Retiring the third law of black hole thermodynamics

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joint work with Christoph Kehle (ETH Zürich)
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Classical thermodynamics

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Third	Impossible to achieve $T = 0$ in finite time

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

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- What is a black hole?
- What is the analogue of temperature T?
- What is the analogue of entropy S?

Crash course on black holes: Spacetimes and Einstein's equations

The setting is Einstein's theory of general relativity. A **spacetime** consists of a 4-manifold $\mathcal{M}^{3+1}$ and a Lorentzian metric g satisfying the **Einstein field equations**

$$Ric(g) - \frac{1}{2}R(g)g = 2T. \qquad \text{(EFE)}$$

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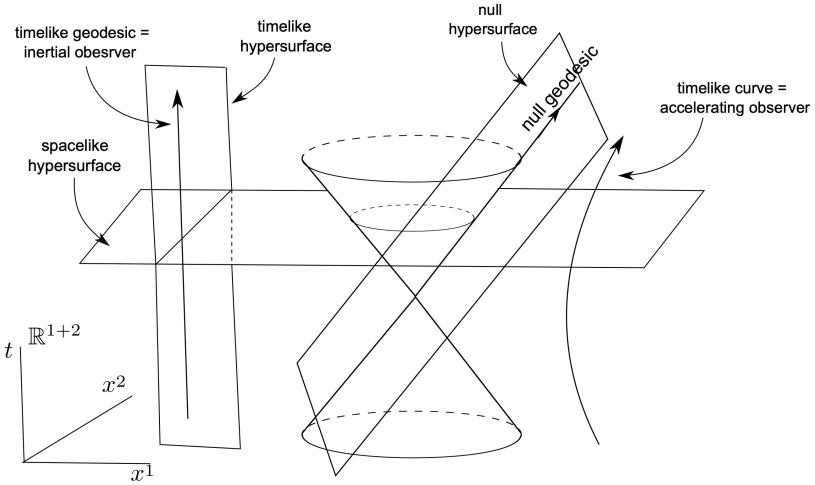
Given a vector $v \in T_p\mathcal{M}$:

- $g(v, v) < 0$, v is *timelike*
- $g(v, v) = 0$, v is *null*
- $g(v, v) > 0$, v is *spacelike*

Curves with null or timelike tangent vector let us define **causality** on a spacetime.

Crash course on black holes: Geometry of Minkowski space



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Examples to come shortly!

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The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be known to exist, much less be located, *without knowing the entire future of the spacetime*.

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The formation, dynamics, and stability of black holes can be rigorously studied using the tools of Lorentzian geometry and hyperbolic PDE.

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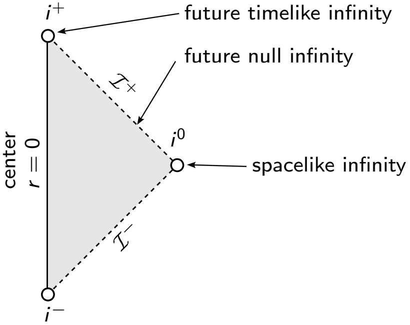
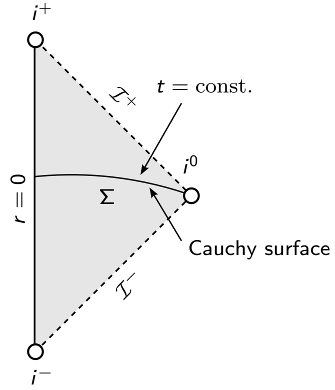
- 45 degree lines correspond to null hypersurfaces in $(3+1)$ dimensions
- steeper than 45: timelike
- shallower than 45: spacelike



Penrose diagram of Minkowski space

Crash course on black holes: Minkowski space



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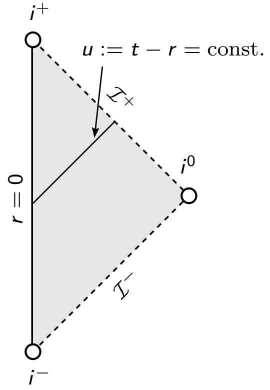

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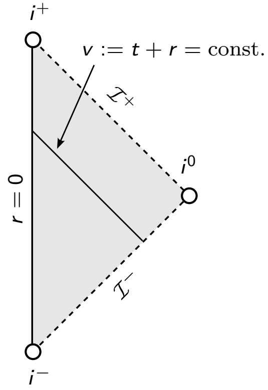

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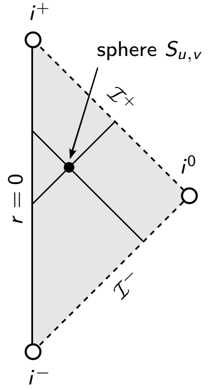

