

Retiring the third law of black hole thermodynamics

Ryan Unger

Department of Mathematics, Princeton University

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joint work with Christoph Kehle (ETH Zürich)

Classical thermodynamics

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Zeroth	T constant throughout body in thermal equilibrium

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Zeroth	T constant throughout body in thermal equilibrium
First	$dE = TdS + \text{work terms } (PdV)$
Second	$dS \geq 0$ in any process
Third	Impossible to achieve $T = 0$ in finite time

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)
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The Four Laws of Black Hole Mechanics

J. M. Bardeen*

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England

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- What is a black hole?
- What is the analogue of temperature T ?
- What is the analogue of entropy S ?

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold \mathcal{M}^{3+1} and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \quad (\text{EFE})$$

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Example: Minkowski space. $\mathcal{M}^{3+1} = \mathbb{R}_{t,x,y,z}^{3+1}$ and

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

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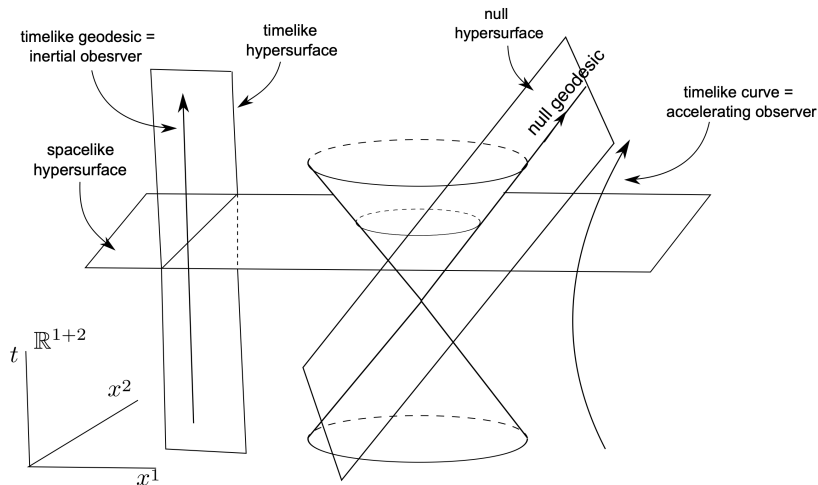
This metric describes the geometry of special relativity with speed of light $c = 1$.

Given a vector $v \in T_p\mathcal{M}$:

- ▶ $g(v, v) < 0$, v is *timelike*
- ▶ $g(v, v) = 0$, v is *null*
- ▶ $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

Crash course on black holes: Geometry of Minkowski space



Crash course on black holes: What is a black hole?

Asymptotic flatness: g approaches the Minkowski metric “at infinity.”

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$$\mathcal{BH} \subset \mathcal{M}$$

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Examples to come shortly!

Crash course on black holes: Teleology



The black hole region \mathcal{BH} and event horizon \mathcal{H}^+ of a spacetime are **teleological** notions—they can't be known to exist, much less be located, *without knowing the entire future of the spacetime*.

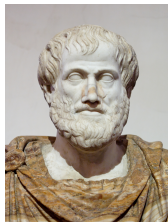
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The formation, dynamics, and stability of black holes can be rigorously studied using the tools of Lorentzian geometry and hyperbolic PDE.

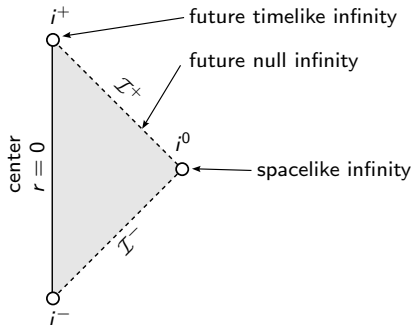
Crash course on black holes: Penrose diagrams

Penrose diagrams: A $(1 + 1)$ -dimensional diagrammatic representation of spacetimes. This is particularly useful for visualizing causality.

