho}(p)$, with a constant independent of p. Let ψ be a nonnegative cutoff function for this ball and define

$G = \psi Q$. Then at point where Q and ψ are smooth and positive, we have by (A.3)

$$
\begin{aligned}
\Delta G &= G\Delta\psi + 2\langle\nabla\psi,\nabla G\rangle + \psi\Delta Q \\
&\geq \frac{\Delta\psi}{\psi}G + 2\frac{\langle\nabla\psi,\nabla G\rangle}{\psi} - 2\frac{|\nabla\psi|^2 G}{\psi^2} + \frac{n}{2(n-1)}\frac{|\nabla G|^2}{G} + \frac{n}{2(n-1)}\frac{\langle\nabla\psi,\nabla G\rangle}{\psi} \\
&\quad - \frac{n}{n-1}\frac{\langle\nabla\psi,\nabla G\rangle}{\psi} - \frac{\langle\nabla v,\nabla G\rangle}{Q}\left(\frac{2(n-2)}{n-1}Q + \frac{2h}{n-1}\right) \\
&\quad + \langle\nabla v,\nabla\psi\rangle\left(\frac{2(n-2)}{n-1}Q + \frac{2h}{n-1}\right) + \frac{2}{n-1}\frac{G^2}{\psi} + \frac{2h^2}{n-1}\psi \\
&\quad - \left(\frac{4h}{n-1} + 2(n-1)K\right)G + 2\langle\nabla v,\nabla h\rangle.
\end{aligned} \tag{A.6}
$$

Let $L = \operatorname{Lip} h$. We first estimate

$$
\begin{aligned}
2\langle\nabla v,\nabla h\rangle &\geq -2|\langle\nabla v,\nabla h\rangle| \\
&\geq -2|\nabla v||\nabla h| \\
&\geq -2LG^{1/2}.
\end{aligned}
$$

We also have

$$|\langle\nabla v,\nabla\psi\rangle| \leq |\nabla\psi|\psi^{-1/2}G^{1/2}.$$

Using these in (A.6) and simplifying yields

$$
\begin{aligned}
\Delta G &\geq \frac{\Delta\psi}{\psi}G + \frac{n-2}{n-1}\frac{\langle\nabla\psi,\nabla G\rangle}{\psi} - \frac{3n-4}{2(n-1)}\frac{|\nabla\psi|^2 G}{\psi^2} + \frac{n}{2(n-1)}\frac{|\nabla G|^2}{G} \\
&\quad - \left(\frac{4h}{n-1} + 2(n-1)K\right)G - \frac{\langle\nabla v,\nabla G\rangle}{Q}\left(\frac{2(n-2)}{n-1}Q + \frac{2h}{n-1}\right) \\
&\quad - \frac{2(n-2)}{n-1}|\nabla\psi|\psi^{-3/2}G^{3/2} + \frac{2}{n-1}\frac{G^2}{\psi} + \frac{2h^2}{n-1}\psi - 2LG^{1/2} \\
&\quad - \frac{2H_2}{n-1}|\nabla\psi|\psi^{-1/2}G^{1/2},
\end{aligned} \tag{A.7}
$$

where $H_1 = \max\{0, \sup h\}$. Since $G \to 0$ at the boundary, it attains a maximum at an interior point x. Suppose for now that G is smooth in a neighborhood of x. This is of course not always the case (since G is nonsmooth on the cut locus of p) and this issue will be dealt with later. By the second derivative test, we have

$$\Delta G(x) \leq 0, \quad \nabla G(x) = 0.$$

If $G(x) > 0$, then $\psi(x) > 0$. We multiply (A.7) by $(n-1)\psi$ and obtain, at x,

$$
\begin{aligned}
0 &\geq (n-1)\Delta\psi\, G - \frac{3n-4}{2}\psi^{-1}|\nabla\psi|^2 G - (4h + 2(n-1)^2 K)G\psi + 2G^2 \\
&\quad - 2(n-2)|\nabla\psi|\psi^{-1/2}G^{3/2} - 2H_1|\nabla\psi|\psi^{1/2}G^{1/2} + 2h^2\psi^2 - 2(n-1)\psi LG^{1/2}.
\end{aligned} \tag{A.8}
$$