Crash course on black holes: The Schwarzschild solution

- Mass $M > 0$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- Solves the **Einstein vacuum** equations
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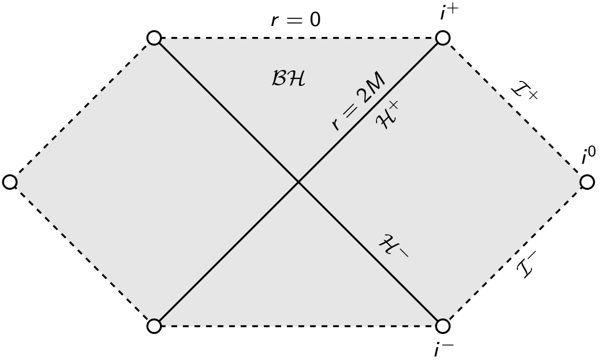

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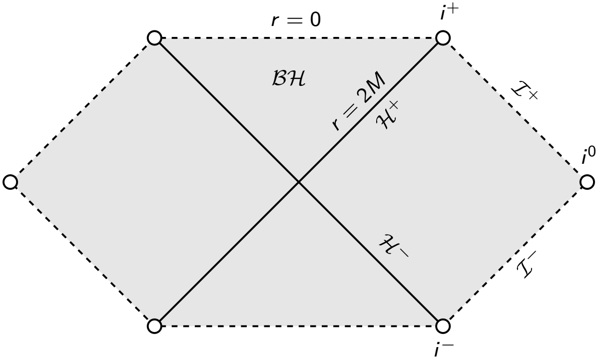

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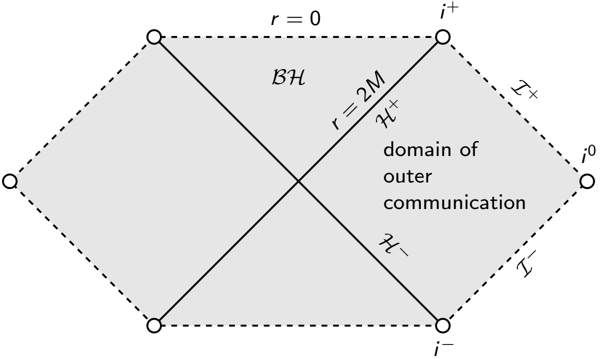

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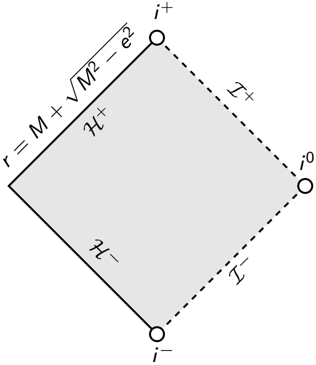

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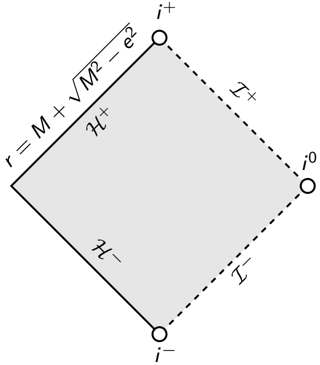

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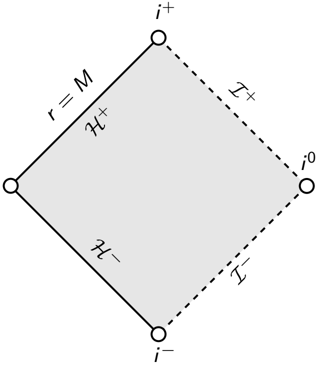

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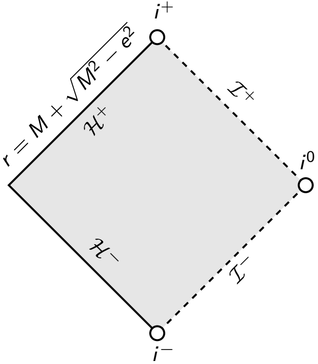

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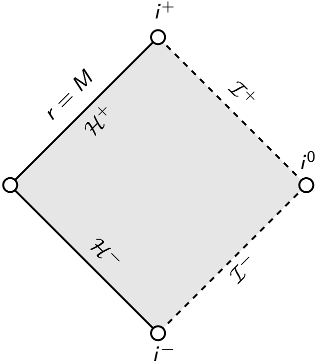

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Superextremal $|e| > M$ does not contain a black hole!

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- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Retiring the third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

Retiring the third law

Conjecture (The third law, BCH ’73, Israel ’86).

A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.

