Nonlinear stability of extremal Reissner–Nordström black holes in spherical symmetry

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Abstract

In this paper, we prove the codimension-one nonlinear asymptotic stability of the extremal Reissner–Nordström family of black holes in the spherically symmetric Einstein–Maxwell-neutral scalar field model, up to and including the event horizon.

More precisely, we show that there exists a teleologically defined, codimension-one "submanifold" $\mathfrak{M}_{\mathrm{stab}}$ of the moduli space of spherically symmetric characteristic data for the Einstein–Maxwell-scalar field system lying close to the extremal Reissner–Nordström family, such that any data in $\mathfrak{M}_{\mathrm{stab}}$ evolve into a solution with the following properties as time goes to infinity: (i) the metric decays to a member of the extremal Reissner–Nordström family uniformly up to the event horizon, (ii) the scalar field decays to zero pointwise and in an appropriate energy norm, (iii) the first translation-invariant ingoing null derivative of the scalar field is approximately constant on the event horizon $\mathcal{H}^+$, (iv) for "generic" data, the second translation-invariant ingoing null derivative of the scalar field grows linearly along the event horizon. Due to the coupling of the scalar field to the geometry via the Einstein equations, suitable components of the Ricci tensor exhibit non-decay and growth phenomena along the event horizon.

Points (i) and (ii) above reflect the "stability" of the extremal Reissner–Nordström family and points (iii) and (iv) verify the presence of the celebrated *Aretakis instability* [Are11b] for the linear wave equation on extremal Reissner–Nordström black holes in the full nonlinear Einstein–Maxwell-scalar field model.

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1 Introduction

Extremal black holes are special solutions of Einstein's equations of general relativity which have absolute zero temperature in the celebrated thermodynamic analogy of black hole mechanics. The simplest examples of extremal black holes are given by the *extremal Reissner–Nordström* (ERN) metrics

$$g_{\mathrm{ERN}} \doteq -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2} dr^2 + r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2), \tag{1.1}$$

where M is a positive parameter known as the *mass*. The metric (1.1) solves the *Einstein–Maxwell* equations,

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 2F_{\mu\alpha}F^{\alpha}{}_{\nu} - \tfrac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \quad \text{and} \quad \nabla_\mu F^{\mu\nu}=0, \tag{1.2}$$

and is spherically symmetric, asymptotically flat, and static, with time-translation Killing vector field $T \doteq \partial_t$.

The extremal Reissner–Nordström metrics (1.1) arise as an exceptional one-parameter subfamily of the full *Reissner–Nordström family* [Rei16; Nor18] of solutions to the Einstein–Maxwell equations,

$$g_{M,e} \doteq -\left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right) dt^2 + \left(1-\frac{2M}{r}+\frac{e^2}{r^2}\right)^{-1} dr^2 + r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2), \tag{1.3}$$

where e is a real parameter representing the *charge* of the electromagnetic field. The extremal case corresponds to the parameter values $|e|=M$. For the parameter range $|e|\le M$, the metric $g_{M,e}$ describes a black hole spacetime. When $|e|<M$, the solution is called *subextremal*, and the $|e|=M$ case corresponds to the extremal Reissner–Nordström metric (1.1) above. When $e=0$, $g_{M,e}$ reduces to the celebrated *Schwarzschild solution* [Sch16] of the Einstein vacuum equations. When $|e|>M$, the *superextremal case*, the metric $g_{M,e}$ no longer describes a black hole. The role of superextremality will be discussed in Section 1.3 below.

For any of the Reissner–Nordström black hole spacetimes, the Killing field satisfies

$$\nabla_T T|_{\mathcal{H}^+} = \varkappa T|_{\mathcal{H}^+}, \tag{1.4}$$

where $\mathcal{H}^+$ denotes the *event horizon*, the boundary of the black hole region. The number $\varkappa=\varkappa(M,e)$, called the *surface gravity* of $\mathcal{H}^+$, is given by

$$\varkappa(M,e) \doteq \frac{\sqrt{M^2-e^2}}{(M+\sqrt{M^2-e^2})^2}$$

and quantifies the celebrated *horizon redshift effect*: the null generators of $\mathcal{H}^+$ have exponentially decaying energy, with rate determined by $\varkappa$.

The event horizon of subextremal Reissner–Nordström has $\varkappa>0$, while the event horizon of extremal Reissner–Nordström has $\varkappa=0$, a distinction which has fundamental repercussions for the behavior of perturbations of these spacetimes. In the seminal work [DR05], Dafermos and Rodnianski proved the nonlinear asymptotic stability of the *subextremal* Reissner–Nordström family as solutions of the Einstein–Maxwell equations coupled to a neutral scalar field in spherical symmetry. The horizon redshift effect is central to [DR05] and is a cornerstone of our understanding of linear waves on subextremal Reissner–Nordström black holes without symmetry assumptions [DR09; DR13]. By the work of Dafermos–Holzegel–Rodnianski [DHR] and Blue [Blu08] (see also [Pas19]) for $e=0$ and Giorgi [Gio20a; Gio20b] for $|e|<M$, subextremal Reissner–Nordström is now known to be linearly stable in Einstein–Maxwell theory outside of symmetry.

The theory of linear waves on extremal black holes—and by extension, nonlinear perturbations of extremal black holes—is very different. In a remarkable series of papers [Are11a; Are11b; Are15], Aretakis showed that ingoing null derivatives of solutions to the linear wave equation on extremal Reissner–Nordström—even those arising from well-localized initial data—generically do not decay on the event horizon, and higher derivatives may even grow polynomially in time. This horizon instability for the linear wave equation, which has become known as the *Aretakis instability*, has also been extended to gravitational perturbations [LMRT13; Ape22] of extremal Reissner–Nordström and to axisymmetric linear waves on extremal Kerr [Are12; LR12]. Gajic has recently shown that extremal Kerr is subject to additional stronger instabilities arising from higher azimuthal modes [Gaj23], which will be discussed further in Section 1.4.2 below.

These horizon instabilities (along with the specter of superextremality which we will address in Section 1.3 below) present a substantial obstacle to understanding the moduli space of solutions to the Einstein equations near extremal Reissner–Nordström, Kerr, and Kerr–Newman black holes. In a pioneering numerical study [MRT13], Murata, Reall, and Tanahashi studied spherically symmetric perturbations of extremal Reissner–Nordström in the Einstein–Maxwell-neutral scalar field model and observed that the Aretakis instability for the scalar field is still activated on the dynamical background, but that the geometry is not completely disrupted in the process. These numerical results and rigorous work on nonlinear model problems by the first-named author, Aretakis, and Gajic [Ang16; AAG18b; AAG20b], have given rise to the hope that extremal Reissner–Nordström could be *stable* in spite of the Aretakis instability; see [DHRT, Conjecture IV.2] and the recent essay by Dafermos [Daf24].

1.1 Stability and instability of extremal Reissner–Nordström for the spherically symmetric Einstein–Maxwell-neutral scalar field system

In this paper, we initiate the rigorous study of the (in)stability properties of extremal Reissner–Nordström black holes as solutions to the full nonlinear Einstein field equations. We work with the spherically symmetric Einstein–Maxwell-neutral scalar field model, which is the same model as in Murata–Reall–Tanahashi [MRT13] (and has been used in other influential works in recent years [Daf03; Daf05c; DR05; LO19a]). This system consists of a spherically symmetric, charged spacetime $(\mathcal{M}^{3+1}, g, F)$ together with a spherically symmetric massless scalar field $\phi : \mathcal{M} \to \mathbb{R}$ satisfying the linear wave equation

$$\Box_g \phi = 0 \tag{1.5}$$

on the dynamical background, with total energy-momentum tensor given by

$$T_{\mu\nu} \doteq F_{\mu\alpha}F^{\alpha}{}_{\nu} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi.$$

This is one of the simplest self-gravitating models in which one can entertain dynamical nonlinear perturbations of Reissner–Nordström, as a toy model for the electro-vacuum equations. See already Section 2.1 for the precise definitions and equations of the model.

We now state rough versions of our main theorems; the detailed statements and associated definitions will be presented in Section 3 below.

Theorem I (Codimension-one nonlinear stability of ERN, rough version)**.** *Let $\mathfrak{M}$ denote the moduli space of characteristic data for the spherically symmetric Einstein–Maxwell-neutral scalar field system posed on a bifurcate null hypersurface $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$, as in Fig. 1 below, which lie close to extremal Reissner–Nordström in an appropriate norm. There exists a "codimension-one submanifold" $\mathfrak{M}_{\mathrm{stab}} \subset \mathfrak{M}$ such that any data in $\mathfrak{M}_{\mathrm{stab}}$ evolve into a spacetime with the following properties:*

(i) Future null infinity $\mathcal{I}^+$ is complete and the causal past of future null infinity, $J^-(\mathcal{I}^+)$, is bounded by a regular event horizon $\mathcal{H}^+$, which itself bounds a nonempty black hole region $\mathcal{BH} \doteq \mathcal{M} \setminus J^-(\mathcal{I}^+)$.

(ii) The metric remains close to the initial extremal Reissner–Nordström metric in the domain of outer communication and appropriately defined energy fluxes and pointwise C^1 norms of the scalar field ϕ are bounded in terms of their initial data on $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$.

(iii) The metric decays polynomially in C^0 (as an appropriate notion of "time" tends to infinity) to a nearby member of the extremal Reissner–Nordström family, relative to a teleologically defined double null gauge uniformly in the entire domain of outer communication. The renormalized Hawking mass (see already Section 2.1.1) converges uniformly to the final mass of the black hole. The scalar field decays polynomially to zero pointwise and in an appropriate energy norm.

(iv) The spacetime does not contain strictly trapped surfaces. Any marginally trapped surfaces lie on $\mathcal{H}^+$.

Some comments about this statement are in order:

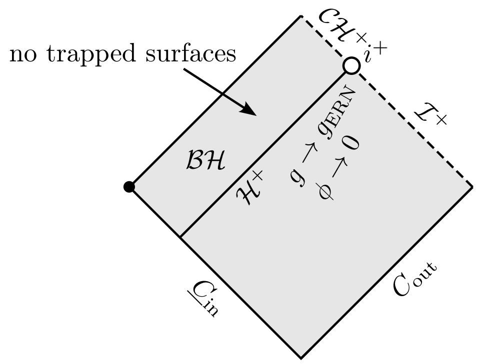

Figure 1: A Penrose diagram showing the maximal development of a solution considered in Theorem I. The Cauchy data ends on the left in the solid point, where it is incomplete (but not singular).

1. Because the extremal Reissner–Nordström family is already codimension-one within the full Reissner–Nordström family, any stability statement for it is necessarily a *positive codimension* statement, with codimension one being sharp. This aspect of the problem is familiar from the proof of nonlinear stability of Schwarzschild outside of symmetry by Dafermos, Holzegel, Rodnianski, and Taylor [DHRT]. For conjectures regarding the regularity of $\mathfrak{M}_{\mathrm{stab}}$ and what happens "on either side" of it, see already Section 1.3.

2. The theorem states that the metric decays in C^0 (and some Christoffel symbols) to that of extremal Reissner–Nordström and that the scalar field decays in C^0 to zero. We will show in Theorem II below that the metric does not necessarily decay to extremal Reissner–Nordström in C^2 (and may grow in C^3) and that the scalar field does not necessarily decay in C^1 (and may grow in C^2).

3. The norms and decay rates for the scalar field ϕ are consistent with the norms and decay rates for spherically symmetric solutions of the linear wave equation on extremal Reissner–Nordström. We do not need to commute in order to close our bootstrap assumptions and hence do not prove sharp decay rates for ϕ everywhere. See already Sections 1.2 and 1.4.1.

4. Since we show that there are no trapped surfaces behind the event horizon, we in fact prove that the maximal development of data in $\mathfrak{M}_{\mathrm{stab}}$ is the *full double null rectangle* depicted in Fig. 1. For the behavior of the scalar field and geometry in the black hole interior, see already Section 1.3.2.

Our second main theorem shows that the Aretakis instability of the event horizon, i.e., non-decay of the first transverse derivative and linear growth of the second transverse derivative, persists in the Einstein–Maxwell-scalar field model. Moreover, in this coupled model, the instability affects the geometry as well.

Before stating the theorem, we recall briefly some features of the geometry of extremal Reissner–Nordström black holes. Let Y denote the coordinate vector field ∂_r in ingoing Eddington–Finkelstein coordinates $(v, r, \vartheta, \varphi)$. This null vector field is translation-invariant and transverse to the event horizon. Moreover, it is canonical in the sense that for *any* spherically symmetric double null coordinates $(u, v, \vartheta, \varphi)$ on extremal Reissner–Nordström with u "ingoing," Y can be written as $(\partial_u r)^{-1}\partial_u$. The YY-component of the Ricci tensor, R_{YY}, vanishes identically on any Reissner–Nordström solution.

Theorem II (Dynamical horizon instability, rough version)**.** *For any initial data lying in $\mathfrak{M}_{\mathrm{stab}}$, the following holds on the event horizon $\mathcal{H}^+$ of its maximal globally hyperbolic development:*

(i) Let $Y \doteq (\partial_u r)^{-1}\partial_u$ denote the gauge-invariant null derivative transverse to $\mathcal{H}^+$, where r is the area-radius of the spacetime, and u is a retarded time coordinate, i.e., increasing towards the black hole. Then R_{YY} and $Y(r\phi)$ are approximately constant on $\mathcal{H}^+$, i.e., do not necessarily decay.

(ii) There exists a relatively open subset of $\mathfrak{M}_{\mathrm{stab}}$ for which the "asymptotic Aretakis charge"

$$H_0[\phi] \doteq \lim_{v\to\infty} Y(r\phi)|_{\mathcal{H}^+}(v) \tag{1.6}$$

is nonvanishing. Here v is an advanced time coordinate on the spacetime such that $v = \infty$ at future timelike infinity i^+. It holds that

$$\lim_{v\to\infty} R_{YY}|_{\mathcal{H}^+}(v) = 2M^{-2}\big(H_0[\phi]\big)^2. \tag{1.7}$$

(iii) If $H_0[\phi] \neq 0$, then there exists a constant $c > 0$ such that

$$\big|\nabla_Y R_{YY}|_{\mathcal{H}^+}(v)\big| \geq cv, \qquad \big|Y^2(r\phi)|_{\mathcal{H}^+}(v)\big| \geq cv. \tag{1.8}$$

As was mentioned before, the problem considered here was previously investigated numerically by Murata, Reall, and Tanahashi in [MRT13]. Theorems I and II rigorously confirm all of their findings about the black hole exterior and event horizon for initial data in $\mathfrak{M}_{\rm stab}$ and make precise the nature of the "fine tuning" required to asymptote to extremality. Their findings about the black hole interior at extremality are also verified by combining Theorem I with the work of Gajic and Luk [GL19]; see already Section 1.3.2.

Remark 1.1. The non-decay and growth of R_{YY} and $\nabla_Y R_{YY}$ along $\mathcal{H}^+$ found in the present paper is distinct from the non-decay and growth of $\underline{\alpha}$ and $\nabla\!\!\!/_Y\underline{\alpha}$ found by Apetroaie in [Ape22] for the generalized Teukolsky system on extremal Reissner–Nordström, where $\underline{\alpha}_{AB} \doteq W(e_A, Y, e_B, Y)$, where W is the Weyl tensor, and $\{e_A\}_{A=1,2}$ span the symmetry spheres. Indeed, $\underline{\alpha}$ vanishes identically for a spherically symmetric metric.

In this paper, we do not pursue the interesting question of which other geometric quantities exhibit instabilities at higher orders of differentiability.

1.2 Overview of the proof

The proof of Theorem I involves a bootstrap argument with a teleologically normalized double null gauge coupled to a discrete modulation argument performed on dyadic timescales. Theorem II is proved by combining the method of characteristics for the wave equation for ϕ with precise estimates on the dynamical degenerate redshift factor along the event horizon $\mathcal{H}^+$. We now describe the proofs in some detail, beginning with the relevant theory for the linear wave equation on extremal Reissner–Nordström.

1.2.1 Review of spherically symmetric linear waves on extremal Reissner–Nordström and the Aretakis instability

Here we briefly review the theory of spherically symmetric solutions to the linear wave equation (1.5) on extremal Reissner–Nordström. Aretakis initiated the study of this problem in [Are11a; Are11b] but we will make use of technical advances made by the first-named author, Aretakis, and Gajic in [AAG18a; AAG18c; AAG20a]. For a brief review of the geometry of extremal Reissner–Nordström, we refer the reader to Section 2.2 of the present paper, [Are11a, Section 2], and the appendix of [Are10].

The general strategy to prove energy decay statements for waves on extremal Reissner–Nordström consists of, as in the subextremal case, deriving a hierarchy of weighted energy boundedness inequalities and time-integrated energy decay estimates. This hierarchy takes the form

$$\begin{aligned}\int_{C(\tau_2)} r^p(\partial_v\psi)^2\,dv + \int_{\underline{C}(\tau_2)} (r-M)^{2-p}\frac{(\partial_u\psi)^2}{-\partial_u r}\,du\qquad&\\ \lesssim \int_{C(\tau_1)} r^p(\partial_v\psi)^2\,dv + \int_{\underline{C}(\tau_1)} (r-M)^{2-p}\frac{(\partial_u\psi)^2}{-\partial_u r}\,du + \text{l.o.t.},&\end{aligned} \tag{1.9}$$

$$\int_{\tau_1}^{\tau_2}\int_{C(\tau)} r^{p-1}(\partial_v\psi)^2\,dvd\tau \lesssim \int_{C(\tau_1)} r^p(\partial_v\psi)^2\,dv + \text{l.o.t.}, \tag{1.10}$$

$$\int_{\tau_1}^{\tau_2}\int_{\underline{C}(\tau)} (r-M)^{3-p}\frac{(\partial_u\psi)^2}{-\partial_u r}\,dud\tau \lesssim \int_{\underline{C}(\tau_1)} (r-M)^{2-p}\frac{(\partial_u\psi)^2}{-\partial_u r}\,du + \text{l.o.t.}, \tag{1.11}$$

where (u, v) denote Eddington–Finkelstein double null coordinates on the domain of outer communication, τ is proper time along a timelike curve Γ with constant area-radius, $\tau_1 \leq \tau_2$, $p \in [0, 3)$ in (1.9), $p \in [1, 3)$ in

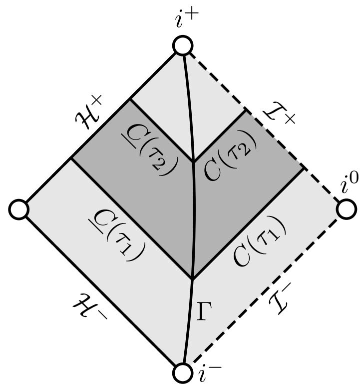

Figure 2: A Penrose diagram of extremal Reissner–Nordström depicting the foliations $C(\tau)$ and $\underline{C}(\tau)$ used in the estimates (1.9)–(1.11). The region of integration in (1.10) and (1.11) is shaded darker.

(1.10) and (1.11), "l.o.t." denotes terms lower in the p-hierarchy, the foliations $C(\tau)$ and $\underline{C}(\tau)$ are defined pictorially in Fig. 2 above, and ψ denotes the *radiation field*

$$\psi \doteq r\phi.$$

Remark 1.2. The range of our horizon and infinity hierarchies is considered for $p \in [0, 3-\delta]$. Specifically, at the horizon, this range is sharp, and it is necessary for us to go beyond $p \geq 2$ in order to simultaneously close the bootstrap argument for both the scalar field and the geometric quantities.

Remark 1.3. The expression $(\partial_u\psi)^2(-\partial_u r)^{-1}du$ is invariant under reparametrization of the double null gauge and represents a *non-degenerate* one-form along $\underline{C}(\tau)$. Indeed, written in ingoing Eddington–Finkelstein coordinates (v, r), it corresponds to $(\partial_r\psi)^2\,dr$ along $\underline{C}(\tau)$, which is manifestly nondegenerate.

The inequality (1.10) is the celebrated *r^p-weighted estimate* of Dafermos and Rodnianski [DR10] and relies only on the asymptotic flatness of the extremal Reissner–Nordström metric. The estimate (1.11) is specific to extremal Reissner–Nordström and can be thought of as a "horizon analogue" of the r^p estimates at infinity: The event horizon $\mathcal{H}^+$ of extremal Reissner–Nordström is located at $r = M$ and hence $r - M$ is a degenerate weight on $\mathcal{H}^+$. The estimate (1.11) states that the time integral of the $(p-1)$-weighted horizon flux is bounded by the initial value of the p-weighted horizon flux. This horizon hierarchy—some special cases of which were introduced in [Are11a] and in full generality was presented in [AAG20a]—replaces the fundamental redshift estimate from [DR09; DR13] which holds for any stationary event horizon with positive surface gravity. Note that the $p = 0$ horizon flux is equivalent to the T-energy, where T is the time-translation Killing field. The duality between r and $(r-M)^{-1}$ is related to the so-called *Couch–Torrence conformal isometry* [CT84] that exchanges $\mathcal{H}^+$ and $\mathcal{I}^+$ in extremal Reissner–Nordström.

Using the pigeonhole principle as in [DR10], (1.9)–(1.11) can be used to prove the energy decay estimate

$$\int_{C(\tau)} r^p(\partial_v\psi)^2\,dv + \int_{\underline{C}(\tau)} (r-M)^{2-p}\frac{(\partial_u\psi)^2}{-\partial_u r}\,du \leq C_\star\tau^{-3+\delta+p}, \tag{1.12}$$

for every $p \in [0, 3-\delta]$ and $\tau \geq \tau_0$, where $C_\star$ is a constant depending on δ and the data at $\tau = \tau_0$. This estimate can then be used to prove pointwise decay of ψ itself. By commuting the wave equation, higher order versions of (1.9)–(1.12) can be proved, but we shall not need to do so in the present paper.

Remark 1.4. Observe that (1.12) for $p = 2$ states that the nondegenerate energy of $\partial_u\psi$ decays only like $\tau^{-1+\delta}$, compared to $\tau^{-3+\delta}$ or $\tau^{-2+\delta}$ in the subextremal case (depending on the assumptions for the decay of the data at infinity—see for example [AAG18a]). Unfortunately, this is in fact sharp for generic data (see [AAG17]) and persists in the nonlinear theory, see already Sections 1.4.1 and 9.3. This slow decay is responsible for many of the technical difficulties we face in the present paper and will be discussed again in Section 1.2.4 below.

While certain energies for ϕ do indeed decay by (1.12), the Aretakis instability states that nondegenerate ingoing null derivatives of ϕ on $\mathcal{H}^+$ *do not decay*, or can even *grow polynomially.* We now briefly explain this mechanism. Let $Y \doteq \partial_r$ in ingoing Eddington–Finkelstein coordinates (v, r).[1] An elementary calculation using the wave equation (1.5) shows that

$$\partial_v \big(Y\psi|_{\mathcal{H}^+} \big) = 0, \tag{1.13}$$

i.e., $Y\psi$ is *constant along* $\mathcal{H}^+$. Therefore, in sharp contrast to the subextremal case, $Y\psi$ does not decay along $\mathcal{H}^+$. This constant is written as $H_0[\phi]$ and is called the (zeroth) *Aretakis charge* of ϕ.

By commuting the wave equation with Y, we can likewise derive an evolution equation for $Y^2\psi$ along $\mathcal{H}^+$. If $H_0[\phi] \neq 0$, we have that

$$\partial_v \big(Y^2\psi|_{\mathcal{H}^+} \big) = -2M^{-2} H_0[\phi] + \text{decaying terms}, \tag{1.14}$$

so upon integrating in v we conclude that $|Y^2\psi| \gtrsim |H_0[\phi]|v$ on $\mathcal{H}^+$ for v large. In fact, we have $|Y^k\psi| \gtrsim |H_0[\phi]|v^{k-1}$ on $\mathcal{H}^+$ for any $k \geq 1$ and v large.

1.2.2 Prescription of seed data and the modulation parameter α

Keeping in mind the ideas of Section 1.2.1, we now turn to the outline of the proofs of Theorems I and II.

We recall the notion of *renormalized Hawking mass*, which is the appropriate analogue of the usual Hawking mass m for solutions of the spherically symmetric Einstein–Maxwell-neutral scalar field model:

$$\varpi \doteq m + \frac{e^2}{2r},$$

where e is the constant charge of the solution. In Reissner–Nordström with parameters M and e, $\varpi = M$.

We refer back to the Penrose diagram of the setup of our main results, Fig. 1. Bifurcate characteristic seed data for the spherically symmetric Einstein–Maxwell-neutral scalar field model on $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$ consists of the restriction of ϕ to $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$, denoted by $\mathring{\phi}$, the area-radius Λ of $C_{\mathrm{out}} \cap \underline{C}_{\mathrm{in}}$, the renormalized Hawking mass ϖ_0 of $C_{\mathrm{out}} \cap \underline{C}_{\mathrm{in}}$, and the constant charge e of the solution.

Fix extremal parameters $M_0 = |e_0|$ and an area-radius $100M_0$ (which lies far outside the horizon located at $r = M_0$). We consider perturbations of the bifurcate cone in the extremal Reissner–Nordström solution with parameters (M_0, e_0) with bifurcation sphere at $r = 100M_0$. We consider the seed data norm

$$\mathfrak{D} \approx |\Lambda - 100M_0| + |\varpi_0 - M_0| + |e - e_0| + \sup_{\underline{C}_{\mathrm{in}}} \big(|\phi| + |\partial_{\hat{u}}\phi| \big) + \sup_{C_{\mathrm{out}}} \big(|\psi| + |r^2 \partial_{\hat{v}}\psi| \big) \tag{1.15}$$

and define a master smallness parameter $\varepsilon \geq \mathfrak{D}$. Here $(\hat{u}, \hat{v})$ are "initial data normalized" coordinates such that $\partial_{\hat{u}} r = -1$ on $\underline{C}_{\mathrm{in}}$ and $\partial_{\hat{v}} r = 1$ on C_{out}. In particular, $\hat{u}$ will be regular across the event horizon.

As in [DHRT], we in fact consider a *family* of initial data which are indexed by a real parameter α such that $\varpi_0 = M_0 + \alpha$. Therefore, for $(\mathring{\phi}, \Lambda, e)$ fixed, the "codimension-one" aspect of Theorem I means that we find (at least one) $\alpha = \alpha_\star$ such that the solution generated by the seed data $(\mathring{\phi}, \Lambda, M_0 + \alpha_\star, e)$ converges to extremal Reissner–Nordström with parameters $M = |e|$ and e. The critical parameter $\alpha_\star$ is determined in evolution and cannot be read off from the initial data. See already Section 1.2.6.

The setup for the initial data is given in Section 3.1.

1.2.3 Setup of the bootstrap argument and the teleological gauge

Our general setup for the bootstrap argument (depicted in Fig. 3 below) is inspired by [DHRT] and the work of Luk–Oh [LO19b] on stability of subextremal Reissner–Nordström in spherical symmetry.

In Fig. 3, the timelike curve Γ has constant area-radius $r = \Lambda$ (from the seed data) and is parametrized by proper time $\tau \geq 1$. The parameter τ_f determines the bootstrap domain $\mathcal{D}_{\tau_f}$ and is sent to infinity in the course of the proof. The double null coordinates (u, v) are teleologically normalized according to the conditions $\partial_u r = -(1 - \frac{2m}{r})$ on the ingoing future boundary of $\mathcal{D}_{\tau_f}$, where m is the Hawking mass of the

[1] The vector field Y can be expressed as $(\partial_u r)^{-1}\partial_u$ in Eddington–Finkelstein double null coordinates (u, v), or any reparametrization thereof.

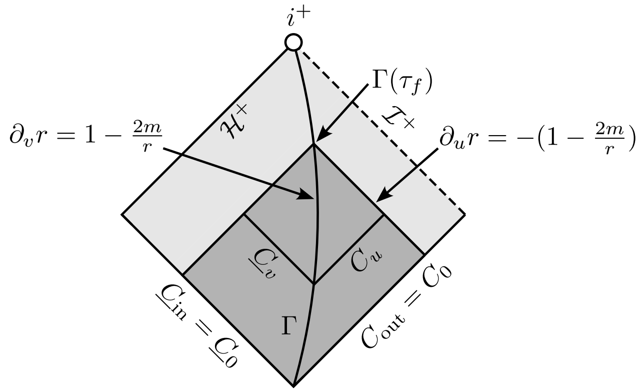

Figure 3: A Penrose diagram of one of the bootstrap domains $\mathcal{D}_{\tau_f} \doteq J^-(\Gamma(\tau_f))$ used in the proof of Theorem I. Here $\Gamma \doteq \{r = \Lambda\}$ is the timelike curve which anchors the bootstrap domains and (u, v) are double null coordinates teleologically normalized as depicted.

spacetime, and $\partial_v r = 1 - \frac{2m}{r}$ on Γ. This latter gauge condition for the v coordinate is nonstandard and marks a crucial difference with the subextremal case in [LO19b; DHRT]. We extend τ to a function on $\mathcal{D}_{\tau_f}$ by setting it equal to proper time along Γ and then declaring it to be constant along ingoing cones to the left of Γ and constant along outgoing cones to the right of Γ.

Since the charge e is conserved, we know a priori that we should aim to converge to extremal Reissner–Nordström with parameters $M \doteq |e|$ and e. We anchor a comparison extremal Reissner–Nordström solution with area-radius function $\bar{r}$ and parameters (M, e) in Eddington–Finkelstein double null gauge to the bootstrap domain $\mathcal{D}_{\tau_f}$ by setting $\bar{r}(\Gamma(\tau_f)) = \Lambda$. Relative to this background solution, we define energy norms motivated by the uncoupled case (recall Section 1.2.1),

$$\mathcal{E}_p(\tau) \doteq \int_{C_{\Gamma^u(\tau)} \cap \mathcal{D}_{\tau_f}} r^p (\partial_v \psi)^2 \, dv + \cdots ,$$
$$\underline{\mathcal{E}}_p(\tau) \doteq \int_{\underline{C}_{\Gamma^u(\tau)} \cap \mathcal{D}_{\tau_f}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\partial_u \bar{r}} \, du + \cdots ,$$

where we have only written the most important terms for now.

Remark 1.5. In the definition of $\mathcal{E}_p$, we use the “dynamical” r and not the background $\bar{r}$. This is because the r^p estimates in spherical symmetry do not generate nonlinear errors and r works just as well as $\bar{r}$ in the far region. The use of $\bar{r} - M$ in $\underline{\mathcal{E}}_p$ is crucial, however.

For the modulation parameter α lying in an appropriate range (to be explained in Section 1.2.6 below), we make the bootstrap assumptions

$$\left| \frac{\partial_u r}{\partial_u \bar{r}} - 1 \right| \lesssim \varepsilon^{3/2} \tau^{-1+\delta}, \quad |r - \bar{r}| \lesssim \varepsilon^{3/2} \tau^{-2+\delta}, \quad |\varpi - M| \lesssim \varepsilon^{3/2} \tau^{-3+\delta} \tag{1.16}$$

on $\mathcal{D}_{\tau_f}$ for the geometry and

$$\mathcal{E}_p(\tau) + \underline{\mathcal{E}}_p(\tau) \lesssim \varepsilon^2 \tau^{-3+\delta+p} \tag{1.17}$$

for $\tau \in [1, \tau_f]$ for the scalar field; compare with (1.12).[2] The main analytic content of Theorem I consists of recovering the bootstrap assumptions (1.16) and (1.17).

The setup for the bootstrap argument is given in Section 3.2, the anchoring procedure is given in Section 3.3, and the bootstrap assumptions are precisely stated in Section 4.1.

[2] We also have bootstrap assumptions for energy fluxes along outgoing cones in the near region and along ingoing cones in the far region, but suppress these at this level of discussion.

1.2.4 Estimates for the geometry: the role of the degenerate redshift

We utilize the well-known quantities

$$\kappa \doteq \frac{\partial_v r}{1-\frac{2m}{r}}, \quad \gamma \doteq \frac{\partial_u r}{1-\frac{2m}{r}},$$

which satisfy good evolution equations in u and v, respectively. The background extremal Reissner–Nordström solution $\bar{r}$ has $\bar{\kappa} = -\bar{\gamma} = 1$ globally. Therefore, the anchoring and gauge conditions ensure that $r(\Gamma(\tau_f)) = \bar{r}(\Gamma(\tau_f))$, $\kappa = \bar{\kappa} = 1$ along $\Gamma \cap \mathcal{D}_{\tau_f}$, and $\gamma = \bar{\gamma} = -1$ along the final ingoing cone in $\mathcal{D}_{\tau_f}$.

The v-equation for γ sees the flux $\mathcal{E}_0$ and hence gives us that $\gamma = -1 + O(\varepsilon^2\tau^{-3+\delta})$, which is the best that can be expected. On the other hand, the u-equation for κ reads $\partial_u\kappa = r\kappa(\partial_u r)^{-1}(\partial_u\phi)^2$, which sees $\underline{\mathcal{E}}_2$, and hence decays much slower.[3] On the other hand, the quantity $(\bar{r}-M)^{2-p}(\kappa-\bar{\kappa})$, for $p \in [0,2]$, obeys a u-evolution equation that sees the flux $\underline{\mathcal{E}}_p$ *when integrating to the left of* Γ, and hence we show that

$$(\bar{r}-M)^{2-p}\kappa = (\bar{r}-M)^{2-p} + O(\varepsilon^2\tau^{-3+\delta+p}) \tag{1.18}$$

to the left of Γ. This hierarchy of decay rates for κ is a fundamental aspect of our geometric estimates.

Because of this, it is important not to waste any powers of $\bar{r}-M$ in the system and our scheme is essentially sharp at the horizon in this regard. Of particular importance are the quantities $1-\frac{2m}{r}$ and $\partial_v r$, which we show admit "Taylor expansions" of the form

$$1-\frac{2m}{r} = \overline{1-\frac{2m}{r}} + \frac{2M}{\bar{r}^3}(\bar{r}-M)(r-\bar{r}) + O(\varepsilon^{3/2}\tau^{-3+\delta}), \tag{1.19}$$

$$\partial_v r = \partial_v\bar{r} + \frac{2M\kappa}{\bar{r}^3}(\bar{r}-M)(r-\bar{r}) + O(\varepsilon^{3/2}\tau^{-3+\delta}). \tag{1.20}$$

In the background solution, the quantities $\overline{1-\frac{2m}{r}}$ and $\partial_v\bar{r}$ vanish quadratically on $\mathcal{H}^+$, so these are to be viewed as expansions in powers of $\bar{r}-M$ with rapidly decaying error.

The linear terms and strong decay of the error terms in (1.19) and (1.20) are important. For example, the function $1-\frac{2m}{r}$ appears naturally in the u-equation for ϖ, which reads $\partial_u\varpi = \frac{1}{2}(1-\frac{2m}{r})r^2(\partial_u r)^{-1}(\partial_u\phi)^2$, and proving $\tau^{-3+\delta}$ decay for this requires taking advantage of the linear term in (1.19). Moreover, the linear term in (1.19) is used to bound the error term in (1.20). The sign of the linear term in (1.20) is crucial: it is positive in the domain of outer communication, which reflects the *global redshift effect* on extremal Reissner–Nordström. However, unlike the subextremal case (compare for instance [LO19b, Lemma 8.19]), the effect is degenerate on the horizon and the rapidly decaying error term becomes important.

The geometric estimates are proved in Section 5.

1.2.5 Integrated local energy decay and the r^p and $(\bar{r}-M)^{2-p}$ energy hierarchies

To recover the bootstrap assumption (1.17) for the scalar field, we follow the general strategy outlined in Section 1.2.1. We prove the estimates

$$\mathcal{E}_p(\tau_2) + \underline{\mathcal{E}}_p(\tau_2) \lesssim \mathcal{E}_p(\tau_1) + \underline{\mathcal{E}}_p(\tau_1) + \text{decaying nonlinear error} \tag{1.21}$$

for $1 \le \tau_1 \le \tau_2$ and $p \in [0, 3-\delta]$, and

$$\int_{\tau_1}^{\tau_2}\big(\mathcal{E}_{p-1}(\tau) + \underline{\mathcal{E}}_{p-1}(\tau)\big)\,d\tau \lesssim \mathcal{E}_p(\tau_1) + \underline{\mathcal{E}}_p(\tau_1) + \text{decaying nonlinear error} \tag{1.22}$$

for $p \in [1, 3-\delta]$. The choice of multiplier vector fields and lower order currents is inspired by work in the uncoupled case and [LO19b]. We utilize a mix of "dynamical" (i.e., defined relative to the dynamical metric) and "background" (i.e., defined relative to the background extremal Reissner–Nordström metric $\bar{r}$) multipliers to minimize the number of error terms. The expansion (1.19) is used crucially to estimate nonlinear errors in the near-horizon region. Once (1.21) and (1.22) have been proved, decay is inferred by a standard application of the pigeonhole principle as in [DR10].

The energy hierarchies are defined in Section 6 and energy decay is proved in Section 7.

[3] In fact, if we only took p up to 2 in our master hierarchy, we would not even have decay for the $p=2$ flux! Note again (recall Remark 1.4) that in the subextremal case, we would have $\tau^{-3+\delta}$ decay for $\kappa-1$ if we take p up to $3-\delta$, and τ^{-2} decay (which is still integrable) if we only take p up to 2.

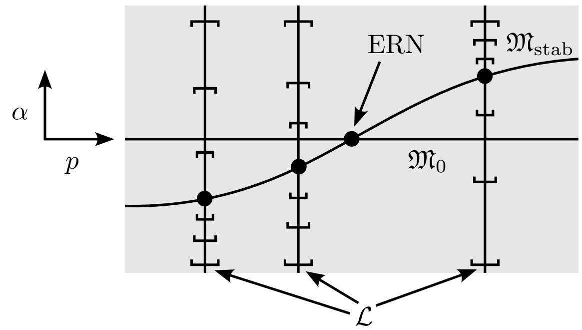

Figure 4: A schematic depiction of our modulation scheme. Let $p \mapsto \mathring{\phi}(p)$ be a one-parameter family of characteristic initial data for the scalar field with $\mathring{\phi}(0) = 0$. We can then consider the plane in $\mathfrak{M}$ parametrized by (p, α), which is what has been depicted. Each p generates a line segment $\mathcal{L}$ in $\mathfrak{M}$ which intersects the "submanifold" $\mathfrak{M}_{\rm stab}$ at least once. On the three $\mathcal{L}$'s depicted here, we have also drawn three of the nested modulation sets $\mathfrak{A}_i$ which converge to $\mathcal{L} \cap \mathfrak{M}_{\rm stab}$. Note that we have drawn $\mathfrak{M}_{\rm stab}$ as a smooth, connected curve here, which is in line with our conjectures in Section 1.3, but we do not prove any such fine structure of it in this paper.

1.2.6 The codimension-one "submanifold" $\mathfrak{M}_{\rm stab}$ and modulation on dyadic timescales

While our bootstrap argument is performed continuously in time, the choice of allowed modulation parameters α is only decided when τ_f is dyadic, i.e., a power of 2. This approach is motivated by the purely dyadic approach of Dafermos–Holzegel–Rodnianski–Taylor in [DHRT22] and turns out to be quite fortuitous compared to the continuous in time modulation theory employed in [DHRT].

In practice, we construct a sequence of nested compact intervals $\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \cdots$ such that $\mathfrak{A}_i$ consists of those α's which we consider for bootstrap time $\tau_f \in [2^i, 2^{i+1})$. These sets are defined implicitly by the requirement that the renormalized Hawking mass ϖ at the point $\Gamma(2^i)$ must lie within $\varepsilon^{3/2}2^{(-3+\delta)i}$ of M for $\alpha \in \mathfrak{A}_i$. As $\tau_f \to \infty$ (and, consequently, $i \to \infty$), we extract (at least one) critical parameter $\alpha_\star$ for which the solution tends to extremality. The stable "submanifold" $\mathfrak{M}_{\rm stab} \subset \mathfrak{M}$ is the collection of all seed data sets of the form $(\mathring{\phi}, \Lambda, M_0 + \alpha_\star, e)$ and is codimension-one in the sense that every "line" of constant $\mathring{\phi}$, Λ, and e intersects it at least once.[4] Refer to Fig. 4 above.

Since we construct $\mathfrak{A}_i$ only after each dyadic time, we may always assume a $\tau^{-3+\delta}$ decay rate for the difference $\varpi - M$, which is sharp. This should be contrasted with [DHRT], where the assumption on angular momentum in the modulation set has a decay rate of τ_f^{-1}, but the *improvement* (i.e., the change in angular momentum along $\mathcal{I}^+$) is required to decay like τ_f^{-2}, which is a significant technical issue. In our work, both the assumption and the improvement have the sharp decay rate $\tau_f^{-3+\delta}$ and the gain is purely in the power of ε ($\varepsilon^{3/2}$ vs. ε^2); see already (8.1). This strong decay rate for $\varpi - M$ is important, in particular because of the absence of redshift in the geometric estimates.