We now make a choice of ψ. Let ϕ be as in Lemma A.9 and define $\psi(y) = \phi(r(y))$, where $r(y) = d(y,p)$. It follows from Corollary A.3 that

$$\begin{aligned}\Delta\psi &= \phi'\Delta r + \phi'' \\ &\geq -C_1(\rho^{-1}\sqrt{K} + \rho^{-2})\end{aligned}$$

and also

$$|\nabla\psi|^2\psi^{-1} \leq C_2\rho^{-2}.$$

Hence (A.8) yields

$$\begin{aligned}0 \geq &-(C_3\rho^{-1}\sqrt{K} + C_4\rho^{-2})G - (4h\psi + 2(n-1)^2K\psi)G - C_5\rho^{-1}G^{3/2} + 2G^2 \\ &- C_6H_1\rho^{-1}G^{1/2} + h^2\psi^2 - 2(n-1)\psi LG^{1/2}. \end{aligned} \tag{A.9}$$

Since x is a maximum point of G and $\psi = 1$ on $B_\rho(p)$,

$$\psi(x)|\nabla v|^2(x) \geq \sup_{B_\rho(p)} |\nabla v|^2.$$

Using the fact that

$$\psi(x)|\nabla v|^2(x) \leq \psi(x) \sup_{B_{2\rho}(p)} |\nabla v|^2,$$

we conclude that

$$\sigma(\rho) \leq \psi(x) \leq 1, \tag{A.10}$$

where σ is defined by

$$\sigma(\rho) = \frac{\sup\limits_{B_\rho(p)} |\nabla v|^2}{\sup\limits_{B_{2\rho}(p)} |\nabla v|^2}.$$

Let $H_2 = \inf h^2$. Applying the estimate (A.10) to (A.9), we have

$$\begin{aligned}0 \geq &-(C_3\rho^{-1}\sqrt{K} + C_4\rho^{-2} - 4H_1\sigma(p) + 2(n-1)^2K)G - C_5\rho^{-1}G^{3/2} + 2G^2 \\ &- C_6H_1\rho^{-1}G^{1/2} + H_2\sigma(p)^2 - 2(n-1)LG^{1/2}. \end{aligned} \tag{A.11}$$

The goal is to turn this into a quadratic inequality for $G(x)$. Let $\varepsilon > 0$ and use Cauchy's inequality to estimate

$$\begin{aligned}-C_5\rho^{-1}G^{3/2} &\geq -\varepsilon G^2 - C_7\varepsilon^{-1}\rho^{-2}G \\ -C_6H_1\rho^{-1}G^{1/2} &\geq -\varepsilon H_1^2 - C_8\varepsilon^{-1}\rho^{-2}G \\ -C_3\rho^{-1}\sqrt{K} &\geq -\varepsilon K - C_9\varepsilon^{-1}\rho^{-2} \\ -2(n-1)LG^{1/2} &\geq -C_{10}\varepsilon G - C_{11}\varepsilon^{-1}L^2.\end{aligned}$$

Note that $C_{10} = 0$ if $L = 0$, and $C_{10} = 1$ if not. By combining all of these with (A.11), we obtain

$$\begin{aligned}0 \geq &-(C_{12}(1+\varepsilon^{-1})\rho^{-2} - 4H_1\sigma(p) + (2(n-1)^2 + \varepsilon)K + C_{10}\varepsilon)G \\ &+ (2-\varepsilon)G^2 + 2H_2\sigma(p)^2 - \varepsilon H_1^2 - C_{11}\varepsilon^{-1}L^2.\end{aligned}$$