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More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:

Counterexample to the third law: detailed statement of our theorem

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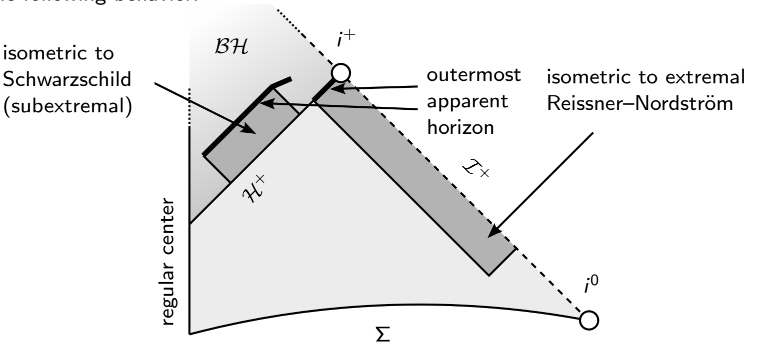

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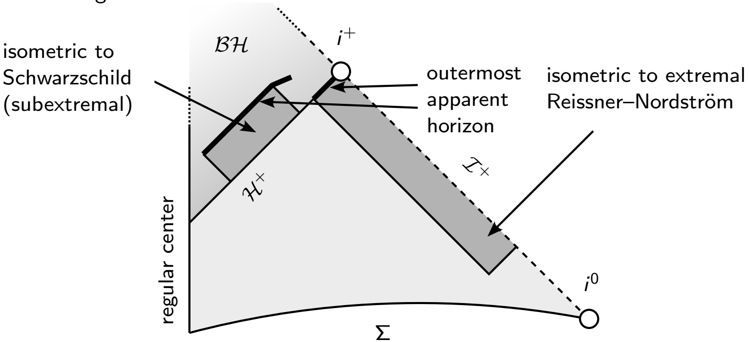

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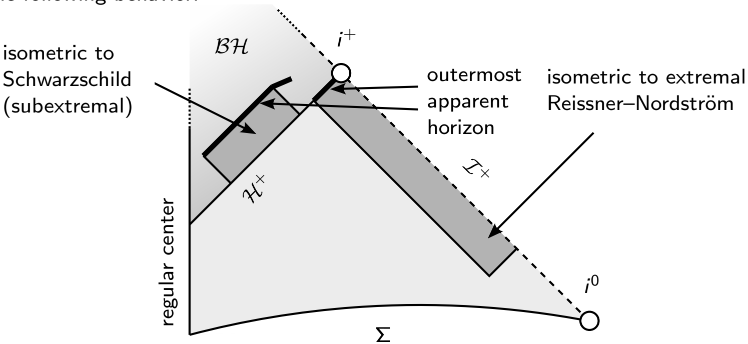

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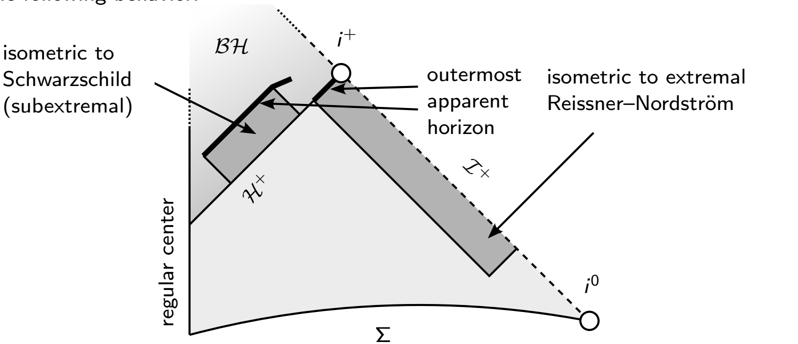

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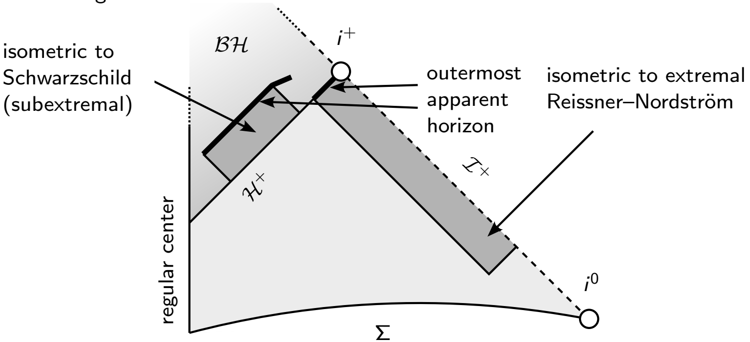

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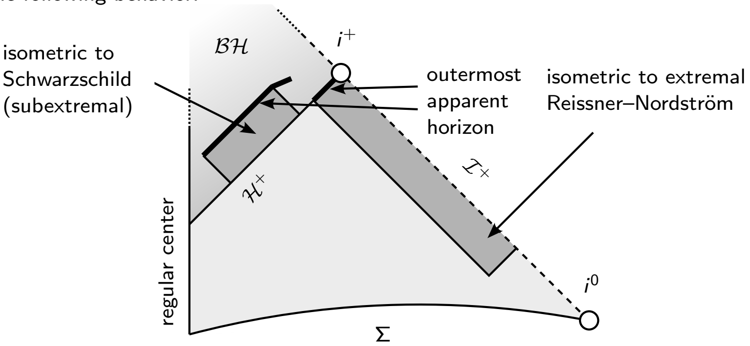
- These solutions are arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example.