We have depicted our modulation scheme in Fig. 4 above. The modulation sets $\mathfrak{A}_i$ are defined in Section 4.1 and the modulation argument is performed in Section 8.1.2. Modulation on dyadic timescales, motivated by [DHRT22], has also been performed by Kádár in [Kád24a; Kád24b].

1.2.7 Construction of the eschatological gauge and background solution

Recall from Section 1.2.3 that the teleologically normalized coordinate u and the background extremal Reissner–Nordström solution $\bar{r}$ depend on the bootstrap time τ_f. Therefore, an important final step when taking $\tau_f \to \infty$ is to prove that we can define a unique "final background solution" which we converge to and an "eschatological gauge" (i.e., final teleological gauge) in which we converge to it. The regularity of the limiting gauge is actually related to the decay assumptions on initial data,[5] and the best that can be

[4] In [DHRT], the codimension-three "submanifold" is characterized in a similar manner via three-planes in the moduli space of seed data sets.

[5] A similar phenomenon occurs in [DHRT].

hoped for given only the finiteness of (1.15) is that the final u coordinate is a C^1 function of the initial data coordinate $\hat{u}$. These limiting arguments are carried out in Sections 8.2 and 8.3.

1.2.8 Absence of trapped surfaces and rigidity of the apparent horizon

After we have constructed the dynamical extremal spacetimes, it remains to prove part (iv) of Theorem I. This argument is originally due to Kommemi (in unpublished work) and a very similar argument appears in [LO19a, Appendix A]. The first aspect of the argument is a proof that any point on the outermost apparent horizon $\mathcal{A}'$ must lie on $\mathcal{H}^+$ because of simple monotonicities to the right of $\mathcal{A}'$ and the fact that r and ϖ both asymptote to M along $\mathcal{H}^+$. The second part involves Taylor expanding $\partial_v r$ in u along $\mathcal{H}^+$ to show that there are no trapped or marginally trapped spheres behind $\mathcal{H}^+$. This second part of the argument is reminiscent of the proof of "Israel's observation" (Proposition 1.1) in [KU22]. We carry out these arguments in Section 8.3.5.

Remark 1.6. These soft arguments, using monotonicity and the wave equation for r, do not directly carry over to the charged scalar field [Kom13] or charged Vlasov [KU24] models.

1.2.9 The Aretakis instability on the dynamical geometry

Using the geometric estimates proved in Section 5, the exact conservation law (1.13) is replaced by the "almost conservation law"

$$\partial_v\big(Y\psi|_{\mathcal{H}^+}\big) = O\big(\varepsilon^3\tau^{-2+\delta}\big).$$

Note that the error is integrable in τ (again, thanks to going up to $p = 3-\delta$ in the hierarchy!) and much better in ε than $Y\psi$. Therefore, integrating this immediately gives (i) of Theorem II. Part (ii) is completely soft and is essentially an immediate consequence of (i). To prove part (iii), we commute the wave equation on the dynamical background with Y and show that the nonlinear error terms are decaying, which results in an estimate entirely analogous to (1.14). The instability of the geometry is obtained by directly inserting this behavior for ϕ into the Einstein equations. We do not prove growth of derivatives higher than second order because this would require proving higher order estimates for the geometry, which is not necessary for the proof of Theorem I. These arguments are carried out in Section 9.1.

1.3 The conjectural picture of the local moduli space

Theorem I makes no statement about the regularity (or even connectedness!) of the stable "submanifold" $\mathfrak{M}_{\mathrm{stab}}$, nor about behavior of solutions arising from data in $\mathfrak{M}\setminus\mathfrak{M}_{\mathrm{stab}}$. In this section, we propose conjectures addressing these issues, motivated by conjectures in [DHRT], [Daf24], and [KU24] by the second- and third-named authors of the present paper.

1.3.1 Asymptotically extremal black holes as a locally separating hypersurface between collapse and dispersion

In the following conjectures, we will write the symbol $\mathfrak{M}$ to mean a moduli space of initial data posed "in the same way" as in our Theorem I, but possibly topologized by a different norm than in Section 3.1 below. In particular, it could be that the following conjectures are only true under stronger asymptotic decay assumptions. However, we will always suppose that $\mathfrak{M}$ carries a Banach space structure, so that we have access to the notion of C^1 submanifolds of $\mathfrak{M}$.

For $\mathfrak{r}\in[0,1]$, we can consider the subset $\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}\subset\mathfrak{M}$ consisting of initial data which form a black hole with asymptotic parameter ratio $|e_\infty|/M_\infty = \mathfrak{r}$.[6] In this notation, the stable set from Theorem I is $\mathfrak{M}^1_{\mathrm{stab}}$ and $\bigcup_{\mathfrak{r}<1}\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}$ was studied by Dafermos and Rodnianski in [DR05].

Conjecture 1 (Regularity of $\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}$)**.** *For every $\mathfrak{r}\in[0,1]$, $\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}$ is a C^1 hypersurface in $\mathfrak{M}$. Moreover, if $\mathfrak{M}_{\mathrm{black}}$ denotes the subset of $\mathfrak{M}$ consisting of seed data which form a black hole in evolution, then $\mathfrak{M}_{\mathrm{black}}$ is foliated by the collection $\{\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}\}$.*

[6] In this model, the charge e is constant and can be read off from initial data, but $M_\infty \doteq \lim_{v\to\infty}\varpi|_{\mathcal{H}^+}$ is only teleologically determined.

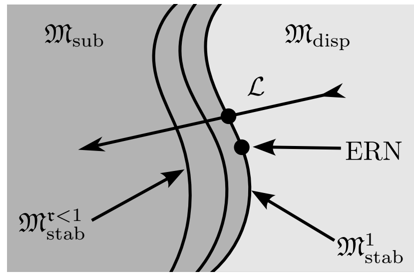

Figure 5: A cartoon depiction of the conjectured structure of a neighborhood of extremal Reissner–Nordström in the moduli space $\mathfrak{M}$ of initial data posed as in Fig. 1. We have suppressed infinitely many dimensions and emphasize the codimension-one property of the submanifolds $\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}$. We have drawn a distinguished point, which is extremal Reissner–Nordström. We have also drawn one of the lines $\mathcal{L}$ from Fig. 4 with the natural orientation given by increasing modulation parameter α. The solid point on $\mathcal{L}$ corresponds to $\alpha = \alpha_\star$ (recall Section 1.2.6). See also Fig. 6 below.

In particular, we conjecture that the stable "submanifold" from Theorem I is in fact a connected, regular hypersurface in $\mathfrak{M}$. Our next conjecture addresses what happens on "either side" of $\mathfrak{M}^{1}_{\mathrm{stab}}$; see Fig. 5 above.

To motivate the following conjecture, it is helpful to consider the Reissner–Nordström family itself for a moment. If we consider characteristic data for the spherically symmetric Einstein–Maxwell system posed on a bifurcate null hypersurface as in Fig. 1 and then vary the renormalized Hawking mass ϖ of the bifurcation sphere, the resulting developments sweep out a portion of the family of Reissner–Nordström solutions. In particular, we can observe a transition from *dispersion within the domain of dependence of the data* (when the parameters are superextremal) to *existence of a black hole* (when the parameters are extremal or subextremal). Extremal Reissner–Nordström is the critical solution in this phase transition. Note that none of these developments contain naked singularities. We conjecture that this critical behavior is preserved when we pass to the spherically symmetric Einstein–Maxwell-neutral scalar field model:

Conjecture 2 (Extremality as a stable critical phenomenon)**.** *Sufficiently close to extremal Reissner–Nordström, $\mathfrak{M} \setminus \mathfrak{M}^{1}_{\mathrm{stab}}$ has two connected components, which we denote by $\mathfrak{M}_{\mathrm{sub}}$ and $\mathfrak{M}_{\mathrm{disp}}$. These sets have the following properties:*

(i) The set $\mathfrak{M}_{\mathrm{sub}}$ consists of seed data leading to the formation of an asymptotically subextremal Reissner–Nordström black hole and is foliated by the hypersurfaces $\mathfrak{M}^{\mathfrak{r}}_{\mathrm{stab}}$ with $\mathfrak{r} < 1$.

(ii) Solutions evolving from seed data in $\mathfrak{M}_{\mathrm{disp}}$ do not form a black hole in the domain of dependence of the initial data hypersurface $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$.

In particular, the "extremal threshold" $\mathfrak{M}^{1}_{\mathrm{stab}}$ delineates the boundary between black hole formation and dispersion in the domain of dependence of $C_{\mathrm{out}} \cup \underline{C}_{\mathrm{in}}$.

This conjecture relates to Fig. 4 and our modulation scheme in the present paper as follows: for $\alpha > \alpha_\star$ (where $\alpha_\star$ is the critical parameter described in Sections 1.2.2 and 1.2.6), we expect the seed data set $(\mathring{\phi}, \Lambda, M_0 + \alpha, e)$ to lie in $\mathfrak{M}_{\mathrm{sub}}$ and if $\alpha < \alpha_\star$, we expect it to lie in $\mathfrak{M}_{\mathrm{disp}}$.

Since charge is conserved in the neutral scalar field model, it is not possible to have solutions with a nontrivial charge and a regular center. Therefore, Conjectures 1 and 2 are necessarily "local near the event horizon" because the ingoing cone $\underline{C}_{\mathrm{in}}$ must terminate at a symmetry sphere with positive area-radius. Therefore, even in the "dispersive" case, the solutions are necessarily geodesically incomplete. The second- and third-named authors of the present paper have shown that extremal Reissner–Nordström black holes can form dynamically from initial data posed on $\mathbb{R}^3$ in the Einstein–Maxwell-charged scalar field and Einstein–Maxwell–charged Vlasov models [KU22; KU24]. Therefore, Conjectures 1 and 2 can be formulated for these models, where data in $\mathfrak{M}_{\mathrm{disp}}$ now really "exist globally" and "disperse" in the literal sense. We refer to the introduction of [KU24] for more details.

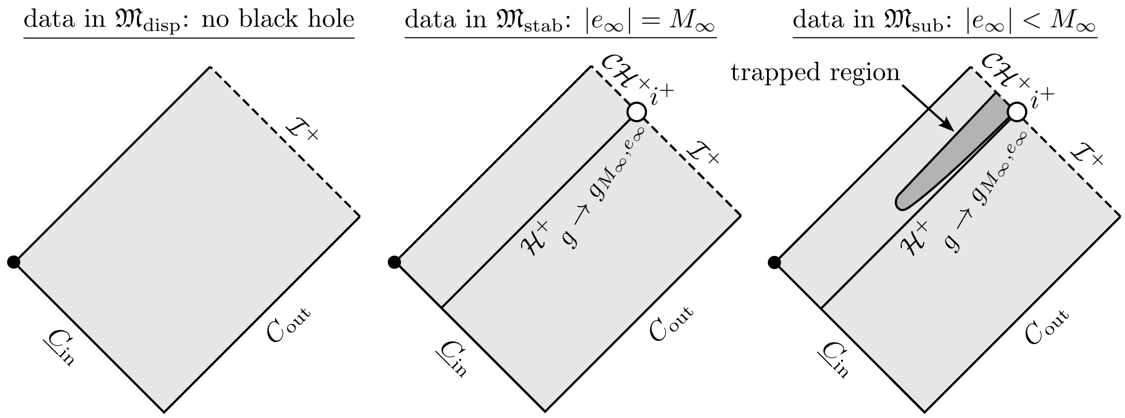

Figure 6: Penrose diagrams depicting evolutions of seed data in the sets $\mathfrak{M}_{\mathrm{disp}}$, $\mathfrak{M}^1_{\mathrm{stab}}$, and $\mathfrak{M}_{\mathrm{sub}}$. One can think of these as arising from a one-parameter family of seed data crossing the extremal threshold in Fig. 5. Spacetimes arising from $\mathfrak{M}_{\mathrm{disp}}$ have incomplete null infinity $\mathcal{I}^+$ because the ingoing cone $\underline{C}_{\mathrm{in}}$ is incomplete and no black hole has formed. As proved in Theorem I, solutions on the critical threshold $\mathfrak{M}^1_{\mathrm{stab}}$ contain no trapped surfaces, but it follows from the work of Dafermos [Daf05c] that solutions arising from $\mathfrak{M}_{\mathrm{sub}}$ have a nonempty trapped region as depicted. The explicit Reissner–Nordström family itself already displays this transition behavior, in which case the trapped region in the third Penrose diagram would intersect the initial data, and the solid point would lie "beyond the Cauchy horizon" in the maximal analytic extension of subextremal Reissner–Nordström.

1.3.2 The black hole interior at extremality

As explained in Section 1.2.8, the black hole regions of the spacetimes evolving from data in $\mathfrak{M}_{\mathrm{stab}}$ are free of trapped surfaces, see Fig. 1. In particular, this shows that the interior is bounded to the future by a smooth outgoing null hypersurface (a trivial, smooth Cauchy horizon) emanating from the final sphere of $\underline{C}_{\mathrm{in}}$ (the solid point in Fig. 1) and a potentially singular ingoing Cauchy horizon $\mathcal{CH}^+$ emanating from i^+. Analogously to the surface gravity of $\mathcal{H}^+$ defined in (1.4), one can define the surface gravity of $\mathcal{CH}^+$ in Reissner–Nordström for $0 < |e| \le M$. In the extremal case $|e| = M$, this surface gravity also vanishes, resulting in the absence of Penrose's *blueshift* instability [Pen68] and, somewhat ironically, renders the extremal Reissner–Nordström Cauchy horizon $\mathcal{CH}^+$ *more* stable than its subextremal variant [Gaj17a; Gaj17b]. Indeed, our decay estimates of Theorem I on the geometry and the scalar field along $\mathcal{H}^+$ satisfy the assumptions in [GL19, Theorem 5.1] and therefore the spacetimes evolving from $\mathfrak{M}_{\mathrm{stab}}$ can be extended in $C^{0,1/2} \cap H^1_{\mathrm{loc}}$ across $\mathcal{CH}^+$ and the Hawking mass remains uniformly bounded. This is in sharp contrast to the subextremal case, where generically the Hawking mass blows up identically along $\mathcal{CH}^+$ [Daf03; Daf05c; LOS23; Gau24] and extensions are believed to be less regular than $C^{0,1/2} \cap H^1_{\mathrm{loc}}$ [LO19a; LO19b]. In the extremal case, the existence of Cauchy data leading to any type of singularity at $\mathcal{CH}^+$ remains open—even for the linear wave equation!

Note that according to Conjecture 2, asymptotically extremal Reissner–Nordström black holes only form from the (manifestly nongeneric) "codimension 1 submanifold" $\mathfrak{M}_{\mathrm{stab}}$, so their more regular Cauchy horizons do not endanger Penrose's strong cosmic censorship conjecture.

1.4 Outlook

We conclude the introduction with some thoughts and comments about future directions and related problems. We refer the reader also to the recent essay by Dafermos [Daf24] about the stability problem for extremal black holes and the introduction of [KU24] for connections with critical collapse.

1.4.1 Observational signatures and late-time tails

In Section 9.2, we prove asymptotics for the scalar field in the near-horizon region, using the energy decay estimates from the proof of Theorem I (in particular the $(\bar{r}-M)^{2-p}$ hierarchy up to $p=3-\delta$). In Section 9.3, we then use this sharp decay estimate to show that the range of the $(\bar{r}-M)^{2-p}$ hierarchy is indeed sharp, as long as the asymptotic Aretakis charge is not too small compared to the master smallness parameter ε of the problem. Namely, we show that the non-degenerate integrated energy is *unbounded* in any infinite area close to the horizon. This is in sharp contrast to the subextremal case, where the celebrated redshift estimate [DR09] implies that this integrated energy is finite near the horizon.

We note that compared to the uncoupled case, the leading order term in the decay for ψ does not only include the asymptotic Aretakis charge $H_0[\phi]$ but also an error of order ε^3 that comes from the geometric estimates. This marks a difference with the uncoupled case and is a manifestation of the failure of the Couch–Torrence conformal isometry in the dynamical case. This creates a marked difference in the proof of the failure of the redshift estimate compared to the uncoupled case, see already Remark 9.9.

In the present paper, we only obtain sharp pointwise estimates for the scalar field near the horizon. It would be an interesting problem to obtain asymptotics for the scalar field everywhere for our asymptotically extremal black holes, depending on the initial assumptions for the behavior of the scalar field at infinity. In the uncoupled case, assuming either compactly supported initial data or sufficiently fast decay so that the so-called *Newman–Penrose constant* vanishes, one can extend the r^p hierarchy to the range of $p\in(0,5)$. After some technical work (commutation and a "time inversion" process), one can show that the radiation field decays like u^{-2} in a region close to $\mathcal{I}^+$, where the constant of the leading order decaying term depends not only on the Newman–Penrose constant but also on the Aretakis charge. This can be considered an *observational signature* for extremal Reissner–Nordström black holes and it would be interesting to extract this signature in our dynamical setting. For more details on observational signatures for extremal Reissner–Nordström black holes, see [AAG18c; AAG18c] and the numerical results of [AKS23].

1.4.2 Outside of symmetry

The present work in symmetry is the first step towards addressing the stability problem for (and, more generally, the structure of moduli space near) extremal black holes in Einstein(–Maxwell) theory outside of symmetry. We wish to consider the *extremal Kerr–Newman* family which consists of axisymmetric, rotating charged black holes with mass M, charge e, and specific angular momentum a satisfying the relation $M^2=e^2+a^2$. When $a=0$, this reduces to extremal Reissner–Nordström and when $e=0$, to extremal Kerr.

In the very slowly rotating case ($|a|\ll M$), we have the following:

Conjecture 3 (Dafermos–Holzegel–Rodnianski–Taylor [DHRT; Daf24])**.** *Conjectures 1 and 2 hold, suitably interpreted, in a neighborhood of very slowly rotating extremal Kerr–Newman, without symmetry.*

Significant progress on linear stability in the subextremal case has been made by Giorgi and Wan in [Gio22; Gio24; Gio23; GW24]). For extremal Reissner–Nordström, a complete understanding of the Teukolsky equation was recently obtained by Apetroaie [Ape22], who showed both decay statements and that a version of the Aretakis instability is present (see also the earlier [LMRT13]).[7] Unfortunately, an understanding of even the linear wave equation on very slowly rotating extremal Kerr–Newman (with $a\neq 0$) remains open. In addition to requiring a detailed understanding of linear theory around very slowly rotating extremal Kerr–Newman, we expect that resolving Conjecture 3 will also rely on a nontrivial understanding of the null structure of the Einstein equations in the near-horizon region, analogous to the nonlinear results of [Ang16; AAG18b; AAG20b] and the present work.

The case of extremal Kerr is considerably more speculative. While mode stability for the linear wave equation has been shown by Teixeira da Costa [TdC20], axisymmetric scalar perturbations have been shown to exhibit a similar non-decay and growth hierarchy as general scalar perturbations of extremal Reissner–Nordström [Are12]. Moreover, even pointwise boundedness for linear waves remains open. Recently, Gajic has shown the existence of stronger *azimuthal instabilities* (i.e., associated to $m\neq 0$) on extremal Kerr

[7] One could also consider this conjecture further restricted to the codimension-three set of data with vanishing final angular momentum (so as to only consider asymptotically Reissner–Nordström solutions, as in [DHRT]), in which case [Ape22] would become all the more relevant.

[Gaj23] (see also the earlier heuristic analysis [CGZ16]). In particular, growth is already triggered at the first order of differentiability. It remains to be seen if these instabilities are compatible with global existence and stability for nonlinear wave equations! For more discussion, we refer to [Daf24].

Acknowledgments

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2 Preliminaries

2.1 The Einstein–Maxwell-uncharged scalar field system in spherical symmetry

2.1.1 Double null gauge

Let $(\mathcal{M}, g)$ be a smooth, connected, time-oriented, four-dimensional Lorentzian manifold. We say that $(\mathcal{M}, g)$ is *spherically symmetric* if $\mathcal{M}$ splits diffeomorphically as $\mathring{\mathcal{Q}} \times S^2$ with metric

$$g = g_{\mathcal{Q}} + r^2 g_{S^2},$$

where $(\mathcal{Q}, g_{\mathcal{Q}})$ is a (1+1)-dimensional Lorentzian spacetime with boundary (corresponding to the initial data hypersurface[8]), $g_{S^2} \doteq d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the round metric on the unit sphere, and r is a nonnegative function on $\mathcal{Q}$ which can be geometrically interpreted as the area-radius of the orbits of the isometric SO(3) action on $(\mathcal{M}, g)$. We assume that $(\mathcal{Q}, g_{\mathcal{Q}})$ admits a *global double-null coordinate system* (u, v) such that the metric g takes the form

$$g = -\Omega^2\, du dv + r^2 g_{S^2} \tag{2.1}$$

for a positive function $\Omega^2 \doteq -2g_{\mathcal{Q}}(\partial_u, \partial_v)$ on $\mathcal{Q}$ and such that ∂_u and ∂_v are future-directed. The constant u and v curves are null in $(\mathcal{Q}, g_{\mathcal{Q}})$ and correspond to null hypersurfaces "upstairs" in the full spacetime $(\mathcal{M}, g)$. We will often refer interchangeably to $(\mathcal{M}, g)$ and the reduced spacetime $(\mathcal{Q}, r, \Omega^2)$.

The double null coordinates (u, v) above are not uniquely defined. For any strictly increasing smooth functions $U, V : \mathbb{R} \to \mathbb{R}$, $(\tilde{u}, \tilde{v}) = (U(u), V(v))$ defines a double null coordinate system on $\mathcal{Q}$ for which $g = -\tilde{\Omega}^2\, d\tilde{u} d\tilde{v} + \tilde{r}^2 g_{S^2}$, where $\tilde{\Omega}^2(\tilde{u}, \tilde{v}) = (U'V')^{-1}\Omega^2(U^{-1}(\tilde{u}), V^{-1}(\tilde{v}))$ and $r(\tilde{u}, \tilde{v}) = r(U^{-1}(\tilde{u}), V^{-1}(\tilde{v}))$.

Recall the *Hawking mass* $m : \mathcal{M} \to \mathbb{R}$, which is defined by the relation

$$1 - \frac{2m}{r} \doteq g(\nabla r, \nabla r)$$

and can be viewed as a function on $\mathcal{Q}$ according to

$$m = \frac{r}{2}\left(1 + \frac{4\partial_u r \partial_v r}{\Omega^2}\right). \tag{2.2}$$

This function is clearly independent of the choice of double null gauge.

We will consider spherically symmetric spacetimes equipped with spherically symmetric electromagnetic fields with constant Coulomb charge $e \in \mathbb{R}$. The field strength tensor takes the simple form

$$F = -\frac{\Omega^2 e}{2r^2} du \wedge dv \tag{2.3}$$

and the charge can be recovered from Gauss' formula

$$e = \frac{1}{4\pi}\int_{\{(u,v)\}\times S^2} \star F$$

[8] In this paper, our spacetimes do not include a center of symmetry $\Gamma \subset \{r = 0\}$.

where $\star$ is the Hodge star operator of $(\mathcal{M},g)$ defined relative to the orientation $du\wedge dv\wedge d\vartheta\wedge d\varphi$.

To see best the good structure of the spherically symmetric Einstein equations, it is very helpful to introduce some shorthand notation. We recall the *renormalized Hawking mass*

$$\varpi \doteq m+\frac{e^2}{2r}, \tag{2.4}$$

the *mass aspect function*

$$\mu \doteq \frac{2m}{r}=\frac{2\varpi}{r}-\frac{e^2}{r^2}, \tag{2.5}$$

and the traditional notation

$$\nu\doteq\partial_u r,\quad \lambda\doteq\partial_v r,\quad \kappa\doteq-\frac{\Omega^2}{4\partial_u r}=\frac{\lambda}{1-\mu},\quad \gamma\doteq-\frac{\Omega^2}{4\partial_v r}=\frac{\nu}{1-\mu}. \tag{2.6}$$

As the degenerate redshift effect of the event horizon of extremal Reissner–Nordström will play an important role in this paper, we define the *redshift factor*

$$\varkappa\doteq\frac{1}{r^2}\left(\varpi-\frac{e^2}{r}\right). \tag{2.7}$$

The connection of this *function* $\varkappa$ with the *constant* $\varkappa$ in (1.4) will be explained in Remark 2.3 below.

2.1.2 The system of equations

Definition 2.1. The *Einstein–Maxwell-neutral scalar field system* consists of a spacetime $(\mathcal{M},g)$ equipped with an electromagnetic field F and a scalar field $\phi:\mathcal{M}\to\mathbb{R}$ satisfying the equations

$$\operatorname{Ric}(g)-\tfrac12 R(g)g=2(T^{\mathrm{EM}}+T^{\mathrm{SF}}), \tag{2.8}$$

$$dF=0,\quad d\star F=0, \tag{2.9}$$

$$\Box_g\phi=0, \tag{2.10}$$

where the energy-momentum tensors are defined by

$$\begin{aligned} T^{\mathrm{EM}}_{\mu\nu}&\doteq F_{\mu\alpha}F^{\alpha}{}_{\nu}-\tfrac14 g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},\\ T^{\mathrm{SF}}_{\mu\nu}&\doteq \partial_\mu\phi\partial_\nu\phi-\tfrac12 g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi. \end{aligned} \tag{2.11}$$

We say that $(\mathcal{M},g,F,\phi)$ is *spherically symmetric* if $(\mathcal{M},g)$ is spherically symmetric as defined in Section 2.1.1, F has the form (2.3), and ϕ is independent of the angular coordinates ϑ and φ. In this case, Einstein's equation (2.8) reduces to the wave equations

$$\partial_u\partial_v r=-\frac{\Omega^2}{4r}-\frac{\partial_u r\partial_v r}{r}+\frac{\Omega^2e^2}{4r^3}, \tag{2.12}$$

$$\partial_u\partial_v\log\Omega^2=\frac{\Omega^2}{2r^2}+\frac{2\partial_u r\partial_v r}{r^2}-\frac{\Omega^2e^2}{r^4}-2\partial_u\phi\partial_v\phi, \tag{2.13}$$

and Raychaudhuri's equations

$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right)=-\frac{r}{\Omega^2}(\partial_u\phi)^2, \tag{2.14}$$

$$\partial_v\left(\frac{\partial_v r}{\Omega^2}\right)=-\frac{r}{\Omega^2}(\partial_v\phi)^2. \tag{2.15}$$

The Maxwell equations (2.9) are automatically satisfied since e is constant. Finally, the wave equation (2.10) is equivalent to

$$\partial_u\partial_v\phi=-\frac{\partial_v r\partial_u\phi}{r}-\frac{\partial_u r\partial_v\phi}{r}. \tag{2.16}$$

We will not actually work with the equations (2.12)–(2.16) as written here and will instead use the quantities (2.4)–(2.7) which satisfy more helpful equations. First, the wave equation (2.12) can be written in the compact form

$$\partial_u\lambda = \partial_v\nu = 2\kappa\nu\varkappa, \tag{2.17}$$

which allows us to write (2.16) as an inhomogeneous wave equation for the *radiation field* $\psi \doteq r\phi$,

$$\partial_u\partial_v\psi = (\partial_u\partial_v r)\phi = 2\kappa\nu\varkappa\phi. \tag{2.18}$$

Using (2.12)–(2.15), we derive the fundamental relations

$$\partial_u\varpi = (1-\mu)\frac{r^2}{2\nu}(\partial_u\phi)^2, \tag{2.19}$$

$$\partial_v\varpi = \frac{r^2}{2\kappa}(\partial_v\phi)^2, \tag{2.20}$$

$$\partial_u\kappa = \frac{r\kappa}{\nu}(\partial_u\phi)^2, \tag{2.21}$$

$$\partial_v\gamma = \frac{r\gamma}{\lambda}(\partial_v\phi)^2. \tag{2.22}$$

Finally, we note that for any ϕ satisfying the wave equation (2.16) and any C^1 vector field $X = X^u\partial_u + X^v\partial_v$ and any C^2 function $h = h(r)$, we have the multiplier identities

$$\partial_v(r^2(\partial_u\phi)^2X^u) + \partial_u(r^2(\partial_v\phi)^2X^v) = r^2\partial_v X^u(\partial_u\phi)^2 + r^2\partial_u X^v(\partial_v\phi)^2 - 2r\left(\nu X^u + \lambda X^v\right)\partial_u\phi\partial_v\phi \tag{2.23}$$

and

$$\partial_u\partial_v(hr\phi^2) - \partial_u(rh'\lambda\phi^2) - \partial_v(rh'\nu\phi^2) + (r\lambda\nu h'' + 2r\kappa\nu\varkappa h' - 2\kappa\varkappa\nu h)\phi^2 - 2hr\partial_u\phi\partial_v\phi = 0. \tag{2.24}$$

2.2 The geometry of Reissner–Nordström

In this section, we briefly review some important geometric aspects of Reissner–Nordström black holes spacetimes. See Fig. 7 below for the Penrose diagram of extremal Reissner–Nordström. Let $M > 0$ and $e \in \mathbb{R}$. Then the Reissner–Nordström metric is written in Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ as

$$g_{M,e} = -D(r)dt^2 + D(r)^{-1}dr^2 + r^2 g_{S^2},$$

where $D(r) \doteq 1 - \frac{2M}{r} + \frac{e^2}{r^2}$. When $|e| \le M$ and $g_{M,e}$ analytically extends to describe a black hole spacetime, these coordinates cover the domain of outer communication of the black hole for $(t, r) \in \mathbb{R} \times (r_+, \infty)$, where $r_+ \doteq M + \sqrt{M^2 - e^2}$. In order to cover the event horizon $\mathcal{H}^+$ located at $r = r_+$, we introduce the *tortoise coordinate*

$$r_*(r) \doteq \int^r \frac{dr'}{D(r')},$$

where we have left the integration constant ambiguous. In the extremal case $|e| = M$, we have

$$r_*(r) = r - M - \frac{M^2}{r-M} + 2M\log(r-M) + \text{constant}, \tag{2.25}$$

which will be used in Section 9.2 below. By defining the advanced time coordinate $v \doteq \frac{1}{2}(t + r_*)$, we bring $g_{M,e}$ into the *ingoing Eddington–Finkelstein* form

$$g_{M,e} = -4Ddv^2 + 4dvdr + r^2 g_{S^2},$$

which is regular across $\mathcal{H}^+$. The vector field $T \doteq \partial_v$ is the time-translation Killing vector field (equal to ∂_t in Schwarzschild coordinates in the domain of outer communication). In these coordinates, the vector field $Y \doteq \partial_r$ is past-directed null and is transverse to $\mathcal{H}^+$. It is also clearly translation-invariant in the sense that

$$\mathcal{L}_T Y = [\partial_v, \partial_r] = 0.$$

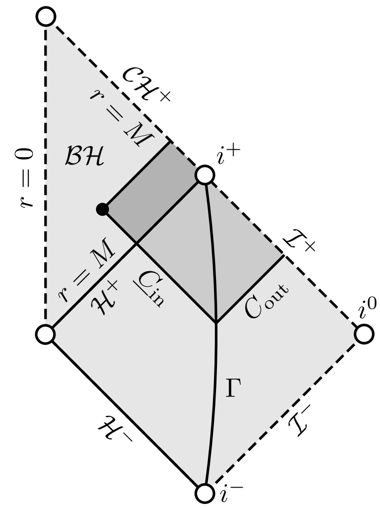

Figure 7: A Penrose diagram of (one period of) the maximally extended extremal Reissner–Nordström solution. The union of the two darker shaded regions is the domain of dependence of the bifurcate null hypersurface $C_{\text{out}} \cup \underline{C}_{\text{in}}$ and represents the solution we are perturbing around in Theorem I. We prove stability of the medium gray colored region.

In order to put the Reissner–Nordström metric in double null gauge, we define the retarded time coordinate $u \doteq \frac{1}{2}(t - r_*)$, so that the metric takes the *Eddington–Finkelstein double null* form

$$g_{M,e} = -4D\, dudv + r^2 g_{S^2}.$$

The area-radius r is now an implicit function of u and v. These coordinates once again only cover the domain of outer communication if $|e| \le M$. The event horizon $\mathcal{H}^+$ formally corresponds to $u = +\infty$ and null infinity $\mathcal{I}^+$ formally corresponds to $v = +\infty$. From the identity $r_* = v - u$, we infer that in these coordinates, $\partial_u r = -D$ and $\partial_v r = D$. Since clearly $\Omega^2 = 4D$ and $D = 1 - \mu$, this implies that

$$\gamma = -1 \quad \text{and} \quad \kappa = 1$$

in these coordinates. Since $\partial_u r = -D$, we have that

$$Y = (\partial_u r)^{-1} \partial_u.$$

Finally, by simply changing the origin of the (u, v) coordinates, we prove:

Lemma 2.2. *Let $M > 0$ and $|e| \le M$ be Reissner–Nordström black hole parameters. Then for any $R_0 \in (r_+, \infty)$ and $(u_0, v_0) \in \mathbb{R}^2$, there exists a unique function $\bar{r} : \mathbb{R}^2 \to (r_+, \infty)$ such that*

$$\bar{g} \doteq -4D(\bar{r})\, dudv + \bar{r}^2(u, v) g_{S^2} \tag{2.26}$$

on $\mathbb{R}^2 \times S^2$ is isometric to the Reissner–Nordström black hole exterior with parameters (M, e) and such that

$$\bar{\varpi} = M, \quad \partial_u \bar{r} = -(1 - \bar{\mu}), \quad \partial_v \bar{r} = 1 - \bar{\mu}, \quad \text{and} \quad \bar{r}(u_0, v_0) = R_0.$$

As mentioned above, the Eddington–Finkelstein double null coordinates defined in this section do not cover the event horizon, but it can be formally attached as the null hypersurface $u = +\infty$. With this understanding, we may extend geometric quantities associated to the metric (2.26) to the horizon by setting

$$\bar{r}(\infty, \cdot) = M, \quad \bar{\lambda}(\infty, \cdot) = 0, \quad \bar{\mu}(\infty, \cdot) = 1, \quad \bar{\varkappa}(\infty, \cdot) = 0.$$

Remark 2.3. In exact Reissner–Nordström, we may write the time-translation Killing vector field relative to a generic double null coordinate system as the *Kodama vector field*

$$T = \frac{2\lambda}{\Omega^2}\partial_u - \frac{2\nu}{\Omega^2}\partial_v, \tag{2.27}$$

as long as u and v are chosen so that T is future-directed timelike for $r > r_+$. One can then compute

$$\nabla_T T = -\varkappa T^u \partial_u + \varkappa T^v \partial_v,$$

where $\varkappa$ is the redshift factor defined in (2.7). Therefore, on the event horizon $\mathcal{H}^+$, where $T^u = 0$,

$$\nabla_T T|_{\mathcal{H}^+} = \varkappa T|_{\mathcal{H}^+}.$$

Compare this identity with (1.4).

2.3 The characteristic initial value problem

2.3.1 Characteristic data and existence in thin slabs

Given $U_0 < U_1$ and $V_0 < V_1$, let

$$\mathcal{C}(U_0, U_1, V_0, V_1) \doteq (\{U_0\} \times [V_0, V_1]) \cup ([U_0, U_1] \times \{V_0\}),$$
$$\mathcal{R}(U_0, U_1, V_0, V_1) \doteq [U_0, U_1] \times [V_0, V_1].$$

We will omit the decoration (U_0, U_1, V_0, V_1) from $\mathcal{C}$ and $\mathcal{R}$ when the implied meaning is clear. A continuous function $f : \mathcal{C} \to \mathbb{R}$ is said to be *smooth* if $f_{\text{out}} \doteq f|_{\{U_0\}\times[V_0,V_1]}$ and $f_{\text{in}} \doteq f|_{[U_0,U_1]\times\{V_0\}}$ are C^∞ functions.

Definition 2.4. A smooth *(bifurcate) characteristic initial data set* for the Einstein–Maxwell-scalar field system consists of a triple of smooth functions $\mathring{r}, \mathring{\Omega}^2, \mathring{\phi} : \mathcal{C} \to \mathbb{R}$ with $\mathring{r}$ and $\mathring{\Omega}^2$ positive, together with a real number e. The functions $\mathring{r}, \mathring{\Omega}^2$, and $\mathring{\phi}$ are assumed to satisfy (2.14) on $[U_0, U_1] \times \{V_0\}$ and (2.15) on $\{U_0\} \times [V_0, V_1]$.

Proposition 2.5. *For any $U_0 < U_1$, $V_0 < V_1$ and $B > 0$, there exists a constant $\varepsilon_{\text{slab}}$ with the following property. Let $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ be a characteristic initial data set of the Einstein–Maxwell-scalar field system on $\mathcal{C}(U_0, U_1, V_0, V_1)$ satisfying*

$$\|\log \mathring{r}\|_{C^1(\mathcal{C})} + \|\log \mathring{\Omega}^2\|_{C^1(\mathcal{C})} + \|\mathring{\phi}\|_{C^1(\mathcal{C})} + |e| \le B.$$

Then there exists a unique, smooth solution (r, Ω^2, ϕ, e) to the spherically symmetric Einstein–Maxwell-scalar field system on the slab

$$\mathcal{D} \doteq \mathcal{R}(U_0, U_0 + \min\{\varepsilon_{\text{slab}}, U_1 - U_0\}, V_0, V_1) \cup \mathcal{R}(U_0, U_1, V_0, V_0 + \min\{\varepsilon_{\text{slab}}, V_1 - V_0\})$$

which extends the initial data. Moreover, the norms $\|\log r\|_{C^k(\mathcal{D})}$, $\|\log \Omega^2\|_{C^k(\mathcal{D})}$, and $\|\phi\|_{C^k(\mathcal{D})}$ are bounded in terms of appropriate higher order initial data norms.

Proof. Local existence for the characteristic initial value problem in small rectangles is obtained by a standard iteration argument, see for instance [KU24, Appendix A]. In order to extend the region of existence to a thin slab, one exploits the null structure exhibited by the system (2.12)–(2.16): the equations can be viewed as linear ODEs in u for the v-derivatives of the unknowns, and vice-versa. See [Luk12] for an elaboration of this principle in a much more complicated setting. Alternatively, one can use the "generalized extension principle" for this model (see [Kom13]) and argue as in [KU24, Proposition 3.17]. □

We also have a natural notion of maximal globally hyperbolic development [CG69; Sbi16] in spherical symmetry, which can be directly realized as a subset of the domain of dependence of $\mathcal{C}$ viewed as a subset of $(1 + 1)$-dimensional Minkowski space.

Proposition 2.6. *Let $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ be a characteristic initial data set of the Einstein–Maxwell-scalar field system on $\mathcal{C}(U_0, U_1, V_0, V_1)$, where U_1 and V_1 are allowed to take the value $+\infty$. Then there exists a set $\mathcal{Q}_{\max} \subset \mathcal{R}(U_0, U_1, V_0, V_1)$ with the following properties:*

1. *$\mathcal{Q}_{\max}$ is globally hyperbolic as a subset of $\mathbb{R}^{1+1}_{u,v}$ with Cauchy surface $\mathcal{C}$.*
2. *The solution (r, Ω^2, ϕ, e) extends uniquely to $\mathcal{Q}_{\max}$.*
3. *$\mathcal{Q}_{\max}$ is maximal with respect to properties 1. and 2.*

For a characterization of the future boundary of $\mathcal{Q}_{\max}$, see [Kom13].

2.3.2 Gauge-normalized seed data

In defining the moduli space of initial data $\mathfrak{M}$ for our main theorem, it will be convenient to parametrize the space of bifurcate characteristic initial data as a linear space in such a way that certain gauge conditions are automatically satisfied on $\mathcal{C}$ in the maximal development. Note that this notion of seed data is distinct from the one used in [KU22] because the gauge condition is different.

Definition 2.7. Let $\mathcal{C}$ be a spherically symmetric bifurcate null hypersurface. A *seed data set* is a quadruple

$$\mathcal{S} \doteq (\mathring{\phi}, r_0, \varpi_0, e),$$

where $\mathring{\phi}$ is a smooth function on $\mathcal{C}$, r_0 is a positive real number, and ϖ_0, e are real numbers.