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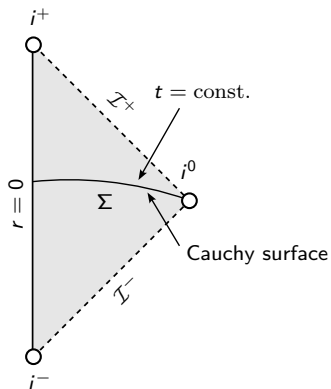
Penrose diagrams: A $(1 + 1)$ -dimensional diagrammatic representation of spacetimes. This is particularly useful for visualizing causality.

- ▶ 45 degree lines correspond to null hypersurfaces in $(3 + 1)$ dimensions
- ▶ steeper than 45: timelike
- ▶ shallower than 45: spacelike

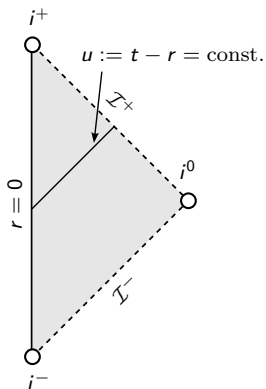


Penrose diagram of Minkowski space

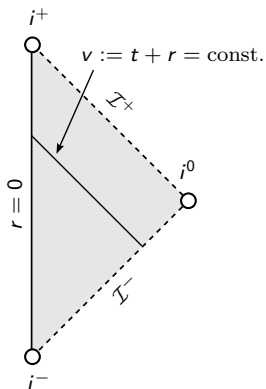
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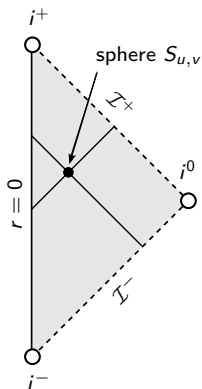
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Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

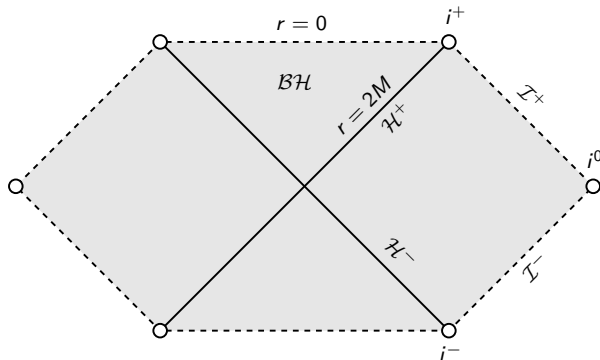
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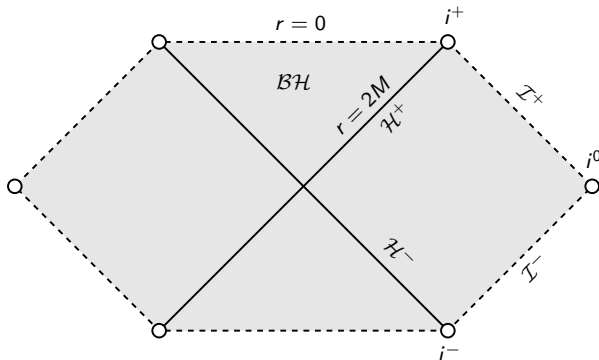


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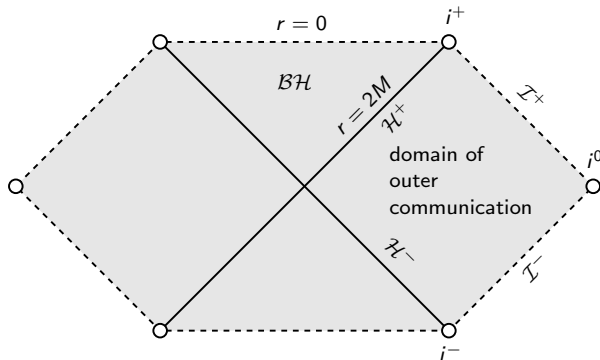
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- Mass $M > 0$ and charge $0 \leq |e| \leq M$