Counterexample to the third law: detailed statement of our theorem

There exist regular solutions of the Einstein–Maxwell-charged scalar field system with the following behavior:



- Spherically symmetric Cauchy data for the Einstein–Maxwell-charged scalar field system which undergo gravitational collapse.
- Forms an exactly Schwarzschild "apparent horizon."
- Forms an exactly extremal Reissner–Nordström event horizon at a later advanced time.
- These solutions are arbitrarily regular: for any $k \in \mathbb{N}$, there exists a C^k example.
- Model satisfies the dominant energy condition.

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- 2-form $F = dA$ (electromagnetism)
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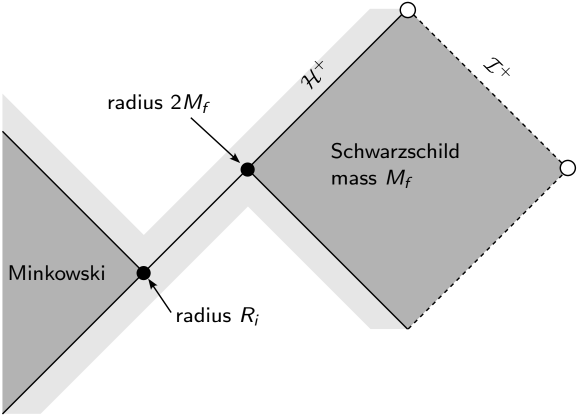
- Scalar field ϕ degree of freedom required to break the rigidity of pure Einstein–Maxwell in spherical symmetry.

Prototype: Minkowski to Schwarzschild gluing



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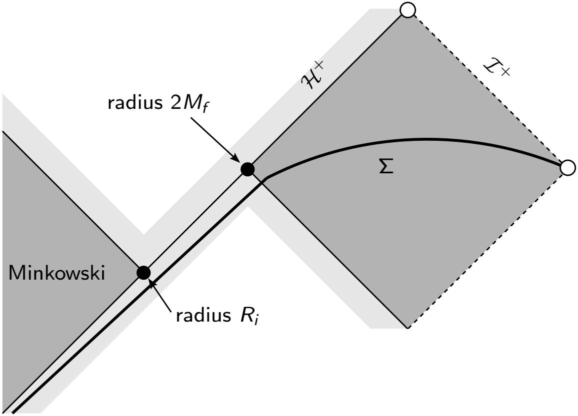

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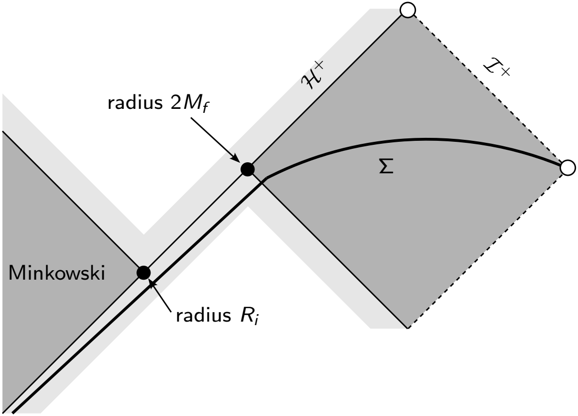

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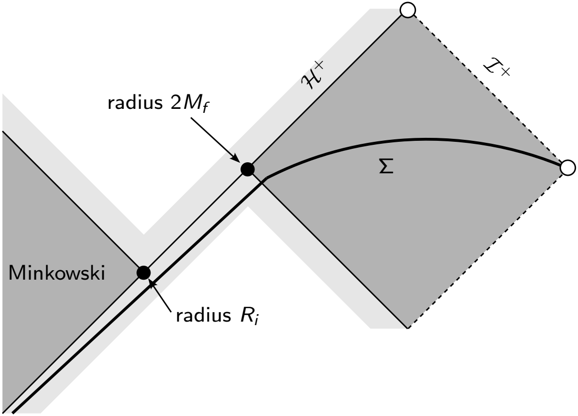