Proposition 2.8. *Let $\mathcal{S} = (\mathring{\phi}, r_0, \varpi_0, e)$ be a seed data set on $\mathcal{C}(U_0, U_1, V_0, V_1)$ with $U_1 - U_0 < r_0$. Then there exists a unique characteristic initial data set $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ on $\mathcal{C}(U_0, U_1, V_0, V_1)$ such that the maximal development $(\mathcal{Q}_{\max}, r, \Omega^2, \phi, e)$ of $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$ has the following properties:*

1. *$r(U_0, V_0) = r_0$,*
2. *$\varpi(U_0, V_0) = \varpi_0$,*
3. *$\nu = -1$ on $[U_0, U_1] \times \{V_0\}$, and*
4. *$\lambda = 1$ on $\{U_0\} \times [V_0, V_1]$.*

We refer to characteristic data obtained from $\mathcal{S}$ in this manner as *gauge-normalized* characteristic data determined by $\mathcal{S}$.

Proof. For $(u, v) \in [U_0, U_1] \times [V_0, V_1]$, set

$$\mathring{r}_{\mathrm{in}}(u) \doteq r_0 - u, \quad \mathring{r}_{\mathrm{out}}(v) \doteq r_0 + v,$$

and then

$$\mathring{\Omega}^2_{\mathrm{in}}(u) \doteq 4 \exp\left(-\int_{U_0}^{u} \mathring{r}_{\mathrm{in}} (\partial_u \mathring{\phi}_{\mathrm{in}})^2 \, du'\right), \quad \mathring{\Omega}^2_{\mathrm{out}}(v) \doteq 4 \exp\left(\int_{V_0}^{v} \mathring{r}_{\mathrm{out}} (\partial_v \mathring{\phi}_{\mathrm{out}})^2 \, dv'\right).$$

Assembling these functions into a characteristic data set $(\mathring{r}, \mathring{\Omega}^2, \mathring{\phi}, e)$, we can immediately see that conditions 1.–4. will be satisfied on the maximal development by virtue of the definitions. □

3 Stability and instability of extremal Reissner–Nordström: setup and statements of the main theorems

In this section, we give the detailed statements of our main results as well as the precise definitions needed to make the statements. In Section 3.1, we define the moduli space of seed data $\mathfrak{M}$ that features in Theorem I. In Section 3.2, we define the bootstrap domains $\mathcal{D}_{\tau_f}$ and the associated teleologically normalized gauges. In Section 3.3, we define the anchored background extremal Reissner–Nordström spacetimes on $\mathcal{D}_{\tau_f}$ and the associated energy hierarchies. Finally, in Section 3.4, we give the precise statements of Theorems I and II.

3.1 Definition of the moduli space of seed data $\mathfrak{M}$

Fix a mass parameter $M_0 > 0$ and let e_0 be a charge parameter satisfying $|e_0| = M_0$. Let

$$U_* \doteq \frac{995}{10} M_0$$

and let

$$\hat{\mathcal{C}} \doteq \mathcal{C}(0, U_*, 0, \infty) = \underline{C}_{\mathrm{in}} \cup C_{\mathrm{out}}$$

denote the bifurcate null hypersurface on which we pose our data. We denote the null coordinates on $\hat{\mathcal{C}}$ by $\hat{u}$ and $\hat{v}$. Later, in the proof of the main theorem, $\hat{u}$ and $\hat{v}$ will be renormalized.

Let $\mathcal{S} = (\mathring{\phi}, \Lambda, \varpi_0, e)$ denote a seed data set on $\hat{\mathcal{C}}$ with bifurcation sphere area-radius r_0 denoted by Λ. We define the *seed data norm*

$$\mathfrak{D}[\mathcal{S}] \doteq |\Lambda - 100M_0| + |\varpi_0 - M_0| + |e - e_0| + \sup_{\underline{C}_{\mathrm{in}}} \left(|\mathring{\phi}_{\mathrm{in}}| + |\partial_{\hat{u}} \mathring{\phi}_{\mathrm{in}}| \right) + \sup_{C_{\mathrm{out}}} \left((1+\hat{v})|\mathring{\phi}_{\mathrm{out}}| + (1+\hat{v})^2 |\partial_{\hat{v}} \mathring{\phi}_{\mathrm{out}}| + (1+\hat{v})^2 |\partial_{\hat{v}}(\hat{v}\mathring{\phi}_{\mathrm{out}})| \right).$$

The seed data norm $\mathfrak{D}$ and many of the constructions and smallness parameters in this paper will implicitly depend on the (fixed) choice of M_0.

Remark 3.1. The unique seed data with $\mathfrak{D} = 0$, namely $(0, 100M_0, M_0, e_0)$, corresponds to extremal Reissner–Nordström with parameters (M_0, e_0) and bifurcation sphere area-radius $100M_0$. Note that since Λ is not fixed to be $100M_0$, there is a one-parameter family of seed data corresponding to the extremal Reissner–Nordström black hole with parameters (M_0, e_0), but all except for the one with $\Lambda = 100M_0$ will have $\mathfrak{D} > 0$. The event horizon for the evolution of $(0, 100M_0, M_0, e_0)$ is located at

$$\hat{u}_{\mathcal{H}^+,0} \doteq 99M_0 \tag{3.1}$$

in the $(\hat{u}, \hat{v})$ gauge determined by Proposition 2.8.

We now show that for $\mathfrak{D}[\mathcal{S}]$ sufficiently small, the associated characteristic data have "no antitrapped spheres of symmetry."

Lemma 3.2. *There exists an $\varepsilon_{\mathrm{loc}} > 0$ depending only on M_0 such that if $\mathcal{S}$ is a seed data set for which*

$$\mathfrak{D}[\mathcal{S}] \le 3\varepsilon_{\mathrm{loc}}, \tag{3.2}$$

then the maximal globally hyperbolic development $(\hat{\mathcal{Q}}_{\max}, r, \Omega^2, \phi, e)$ of $\mathcal{S}$ has $\partial_{\hat{u}} r < 0$ everywhere on $\hat{\mathcal{Q}}_{\max}$.

Remark 3.3. The role of the number 3 in (3.2) is that every seed data set in the moduli space $\mathfrak{M}$, defined below in (3.4), will satisfy (3.2).

Proof of Lemma 3.2. Let $\varepsilon > 0$ and suppose $\mathfrak{D}[\mathcal{S}] \le \varepsilon$. We will show that $\hat{\nu} < 0$ on C_{out} for ε sufficiently small, which is then propagated to the rest of $\hat{\mathcal{Q}}_{\max}$ by the inequality $\partial_{\hat{u}}(\hat{\Omega}^{-2}\hat{\nu}) \le 0$ obtained from Raychaudhuri's equation (2.14). We make the bootstrap assumption

$$\tfrac{1}{2} \le 1 - \mu \le 2 \tag{3.3}$$

on $\{0\} \times [0, \hat{v}_0]$, which is clearly satisfied for ε and $\hat{v}_0$ sufficiently small. Since $\hat{\lambda} = 1$ on C_{out}, $r(0, \hat{v}) = \Lambda + \hat{v}$ and

$$|\partial_{\hat{v}} \varpi(0, \hat{v})| = \left| \tfrac{1}{2}(1-\mu)(0,\hat{v})(\Lambda + \hat{v})^2 (\partial_{\hat{v}} \mathring{\phi}_{\mathrm{out}})^2 \right| \lesssim \varepsilon^2 (1+\hat{v})^{-2}$$

by (2.20) and (3.3). Therefore, $|\varpi(0, \hat{v}) - M_0| \lesssim \varepsilon^2$ for every $\hat{v} \in [0, \hat{v}_0]$, which easily allows us to propagate (3.3) and therefore (3.3) holds on C_{out}. It now follows easily from the relation $\hat{\Omega}^2 = -4\hat{\lambda}\hat{\nu}/(1-\mu)$ that $\hat{\nu} < 0$ on C_{out}. $\square$

We denote by $\mathcal{S}_0$ seed data sets on $\hat{\mathcal{C}}$ for which $\varpi_0 = M_0$. These constitute a codimension-one affine subspace, denoted by $\mathfrak{M}_0$, of the vector space of seed data. Given such an $\mathcal{S}_0 = (\mathring{\phi}, \Lambda, M_0, e) \in \mathfrak{M}_0$ and $\alpha \in \mathbb{R}$, we define the *modulated* seed data set

$$\mathcal{S}_0(\alpha) \doteq (\mathring{\phi}, \Lambda, M_0 + \alpha, e).$$

For $\varepsilon > 0$, we then define

$$\mathcal{L}(\mathcal{S}_0, \varepsilon) \doteq \{\mathcal{S}_0(\alpha) : \alpha \in [-2\varepsilon, 2\varepsilon]\},$$

which is a line segment in the vector space of seed data.

Definition 3.4. Let $M_0 > 0$. The *moduli space of seed data* centered on mass M_0 is the set

$$\mathfrak{M} \doteq \bigcup_{\mathcal{S}_0 \in \mathfrak{M}_0 : \mathfrak{D}[\mathcal{S}_0] \le \varepsilon_{\mathrm{loc}}} \mathcal{L}(\mathcal{S}_0, \varepsilon_{\mathrm{loc}}), \tag{3.4}$$

where $\varepsilon_{\mathrm{loc}}$ is the small parameter from Lemma 3.2. For $0 < \varepsilon \le \varepsilon_{\mathrm{loc}}$, we define the moduli space with *smallness parameter* ε by

$$\mathfrak{M}(\varepsilon) \doteq \bigcup_{\mathcal{S}_0 \in \mathfrak{M}_0 : \mathfrak{D}[\mathcal{S}_0] \le \varepsilon} \mathcal{L}(\mathcal{S}_0, \varepsilon). \tag{3.5}$$

We endow $\mathfrak{M}$ and $\mathfrak{M}(\varepsilon)$ with the metric space topology associated to the norm $\mathfrak{D}$.

3.2 The geometric setup for the statement of stability

In this section, we explain several geometric constructions that will be required to state the precise version of our main theorem below and will be crucial to the proof. In order to explain these constructions, we will be required to make several assumptions, which we will later formalize as bootstrap assumptions in Section 4.1 below. Refer to Fig. 8 for a diagram explaining our gauge choice and notation explained in this section and the following.

Let $\mathcal{S} \in \mathfrak{M}(\varepsilon)$ with $0 < \varepsilon \le \varepsilon_{\mathrm{loc}}$ and denote the maximal development of $\mathcal{S}$ by $(\hat{\mathcal{Q}}_{\max}, r, \Omega^2, \phi, e)$. Since we are aiming to converge to an extremal Reissner–Nordström black hole with charge e (note that this parameter is conserved in the neutral scalar field model), we define the target mass parameter

$$M \doteq |e|.$$

We define the set

$$\Gamma \doteq \{r = \Lambda\}.$$

This is clearly a timelike curve near $\hat{\mathcal{C}}$ for ε sufficiently small and we will verify in the course of the proof of the main theorem that Γ is an intextendible timelike curve in $\hat{\mathcal{Q}}_{\max}$. Assuming for the moment that this is the case, we may parametrize Γ by its proper time τ, which we normalize to start at 1 at $\Gamma \cap \hat{\mathcal{C}}$. We write the components of Γ as $\Gamma(\tau) = (\Gamma^{\hat{u}}(\tau), \Gamma^{\hat{v}}(\tau))$ in the $(\hat{u}, \hat{v})$ coordinates on $\hat{\mathcal{Q}}_{\max}$.

For $\tau_f \in [1, \infty)$ such that $[1, \tau_f]$ lies in the domain of definition Γ, we define

$$\hat{\mathcal{D}}_{\tau_f} \doteq [0, \Gamma^{\hat{u}}(\tau_f)] \times [0, \Gamma^{\hat{v}}(\tau_f)].$$

Assuming that $\hat{\gamma} < 0$ on the final ingoing cone in $\hat{\mathcal{D}}_{\tau_f}$ and $\hat{\kappa} > 0$ on $\Gamma \cap \hat{\mathcal{D}}_{\tau_f}$, we may define strictly increasing functions $\mathfrak{u}_{\tau_f} : [0, \Gamma^{\hat{u}}(\tau_f)] \to \mathbb{R}$ and $\mathfrak{v} : [0, \Gamma^{\hat{v}}(\tau_f)] \to \mathbb{R}$ by

$$\mathfrak{u}_{\tau_f}(\hat{u}) \doteq -\int_0^{\hat{u}} \hat{\gamma}(\hat{u}', \Gamma^{\hat{v}}(\tau_f))\, d\hat{u}', \tag{3.6}$$

$$\mathfrak{v}(\hat{v}) \doteq \int_0^{\hat{v}} \hat{\kappa}(\Gamma^{\hat{u}}((\Gamma^{\hat{v}})^{-1}(\hat{v}')), \hat{v}')\, d\hat{v}', \tag{3.7}$$

which then assemble into a map

$$\begin{aligned} \Phi_{\tau_f} : \hat{\mathcal{D}}_{\tau_f} &\to \mathbb{R}^2 \\ (\hat{u}, \hat{v}) &\mapsto (\mathfrak{u}_{\tau_f}(\hat{u}), \mathfrak{v}(\hat{v})), \end{aligned} \tag{3.8}$$

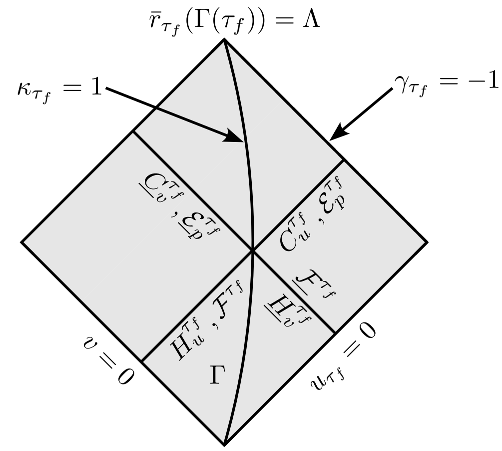

Figure 8: A Penrose diagram showing the gauge conditions, null hypersurfaces, and energies in our bootstrap domain $\mathcal{D}_{\tau_f}$. The function τ measures advanced time to the left of Γ and retarded time to the right of Γ.

which is a diffeomorphism onto its image. We denote the image of Φ_{τ_f} by $\mathcal{D}_{\tau_f}$, which comes equipped with the double null coordinates $(u_{\tau_f}, v) = \Phi_{\tau_f}(\hat{u}, \hat{v})$. Let $\hat{\Phi}_{\tau_f}$ denote the inverse of Φ_{τ_f}. In the (u_{τ_f}, v) coordinate system, the solution $(r, \hat{\Omega}^2, \phi, e)$ is given by $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$, where $r_{\tau_f} \doteq r \circ \hat{\Phi}_{\tau_f}$, $\phi_{\tau_f} \doteq \phi \circ \hat{\Phi}_{\tau_f}$, and

$$\Omega^2_{\tau_f} \doteq \frac{1}{\mathfrak{u}'_{\tau_f} \circ \mathfrak{u}^{-1}_{\tau_f}} \frac{1}{\mathfrak{v}' \circ \mathfrak{v}^{-1}} \hat{\Omega}^2 \circ \hat{\Phi}_{\tau_f}.$$

In the (u_{τ_f}, v) coordinate system, we write the coordinates of Γ as $\Gamma(\tau) = (\Gamma^{u_{\tau_f}}(\tau), \Gamma^v(\tau))$. We will frequently omit the subscript τ_f on u_{τ_f} and $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$ when it is clear that τ_f has been fixed.

By a slight abuse of notation, we define a continuous function τ on $\mathcal{D}_{\tau_f}$ implicitly by

$$\tau(u_{\tau_f}, v) \doteq \begin{cases} \tau : \Gamma^{u_{\tau_f}}(\tau) = u_{\tau_f} & \text{if } r(u_{\tau_f}, v) \geq \Lambda \\ \tau : \Gamma^v(\tau) = v & \text{if } r(u_{\tau_f}, v) < \Lambda \end{cases}. \tag{3.9}$$

The function τ measures (approximately) Bondi time in the region near null infinity and (approximately) Eddington–Finkelstein time near the event horizon. This will be made precise in Lemma 5.6 below.

We define four classes of null hypersurfaces in $\mathcal{D}_{\tau_f}$:

$$C_u^{\tau_f} \doteq (\{u\} \times [0, \Gamma^v(\tau)]) \cap \{r \geq \Lambda\}, \qquad \underline{C}_v^{\tau_f} \doteq ([0, \Gamma^{u_{\tau_f}}(\tau_f)] \times \{v\}) \cap \{r \leq \Lambda\},$$
$$H_u^{\tau_f} \doteq (\{u\} \times [0, \Gamma^v(\tau)]) \cap \{r \leq \Lambda\}, \qquad \underline{H}_v^{\tau_f} \doteq ([0, \Gamma^{u_{\tau_f}}(\tau_f)] \times \{v\}) \cap \{r \geq \Lambda\}.$$

We shall often suppress the dependence of these hypersurfaces on τ_f. It is also convenient to write

$$C^{\tau_f}(\tau) \doteq C^{\tau_f}_{\Gamma^{u_{\tau_f}}(\tau)} \quad \text{and} \quad \underline{C}^{\tau_f}(\tau) = \underline{C}^{\tau_f}_{\Gamma^v(\tau)}.$$

For the seed data sets we will ultimately consider, Γ exists and remains timelike for all $\tau \in [1, \infty)$. We then define a number

$$\hat{u}_{\mathcal{H}^+} \doteq \lim_{\tau \to \infty} \Gamma^{\hat{u}}(\tau). \tag{3.10}$$

Since $\tau \mapsto \Gamma^{\hat{u}}(\tau)$ is monotone increasing, the existence of this limit is automatic and we will show the strict inequality

$$\hat{u}_{\mathcal{H}^+} < U_*,$$

see already Lemma 8.9. We then set

$$\hat{\mathcal{D}}_\infty \doteq [0, \hat{u}_{\mathcal{H}^+}) \times [0, \infty).$$

We will also show that there exists a surjective, strictly increasing C^1 function $u_\infty : [0, \hat{u}_{\mathcal{H}^+}) \to [0, \infty)$, such that, if we use (u_∞, v) as coordinates on $\hat{\mathcal{D}}_\infty$, then $\partial_{u_\infty} r \to -1$ at null infinity $\mathcal{I}^+$. We denote $\hat{\mathcal{D}}_\infty$ by $\mathcal{D}_\infty$ under this change of coordinates.

Finally, our geometric estimates will imply that for any $R \in (M, \infty)$, the set $\Gamma_R \doteq \{r = R\}$ is a timelike curve in $\mathcal{D}_{\tau_f}$ for any $\tau_f \in [1, \infty]$ (or possibly empty). For any given v, the intersection $(\mathbb{R} \times \{v\}) \cap \Gamma_R$ is either empty or consists of a single point, which we then denote by $(u^R(v), v)$. Likewise, for any given u, the intersection $(\{u\} \times \mathbb{R}) \cap \Gamma_R$ is either empty or consists of a single point, which we then denote by $(u, v^R(u))$.

3.3 Anchored extremal Reissner–Nordström solutions and definitions of the energies

Let (r, Ω^2, ϕ, e) be a spherically symmetric solution of the Einstein–Maxwell-scalar field system defined on a coordinate rectangle $\mathcal{D}_{\tau_f}$ with gauge conditions as explained in Section 3.2.

First, assume $\tau_f < \infty$. We define the *τ_f-anchored background solution* to be the extremal Reissner–Nordström metric $(\bar{r}_{\tau_f}, \bar{\Omega}^2_{\tau_f})$ with parameters $M = |e|$ in Eddington–Finkelstein double null form (2.26) which is uniquely determined by Lemma 2.2 according to

$$\bar{r}_{\tau_f}(\Gamma(\tau_f)) = \Lambda.$$

Given the anchored background solution we now adopt the following notation:

- Barred quantities such as $\bar{\lambda}_{\tau_f}, \bar{\varpi}_{\tau_f} = M$, or $\bar{\kappa}_{\tau_f} = 1$ correspond to those of $(\bar{r}_{\tau_f}, \bar{\Omega}^2_{\tau_f})$.
- Differences are denoted with a tilde, such as $\tilde{r}_{\tau_f} \doteq r_{\tau_f} - \bar{r}_{\tau_f}$, $\tilde{\varpi}_{\tau_f} \doteq \varpi_{\tau_f} - M$, or $\tilde{\gamma}_{\tau_f} = \gamma_{\tau_f} + 1$.

In the proof of Theorem I, we will send $\tau_f \to \infty$ and thus need to extend this definition to the case when $\tau_f = \infty$. Instead of trying to anchor directly "at $\tau_f = \infty$," it is much easier to simply anchor the background solution at $\Gamma \cap \hat{\mathcal{C}} = \{(0, 0)\}$. Therefore, we define the *∞-anchored background solution* to be the unique extremal Reissner–Nordström metric $(\bar{r}_\infty, \bar{\Omega}^2_\infty)$ with parameters $M = |e|$ in Eddington–Finkelstein double null form (2.26) with the property that

$$\bar{r}_\infty(0, 0) = \bar{r}_\star \doteq \lim_{\tau_f \to \infty} \bar{r}_{\tau_f}(0, 0),$$

see already Fig. 9. We will show below in Proposition 8.13 that this limit actually exists. We will also show that this background solution is anchored "at $\tau_f = \infty$" in the sense that

$$\lim_{\tau_f \to \infty} \bar{r}_\infty(\Gamma(\tau_f)) = \Lambda,$$

see already (8.17).

We may now define the fundamental weighted energy norms for the scalar field. Some of the norms will depend explicitly on the background solution $\bar{r}_{\tau_f}$ in a nontrivial manner.

Definition 3.5. Let $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$ be defined on $\mathcal{D}_{\tau_f}$ with teleologically normalized coordinates (u_{τ_f}, v), where $\tau_f \in [1, \infty]$. Let $\bar{r}_{\tau_f}$ be the associated τ_f-anchored background solution. Let

$$\psi_{\tau_f} \doteq r_{\tau_f} \phi_{\tau_f}$$

denote the *radiation field* of ϕ_{τ_f}. For $\tau, \tau' \in [1, \tau_f]$, $p \in [0, 3)$, and $(u, v) \in \mathcal{D}_{\tau_f}$, we define:

1. The $(\bar{r} - M)^{2-p}$-weighted flux to the horizon:

$$\underline{\mathcal{E}}_p^{\tau_f}(\tau) \doteq \begin{cases} \displaystyle\int_{\underline{C}^{\tau_f}(\tau)} \left((\bar{r}_{\tau_f} - M)^{2-p} \left[\frac{(\partial_{u_{\tau_f}} \psi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}} + \frac{(\partial_{u_{\tau_f}} \phi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}} \right] - \frac{\bar{\nu}_{\tau_f} \phi_{\tau_f}^2}{(\bar{r}_{\tau_f} - M)^p} \right) du_{\tau_f} & \text{for } p \in [0, 1) \\ \displaystyle\int_{\underline{C}^{\tau_f}(\tau)} \left((\bar{r}_{\tau_f} - M)^{2-p} \left[\frac{(\partial_{u_{\tau_f}} \psi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}} + \frac{(\partial_{u_{\tau_f}} \phi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}} \right] - \bar{\nu}_{\tau_f} \phi_{\tau_f}^2 \right) du_{\tau_f} & \text{for } p \in [1, 3) \end{cases}.$$

2. The r^p-weighted flux to null infinity:

$$\mathcal{E}_p^{\tau_f}(\tau) \doteq \begin{cases} \displaystyle\int_{C^{\tau_f}(\tau)} \left(r_{\tau_f}^p(\partial_v\psi_{\tau_f})^2 + r_{\tau_f}^{p+2}(\partial_v\phi_{\tau_f})^2 + r_{\tau_f}^p\phi_{\tau_f}^2\right) dv & \text{for } p\in[0,1) \\ \displaystyle\int_{C^{\tau_f}(\tau)} \left(r_{\tau_f}^p(\partial_v\psi_{\tau_f})^2 + r_{\tau_f}^2(\partial_v\phi_{\tau_f})^2 + \phi_{\tau_f}^2\right) dv & \text{for } p\in[1,3) \end{cases}.$$

3. The energy flux along outgoing cones in the near region:

$$\mathcal{F}^{\tau_f}(u,\tau') \doteq \int_{H_u\cap\{\tau\geq\tau'\}} \left((\partial_v\phi_{\tau_f})^2 + \bar{\lambda}_{\tau_f}\phi_{\tau_f}^2\right) dv.$$

4. The energy flux along ingoing cones in the far region:

$$\underline{\mathcal{F}}^{\tau_f}(v,\tau') \doteq \int_{\underline{H}_v\cap\{\tau\geq\tau'\}} \left(r_{\tau_f}^2(\partial_{u_{\tau_f}}\phi_{\tau_f})^2 + \phi_{\tau_f}^2\right) du_{\tau_f}.$$

3.4 Detailed statements of the main theorems

We can now state our main theorems using the notation and definitions from Sections 3.1 to 3.3.

3.4.1 Nonlinear stability

Theorem 1 (Stability of extremal Reissner–Nordström in spherical symmetry). *Let $M_0>0$, $e_0\in\mathbb{R}$ with $|e_0|=M_0$, and let δ be an arbitrary parameter satisfying*

$$0<\delta<\frac{1}{100}. \tag{3.11}$$

There exists a number $\varepsilon_{\mathrm{stab}}(M_0,\delta)>0$, a set $\mathfrak{M}_{\mathrm{stab}}\subset\mathfrak{M}$, and a constant $C(M_0,\delta)$ (which is implicit in the notation $\lesssim$ below) with the following properties:

1. *$\mathfrak{M}_{\mathrm{stab}}$ is "codimension-one" inside of $\mathfrak{M}(\varepsilon)$: For every $0<\varepsilon\leq\varepsilon_{\mathrm{stab}}$ and $\mathcal{S}_0\in\mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0]\leq\varepsilon$, it holds that*

$$\mathfrak{M}_{\mathrm{stab}}\cap\mathcal{L}(\mathcal{S}_0,\varepsilon)\neq\emptyset. \tag{3.12}$$

2. *Existence of a black hole region: Let $(\hat{\mathcal{Q}}_{\max},r,\Omega^2,\phi,e)$ be the maximal development of a seed data set in the intersection (3.12). Then $\hat{\mathcal{Q}}_{\max}=[0,U_*]\times[0,\infty)$. There exists a $\hat{u}_{\mathcal{H}^+}\in(0,U_*)$ such that $r(\hat u,\hat v)\to\infty$ as $\hat v\to\infty$ for every for every $\hat u\in[0,\hat u_{\mathcal{H}^+})$ and $r(\hat u_{\mathcal{H}^+},\hat v)\to|e|$ as $\hat v\to\infty$. Therefore, $[0,\hat u_{\mathcal{H}^+})\times\{\hat v=\infty\}$ may be regarded as future null infinity $\mathcal{I}^+$, there exists a nonempty black hole region*

$$\mathcal{BH}\doteq\hat{\mathcal{Q}}_{\max}\setminus J^-(\mathcal{I}^+)=[\hat u_{\mathcal{H}^+},U_*]\times[0,\infty),$$

and

$$\mathcal{H}^+\doteq\partial J^-(\mathcal{I}^+)=\{\hat u_{\mathcal{H}^+}\}\times[0,\infty)$$

is the event horizon. Moreover, future null infinity is complete in the sense of Christodoulou [Chr99]. There exist C^1 double null coordinates (u_∞,v) on the domain of outer communication $[0,\hat u_{\mathcal{H}^+})\times[0,\infty)$ such that u_∞ is Bondi normalized, i.e., the event horizon $\mathcal{H}^+$ can be formally regarded as $\{u_\infty=\infty\}$ and $\partial_{u_\infty}r\to-1$ along any outgoing cone in the domain of outer communication.

3. *Orbital stability: There exists an ∞-anchored extremal Reissner–Nordström solution $\bar r_\infty$ in the (u_∞,v) coordinates whose parameters satisfy*

$$|M-M_0|+|e-e_0|\lesssim\varepsilon. \tag{3.13}$$

Relative to this background solution, the $p=3-\delta$ energy of the scalar field is bounded by its initial value,

$$\sup_{\tau\in[1,\infty)}\left(\mathcal{E}_{3-\delta}^{\infty}(\tau)+\underline{\mathcal{E}}_{3-\delta}^{\infty}(\tau)\right)\lesssim\mathcal{E}_{3-\delta}^{\infty}(1)+\underline{\mathcal{E}}_{3-\delta}^{\infty}(1), \tag{3.14}$$

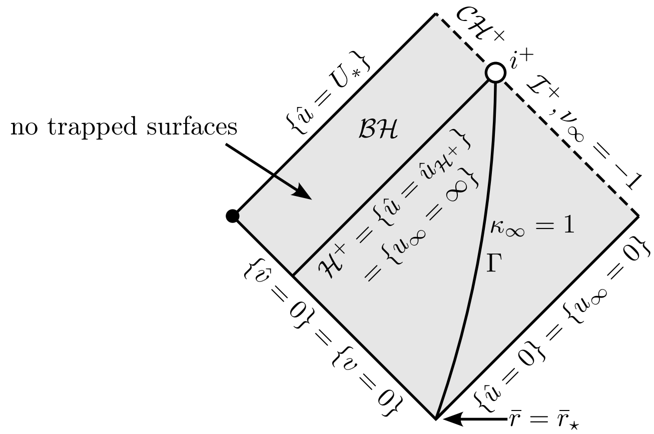

Figure 9: A Penrose diagram depicting the maximal development of seed data lying in $\mathfrak{M}_{\mathrm{stab}}$.

and the scalar field is pointwise bounded in C^1 in the domain of outer communication in terms of its initial values,

$$\sup_{J^-(\mathcal{I}^+)} \left(|r\phi| + |r^2\partial_{\hat{v}}\psi| + |r^2\partial_{\hat{v}}\phi| + \left|\frac{\partial_{\hat{u}}\psi}{-\hat{\nu}}\right| + \left|r\frac{\partial_{\hat{u}}\phi}{-\hat{\nu}}\right| \right)$$
$$\lesssim \sup_{J^-(\mathcal{I}^+)\cap\hat{\mathcal{C}}} \left(|r\phi| + |r^2\partial_{\hat{v}}\psi| + |r^2\partial_{\hat{v}}\phi| + \left|\frac{\partial_{\hat{u}}\psi}{-\hat{\nu}}\right| + \left|r\frac{\partial_{\hat{u}}\phi}{-\hat{\nu}}\right| \right). \quad (3.15)$$

The right-hand side of (3.14) *is* $\lesssim \varepsilon^2$ *and the right-hand side of* (3.15) *is* $\lesssim \varepsilon$.

4. <u>*Asymptotic stability:*</u> *The geometry decays towards the ∞-anchored extremal Reissner–Nordström solution in the following sense:*

$$|\gamma_\infty + 1| \lesssim \varepsilon^2 r^{-1}\tau^{-3+\delta} \quad (3.16)$$

holds on $\mathcal{D}_\infty \cap \{r \geq \Lambda\}$,

$$\left|\frac{\nu_\infty}{\bar{\nu}_\infty} - 1\right| \lesssim \varepsilon^2\tau^{-1+\delta} \quad (3.17)$$

holds on $\mathcal{D}_\infty$, and

$$|r - \bar{r}_\infty| \lesssim \varepsilon^2\tau^{-2+\delta}, \quad |\lambda_\infty - \bar{\lambda}_\infty| \lesssim \varepsilon^2\tau^{-2+\delta}, \quad |\kappa_\infty - 1| \lesssim \varepsilon^2\tau^{-1+\delta}, \quad |\varpi - M| \lesssim \varepsilon^2\tau^{-3+\delta} \quad (3.18)$$

hold on $\mathcal{D}_\infty$, up to and including the event horizon $\mathcal{H}^+$. The scalar field decays to zero in energy norm,

$$\underline{\mathcal{E}}_p^\infty(\tau) + \mathcal{E}_p^\infty(\tau) \lesssim \varepsilon^2\tau^{-3+\delta+p} \quad (3.19)$$

for every $\tau \in [1,\infty)$ and $p \in [0, 3-\delta]$ and its amplitude decays to zero pointwise,

$$|(\bar{r}_\infty - M)^{1/2}\phi| \lesssim \varepsilon\tau^{-3/2+\delta/2}, \quad |\psi| \lesssim \varepsilon\tau^{-1+\delta/2} \quad (3.20)$$

on $\mathcal{D}_\infty$, up to and including the event horizon $\mathcal{H}^+$.

5. <u>*Absence of trapped surfaces:*</u> *On the spacetime $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2)$, it holds that $\hat{\nu} < 0$, i.e., there exist no antitrapped spheres of symmetry. Moreover, we have the following dichotomy:*

   (a) $\hat{\lambda} \geq 0$ on $\hat{\mathcal{Q}}_{\max}$, i.e., there exist no (strictly) trapped symmetry spheres.

   (b) If $\lambda(\hat{u}_0, v_0) = 0$ for some $(\hat{u}_0, \hat{v}_0) \in \hat{\mathcal{Q}}_{\max}$, then $\hat{u}_0 = \hat{u}_{\mathcal{H}^+}$ and $\phi(\hat{u}_{\mathcal{H}^+}, \hat{v}) = 0$ for all $v \in [v_0, \infty)$.

Remark 3.6. By [KU22, Proposition B.1], $\hat{\mathcal{Q}}_{\max} \times S^2$ with the $(3+1)$-dimensional metric (2.1) does not contain *any* trapped surfaces (i.e., without restriction to spherical symmetry).

The proof of Theorem 1 is given in Section 8 and we will now briefly indicate to the reader where the various parts are shown. The set $\mathfrak{M}_{\rm stab}$ is defined in Section 8.1.3, where the codimension-one property (3.12) is also proved. Part 2. is proved in Proposition 8.16, while the eschatological gauge (u_∞, v) is constructed in Section 8.2.1. The culmination of the proofs of Points 3.–5. is given in Section 8.3.5. Points 3. and 4. rely in particular on estimates proved in Sections 5 to 7.

3.4.2 The Aretakis instability

Let

$$Y \doteq \hat{\nu}^{-1}\partial_{\hat{u}}$$

denote the gauge-invariant null derivative which is transverse to the event horizon $\mathcal{H}^+$, analogous to ∂_r in ingoing Eddington–Finkelstein coordinates (v, r) in Reissner–Nordström (recall Section 2.2).

Theorem 2 (The Aretakis instability for dynamical extremal horizons)**.** *Let $\mathfrak{M}_{\rm stab}$ denote the subset of the moduli space $\mathfrak{M}$ given by Theorem 1 consisting of seed data asymptotically converging to extremal Reissner–Nordström in evolution. Then the following holds:*

i) For any solution $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ arising from $\mathcal{S} \in \mathfrak{M}_{\rm stab}$, the "asymptotic Aretakis charge"

$$H_0[\phi] \doteq \lim_{\hat{v}\to\infty} Y\psi|_{\mathcal{H}^+}$$

exists and it holds that

$$|Y\psi|_{\mathcal{H}^+}(\hat{v}) - H_0[\phi]| \lesssim \varepsilon^3(1+\hat{v})^{-1+\delta}, \tag{3.21}$$

$$\big|R_{YY}|_{\mathcal{H}^+}(\hat{v}) - 2M^{-2}\big(H_0[\phi]\big)^2\big| \lesssim \varepsilon^2(1+\hat{v})^{-1+\delta/2}, \tag{3.22}$$

where $\varepsilon \geq \mathfrak{D}[\mathcal{S}]$.

ii) The set

$$\mathfrak{M}^{\neq 0}_{\rm stab} \doteq \{\mathcal{S} \in \mathfrak{M}_{\rm stab} : H_0[\phi] \neq 0\}$$

has nonempty interior as a subset of $\mathfrak{M}_{\rm stab}$.

iii) For any solution arising from data lying in $\mathfrak{M}^{\neq 0}_{\rm stab}$, it holds that

$$\big|Y^2(r\phi)|_{\mathcal{H}^+}(\hat{v})\big| \gtrsim |H_0[\phi]|\hat{v}, \tag{3.23}$$

$$\big|\nabla_Y R_{YY}|_{\mathcal{H}^+}(\hat{v})\big| \gtrsim \big(H_0[\phi]\big)^2\hat{v} \tag{3.24}$$

for $\hat{v} \gtrsim 1 + |\varepsilon H_0[\phi]^{-1}|^{1/(1-\delta)}$.

The proof of this theorem is given in Section 9.1.

Remark 3.7. In this work, we do not prove pointwise decay of $\partial_v\phi$ as it is not required to close our energy estimates. Assuming the decay rate $\partial_v\phi|_{\mathcal{H}^+} = O(v^{-2})$ as in the uncoupled case, we would have the following: Relative to the null frame $\{e_1, e_2, e_3, e_4\}$ with $e_1 = \partial_\vartheta$, $e_2 = \partial_\varphi$, $e_3 = Y$, and $e_4 = \hat{\kappa}^{-1}\partial_{\hat{v}}$ (so that e_3 and e_4 are gauge invariant and $g(e_3, e_4) = -4$), the only components of the Riemann tensor that do not necessarily converge to those of extremal Reissner–Nordström along $\mathcal{H}^+$ are R_{1313}, R_{2323}, and appropriate permutations of indices (i.e., $\underline{\alpha}$ computed relative to the Riemann tensor). The only components of the covariant derivative of the Riemann tensor that are allowed to not decay or grow are $\nabla_3 R_{1313}$, $\nabla_3 R_{2323}$, and appropriate permutations of indices. At higher orders of differentiation, other components should also display non-decay and growth properties.

Remark 3.8. Using the Einstein equations, (3.21), and (3.23), one can also prove

$$\big|Y\log\kappa|_{\mathcal{H}^+}(v) - M^{-2}\big(H_0[\phi]\big)^2\big| \lesssim \varepsilon^2(1+v)^{-1+\delta/2} \tag{3.25}$$

$$\big|Y^2\log\kappa|_{\mathcal{H}^+}(v)\big| \gtrsim (H_0[\phi])^2\, v, \tag{3.26}$$

where we have interpreted κ as being written in the "mixed" coordinate system $(\hat{u}, v)$ which still extends smoothly to $\mathcal{H}^+$. In this coordinate system, we have that $|\log\kappa| \lesssim \varepsilon^2(1+v)^{-1+\delta}$ up to $\mathcal{H}^+$ by (3.18), so (3.25) and (3.26) represent yet another aspect of the extremal horizon instability.

4 Setup for the proof of stability

In this section, we set up the proof of Theorem 1 and discuss the logic of the proof. In Section 4.1, we state the bootstrap assumptions for the proof of Theorem 1, which also involves defining a sequence of sets used in our codimension-one modulation argument. In Section 4.2, we show that the bootstrap set is nonempty by virtue of the local existence theory. Finally, in Section 4.3, we state two propositions that encode the main analytic content of the proof of Theorem 1—improvability of the bootstrap assumptions.

4.1 The bootstrap and modulation parameter sets

We define in this section two sets of parameters: a bootstrap set $\mathfrak{B}$ containing the τ_f's for which we assume the solution exists on $\mathcal{D}_{\tau_f}$ and satisfies certain properties, and a sequence of compact intervals

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots$$

of α parameters which are used in the modulation argument to hit extremality.

We first set

$$\mathfrak{A}_0 \doteq [|e| - |e_0| - \varepsilon^{3/2}, |e| - |e_0| + \varepsilon^{3/2}], \tag{4.1}$$

but the sets $\mathfrak{A}_i$ for $i \geq 1$ will only be properly defined in the course of the proof of Theorem 1; see already Section 8.1.2.[9] We now briefly indicate their construction to make the purpose of the bootstrap assumption 1. below clear. We define continuous functions

$$\begin{aligned} \Pi_i : \mathfrak{A}_i &\to \mathbb{R} \\ \alpha &\mapsto \varpi(\Gamma(L_i)) - M = \tilde{\varpi}(\Gamma(L_i)), \end{aligned} \tag{4.2}$$

where $L_i \doteq 2^i$, which will be assumed to satisfy the fundamental estimate

$$\sup_{\mathfrak{A}_i} |\Pi_i| \leq \varepsilon^{3/2} L_i^{-3+\delta}. \tag{4.3}$$

Of course, Π_i is only defined if the solution corresponding to $\mathcal{S}_0(\alpha)$ for $\alpha \in \mathfrak{A}_i$ exists until the time $\tau = L_i$. Assuming that this is the case, we will then inductively define

$$\mathfrak{A}_{i+1} = [\alpha_{i+1}^-, \alpha_{i+1}^+], \tag{4.4}$$

where $\alpha_{i+1}^\pm \in \mathrm{int}(\mathfrak{A}_i)$ are chosen according to a simple algorithm ensuring that the improved estimate

$$\sup_{\mathfrak{A}_{i+1}} |\Pi_{i+1}| \leq \varepsilon^{3/2} L_{i+1}^{-3+\delta}$$

holds; see already Lemma 8.4.