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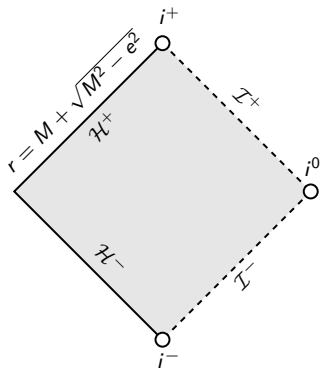
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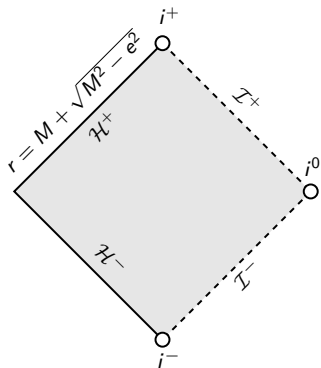
Subextremal $|e| < M$

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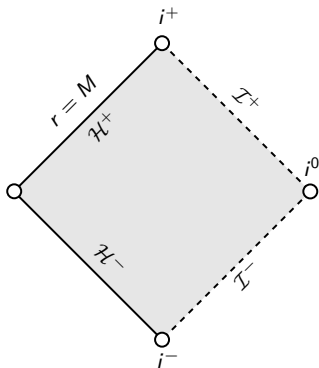
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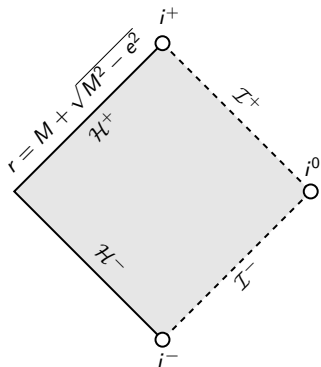
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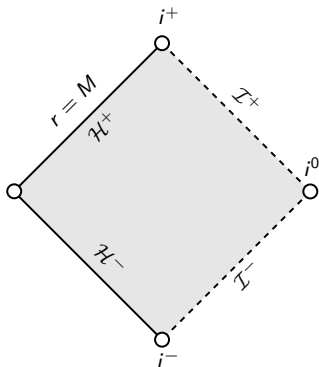
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Extremal $|e| = M$

Superextremal $|e| > M$ does not contain a black hole!

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- ▶ **Subextremal horizon:** $\kappa > 0$
- ▶ **Extremal horizon:** $\kappa \equiv 0$

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- ▶ Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Retiring the third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Retiring the third law

Conjecture (The third law, BCH '73, Israel '86).

*A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain *regular* and obey the *weak energy condition*.*

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Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the “third law of black hole thermodynamics” is false.

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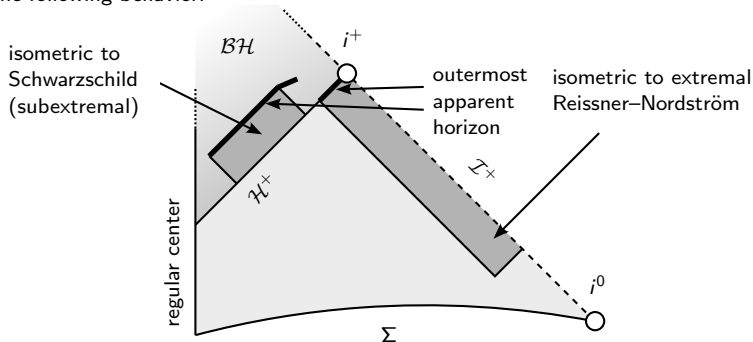
Theorem (Kehle–U. '22).

Subextremal black holes can become extremal in finite time, evolving from regular initial data. In particular, the “third law of black hole thermodynamics” is false.