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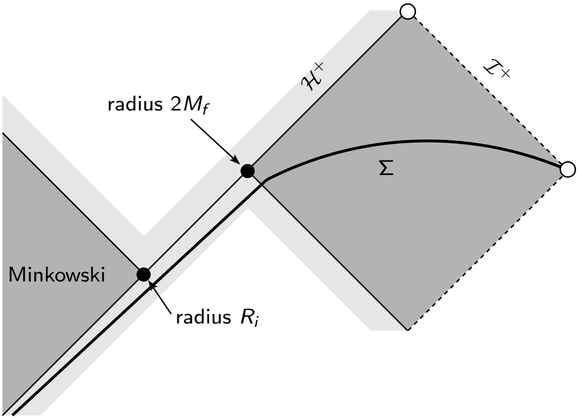

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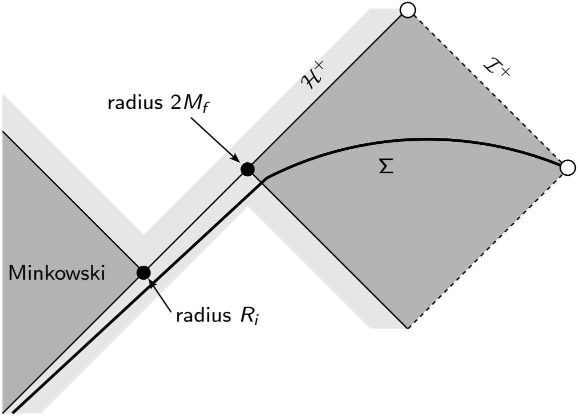

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The present problem is fundamentally **nonlinear** and **nonperturbative**, but is tractable because of the symmetry assumption.

Toy model: gluing for the linear wave equation

We now fix a spacetime $(\mathcal{M}^{3+1}, g)$ and consider the **linear wave equation**

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Given two domains $\mathfrak{R}_1, \mathfrak{R}_2 \subset \mathcal{M}$ and functions

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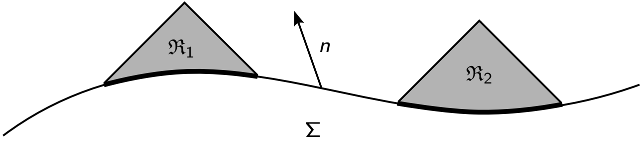
Spacelike gluing for the linear wave equation

Consider first the case of two spacelike separated regions:



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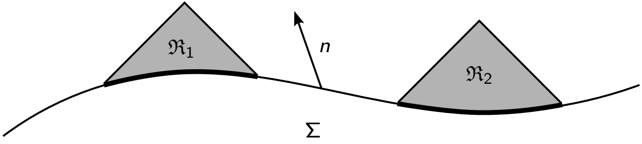

In this case the gluing problem for $(*)$ is trivial:

Proposition (Cauchy problem for the wave equation).

Given a spacelike hypersurface $\Sigma \subset (\mathcal{M}, g)$ and functions $\phi_0, \phi_1 \in C^\infty(\Sigma)$, there exists a unique function $\phi : \mathcal{M} \to \mathbb{R}$ solving

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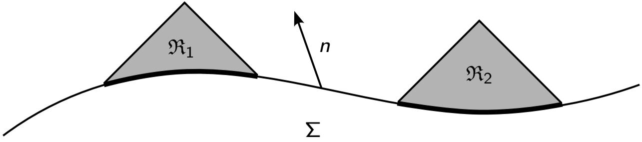

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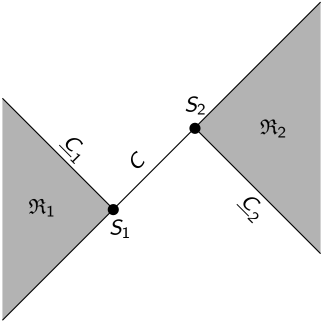
Analogue for the Einstein equations is a highly nontrivial problem in elliptic PDE, pioneered by Corvino–Schoen '06.

Characteristic/null gluing for the linear wave equation



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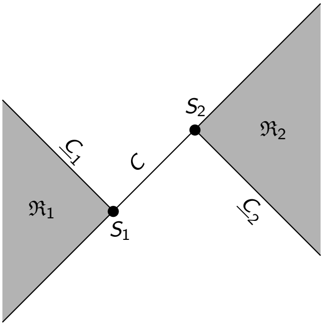

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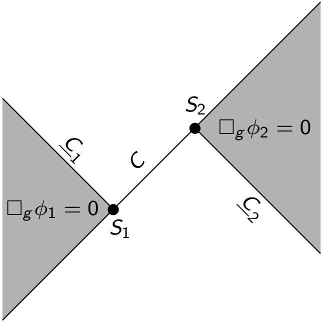

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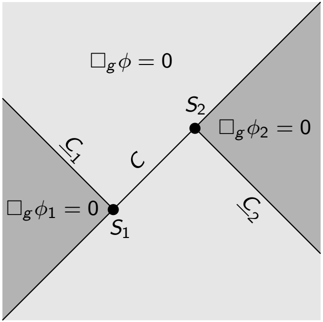