It is convenient to define the function $I(\tau_f) \doteq \lfloor \log_2 \tau_f \rfloor$, i.e., the largest integer such that $2^{I(\tau_f)} \leq \tau_f$.

Definition 4.1. Let $A \geq 1$, $0 < \varepsilon \leq \varepsilon_{\mathrm{loc}}$, and $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \leq \varepsilon$. Then $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$ denotes the set of $\tau_f \in [1, \infty)$ such that:

1. For every $i \in \{0, 1, \ldots, I(\tau_f)\}$, there exist numbers $\alpha_i^\pm \in [|e| - |e_0| - \varepsilon^{3/2}, |e| - |e_0| + \varepsilon^{3/2}]$, which may depend on $\mathcal{S}_0$ and ε, but are independent of A and τ_f, with $\alpha_i^- < \alpha_i^+$ and $\alpha_0^\pm = |e| - |e_0| \pm \varepsilon^{3/2}$, such that the nesting condition $\mathfrak{A}_{i+1} \subset \mathfrak{A}_i$ holds, where $\mathfrak{A}_i \doteq [\alpha_i^-, \alpha_i^+]$.

Given $\alpha \in \mathfrak{A}_{I(\tau_f)}$, let $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ denote the maximal development of the modulated seed data $\mathcal{S}_0(\alpha)$ in the initial data gauge $(\hat{u}, \hat{v})$ determined by Proposition 2.8 and Lemma 3.2.

[9] This is because the sets $\mathfrak{A}_i$ for $i \geq 1$ are defined teleologically, i.e., they cannot be directly read off from the initial data. Their existence can only be inferred in the context of our continuity argument, in particular, the argument requires proving quantitative decay rates for $\tilde{\varpi} = \varpi - M$.

2. For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, there exists a timelike curve $\Gamma : [1, \tau_f] \to \hat{\mathcal{Q}}_{\max}$, which is the unique smooth solution of the ODE
$$\frac{d}{d\tau}(\Gamma^{\hat{u}}, \Gamma^{\hat{v}}) = \left(\frac{\sqrt{1-\mu}}{-2\hat{\nu}}, \frac{\sqrt{1-\mu}}{2\hat{\lambda}}\right)\Big|_{\Gamma(\tau)},$$
with initial condition $\Gamma(1) = (0, 0)$.

By global hyperbolicity of $\hat{\mathcal{Q}}_{\max}$, point 2. implies that for every $\alpha \in \mathfrak{A}_{I(\tau_f)}$,
$$\hat{\mathcal{D}}_{\tau_f} \doteq [0, \Gamma^{\hat{u}}(\tau_f)] \times [0, \Gamma^{\hat{v}}(\tau_f)] \subset \hat{\mathcal{Q}}_{\max}.$$

3. For every $i \in \{0, 1, \dots, I(\tau_f)\}$, the map Π_i (recall (4.2)) is defined on $\mathfrak{A}_i$,
$$\Pi_i : \mathfrak{A}_i \to [-\varepsilon^{3/2} L_i^{-3+\delta}, \varepsilon^{3/2} L_i^{-3+\delta}]$$
is surjective, and $\Pi_i(\alpha_i^{\pm}) = \pm\varepsilon^{3/2} L_i^{-3+\delta}$.

4. For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, $\hat{\gamma}$ is strictly negative on $[0, \Gamma^{\hat{u}}(\tau_f)] \times \{\Gamma^{\hat{v}}(\tau_f)\}$ and $\hat{\kappa}$ is strictly positive on Γ. Therefore, the teleologically normalized coordinates (u_{τ_f}, v) are defined on $\hat{\mathcal{D}}_{\tau_f}$.

On $\mathcal{D}_{\tau_f}$, let $(\bar{r}_{\tau_f}, \bar{\Omega}^2_{\tau_f})$ be the τ_f-anchored extremal Reissner–Nordström solution as defined in Section 3.3.

5. For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, the following bootstrap assumptions for the geometry hold on $\mathcal{D}_{\tau_f}$:
$$\left|\frac{\nu_{\tau_f}}{\bar{\nu}_{\tau_f}} - 1\right| \le A^3 \varepsilon^{3/2} \tau^{-1+\delta}, \tag{4.5}$$
$$|\tilde{r}_{\tau_f}| \le A^2 \varepsilon^{3/2} \tau^{-2+\delta}, \tag{4.6}$$
$$|\tilde{\varpi}_{\tau_f}| \le A \varepsilon^{3/2} \tau^{-3+\delta}. \tag{4.7}$$

6. For every $\alpha \in \mathfrak{A}_{I(\tau_f)}$, the following bootstrap assumptions for the scalar field hold:
$$\mathcal{E}_p^{\tau_f}(\tau) \le A\varepsilon^2 \tau^{-3+\delta+p}, \tag{4.8}$$
$$\underline{\mathcal{E}}_p^{\tau_f}(\tau) \le A\varepsilon^2 \tau^{-3+\delta+p}, \tag{4.9}$$
$$\mathcal{F}^{\tau_f}(u, \tau) \le A\varepsilon^2 \tau^{-3+\delta}, \tag{4.10}$$
$$\underline{\mathcal{F}}^{\tau_f}(v, \tau) \le A\varepsilon^2 \tau^{-3+\delta} \tag{4.11}$$
for every $\tau \in [1, \tau_f]$, $(u, v) \in \mathcal{D}_{\tau_f}$, and $p \in [0, 3-\delta]$.

4.2 Nonemptiness of the bootstrap set

We begin the continuity argument with the following simple consequence of the local existence theory:

Proposition 4.2. *For any $A \ge 1$ and ε sufficiently small,*
$$\{\mathcal{S}_0(\alpha) : \alpha \in \mathfrak{A}_0\} \subset \mathcal{L}(\mathcal{S}_0, \varepsilon) \tag{4.12}$$
and $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$ is nonempty.

Proof. Observe that $|\alpha| \le |e - e_0| + \varepsilon^{3/2} \le 2\varepsilon$ for $\alpha \in \mathfrak{A}_0$ and $\varepsilon \le 1$ by the definition of $\mathfrak{D}$, which verifies (4.12). Points 1. and 3. of Definition 4.1 are trivial for $\tau_f \in [1, 2)$. By local well-posedness and continuous dependence on initial data, every element of $\mathcal{S}_0(\alpha)$ extends to a neighborhood of the initial data hypersurface $\hat{\mathcal{C}}$ which is uniform near $(0, 0)$. Since the solution is close to Reissner–Nordström in this small region by Cauchy stability, the bootstrap assumptions 2. and 4.–6. are automatically satisfied for A sufficiently large, ε sufficiently small, and $|\tau_f - 1|$ sufficiently small. Therefore, $\tau_f \in \mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$ for $|\tau_f - 1|$ sufficiently small. □

4.3 Improving the bootstrap assumptions—the main estimates

The main analytic content of the proof of Theorem 1 consists of the following two propositions, which will be used to show that the bootstrap set is *open*.

Proposition 4.3 (Improving the bootstrap assumptions for the geometry)**.** *There exist positive constants A_0 and ε_0 depending only on M_0 and δ, such that if $A = A_0$, $\varepsilon \le \varepsilon_0$, $\tau_f \in \mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, then the estimates*

$$\left|\frac{\nu_{\tau_f}}{\bar{\nu}_{\tau_f}} - 1\right| \le \tfrac{1}{2}A^3\varepsilon^{3/2}\tau^{-1+\delta}, \tag{4.13}$$

$$|\tilde{r}_{\tau_f}| \le \tfrac{1}{2}A^2\varepsilon^{3/2}\tau^{-2+\delta}, \tag{4.14}$$

$$|\tilde{\varpi}_{\tau_f}| \le \tfrac{1}{2}A\varepsilon^{3/2}\tau^{-3+\delta} \tag{4.15}$$

hold on $\mathcal{D}_{\tau_f}$.

This proposition is proved in Section 5.3.4.

Proposition 4.4 (Improving the bootstrap assumptions for the scalar field)**.** *There exist positive constants A_0 and ε_0 depending only on M_0 and δ, such that if $A = A_0$, $\varepsilon \le \varepsilon_0$, $\tau_f \in \mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, then the estimates*

$$\mathcal{E}_p^{\tau_f}(\tau) \le \tfrac{1}{2}A\varepsilon^2\tau^{-3+\delta+p}, \tag{4.16}$$

$$\underline{\mathcal{E}}_p^{\tau_f}(\tau) \le \tfrac{1}{2}A\varepsilon^2\tau^{-3+\delta+p}, \tag{4.17}$$

$$\mathcal{F}^{\tau_f}(u,\tau) \le \tfrac{1}{2}A\varepsilon^2\tau^{-3+\delta}, \tag{4.18}$$

$$\underline{\mathcal{F}}^{\tau_f}(v,\tau) \le \tfrac{1}{2}A\varepsilon^2\tau^{-3+\delta} \tag{4.19}$$

hold for every $\tau \in [1, \tau_f]$, $(u, v) \in \mathcal{D}_{\tau_f}$, and $p \in [0, 3-\delta]$.

This proposition is proved in Section 7.1 and relies on energy estimates proved in Section 6.

Remark 4.5. The modulation argument for ϖ, i.e., the construction of the sets $\mathfrak{A}_i$, takes place in Section 8.1.2 when we show that $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$ is *closed*. It relies crucially on an estimate for $\tilde{\varpi}_{\tau_f}$ that will be proved in the course of the proof of Proposition 4.3. See already Lemma 5.14.

5 Estimates for the geometry

In this section, we prove estimates for the geometry in a bootstrap domain $\mathcal{D}_{\tau_f}$. In Section 5.2, we estimate $\tilde{\kappa}_{\tau_f}$, $\tilde{\gamma}_{\tau_f}$, $\tilde{\kappa}_{\tau_f}$, the behavior of Γ and τ, and provide the fundamental Taylor expansions for $\tilde{\lambda}_{\tau_f}$ and $\tilde{\nu}_{\tau_f}$. In Section 5.3, we estimate $\tilde{r}_{\tau_f}$, $\nu_{\tau_f}/\bar{\nu}_{\tau_f}$, and $\tilde{\varpi}_{\tau_f}$. We improve the bootstrap assumptions for the geometry (Proposition 4.3) in Section 5.3.4.

5.1 Conventions for Sections 5 to 7

In this section, Section 6, and Section 7, we will fix τ_f and essentially exclusively work in the teleological coordinate system (u_{τ_f}, v). Therefore, for ease of reading, we will omit the subscript τ_f on u_{τ_f}, the solution $(r_{\tau_f}, \Omega^2_{\tau_f}, \phi_{\tau_f}, e)$, the background extremal Reissner–Nordström solution $\bar{r}_{\tau_f}$, and the energies. In order to keep track of the constant A appearing in the bootstrap assumptions (4.5)–(4.11), the notations $a \lesssim b$, $a \gtrsim b$, and $a \sim b$ mean that the estimate does not depend on u, v, α, A, ε, τ, or τ_f. We will omit the decoration $(\mathcal{S}_0, \varepsilon, A)$ on the bootstrap set $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A)$. We will also often absorb powers of the bootstrap constant A by sacrificing a little bit of ε without comment, for example, $A\varepsilon^{3/2} \lesssim \varepsilon$ for ε sufficiently small. Note that A will be fixed at the end of Section 7.1 depending only on M_0 and δ.

5.2 Initial estimates

5.2.1 Basic consequences of the bootstrap assumptions

Lemma 5.1. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\tfrac{1}{2}M \leq \varpi \leq 2M, \tag{5.1}$$

$$\tfrac{1}{2} \leq \frac{\nu}{\bar{\nu}} \leq 2, \tag{5.2}$$

$$\lambda > 0 \tag{5.3}$$

in $\mathcal{D}_{\tau_f}$ and

$$1 - \mu \geq \tfrac{3}{4}, \tag{5.4}$$

$$\lambda \geq \tfrac{1}{2}, \tag{5.5}$$

$$-2 \leq \nu \leq -\tfrac{1}{2}, \tag{5.6}$$

in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$.

Proof. The bounds (5.1) follow directly from (4.15). For (5.4) we notice that

$$1 - \mu = 1 - \frac{2\varpi}{r} + \frac{e^2}{r^2} \geq 1 - \frac{4M}{\Lambda} \geq \frac{1}{2}.$$

For (5.5) we first observe that the monotonicity of (2.21) implies that $\kappa \geq 1$ in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$. Therefore, $\lambda = \kappa(1-\mu) \geq \frac{1}{2}$ implies (5.5). By monotonicity of Raychaudhuri's equation (2.15), $\lambda > 0$ throughout $\mathcal{D}_{\tau_f}$. Finally, to show (5.6) we note that $\bar{\nu} = -(1-\bar{\mu})$ and thus $-1 \leq \bar{\nu} \leq -\frac{3}{4}$ in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$ using (4.6). The estimate (5.6) follows then from (4.5). □

5.2.2 Estimates for $\tilde{\kappa}$ and $\tilde{\gamma}$

We now use the bootstrap assumptions for the energy decay of ϕ to derive bounds for $\tilde{\kappa}$ and $\tilde{\gamma}$.

Lemma 5.2. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$|\tilde{\kappa}| \lesssim A\varepsilon^2 \tau^{-1+\delta}, \tag{5.7}$$

$$(\bar{r} - M)|\tilde{\kappa}| \lesssim A\varepsilon^2 \tau^{-2+\delta}, \tag{5.8}$$

$$(\bar{r} - M)^2|\tilde{\kappa}| \lesssim A\varepsilon^2 \tau^{-3+\delta} \tag{5.9}$$

in $\mathcal{D}_{\tau_f} \cap \{r \leq \Lambda\}$ and

$$|\tilde{\kappa}| \lesssim A\varepsilon^2 \tau^{-3+\delta} \tag{5.10}$$

in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$.

Proof. Estimate for $r(u,v) \leq \Lambda$: From (2.21) and $\kappa = \tilde{\kappa} + 1$ we have

$$\partial_u \tilde{\kappa} = \frac{r(\tilde{\kappa}+1)}{\nu}(\partial_u \phi)^2.$$

Integrating to the future from Γ, where $\tilde{\kappa} = 0$ by our gauge choice, and multiplying by $(\bar{r} - M)^{2-p}$ for $p \in \{0,1,2\}$ gives

$$(\bar{r} - M)^{2-p}|\tilde{\kappa}(u,v)| \leq (\bar{r} - M)^{2-p} \int_{\underline{C}_v \cap \{u' \leq u\}} \frac{|\tilde{\kappa}|+1}{-\nu} r (\partial_u \phi)^2 du \lesssim \int_{\underline{C}_v \cap \{u' \leq u\}} (\bar{r} - M)^{2-p} \frac{|\tilde{\kappa}|+1}{-\bar{\nu}} (\partial_u \phi)^2 du,$$

where we used that $\nu \sim \bar{\nu}$ by (4.5) and that $\bar{\nu} < 0$ on $\mathcal{D}_f$ in order to move the factor $(\bar{r} - M)^{2-p}$ inside the integral. Applying Grönwall's lemma then gives

$$(\bar{r} - M)^{2-p}|\tilde{\kappa}(u,v)| \lesssim \underline{\mathcal{E}}_p(\tau) \exp\left(\underline{\mathcal{E}}_2(\tau)\right) \lesssim A\varepsilon^2 \tau^{-3+\delta+p},$$

from which (5.7)–(5.9) follow.

Estimate for $r(u,v) \geq \Lambda$: We integrate (2.21) backwards from Γ using an integrating factor and the gauge condition to obtain

$$\log \kappa(u,v) = \int_{\underline{H}_v \cap \{u' \geq u\}} r \left(\frac{\partial_u \phi}{-\bar{\nu}}\right)^2 \left(\frac{\bar{\nu}}{\nu}\right) (-\bar{\nu})\, du' \lesssim \underline{\mathcal{F}}(v, \tau(u,v)), \tag{5.11}$$

using (4.5). The estimate (5.10) follows now from (4.11). □

Note that the estimates (5.7) and (5.10) imply that

$$\tfrac{1}{2} \leq \kappa \leq 2, \tag{5.12}$$

$$\lambda \leq 2 \tag{5.13}$$

in $\mathcal{D}_{\tau_f}$.

Lemma 5.3. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$|\tilde{\gamma}| \lesssim A\varepsilon^2 r^{-1} \tau^{-3+\delta}, \tag{5.14}$$

$$|\tilde{\gamma}| \lesssim A\varepsilon^2 r^{-3/2} \tau^{-5/2+\delta} \tag{5.15}$$

in $\mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$.

Proof. From (2.22) and $\gamma = \tilde{\gamma} - 1$ we have

$$\partial_v \tilde{\gamma} = \frac{r(\tilde{\gamma} - 1)}{\lambda} (\partial_v \phi)^2.$$

Integrating from $\underline{C}_{\Gamma^v(\tau_f)}$ and multiplying by r^{1+p} for $p \in \{0, 1/2\}$ gives for $(u,v) \in \mathcal{D}_{\tau_f} \cap \{r \geq \Lambda\}$ that

$$|r^{1+p} \tilde{\gamma}(u,v)| \leq r^{1+p} \int_{C_u \cap \{v \leq v'\}} r \frac{|\tilde{\gamma}| + 1}{\lambda} (\partial_v \phi)^2\, dv' \leq \int_{C_u \cap \{v \leq v'\}} r^{p+2} \frac{|\tilde{\gamma}| + 1}{\lambda} (\partial_v \phi)^2\, dv',$$

where we used (5.5) to move the r weight into the integral. Applying Grönwall's inequality as before and noting that $\lambda \sim 1$ gives

$$|r^{1+p} \tilde{\gamma}(u,v)| \lesssim \mathcal{E}_p(\tau) \exp\big(\mathcal{E}_0(\tau)\big)$$

for $p \in \{0, 1/2\}$. This shows (5.14) and (5.15) using bootstrap assumption (4.8). □

Remark 5.4. The estimate (5.15), with its integrable τ-weight *and* integrable r-weight, is important in controlling the gauge Φ_{τ_f} as $\tau_f \to \infty$. See already Section 8.2.2.

5.2.3 Properties of Γ and τ

In this section, we derive some basic properties of the timelike curve $\Gamma = \{r = \Lambda\}$ and the associated function $\tau(u,v)$ defined by (3.9). Recall that τ was also used to denote the proper time parametrization of Γ, defined via the condition

$$g(\dot{\Gamma}, \dot{\Gamma}) = -1, \tag{5.16}$$

where $\dot{}$ is used to denote $\frac{d}{d\tau}$. We write the components of Γ with this parametrization in the teleological coordinate chart (u,v) as $\Gamma^u(\tau)$ and $\Gamma^v(\tau)$.

Lemma 5.5. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\dot{\Gamma}^u(\tau) \sim \dot{\Gamma}^v(\tau) \sim 1 \tag{5.17}$$

for $\tau \in [1, \tau_f]$.

Proof. Since r is constant along Γ, we have $\nu\dot{\Gamma}^u + \lambda\dot{\Gamma}^v = 0$ on Γ. By the gauge condition $\lambda = \kappa(1-\mu)|_\Gamma = 1-\mu|_\Gamma \sim 1$ and (5.6), it follows that $\dot{\Gamma}^u \sim \dot{\Gamma}^v$. The proper time condition (5.16) can be written as

$$\frac{4\nu\lambda}{1-\mu}\dot{\Gamma}^u\dot{\Gamma}^v = -1,$$

which implies that $\dot{\Gamma}^u\dot{\Gamma}^v \sim 1$ by (5.4). Now (5.17) easily follows. □

The following lemma relates the "time function" τ (which was defined in (3.9)) to (approximate) Bondi time u in the far region and (approximate) ingoing Eddington–Finkelstein time v in the near-horizon region.

Lemma 5.6. *For any $A \ge 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\tau(u_1, v) - \tau(u_2, v) \sim u_1 - u_2 \tag{5.18}$$

for $(u_1, v), (u_2, v) \in \mathcal{D}_{\tau_f} \cap \{r \ge \Lambda\}$ and

$$\tau(u, v_1) - \tau(u, v_2) \sim v_1 - v_2 \tag{5.19}$$

for $(u, v_1), (u, v_2) \in \mathcal{D}_{\tau_f} \cap \{r \le \Lambda\}$. Moreover, for any $\eta > 1$ and $1 \le \tau_1 \le \tau_2 \le \tau_f$, it holds that

$$\int_{\underline{H}_v\cap\{\tau_1\le\tau\le\tau_2\}} \tau^{-\eta}\, du \lesssim_\eta \tau_1^{-\eta+1}, \tag{5.20}$$

$$\int_{H_u\cap\{\tau_1\le\tau\le\tau_2\}} \tau^{-\eta}\, dv \lesssim_\eta \tau_1^{-\eta+1}. \tag{5.21}$$

Proof. This is immediate from (5.17), the fundamental theorem of calculus, and the change of variables formula. □

Next, we show that the v and $\hat{v}$ coordinates are comparable.

Lemma 5.7. *For $\tau_f \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_{I(\tau_f)}$, ε sufficiently small depending on A, and $\hat{v}_1, \hat{v}_2 \in [0, \Gamma^{\hat{v}}(\tau_f)]$, it holds that*

$$\mathfrak{v}(\hat{v}_2) - \mathfrak{v}(\hat{v}_1) \sim \hat{v}_2 - \hat{v}_1. \tag{5.22}$$

Moreover, on $\hat{\mathcal{D}}_{\tau_f}$ it holds that

$$\hat{\kappa} \sim 1. \tag{5.23}$$

Proof. Recall the definition of the map $\hat{v} \mapsto \mathfrak{v}(\hat{v})$ in (3.7). The derivative of the inverse map $v \mapsto \mathfrak{v}^{-1}(v)$ is given by $\lambda(0, v)$, so (5.22) follows from (5.5), (5.13), and the fundamental theorem of calculus, and (5.23) follows from the identity

$$\hat{\kappa}(\hat{u}, \hat{v}) = \frac{\kappa(\mathfrak{u}(\hat{u}), \mathfrak{v}(\hat{v}))}{\lambda(0, \mathfrak{v}(\hat{v}))}$$

and (5.12). □

5.2.4 Taylor expansions of $\widetilde{1-\mu}$, $\partial_u\tilde{r}$, and $\partial_v\tilde{r}$

We now derive Taylor expansions for $\widetilde{1-\mu}$, $\partial_u\tilde{r}$, and $\partial_v\tilde{r}$ in terms of tilde quantities. The precise form of the terms linear in $\tilde{r}$ will be crucial for arguments later in the paper.

Lemma 5.8. *For any $A \ge 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\left|\widetilde{1-\mu} - \frac{2M}{\bar{r}^3}(\bar{r} - M)\tilde{r}\right| \lesssim \frac{|\tilde{r}|^2}{\bar{r}^3} + \frac{|\tilde{\varpi}|}{\bar{r}}, \tag{5.24}$$

$$|\widetilde{1-\mu}| \lesssim A^2\varepsilon^{3/2}\tau^{-2+\delta} \tag{5.25}$$

in $\mathcal{D}_{\tau_f}$.

Proof. Using (4.6), we immediately derive the Taylor expansion

$$\frac{1}{r^\eta}=\frac{1}{\bar r^\eta}-\frac{\eta\tilde r}{\bar r^{\eta+1}}+O_\eta\left(\frac{|\tilde r|^2}{\bar r^{\eta+2}}\right) \tag{5.26}$$

for any $\eta>0$. Using this, we then compute

$$\begin{aligned}\widetilde{1-\mu}&=-\frac{2\varpi}{r}+\frac{2M}{\bar r}+\frac{e^2}{r^2}-\frac{e^2}{\bar r^2}=-\frac{2\tilde\varpi}{r}-2M\left(\frac1r-\frac1{\bar r}\right)+e^2\left(\frac{1}{r^2}-\frac{1}{\bar r^2}\right)\\&=-\frac{2\tilde\varpi}{r}+\frac{2M\tilde r}{\bar r^2}+O\left(\frac{|\tilde r|^2}{\bar r^3}\right)-\frac{2e^2\tilde r}{\bar r^3}+O\left(\frac{|\tilde r|^2}{\bar r^4}\right)=-\frac{2\tilde\varpi}{r}+\frac{2M}{\bar r^3}(\bar r-M)\tilde r+O\left(\frac{|\tilde r|^2}{\bar r^3}\right),\end{aligned}$$

which gives (5.24). We then immediately obtain (5.25) from the bootstrap assumptions. □

Lemma 5.9. *For any $A\ge1$, ε sufficiently small, $\tau_f\in\mathfrak B$, and $\alpha\in\mathfrak A_{I(\tau_f)}$, we have the expansions*

$$\partial_u\tilde r=\frac{2M\gamma}{\bar r^3}(\bar r-M)\tilde r+E_u\quad in\ \mathcal D_{\tau_f}\cap\{r\ge\Lambda\}, \tag{5.27}$$

$$\partial_v\tilde r=\frac{2M\kappa}{\bar r^3}(\bar r-M)\tilde r+E_v\quad in\ \mathcal D_{\tau_f}, \tag{5.28}$$

where the error terms satisfy the estimates

$$|E_u|\lesssim A\varepsilon^{3/2}\bar r^{-1}\tau^{-3+\delta}, \tag{5.29}$$

$$|E_v|\lesssim A\varepsilon^{3/2}\tau^{-3+\delta}. \tag{5.30}$$

Proof. Expansion for $\partial_u\tilde r$: We compute

$$\partial_u\tilde r=\tilde\nu=\gamma(1-\mu)-\bar\gamma(\overline{1-\mu})=\tilde\gamma(\overline{1-\mu})+\gamma(\widetilde{1-\mu})=\tilde\gamma\left(1-\frac{M}{\bar r}\right)^2+\gamma\left[\frac{2M}{\bar r^3}(\bar r-M)\tilde r+O\left(\frac{|\tilde r|^2}{\bar r^3}+\frac{|\tilde\varpi|}{\bar r}\right)\right],$$

which implies (5.27) with error term estimated by

$$|E_u|\lesssim\frac{|\tilde r|^2}{\bar r^3}+\frac{|\tilde\varpi|}{\bar r}+|\tilde\gamma|\lesssim A\varepsilon^{3/2}\bar r^{-1}\tau^{-3+\delta}$$

by the bootstrap assumptions and (5.14).

Expansion for $\partial_v\tilde r$: We compute

$$\partial_v\tilde r=\tilde\lambda=\kappa(1-\mu)-\bar\kappa(\overline{1-\mu})=\tilde\kappa(\overline{1-\mu})+\kappa(\widetilde{1-\mu})=\frac{(\bar r-M)^2}{\bar r^2}\tilde\kappa+\kappa\left[\frac{2M}{\bar r^3}(\bar r-M)\tilde r+O\left(\frac{|\tilde r|^2}{\bar r^3}+\frac{|\tilde\varpi|}{\bar r}\right)\right], \tag{5.31}$$

which implies (5.28) with error term estimated by

$$|E_v|\lesssim\frac{|\tilde r|^2}{\bar r^3}+\frac{|\tilde\varpi|}{\bar r}+\frac{(\bar r-M)^2}{\bar r^2}|\tilde\kappa|\lesssim A\varepsilon^{3/2}\tau^{-3+\delta}$$

by the bootstrap assumptions and (5.9). □

Lemma 5.10. *For any $A\ge1$, ε sufficiently small, $\tau_f\in\mathfrak B$, and $\alpha\in\mathfrak A_{I(\tau_f)}$, it holds that*

$$|\tilde\varkappa|\lesssim A\varepsilon^{3/2}r^{-2}\tau^{-3+\delta}+A^2\varepsilon^{3/2}r^{-3}\tau^{-2+\delta} \tag{5.32}$$

in $\mathcal D_{\tau_f}$.

Proof. This follows directly from the definition of $\varkappa$, (2.7), and the bootstrap assumptions (4.6) and (4.7). □

5.3 Improving the bootstrap assumptions for the geometry

5.3.1 Estimates for $\tilde r$

In the following lemma, we estimate $\tilde r$ in three stages: Along Γ, we estimate $\tilde r$ by using the estimates (5.10) and (5.14) for $\tilde\kappa$ and $\tilde\gamma$, for $r \le \Lambda$ we use the equation (5.28), which is *redshifted* backwards in v (see already Remark 5.12), and for $r \ge \Lambda$ we use the equation (5.27).

Lemma 5.11. *For any $A \ge 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$|\tilde r| \lesssim A\varepsilon^{3/2}\tau^{-2+\delta}, \tag{5.33}$$

$$|\tilde\lambda| \lesssim A\varepsilon^{3/2}\tau^{-2+\delta} \tag{5.34}$$

in $\mathcal{D}_{\tau_f}$.

Proof. Estimate along Γ: Using the relation $\nu\dot\Gamma^u + \lambda\dot\Gamma^v = 0$ along Γ, we compute

$$\frac{d}{d\tau}\tilde r(\Gamma(\tau)) = -\bar\nu\dot\Gamma^u - \bar\lambda\dot\Gamma^v = \left(\frac{\bar\lambda}{\lambda}\nu - \bar\nu\right)\dot\Gamma^u = (\overline{1-\mu})\left(\frac{\bar\kappa}{\kappa} - \frac{\bar\gamma}{\gamma}\right)\gamma\dot\Gamma^u.$$

Using (5.10) and (5.14), we estimate

$$\left|\frac{\bar\kappa}{\kappa} - \frac{\bar\gamma}{\gamma}\right| \lesssim A\varepsilon^2\tau^{-3+\delta}$$

along Γ and therefore conclude

$$\left|\tilde r|_\Gamma\right| \lesssim A\varepsilon^2\tau^{-2+\delta} \tag{5.35}$$

by the fundamental theorem of calculus and the anchoring condition $\tilde r(\Gamma(\tau_f)) = 0$.

Estimate for $r \le \Lambda$: We view (5.28) as an ODE for $\tilde r$ in v. For $(u,v),(u,v_2) \in \mathcal{D}_{\tau_f} \cap \{r \le \Lambda\}$ we use an integrating factor to write

$$\begin{aligned}\tilde r(u,v) &= \exp\left(-\int_v^{v_2} \frac{2M\kappa}{\bar r^3}(\bar r - M)\,dv'\right)\left[\tilde r(u,v_2) - \int_v^{v_2}\exp\left(\int_{v'}^{v_2}\frac{2M\kappa}{\bar r^3}(\bar r - M)\,dv''\right)E_v\,dv'\right] \\ &= \exp\left(-\int_v^{v_2} \frac{2M\kappa}{\bar r^3}(\bar r - M)\,dv'\right)\tilde r(u,v_2) - \int_v^{v_2}\exp\left(-\int_v^{v'}\frac{2M\kappa}{\bar r^3}(\bar r - M)\,dv''\right)E_v\,dv'.\end{aligned} \tag{5.36}$$

For $(u,v) \in \mathcal{D}_{\tau_f} \cap \{r \le \Lambda\}$, let $v_2 = v^\Lambda(u)$. Then (5.36), (5.35), (5.30), and (5.21) imply

$$|\tilde r(u,v)| \lesssim A\varepsilon^2\tau^{-2+\delta}(u,v) + A\varepsilon^{3/2}\tau^{-2+\delta}(u,v) \tag{5.37}$$

after observing that the integrating factor is nonnegative and hence the exponentials are bounded by 1.

Estimate for $r \ge \Lambda$: We view (5.27) as an ODE for $\tilde r$ in u. For $(u,v),(u_2,v) \in \mathcal{D}_{\tau_f} \cap \{r \ge \Lambda\}$ we use an integrating factor to write

$$\tilde r(u,v) = \exp\left(-\int_u^{u_2}\frac{2M\gamma}{\bar r^3}(\bar r - M)\,du'\right)\left[\tilde r(u_2,v) - \int_u^{u_2}\exp\left(\int_{u'}^{u_2}\frac{2M\gamma}{\bar r^3}(\bar r - M)\,du''\right)E_u\,du'\right]. \tag{5.38}$$

For $(u,v) \in \mathcal{D}_{\tau_f} \cap \{r \ge \Lambda\}$ let $u_2 = u^\Lambda(v)$. By (5.6), the integrating factor is bounded:

$$\int_u^{u_2}\frac{2M(-\gamma)}{\bar r^3}(\bar r - M)\,du' \lesssim \int_{\underline{H}_v}\bar r^{-2}\,du' \lesssim 1. \tag{5.39}$$

It then follows from (5.38), (5.35), (5.29), and (5.20) that

$$|\tilde r(u,v)| \lesssim A\varepsilon^2\tau^{-2+\delta}(u,v) + A\varepsilon^{3/2}\tau^{-2+\delta}(u,v). \tag{5.40}$$

The estimates (5.35), (5.37), and (5.40) give (5.33) as desired.

Finally, (5.34) now follows from (5.28) and (5.30). □

Remark 5.12. The good sign of the integrating factor in (5.36) is due to the *global redshift effect* present on extremal Reissner–Nordström. To exploit this, it was crucial that we integrated the ODE (5.28) *backwards* in v. Indeed, integrating (5.28) forwards in time results in a bad *global blueshift*, despite the absence of the horizon redshift effect! Compare with [LO19b, Lemma 8.19].

5.3.2 Estimates for $\nu/\bar{\nu}$

Lemma 5.13. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\left|\frac{\nu}{\bar{\nu}} - 1\right| \lesssim A^2\varepsilon^{3/2}\tau^{-1+\delta} \tag{5.41}$$

on $\mathcal{D}_{\tau_f}$.

Proof. Estimate for $r \geq \Lambda$: We obtain directly from (5.27) and the bootstrap assumptions that

$$|\tilde{\nu}| \lesssim \bar{r}^{-2}|\tilde{r}| + A\varepsilon^{3/2}\bar{r}^{-1}\tau^{-3+\delta} \lesssim A^2\varepsilon^{3/2}\bar{r}^{-1}\tau^{-2+\delta}, \tag{5.42}$$

which implies (5.41) for $r \geq \Lambda$ after dividing by $|\bar{\nu}| \sim 1$.

Estimate for $r \leq \Lambda$: From the wave equation (2.17) we obtain

$$\left|\partial_v \log\left(\frac{\nu}{\bar{\nu}}\right)\right| = 2|\widetilde{\kappa\varkappa}| = 2|\kappa\tilde{\varkappa}| + 2|\tilde{\kappa}\bar{\varkappa}| \lesssim A^2\varepsilon^{3/2}\tau^{-2+\delta} + (\bar{r} - M)|\tilde{\kappa}| \lesssim A^2\varepsilon^{3/2}\tau^{-2+\delta} \tag{5.43}$$

by (5.32) and (5.8). By (5.42), we have

$$\left|\log\left(\frac{\nu}{\bar{\nu}}\right)\Big|_\Gamma\right| \lesssim |\tilde{\nu}|_\Gamma| \lesssim A^2\varepsilon^{3/2}\tau^{-2+\delta}$$

so integrating (5.43) backwards from Γ shows (5.41). $\square$

5.3.3 Estimates for $\tilde{\varpi}$

Recall the dyadic index $I(\tau_f)$ which was defined to be the largest integer such that $L_{I(\tau_f)} = 2^{I(\tau_f)} \leq \tau_f$. Using the energy decay bootstrap assumptions for ϕ, we now show that ϖ remains close to its value at $\Gamma(L_{I(\tau_f)})$. Note that the power of ε in (5.44) below is strictly better than in the bootstrap assumption (4.7). This is fundamental for our modulation argument, see already Section 8.1.2 below.

Lemma 5.14. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, and $\alpha \in \mathfrak{A}_{I(\tau_f)}$, it holds that*

$$|\tilde{\varpi} - \Pi_{I(\tau_f)}(\alpha)| \lesssim A\varepsilon^2\tau^{-3+\delta} \tag{5.44}$$

on $\mathcal{D}_{\tau_f}$, where $\Pi_{I(\tau_f)} \doteq \tilde{\varpi}(\Gamma(L_{I(\tau_f)}))$.

Proof. Estimate for $u = \Gamma^u(L_{I(\tau_f)})$: We write $I = I(\tau_f)$ for short. By (2.20) and (5.12), we have

$$|\partial_v(\tilde{\varpi} - \Pi_I(\alpha))| \lesssim r^2(\partial_v\phi)^2. \tag{5.45}$$

Integrating this estimate forwards in v, we obtain from the bootstrap assumptions

$$|\tilde{\varpi} - \Pi_I(\alpha)|(\Gamma^u(L_I), v) \lesssim \mathcal{E}_0(L_I) \lesssim A\varepsilon^2 L_I^{-3+\delta} \lesssim A\varepsilon^2\tau_f^{-3+\delta} \tag{5.46}$$

for every $v \geq \Gamma^v(L_I)$. On the other hand, integrating (5.45) backwards in v, we obtain from the bootstrap assumptions and (5.21)

$$|\tilde{\varpi} - \Pi_I(\alpha)|(\Gamma^u(L_I), v) \lesssim \mathcal{F}(\Gamma^u(L_I), \tau(\Gamma^u(L_I, v)) \lesssim A\varepsilon^2\tau(\Gamma^u(L_I), v)^{-3+\delta} \tag{5.47}$$

for every $v \leq \Gamma^v(L_I)$.

Estimate for $u < \Gamma^u(L_{I(\tau_f)})$: By (2.19), the bootstrap assumptions, and (5.24), we estimate

$$\begin{aligned}|\partial_u(\tilde{\varpi} - \Pi_I(\alpha))| &\lesssim r^2(\overline{1-\mu})\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu}) + r^2|\widetilde{1-\mu}|\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu}) \\ &\lesssim r^2(\overline{1-\mu})\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu}) + A^2\varepsilon^{3/2}\tau^{-2+\delta}\left(\frac{\bar{r}-M}{\bar{r}^3}\right)\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu}) \\ &\quad + A\varepsilon^{3/2}\tau^{-3+\delta}r^2\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu}).\end{aligned} \tag{5.48}$$

If $v \geq \Gamma^v(L_I)$, then $r(u,v) \geq \Lambda$ and by integrating (5.48) and using (5.46) and the bootstrap assumptions we have

$$|\tilde{\varpi} - \Pi_I(\alpha)|(u,v) \lesssim A\varepsilon^2\tau_f^{-3+\delta} + \underline{\mathcal{F}}(v,\tau(u,v)) \lesssim A\varepsilon^2\tau(u,v)^{-3+\delta}. \tag{5.49}$$

On the other hand, if $v < \Gamma^v(L_I)$, we consider first the case $r(u,v) \leq \Lambda$. Integrating (5.48) backwards from $(\Gamma^u(L_I), v)$ and using (5.46) and the bootstrap assumptions, we have

$$\begin{aligned}|\tilde{\varpi} - \Pi_I(\alpha)|(u,v) &\lesssim A\varepsilon^2\tau(\Gamma^u(L_I),v)^{-3+\delta} + \underline{\mathcal{E}}_0(\tau(u,v)) \\ &\quad + A^2\varepsilon^{3/2}\tau(u,v)^{-2+\delta}\underline{\mathcal{E}}_1(\tau(u,v)) + A\varepsilon^{3/2}\tau(u,v)^{-3+\delta}\underline{\mathcal{E}}_2(\tau(u,v)) \\ &\lesssim A\varepsilon^2\tau(\Gamma^u(L_I),v)^{-3+\delta} + A\varepsilon^2\tau(u,v)^{-3+\delta} + A^3\varepsilon^{7/2}\tau(u,v)^{-4+2\delta} \\ &\lesssim A\varepsilon^2\tau(u,v)^{-3+\delta},\end{aligned} \tag{5.50}$$

as desired, where we note that $\tau(\Gamma^u(L_I),v) = \tau(u,v)$. If instead $r(u,v) \geq \Lambda$, then the ingoing null cone emanating from (u,v) will strike Γ at the point $(u_\star,v) = (u^\Lambda(v),v)$. We first apply the estimate (5.50) at $(u_\star,v)$ to obtain

$$|\tilde{\varpi} - \Pi_I(\alpha)|(u_\star,v) \lesssim A\varepsilon^2\tau(u_\star,v)^{-3+\delta}.$$

Integrating (5.48) further backwards towards (u,v), we finally obtain

$$\begin{aligned}|\tilde{\varpi} - \Pi_{I(\tau_f)}(\alpha)|(u,v) &\lesssim A\varepsilon^2\tau(u_\star,v)^{-3+\delta} + \underline{\mathcal{F}}(v,\tau(u,v)) \\ &\lesssim A\varepsilon^2\tau(u_\star,v)^{-3+\delta} + A\varepsilon^2\tau(u,v)^{-3+\delta} \\ &\lesssim A\varepsilon^2\tau(u,v)^{-3+\delta},\end{aligned}$$

as desired.