More precisely, there exist **regular** solutions of the Einstein–Maxwell–charged scalar field system (which satisfies the **weak energy condition**) with the following behavior:

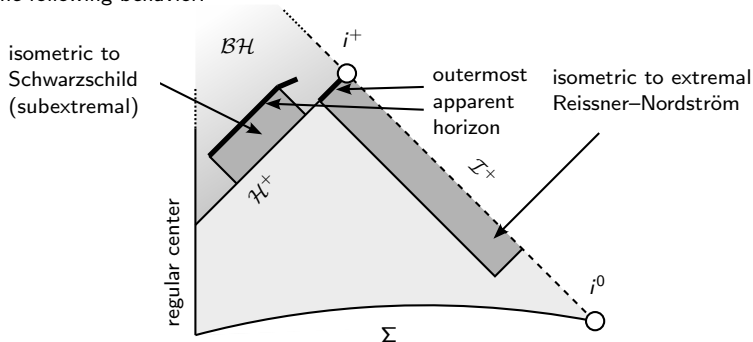
Counterexample to the third law: detailed statement of our theorem

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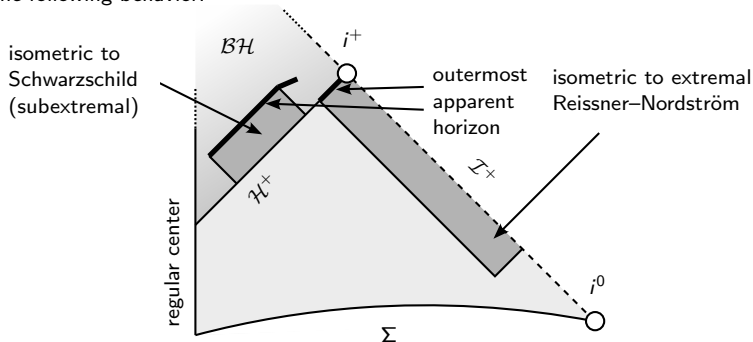
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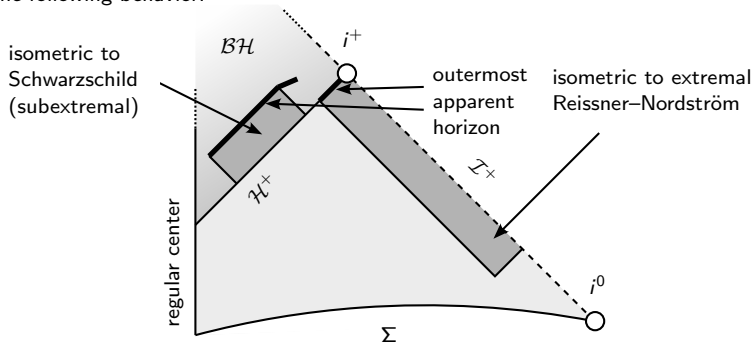
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- ▶ Forms an exactly Schwarzschild “apparent horizon.”

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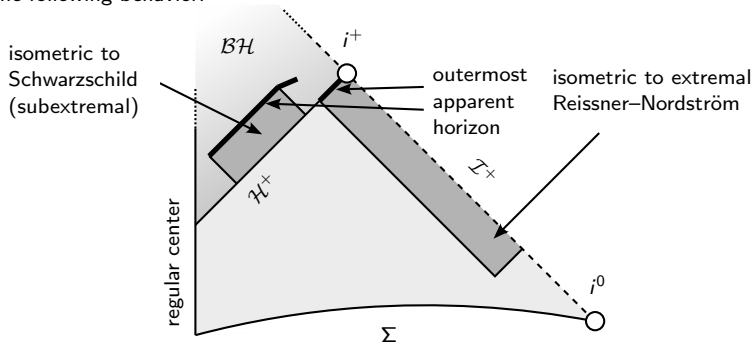
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- ▶ Forms an exactly Schwarzschild “apparent horizon.”
- ▶ Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.
- ▶ These solutions are arbitrarily **regular**: for any $k \in \mathbb{N}$, there exists a C^k example.

Einstein–Maxwell-charged scalar field system

- ▶ Lorentzian manifold (\mathcal{M}^{3+1}, g)
- ▶ 2-form $F = dA$ (electromagnetism)
- ▶ Charged (complex) scalar field ϕ

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- Lorentzian manifold (\mathcal{M}^{3+1}, g)
- 2-form $F = dA$ (electromagnetism)
- Charged (complex) scalar field ϕ
- Equations:

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = 2 \left(T_{\mu\nu}^{\text{EM}} + T_{\mu\nu}^{\text{CSF}} \right) \quad (1)$$

$$\nabla^\mu F_{\mu\nu} = 2\epsilon \operatorname{Im}(\phi \overline{D_\nu \phi}) \quad (2)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0 \quad (3)$$

$$T_{\mu\nu}^{\text{EM}} = g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \quad (4)$$

$$T_{\mu\nu}^{\text{CSF}} = \operatorname{Re}(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi} \quad (5)$$

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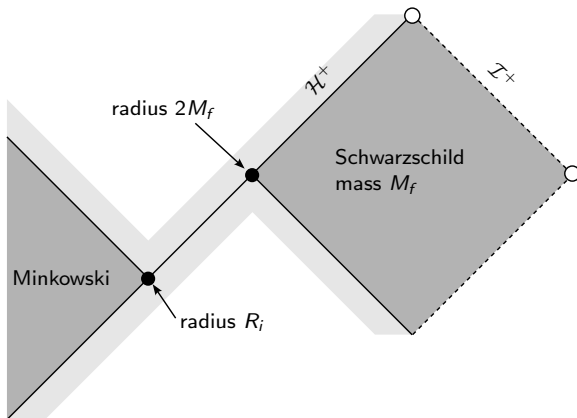
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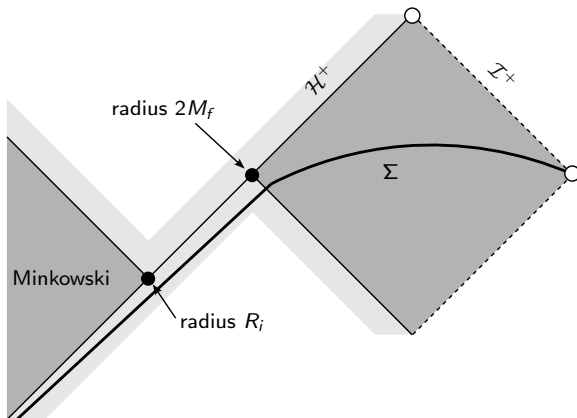
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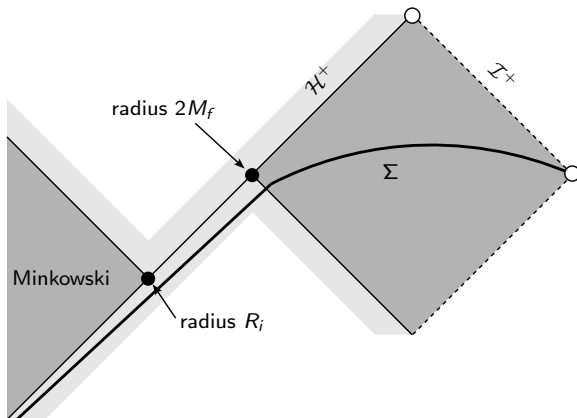
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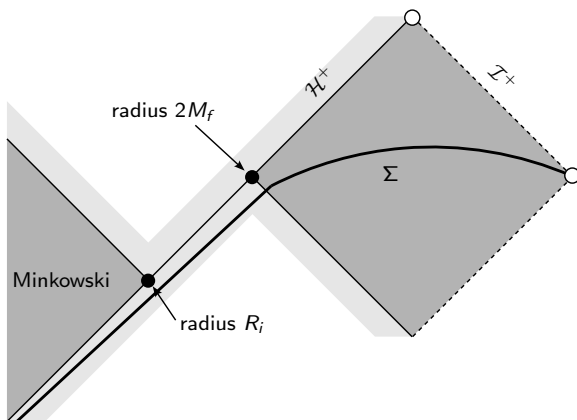


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The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Toy model: gluing for the linear wave equation

We now fix a spacetime (\mathcal{M}^{3+1}, g) and consider the **linear wave equation**