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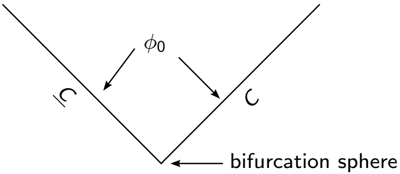

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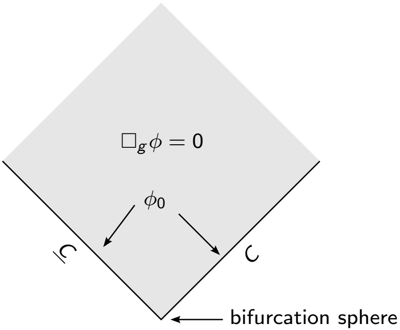

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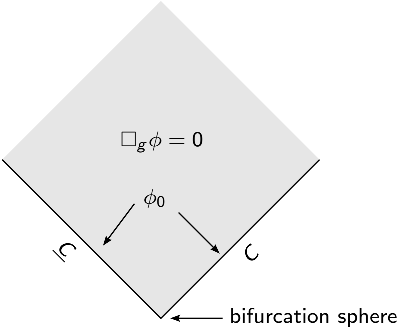

Characteristic data for the wave equation does not involve derivatives.

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- $(*)$ lets us compute $\partial_u^i \phi$ on C to all orders

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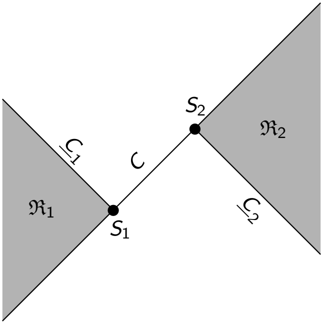
These equations have become known as the *null "constraints"* and are very distinct from the "usual" constraint equations for Einstein.

Back to characteristic gluing



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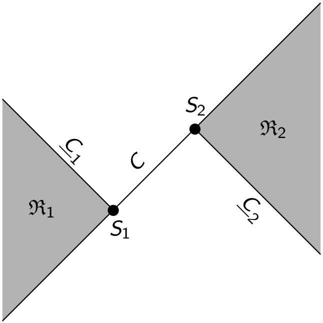

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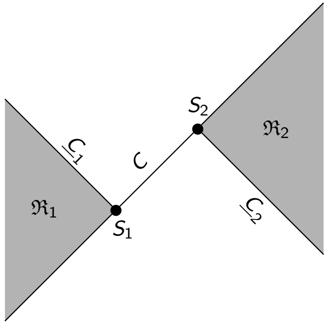

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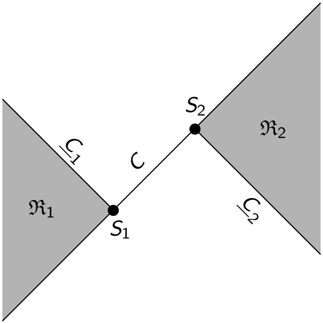

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- Initial/boundary value problem for a system of coupled ODEs where you are allowed to (almost) freely prescribe the potential

Einstein-scalar field in spherical symmetry

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Einstein-scalar field in spherical symmetry

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- Raychaudhuri's equations (constraints)

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = -\frac{r}{\Omega^2} (\partial_u \phi)^2 \tag{9}$$
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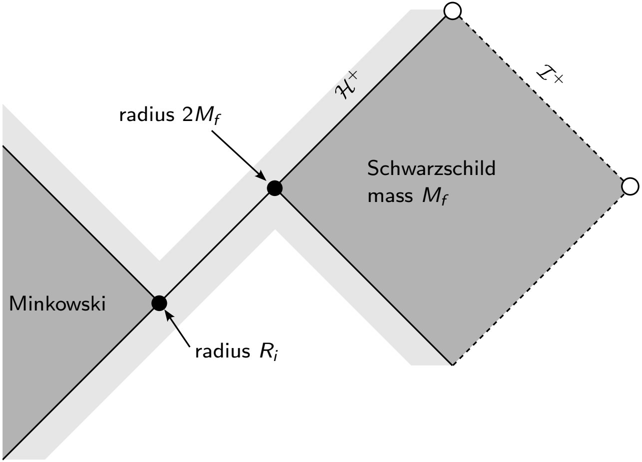
Minkowski to Schwarzschild gluing