Estimate for $u > \Gamma^u(L_{I(\tau_f)})$: This is similar to the case $u < \Gamma^u(L_{I(\tau_f)})$ and is left to the reader. □

Remark 5.15. It is clear from the proof that the estimate (5.44) also holds with $I(\tau_f)$ replaced by $I(\tau_f)-1$ if $I(\tau_f) \geq 1$. This observation will be used later in Section 8.1.2 when we perform the modulation argument for ϖ.

5.3.4 The proof of Proposition 4.3

We now put all of the estimates proved in this section together and improve the bootstrap assumptions for the geometry, (4.5)–(4.7).

Proof of Proposition 4.3. To summarize: by Lemmas 5.11, 5.13, and 5.14, we have the estimates

$$\begin{aligned}|\tilde{r}| &\lesssim A\varepsilon^{3/2}\tau^{-2+\delta}, \\ \left|\frac{\nu}{\bar{\nu}} - 1\right| &\lesssim A^2\varepsilon^{3/2}\tau^{-1+\delta}, \\ |\tilde{\varpi} - \Pi_{I(\tau_f)}(\alpha)| &\lesssim A\varepsilon^2\tau^{-3+\delta}\end{aligned}$$

in $\mathcal{D}_{\tau_f}$, where the implicit constant <u>does not</u> depend on A. Therefore, by choosing A sufficiently large, the first two estimates imply

$$\begin{aligned}|\tilde{r}| &\leq \tfrac{1}{2}A^2\varepsilon^{3/2}\tau^{-2+\delta}, \\ \left|\frac{\nu}{\bar{\nu}} - 1\right| &\leq \tfrac{1}{2}A^3\varepsilon^{3/2}\tau^{-1+\delta}\end{aligned}$$

as desired. In order to improve the bootstrap assumption for $\tilde{\varpi}$, we use the definition of the modulation sets (4.1) and (4.3). Namely, since $|\Pi_{I(\tau_f)}| \leq \varepsilon^{3/2}L_{I(\tau_f)}^{-3+\delta}$, we may estimate

$$|\tilde{\varpi}| \leq |\tilde{\varpi} - \Pi_{I(\tau_f)}(\alpha)| + |\Pi_{I(\tau_f)}(\alpha)| \lesssim A\varepsilon^2\tau^{-3+\delta} + \varepsilon^{3/2}\tau_f^{-3+\delta} \lesssim (A\varepsilon^{1/2}+1)\varepsilon^{3/2}\tau^{-3+\delta}.$$

Therefore, by choosing A sufficiently large and ε correspondingly sufficiently small, we have

$$|\tilde{\varpi}| \leq \tfrac{1}{2}A\varepsilon^{3/2}\tau^{-3+\delta}$$

in $\mathcal{D}_{\tau_f}$ as desired. □

6 Energy estimates for the scalar field

In this section, we derive the fundamental hierarchies of energy estimates for the scalar field which were used to improve the estimates for the geometry in Section 5. In Section 6.1, we prove a basic degenerate energy boundedness statement for ϕ which is analogous to the usual T-energy estimate on extremal Reissner–Nordström. In Section 6.2, we prove Morawetz estimates for ϕ, i.e., weighted spacetime L^2 bounds for ϕ and its derivatives. In Section 6.3, we prove $(\bar{r}-M)^{2-p}$-weighted energy estimates for $\partial_u\psi$ in the near-horizon region, where $\psi \doteq r\phi$. Finally, in Section 6.4 we prove r^p-weighted energy estimates for $\partial_v\psi$ in the far region.

In this section, we adopt the notational conventions outlined in Section 5.1. We will also often use the geometric bootstrap assumptions and their consequences, such as Lemma 5.1, without comment.

The energy estimates in this section (except for Lemma 6.2) will take place on a region $\mathcal{R}\subset\mathcal{D}_{\tau_f}$ defined as follows: Let $\tau_f\in\mathfrak{B}$, $(u_1,v_1),(u_2,v_2)\in\Gamma\cap\mathcal{D}_{\tau_f}$, $u_2'>u_2$, and $v_2'>v_2$, where (u_2,v_2) is to the future of (u_1,v_1) We then define

$$\mathcal{R}\doteq[u_1,u_2]\times[v_1,v_2']\cup[u_1,u_2']\times[v_1,v_2]$$

and

$$\mathcal{R}_{\le\Lambda}\doteq\mathcal{R}\cap\{r\le\Lambda\},\quad \mathcal{R}_{\ge\Lambda}\doteq\mathcal{R}\cap\{r\ge\Lambda\}.$$

The null segments constituting $\partial\mathcal{R}$ are numbered I–VI as depicted in Fig. 10 below. We define τ_1 to be the value of τ on $\mathrm{I}\cup\mathrm{II}$ and τ_2 to be the value of τ on $\mathrm{IV}\cup\mathrm{V}$.

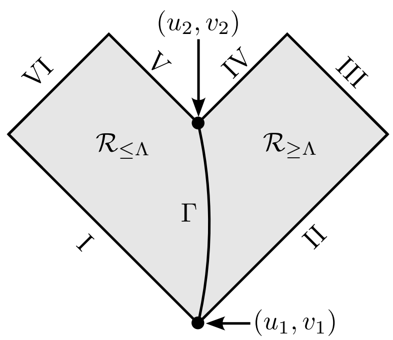

Figure 10: A Penrose diagram depicting the region $\mathcal{R}$ and the hypersurfaces I–VI used in the energy estimates in this section.

6.1 The degenerate energy estimate

We begin with an energy boundedness statement (with a small, decaying nonlinear error term) for an energy which degenerates quadratically in $\partial_u\phi$ at $\bar{r}=M$.

Proposition 6.1. *For any $A\ge 1$, ε sufficiently small, $\tau_f\in\mathfrak{B}$, $\alpha\in\mathfrak{A}_{I(\tau_f)}$, $(u_1,v_1),(u_2,v_2)\in\Gamma\cap\mathcal{D}_{\tau_f}$, $u_2'>u_2$, and $v_2'>v_2$, where (u_2,v_2) is to the future of (u_1,v_1), it holds that*

$$\begin{aligned}&\int_{\mathrm{III}}\left(r^2(\partial_u\phi)^2+\phi^2\right)du+\int_{\mathrm{IV}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv+\int_{\mathrm{V}}\left((\bar{r}-M)^2\frac{(\partial_u\phi)^2}{-\bar{\nu}}-\bar{\nu}\phi^2\right)du\\&+\int_{\mathrm{VI}}\left(r^2(\partial_v\phi)^2+\bar{\lambda}\phi^2\right)dv\lesssim\int_{\mathrm{I}}\left((\bar{r}-M)^2\frac{(\partial_u\phi)^2}{-\bar{\nu}}-\bar{\nu}\phi^2\right)du+\int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv+\varepsilon^3\tau_1^{-4+2\delta},\end{aligned}\tag{6.1}$$

where the hypersurfaces I–VI *are as depicted in Fig. 10.*

We prove this estimate by using the *Kodama vector field* [Kod80]

$$K\doteq\frac{1-\mu}{2}\left(\frac{1}{\lambda}\partial_v-\frac{1}{\nu}\partial_u\right)\tag{6.2}$$

to derive an energy identity on $\partial\mathcal{R}$. On Reissner–Nordström, K is exactly the time-translation Killing vector field. Of course, K is no longer Killing on a general spherically symmeytric solution of the Einstein–Maxwell-scalar field system, but it has the remarkable property that the current $T^{\mathrm{SF}}_{\mu\nu}K^\nu$ is divergence-free, where T^{SF} is the energy-momentum tensor of the scalar field (2.11). This property is expressed more plainly by the identity (6.3) below. In Remark 6.3 below, we sketch a proof that avoids the use of K.

We will augment the energy identity arising from K with a "null Lagrangian" current [Chr00] generated by a function σ on $\mathcal{D}_{\tau_f}$ which allows us to simultaneously estimate zeroth order fluxes of ϕ. This current is analogous to the two-form ϖ used in [DHRT22; HMVR24] and can be thought of as encoding a Hardy inequality on each face of $\partial\mathcal{R}$ *without boundary terms.*

Proof of Proposition 6.1. We apply the general multiplier identity (2.23) to the Kodama vector field (6.2) and add the trivial identity $\partial_u\partial_v(\sigma\phi^2)-\partial_v\partial_u(\sigma\phi^2)=0$, where σ is an arbitrary smooth function on $\mathcal{D}_{\tau_f}$, to obtain

$$\partial_u\left(\kappa^{-1}r^2(\partial_v\phi)^2+\partial_v(\sigma\phi^2)\right)+\partial_v\left(-\gamma^{-1}r^2(\partial_u\phi)^2-\partial_u(\sigma\phi^2)\right)=0. \tag{6.3}$$

Integrating this relation over the region $\mathcal{R}$ by parts, we obtain the energy identity

$$\begin{aligned}\int_{\mathrm{III}}\left((1-\mu)r^2\frac{(\partial_u\phi)^2}{-\nu}-\partial_u(\sigma\phi^2)\right)du+\int_{\mathrm{IV}}\left(\frac{r^2}{\kappa}(\partial_v\phi)^2+\partial_v(\sigma\phi^2)\right)dv&\\ +\int_{\mathrm{V}}\left((1-\mu)r^2\frac{(\partial_u\phi)^2}{-\nu}-\partial_u(\sigma\phi^2)\right)du+\int_{\mathrm{VI}}\left(\frac{r^2}{\kappa}(\partial_v\phi)^2+\partial_v(\sigma\phi^2)\right)dv&\\ =\int_{\mathrm{I}}\left((1-\mu)r^2\frac{(\partial_u\phi)^2}{-\nu}-\partial_u(\sigma\phi^2)\right)du+\int_{\mathrm{II}}\left(\frac{r^2}{\kappa}(\partial_v\phi)^2+\partial_v(\sigma\phi^2)\right)dv.&\end{aligned} \tag{6.4}$$

For a small constant $\eta>0$ to be determined, we set $\sigma\doteq\eta(\bar r-M)$.

If I_u denotes the du integrand in (6.4), we then compute

$$I_u=r^2(1-\bar\mu)\frac{\bar\nu}{\nu}\frac{(\partial_u\phi)^2}{-\bar\nu}+r^2(\widetilde{1-\mu})\frac{\bar\nu}{\nu}\frac{(\partial_u\phi)^2}{-\bar\nu}-\eta\bar\nu\phi^2-2\eta(\bar r-M)\phi\partial_u\phi.$$

For η sufficiently small, using (5.24) and Young's inequality, we estimate

$$I_u\lesssim r^2(1-\bar\mu)\frac{(\partial_u\phi)^2}{-\bar\nu}+A\varepsilon^{3/2}\tau^{-2+\delta}(\bar r-M)r\frac{(\partial_u\phi)^2}{-\bar\nu}+A\varepsilon^{3/2}\tau^{-3+\delta}r^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2, \tag{6.5}$$

$$I_u\gtrsim r^2(1-\bar\mu)\frac{(\partial_u\phi)^2}{-\bar\nu}-A\varepsilon^{3/2}\tau^{-2+\delta}(\bar r-M)r\frac{(\partial_u\phi)^2}{-\bar\nu}-A\varepsilon^{3/2}\tau^{-3+\delta}r^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2 \tag{6.6}$$

on $\mathcal{D}_{\tau_f}$. For the dv integrand I_v, we readily estimate

$$I_v\sim r^2(\partial_v\phi)^2+\bar\lambda\phi^2 \tag{6.7}$$

on $\mathcal{D}_{\tau_f}$ for η sufficiently small.

Inserting the estimates (6.5)–(6.7) into (6.4), we obtain

$$\begin{aligned}\int_{\mathrm{III}}\left(r^2(\partial_u\phi)^2+\phi^2\right)du+\int_{\mathrm{IV}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv+\int_{\mathrm{V}}\left((\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du&\\ +\int_{\mathrm{VI}}\left(r^2(\partial_v\phi)^2+\bar\lambda\phi^2\right)dv\lesssim\int_{\mathrm{I}}\left((\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du+\int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv+A\varepsilon^{3/2}E,&\end{aligned} \tag{6.8}$$

where the error term E is given by

$$E\doteq\int_{\mathrm{I}}\tau^{-2+\delta}(\bar r-M)\frac{(\partial_u\phi)^2}{-\bar\nu}\,du+\int_{\mathrm{I}}\tau^{-3+\delta}\frac{(\partial_u\phi)^2}{-\bar\nu}\,du+\int_{\mathrm{V}}\tau^{-2+\delta}(\bar r-M)\frac{(\partial_u\phi)^2}{-\bar\nu}\,du+\int_{\mathrm{V}}\tau^{-3+\delta}\frac{(\partial_u\phi)^2}{-\bar\nu}\,du.$$

Using the bootstrap assumption (4.9) with $p=1$ and $p=2$, we estimate

$$E\lesssim\varepsilon^3\tau_1^{-4+2\delta}+\varepsilon^3\tau_2^{-4+2\delta},$$

which when inserted into (6.8) completes the proof of (6.1). $\square$

With essentially the same proof, we also have:

Lemma 6.2. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_{I(\tau_f)}$, $(u_1, v_1), (u_2, v_2) \in \Gamma \cap \mathcal{D}_{\tau_f}$, where (u_2, v_2) is to the future of (u_1, v_1), it holds that*

$$\int_{\mathrm{iii}} \big(r^2(\partial_u\phi)^2 + \phi^2\big)\,du + \int_{\mathrm{iv}} \big(r^2(\partial_v\phi)^2 + \bar\lambda\phi^2\big)\,dv \lesssim \int_{\mathrm{i}} \left((\bar r - M)^2 \frac{(\partial_u\phi)^2}{-\bar\nu} - \bar\nu\phi^2\right) du + \int_{\mathrm{ii}} \big(r^2(\partial_v\phi)^2 + \phi^2\big)\,dv + \varepsilon^3\tau_1^{-4+2\delta}, \quad (6.9)$$

where the hypersurfaces i–iv *are as depicted in Fig. 11.*

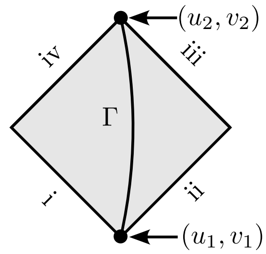

Figure 11: A Penrose diagram depicting the hypersurfaces i–iv used in Lemma 6.2.

Remark 6.3. The Kodama vector field can be avoided by instead using

$$\bar T \doteq \frac{1}{2}(\partial_v + \partial_u) = \frac{1-\bar\mu}{2}\left(\frac{1}{\bar\lambda}\partial_v - \frac{1}{\bar\nu}\partial_u\right),$$

the time-translation Killing vector field of the comparison extremal Reissner–Nordström solution. In this remark, we sketch how to use $\bar T$ to prove the estimate (6.1) (with $-4+2\delta$ replaced by $-4+3\delta$).

For $\bar T$, the identity (2.23) yields

$$\partial_u\big(r^2(\partial_v\phi)^2\big) + \partial_v\big(r^2(\partial_u\phi)^2\big) = -2r(\nu+\lambda)\partial_u\phi\partial_v\phi.$$

We now explain how to estimate the bulk error term

$$\iint_{\mathcal{R}} r|\lambda+\nu||\partial_u\phi||\partial_v\phi|\,dudv.$$

Estimate in $\mathcal{R}_{\leq\Lambda}$: Using the definitions, we have

$$\lambda + \nu = (1-\bar\mu)\left(\kappa - \frac{\nu}{\bar\nu}\right) + (\widetilde{1-\mu})\kappa,$$

which by (4.5), (5.9), and (5.25) implies

$$|\lambda+\nu| \lesssim A^3\varepsilon^{3/2}\tau^{-1+\delta}(1-\bar\mu) + A^2\varepsilon^{3/2}\tau^{-2+\delta}. \quad (6.10)$$

Using this estimate and Young's inequality yields

$$\iint_{\mathcal{R}_{\leq\Lambda}} r|\lambda+\nu||\partial_u\phi||\partial_v\phi|\,dudv \lesssim A^3\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}} \left((\bar r - M)^4\frac{(\partial_u\phi)^2}{-\bar\nu} - \bar\nu(\partial_v\phi)^2\right) dudv + A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}} \tau^{-4+2\delta}\frac{(\partial_u\phi)^2}{-\bar\nu}\,dudv.$$

The first term can be absorbed by the Morawetz bulk (see already Proposition 6.4) and the second term can be estimated using the bootstrap assumption (4.9),

$$A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}}\tau^{-4+2\delta}\frac{(\partial_u\phi)^2}{-\bar{\nu}}\,dudv\lesssim A^2\varepsilon^{3/2}\int_{\tau_1}^{\tau_2}\tau^{-4+2\delta}\underline{\mathcal{E}}_2(\tau)\,d\tau\lesssim\varepsilon^3\int_{\tau_1}^{\tau_2}\tau^{-5+3\delta}\,d\tau\lesssim\varepsilon^3\tau_1^{-4+3\delta}.$$

Estimate in $\mathcal{R}_{\geq\Lambda}$: The estimate (6.10) implies $|\lambda+\nu|\lesssim A^3\varepsilon^{3/2}\tau^{-1+\delta}$ in $\mathcal{R}_{\geq\Lambda}$. Therefore, we may estimate

$$\iint_{\mathcal{R}_{\geq\Lambda}}r|\lambda+\nu||\partial_u\phi||\partial_v\phi|\,dudv\lesssim A^3\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}}r^{1-\eta}(\partial_u\phi)^2\,dudv+A^3\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}}\tau^{-1+\delta}r^{1+\eta}(\partial_v\phi)^2\,dudv,$$

where $\eta>0$ is the parameter in the Morawetz estimate in Proposition 6.4. The first term can again be absorbed into the Morawetz bulk and the second term can be estimated by using the bootstrap assumption (4.8),

$$A^3\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}}r^{1+\eta}(\partial_v\phi)^2\,dudv\lesssim A^3\varepsilon^{3/2}\int_{\tau_1}^{\tau_2}\tau^{-2+2\delta}\mathcal{E}_0(\tau)\,d\tau\lesssim\varepsilon^3\tau_1^{-4+3\delta},$$

which completes the proof.

6.2 Integrated local energy decay

In this section, we prove the following *Morawetz estimate* for the scalar field:

Proposition 6.4. *Fix $\eta\in(0,1)$. Then for any $A\geq 1$, ε sufficiently small, $\tau_f\in\mathfrak{B}$, $\alpha\in\mathfrak{A}_{I(\tau_f)}$, $(u_1,v_1),(u_2,v_2)\in\Gamma\cap\mathcal{D}_{\tau_f}$, $u_2'>u_2$, and $v_2'>v_2$, where (u_2,v_2) is to the future of (u_1,v_1), it holds that*

$$\iint_{\mathcal{R}}\left(\left(1-\frac{M}{\bar{r}}\right)^4r^{1-\eta}\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2+r^{1-\eta}(\partial_v\phi)^2+\left(1-\frac{M}{\bar{r}}\right)^2\frac{\phi^2}{r^{1+\eta}}\right)(-\bar{\nu})dudv$$
$$\lesssim\int_{\mathrm{I}}\left((\bar{r}-M)^2\frac{(\partial_u\phi)^2}{-\bar{\nu}}-\bar{\nu}\phi^2\right)du+\int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv+\varepsilon^3\tau_1^{-4+2\delta}.\quad(6.11)$$

The proof of this estimate will be given in Section 6.2.2 below after a series of lemmas.

6.2.1 Near-horizon Hardy inequalities I

In this section, we show how to control the zeroth order term on the left-hand side of (6.11) by other quantities which will naturally appear later in Section 6.2.2. We begin with the following identity, which is proved by directly expanding the term in square brackets and integrating by parts (see also Lemma 8.30 in [LO19b]).

Lemma 6.5. *For any $[u_1,u_2]\times\{v\}\subset\mathcal{D}_{\tau_f}$, $f:\mathcal{D}_{\tau_f}\to\mathbb{R}$, and $\alpha\in\mathbb{R}$, it holds that*

$$\frac{(\alpha+1)^2}{4}\int_{u_1}^{u_2}(\bar{r}-M)^\alpha f^2\,(-\bar{\nu})du+\int_{u_1}^{u_2}\frac{(\bar{r}-M)^\alpha}{-\bar{\nu}}\left[(\bar{r}-M)\partial_uf-\frac{\alpha+1}{2}(-\bar{\nu})f\right]^2du$$
$$=\int_{u_1}^{u_2}(\bar{r}-M)^{\alpha+2}\left(\frac{\partial_uf}{-\bar{\nu}}\right)^2(-\bar{\nu})du+\frac{\alpha+1}{2}[(\bar{r}-M)^{\alpha+1}f^2](u_1,v)-\frac{\alpha+1}{2}[(\bar{r}-M)^{\alpha+1}f^2](u_2,v).\quad(6.12)$$

We will use this identity several times later in Section 6.3.1, but for now use it to control the zeroth order bulk in (6.11):

Lemma 6.6. *With hypotheses as in Proposition 6.4, it holds that*

$$\iint_{\mathcal{R}_{\leq\Lambda}}(\bar{r}-M)^2\phi^2\,(-\bar{\nu})dudv\lesssim\iint_{\mathcal{R}_{\leq\Lambda}}(\bar{r}-M)^4\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu})dudv+\int_\Gamma\phi^2\,ds,\quad(6.13)$$

where ds is the arc length measure of Γ with respect to the Euclidean metric du^2+dv^2.

Proof. Let $v \in [v_1, v_2]$. We apply the identity (6.12) with $f = \phi$ and $\alpha = 2$ along the segment

$$\underline{S}_v \doteq [u^\Lambda(v), u_2'] \times \{v\} \subset \mathcal{R}_{\leq\Lambda} \tag{6.14}$$

to obtain the inequality

$$\int_{\underline{S}_v} (\bar{r} - M)^2 \phi^2 \, (-\bar{\nu}) du \lesssim \int_{\underline{S}_v} (\bar{r} - M)^4 \left(\frac{\partial_u \phi}{-\bar{\nu}} \right)^2 (-\bar{\nu}) du + \phi^2(u^\Lambda(v), v), \tag{6.15}$$

where we dropped the boundary term at (u_2', v) due to its favorable sign. Integrating this inequality now over $v \in [v_1, v_2]$ and using the fact that dv along Γ is proportional to ds by (5.17), we obtain (6.13). $\square$

6.2.2 Proof of the Morawetz estimate

The proof of Proposition 6.4 involves exploiting the following identities and using our energy boundedness statement Proposition 6.1 to control flux terms with unfavorable signs.

Lemma 6.7 (Morawetz identities)**.** *Let $f : (0, \infty) \to \mathbb{R}$ be a C^2 function and let ϕ be a solution of (2.16). Then the following identities hold, where $f = f(r)$ and $' = \frac{d}{dr}$:*

$$\begin{aligned} r^2 f' \left(\frac{\lambda}{-\gamma} (\partial_u \phi)^2 - \frac{\nu}{\kappa} (\partial_v \phi)^2 \right) + 4rf(1-\mu)\partial_u\phi\partial_v\phi + \frac{2fr^3}{\kappa(-\nu)} (\partial_u \phi)^2 (\partial_v \phi)^2 \\ = \partial_v \left(\frac{r^2 f}{-\gamma} (\partial_u \phi)^2 \right) - \partial_u \left(\frac{r^2 f}{\kappa} (\partial_v \phi)^2 \right), \end{aligned} \tag{6.16}$$

$$\begin{aligned} - r^2 f' \big(\lambda (\partial_u \phi)^2 - \nu (\partial_v \phi)^2 \big) - 2fr(\lambda - \nu - 2)\partial_u\phi\partial_v\phi + 2\big(rf''\lambda + 2(rf' - f)\kappa\varkappa\big)(-\nu)\phi^2 \\ = \partial_u \big(r^2 f (\partial_v \phi)^2 - 2rf'\lambda\phi^2 + \partial_v(rf\phi^2) \big) - \partial_v \big(r^2 f (\partial_u \phi)^2 + 2rf'\nu\phi^2 - \partial_u(rf\phi^2) \big). \end{aligned} \tag{6.17}$$

Proof. To prove (6.16), we apply the general multiplier identity (2.23) to the vector field

$$-f(r) \left(\frac{1}{\gamma} \partial_u + \frac{1}{\kappa} \partial_v \right). \tag{6.18}$$

To prove (6.17), we first apply (2.23) to the vector field

$$-f(r) \left(\partial_v - \partial_u \right) \tag{6.19}$$

to derive the identity

$$r^2 f' \big(\lambda (\partial_u \phi)^2 - \nu (\partial_v \phi)^2 \big) + 2rf(\lambda - \nu)\partial_u\phi\partial_v\phi = \partial_v \big(r^2 f (\partial_u \phi)^2 \big) - \partial_u \big(r^2 f (\partial_v \phi)^2 \big). \tag{6.20}$$

To obtain (6.17) we now add the zeroth order current identity (2.24) with $h = -2f$ to (6.20). $\square$

Remark 6.8. The vector field (6.18) is the same as used in [LO19b, Lemma 8.24] and the "modified" identity (6.17) is inspired by the identity (8.134) in [LO19b].

Remark 6.9. The vector fields used in these identities are variants of the usual Morawetz vector field $f(r)\partial_{r_\star}$ used in [DR09], for instance. We use (6.18) because for the choice $f = -r^{-3}$, the cross term in (6.16) (the middle term on the left-hand side) can be immediately absorbed into the good bulk using Young's inequality (see also [Are11a, Proposition 9.3.1]). One might worry about the quartic nonlinear error term, but it turns out to have a favorable sign with this choice of f! The point of the "modified" Morawetz identity (6.17) is twofold: it introduces a (potentially) good zeroth order term and (potentially) improves the r-weight on the cross term. We will show that both of these good properties are true for a well-chosen f below. However, this choice of f would produce a bad quartic error term in (6.16), so we use the "background" vector field (6.19) to derive the modified identity. Note that we could also use $f = -r^{-3}$ in (6.20), but dealing with the cross term would take a bit more work.

The proof of Proposition 6.4 now proceeds in four steps:

1. Use (6.20) with $f = -r^{-3}$ in $\mathcal{R}$ to control $\partial\phi$ in spacetime L^2, but with a suboptimal r-weight.
2. Use (6.20) with $f = -r^{-3}$ in $\mathcal{R}_{\geq\Lambda}$ to control $\partial\phi$ in L^2 along the timelike curve Γ.
3. Use (6.17) with $f = -1 + R^{-1}\chi r^{-\eta}$, where $R \geq \Lambda$ is a large constant and χ is an appropriately chosen cutoff, to improve the r-weights for $\partial\phi$ and to estimate ϕ in spacetime L^2 for $r \geq \Lambda$.
4. Combine these estimates with the Hardy inequality (6.13) to conclude (6.11).

Lemma 6.10. *With hypotheses as in Proposition 6.4, it holds that*

$$\iint_{\mathcal{R}} \frac{1}{r^2}\left(\left(1-\frac{M}{\bar r}\right)^4\left(\frac{\partial_u\phi}{-\bar\nu}\right)^2 + (\partial_v\phi)^2\right)(-\bar\nu)dudv$$
$$\lesssim \int_{\mathrm{I}}\left((\bar r - M)^2\left(\frac{\partial_u\phi}{-\bar\nu}\right)^2 + \phi^2\right)(-\bar\nu)du + \int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv + \varepsilon^3\tau_1^{-4+2\delta}. \tag{6.21}$$

Proof. We integrate (6.16) over $\mathcal{R}$ with the choice $f(r) = -r^{-3}$.

Estimating the fluxes: For this choice of f, the integrands in the flux terms on the right-hand side of (6.16) are bounded by the integrands in the fluxes of the Kodama energy identity (6.4). Therefore, by the same reasoning as in the proof of Proposition 6.1, we may estimate

$$\iint_{\mathcal{R}} B\,(-\bar\nu)dudv \lesssim \int_{\mathrm{I}}\left((\bar r - M)^2\frac{(\partial_u\phi)^2}{-\bar\nu} - \bar\nu\phi^2\right)du + \int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv + \varepsilon^3\tau_1^{-4+2\delta} \tag{6.22}$$

where

$$B \doteq 3r^{-2}\left(\lambda(1-\mu)\left(\frac{\partial_u\phi}{-\nu}\right)^2 + \kappa^{-1}(\partial_v\phi)^2\right) - 4r^{-2}(1-\mu)\left(\frac{\partial_u\phi}{-\nu}\right)\partial_v\phi + \frac{2}{\kappa\nu^2}(\partial_u\phi)^2(\partial_v\phi)^2.$$

Estimating the bulk: By Young's inequality, the sign of the final term in B, (5.2), and (5.12), we observe that

$$B \geq r^{-2}\lambda(1-\mu)\left(\frac{\partial_u\phi}{-\nu}\right)^2 + r^{-2}\kappa^{-1}(\partial_v\phi)^2 \gtrsim r^{-2}\lambda(1-\mu)\left(\frac{\partial_u\phi}{-\bar\nu}\right)^2 + r^{-2}(\partial_v\phi)^2.$$

Using the expansions (5.24) and (5.28), we derive

$$\left|\lambda(1-\mu) - (1-\bar\mu)^2\right| \lesssim A^2\varepsilon^{3/2}\left((1-\bar\mu)\tau^{-2+\delta} + (1-\bar\mu)^{1/2}\tau^{-5+2\delta} + \tau^{-6+2\delta}\right)$$

on $\mathcal{D}_{\tau_f}$, and, therefore,

$$\iint_{\mathcal{R}}\left[(\partial_v\phi)^2 + (1-\bar\mu)^2\left(\frac{\partial_u\phi}{-\bar\nu}\right)^2\right]r^{-2}(-\bar\nu)dudv \lesssim \iint_{\mathcal{R}} B(-\bar\nu)dudv + A^2\varepsilon^{3/2}E, \tag{6.23}$$

with error term

$$E \doteq \iint_{\mathcal{R}} r^{-2}\left((1-\bar\mu)\tau^{-2+\delta} + (1-\bar\mu)^{1/2}\tau^{-5+2\delta} + \tau^{-6+2\delta}\right)\left(\frac{\partial_u\phi}{-\bar\nu}\right)^2(-\bar\nu)dudv \doteq E_{\geq\Lambda} + E_{\leq\Lambda},$$

where the notation $E_{\geq\Lambda}$ means the integral is restricted to $\mathcal{R}_{\geq\Lambda}$ and similarly for $E_{\leq\Lambda}$. By (5.4) and (5.5), $E_{\geq\Lambda}$ can be absorbed into the integral of B in (6.23). To estimate $E_{\leq\Lambda}$, we use the bootstap assumptions for ϕ:

$$\iint_{\mathcal{R}\cap\{r\leq\Lambda\}}(1-\bar{\mu})\tau^{-2+\delta}\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu})dudv\lesssim\int_{\tau_1}^{\tau_2}\tau^{-2+\delta}\underline{\mathcal{E}}_0(\tau)\,d\tau\lesssim A\varepsilon^2\tau_1^{-4+2\delta},\tag{6.24}$$

$$\iint_{\mathcal{R}\cap\{r\leq\Lambda\}}(1-\bar{\mu})^{1/2}\tau^{-5+\delta}\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu})dudv\lesssim\int_{\tau_1}^{\tau_2}\tau^{-5+\delta}\underline{\mathcal{E}}_1(\tau)\,d\tau\lesssim A\varepsilon^2\tau_1^{-6+2\delta},\tag{6.25}$$

$$\iint_{\mathcal{R}\cap\{r\leq\Lambda\}}\tau^{-6+\delta}\left(\frac{\partial_u\phi}{-\bar{\nu}}\right)^2(-\bar{\nu})dudv\lesssim\int_{\tau_1}^{\tau_2}\tau^{-6+\delta}\underline{\mathcal{E}}_2(\tau)\,d\tau\lesssim A\varepsilon^2\tau_1^{-6+2\delta}.\tag{6.26}$$

Combining (6.22)–(6.26) yields (6.21), as desired. □

Next, we state a simple consequence of Stokes' theorem in the plane.

Lemma 6.11. *Let j^u, j^v, and Q be smooth functions on $\mathcal{R}_{\geq\Lambda}$ satisfying $\partial_u j^u+\partial_v j^v=Q$. Then the divergence identity*

$$\iint_{\mathcal{R}_{\geq\Lambda}}Q\,dudv=\int_{\mathrm{IV}}j^u\,dv-\int_{\mathrm{II}}j^u\,dv+\int_{\mathrm{III}}j^v\,du+\int_\Gamma(n^u_\Gamma j^u+n^v_\Gamma j^v)ds$$

holds, where

$$n_\Gamma\doteq\big((\dot\Gamma^u)^2+(\dot\Gamma^v)^2\big)^{-1/2}(\dot\Gamma^v\partial_u-\dot\Gamma^u\partial_v)$$

is the outward-pointing normal to $\mathcal{R}_{\geq\Lambda}$ along Γ with respect to the flat Euclidean metric du^2+dv^2 on $\mathcal{R}_{\geq\Lambda}$, $\dot{}$ is used to denote $\frac{d}{d\tau}$, and ds is the Euclidean arc length measure of Γ.

Lemma 6.12. *With hypotheses as in Proposition 6.4, it holds that*

$$\int_\Gamma\big((\partial_u\phi)^2+(\partial_v\phi)^2\big)ds\lesssim\int_{\mathrm{I}}\left((\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du+\int_{\mathrm{II}}\big(r^2(\partial_v\phi)^2+\phi^2\big)\,dv+\varepsilon^3\tau_1^{-4+2\delta}.\tag{6.27}$$

Proof. We integrate (6.16) over $\mathcal{R}_{\geq\Lambda}$ with the choice $f=-r^{-3}$. As in the proof of Lemma 6.10, the fluxes along the null hypersurfaces II, III, and IV can be estimated using Proposition 6.1. The bulk terms can be estimated using (6.21). Observe that the contribution along Γ is given by

$$I_\Gamma\doteq\int_\Gamma\left(-\frac{1}{\kappa r}(\partial_v\phi)^2n^u_\Gamma-\frac{1}{\gamma r}(\partial_u\phi^2)n^v_\Gamma\right)ds.\tag{6.28}$$

Since Γ is uniformly timelike (recall the estimate (5.17)), we have $n^u_\Gamma\sim-n^v_\Gamma\sim1$ and therefore (using also that $\kappa\sim1$ and $\gamma\sim-1$ on Γ)

$$I_\Gamma\sim\int_\Gamma\big((\partial_u\phi)^2+(\partial_v\phi)^2\big)ds.\tag{6.29}$$

This completes the proof of (6.27). □

Lemma 6.13. *With hypotheses as in Proposition 6.4, it holds that*

$$\begin{aligned}\iint_{\mathcal{R}_{\geq\Lambda}}&\left(r^{1-\eta}(\partial_u\phi)^2+r^{1-\eta}(\partial_v\phi)^2+\frac{\phi^2}{r^{1+\eta}}\right)(-\bar\nu)dudv+\int_\Gamma\phi^2\,ds\\&\lesssim_\eta\int_{\mathrm{I}}\left((\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du+\int_{\mathrm{II}}\big(r^2(\partial_v\phi)^2+\phi^2\big)\,dv+\varepsilon^3\tau_1^{-4+2\delta}.\end{aligned}\tag{6.30}$$

Proof. We integrate (6.17) over $\mathcal{R}\cap\{r\geq\Lambda\}$ with the choice $f(r)=-1+R^{-1}\chi(r)r^{-\eta}$, where $R\geq\Lambda$ is a large constant to be determined and χ is a cutoff function such that $\chi(r)=0$ for $r\leq R$, $\chi(r)=1$ for $r\geq2R$, and such that $|\chi'|\lesssim R^{-1}$ and $|\chi''|\lesssim R^{-2}$.

By Lemma 6.11, we have

$$\iint_{\mathcal{R}_{\geq\Lambda}}(F_1+F_2+Z\phi^2)\,(-\nu)dudv-\int_\Gamma(n^u_\Gamma j^u+n^v_\Gamma j^v)ds=\int_{\mathrm{IV}}j^u\,dv-\int_{\mathrm{II}}j^u\,dv+\int_{\mathrm{III}}j^v\,du,\tag{6.31}$$

where the bulk terms are given by

$$F_1 \doteq R^{-1}r^{1-\eta}(\eta\chi - \chi' r)\big((\partial_v\phi)^2 - \nu\lambda(\partial_u\phi)^2\big),$$

$$F_2 \doteq 2(1-R^{-1}\chi r^{-\eta})r(\lambda-\nu-2)\left(\frac{\partial_u\phi}{-\nu}\right)\partial_v\phi,$$

$$Z \doteq R^{-1}\lambda\left(\chi'' r^{1-\eta} - 2\eta\chi' r^{-\eta} + \eta(\eta+1)\chi r^{-1-\eta}\right) + 2\left(1 + R^{-1}\chi' r^{1-\eta} - (\eta+1)R^{-1}\chi r^{-\eta}\right)2\kappa\varkappa,$$

and the flux terms are given by

$$j^u \doteq r^2 f(\partial_v\phi)^2 + rf'\lambda\phi^2 + \lambda f\phi^2 + 2rf\phi\partial_v\phi, \quad j^v \doteq -r^2 f(\partial_u\phi)^2 - rf'\nu\phi^2 + \nu f\phi^2 + 2rf\phi\partial_u\phi.$$

Estimate for the zeroth order bulk: We claim that

$$Z = 2\kappa\varkappa + \eta(\eta+1)R^{-1}r^{-1-\eta}\chi - 2R^{-1}(\eta+1)\kappa\varkappa r^{-\eta}\chi + 2R^{-1}(\kappa\varkappa r^{1-\eta} - \eta r^{-\eta})\chi' + R^{-1}r^{1-\eta}\chi'' \gtrsim R^{-1}r^{-1-\eta} \tag{6.32}$$

on $\mathcal{R}_{\ge\Lambda}$ for R sufficiently large. For $r \le R$, only the first term is nonzero and since $\varkappa \sim r^{-2}$ for $r \ge \Lambda$, we have $Z \gtrsim R^{-1}r^{-1} \ge R^{-1}r^{-1-\eta}$. For $R \le r \le 2R$, we estimate

$$Z \gtrsim r^{-2} + R^{-1}r^{-1-\eta}\chi - R^{-2-\eta} \gtrsim r^{-2} \gtrsim R^{-1}r^{-1-\eta}$$

for R sufficiently large. For $r \ge 2R$, only the first two terms are nonzero and (6.32) is evidently true. We therefore have

$$\iint_{\mathcal{R}_{\ge\Lambda}} Z\phi^2\,(-\nu)\,du\,dv \gtrsim R^{-1}\iint_{\mathcal{R}_{\ge\Lambda}} r^{-1-\eta}\phi^2\,(-\bar\nu)\,du\,dv. \tag{6.33}$$

Estimates for the first order bulks: Let $F_0 \doteq r^{-2}\left((\partial_u\phi)^2 + (\partial_v\phi)^2\right)$. This quantity has the significance that its integral over $\mathcal{R}\cap\{r \ge \Lambda\}$ can be estimated by Lemma 6.10. It is clear that $F_1 \gtrsim R^{-1}r^{1-\eta}\left((\partial_u\phi)^2 + (\partial_v\phi)^2\right)$ for $r \ge 2R$. For F_2, we estimate

$$|F_2| \lesssim r|\lambda-\nu-2||\partial_u\phi||\partial_v\phi| \le r\left(|\bar\lambda - \bar\nu - 2| + |\tilde\lambda| + |\tilde\nu|\right)|\partial_u\phi||\partial_v\phi| \lesssim \left(1 + A^2\varepsilon^{3/2}r\tau^{-2+\delta}\right)|\partial_u\phi||\partial_v\phi|$$

using (5.34) and (5.42). For $r \le R^2$, we have $|\partial_u\phi||\partial_v\phi| \lesssim R^4F_0$ and for $r \ge R^2$, we have

$$|\partial_u\phi||\partial_v\phi| \le R^{-2+2\eta}r^{1-\eta}|\partial_u\phi||\partial_v\phi| \lesssim R^{-1+\eta}F_1.$$

Therefore, for any $b>0$ we may choose R sufficiently large that

$$|F_2| \lesssim R^4F_0 + b\mathbf{1}_{\{r\ge 2R\}}F_1 + A^2\varepsilon^{3/2}r\tau^{-2+\delta}|\partial_u\phi||\partial_v\phi|.$$

Choosing b sufficiently small to absorb the middle term, we therefore have that

$$\begin{aligned}\iint_{\mathcal{R}_{\ge\Lambda}}(F_1+F_2)\,(-\nu)\,du\,dv \gtrsim R^{-3}&\iint_{\mathcal{R}_{\ge\Lambda}} r^{1-\eta}\left((\partial_u\phi)^2 + (\partial_v\phi)^2\right)du\,dv\\ &- R^4\cdot(\text{LHS of (6.21)}) - A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\ge\Lambda}} r\tau^{-2+\delta}|\partial_u\phi||\partial_v\phi|\,du\,dv\end{aligned} \tag{6.34}$$

by Lemma 6.10 for R sufficiently large. At this point, we fix R and treat it as an implicit constant. For the final term on the right-hand side of this inequality, we use Young's inequality with a parameter $c>0$ and the bootstrap assumptions for ϕ to estimate

$$\begin{aligned}\iint_{\mathcal{R}_{\ge\Lambda}} r\tau^{-2+\delta}|\partial_u\phi||\partial_v\phi|\,du\,dv &\lesssim c\iint_{\mathcal{R}_{\ge\Lambda}}(\partial_u\phi)^2\,du\,dv + c^{-1}\iint_{\mathcal{R}_{\ge\Lambda}}\tau^{-4+2\delta}r^2(\partial_v\phi)^2\,du\,dv\\ &\lesssim c\iint_{\mathcal{R}_{\ge\Lambda}}(\partial_u\phi)^2\,du\,dv + c^{-1}\int_{\tau_1}^{\tau_2}\tau^{-4+2\delta}\mathcal{E}_0(\tau)\,d\tau\\ &\lesssim c\iint_{\mathcal{R}_{\ge\Lambda}}(\partial_u\phi)^2\,du\,dv + c^{-1}A\varepsilon^2\tau_1^{-6+3\delta}.\end{aligned}$$

Choosing c sufficiently small, we can absorb the first term on the right-hand side into the good bulk on the right-hand side of (6.34) and conclude

$$\iint_{\mathcal{R}_{\geq\Lambda}} (F_1+F_2)\,(-\nu)dudv \gtrsim \iint_{\mathcal{R}_{\geq\Lambda}} r^{1-\eta}\left((\partial_u\phi)^2+(\partial_v\phi)^2\right)dudv - (\text{LHS of (6.21)}) - \varepsilon^3\tau_1^{-6+3\delta}. \quad (6.35)$$

Estimates for the null fluxes: As in the proof of Lemma 6.10, the terms on the right-hand side of (6.31) are all estimated using Proposition 6.1 by

$$\text{RHS of (6.31)} \lesssim \int_{\mathrm{I}}\left((\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du + \int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv + \varepsilon^3\tau(u_1,v_1)^{-3+\delta}. \quad (6.36)$$

Estimate for the flux along Γ: On Γ, we have by definition

$$j^v|_\Gamma = r^2(\partial_u\phi)^2-\nu\phi^2-2r\phi\partial_u\phi,\quad j^u|_\Gamma = -r^2(\partial_v\phi)^2-\lambda\phi^2-2r\phi\partial_v\phi.$$

Therefore, using again that $n_\Gamma^u\sim -n_\Gamma^v\sim 1$, we have

$$-\int_\Gamma (n_\Gamma^u j^u + n_\Gamma^v j^v)ds \gtrsim \int_\Gamma \phi^2\,ds - \int_\Gamma\left((\partial_u\phi)^2+(\partial_v\phi)^2\right)ds. \quad (6.37)$$

Combining (6.31), (6.33), (6.35), (6.36), (6.37), and (6.27), we conclude (6.30). □

Proof of Proposition 6.4. Combine (6.13), (6.21), and (6.30). □

6.3 The $\mathcal{H}^+$-localized hierarchy

In this section, we prove the $(\bar r-M)^{2-p}$-hierarchy of weighted energy estimates in the near-horizon region. For the linear wave equation on extremal Reissner–Nordström, this hierarchy was introduced by Aretakis for $p=0,1$, and 2 in [Are11a] and for $p\in[0,3)$ (for the $\ell=0$ mode) by the first-named author, Aretakis, and Gajic in [AAG20a].

