Retiring the third law of black hole thermodynamics

Ryan Unger

Department of Mathematics, Princeton University

Cambridge Friday GR seminar
January 2023

joint work with Christoph Kehle (ETH Zürich)
arXiv:2211.15742

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between **classical** black hole dynamics and classical thermodynamics.

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between **classical** black hole dynamics and classical thermodynamics.

Fundamentally, these are statements about Einstein gravity coupled to classical matter fields (and are already interesting in vacuum!).

Black hole thermodynamics

Law	Classical thermodynamics	Black holes

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	surface gravity $\kappa \not\to 0$ in finite advanced time

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	surface gravity $\kappa \not\to 0$ in finite advanced time

- Hawking radiation—interpret $\kappa \propto T$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	surface gravity $\kappa \not\to 0$ in finite advanced time

- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$

Black hole thermodynamics

Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	surface gravity $\kappa \not\to 0$ in finite advanced time

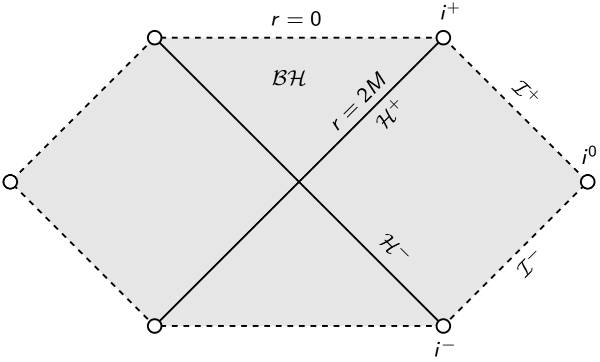
- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$
- These identifications are further justified physically using QFT, but are **formal analogies** in classical GR.

Black hole thermodynamics

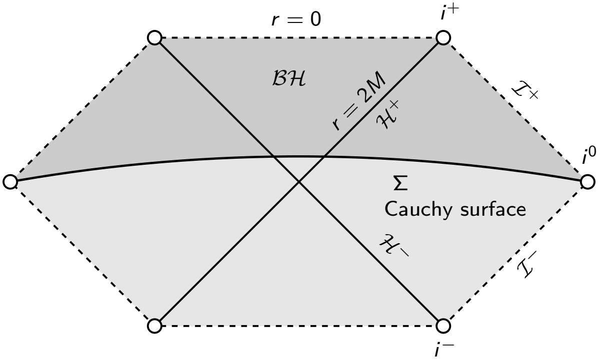
Law	Classical thermodynamics	Black holes
Zeroth	T constant in equilibrium	surface gravity κ constant on stationary horizon
First	$dE = TdS + \cdots$	$dM = \kappa dA + \cdots$
Second	$dS \geq 0$	$dA \geq 0$
Third	$T \not\to 0$ in finite process	surface gravity $\kappa \not\to 0$ in finite advanced time

- Hawking radiation—interpret $\kappa \propto T$
- Bekenstein entropy—interpret $A \propto S$
- These identifications are further justified physically using QFT, but are **formal analogies** in classical GR.
- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Refresher on Schwarzschild

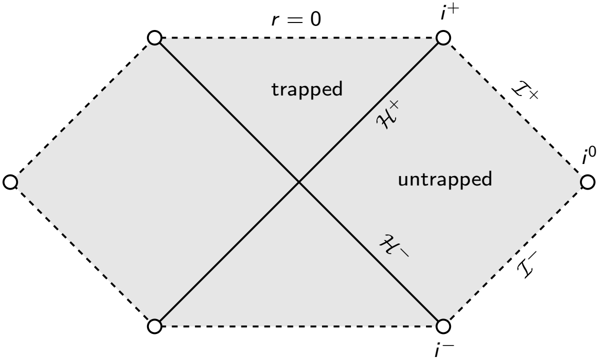


Refresher on Schwarzschild



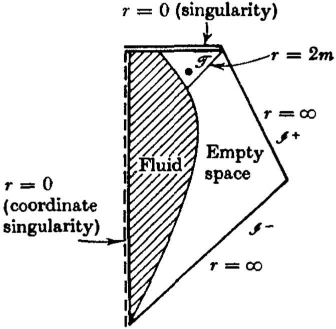
Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface Σ as depicted here.

Refresher on Schwarzschild



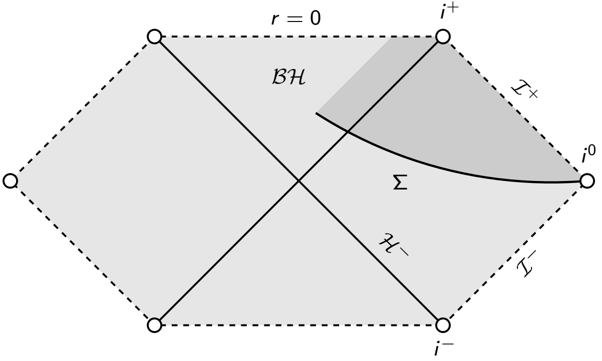
The black hole interior is foliated by **trapped surfaces** (both future null expansions negative).

Refresher on gravitational collapse

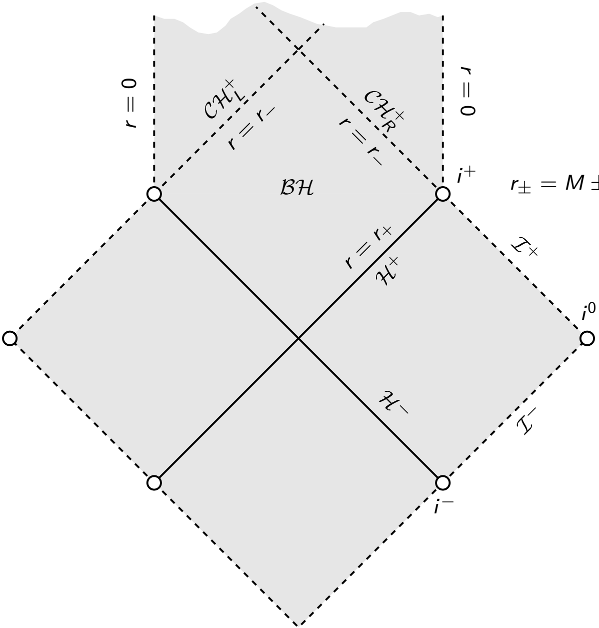


Penrose diagram of gravitational collapse. One-ended Cauchy data!