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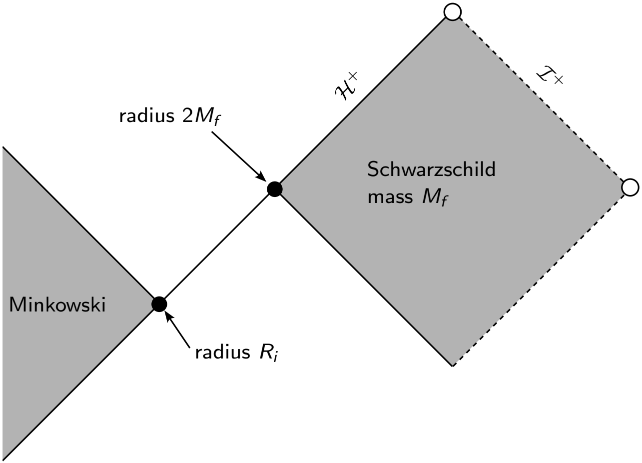

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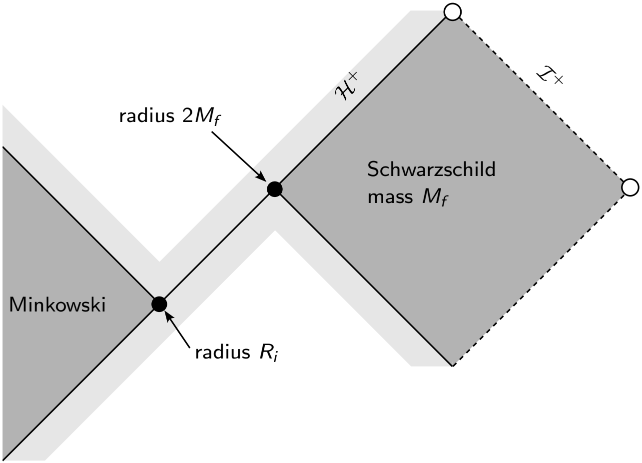
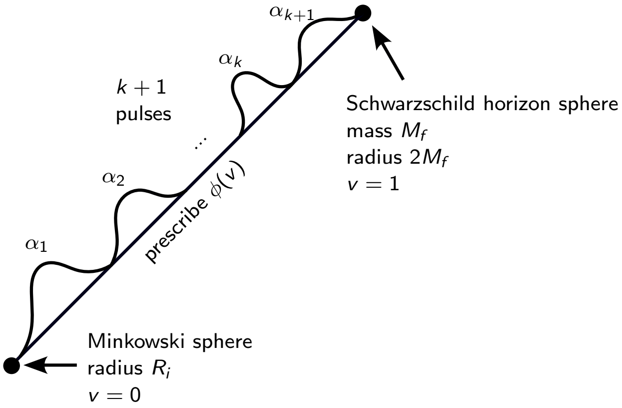


Minkowski to Schwarzschild gluing

Theorem (Kehle–U. '22).
For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



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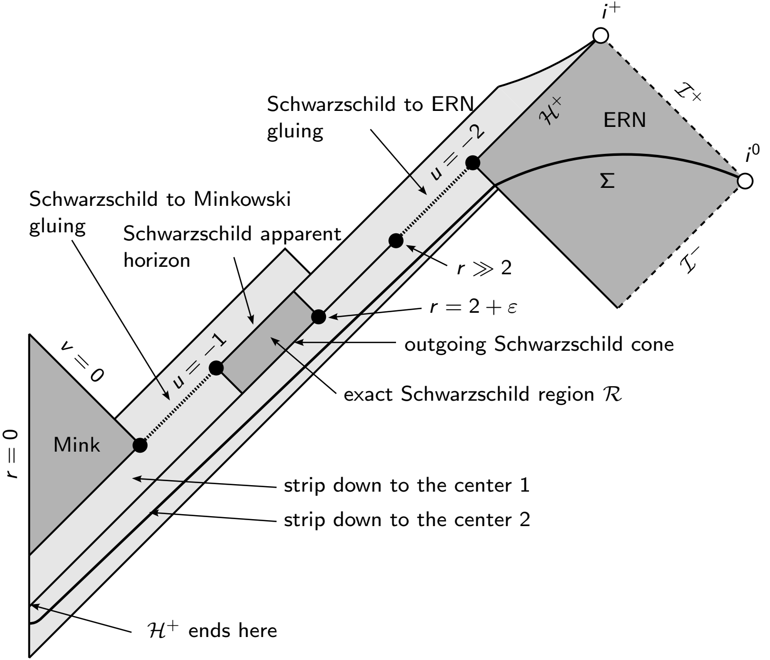
  is odd
- **Borsuk–Ulam theorem:** there exists α_* such that

$$\Big(\partial_u \phi_{\alpha_*}(1), \ldots, \partial_u^k \phi_{\alpha_*}(1)\Big) = 0.$$

Disproof of the third law



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This would follow from characteristic gluing, in vacuum, of a Minkowski sphere to a Kerr exterior sphere.