In order to give the cleanest statement, it is convenient to introduce the following operation: given a real number x, let $x_\star = x$ if $x\in[0,1)$, and $x_\star=0$ otherwise.

Proposition 6.14. *For any $p\in(0,3)$, $A\geq 1$, ε sufficiently small, $\tau_f\in\mathfrak{B}$, $\alpha\in\mathfrak{A}_{I(\tau_f)}$, $(u_1,v_1),(u_2,v_2)\in\Gamma\cap\mathcal{D}_{\tau_f}$, $u_2'>u_2$, and $v_2'>v_2$, where (u_2,v_2) is to the future of (u_1,v_1), it holds that*

$$\begin{aligned}\int_{\mathrm{V}}(\bar r-M)^{2-p}\left(\frac{(\partial_u\psi)^2}{-\bar\nu}+\frac{(\partial_u\phi)^2}{-\bar\nu}\right)du + \int_{\mathrm{V}}(\bar r-M)^{-p_\star}\phi^2\,(-\bar\nu)du&\\ + \iint_{\mathcal{R}_{\leq\Lambda}}(\bar r-M)^{3-p}\left(\frac{(\partial_u\psi)^2}{-\bar\nu}+\frac{(\partial_u\phi)^2}{-\bar\nu}\right)dudv + \iint_{\mathcal{R}_{\leq\Lambda}}(\bar r-M)^{-(p-1)_\star}\phi^2\,(-\bar\nu)dudv&\\ \lesssim \int_{\mathrm{I}}(\bar r-M)^{2-p}\left(\frac{(\partial_u\psi)^2}{-\bar\nu}+\frac{(\partial_u\phi)^2}{-\bar\nu}\right)du + \int_{\mathrm{I}}\phi^2\,(-\bar\nu)du + \int_{\mathrm{II}}\left(r^2(\partial_v\phi)^2+\phi^2\right)dv + \varepsilon^3\tau_1^{-4+2\delta+p}.& \quad (6.38)\end{aligned}$$

The proof of this estimate will be given in Section 6.3.2 below after a series of lemmas.

6.3.1 Near-horizon Hardy inequalities II

We will again require several inequalities to handle lower order terms.

Lemma 6.15. *Let $\alpha>-1$. With hypotheses as in Proposition 6.14, it holds that*

$$\iint_{\mathcal{R}_{\leq\Lambda}}(\bar r-M)^\alpha\psi^2\,(-\bar\nu)dudv \lesssim \iint_{\mathcal{R}_{\leq\Lambda}}(\bar r-M)^{\alpha+2}\frac{(\partial_u\psi)^2}{-\bar\nu}dudv + \int_\Gamma\phi^2\,ds, \quad (6.39)$$

$$\int_{\underline{S}_v}(\bar r-M)^\alpha\psi^2\,(-\bar\nu)du \lesssim \int_{\underline{S}_v}\left((\bar r-M)^{\alpha+2}\frac{(\partial_u\psi)^2}{-\bar\nu}+(\bar r-M)^2\frac{(\partial_u\phi)^2}{-\bar\nu}-\bar\nu\phi^2\right)du, \quad (6.40)$$

where $\underline{S}_v$ is defined in (6.14).

Proof. To prove (6.39), apply (6.12) with $f = \psi$ and argue as in the proof of Lemma 6.6. To prove (6.40), let $\chi = \chi(r)$ be a cutoff function such that $\chi(r) = 1$ for $r \leq \frac{1}{3}\Lambda$ and $\chi(r) = 0$ for $r \geq \frac{2}{3}\Lambda$ and apply (6.12) to $f = \chi\psi$. The cutoff kills the boundary term along Γ (which has an unfavorable sign) and by adding $\int_{\underline{S}_v} \phi^2 (-\bar{\nu}) du$ to both sides of the resulting inequality, we conclude (6.40). $\square$

Lemma 6.16. *With hypotheses as in Proposition 6.14, it holds that*

$$\int_{\underline{S}_v} (\bar{r} - M)^{2-p} \frac{(\partial_u \phi)^2}{-\bar{\nu}} \, du \lesssim \int_{\underline{S}_v} (\bar{r} - M)^{2-p} \left(\frac{(\partial_u \phi)^2}{-\bar{\nu}} - \bar{\nu}\psi^2 \right) du, \tag{6.41}$$

$$\iint_{\mathcal{R}_{\leq \Lambda}} (\bar{r} - M)^{2-p} \frac{(\partial_u \phi)^2}{-\bar{\nu}} \, dudv \lesssim \iint_{\mathcal{R}_{\leq \Lambda}} (\bar{r} - M)^{2-2} \left(\frac{(\partial_u \phi)^2}{-\bar{\nu}} - \bar{\nu}\psi^2 \right) dudv. \tag{6.42}$$

Proof. This follows immediately from the trivial estimate $(\partial_u \phi)^2 \lesssim (\partial_u \psi)^2 + \bar{\nu}^2 \psi^2$ in $\mathcal{R}_{\leq \Lambda}$. $\square$

6.3.2 Proof of the $(\bar{r} - M)^{2-p}$ estimates

We begin with a simple consequence of Stokes' theorem in the plane.

Lemma 6.17. *Let j^u, j^v, and Q be smooth functions on $\mathcal{R}_{\leq \Lambda}$ satisfying $\partial_u j^u + \partial_v j^v = Q$. Then the divergence identity*

$$\iint_{\mathcal{R}_{\leq \Lambda}} Q \, dudv = \int_{\mathrm{V}} j^v \, du - \int_{\mathrm{I}} j^v \, du + \int_{\mathrm{VI}} j^u \, dv - \int_{\Gamma} (n^u_\Gamma j^u + n^v_\Gamma j^v) ds$$

holds, where

$$n_\Gamma \doteq \left((\dot{\Gamma}^u)^2 + (\dot{\Gamma}^v)^2 \right)^{-1/2} (\dot{\Gamma}^v \partial_u - \dot{\Gamma}^u \partial_v)$$

is as in Lemma 6.11.

We now prove the main estimate of Proposition 6.14.

Lemma 6.18. *With hypotheses as in Proposition 6.14, it holds that*

$$\begin{aligned} \int_{\mathrm{V}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du + \iint_{\mathcal{R}_{\leq \Lambda}} (\bar{r} - M)^{3-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, dudv \lesssim \int_{\mathrm{I}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du \\ + \int_{\mathrm{I}} \left((\bar{r} - M)^2 \frac{(\partial_u \phi)^2}{-\bar{\nu}} - \bar{\nu}\phi^2 \right) du + \int_{\mathrm{II}} \left(r^2 (\partial_v \phi)^2 + \phi^2 \right) dv + \varepsilon^3 \tau_1^{-4+2\delta+p} \end{aligned} \tag{6.43}$$

Proof. We evaluate the expression

$$\iint_{\mathcal{R}_{\leq \Lambda}} \partial_v \left((\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \right) dudv$$

first by integrating by parts (using Lemma 6.17) and then by using the wave equation (2.18). This leads to the identity

$$\begin{aligned} &\int_{\mathrm{V}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du - \int_{\mathrm{I}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du + \int_{\Gamma} n^v_\Gamma (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, ds \\ &\quad = - \iint_{\mathcal{R}_{\leq \Lambda}} \left(\frac{2M + (p-2)\bar{r}}{\bar{r}^3} \right) (\bar{r} - M)^{3-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, dudv - \iint_{\mathcal{R}_{\leq \Lambda}} \frac{4\kappa\nu}{r\bar{\nu}} (\bar{r} - M)^{2-p} \varkappa \psi \partial_u \psi \, dudv \end{aligned}$$

Since $p \in (0, 3)$, $2M + (p-2)\bar{r} > 0$ for $\bar{r}$ close to M, so we may add a large multiple of the Morawetz estimate (6.11) and use additionally (6.27) and (6.30) to estimate

$$\begin{aligned} &\int_{\mathrm{V}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du + \iint_{\mathcal{R}_{\leq \Lambda}} (\bar{r} - M)^{3-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, dudv \\ &\qquad \lesssim \int_{\mathrm{I}} (\bar{r} - M)^{2-p} \frac{(\partial_u \psi)^2}{-\bar{\nu}} \, du + \iint_{\mathcal{R}_{\leq \Lambda}} (\bar{r} - M)^{2-p} |\varkappa| |\psi| |\partial_u \psi| \, dudv + \text{RHS of (6.11)} \end{aligned} \tag{6.44}$$

Using Young's inequality with a parameter $b > 0$, the geometric estimate (5.32), and the observation that $0 < \bar{\varkappa} \lesssim \bar{r} - M$ in $\mathcal{R}_{\leq\Lambda}$, we estimate the error term on the right-hand side by

$$\iint_{\mathcal{R}_{\leq\Lambda}} (\bar{r}-M)^{2-p}|\varkappa||\psi||\partial_u\psi|\, dudv \lesssim b \iint_{\mathcal{R}_{\leq\Lambda}} (\bar{r}-M)^{3-p}\frac{(\partial_u\psi)^2}{-\bar{\nu}}\, dudv + b^{-1}\iint_{\mathcal{R}_{\leq\Lambda}} (\bar{r}-M)^{3-p}\psi^2\,(-\bar{\nu})dudv$$
$$+ A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}} \tau^{-2+\delta}(\bar{r}-M)^{2-p}\left(b\frac{(\partial_u\psi)^2}{-\bar{\nu}} + b^{-1}(-\bar{\nu})\psi^2\right) dudv. \quad (6.45)$$

The first term can be absorbed into the bulk on the right-hand side of (6.44) if b is chosen sufficiently small. The second term may be estimated by the Hardy inequality (6.39),

$$\iint_{\mathcal{R}_{\leq\Lambda}} (\bar{r}-M)^{3-p}\psi^2\,(-\bar{\nu})dudv \lesssim \iint_{\mathcal{R}_{\leq\Lambda}} (\bar{r}-M)^{5-p}\frac{(\partial_u\psi)^2}{-\bar{\nu}}\, dudv + \int_\Gamma \phi^2\, ds,$$

and since $5 - p > 3 - p$, we may use (6.27) and add a large multiple of the Morawetz estimate (6.11) to handle the error. The third term in (6.45) is estimated using the bootstrap assumption (4.9),

$$A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}} \tau^{-2+\delta}(\bar{r}-M)^{2-p}\frac{(\partial_u\psi)^2}{-\bar{\nu}}\, dudv \lesssim A^2\varepsilon^{3/2}\int_{\tau_1}^{\tau_2}\tau^{-2+\delta}\underline{\mathcal{E}}_p(\tau)\, d\tau \lesssim \varepsilon^3\tau_1^{-4+2\delta+p}. \quad (6.46)$$

The fourth term in (6.45) is estimated by first applying the Hardy inequality (6.40) and then the bootstrap assumption (4.9),

$$\begin{aligned} &A^2\varepsilon^{3/2}\iint_{\mathcal{R}_{\leq\Lambda}} \tau^{-2+\delta}(\bar{r}-M)^{2-p}\psi^2\,(-\bar{\nu})dudv \\ &\quad\lesssim A^2\varepsilon^{3/2}\int_{v_1}^{v_2}\tau^{-2+\delta}\int_{\underline{S}_v}\left((\bar{r}-M)^{4-p}\frac{(\partial_u\psi)^2}{-\bar{\nu}} + (\bar{r}-M)^2\frac{(\partial_u\phi)^2}{-\bar{\nu}} - \bar{\nu}\phi^2\right) dudv \\ &\quad\lesssim A^2\varepsilon^{3/2}\int_{\tau_1}^{\tau_2}\tau^{-2+\delta}\underline{\mathcal{E}}_{\max\{p-2,0\}}(\tau)\, d\tau \lesssim \varepsilon^3\tau_1^{-4+2\delta+\max\{p-2,0\}} \lesssim \varepsilon^3\tau_1^{-4+2\delta+p}. \quad (6.47)\end{aligned}$$

Combining (6.44)–(6.47) proves (6.43). □

Proof of Proposition 6.14. Combine (6.43) and (6.39)–(6.42). □

6.4 The $\mathcal{I}^+$-localized hierarchy

In this section, we prove Dafermos and Rodnianski's hierarchy of r^p-weighed energy estimates in the far region. Recall the notation $x_\star = \mathbf{1}_{[0,1)}x$ from Section 6.3.

Proposition 6.19. *For any $p \in (0, 3)$, $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_{I(\tau_f)}$, $(u_1, v_1), (u_2, v_2) \in \Gamma \cap \mathcal{D}_{\tau_f}$, $u_2' > u_2$, and $v_2' > v_2$, where (u_2, v_2) is to the future of (u_1, v_1), it holds that*

$$\begin{aligned}&\int_{\mathrm{IV}}\left(r^p(\partial_v\psi)^2 + r^{p_\star+2}(\partial_v\phi)^2 + r^{p_\star}\phi^2\right) dv + \iint_{\mathcal{R}_{\geq\Lambda}}\left(r^{p-1}(\partial_v\psi)^2 + r^{(p-1)_\star+2}(\partial_v\phi)^2 + r^{(p-1)_\star}\phi^2\right) dudv \\ &\quad\lesssim \iint_{\mathrm{II}}\left(r^p(\partial_v\psi)^2 + r^2(\partial_v\phi)^2 + \phi^2\right) dv + \int_{\mathrm{I}}\left((\bar{r}-M)^2\frac{(\partial_u\phi)^2}{-\bar{\nu}} - \bar{\nu}\phi^2\right) du + \varepsilon^3\tau_1^{-4+2\delta}. \quad (6.48)\end{aligned}$$

The proof of this estimate will be given in Section 6.4.2 below after a series of lemmas.

6.4.1 r^p-weighted Hardy inequalities

We will again require several inequalities to handle lower order terms. The following identity is proved by expanding the term in square bracket and integrating by parts (see also [LO19b, Lemma 8.30]).

Lemma 6.20. *For any $\{u\} \times [v_1, v_2] \subset \mathcal{D}_{\tau_f}$, $f : \mathcal{D}_{\tau_f} \to \mathbb{R}$, and $\alpha \in \mathbb{R}$, it holds that*

$$\frac{(\alpha+1)^2}{4} \int_{v_1}^{v_2} r^\alpha f^2 \, \lambda dv + \int_{v_1}^{v_2} \frac{r^\alpha}{\lambda} \left[r\partial_v f + \frac{\alpha+1}{2} \lambda f \right]^2 dv$$
$$= \int_{v_1}^{v_2} \frac{r^{\alpha+2}}{\lambda} (\partial_v f)^2 \, dv + \frac{\alpha+1}{2} (r^{\alpha+1} f^2)(u, v_2) - \frac{\alpha+1}{2} (r^{\alpha+1} f^2)(u, v_1). \quad (6.49)$$

Lemma 6.21. *Let $\alpha < -1$. With hypotheses as in Proposition 6.19, it holds that*

$$\iint_{\mathcal{R}_{\geq \Lambda}} r^\alpha \psi^2 \, dudv \lesssim \iint_{\mathcal{R}_{\geq \Lambda}} r^{\alpha+2} (\partial_v \psi)^2 \, dudv + \int_\Gamma \phi^2 \, ds, \quad (6.50)$$

$$\int_{S_u} r^\alpha \psi^2 \, dv \lesssim \int_{S_u} \left(r^{\alpha+2} (\partial_v \psi)^2 + \phi^2 \right) dv, \quad (6.51)$$

where

$$S_u \doteq \{u\} \times [v^\Lambda(u), v_2'].$$

Proof. To prove (6.50), apply (6.49) with $f = \psi$ to derive

$$\int_{S_u} r^\alpha \psi^2 \, dv \lesssim \int_{S_u} r^{\alpha+2} (\partial_v \psi)^2 \, dv + (r^{\alpha+1} \psi^2)(u, v^\Lambda(u)),$$

where we have used that the boundary term at v_2' has a favorable sign. Now (6.50) is obtained by integrating this inequality in u. To prove (6.51), $\chi = \chi(r)$ be a cutoff function such that $\chi(r) = 1$ for $r \geq 3\Lambda$ and $\chi(r) = 0$ for $r \leq 2\Lambda$ and apply (6.49) to $f = \chi\psi$. The cutoff kills the boundary term along Γ and by adding $\int_{S_u} \phi^2 \, dv$ to both sides of the resulting inequality, we conclude (6.51). □

Lemma 6.22. *Let $\alpha < -1$. With hypotheses as in Proposition 6.19, it holds that*

$$\int_{S_u} r^{\alpha+2} (\partial_v \phi)^2 \, dv \lesssim \int_{\underline{S}_v} \left(r^\alpha (\partial_v \psi)^2 + r^{\alpha-2} \psi^2 \right) dv, \quad (6.52)$$

$$\iint_{\mathcal{R}_{\geq \Lambda}} r^{\alpha+2} (\partial_v \phi)^2 \, dudv \lesssim \iint_{\mathcal{R}_{\geq \Lambda}} \left(r^\alpha (\partial_v \psi)^2 + r^{\alpha-2} \psi^2 \right) dudv. \quad (6.53)$$

Proof. This follows immediately from the trivial estimate $(\partial_v \phi)^2 \lesssim r^{-2} (\partial_v \psi)^2 + r^{-4} \psi^2$ in $\mathcal{R}_{\leq \Lambda}$. □

6.4.2 Proof of the r^p estimates

We now prove the main estimate of Proposition 6.19.

Lemma 6.23. *With hypotheses as in Proposition 6.19, it holds that*

$$\int_{\mathrm{IV}} r^p (\partial_v \psi)^2 \, dv + \iint_{\mathcal{R}_{\geq \Lambda}} r^{p-1} (\partial_v \psi)^2 \, dudv \lesssim \int_{\mathrm{II}} r^p (\partial_v \psi)^2 \, dv$$
$$+ \int_{\mathrm{I}} \left((\bar{r} - M)^2 \frac{(\partial_u \phi)^2}{-\bar{\nu}} - \bar{\nu} \phi^2 \right) du + \int_{\mathrm{II}} \left(r^2 (\partial_v \phi)^2 + \phi^2 \right) dv + \varepsilon^3 \tau(u_1, v_1)^{-4+2\delta}. \quad (6.54)$$

Proof. We evaluate the expression

$$\iint_{\mathcal{R}_{\geq \Lambda}} \partial_u \left(r^p (\partial_v \psi)^2 \right) dudv$$

first by integrating by parts and then by using the wave equation (2.18). This leads to the identity

$$\int_{\mathrm{IV}} r^p (\partial_v \psi)^2 \, dv - \int_{\mathrm{II}} r^p (\partial_v \psi)^2 \, dv + \int_\Gamma n_\Gamma^u r^p (\partial_v \psi)^2 \, ds$$
$$= - \iint_{\mathcal{R}_{\geq \Lambda}} p r^{p-1} (\partial_v \psi)^2 \, (-\nu) dudv + \iint_{\mathcal{R}_{\geq \Lambda}} 4\kappa\nu\varkappa r^{p-1} \psi \partial_v \psi \, dudv. \quad (6.55)$$

We estimate the second term by

$$\left|\iint_{\mathcal{R}_{\geq\Lambda}} 4\kappa\nu\varkappa r^{p-1}\psi\partial_v\psi\, dudv\right| \lesssim \iint_{\mathcal{R}_{\geq\Lambda}} r^{p-3}\psi\partial_v\psi \lesssim b\iint_{\mathcal{R}_{\geq\Lambda}} r^{p-1}(\partial_v\psi)^2\, dudv + b^{-1}\iint_{\mathcal{R}_{\geq\Lambda}} r^{p-5}\psi^2\, dudv$$

where $b > 0$ is an arbitrary constant. For b sufficiently small, this can be absorbed by the good bulk on the right-hand side of (6.55). To estimate the lower order term, we apply Hardy's inequality (6.50) with $\alpha = p - 5 < -1$ to obtain

$$\iint_{\mathcal{R}_{\geq\Lambda}} r^{p-5}\psi^2\, dudv \lesssim \iint_{\mathcal{R}_{\geq\Lambda}} r^{p-3}(\partial_v\psi)^2\, dv + \int_\Gamma \phi^2\, ds.$$

Since $p - 3 < p - 1$, for any sufficiently small $c > 0$ we have the estimate

$$\iint_{\mathcal{R}_{\geq\Lambda}} r^{p-3}(\partial_v\psi)^2\, dv \lesssim c\iint_{\mathcal{R}_{\geq\Lambda}} r^{p-1}(\partial_v\psi)^2\, dv + c^{-1}\iint_{\mathcal{R}_{\geq\Lambda}} \big(r^{1-\eta}(\partial_v\phi)^2 + r^{-1-\eta}\phi^2\big)\, dudv. \tag{6.56}$$

Combining (6.55) and (6.56) with c sufficiently small and applying the Morawetz estimates (6.27) and (6.30) yields (6.54) as desired. □

Proof of Proposition 6.19. Combine (6.54) and (6.50)–(6.53). □

7 Boundedness and decay of the scalar field

In this section, we prove boundedness and decay estimates for the scalar field ϕ. In Section 7.1, we use a standard pigeonhole principle argument to prove decay of the energies $\mathcal{E}_p$, $\underline{\mathcal{E}}_p$, $\mathcal{F}$, and $\underline{\mathcal{F}}$, which proves Proposition 4.4. In Section 7.2, we then use these energy estimates to prove pointwise decay and boundedness for ϕ and its derivatives.

In this section, we adopt the notational conventions outlined in Section 5.1. We will also often use the geometric bootstrap assumptions and their consequences, such as Lemma 5.1, without comment.

7.1 Improving the bootstrap assumptions for the scalar field: the proof of Proposition 4.4

Recall the definition of the energies $\mathcal{E}_p$, $\underline{\mathcal{E}}_p$, $\mathcal{F}$, and $\underline{\mathcal{F}}$ from Section 3.3. With this compact notation, we can summarize the content of Propositions 6.1, 6.14, and 6.19 as:

Proposition 7.1. *For any $A \geq 1$, ε sufficiently small, $\tau_f \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_{I(\tau_f)}$, and $1 \leq \tau_1 \leq \tau_2 \leq \tau_f$, it holds that*

$$\sup_{\tau\in[\tau_1,\tau_2]} \big(\mathcal{E}_p(\tau) + \underline{\mathcal{E}}_p(\tau)\big) \lesssim \mathcal{E}_p(\tau_1) + \underline{\mathcal{E}}_p(\tau_1) + \varepsilon^3\tau_1^{-4+2\delta+p} \tag{7.1}$$

for every $p \in [0, 3-\delta]$ and

$$\int_{\tau_1}^{\tau_2} \big(\mathcal{E}_{p-1}(\tau) + \underline{\mathcal{E}}_{p-1}(\tau)\big) d\tau \lesssim \mathcal{E}_p(\tau_1) + \underline{\mathcal{E}}_p(\tau_1) + \varepsilon^3\tau_1^{-4+2\delta+p} \tag{7.2}$$

for every $p \in [1, 3-\delta]$.

We shall require the following interpolation lemma:

Lemma 7.2. *Let $0 \leq p_1 < p < p_2 \leq 3-\delta$ and $\tau \in [1, \tau_f]$. It then holds that*

$$\mathcal{E}_p(\tau) \lesssim \big(\mathcal{E}_{p_1}(\tau)\big)^{\frac{p_2-p}{p_2-p_1}} \big(\mathcal{E}_{p_2}(\tau)\big)^{\frac{p-p_1}{p_2-p_1}},$$
$$\underline{\mathcal{E}}_p(\tau) \lesssim \big(\underline{\mathcal{E}}_{p_1}(\tau)\big)^{\frac{p_2-p}{p_2-p_1}} \big(\underline{\mathcal{E}}_{p_2}(\tau)\big)^{\frac{p-p_1}{p_2-p_1}}.$$

Proof. This is a consequence of the following general inequality for a nonnegative function w on a measure space (X,μ), which is immediately obtained from Hölder's inequality:

$$\int_X w^p\,d\mu \le \left(\int_X w^{p_1}\,d\mu\right)^{\frac{p_2-p}{p_2-p_1}}\left(\int_X w^{p_2}\,d\mu\right)^{\frac{p-p_1}{p_2-p_1}}. \qquad \square$$

The last preparatory result we shall require is an estimate for the initial energy in terms of our seed data norm $\mathfrak{D}$.

Lemma 7.3. *For any $A\ge 1$, ε sufficiently small, $\tau_f\in\mathfrak{B}$, and $\alpha\in\mathfrak{A}_{I(\tau_f)}$, it holds that*

$$\mathcal{E}_p(1)+\underline{\mathcal{E}}_p(1)\lesssim\big(\mathfrak{D}[\mathcal{S}_0]\big)^2. \tag{7.3}$$

Proof. Because $\lambda\sim 1$ on C_0 and $\nu\sim\bar\nu$ on $\underline{C}_0$, it follows from the definition of $\mathfrak{D}[\mathcal{S}_0]$ that

$$|r\phi|+r^2|\partial_v\phi|+r^2|\partial_v\psi|\lesssim\mathfrak{D}[\mathcal{S}_0] \tag{7.4}$$

on C_0 and

$$|\phi|+\frac{|\partial_u\phi|}{-\bar\nu}+\frac{|\partial_u\psi|}{-\bar\nu}\lesssim\mathfrak{D}[\mathcal{S}_0] \tag{7.5}$$

on $\underline{C}_0$. The bound (7.3) for $\mathcal{E}_p(1)$ is now immediate. The bound for $\underline{\mathcal{E}}_p(1)$ follows once we observe that

$$\int_{\underline{C}_0}(\bar r-M)^{2-p}(-\bar\nu)du\le\int_M^\Lambda(\bar r-M)^{2-p}\,d\bar r\lesssim 1$$

for $p\in[0,3-\delta]$. $\square$

Proof of Proposition 4.4. First, we apply (7.1) to $[1,\tau_f]$ and use (7.3) to infer

$$\sup_{\tau\in[1,\tau_f]}\big(\mathcal{E}_p(\tau)+\underline{\mathcal{E}}_p(\tau)\big)\lesssim\big(\mathfrak{D}[\mathcal{S}_0]\big)^2+\varepsilon^3\lesssim\varepsilon^2 \tag{7.6}$$

for every $p\in[0,3-\delta]$ by definition of ε. We assume $I(\tau_f)\ge 1$, else (7.6) already suffices. For each $i\in\{0,\dots,I(\tau_f)-1\}$, we deduce from (7.6) for $p=3-\delta$ and (7.2) for $p=3-\delta$ on the interval $[L_i,L_{i+1}]$, together with the pigeonhole principle, the existence of a $\tau_i'\in[L_i,L_{i+1}]$ such that

$$\mathcal{E}_{2-\delta}(\tau_i')+\underline{\mathcal{E}}_{2-\delta}(\tau_i')\lesssim(L_{i+1}-L_i)^{-1}\left(\varepsilon^2+\varepsilon^3L_i^{-1+\delta}\right)\lesssim\varepsilon^2L_i^{-1}. \tag{7.7}$$

Using now (7.1) for $p=2-\delta$ and the dyadicity of the sequence L_i, we upgrade (7.7) to the statement that

$$\mathcal{E}_{2-\delta}(\tau)+\underline{\mathcal{E}}_{2-\delta}(\tau)\lesssim\varepsilon^2\tau^{-1} \tag{7.8}$$

for every $\tau\in[1,\tau_f]$. Using Lemma 7.2 to interpolate between (7.6) for $p=3-\delta$ and (7.8), we obtain

$$\mathcal{E}_2(\tau)+\underline{\mathcal{E}}_2(\tau)\lesssim\varepsilon^2\tau^{-1+\delta} \tag{7.9}$$

for every $\tau\in[1,\tau_f]$. We now apply (7.9) and (7.2) with $p=2$ to the intervals $[L_i,L_{i+1}]$ with $i\in\{0,\dots,I(\tau_f)-1\}$ to find $\tau_i''\in[L_i,L_{i+1}]$ such that

$$\mathcal{E}_1(\tau_i'')+\underline{\mathcal{E}}_1(\tau_i'')\lesssim(L_{i+1}-L_i)^{-1}\big(\mathcal{E}_2(L_i)+\underline{\mathcal{E}}_2(L_i)+\varepsilon^3L_i^{-2+2\delta}\big)\lesssim\varepsilon^2L_i^{-2+\delta}.$$

Using (7.6), this is again immediately upgraded to

$$\mathcal{E}_1(\tau)+\underline{\mathcal{E}}_1(\tau)\lesssim\varepsilon^2\tau^{-2+\delta}$$

for every $\tau\in[1,\tau_f]$. Repeating this argument once more, we conclude

$$\mathcal{E}_0(\tau)+\underline{\mathcal{E}}_0(\tau)\lesssim\varepsilon^2\tau^{-3+\delta}$$

for every $\tau \in [1, \tau_f]$. Interpolating between this estimate and (7.6) for $p = 3 - \delta$, we finally have

$$\mathcal{E}_p(\tau) + \underline{\mathcal{E}}_p(\tau) \lesssim \varepsilon^2 \tau^{-3+\delta+p} \tag{7.10}$$

for every $p \in [0, 3 - \delta]$ and $\tau \in [1, \tau_f]$.

Applying now Lemma 6.2, we estimate

$$\mathcal{F}(u, \tau) \lesssim \varepsilon^2 \tau^{-3+\delta}, \quad \underline{\mathcal{F}}(v, \tau) \lesssim \varepsilon^2 \tau^{-3+\delta} \tag{7.11}$$

for any $(u, v) \in \mathcal{D}_{\tau_f}$ and $\tau \in [1, \tau_f]$

For A sufficiently large, (7.10) implies (4.16) and (4.17) and (7.11) implies (4.18) and (4.19). □

7.2 Pointwise estimates

In this section we will prove several pointwise estimates for the scalar field ϕ and its derivatives, with the energy decay estimates of the previous section as a starting point. We shall also use the method of characteristics to bound derivatives of ϕ. At this point, we fix $A = A_0$ sufficiently large that both Propositions 4.3 and 4.4 hold.

Proposition 7.4. *For any ε sufficiently small, $\tau_f \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_{I(\tau_f)}$, and $1 \le \tau_1 \le \tau_2 \le \tau_f$, we have the pointwise decay estimates*

$$|(\bar{r} - M)^{1/2} \phi| \lesssim \varepsilon \tau^{-3/2+\delta/2}, \tag{7.12}$$

$$|\psi| \lesssim \varepsilon \tau^{-1+\delta/2} \tag{7.13}$$

and the pointwise boundedness estimates

$$|r^2 \partial_v \psi| \lesssim \varepsilon, \tag{7.14}$$

$$|r^2 \partial_v \phi| \lesssim \varepsilon, \tag{7.15}$$

$$\left| \frac{\partial_u \psi}{-\nu} \right| \lesssim \varepsilon, \tag{7.16}$$

$$\left| r \frac{\partial_u \phi}{-\nu} \right| \lesssim \varepsilon \tag{7.17}$$

on $\mathcal{D}_{\tau_f}$.

Proof. PROOF OF (7.12) AND (7.13) FOR $r \le \Lambda$: Let $\chi_1 = \chi_1(r)$ be a smooth cutoff satisfying $\chi_1(r) = 1$ for $r \le \Lambda - M$ and $\chi_1(r) = 0$ for $r \ge \Lambda$. By the same argument as in Lemma 5.5, the set $\{r = \Lambda - M\}$ is a timelike curve in $\mathcal{D}_{\tau_f}$. For $(u, v) \in \mathcal{D}_{\tau_f}$ with $r(u, v) \le \Lambda - M$, the segment $[u^\Lambda(v), u] \times \{v\}$ is entirely contained in $\mathcal{D}_{\tau_f}$. Therefore, for $\beta \in \{0, 1\}$, we may write

$$(\bar{r} - M)^{1-\beta} \phi^2(u, v) = \int_{u^\Lambda(v)}^{u} \left(\chi_1' \nu (\bar{r} - M)^{1-\beta} \phi^2 + (1 - \beta) \chi_1 \nu \phi^2 + 2 \chi_1 (\bar{r} - M)^{1-\beta} \phi \partial_u \phi \right) du'.$$

The first and second terms can both be estimated by $\underline{\mathcal{E}}_0(\tau(u, v))$ and for the third we use Cauchy–Schwarz,

$$\begin{aligned} \int_{u^\Lambda(v)}^{u} \chi_1 (\bar{r} - M)^{1-\beta} |\phi| |\partial_u \phi| \, du' &\lesssim \left(\int_{u^\Lambda(v)}^{u} \phi^2 \, (-\bar{\nu}) du' \right)^{1/2} \left(\int_{u^\Lambda(v)}^{u} (\bar{r} - M)^{2-2\beta} \frac{(\partial_u \phi)^2}{-\bar{\nu}} \, du' \right)^{1/2} \\ &\lesssim \left(\underline{\mathcal{E}}_0(\tau(u, v)) \underline{\mathcal{E}}_{2\beta}(\tau(u, v)) \right)^{1/2} \lesssim \varepsilon^2 \tau^{-3+\beta+\delta}(u, v). \end{aligned}$$

For $(u, v) \in \{\Lambda - M \le r \le \Lambda\} \cap \mathcal{D}_{\tau_f}$, define

$$v^*(u) \doteq \begin{cases} v^{\Lambda - M}(u) & \text{if } r(u, 0) \ge \Lambda - M \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\{u\} \times [v^*(u), v] \subset \mathcal{D}_{\tau_f}$, $0 \le v - v^*(u) \lesssim 1$ and $\tau(u, v^*(u)) \sim \tau(u, v)$. Therefore,

$$|\phi(u,v)| \le |\phi(u, v^*(u))| + \int_{v^*(u)}^{v} |\partial_v \phi|\, dv' \lesssim \varepsilon \tau^{-3/2+\delta/2} + \big(\mathcal{F}(u, \tau(u, v^*(u)))\big)^{1/2} \lesssim \varepsilon \tau^{-3/2+\delta/2}(u,v),$$

where we used that $(u, v^*(u)) \in \{r \le \Lambda - M\} \cup \underline{C}_0$ and the estimate (7.11). This proves (7.12) and (7.13) for all $r \le \Lambda$.

Proof of (7.12) and (7.13) for $r \ge \Lambda$: Let $\chi_2 = \chi_2(r)$ be a smooth cutoff satisfying $\chi_2(r) = 1$ for $r \ge \Lambda + M$ and $\chi_2(r) = 0$ for $r \le \Lambda$. By the same argument as in Lemma 5.5, the set $\{r = \Lambda + M\}$ is a timelike curve in $\mathcal{D}_{\tau_f}$. For $(u,v) \in \mathcal{D}_{\tau_f}$ with $r(u,v) \ge \Lambda + M$, the segment $\{u\} \times [v^\Lambda(u), v]$ is entirely contained in $\mathcal{D}_{\tau_f}$. Therefore, for $\beta \in \{0,1\}$, we have

$$\begin{aligned}
\big|r^{-\beta}\psi^2(u,v)\big| &= \left| \int_{v^\Lambda(u)}^{v} \big(\chi_2' \lambda r^{-\beta}\psi^2 - \beta \chi_2 r^{-\beta-1}\lambda \psi^2 + 2\chi_2 r^{-\beta}\psi \partial_v \psi\big)\, dv' \right| \\
&\lesssim \mathcal{E}_0(\tau(u,v)) + \big(\mathcal{E}_0(\tau(u,v))\mathcal{E}_{2-2\beta}(\tau(u,v))\big)^{1/2} \lesssim \varepsilon^2 \tau^{-2-\beta+\delta}(u,v),
\end{aligned}$$

which proves (7.12) and (7.13) for $r \ge \Lambda + M$. The region $\Lambda \le r \le \Lambda + M$ is handled completely analogously to $\Lambda - M \le r \le \Lambda$ and is omitted. This completes the proofs of (7.12) and (7.13).

Proof of (7.14): Using (2.18), we compute

$$\partial_u(r^2 \partial_v \psi) = \frac{2\nu}{r}(r^2 \partial_v \psi) + 2r\kappa\nu\varkappa\psi$$

which can be solved for

$$(r^2\partial_v\psi)(u,v) = \frac{r^2(u,v)}{r^2(0,v)}(r^2\partial_v\psi)(0,v) + r^2(u,v)\int_0^u \left(\frac{2\kappa\nu\varkappa}{r}\psi\right)(u',v)\, du' \tag{7.18}$$

using an integrating factor (note that $\exp(\int_{u_1}^{u_2} \frac{2\nu}{r}(u',v)\, du') = r^2(u_2,v)/r^2(u_1,v)$). Since $r(u_2,v) \le r(u_1,v)$ for $u_1 \le u_2$ and

$$r^2(u,v)\int_0^u r^{-3}(u',v)\,(-\nu) du' \lesssim 1,$$

we readily obtain (7.14) from (7.4), (7.13), (7.18), and the geometric estimates.

Proof of (7.15): This follows immediately from the identity

$$r^2 \partial_v \phi = r\partial_v \psi - \lambda \phi$$

and the previously proved estimates.