If $\varepsilon < 2$, this is an upward-facing parabola for $G(x)$. Since the parabola has a nonnegative value, the roots are automatically real, and G must lie between them. We have the estimate from above

$$G(x) \le \frac{b + \sqrt{b^2 - 4(2-\varepsilon)c}}{4 - 2\varepsilon}, \tag{A.12}$$

where

$$b = -(C_{12}(1+\varepsilon^{-1})\rho^{-2} - 4H_1\sigma(p) + (2(n-1)^2 + \varepsilon)K + C_{10}\varepsilon),$$
$$c = 2H_2\sigma(p)^2 - \varepsilon H_1^2 - C_{11}\varepsilon^{-1}L^2.$$

Then, since

$$\sup_{B_\rho(p)} |\nabla v|^2 = G(x),$$

(A.12) implies the estimate

$$\frac{|\nabla u|^2}{u^2} \le \frac{b + \sqrt{b^2 - 4(2-\varepsilon)c}}{4 - 2\varepsilon} \tag{A.13}$$

in all of $B_\rho(p)$. Since the RHS is independent of p, this completes the proof when x is not in the cut locus of p.

To deal with the cut locus we use Calabi's trick. Choose a segment γ from p to x. Let $\delta > 0$, and choose p_δ on γ such that $d(p, p_\delta) = \delta$. By Lemma A.8 the cut locus of p_δ is bounded away from γ, so x is a regular point for the new distance function $r_\delta(y) = d(y, p_\delta)$. Note that

$$r_\delta(y) + \delta \ge r(y)$$

and

$$r_\delta(x) + \delta = r(x).$$

Since ϕ is decreasing, if we define $\psi_\delta(y) = \phi(r_\delta(y) + \delta)$, then

$$\psi_\delta(x) = \psi(x), \quad \psi_\delta(y) \le \psi(y).$$

Thus x is also a maximum for the function $G_\delta = \psi_\delta Q$. Now we have (A.8) with δ, since the derivation did not require a choice of cutoff function.

$$\begin{aligned} 0 \ge{}& (n-1)\Delta\psi_\delta\, G_\delta - \frac{3n-4}{2}\psi_\delta^{-1}|\nabla\psi_\delta|^2 G_\delta - (4h + 2(n-1)^2K)G_\delta\psi + 2G_\delta^2 \\ &- 2(n-2)|\nabla\psi_\delta|\psi_\delta^{-1/2}G_\delta^{3/2} - 2H_1|\nabla\psi_\delta|\psi_\delta^{1/2}G_\delta^{1/2} + 2h^2\psi_\delta^2 - 2(n-1)\psi_\delta L G^{1/2}. \end{aligned} \tag{A.14}$$

We also have

$$|\nabla\psi_\delta|^2\psi_\delta^{-1} \le C_2\rho^{-2},$$

but the estimate of $\Delta\psi_\delta$ changes a little bit to

$$\Delta\psi_\delta \ge -C_1\left(\rho^{-1}\sqrt{K} + (\rho-\delta)^{-1} + \rho^{-2}\right).$$

Using this, the proof proceeds as before, but now with δ terms in the RHS of (A.12) for $G_\delta(x)$. These disappear as $\delta \downarrow 0$. So letting $\delta \downarrow 0$, we find that $G(x)$ satisfies (A.12) even if x is in the cut locus of p.

As a bonus, we show how to derive the sharp inequality (A.1) from (A.13). When $h = -\lambda$ is a constant, then $H_1 = \lambda$, $H_2 = \lambda^2$, $L = 0$, and $C_{10} = 0$. So we have

$$\frac{|\nabla u|^2}{u^2} \leq \frac{b + \sqrt{b^2 - 4(2-\varepsilon)\lambda^2(2\sigma(p)^2 - \varepsilon)}}{4 - 2\varepsilon},$$

with

$$b = -(C_{12}(1+\varepsilon^{-1})\rho^{-2} - 4\lambda\sigma(p) + (2(n-1)^2 + \varepsilon)K).$$

We may let $\rho \to \infty$, then $\sigma(\rho) \to 1$. Since this gets rid of the ε^{-1} term, we may then let $\varepsilon \downarrow 0$ to conclude (A.1). $\square$

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