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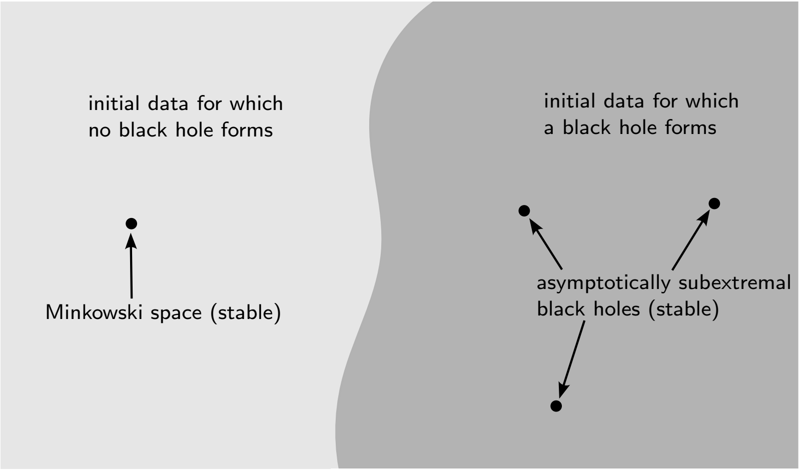

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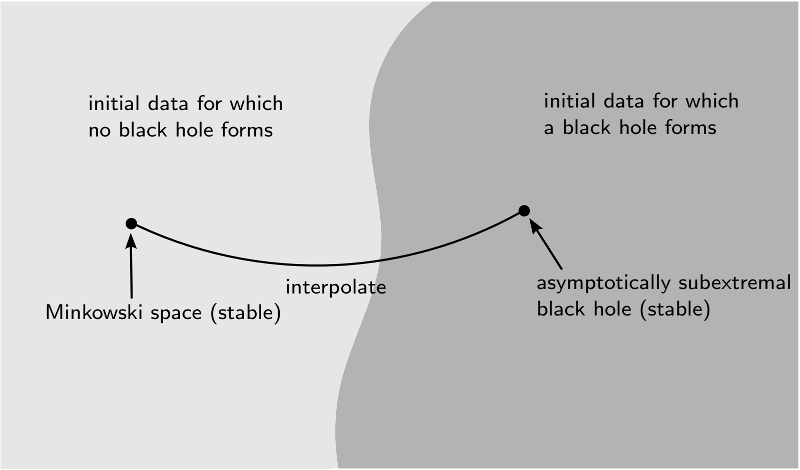

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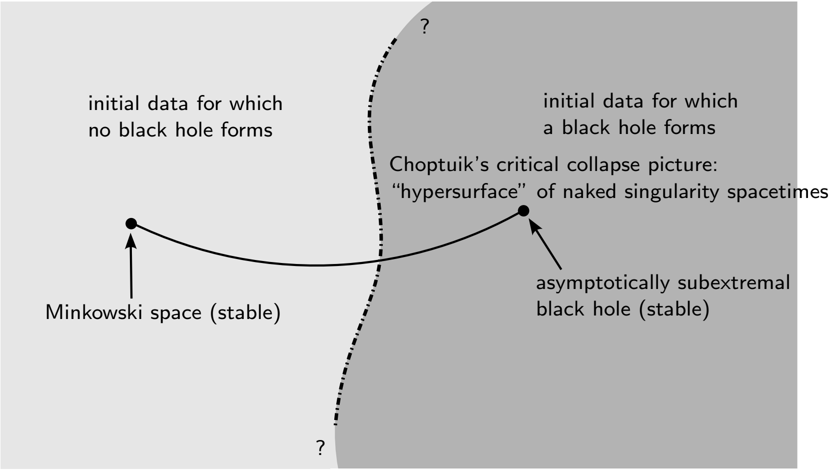

<u>Critical behavior:</u> naked singularities expected when a "low energy" black hole forms.

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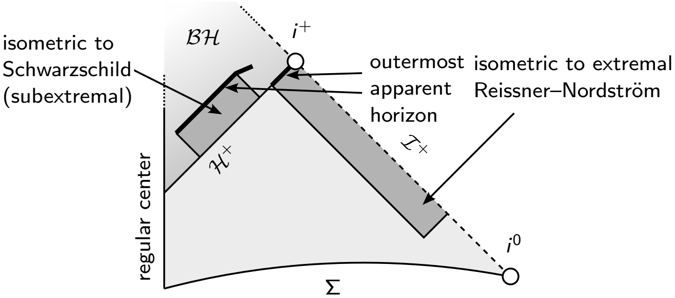
Critical behavior: the very black-holeness of certain spacetimes is unstable.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(subextremal)
$\mathcal{BH}$
i^+
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Reissner–Nordström
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regular center
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Σ

Thank you!