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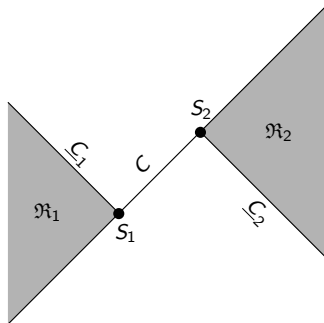
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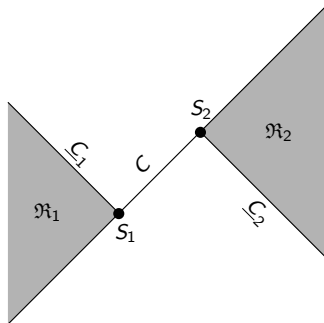
$$\phi|_{\mathfrak{R}_1} = \phi_1$$

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Characteristic/null gluing for the linear wave equation



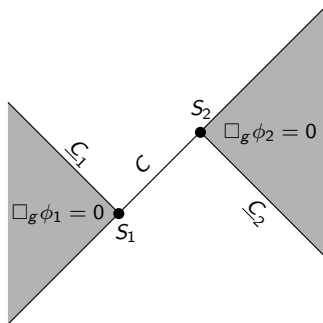
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Analogy:

- ▶ \mathfrak{R}_1 is Minkowski
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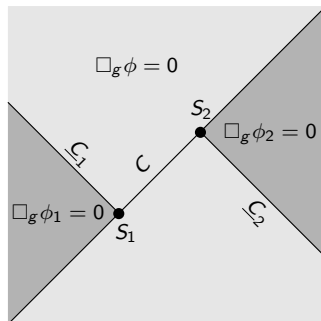
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The characteristic initial value problem

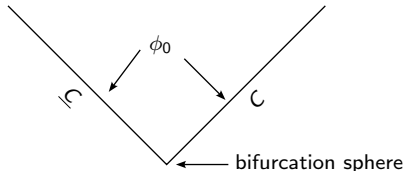
Bifurcate null hypersurface: $C \cup \underline{C}$ where C and \underline{C} meet transversally, $C \cap \underline{C} \approx S^2$

Proposition (Characteristic IVP for the wave equation).

Given a bifurcate null hypersurface $C \cup \underline{C}$ and $\phi_0 : C \cup \underline{C} \rightarrow \mathbb{R}$, there exists a unique function ϕ defined to the future of $C \cup \underline{C}$ such that

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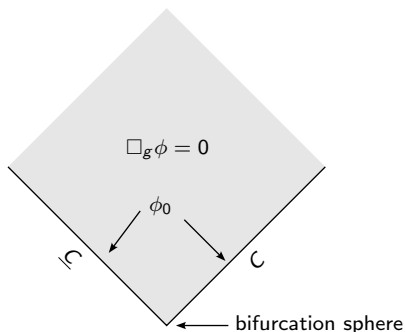
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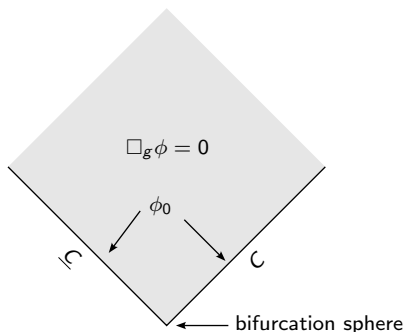
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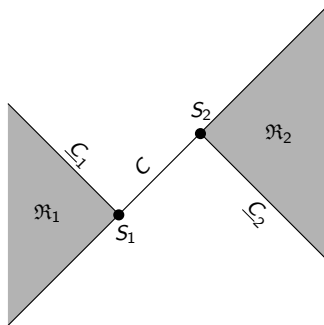
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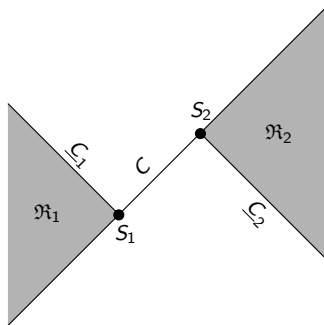
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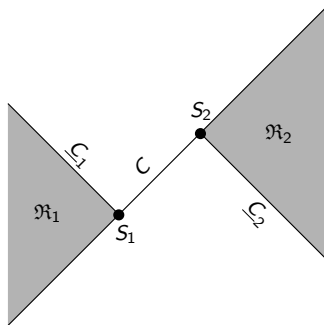
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- ▶ Remarkably, the presence of such conservation laws at the linear level is completely circumvented by our fully nonperturbative mechanism.