Refresher on gravitational collapse

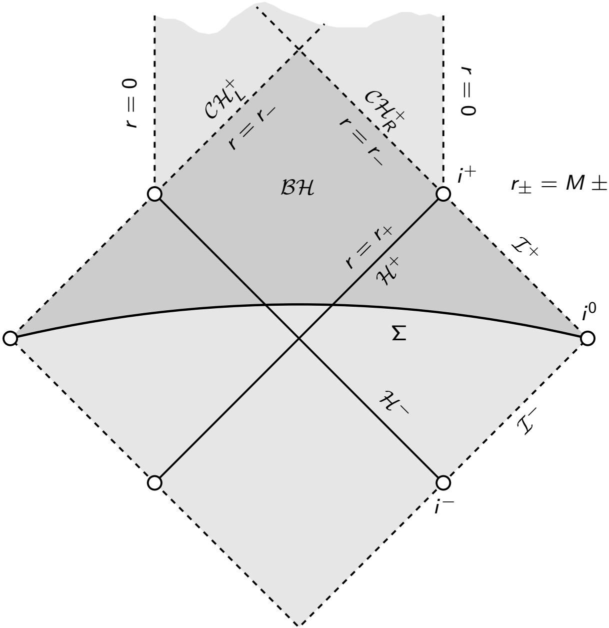


Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



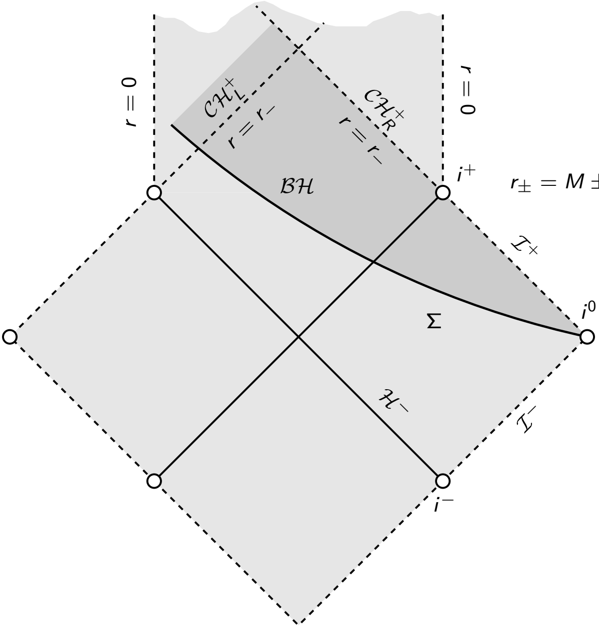
$r_\pm = M \pm \sqrt{M^2 - e^2}$

Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



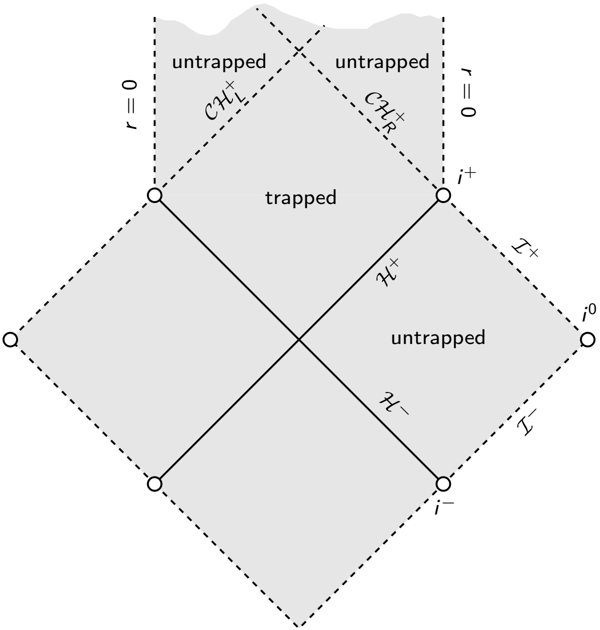
$r_\pm = M \pm \sqrt{M^2 - e^2}$

Refresher on subextremal Reissner–Nordström: $0 < |e| < M$

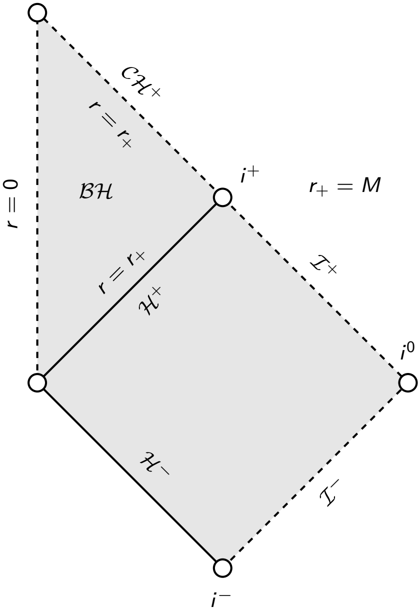


$r_\pm = M \pm \sqrt{M^2 - e^2}$

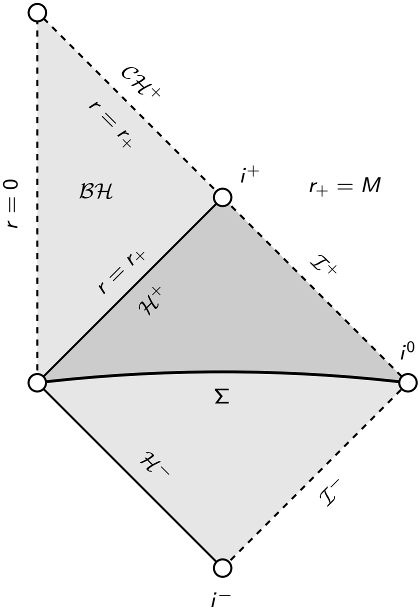
Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



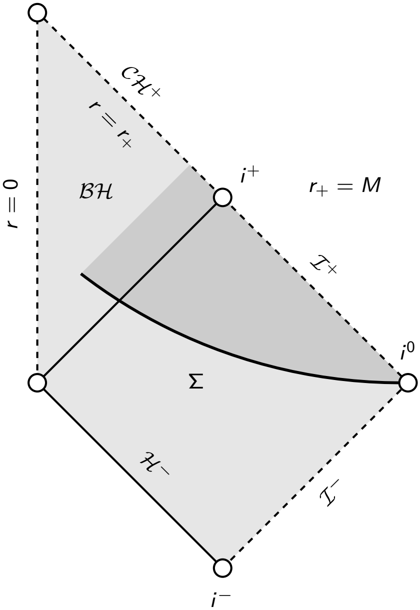
Refresher on extremal Reissner–Nordström: $0 < |e| = M$



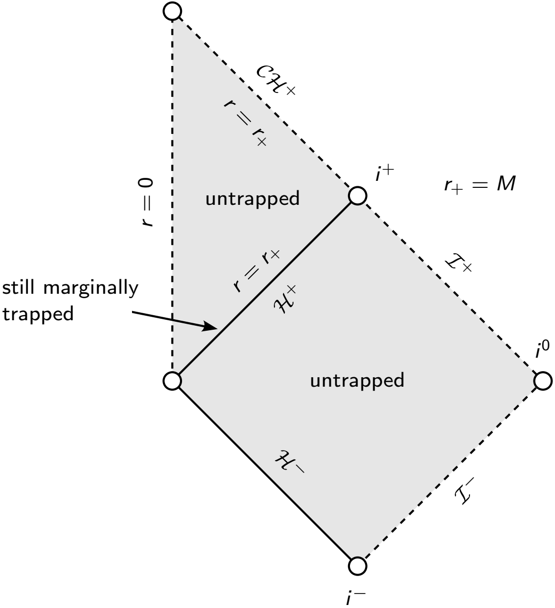
Refresher on extremal Reissner–Nordström: $0 < |e| = M$



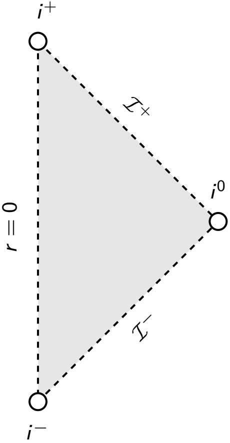
Refresher on extremal Reissner–Nordström: $0 < |e| = M$



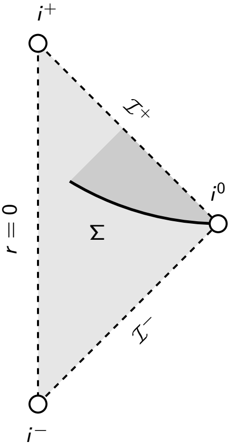
Refresher on extremal Reissner–Nordström: $0 < |e| = M$



Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



Refresher on superextremal Reissner–Nordström: $0 < M < |e|$



Surface gravity of Reissner–Nordström

- RN with mass M and charge e, $|e| \leq M$, has

$$\kappa = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Subextremal:** $\kappa > 0$
- **Extremal:** $\kappa = 0$

The third law

Original formulation of Bardeen–Carter–Hawking:

> Extending the analogy even further one would postulate:
>
> *The Third Law*
>
> It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Difficult to interpret κ dynamically!