Proof of (7.16): Using (2.17) and (2.18), we compute

$$\partial_v\left(\frac{\partial_u \psi}{-\nu}\right) = -2\kappa\varkappa\left(\frac{\partial_u\psi}{-\nu}\right) - 2\kappa\varkappa\phi, \tag{7.19}$$

which can be solved for

$$\left(\frac{\partial_u\psi}{-\nu}\right)(u,v) = \exp\left(-\int_0^v 2\kappa\varkappa\, dv'\right)\left(\frac{\partial_u\psi}{-\nu}\right)(u,0) - \int_0^v \exp\left(-\int_{v'}^{v} 2\kappa\varkappa\, dv''\right) 2\kappa\varkappa\phi\, dv' \tag{7.20}$$

using an integrating factor, where the integrands are evaluated at constant u. Using the simple observation that $\kappa\bar{\varkappa} \ge 0$ on $\mathcal{D}_{\tau_f}$ and the decay estimate (5.32), we have

$$\exp\left(-\int_{v_1}^{v_2} 2\kappa\varkappa\, dv'\right) \le \exp\left(-\int_{v_1}^{v_2} 2\kappa\tilde{\varkappa}\, dv'\right) \lesssim 1 + \int_{v_1}^{v_2} |\tilde{\varkappa}|\, dv' \lesssim 1 + \varepsilon^{3/2} v_1^{-1+\delta} \lesssim 1$$

for ε sufficiently small. Next, we estimate

$$\begin{aligned}
\int_0^v |\varkappa||\phi|\, dv' &\lesssim \int_0^v r^{-3}\big((\bar{r} - M)|\phi| + |\tilde{\varkappa}||\phi|\big)\, dv' \\
&\lesssim \varepsilon \int_0^v \big(r^{-3}\tau^{-3/2+\delta/2} + r^{-2}\tau^{-3+\delta} + r^{-3}\tau^{-2+\delta}\big)\, dv' \lesssim \varepsilon
\end{aligned}$$

using (7.12) and (5.32) again. Therefore, (7.20) and (7.5) yield (7.16) as desired.

PROOF OF (7.17): This follows immediately from the identity

$$r\frac{\partial_u\phi}{-\nu} = \frac{\partial_u\psi}{-\nu} + \phi$$

and the previously proved estimates. □

8 The proof of nonlinear stability, Theorem 1

In this section, we complete the proof of Theorem 1. As the proof will involve repeatedly updating the gauge as $\tau_f \to \infty$, we will now reintroduce the τ_f sub- and superscripts on various relevant quantities. Also, as in Section 7.2, we fix $A = A_0$ so that Propositions 4.3 and 4.4 hold.

We now briefly recall the notation for our gauges and refer the reader back to Sections 3.2 and 3.3 for the precise definitions. The coordinates $(\hat{u}, \hat{v})$ refer to the initial data normalized coordinates on the maximal development $\hat{\mathcal{Q}}_{\max}$, which are transformed into the teleologically normalized coordinates (u_{τ_f}, v) by the diffeomoprhism $\Phi_{\tau_f} = (\mathfrak{u}_{\tau_f}, \mathfrak{v}) : \hat{\mathcal{D}}_{\tau_f} \to \mathcal{D}_{\tau_f}$. The inverse of Φ_{τ_f} is denoted by $\hat{\Phi}_{\tau_f}$. Note that the background extremal Reissner–Nordström solutions $\bar{r}_{\tau_f}$ are defined on $\mathcal{D}_{\tau_f}$ in the coordinates (u_{τ_f}, v). Therefore, part of the proof of Theorem 1 will be to show that the pullbacks $\bar{r}_{\tau_f} \circ \Phi_{\tau_f}$ converge on (an appropriate subset of) $\hat{\mathcal{Q}}_{\max}$.

In Section 8.1, we carry out the main continuity argument of the paper by showing that the bootstrap set $\mathfrak{B}$ is open and closed. In Section 8.2, we prove estimates comparing the gauges Φ_{τ_f} and background solutions $\bar{r}_{\tau_f}$ for different values of τ_f. In Section 8.3, we extract the limiting comparison solution $\bar{r}_\infty$ as $\tau_f \to \infty$ and show that the energy hierarchies extend to the limit. Finally, we complete the proof of Theorem 1 in Section 8.3.5.

8.1 The continuity argument

Recall the definition of the bootstrap set $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A_0)$ from Section 4.1. We now have the following fundamental statement:

Proposition 8.1. *Let M_0 and δ be as in the statement of Theorem 1. There exists an $\varepsilon_{\mathrm{stab}}(M_0, \delta) > 0$ such that if $\varepsilon \le \varepsilon_{\mathrm{stab}}$ and $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \le \varepsilon$, then $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A_0) = [1, \infty)$, where A_0 is the constant for which Propositions 4.3 and 4.4 hold.*

Proof. We infer this statement by proving that $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A_0) \subset [1, \infty)$ is nonempty, open, and closed for ε sufficiently small. Nonemptiness was proved in Proposition 4.2 above. Openness will be proved in Section 8.1.1 below. Closedness will be proved in Section 8.1.2 below. □

Remark 8.2. The number $\varepsilon_{\mathrm{stab}}$ will be restricted one final time in Proposition 8.17 below.

8.1.1 The proof of openness

We begin by showing that the bootstrap region always stops short of the future boundary of the initial data hypersurface $\hat{\mathcal{C}}$ which was defined in Section 3.1. See Fig. 12.

Lemma 8.3. *There exists a constant $\theta \in (0, 1)$ such that for ε sufficiently small and $\tau_f \in \mathfrak{B}(\mathcal{S}_0, \varepsilon, A_0)$, it holds that $\Gamma^{\hat{u}}(\tau_f) \le \theta U_*$.*

Proof. Since $\partial_{\hat{u}} r = -1$ on $\underline{C}_{\mathrm{in}}$, we see that $r(\Gamma^{\hat{u}}(\tau_f), 0) = \Lambda - \Gamma^{\hat{u}}(\tau_f)$. Using (4.6), we then estimate

$$\Gamma^{\hat{u}}(\tau_f) = \Lambda - r(\Gamma^{\hat{u}}(\tau_f), 0) = \Lambda - \bar{r}_{\tau_f} \circ \Phi_{\tau_f}(\Gamma^{\hat{u}}(\tau_f), 0) - \tilde{r} \circ \Phi_{\tau_f}(\Gamma^{\hat{u}}(\tau_f), 0) = 99M_0 + O(\varepsilon),$$

which is quantitatively strictly smaller than $U_* = \frac{995}{10} M_0$ for ε sufficiently small. □

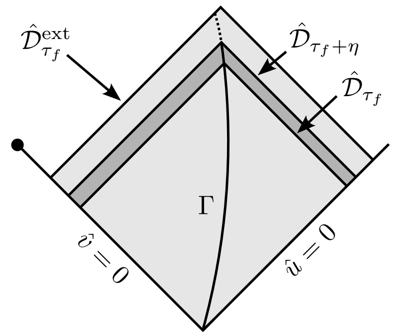

Figure 12: A Penrose diagram of the openness argument in the proof of Proposition 8.1. The extension $\hat{\mathcal{D}}^{\rm ext}_{\tau_f}$ avoids the solid black point by Lemma 8.3.

Proof that $\mathfrak{B}$ *is open.* Let $\tau_f \in \mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$. We show that $\tau_f+\eta \in \mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$ for $\eta>0$ sufficiently small.

Recall that at bootstrap time τ_f, the solution $(r,\hat{\Omega}^2,\phi,e)$ is assumed to exist on the rectangle

$$\hat{\mathcal{D}}_{\tau_f} \doteq [0,\Gamma^{\hat{u}}(\tau_f)]\times[0,\Gamma^{\hat{v}}(\tau_f)]$$

in the maximal development $\hat{\mathcal{Q}}_{\rm max}$ in the "initial data coordinates" $(\hat{u},\hat{v})$. By Proposition 2.5 and Lemma 8.3, there exists a small number $\sigma>0$ such that

$$\hat{\mathcal{D}}^{\rm ext}_{\tau_f} \doteq [0,\Gamma^{\hat{u}}(\tau_f)+\sigma]\times[0,\Gamma^{\hat{v}}(\tau_f)+\sigma]\subset\hat{\mathcal{Q}}_{\rm max}.$$

Since Γ being a timelike curve is an open condition, $\Gamma|_{[0,\tau_f]}$ extends to an inextendible timelike curve in $\hat{\mathcal{D}}^{\rm ext}_{\tau_f}$ for σ sufficiently small (refer also to the quantitative estimates (5.17)). Again by continuity, (5.4), (5.6), and (5.12) imply that $\hat{\kappa}>0$ and $\hat{\gamma}<0$ in $\hat{\mathcal{D}}^{\rm ext}_{\tau_f}\cap\{r\ge\Lambda\}$ for σ sufficiently small. Therefore, $\hat{\mathcal{D}}_{\tau_f+\eta}\subset\hat{\mathcal{Q}}_{\rm max}$ for η sufficiently small and the map $\Phi_{\tau_f+\eta}$ exists, which allows us to equip $\hat{\mathcal{D}}_{\tau_f+\eta}$ with teleologically normalized coordinates $(u_{\tau_f+\eta},v)=\Phi_{\tau_f+\eta}(\hat{u},\hat{v})$. By direct inspection of the definition (3.6), we see that $\mathfrak{u}_{\tau_f+\eta}\to\mathfrak{u}_{\tau_f}$ smoothly on $[0,\Gamma^{\hat{u}}(\tau_f)]$ as $\eta\to0$. Moreover, letting $\bar{r}_{\tau_f+\eta}$ and $\bar{r}_{\tau_f}$ denote the respective anchored extremal Reissner–Nordström backgrounds, we also have that $\bar{r}_{\tau_f+\eta}\circ\Phi_{\tau_f+\eta}\to\bar{r}_{\tau_f}\circ\Phi_{\tau_f}$ smoothly on $\hat{\mathcal{D}}_{\tau_f}$ as $\eta\to0$.

The step function $I(\tau)=\lfloor\log_2\tau\rfloor$ has the property that $I(\tau_f+\eta)=I(\tau_f)$ for η sufficiently small. It trivially follows that Π_i is defined on $\mathfrak{A}_i$ for every $i\in\{0,\dots,I(\tau_f+\eta)\}$. These soft arguments show that points 1.–4. of the definition of $\mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$ are satisfied for $\tau_f+\eta$ if η is chosen sufficiently small.

We now invoke Propositions 4.3 and 4.4: The bootstrap assumptions (4.5)–(4.11) hold on $\mathcal{D}_{\tau_f}$ with quantitatively strictly better constants $(1\mapsto\frac12)$. Using now the smoothness of the limits $\Phi_{\tau_f+\eta}\to\Phi_{\tau_f}$ and $\bar{r}_{\tau_f+\eta}\circ\Phi_{\tau_f+\eta}\to\bar{r}_{\tau_f}\circ\Phi_{\tau_f}$, it is now immediate that the bootstrap assumptions (4.5)–(4.11) hold on $\mathcal{D}_{\tau_f}$ for η sufficiently small. Therefore, $\tau_f+\eta\in\mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$. □

8.1.2 The proof of closedness: modulation of ϖ

Proof that $\mathfrak{B}$ *is closed.* Let $\tau_f^n\in\mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$ be a strictly increasing sequence of times with finite limit τ_f^∞ as $n\to\infty$. We aim to show that $\tau_f^\infty\in\mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0)$.

The case when τ_f^∞ is not dyadic: We first argue that $\Gamma(\tau_f^n)$ has a limit point in the $(\hat{u},\hat{v})$-plane as $n\to\infty$. Since Γ is timelike, the sequences $\Gamma^{\hat{u}}(\tau_f^n)$ and $\Gamma^{\hat{v}}(\tau_f^n)$ are strictly monotone increasing. By Lemma 8.3, $\Gamma^{\hat{u}}(\tau_f^n)$ converges to a number $\hat{u}_\star<U_*$. By Lemma 5.7 and (5.19), $\Gamma^{\hat{v}}(\tau_f^n)$ converges to a finite number $\hat{v}_\star$. Therefore, the set $\hat{\mathcal{D}}_\star\doteq[0,\hat{u}_\star)\times[0,\hat{v}_\star)$ is contained in $\hat{\mathcal{Q}}_{\rm max}$.

We now show that the closure of $\hat{\mathcal{D}}_\star$ is contained in $\hat{\mathcal{Q}}_{\rm max}$, i.e., that the solution $(r,\hat{\Omega}^2,\phi,e)$ extends to the closure of $\hat{\mathcal{D}}_\star$ in C^∞. By the bootstrap assumptions, Lemma 5.7, (5.5), (5.13), (5.32), (7.13), and (7.15), we immediately obtain the estimates $1\lesssim r\lesssim1+\tau_f^\infty$, $0<\hat{\lambda}\lesssim1$, $\hat{\kappa}\sim1$, $|\phi|\lesssim1$, and $|\partial_{\hat{v}}\phi|\lesssim1$ on $\hat{\mathcal{D}}_\star$. Using these estimates and (2.17), we have that $|\log(-\hat{\nu})|\lesssim1+\tau_f^\infty$ on $\hat{\mathcal{D}}_\star$. We now use the gauge-invariant estimate

(7.17) to estimate $|\partial_{\hat{u}}\phi| \lesssim 1 + \tau_f^\infty$ on $\hat{\mathcal{D}}_\star$. Using the identity $\hat{\Omega}^2 = -4\hat{\kappa}\hat{\nu}$, we also have $1 \lesssim \hat{\Omega}^2 \lesssim 1 + \tau_f^\infty$ on $\hat{\mathcal{D}}_\star$. Now by integrating (2.13) in $\hat{u}$ and $\hat{v}$, we have that $|\partial_{\hat{u}}\log\hat{\Omega}^2| + |\partial_{\hat{v}}\log\hat{\Omega}^2| \lesssim 1 + (\tau_f^\infty)^2$ on $\hat{\mathcal{D}}_\star$.

These arguments imply that $(r, \hat{\Omega}^2, \phi, e)$ extends to the closure of $\hat{\mathcal{D}}_\star$ in C^1. Standard propagation of regularity results now imply that this extension is actually C^∞. Standard continuity arguments, such as those used in the proof of openness, imply that parts 2. and 4.–6. of Definition 4.1 hold on $\overline{\hat{\mathcal{D}}_\star} = \hat{\mathcal{D}}_{\tau_f^\infty}$. Since τ_f^∞ is not dyadic, i.e., not a power of 2, parts 1. and 3. of Definition 4.1 are automatically inherited from τ_f^n for n sufficiently large.

THE CASE WHEN τ_f^∞ IS DYADIC: By the same arguments as in the previous case, the solution extends to $\hat{\mathcal{D}}_{\tau_f^\infty}$ and parts 2. and 4.–6. of Definition 4.1 are satisfied. Since $I \doteq I(\tau_f^\infty) > I(\tau_f^n)$ for any finite n, we must now construct the set $\mathfrak{A}_I$ and ensure that parts 1. and 3. of Definition 4.1 hold. Note that $I \geq 1$.

By assumption, there exist numbers $\alpha_i^- < \alpha_i^+$ for every $i \in \{0, \ldots, I-1\}$ such that $\mathfrak{A}_i \doteq [\alpha_i^-, \alpha_i^+]$ are nested, i.e., $\mathfrak{A}_{I-1} \subset \mathfrak{A}_{I-2} \subset \cdots \subset \mathfrak{A}_0$,

$$\Pi_i : \mathfrak{A}_i \to [-\varepsilon^{3/2}L_i^{-3+\delta}, \varepsilon^{3/2}L_i^{-3+\delta}]$$

is surjective, and $\Pi_i(\alpha_i^\pm) = \pm\varepsilon^{3/2}L_i^{-3+\delta}$. Clearly, the map $\Pi_I(\alpha) \doteq \tilde{\varpi}(\Gamma(L_I))$ is defined on $\mathfrak{A}_{I-1}$ since the solution exists on $\mathcal{D}_{\tau_f^\infty}$ for every $\alpha \in \mathfrak{A}_{I-1}$. We apply Lemma 8.4 below with $f_1 = \Pi_{I-1}$, $[x_1^-, x_2^+] = \mathfrak{A}_{I-1}$, $c_1 = \varepsilon^{3/2}L_{I-1}^{-3+\delta}$, $f_2 = \Pi_I$, and $c_2 = \varepsilon^{3/2}L_I^{-3+\delta}$. In order to verify the assumption (8.2), we estimate, using (5.44) and Remark 5.15,

$$|\Pi_I(\alpha) - \Pi_{I-1}(\alpha)| \leq C\varepsilon^2 L_I^{-3+\delta}$$

for every $\alpha \in \mathfrak{A}_{I-1}$, where C is a constant that does not depend on τ_f^∞. We now observe that

$$\varepsilon^{3/2}L_{I-1}^{-3+\delta} - \varepsilon^{3/2}L_I^{-3+\delta} = (2^{3+\delta} - 1)\varepsilon^{3/2}L_I^{-3+\delta} > C\varepsilon^2 L_I^{-3+\delta} \tag{8.1}$$

for ε sufficiently small, which verifies (8.2). We may now take $\alpha_I^\pm = x_2^\pm$ and Lemma 8.4 implies that

$$\Pi_I : \mathfrak{A}_I \to [-\varepsilon^{3/2}L_I^{-3+\delta}, \varepsilon^{3/2}L_I^{-3+\delta}]$$

is surjective, where $\mathfrak{A}_I := [\alpha_I^-, \alpha_I^+]$, as desired. □

Lemma 8.4. *Let $0 < c_2 < c_1 < 0$ and let $f_1 : [x_1^-, x_1^+] \to [-c_1, c_1]$ be a continuous surjective function satisfying $f_1(x_1^\pm) = \pm c_1$. Let $f_2 : [x_1^-, x_1^+] \to \mathbb{R}$ be a continuous function satisfying the estimate*

$$\sup_{[x_1^-, x_1^+]} |f_2 - f_1| < c_1 - c_2. \tag{8.2}$$

Then there exists an interval $[x_2^-, x_2^+] \subset (x_1^-, x_1^+)$ such that $f_2 : [x_2^-, x_2^+] \to [-c_2, c_2]$ is surjective with $f_2(x_2^\pm) = \pm c_2$.

Proof. We observe that

$$f_2(x_1^+) \geq f_1(x_1^+) - |f_2(x_1^+) - f_1(x_1^+)| > c_1 - (c_1 - c_2) = c_2$$

and similarly, $f_2(x_1^-) < -c_2$. It follows from the intermediate value theorem that

$$x_2^- \doteq \min f_2^{-1}(-c_2) \quad \text{and} \quad x_2^+ \doteq \max f_2^{-1}(c_2)$$

exist and satisfy $x_1^- < x_2^- < x_2^+ < x_1^+$. Surjectivity of f_2 on $[x_2^-, x_2^+]$ follows again from the intermediate value theorem. □

8.1.3 The definition of the stable manifold $\mathfrak{M}_{\mathrm{stab}}$

Definition 8.5. Let $\varepsilon \leq \varepsilon_{\mathrm{stab}}$. For $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \leq \varepsilon$, we have $\mathfrak{B}(\mathcal{S}_0, \varepsilon, A_0) = [1, \infty)$ by Proposition 8.1. Therefore, there exists a sequence of nested, compact, nonempty intervals $\{\mathfrak{A}_i\}_{i\geq 0}$ as in Definition 4.1. Note that there may be multiple such $\{\mathfrak{A}_i\}_{i\geq 0}$ for which Definition 4.1 holds; we shall say that such an $\{\mathfrak{A}_i\}_{i\geq 0}$ is *consistent* with this definition. See already Remark 8.7. We define

$$\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0, \varepsilon) \doteq \bigcup_{\{\mathfrak{A}_i\}_{i\geq 0}\ \text{consistent}} \left\{ \mathcal{S}_0(\alpha_\star) \in \mathcal{L}(\mathcal{S}_0, \varepsilon) : \alpha_\star \in \bigcap_{i\geq 0} \mathfrak{A}_i \right\} \tag{8.3}$$

and

$$\mathfrak{M}_{\mathrm{stab}} \doteq \bigcup_{\varepsilon\in[0,\varepsilon_{\mathrm{stab}}]} \left(\bigcup_{\mathcal{S}_0\in\mathfrak{M}_0:\mathfrak{D}[\mathcal{S}_0]\le\varepsilon} \mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)\right).$$

This set $\mathfrak{M}_{\mathrm{stab}}$ is the stable "submanifold" of seed data referred to in the statement of Theorem 1. We immediately infer the codimension-one property of $\mathfrak{M}_{\mathrm{stab}}$, recall (3.12).

Proposition 8.6. *Let $\varepsilon \le \varepsilon_{\mathrm{stab}}$. For $\mathcal{S}_0 \in \mathfrak{M}_0$ with $\mathfrak{D}[\mathcal{S}_0] \le \varepsilon$, it holds that*

$$\mathfrak{M}_{\mathrm{stab}} \cap \mathcal{L}(\mathcal{S}_0,\varepsilon) = \mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon) \neq \emptyset. \tag{8.4}$$

Proof. The equality in (8.4) follows immediately from the definitions of $\mathfrak{M}_{\mathrm{stab}}$, $\mathcal{L}(\mathcal{S}_0,\varepsilon)$, and $\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)$. The nonemptiness of $\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)$ follows from the existence of at least one valid sequence of modulation sets $\{\mathfrak{A}_i\}_{i\ge0}$ per Definition 4.1 and Proposition 8.1. □

Remark 8.7. We only prove that the intersection $\bigcap_{i\ge0}\mathfrak{A}_i$ is nonempty, not that it only contains one element. Moreover, as was already mentioned, there might be multiple $\{\mathfrak{A}_i\}_{i\ge0}$ consistent with Definition 4.1. (The construction of Lemma 8.4 gives one such choice, which is algorithmic but not necessarily unique in general.) In (8.3), we consider all possible descending chains of modulation sets which are consistent with Definition 4.1. A priori, this could also result in the set $\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)$ containing more than one element. One consequence of the conjectures in Section 1.3.1 would be that $\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)$ contains a single seed data set and hence $\mathfrak{M}_{\mathrm{stab}}$ is *exactly* codimension one.

8.2 Estimates for the gauge and background changes

8.2.1 Extension of the solution to $\hat{\mathcal{D}}_\infty$ and the eschatological gauge Φ_∞

We now introduce the following convention:

For the remainder of this paper, we consider without further comment solutions arising from seed data in $\mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0,\varepsilon)$. For such a solution, all of the estimates proved in Sections 5 to 7 hold for every $\tau_f \ge 1$.

Since $\mathfrak{B}(\mathcal{S}_0,\varepsilon,A_0) = [1,\infty)$ by Proposition 8.1, we have

$$\hat{\mathcal{D}}_\infty \doteq \bigcup_{\tau_f\ge1} \hat{\mathcal{D}}_{\tau_f} \subset \hat{\mathcal{Q}}_{\max}.$$

In this section, we show that the *eschatological gauge* Φ_∞ is well-defined and C^1 on $\hat{\mathcal{D}}_\infty$, which is a part of Part 2. of the statement of Theorem 1.

Proposition 8.8. *For $(\hat{u},\hat{v}) \in \hat{\mathcal{D}}_\infty$, the limit*

$$\Phi_\infty(\hat{u},\hat{v}) \doteq \lim_{\tau_f\to\infty} \Phi_{\tau_f}(\hat{u},\hat{v})$$

exists and defines a C^1 diffeomorphism $\Phi_\infty : \hat{\mathcal{D}}_\infty \to [0,\infty)\times[0,\infty)$.

First, we observe the following immediate consequence of Lemma 8.3:

Lemma 8.9. *The limit $\hat{u}_{\mathcal{H}^+} \doteq \lim_{\tau\to\infty}\Gamma^{\hat{u}}(\tau)$ exists and satisfies $\hat{u}_{\mathcal{H}^+} \le \theta U_*$, where $\theta\in(0,1)$ is the constant from Lemma 8.3. Furthermore, $\hat{\mathcal{D}}_\infty = [0,\hat{u}_{\mathcal{H}^+})\times[0,\infty)$.*

Next, we show that the maps Φ_{τ_f} are Cauchy in C^1.

Lemma 8.10. *For any $\hat{u}_0 \in [0,\hat{u}_{\mathcal{H}^+})$, $\bar{\tau}_f$ sufficiently large that $\Gamma^{\hat{u}}(\bar{\tau}_f) \ge \hat{u}_0$, and $\tau_f \ge \bar{\tau}_f$, it holds that*

$$\|\Phi_{\bar{\tau}_f} - \Phi_{\tau_f}\|_{C^1([0,\hat{u}_0]\times[0,\infty))} \lesssim_{\hat{u}_0} \varepsilon^2\bar{\tau}_f^{-1}. \tag{8.5}$$

Proof. By definition and (2.22) (applied twice), we have that

$$\begin{aligned}
\mathfrak{u}'_{\bar\tau_f}(\hat u) - \mathfrak{u}'_{\tau_f}(\hat u) &= \hat\gamma(\hat u, \Gamma^{\hat v}(\tau_f)) - \hat\gamma(\hat u, \Gamma^{\hat v}(\bar\tau_f))\\
&= \hat\gamma(\hat u, \Gamma^{\hat v}(\bar\tau_f)) \left[1 - \exp\left(\int_{\Gamma^{\hat v}(\bar\tau_f)}^{\Gamma^{\hat v}(\tau_f)} \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right)\right]\\
&= \frac{1}{1-\mu(\hat u,0)} \exp\left(\int_0^{\Gamma^{\hat v}(\bar\tau_f)} \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right) \left[1 - \exp\left(\int_{\Gamma^{\hat v}(\bar\tau_f)}^{\Gamma^{\hat v}(\tau_f)} \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right)\right]
\end{aligned}$$

for $\hat u \in [0, \hat u_0]$. As the solution exists and is smooth on $\hat{\mathcal{D}}_{\bar\tau_f}$, we have that

$$\left|\frac{1}{1-\mu(\hat u,0)} \exp\left(\int_0^{\Gamma^{\hat v}(\bar\tau_f)} \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right)\right| \le C(\hat u_0).$$

To see this, note that $1-\mu(\hat u, 0)$ is nonvanishing for $\hat u \in [0, \hat u_{\mathcal{H}^+})$ by (5.3). The integral over $r \le \Lambda$ is estimated softly by the compactness of the region $\{0 \le \hat u \le \hat u_0\} \cap \{r \le \Lambda\}$ and for $r \ge \Lambda$ by (4.8). Using the pointwise estimate (7.15), we now estimate

$$\left|1 - \exp\left(\int_{\Gamma^{\hat v}(\bar\tau_f)}^{\Gamma^{\hat v}(\tau_f)} \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right)\right| \lesssim \int_{\Gamma^{\hat v}(\bar\tau_f)}^{\Gamma^{\hat v}(\tau_f)} \varepsilon^2 r(\hat u, \hat v)^{-3}\, d\hat v \lesssim \varepsilon^2 \bar\tau_f^{-2}.$$

Putting these estimates together yields $|\mathfrak{u}'_{\bar\tau_f} - \mathfrak{u}'_{\tau_f}| \lesssim_{\hat u_0} \varepsilon^2 \bar\tau_f^{-1}$ and integrating once yields (8.5) as desired. □

Proof of Proposition 8.8. The existence and regularity of the map $\Phi_\infty = (\mathfrak{u}_\infty, \mathfrak{v})$ is immediate from the estimate (8.5). We now show that $\mathfrak{u}_\infty : [0, \hat u_{\mathcal{H}^+}) \to [0,\infty)$ is a diffeomorphism. First, $\mathfrak{u}_\infty$ is strictly increasing because

$$\frac{d\mathfrak{u}_\infty}{d\hat u} = -\lim_{\hat v \to \infty} \hat\gamma(\hat u, \hat v) = \frac{1}{1-\mu(\hat u,0)} \exp\left(\int_0^\infty \frac{r}{\hat\lambda}(\partial_{\hat v}\phi)^2\, d\hat v\right) > 0$$

and $\mathfrak{u}_\infty$ is surjective onto $[0,\infty)$ because $\mathfrak{u}_\infty(0) = 0$ and $\mathfrak{u}_{\tau_f}(\Gamma^{\hat u}(\tau_f)) \sim \tau_f$ as $\tau \to \infty$ by Lemma 5.6. Finally, by Lemma 5.7 and its proof, $\mathfrak{v} : [0,\infty) \to [0,\infty)$ is a diffeomorphism. □

We apply the coordinate transformation $(u_\infty, v) = \Phi_\infty(\hat u, \hat v)$ to $\hat{\mathcal{D}}_\infty$, after which we denote it by $\mathcal{D}_\infty$. However, since Φ_∞ is only C^1 in $\hat u$, the solution $(r, \hat\Omega^2, \phi, e)$ will only be C^1 in u_∞ when expressed in the coordinates (u_∞, v). Moreover, the solution expressed in (u_∞, v) coordinates will satisfy the estimate

$$|\gamma_\infty + 1| \lesssim \varepsilon^2 r^{-3/2} \tau^{-2+\delta} \tag{8.6}$$

in $\mathcal{D}_\infty \cap \{r \ge \Lambda\}$ by simply passing to the limit in (5.15).

8.2.2 Uniform C^1 convergence of $\Phi_{\bar\tau_f} \circ \hat\Phi_{\tau_f}$ to the identity as $\bar\tau_f \to \infty$

As we saw in the proof of Lemma 8.10, the difference of the coordinate changes $\hat u \mapsto u_{\bar\tau_f}$ and $\hat u \mapsto u_{\tau_f}$ does not satisfy a uniform estimate up to the horizon as $\bar\tau_f \to \infty$. In the following proposition, we prove that $u_{\bar\tau_f} \mapsto u_{\tau_f}$, however, is well-behaved in the entire domain of outer communication as $\bar\tau_f \to \infty$.

Proposition 8.11. *Let $1 \le \bar\tau_f < \tau_f \le \infty$ and set $\Psi_{\bar\tau_f,\tau_f} \doteq \Phi_{\bar\tau_f} \circ \hat\Phi_{\tau_f}$. Then the following estimates hold:*

$$\sup_{J^-(\Gamma(\tau_f)) \cap \mathcal{D}_{\tau_f}} |\Psi_{\bar\tau_f,\tau_f} - \mathrm{id}\,| \lesssim \varepsilon^2 \bar\tau_f^{-1/2}, \tag{8.7}$$

$$\sup_{J^-(\Gamma(\tau_f)) \cap \mathcal{D}_{\tau_f}} |d(\Psi_{\bar\tau_f,\tau_f} - \mathrm{id})| \lesssim \varepsilon^2 \bar\tau_f^{-3/2}. \tag{8.8}$$

Proof. From the definitions, we have $\Psi_{\bar\tau_f,\tau_f} = \mathfrak{g}_{\bar\tau_f,\tau_f} \times \mathrm{id}$, where

$$\mathfrak{g}_{\bar\tau_f,\tau_f}(u_{\tau_f}) \doteq -\int_0^{u_{\tau_f}} \gamma_{\tau_f}(u'_{\tau_f}, \Gamma^v(\bar\tau_f))\, du'_{\tau_f} = u_{\tau_f} - \int_0^{u_{\tau_f}} \big(\gamma_{\tau_f}(u'_{\tau_f}, \Gamma^v(\bar\tau_f)) + 1\big)\, du'_{\tau_f}. \tag{8.9}$$

By applying the estimate (5.15), we have that

$$|\mathfrak{g}'_{\bar\tau_f,\tau_f} - 1| \le |\tilde\gamma_{\tau_f}| \lesssim \varepsilon^2 r^{-3/2} \tau^{-2+\delta}, \tag{8.10}$$

where the right-hand side is evaluated on $[0, \Gamma^{u_{\tau_f}}(\bar\tau_f)] \times \{\Gamma^v(\bar\tau_f)\}$. On $\{v = \Gamma^v(\bar\tau_f)\} \cap \{\tau \le \frac12 \bar\tau_f\}$, it is easy to see that $r \gtrsim \bar\tau_f$. By considering separately the two regions $\tau \ge \frac12\bar\tau_f$ and $\tau \le \frac12\bar\tau_f$ on $[0, \Gamma^{u_{\tau_f}}(\bar\tau_f)] \times \{\Gamma^v(\bar\tau_f)\}$, we use (8.10) to estimate

$$|\mathfrak{g}'_{\bar\tau_f,\tau_f} - 1| \lesssim \varepsilon^2 \bar\tau_f^{-3/2}. \tag{8.11}$$

This now readily implies (8.7) and (8.8). $\square$

8.2.3 Differences of the background solutions at different bootstrap times

Next, we compare $\bar r_{\bar\tau_f}$ and $\bar r_{\tau_f}$ for two late bootstrap times. Since these are defined on different bootstrap domains with different coordinate systems, we need to pull them back onto the same domain. If we pull both back to $\hat{\mathcal{D}}_\infty$, we would have to contend with the poor estimate (8.5). Instead, we pull $\bar r_{\bar\tau_f}$ back to the "later" coordinate system (u_{τ_f}, v), which takes advantage of the good estimate (8.8).

Lemma 8.12. *For any $\bar\tau \ge 1$ sufficiently large and $\bar\tau_f < \tau_f < \infty$, it holds that*

$$\sup_{J^-(\Gamma(\tau_f))\cap\mathcal{D}_{\tau_f}} |\bar r_{\tau_f} - \bar r_{\bar\tau_f} \circ \Psi_{\bar\tau_f,\tau_f}| \lesssim \varepsilon^2 \bar\tau_f^{-1/2}. \tag{8.12}$$

Proof. Let $\varrho \doteq \bar r_{\tau_f} - \bar r_{\bar\tau_f} \circ \Psi_{\bar\tau_f,\tau_f}$. We differentiate in u_{τ_f} to obtain

$$\begin{aligned}\partial_{u_{\tau_f}}\varrho &= \bar\nu_{\tau_f} - (\bar\nu_{\bar\tau_f} \circ \Psi_{\bar\tau_f,\tau_f})\mathfrak{g}'_{\bar\tau_f,\tau_f} = -(1 - \bar\mu_{\tau_f}) + (1 - \bar\mu_{\bar\tau_f} \circ \Psi_{\bar\tau_f,\tau_f})\mathfrak{g}'_{\bar\tau_f,\tau_f} \\ &= \left(1 - \frac{M}{\bar r_{\bar\tau_f} \circ \Psi_{\bar\tau_f,\tau_f}}\right)^2 - \left(1 - \frac{M}{\bar r_{\tau_f}}\right)^2 + O(\varepsilon^2 \bar\tau_f^{-3/2})\end{aligned}$$

where we used (8.11) and the Eddington–Finkelstein gauge condition for $\bar r_{\bar\tau_f}$ and $\bar r_{\tau_f}$. Under the bootstrap assumption that $|\varrho| \le \varepsilon \bar\tau_f^{-1/3}$ with $\bar\tau_f$ large, we may Taylor expand to obtain

$$\partial_{u_{\tau_f}}\varrho = -\frac{2M}{\bar r_{\tau_f}^3}(\bar r_{\tau_f} - M)\varrho + O(\bar r_{\tau_f}^{-3}\varrho^2) + O(\varepsilon^2 \bar\tau_f^{-3/2}). \tag{8.13}$$

By (5.35), we estimate $|\varrho(\Gamma(\bar\tau_f))| \lesssim \varepsilon^2 \bar\tau_f^{-3+\delta}$, which for $\bar\tau_f$ large shows that the bootstrap assumption is satisfied near $\Gamma(\bar\tau_f)$. Integrating (8.13) backwards from $\Gamma(\bar\tau_f)$ (with the help of an integrating factor, see also (5.39)), we estimate

$$|\varrho(u_{\tau_f}, \Gamma^v(\bar\tau_f))| \lesssim \varepsilon^2 \bar\tau_f^{-3+\delta} + \int_{u_{\tau_f}}^{\Gamma^{u_{\tau_f}}(\bar\tau_f)} \big(\bar r_{\tau_f}^{-3}\varrho^2 + \varepsilon^2 \bar\tau_f^{-3/2}\big)\, du'_{\tau_f} \lesssim \varepsilon^2 \bar\tau_f^{-1/2} + \varepsilon^2 \bar\tau_f^{-2/3} \lesssim \varepsilon^2 \bar\tau_f^{-1/2}. \tag{8.14}$$

For $\bar\tau_f$ sufficiently large, this improves the bootstrap assumption on ϱ and therefore (8.14) holds for $u_{\tau_f} \in [0, \Gamma^{u_{\tau_f}}(\bar\tau_f)]$.

Next, similarly to (8.13), we derive the equation

$$\partial_v \varrho = \frac{2M}{\bar r_{\tau_f}^3}(\bar r_{\tau_f} - M)\varrho + O(\bar r_{\tau_f}^{-3}\varrho^2).$$

This can be integrated backwards from $[0, \Gamma^{u_{\tau_f}}(\bar\tau_f)] \times \{\Gamma^v(\bar\tau_f)\}$ and by using (8.14) and the good sign of the coefficient of the linear term on the right-hand side, we obtain (8.12) as desired. $\square$

8.3 Taking $\tau_f \to \infty$ and completing the proof of Theorem 1

With estimates for the gauges and background solutions in hand, we can now justify passing to the limit $\tau_f \to \infty$ in the constructions and estimates established in Sections 4 to 7. Once the "final" estimates on $\mathcal{D}_\infty$ have been established, we can infer the existence of the event horizon $\mathcal{H}^+$ and complete the proof of Theorem 1.

8.3.1 Extracting the final anchored background solution $\bar{r}_\infty$

Proposition 8.13. *The limit*

$$\bar{r}_\star \doteq \lim_{\tau_f \to \infty} \bar{r}_{\tau_f}(0,0) \tag{8.15}$$

exists and satisfies

$$|\bar{r}_\star - \Lambda| \lesssim \varepsilon^2. \tag{8.16}$$

Let $\bar{r}_\infty$ denote the ∞-anchored background solution on $\mathcal{D}_\infty$ with bifurcation sphere area-radius $\bar{r}_\infty(0,0) = \bar{r}_\star$, as defined in Section 3.3. Then it holds that

$$\sup_{J^-(\Gamma(\tau_f))\cap\mathcal{D}_\infty} |\bar{r}_\infty - \bar{r}_{\tau_f} \circ \Psi_{\tau_f,\infty}| \lesssim \varepsilon^2 \tau_f^{-1/2}. \tag{8.17}$$

Proof. The existence of the limit (8.15) follows immediately from (8.12) evaluated at $(0,0)$ and then the estimate (8.16) follows from (5.35). To prove (8.17), first observe that by (8.12), the limit

$$\mathfrak{r}(\hat{u},\hat{v}) \doteq \lim_{\tau_f \to \infty} \bar{r}_{\tau_f} \circ \Phi_{\tau_f}$$

exists on $\hat{\mathcal{D}}_\infty$ and satisfies

$$\sup_{J^-(\Gamma(\tau_f))\cap\hat{\mathcal{D}}_\infty} |\mathfrak{r} - r_{\tau_f} \circ \Phi_{\tau_f}| \lesssim \varepsilon^{3/2} \tau_f^{-1/2}. \tag{8.18}$$

We claim that $\bar{r}_\infty = \mathfrak{r} \circ \hat{\Phi}_\infty$, whence (8.17) follows from (8.18). As $\bar{r}_\infty(0,0) = \bar{r}_\star = \mathfrak{r} \circ \hat{\Phi}_\infty(0,0)$, it suffices to show that $\mathfrak{r} \circ \hat{\Phi}_\infty$ satisfies the Eddington–Finkelstein gauge conditions in the coordinates (u_∞, v). Indeed,

$$\partial_{u_\infty}(\bar{r}_{\tau_f} \circ \Psi_{\tau_f,\infty}) = -(1 - \bar{\mu}_{\tau_f} \circ \Psi_{\tau_f,\infty})\mathfrak{g}'_{\tau_f,\infty} = -\left(1 - \frac{M}{\bar{r}_{\tau_f} \circ \Psi_{\tau_f,\infty}}\right)^2 \mathfrak{g}'_{\tau_f,\infty} \tag{8.19}$$

by the definitions. Using (8.11), we see that this converges uniformly to $-(1 - M/\mathfrak{r} \circ \hat{\Phi}_\infty)^2$ as $\tau_f \to \infty$, which verifies the Eddington–Finkelstein gauge condition for $\partial_{u_\infty}(\mathfrak{r} \circ \hat{\Phi}_\infty)$. The corresponding argument for $\partial_v(r \circ \hat{\Phi}_\infty)$ is trivial, which completes the proof. □

Remark 8.14. Note that $\bar{r}_\infty$ is smooth on $\mathcal{D}_\infty$ in the (u_∞, v) coordinate system, but $\bar{r}_\infty \circ \Phi_\infty$ is only C^1 in $\hat{u}$ on $\hat{\mathcal{D}}_\infty$.