Einstein-scalar field in spherical symmetry

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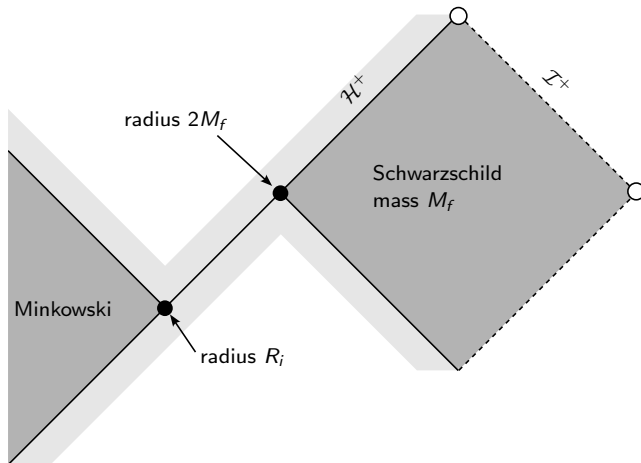
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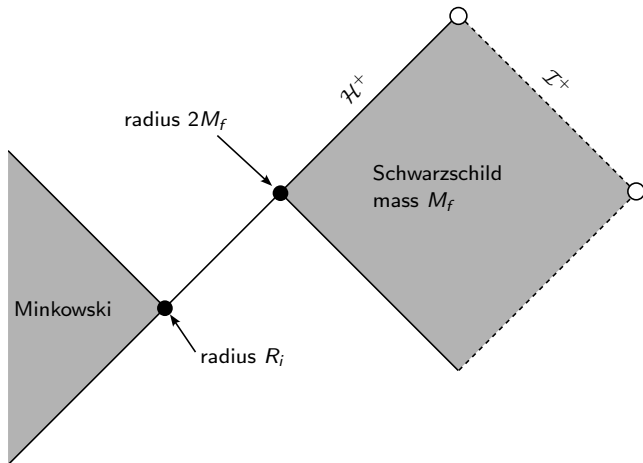
$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2 \quad (9)$$

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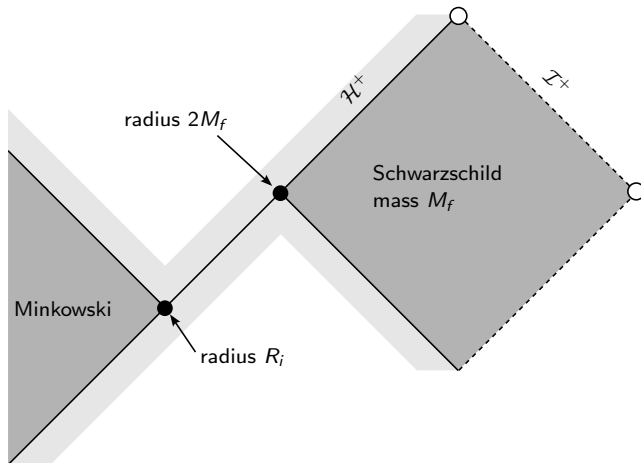
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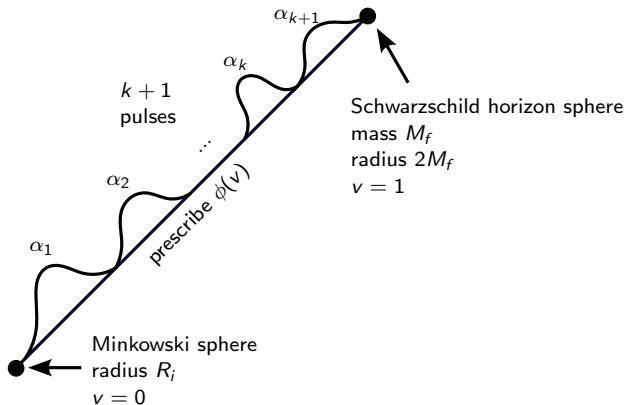
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Theorem (Kehle–U. '22).