The third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Difficult to interpret κ dynamically!

Statement revised by Israel '86:

VOLUME 57, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JULY 1986

Third Law of Black-Hole Dynamics: A Formulation

W. Israel[(a)]
Research Institute for Fundamental Physics, Yukawa Hall, Kyoto University, Kyoto 606, Japan
(Received 19 May 1986)

The third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Difficult to interpret κ dynamically!

Statement revised by Israel '86:

VOLUME 57, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JULY 1986

Third Law of Black-Hole Dynamics: A Formulation and Proof

W. Israel[a]
Research Institute for Fundamental Physics, Yukawa Hall, Kyoto University, Kyoto 606, Japan
(Received 19 May 1986)

Israel's formulation of the third law

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

Israel's formulation of the third law

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."

Israel's formulation of the third law

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.

Israel's formulation of the third law

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.

Israel's formulation of the third law

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.
   - If singularities allowed, counterexample using massive dust shell. (Farrugia–Hajicek '79)

Israel's formulation of the third law

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.
   - If singularities allowed, counterexample using massive dust shell. (Farrugia–Hajicek '79)
3. Weak energy condition must be enforced.

Israel's formulation of the third law

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.
   - If singularities allowed, counterexample using massive dust shell. (Farrugia–Hajicek '79)
3. Weak energy condition must be enforced.
   - If no energy condition, counterexample using charged null dust. (Sullivan–Israel '80)

Israel's formulation of the third law

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.
   - If singularities allowed, counterexample using massive dust shell. (Farrugia–Hajicek '79)
3. Weak energy condition must be enforced.
   - If no energy condition, counterexample using charged null dust. (Sullivan–Israel '80)
4. Extremizing associated with "losing trapped surfaces."

Israel's formulation of the third law

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

1. "Finite advanced time" replaces "finite sequence of operations."
   - Brings the third law onto a similar footing as the second law, no quasistationary assumption.
2. "Stress-energy tensor stays bounded" is a regularity condition.
   - If singularities allowed, counterexample using massive dust shell. (Farrugia–Hajicek '79)
3. Weak energy condition must be enforced.
   - If no energy condition, counterexample using charged null dust. (Sullivan–Israel '80)
4. Extremizing associated with "losing trapped surfaces."
   - This parenthetical remark is related to the "mistake" in Israel's paper and we will discuss it later.

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).
A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).
A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Theorem (Kehle–U. '22).
Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

Retiring the third law

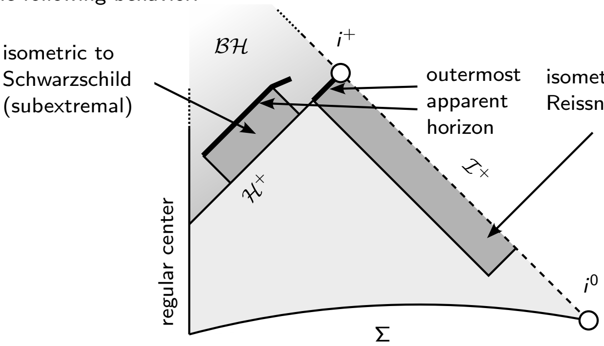
Conjecture (The third law, BCH '73, Israel '86).
A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

Theorem (Kehle–U. '22).
Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the "third law of black hole thermodynamics" is false.

More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field (self-gravitating charged massless scalar field) system with the following behavior:

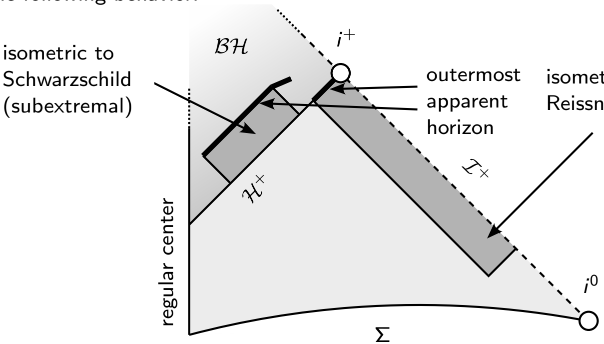
Counterexample to the third law: detailed statement of our theorem

There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



Counterexample to the third law: detailed statement of our theorem

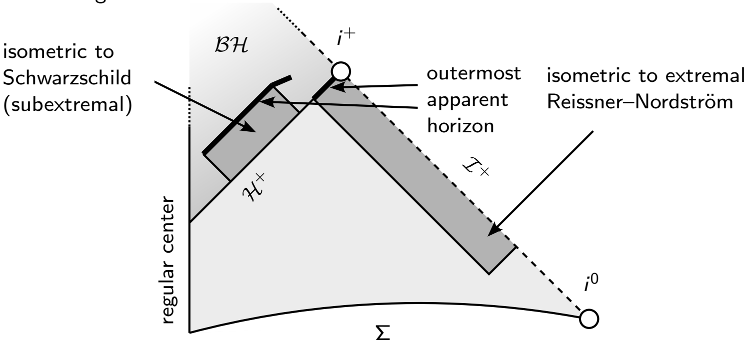
There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.

Counterexample to the third law: detailed statement of our theorem

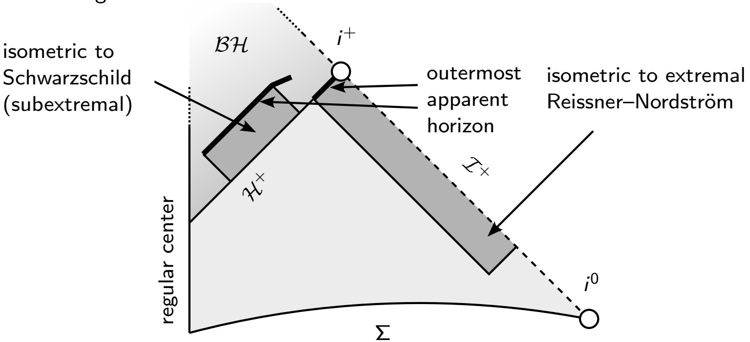
There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.
- Forms an exactly Schwarzschild "apparent horizon."

Counterexample to the third law: detailed statement of our theorem

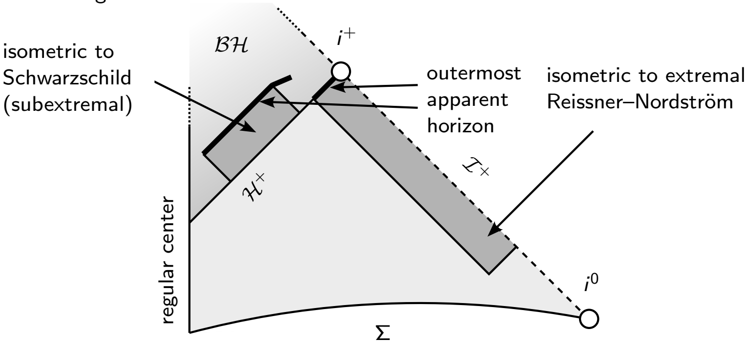
There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.
- Forms an exactly Schwarzschild "apparent horizon."
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.

Counterexample to the third law: detailed statement of our theorem

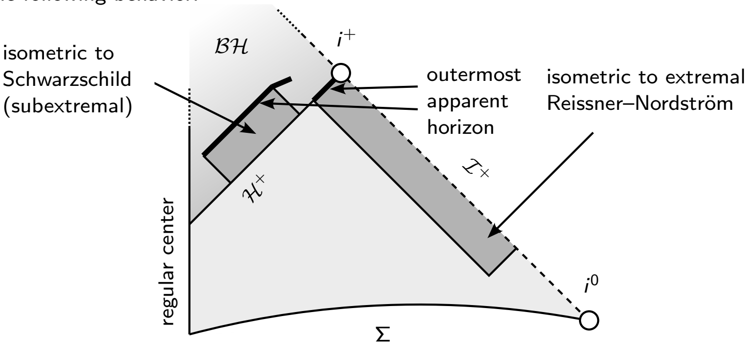
There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.
- Forms an exactly Schwarzschild "apparent horizon."
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.
- These solutions are arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example.

Counterexample to the third law: detailed statement of our theorem

There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.
- Forms an exactly Schwarzschild "apparent horizon."
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.
- These solutions are arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example.
- Model satisfies the dominant energy condition.

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) massless scalar field ϕ

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) massless scalar field ϕ
- Equations:

$$
\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) && (1)\\
\nabla^{\mu} F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu\phi}) && (2)\\
g^{\mu\nu} D_\mu D_\nu \phi &= 0 && (3)\\
T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} && (4)\\
T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} && (5)
\end{aligned}
$$

- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ).

Einstein–Maxwell-charged scalar field system

- Lorentzian manifold $(\mathcal{M}^{3+1}, g)$
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) massless scalar field ϕ
- Equations:

$$
\begin{aligned}
R_{\mu\nu}(g) - \tfrac{1}{2}R(g)g_{\mu\nu} &= 2\left(T^{\mathrm{EM}}_{\mu\nu} + T^{\mathrm{CSF}}_{\mu\nu}\right) && (1)\\
\nabla^{\mu} F_{\mu\nu} &= 2\mathfrak{e}\,\mathrm{Im}(\phi\overline{D_\nu\phi}) && (2)\\
g^{\mu\nu} D_\mu D_\nu \phi &= 0 && (3)\\
T^{\mathrm{EM}}_{\mu\nu} &= g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} && (4)\\
T^{\mathrm{CSF}}_{\mu\nu} &= \mathrm{Re}(D_\mu\phi\overline{D_\nu\phi}) - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} && (5)
\end{aligned}
$$

- In an appropriate gauge, this is a nonlinear hyperbolic system for (g, F, ϕ).
- Spherical symmetry, ϕ degree of freedom breaks rigidity from Birkhoff.

Losing trapped surfaces and the implicit assumption of connectivity

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu}k^\mu k^\nu.$$

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu}k^\mu k^\nu .$$

- Extremal Reissner–Nordström (and Kerr) have no trapped surfaces, but their subextremal versions do.

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu} k^\mu k^\nu .$$

- Extremal Reissner–Nordström (and Kerr) have no trapped surfaces, but their subextremal versions do.
- In his paper, Israel seems to have implicitly assumed that a regular solution will have a connected outermost apparent horizon.

> Infractions can result from the absorption of infinitesimally thin, massive shells,[5] which force the apparent horizon to jump outward discontinuously

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu} k^\mu k^\nu .$$

- Extremal Reissner–Nordström (and Kerr) have no trapped surfaces, but their subextremal versions do.
- In his paper, Israel seems to have implicitly assumed that a regular solution will have a connected outermost apparent horizon.

Infractions can result from the absorption of infinitesimally thin, massive shells,[5] which force the apparent horizon to jump outward discontinuously

But this is not true. The outermost apparent horizon can jump in smooth spacetimes.

Losing trapped surfaces and the implicit assumption of connectivity

- Weak energy condition $\implies$ trapped surfaces persist in evolution once formed.
- Follows immediately from Raychaudhuri's equation

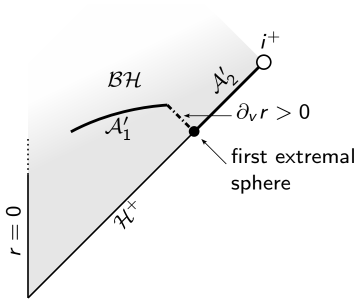
$$\frac{d\theta}{d\lambda} = -\tfrac{1}{2}\theta^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu}k^\mu k^\nu .$$

- Extremal Reissner–Nordström (and Kerr) have no trapped surfaces, but their subextremal versions do.
- In his paper, Israel seems to have implicitly assumed that a regular solution will have a connected outermost apparent horizon.

  > Infractions can result from the absorption of infinitesimally thin, massive shells,[5] which force the apparent horizon to jump outward discontinuously

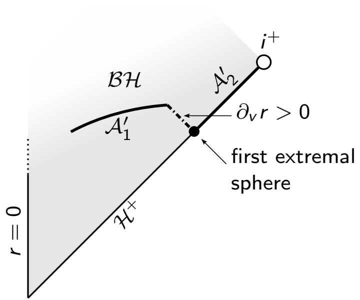
  But this is not true. The outermost apparent horizon can jump in smooth spacetimes.
- Under this implicit assumption, Israel concluded the third law to be true on the basis of Raychaudhuri's equation.

Israel's paper reinterpreted



Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

Israel's paper reinterpreted



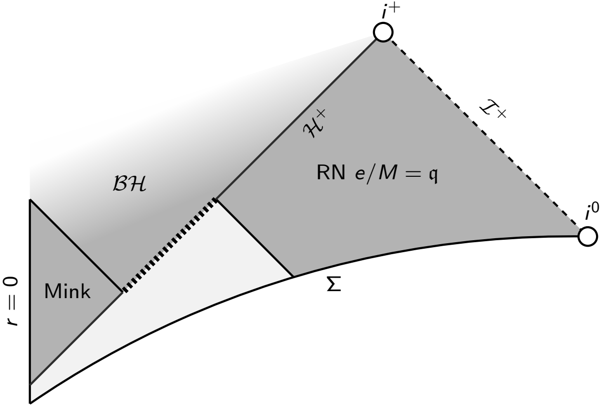
Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

However, this is a feature, not a glitch!

Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

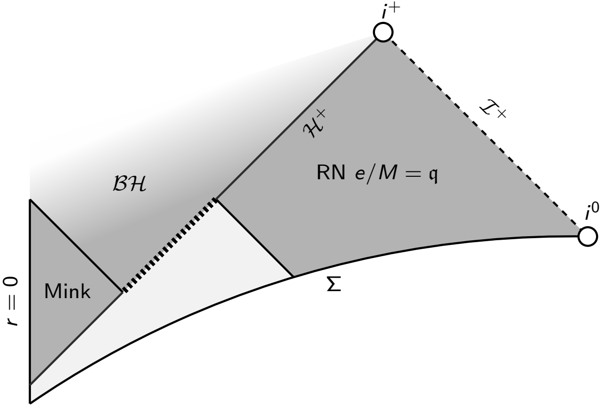
For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:



Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:

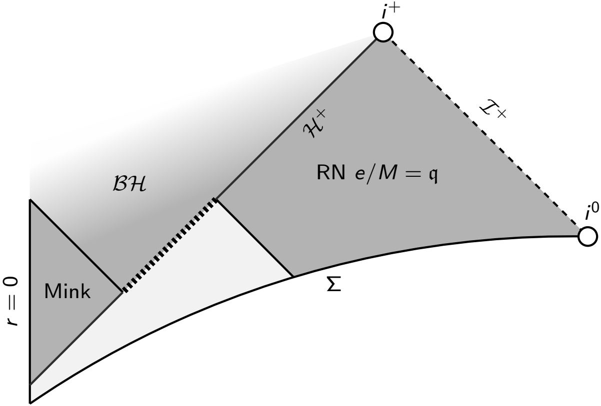


- From the point of view of our construction, the extremal case is exactly the same as the subextremal case!

Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:

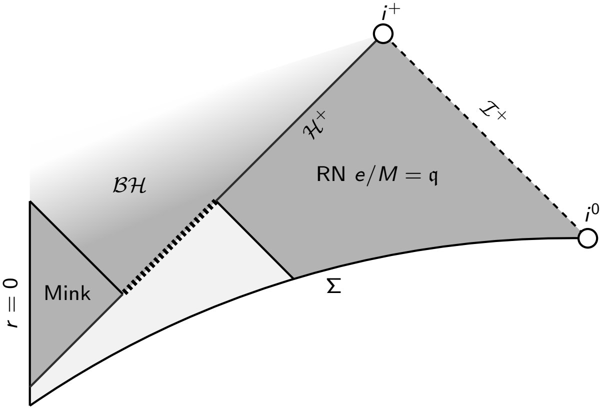


- From the point of view of our construction, the extremal case is exactly the same as the subextremal case!
- $|\mathfrak{q}| \to 1$ represents a regular limit.

Gravitational collapse for any charge to mass ratio

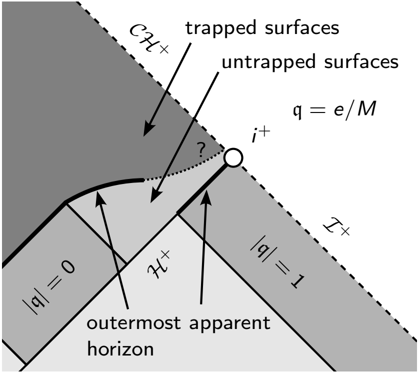
Theorem (Kehle–U.).

For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:

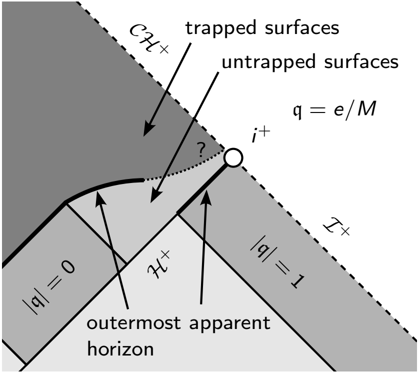


- From the point of view of our construction, the extremal case is exactly the same as the subextremal case!
- $|\mathfrak{q}| \to 1$ represents a regular limit.
- $T = 0$ is not fundamentally different than $T > 0$ in these examples.

Interior structure of third law violating solutions

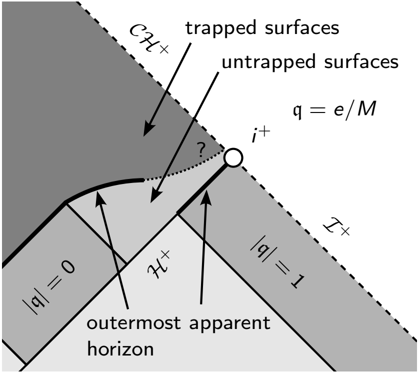


Interior structure of third law violating solutions



- Consistent with Israel's observation: the outermost apparent horizon becomes disconnected.

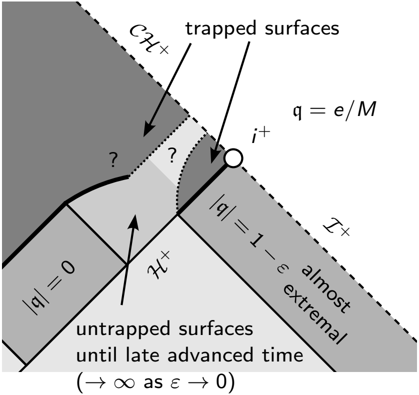
Interior structure of third law violating solutions



- Consistent with Israel's observation: the outermost apparent horizon becomes disconnected.
- Trapped surfaces persist for all time and are found at any retarded time in the black hole interior.

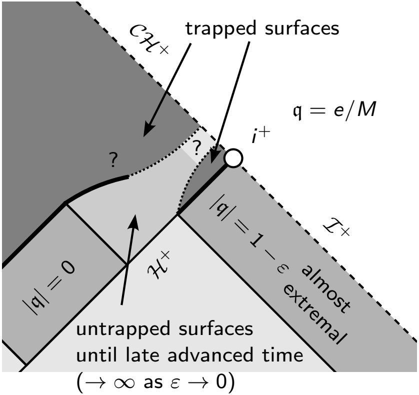
So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



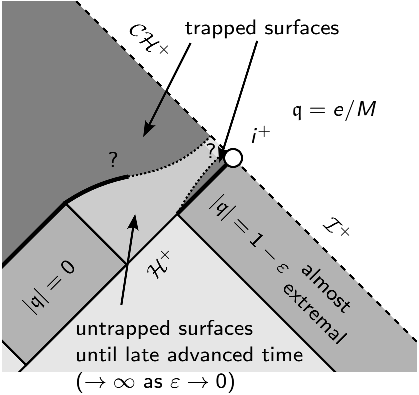
So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



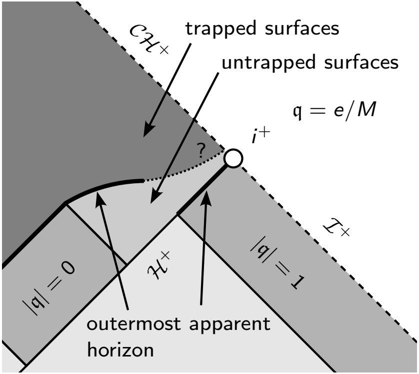
So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

- Keeping a horizon at exactly constant temperature, any temperature, is exceptional.

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

- Keeping a horizon at exactly constant temperature, any temperature, is exceptional.
- For example, the set of solutions which become exactly Schwarzschild after finite time is infinite codimension in the moduli space of solutions.

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

- Keeping a horizon at exactly constant temperature, any temperature, is exceptional.
- For example, the set of solutions which become exactly Schwarzschild after finite time is infinite codimension in the moduli space of solutions.
- In view of our results, forming $T = 0$ is not different or more difficult than forming fixed $T > 0$.

On the (non)genericity of our examples

> *The Third Law*
>
> It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

> A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

- Keeping a horizon at exactly constant temperature, any temperature, is exceptional.
- For example, the set of solutions which become exactly Schwarzschild after finite time is infinite codimension in the moduli space of solutions.
- In view of our results, forming $T = 0$ is not different or more difficult than forming fixed $T > 0$.
- One might try to save the third law by demanding a process that is stable under perturbations.

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

- Keeping a horizon at exactly constant temperature, any temperature, is exceptional.
- For example, the set of solutions which become exactly Schwarzschild after finite time is infinite codimension in the moduli space of solutions.
- In view of our results, forming $T = 0$ is not different or more difficult than forming fixed $T > 0$.
- One might try to save the third law by demanding a process that is stable under perturbations.
- However, such a reformulation would be completely trivial (and has nothing to do with $T = 0$).

The third law and supercharging

Bardeen–Carter–Hawking:

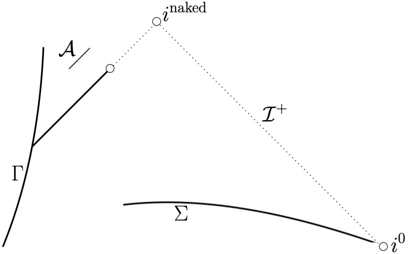
Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

The third law and supercharging

Bardeen–Carter–Hawking:

> Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

- Charging (resp., spinning) up a black hole to create a naked singularity as in superextremal RN (resp., Kerr) is known as **supercharging** (resp., **overspinning**).

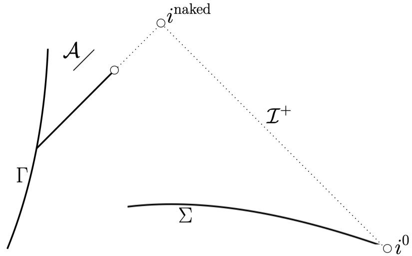


The third law and supercharging

Bardeen–Carter–Hawking:

> Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

- Charging (resp., spinning) up a black hole to create a naked singularity as in superextremal RN (resp., Kerr) is known as **supercharging** (resp., **overspinning**).