For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



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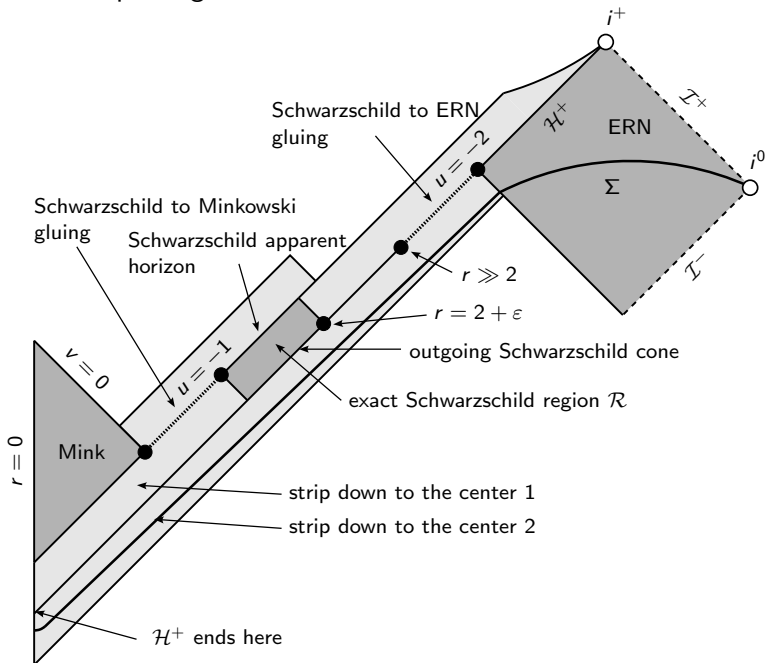
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is odd

- ▶ **Borsuk–Ulam theorem:** there exists α_* such that

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Blueprint to disproving the third law



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Theorem (Kehle–U. forthcoming).

For any $a/M \ll 1$, Minkowski space can be characteristically glued to the event horizon of a Kerr black hole with parameters M and a .

Future work: critical behavior near extremality

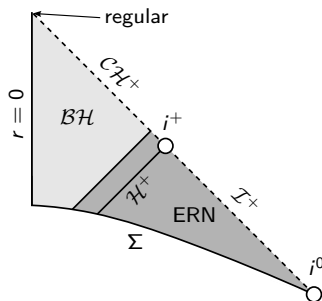
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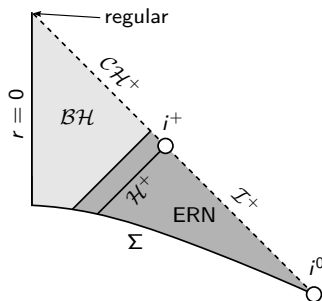
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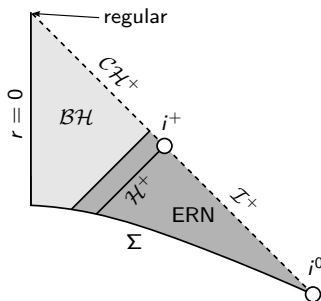


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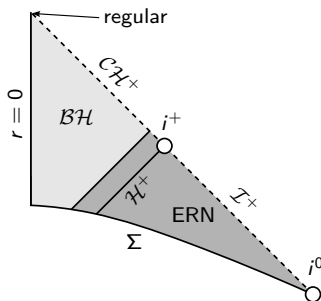
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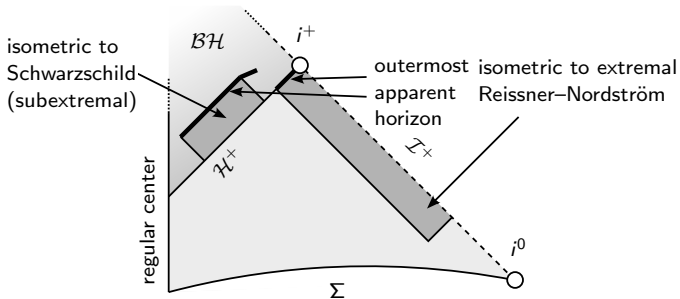
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