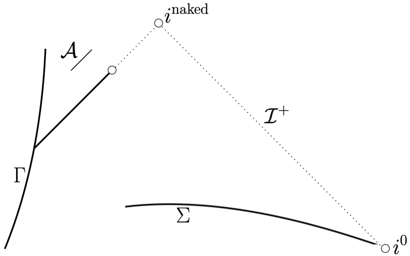
- This is a doomed endeavor because the naked singularity in extremal RN/Kerr is dynamically inaccessible.

The third law and supercharging

Bardeen–Carter–Hawking:

Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

- Charging (resp., spinning) up a black hole to create a naked singularity as in superextremal RN (resp., Kerr) is known as **supercharging** (resp., **overspinning**).



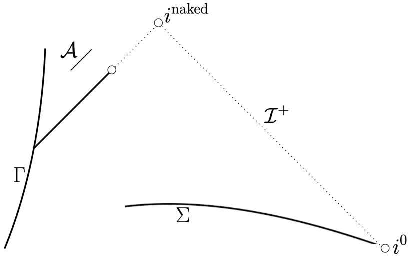
- This is a doomed endeavor because the naked singularity in extremal RN/Kerr is dynamically inaccessible.
- Supercharging was definitively disproved in the EMCSF model in spherical symmetry by Kommemi '13.

The third law and supercharging

Bardeen–Carter–Hawking:

> Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

- Charging (resp., spinning) up a black hole to create a naked singularity as in superextremal RN (resp., Kerr) is known as **supercharging** (resp., **overspinning**).



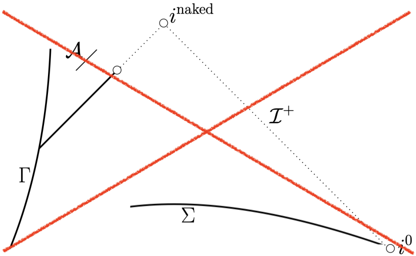
- This is a doomed endeavor because the naked singularity in extremal RN/Kerr is dynamically inaccessible.
- Supercharging was definitively disproved in the EMCSF model in spherical symmetry by Kommemi '13.
- Supercharging is impossible, but not because of a third law obstruction!

The third law and supercharging

Bardeen–Carter–Hawking:

Another reason for believing the third law is that if one could reduce κ to zero by a finite sequence of operations, then presumably one could carry the process further, thereby creating a naked singularity.

- Charging (resp., spinning) up a black hole to create a naked singularity as in superextremal RN (resp., Kerr) is known as **supercharging** (resp., **overspinning**)



- This is a doomed endeavor because the naked singularity in extremal RN/Kerr is dynamically inaccessible
- Supercharging was definitively disproved in the EMCSF model in spherical symmetry by Kommemi '13
- Supercharging is impossible, but not because of a third law obstruction!

Strategy of the construction

Steps in the proof:

Strategy of the construction

Steps in the proof:

1. Construct spacetime teleologically.

Strategy of the construction

Steps in the proof:

1. Construct spacetime teleologically.
   - Teleological construction allows us to locate the event horizon.

Strategy of the construction

Steps in the proof:

1. Construct spacetime teleologically.
   - Teleological construction allows us to locate the event horizon.
2. Extract a Cauchy hypersurface Σ and deduce that the teleologically constructed spacetime arises *dynamically* from initial data.

Strategy of the construction

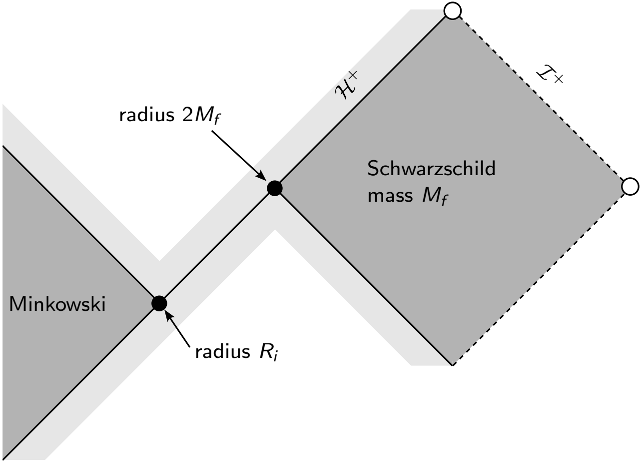
Steps in the proof:

1. Construct spacetime teleologically.
   - Teleological construction allows us to locate the event horizon.
2. Extract a Cauchy hypersurface Σ and deduce that the teleologically constructed spacetime arises *dynamically* from initial data.

The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be located *without knowing the entire future of the spacetime*.

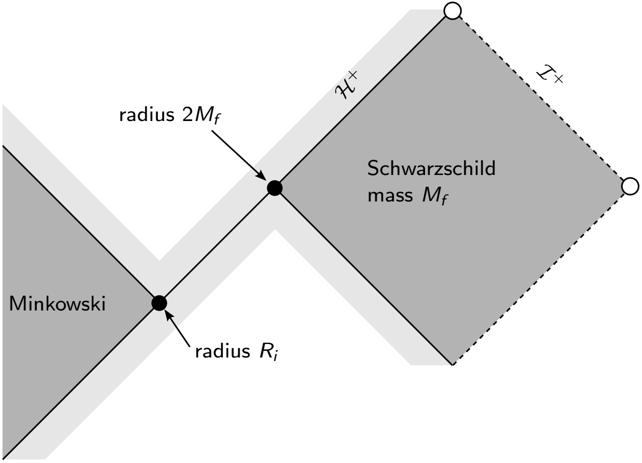
Prototype: Minkowski to Schwarzschild gluing

The type of Penrose diagram we want:



Prototype: Minkowski to Schwarzschild gluing

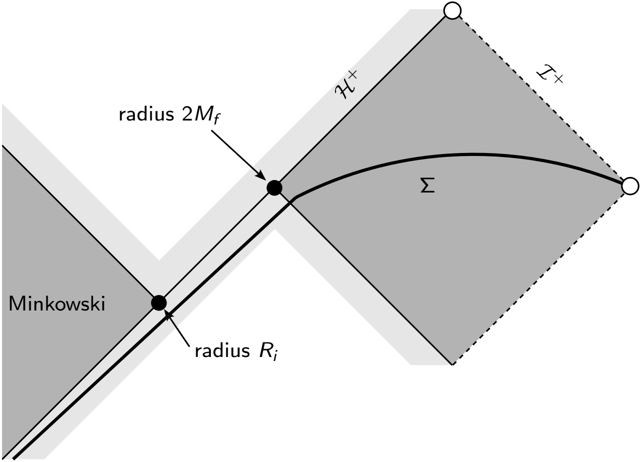
The type of Penrose diagram we want:



We want to **glue** solutions of the Einstein-scalar field system in spherical symmetry.

Prototype: Minkowski to Schwarzschild gluing

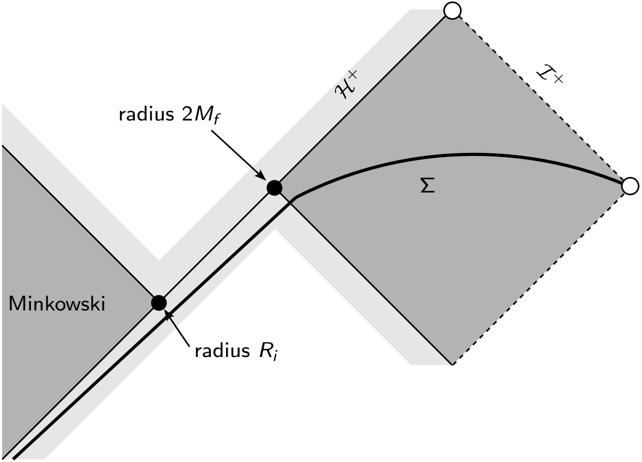
The type of Penrose diagram we want:



We want to **glue** solutions of the Einstein-scalar field system in spherical symmetry.

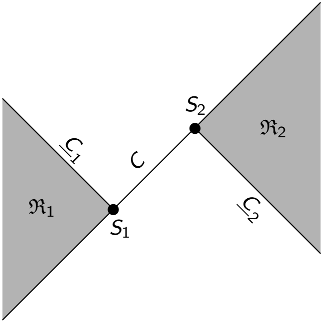
Prototype: Minkowski to Schwarzschild gluing

The type of Penrose diagram we want:

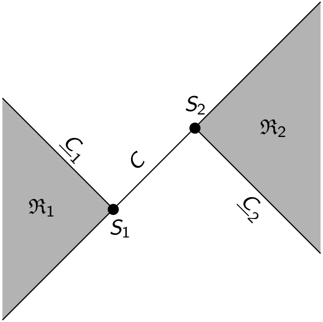


We want to **glue** solutions of the Einstein-scalar field system in spherical symmetry.
Characteristic gluing for the vacuum Einstein equations recently initiated by Aretakis–Czimek–Rodnianski, but in a different regime than required for this problem.

Characteristic/null gluing for the linear wave equation



Characteristic/null gluing for the linear wave equation

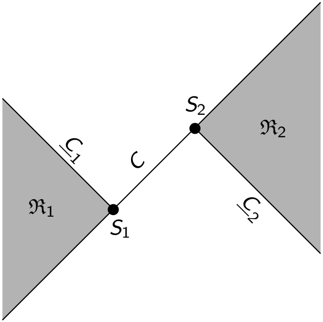


Question: Given functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}, \quad \phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $\Box_g \phi_i = 0$,

Characteristic/null gluing for the linear wave equation



Question: Given functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}, \quad \phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

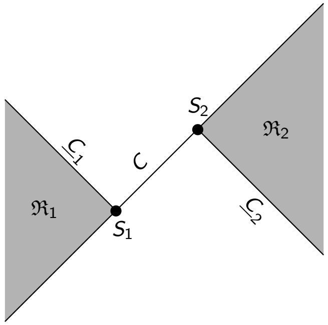
solving $\Box_g \phi_i = 0$, when does there exist a function

$$\phi : \mathcal{M} \to \mathbb{R}$$

solving $\Box_g \phi = 0$, such that

$$\phi|_{\mathfrak{R}_i} = \phi_i \,?$$

Characteristic/null gluing for the linear wave equation



Question: Given functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}, \quad \phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $\Box_g \phi_i = 0$, when does there exist a function

$$\phi : \mathcal{M} \to \mathbb{R}$$

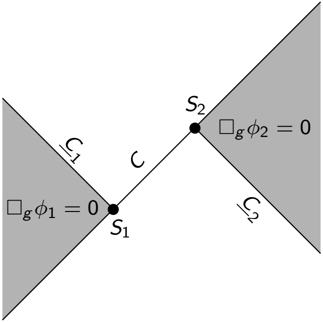
solving $\Box_g \phi = 0$, such that

$$\phi|_{\mathfrak{R}_i} = \phi_i \,?$$

Analogy:

- $\mathfrak{R}_1$ is Minkowski
- $\mathfrak{R}_2$ is Schwarzschild
- C will be the event horizon $\mathcal{H}^+$

Characteristic/null gluing for the linear wave equation



Question: Given functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}, \quad \phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $\Box_g \phi_i = 0$, when does there exist a function

$$\phi : \mathcal{M} \to \mathbb{R}$$

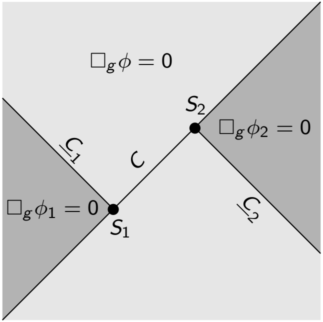
solving $\Box_g \phi = 0$, such that

$$\phi|_{\mathfrak{R}_i} = \phi_i \,?$$

Analogy:

- $\mathfrak{R}_1$ is Minkowski
- $\mathfrak{R}_2$ is Schwarzschild
- C will be the event horizon $\mathcal{H}^+$

Characteristic/null gluing for the linear wave equation



Question: Given functions

$$\phi_1 : \mathfrak{R}_1 \to \mathbb{R}, \quad \phi_2 : \mathfrak{R}_2 \to \mathbb{R}$$

solving $\Box_g \phi_i = 0$, when does there exist a function

$$\phi : \mathcal{M} \to \mathbb{R}$$

solving $\Box_g \phi = 0$, such that

$$\phi|_{\mathfrak{R}_i} = \phi_i\,?$$

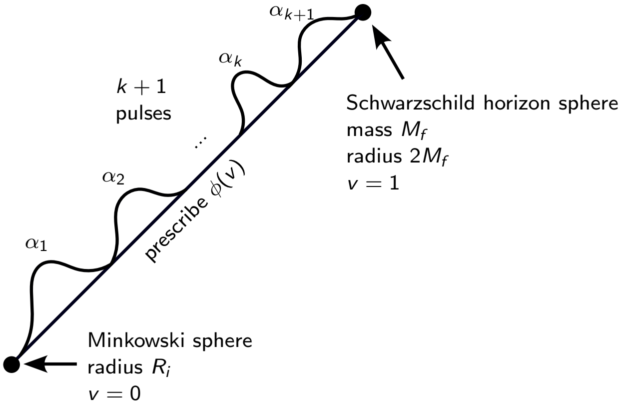
Analogy:

- $\mathfrak{R}_1$ is Minkowski
- $\mathfrak{R}_2$ is Schwarzschild
- C will be the event horizon $\mathcal{H}^+$

Minkowski to Schwarzschild gluing

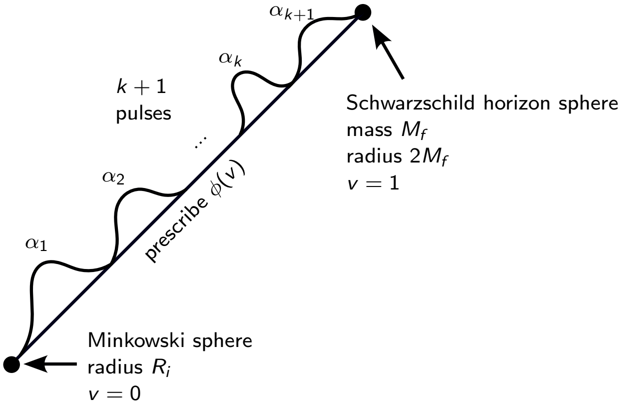
Theorem (Kehle–U. '22).

For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



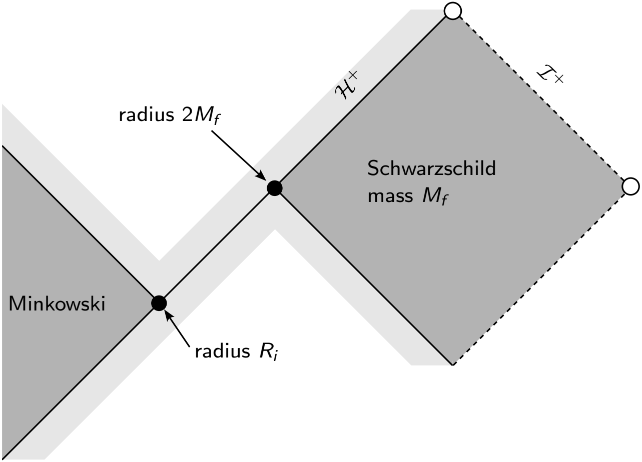
Theorem (Kehle–U. '22).

For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.

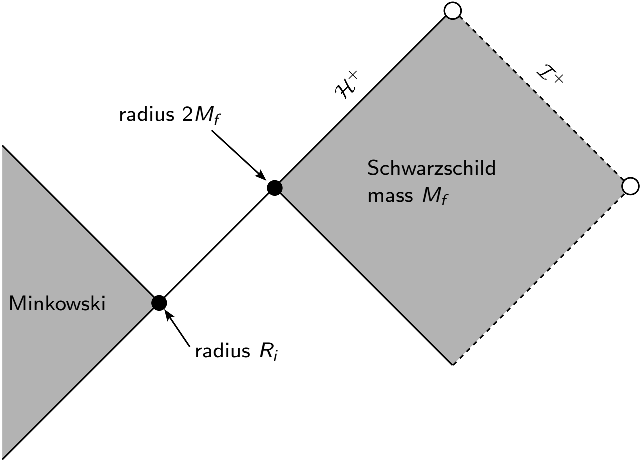


Pulse amplitudes α_j are chosen using the Borsuk–Ulam theorem, making use of the symmetry $\phi \mapsto -\phi$ of the equations.

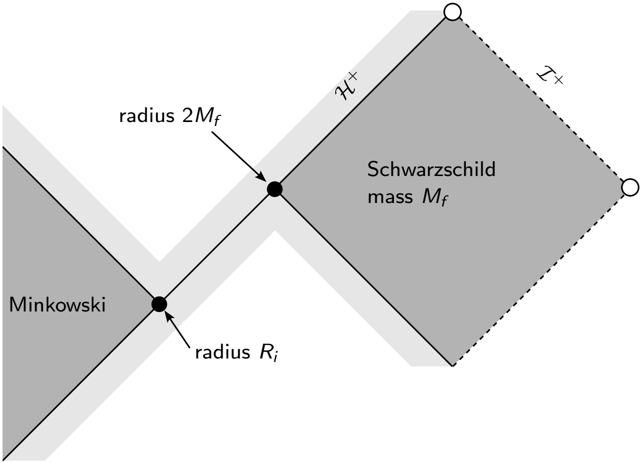
Minkowski to Schwarzschild gluing



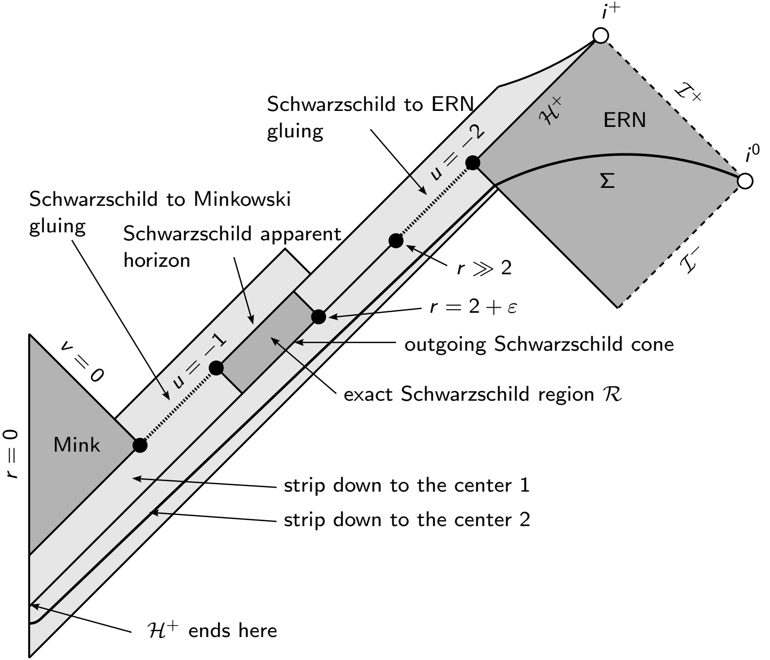
Minkowski to Schwarzschild gluing



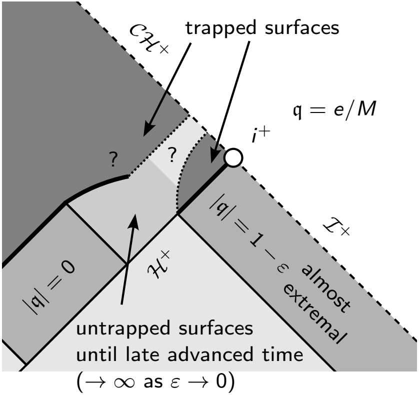
Minkowski to Schwarzschild gluing



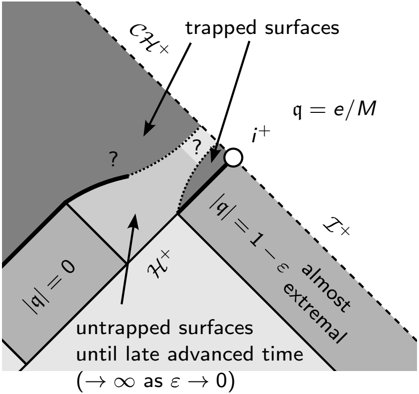
Disproof of the third law



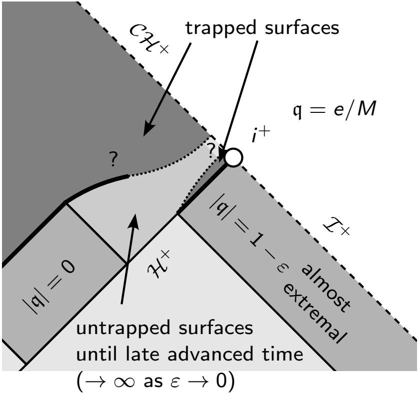
So what does happen if you “go past extremality”?



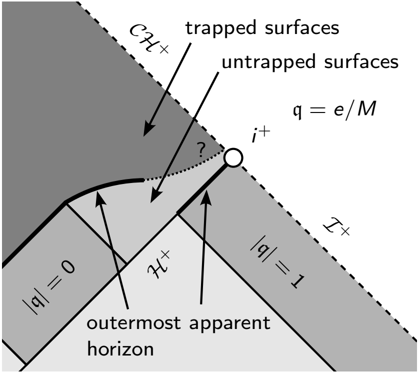
So what does happen if you "go past extremality"?



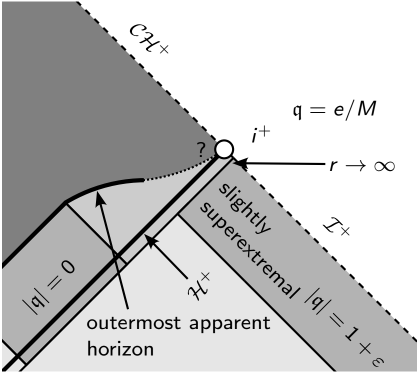
So what does happen if you "go past extremality"?



So what does happen if you "go past extremality"?



So what does happen if you "go past extremality"?



Critical behavior: We expect that there are smooth 1-parameter families of solutions passing through extremality where the event horizon "jumps" as soon as the exterior becomes superextremal. There is **no naked singularity.**

The vacuum case

We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

The vacuum case

We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, already in vacuum, the "third law of black hole thermodynamics" is false.

The vacuum case

We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, already in vacuum, the "third law of black hole thermodynamics" is false.

This would follow from characteristic gluing, in vacuum, of a Minkowsi sphere to a Kerr exterior sphere.

The vacuum case

We believe that the charged scalar field is not necessary to provide a counterexample to the third law:

Conjecture.
There exist Cauchy data for the Einstein vacuum equations

$$R_{\mu\nu} = 0$$

which undergo gravitational collapse and form an exactly Schwarzschild apparent horizon, only for the spacetime to form an exactly extremal Kerr event horizon at a later advanced time. In particular, already in vacuum, the "third law of black hole thermodynamics" is false.

This would follow from characteristic gluing, in vacuum, of a Minkowsi sphere to a Kerr exterior sphere.

Thank you!