8.3.2 Convergence of the energy hierarchies

Proposition 8.15. *For any $\tau \in [1,\infty)$ and $p \in [0, 3-\delta]$, it holds that*

$$\lim_{\tau_f \to \infty} \mathcal{E}_p^{\tau_f}(\tau) = \mathcal{E}_p^{\infty}(\tau),$$
$$\lim_{\tau_f \to \infty} \underline{\mathcal{E}}_p^{\tau_f}(\tau) = \underline{\mathcal{E}}_p^{\infty}(\tau).$$

Proof. We give the proof for the $\partial_u\phi$ term in $\underline{\mathcal{E}}_p^{\tau_f}(\tau)$ as every other term in any of the energies is either similarly or less difficult. We use the change of variables formula to write the du_{τ_f} integral as a du_∞ integral:

$$\begin{aligned}\int_{\underline{C}^{\tau_f}(\tau)} (\bar{r}_{\tau_f} - M)^{2-p} \frac{(\partial_{u_{\tau_f}}\phi_{\tau_f})^2}{-\bar{\nu}_{\tau_f}}\, du_{\tau_f} &= \int_{\underline{C}^{\tau_f}(\tau)} (\bar{r}_{\tau_f} - M)^{2-p} \left(\frac{\partial_{u_{\tau_f}}\phi_{\tau_f}}{-\bar{\nu}_{\tau_f}}\right)^2 (-\bar{\nu}_{\tau_f}) du_{\tau_f} \\ &= \int_{[\Gamma^{u_\infty}(\tau),\Gamma^{u_\infty}(\tau_f)]\times\{v\}} \frac{(\bar{r}_{\tau_f} \circ \Psi_{\tau_f,\infty} - M)^{4-p}}{(\bar{r}_{\tau_f} \circ \Psi_{\tau_f,\infty})^2} \left(\frac{\partial_{u_\infty}\phi_\infty}{(1-\bar{\mu}_{\tau_f}) \circ \Psi_{\tau_f,\infty}}\right)^2 du_\infty, \end{aligned} \tag{8.20}$$

where $v = (\Gamma^v)^{-1}(\tau)$. We shall use the dominated convergence theorem to prove that this integral converges to

$$\int_{[\Gamma^{u_\infty}(\tau),\infty)\times\{v\}} \frac{(\bar r_\infty - M)^{4-p}}{\bar r_\infty^2}\left(\frac{\partial_{u_\infty}\phi_\infty}{1-\bar\mu_\infty}\right)^2 du_\infty = \int_{\underline{C}^\infty(\tau)} (\bar r_\infty - M)^{2-p}\frac{(\partial_{u_\infty}\phi_\infty)^2}{-\bar\nu_\infty}\, du_\infty \tag{8.21}$$

as $\tau_f \to \infty$.

It is clear from Proposition 8.13 that the integrand in (8.20) converges to the integrand in (8.21) pointwise, so we just have to prove a uniform bound by an L^1 function of u_∞. For the scalar field, we estimate

$$\left|\frac{\partial_{u_\infty}\phi_\infty}{1-\bar\mu_\infty}\right| = \lim_{\tau_f\to\infty}\left|\frac{\partial_{u_\infty}\phi_\infty}{(1-\bar\mu_{\tau_f})\circ\Psi_{\tau_f,\infty}}\right| = \lim_{\tau_f\to\infty}\left|\mathfrak{g}'_{\tau_f,\infty}\left(\frac{\partial_{u_{\tau_f}}\phi_{\tau_f}}{-\bar\nu_{\tau_f}}\right)\circ\Psi_{\tau_f,\infty}\right| \lesssim \varepsilon \tag{8.22}$$

by (7.17) and (8.11). To estimate the degenerate factor on the horizon, we note that

$$\partial_{u_{\tau_f}}\left((\bar r_{\tau_f} - M)^{-1}\right) = \bar r_{\tau_f}^{-2},$$

from which it readily follows that

$$\bar r_{\tau_f}(u_{\tau_f}, v) - M \lesssim \left(1 + u_{\tau_f} - u^\Lambda_{\tau_f}(v)\right)^{-1} \tag{8.23}$$

for $(u_{\tau_f}, v) \in \mathcal{D}_{\tau_f} \cap \{r \le \Lambda\}$. Therefore, combining (8.22) and (8.23), we have

$$\left|\frac{(\bar r_{\tau_f}\circ\Psi_{\tau_f,\infty} - M)^{4-p}}{(\bar r_{\tau_f}\circ\Psi_{\tau_f,\infty})^2}\left(\frac{\partial_{u_\infty}\phi_\infty}{(1-\bar\mu_{\tau_f})\circ\Psi_{\tau_f,\infty}}\right)^2\right| \lesssim \varepsilon^2\left(1 + \mathfrak{g}_{\tau_f,\infty}(u_\infty) - u^\Lambda_{\tau_f}(v)\right)^{p-4},$$

which is integrable as $p - 4 < -1$ (using also (8.11) to say that $\mathfrak{g}_{\tau_f,\infty}(u_\infty) - u^\Lambda_{\tau_f}(v) \sim u_\infty$ for u_∞ large). $\square$

By Propositions 8.1 and 8.15, the estimates (4.8)–(4.11) hold for $\tau_f = \tau_\infty$ and every $\tau \in [1,\infty)$. This proves (3.19) of Theorem 1.

8.3.3 Global structure of $\hat{\mathcal{Q}}_{\max}$

We now prove the existence of a black hole region and a regular event horizon in the maximal development $(\hat{\mathcal{Q}}_{\max}, r, \hat\Omega^2, \phi, e)$ of seed data lying in $\mathfrak{M}_{\rm stab}$.

Proposition 8.16. *The maximal development of any seed data lying in $\mathfrak{M}_{\rm stab}$ has $\hat{\mathcal{Q}}_{\max} = [0, U_*] \times [0,\infty)$ and there exists a $\hat u_{\mathcal{H}^+} \in (0, U_*)$ satisfying*

$$|\hat u_{\mathcal{H}^+} - \hat u_{\mathcal{H}^+,0}| \lesssim \varepsilon, \tag{8.24}$$

where $\hat u_{\mathcal{H}^+,0}$ was defined in (3.1), such that $[0, \hat u_{\mathcal{H}^+}] \times [0,\infty) \subset \hat{\mathcal{Q}}_{\max}$,

$$\lim_{\hat v\to\infty} r(\hat u, \hat v) = \infty \tag{8.25}$$

for every $\hat u \in [0, \hat u_{\mathcal{H}^+})$, and

$$\lim_{\hat v\to\infty} r(\hat u_{\mathcal{H}^+}, \hat v) = \lim_{\hat v\to\infty} \varpi(\hat u_{\mathcal{H}^+}, \hat v) = M = |e|. \tag{8.26}$$

Therefore, $[0, \hat u_{\mathcal{H}^+}) \times \{\hat v = \infty\}$ may be regarded as future null infinity $\mathcal{I}^+$ and

$$\mathcal{H}^+ \doteq J^-(\mathcal{I}^+) = \{\hat u_{\mathcal{H}^+}\} \times [0,\infty)$$

is the event horizon. The black hole region is

$$\mathcal{BH} \doteq \hat{\mathcal{Q}}_{\max} \setminus J^-(\mathcal{I}^+) = [\hat u_{\mathcal{H}^+}, U_*] \times [0,\infty) \neq \emptyset$$

and future null infinity is complete in the sense of Christodoulou [Chr99].

Proof. By the geometric estimates, r is bounded from below on $\hat{\mathcal{D}}_\infty$ and $\hat{\lambda} > 0$ by (5.3). It follows from the extension principle[10] [Daf05a] that $\overline{\hat{\mathcal{D}}_\infty} \subset \hat{\mathcal{Q}}_{\max}$, where the closure is taken in the $(\hat{u},\hat{v})$-plane. By Lemma 8.9, $\overline{\hat{\mathcal{D}}_\infty} \setminus \hat{\mathcal{D}}_\infty = \{\hat{u}_{\mathcal{H}^+}\} \times [0,\infty)$.

By (4.6) and (8.17), it holds that

$$|r - \bar{r}_\infty \circ \Phi_\infty| \lesssim \varepsilon^{3/2}\tau^{-2+\delta}$$

on $\hat{\mathcal{D}}_\infty$ and since

$$\lim_{\hat{u}\to\hat{u}_{\mathcal{H}^+}} \bar{r}_\infty \circ \Phi_\infty(\hat{u},\hat{v}) = \lim_{u_\infty\to\infty} \bar{r}_\infty(u_\infty, \mathfrak{v}(\hat{v})) = M$$

by the geometry of extremal Reissner–Nordström, we conclude that

$$|r(\hat{u}_{\mathcal{H}^+},\hat{v}) - M| \lesssim \varepsilon^{3/2}\tau^{-2+\delta}(\hat{u}_{\mathcal{H}^+},\hat{v}) \tag{8.27}$$

for every $\hat{v} \in [0,\infty)$. Furthermore, by passing to the limit in (4.7), we have

$$|\varpi - M| \lesssim \varepsilon^{3/2}\tau^{-3+\delta} \tag{8.28}$$

on $\overline{\hat{\mathcal{D}}_\infty}$. Now (8.27) and (8.28) imply (8.26).

Evaluating (8.27) at $\hat{v} = 0$ and using the gauge condition $\hat{\nu}(\hat{u},0) = -1$, we find that

$$\hat{u}_{\mathcal{H}^+} - \hat{u}_{\mathcal{H}^+,0} = \hat{u}_{\mathcal{H}^+} - (M - \Lambda) + O(\varepsilon) = r(\hat{u}_{\mathcal{H}^+},0) - M + O(\varepsilon) = O(\varepsilon),$$

which verifies (8.24).

To show that $\hat{\mathcal{Q}}_{\max}$ contains the rectangle $(\hat{u}_{\mathcal{H}^+}, U_*] \times [0,\infty)$, we use the logically independent fact that $\hat{\lambda} \geq 0$ everywhere on $\hat{\mathcal{Q}}_{\max}$, which will be shown in Section 8.3.5 below. Since r is bounded below on $[0, U_*] \times \{0\}$, is it bounded below on $\hat{\mathcal{Q}}_{\max}$. The extension principle [Daf05a] therefore implies that $\hat{\mathcal{Q}}_{\max} = [0, U_*] \times [0,\infty)$.

Finally, since r is bounded on $\mathcal{H}^+$, completeness of $\mathcal{I}^+$ follows from the work of Dafermos [Daf05b] (see also [Kom13]). □

This completes the proof of Part 2. of Theorem 1.

8.3.4 The final geometric estimates

In this section, we record the final geometric estimates in the eschatological gauge (u_∞, v), with sharp improvements in ε. This will prove (3.16)–(3.18) of Theorem 1.

While the (u_∞, v) coordinates do not cover the event horizon $\mathcal{H}^+$, we can formally attach it as the $u_\infty = \infty$ limiting curve (as is done in [DR05], for instance). We write this "extended" manifold as $\mathcal{D}_\infty \cup \mathcal{H}^+$. On $\mathcal{D}_\infty \cup \mathcal{H}^+$, we can extend certain background quantities to $\mathcal{H}^+$ as explained in Section 2.2.

Proposition 8.17. *If $\varepsilon_{\text{stab}}$ is sufficiently small, then in the eschatological gauge (u_∞, v) with background solution $\bar{r}_\infty$, it holds that*

$$|\gamma_\infty + 1| \lesssim \varepsilon^2 r^{-1}\tau^{-3+\delta} \tag{8.29}$$

on $\mathcal{D}_\infty \cap \{r \geq \Lambda\}$,

$$\left|\frac{\nu_\infty}{\bar{\nu}_\infty} - 1\right| \lesssim \varepsilon^2\tau^{-1+\delta} \tag{8.30}$$

on $\mathcal{D}_\infty$, and

$$|r - \bar{r}_\infty| \lesssim \varepsilon^2\tau^{-2+\delta}, \tag{8.31}$$

$$|\lambda_\infty - \bar{\lambda}_\infty| \lesssim \varepsilon^2\tau^{-2+\delta}, \tag{8.32}$$

$$|\kappa_\infty - 1| \lesssim \varepsilon^2\tau^{-1+\delta}, \tag{8.33}$$

$$|\varpi - M| \lesssim \varepsilon^2\tau^{-3+\delta}, \tag{8.34}$$

$$|(1-\mu) - (1-\bar{\mu}_\infty)| \lesssim \varepsilon^2\tau^{-2+\delta}, \tag{8.35}$$

$$|\varkappa - \bar{\varkappa}_\infty| \lesssim \varepsilon^2 r^{-2}\tau^{-3+\delta} + \varepsilon^2 r^{-3}\tau^{-2+\delta} \tag{8.36}$$

[10]Note that since we actually already have a pointwise estimate for ϕ on $\hat{\mathcal{D}}_\infty$ by (7.12), one could give a much quicker argument for the extension.

on $\mathcal{D}_\infty \cup \mathcal{H}^+$.

Remark 8.18. Note that compared to the estimates proved in Section 5, the estimates (8.30), (8.31), (8.32), (8.34), (8.35), and (8.36) have factors of ε^2 instead of $\varepsilon^{3/2}$. This improvement is the main content of the following proof.

Proof of Proposition 8.17. The estimates (8.29) and (8.33) are obtained by simply passing to the limit in Lemmas 5.2 and 5.3. To prove (8.34), we revisit the estimate (5.44) and apply it to $\alpha = \alpha_\star$. Since $\alpha_\star \in \bigcap_{i\geq 0}\mathfrak{A}_i$, it holds that $|\Pi_{I(L_i)}(\alpha_\star)| \leq \varepsilon^{3/2}L_i^{-3+\delta}$ for every $i \geq 0$. Therefore, by (5.44) we have $|\varpi - M| \lesssim \varepsilon^{3/2}L_i^{-3+\delta} + \varepsilon^2\tau^{-3+\delta}$ on $\mathcal{D}_{L_i}$. Sending $i \to \infty$ proves (8.34) on $\mathcal{D}_\infty$.

We now prove the remaining estimates via a bootstrap argument in the (u_∞, v) gauge on $\mathcal{D}_\infty$. Let $\check{\mathfrak{B}}$ denote the set of $\tau_f \in [1,\infty)$ for which

$$|r - \bar{r}_\infty| \leq \check{A}\varepsilon^2\tau^{-2+\delta}, \tag{8.37}$$

$$\left|\frac{\nu_\infty}{\bar{\nu}_\infty} - 1\right| \leq \check{A}^2\varepsilon^2\tau^{-1+\delta}, \tag{8.38}$$

where $\check{A} \geq 1$ is a constant to be determined.

Let $\tau_f \in \check{\mathfrak{B}}$. Using these bootstrap assumptions and (8.34), we improve the Taylor expansions (5.27) and (5.28) to

$$\partial_u(r - \bar{r}_\infty) = \frac{2M\gamma_\infty}{\bar{r}_\infty^3}(\bar{r}_\infty - M)(r - \bar{r}_\infty) + O\big(\varepsilon^2\bar{r}_\infty^{-1}\tau^{-3+\delta}\big)$$

in $\mathcal{D}_\infty \cap \{\tau \leq \tau_f\} \cap \{r \geq \Lambda\}$ and

$$\partial_v(r - \bar{r}_\infty) = \frac{2M\kappa_\infty}{\bar{r}_\infty^3}(\bar{r}_\infty - M)(r - \bar{r}_\infty) + O\big(\varepsilon^2\tau^{-3+\delta}\big)$$

in $\mathcal{D}_\infty \cap \{\tau \leq \tau_f\}$. By passing to the limit in (5.35), we have

$$\big|(r - \bar{r}_\infty)|_\Gamma\big| \lesssim \varepsilon^2\tau^{-2+\delta}.$$

By now repeating the arguments of Lemmas 5.11 and 5.13, we improve the bootstrap assumptions (8.37) and (8.38) for $\check{A}$ chosen sufficiently large. Therefore, $\check{\mathfrak{B}} = [1,\infty)$ and the estimates (8.30) and (8.31) hold.

Finally, (8.35) and (8.36) are proved by repeating the arguments of Lemmas 5.8 and 5.10 with our improved estimates at hand. □

Using (8.34), we can prove a sharp-in-ε estimate for the final modulation parameter $\alpha_\star$. The following result can be compared with [DHRT, Remark 6.3.5].

Proposition 8.19. *Let $\mathcal{S}_0(\alpha_\star) = (\mathring{\phi}, r_0, M_0 + \alpha_\star, e) \in \mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0, \varepsilon)$. Then*

$$\alpha_\star = |e| - |e_0| + O(\varepsilon^2) \tag{8.39}$$

Proof. By (8.34), we have

$$\alpha_\star = \varpi(0,0) - M_0 = \varpi(0,0) - M - (M_0 - M) = |e| - |e_0| + O(\varepsilon^2).$$ □

8.3.5 Putting everything together

Everything is now in place to complete the proof of nonlinear stability of Reissner–Nordström, Theorem 1.

Proof of Theorem 1. Part 1. of the theorem is a direct restatement of Proposition 8.6 above. Part 2. of the theorem is a combination of Propositions 8.8 and 8.16, and (8.29) implies that $\nu_\infty \to -1$ at $\mathcal{I}^+$. Orbital stability of the parameters, (3.13), follows immediately from the definition of $\mathfrak{D}$ and the conservation of charge e in the neutral scalar field model.

Proof of orbital stability for the $p = 3 - \delta$ energy: Unfortunately, this does not immediately follow from Proposition 8.15 applied to $p = 3 - \delta$ since the smallness parameter $\varepsilon \geq \mathfrak{D}[\mathcal{S}_0]$ only bounds the energy $\mathcal{E}^\infty_{3-\delta}(1)+\underline{\mathcal{E}}^\infty_{3-\delta}(1)$ from *above* (recall Lemma 7.3). Furthermore, the estimates (6.38) and (6.48) for $p=3-\delta$ have nonlinear errors that depend on ε and are not bounded by $\mathcal{E}^\infty_{3-\delta}(1)+\underline{\mathcal{E}}^\infty_{3-\delta}(1)$ as the estimates are currently written. We now sketch how to remove this deficiency. Working on $\mathcal{D}_\infty$ in the eschatological gauge (u_∞, v), we define a bootstrap set $\breve{\mathfrak{B}}$ consisting of $\tau_f \in [1,\infty)$ such that the bootstrap assumptions (4.5)–(4.11) hold with ε replaced by $\breve{\varepsilon} \doteq \big(\mathcal{E}^\infty_{3-\delta}(1)+\underline{\mathcal{E}}^\infty_{3-\delta}(1)\big)^{1/2}$ and A replaced by a possibly larger constant $\breve{A}$. We now repeat *all* of the arguments in Section 5, Section 6, and Section 7.1 with τ_f replaced by ∞ and ε by $\breve{\varepsilon}$ to show that $\breve{\mathfrak{B}}=[1,\infty)$. Note that the only instance (besides the bootstrap assumptions) where ε enters into the proofs of the estimates in Section 5, Section 6, and Section 7.1, as they are currently written, is in (7.3), which is now replaced by the definition of $\breve{\varepsilon}$. We therefore have

$$\mathcal{E}^\infty_p(\tau)+\underline{\mathcal{E}}^\infty_p(\tau) \lesssim \breve{\varepsilon}^2\tau^{-3+\delta+p} = \big(\mathcal{E}^\infty_{3-\delta}(1)+\underline{\mathcal{E}}^\infty_{3-\delta}(1)\big)\tau^{-3+\delta+p}$$

for every $\tau\in[1,\infty)$ and $p\in[0,3-\delta]$. For $p=3-\delta$, this estimate gives (3.14), as desired.

Proof of orbital stability for the pointwise C^1 norm: Revisiting the proofs of (7.13)–(7.17), we observe that the estimates only depend on energies and pointwise norms coming from initial data of the solution restricted to $J^-(\mathcal{I}^+)\cap\hat{\mathcal{C}}$. Therefore, the ε on the right-hand side of the estimates (7.13)–(7.17) can be replaced by the right-hand side of (3.15).

Part 4. of the theorem follows from Proposition 8.15, Proposition 8.17, (7.12), and (7.13).

Proof of the absence of trapped surfaces: We first recall the notions of *apparent horizon*

$$\mathcal{A} \doteq \{(\hat{u},\hat{v})\in\hat{\mathcal{Q}}_{\max} : \lambda(\hat{u},\hat{v})=0\},$$

outermost apparent horizon

$$\mathcal{A}' \doteq \{(\hat{u},\hat{v})\in\hat{\mathcal{Q}}_{\max} : \lambda(\hat{u},\hat{v})=0 \text{ and } \lambda(\hat{u}',\hat{v})>0 \text{ for every } \hat{u}'<\hat{u}\}$$

and the set[11]

$$\mathcal{R} \doteq \{(\hat{u},\hat{v})\in\hat{\mathcal{Q}}_{\max} : \lambda(\hat{u},\hat{v})>0\}.$$

We observe from (2.19) and (2.20) that if $(\hat{u},\hat{v})\in\mathcal{R}\cup\mathcal{A}$, then we have the monotonicities

$$\partial_u\varpi(\hat{u},\hat{v})\le 0 \quad\text{and}\quad \partial_v\varpi(\hat{u},\hat{v})\ge 0, \tag{8.40}$$

Proof that $\mathcal{A}'\subset\mathcal{H}^+$: Let $(\hat{u}_0,\hat{v}_0)\in\mathcal{A}'$.[12] By the monotonicities (8.40) and the outermost property of $(\hat{u}_0,\hat{v}_0)$, it follows that $M\ge\varpi(\hat{u}_0,\hat{v}_0)$ and $M\ge r(\hat{u}_0,\hat{v}_0)$. We also have $1-\mu(\hat{u}_0,\hat{v}_0)=0$, which can be solved for $r(\hat{u}_0,\hat{v}_0)=\varpi(\hat{u}_0,\hat{v}_0)\pm\sqrt{\varpi(\hat{u}_0,\hat{v}_0)^2-M^2}$, which implies that $\varpi(\hat{u}_0,\hat{v}_0)\ge M$. Hence, $\varpi(\hat{u}_0,\hat{v}_0)=M$ and $r(\hat{u}_0,\hat{v}_0)=M$. Since $\nu<0$ on $\hat{\mathcal{Q}}_{\max}$ (recall Lemma 3.2), this in turn implies that $(\hat{u}_0,\hat{v}_0)\in\mathcal{H}^+$ and that $\{\hat{u}_{\mathcal{H}^+}\}\times[\hat{v}_0,\infty)\subset\mathcal{A}'$. By Raychaudhuri's equation (2.15), it follows that $\partial_v\phi$ vanishes identically on $\{\hat{u}_{\mathcal{H}^+}\}\times[\hat{v}_0,\infty)$. Since ϕ decays along $\mathcal{H}^+$ by (3.20), ϕ itself vanishes on $\{\hat{u}_{\mathcal{H}^+}\}\times[\hat{v}_0,\infty)$.

Proof that $\lambda>0$ behind $\mathcal{H}^+$: Suppose $\lambda(\hat{u}_0,\hat{v}_0)\le 0$. By the argument in the previous paragraph, $(\hat{u}_{\mathcal{H}^+},\hat{v}_0)\in\mathcal{A}'$ and it holds that $r(\hat{u}_{\mathcal{H}^+},\hat{v}_0)=\varpi(\hat{u}_{\mathcal{H}^+},\hat{v}_0)=M$. Therefore, by (2.17), $\partial_u\lambda(\hat{u}_{\mathcal{H}^+},\hat{v}_0)=0$. Because $\partial_u\varpi(\hat{u}_{\mathcal{H}^+},\hat{v}_0)=0$, we may compute

$$\partial_u^2\lambda(\hat{u}_{\mathcal{H}^+},\hat{v}_0)=\frac{2e^2\kappa\nu^2}{r^4}(\hat{u}_{\mathcal{H}^+},\hat{v}_0)>0.$$

If $\hat{u}_0>\hat{u}_{\mathcal{H}^+}$, it follows from Taylor's theorem that there exists a $\hat{u}_*\in(\hat{u}_{\mathcal{H}^+},\hat{u}_0]$ such that $\lambda(\hat{u}_*,\hat{v}_0)=0$ and $\lambda\ge 0$ on $[\hat{u}_{\mathcal{H}^+},\hat{u}_*]\times\{\hat{v}_0\}$. We may then argue again using monotonicity that $M\ge\varpi(\hat{u}_*,\hat{v}_0)$ and $M\ge r(\hat{u}_*,\hat{v}_0)$. The argument in the previous paragraph now implies that $\varpi(\hat{u}_*,\hat{v}_0)=r(\hat{u}_*,\hat{v}_0)=M$, which contradicts the assumption that $\hat{u}_0>\hat{u}_{\mathcal{H}^+}$.

This completes the proof of Theorem 1. □

[11] Sometimes called the "regular region."

[12] This hypothesis could be empty. In this case, the hypothesis of part 5. of the theorem is itself empty.

9 Fine properties of the scalar field on and near the event horizon

In this section, we use the decay estimates for the geometry and the scalar field derived in the previous section to further analyze the behavior of the scalar field on and near the event horizon $\mathcal{H}^+$ of the solutions given by Theorem 1. In Section 9.1, we investigate the Aretakis instability on the dynamical geometry, which proves Theorem 2. In particular, we prove that each of the solutions given by Theorem 1 has an "asymptotic Aretakis charge" $H_0[\phi]$, which while not actually constant, is "almost" constant in a quantitative sense. In Section 9.2, we derive sharp pointwise asymptotics for ψ near $\mathcal{H}^+$. Finally, in Section 9.3, we show that if the final Aretakis charge $H_0[\phi]$ is "quantitatively nonvanishing," then no nondegenerate integrated energy decay statement is true near $\mathcal{H}^+$, which proves the sharpness of the horizon hierarchy proved in Section 6. For a discussion of these results, we refer the reader back to Section 1.4.1.

In this section, we will always work with one of the solutions $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ arising from seed data lying in $\mathfrak{M}_{\text{stab}}$. In particular, the conclusions of Theorem 1 hold for this solution.

9.1 Proof of the Aretakis instability, Theorem 2

Recall the gauge-invariant vector field

$$Y \doteq \nu^{-1}\partial_u,$$

which equals ∂_r in standard ingoing Eddington–Finkelstein coordinates (v, r) in Reissner–Nordström. For a general spherically symmetric spacetime, we derive at once the commutation formula

$$[\partial_v, Y] = -\frac{\partial_u\partial_v r}{\nu} Y,$$

which will be useful in deriving equations for $\partial_v(Y\psi)$ and $\partial_v(Y^2\psi)$, where, as in Section 6, $\psi \doteq r\phi$.

Lemma 9.1. *Let $(\mathcal{Q}, r, \Omega^2, \phi, e)$ be a solution of the spherically symmetric Einstein–Maxwell-scalar field system. Then*

$$\partial_v(Y\psi) + (2\kappa\varkappa)Y\psi = 2\kappa\varkappa\,\phi. \tag{9.1}$$

Proof. Using the commutation formula and (2.16), we compute

$$\partial_v(Y\psi) = [\partial_v, Y]\psi + Y(\partial_v\psi) = -\frac{\partial_u\partial_v r}{\nu}Y\psi + \frac{\partial_u\partial_v r}{\nu}\phi.$$

By (2.12), we conclude (9.1). □

Lemma 9.2. *Let $(\mathcal{Q}, r, \Omega^2, \phi, e)$ be a solution of the spherically symmetric Einstein–Maxwell-scalar field system. Then*

$$\partial_v(Y^2\psi) + (4\kappa\varkappa)Y^2\psi = -\frac{2\kappa e^2}{r^4}Y\psi + E, \tag{9.2}$$

where

$$E \doteq \frac{2\kappa e^2}{r^3}\phi + (1-\mu)r\kappa(Y\phi)^3 + 2\kappa\varkappa\left(r^2(Y\phi)^3 - Y\phi\right). \tag{9.3}$$

Proof. We rewrite (9.1) as

$$\partial_v(Y\psi) = \frac{\partial_u\partial_v r}{\nu}(\phi - Y\psi)$$

and apply again the commutation formula to obtain

$$\begin{aligned}
\partial_v(Y^2\psi) &= [\partial_v, Y]Y\psi + Y\partial_v(Y\psi)\\
&= -\frac{\partial_u\partial_v r}{\nu}Y^2\psi + Y\left[\frac{\partial_u\partial_v r}{\nu}(\phi - Y\psi)\right]\\
&= -\frac{\partial_u\partial_v r}{\nu}Y^2\psi + Y\left(\frac{\partial_u\partial_v r}{\nu}\right)(\phi - Y\psi) + \frac{\partial_u\partial_v r}{\nu}Y(\phi - Y\psi),\\
&= -2\frac{\partial_u\partial_v r}{\nu}Y^2\psi + \frac{\partial_u\partial_v r}{\nu}Y\phi + Y\left(\frac{\partial_u\partial_v r}{\nu}\right)(\phi - Y\psi)
\end{aligned}$$

Using (2.17), (2.19), and (2.21), we compute

$$Y\left(\frac{\partial_u\partial_v r}{\nu}\right) = 2r\kappa\varkappa(Y\phi)^2 - \frac{4\kappa\varkappa}{r} + (1-\mu)\kappa(Y\phi)^2 + \frac{2\kappa e^2}{r^4}.$$

By grouping terms appropriately, we arrive at (9.2) and (9.3). □

Proof of Theorem 2. We work in one of the spacetimes $(\hat{\mathcal{Q}}_{\max}, r, \hat{\Omega}^2, \phi, e)$ given by Theorem 1 with the initial data normalized coordinates $(\hat{u}, \hat{v})$. Recall that the event horizon $\mathcal{H}^+$ is located at $\hat{u} = \hat{u}_{\mathcal{H}^+}$.

Almost-conservation of $Y\psi|_{\mathcal{H}^+}$: We use an integrating factor to solve (9.1) on $\mathcal{H}^+$:

$$Y\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}_2) = \exp\left(-\int_{\hat{v}_1}^{\hat{v}_2} 2\hat{\kappa}\varkappa\, d\hat{v}'\right) Y\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}_1) + \int_{\hat{v}_1}^{\hat{v}_2} \exp\left(-\int_{\hat{v}'}^{\hat{v}_2} 2\hat{\kappa}\varkappa\, d\hat{v}''\right) 2\hat{\kappa}\varkappa\,\phi\, d\hat{v}'$$

for every $0 \le \hat{v}_1 \le \hat{v}_2$, where every quantity on the right-hand side is evaluated at $\hat{u} = \hat{u}_{\mathcal{H}^+}$. By integrating (8.36) along $\mathcal{H}^+$ and using (5.23), we find

$$\int_{\hat{v}_1}^{\hat{v}_2} |\hat{\kappa}\varkappa|_{\mathcal{H}^+}|\, d\hat{v}' \lesssim \varepsilon^2(1+\hat{v}_1)^{-1+\delta}, \quad \exp\left(-\int_{\hat{v}_1}^{\hat{v}_2} 2\hat{\kappa}\varkappa|_{\mathcal{H}^+}\, d\hat{v}'\right) = 1 + O\big(\varepsilon^2(1+\hat{v}_1)^{-1+\delta}\big).$$

Using these, (3.20), and (3.15), we estimate

$$|Y\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}_2) - Y\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}_1)| \lesssim \varepsilon^2(1+\hat{v}_1)^{-1+\delta}|Y\psi(u_{\mathcal{H}^+}, \hat{v}_1)| + \int_{\hat{v}_1}^{\hat{v}_2} |\hat{\kappa}\varkappa||\phi|\, d\hat{v}' \lesssim \varepsilon^3(1+\hat{v}_1)^{-1+\delta}. \tag{9.4}$$

By an elementary Cauchy sequence argument, the limit

$$H_0[\phi] \doteq \lim_{\hat{v}\to\infty} Y\psi(\hat{u}_{\mathcal{H}^+}, \hat{v})$$

exists and the estimate (3.21) holds.

Proof that $\mathfrak{M}^{\neq 0}_{\mathrm{stab}}$ has nonempty interior: Let $\mathring{\varphi}$ be a compactly supported smooth function on $\hat{\mathcal{C}}$ such that

$$\frac{d}{d\hat{u}}\big((100M_0 - \hat{u})\mathring{\varphi}_{\mathrm{in}}\big)\Big|_{\hat{u}=\hat{u}_{\mathcal{H}^+,0}} \neq 0. \tag{9.5}$$

Let $0 < \varepsilon \le \varepsilon_{\mathrm{stab}}$ and set $\mathcal{S}_0' \doteq (c\varepsilon\mathring{\varphi}, 100M_0, M_0, e_0)$, where $c > 0$ is a constant chosen so that $\mathfrak{D}[\mathcal{S}_0'] \le \frac{1}{2}\varepsilon$. In particular, c is independent of ε. For $\eta > 0$ small, we now set

$$\mathfrak{O}_\eta \doteq \{\mathcal{L}(\mathcal{S}_0, \varepsilon) : \mathcal{S}_0 \in \mathfrak{M}_0, \mathfrak{D}[\mathcal{S}_0 - \mathcal{S}_0'] < \eta\}.$$

Clearly, $\mathfrak{O}_\eta$ is open in $\mathfrak{M}$ and we claim that for ε and η sufficiently small, $\emptyset \neq \mathfrak{O}_\eta \cap \mathfrak{M}_{\mathrm{stab}} \subset \mathfrak{M}^{\neq 0}_{\mathrm{stab}}$.

For η small and $\mathcal{S}_0 \in \mathfrak{O}_\eta$, Theorem 1 applies to $\mathcal{L}(\mathcal{S}_0, \varepsilon)$. Let $\mathcal{S} \in \mathfrak{M}_{\mathrm{stab}}(\mathcal{S}_0, \varepsilon)$, let $H_0[\phi]$ be the associated asymptotic Aretakis charge. By (8.24) and (9.5), there exists a constant $c' > 0$ such that

$$|Y\psi(\hat{u}_{\mathcal{H}^+}, 0)| \gtrsim \varepsilon - c'\eta^{1/2} \gtrsim \varepsilon$$

for ε and η sufficiently small. By (9.4), $H_0[\phi]$ differs from $Y\psi(\hat{u}_{\mathcal{H}^+}, 0)$ by an $O(\varepsilon^3)$ quantity, and is therefore nonzero for ε sufficiently small. As this applies for every $\mathcal{S}_0 \in \mathfrak{O}_\eta$, we have proved that $\emptyset \neq \mathfrak{O}_\eta \cap \mathfrak{M}_{\mathrm{stab}} \subset \mathfrak{M}^{\neq 0}_{\mathrm{stab}}$, which shows that $\mathfrak{M}^{\neq 0}_{\mathrm{stab}}$ has nonempty interior in the subspace topology.

Linear growth of $Y^2\psi|_{\mathcal{H}^+}$: We use an integrating factor to solve (9.2) on $\mathcal{H}^+$:

$$Y^2\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}) = \exp\left(-\int_0^{\hat{v}} 4\hat{\kappa}\varkappa\, d\hat{v}'\right) Y\psi(\hat{u}_{\mathcal{H}^+}, 0) + \int_0^{\hat{v}} \exp\left(-\int_{\hat{v}'}^{\hat{v}_2} 4\hat{\kappa}\varkappa\, d\hat{v}''\right)\left(-\frac{2\hat{\kappa}e^2}{r^4}Y\psi + E\right) d\hat{v}'.$$

The first term is $O(\varepsilon)$ by our assumption on the initial data and by (8.35), (8.36), (3.20), and (3.15), we have

$$\big|E|_{\mathcal{H}^+}\big| \lesssim \varepsilon\tau^{-1+\delta/2}.$$

Note that the worst decaying term in E (by far) comes from the zeroth order term, for which we only have the nonintegrable decay estimate (3.20). Using again the geometric estimates on the horizon and (9.4), we have

$$\int_0^{\hat{v}} \frac{2\hat{\kappa}e^2}{r^4} Y\psi \, d\hat{v}' = \int_0^{\mathfrak{v}(\hat{v})} \frac{2\kappa e^2}{r^4} Y\psi \, dv' = \frac{2}{M^2} H_0[\phi]\hat{v} + O\big(\varepsilon^3(1+\hat{v})^\delta\big).$$

Putting these estimates together, we have

$$\left| Y^2\psi(\hat{u}_{\mathcal{H}^+}, \hat{v}) + \frac{2}{M^2} H_0[\phi]\hat{v} \right| \lesssim \varepsilon(1+\hat{v})^{\delta/2} + \varepsilon^3(1+\hat{v})^\delta,$$

which implies (3.23).

Behavior of the Ricci tensor along $\mathcal{H}^+$: By trace-reversing the Einstein equation, we find

$$R_{\mu\nu} = 2T^{\mathrm{EM}}_{\mu\nu} + 2\partial_\mu\phi\partial_\nu\phi$$

and using (2.3) we compute

$$T^{\mathrm{EM}} = \frac{\hat{\Omega}^2 e^2}{4r^4}(d\hat{u} \otimes d\hat{v} + d\hat{v} \otimes d\hat{u}) + \frac{e^2}{2r^2} g_{S^2}.$$

Therefore,

$$R_{\mu\nu} Y^\mu Y^\nu = 2(Y\phi)^2 = 2r^{-2}(Y\psi)^2 - 4r^{-3}\psi Y\psi + 2r^{-4}\psi^2,$$

from which (3.22) follows readily.

Next, we compute

$$\nabla_\rho R_{\mu\nu} = 2\nabla_\rho T^{\mathrm{EM}}_{\mu\nu} + 2\nabla_\rho\nabla_\mu\phi\nabla_\nu\phi + 2\nabla_\mu\phi\nabla_\rho\nabla_\nu\phi.$$

Using again the form of T^{EM}, we have $\nabla_\rho T^{\mathrm{EM}}_{\mu\nu} Y^\rho Y^\mu Y^\nu = 0$. Since

$$\Gamma^{\hat{u}}_{\hat{u}\hat{u}} = \partial_{\hat{u}} \log \hat{\Omega}^2 = \partial_{\hat{u}} \log(-4\hat{\kappa}\hat{\nu}) = \hat{\kappa}^{-1}\partial_{\hat{u}}\hat{\kappa} + \hat{\nu}^{-1}\partial_{\hat{u}}\hat{\nu},$$

it holds that

$$\begin{aligned}
\nabla_\rho\nabla_\mu\phi \, Y^\rho Y^\mu &= \hat{\nu}^{-2}\partial_{\hat{u}}^2\phi - \hat{\nu}^{-2}\big(\hat{\kappa}^{-1}\partial_{\hat{u}}\hat{\kappa} + \hat{\nu}^{-1}\partial_{\hat{u}}\hat{\nu}\big)\partial_{\hat{u}}\phi \\
&= \hat{\nu}^{-2}\partial_{\hat{u}}(\hat{\nu}Y\phi) - \hat{\nu}^{-1}\big(r\hat{\nu}(Y\phi)^2 + \hat{\nu}^{-1}\partial_{\hat{u}}\hat{\nu}\big)Y\phi \\
&= Y^2\phi - r(Y\phi)^3
\end{aligned}$$

and hence

$$\nabla_\rho R_{\mu\nu} Y^\rho Y^\mu Y^\nu = 4Y\phi Y^2\phi - 4r(Y\phi)^4,$$

from which (3.24) follows readily. □

9.2 Sharp decay for the scalar field near the horizon

In this section, we work in the eschatological gauge (u_∞, v) on the domain $\mathcal{D}_\infty$ with the final anchored extremal Reissner–Nordström solution $\bar{r}_\infty$. Given a number $C_2 > 0$, we introduce the following additional smallness assumption on the initial data:

$$\sup_{\underline{C}_{\mathrm{in}}} |Y^2\psi| \le C_2\varepsilon. \tag{9.6}$$

This will be used to perform a Taylor expansion of $Y\psi$ near the event horizon on $\underline{C}_{\mathrm{in}}$.

We require the following lemma (see also (2.25)).

Lemma 9.3. *On $\mathcal{D}_\infty \cap \{\bar{r}_\infty \le \frac{1}{2}\Lambda\}$, it holds that*

$$\frac{1}{\bar{r}_\infty - M} \sim u_\infty - v. \tag{9.7}$$

Proof. We have the identity

$$v-u_\infty=\int_{\bar r_\star}^{\bar r_\infty}\frac{d\bar r'}{D(\bar r')}=-\frac{M^2}{\bar r_\infty-M}+\frac{M^2}{\bar r_\star-M}+2M\log\left(\frac{\bar r_\infty-M}{\bar r_\star-M}\right)+\bar r_\infty-\bar r_\star \tag{9.8}$$

on $\mathcal{D}_\infty$, where $\bar r_\star$ was defined in (8.15) and $D(r)\doteq(1-M/r)^2$. Indeed, this is trivially true at the bifurcation sphere $(u_\infty,v)=(0,0)$ by definition of $\bar r_\star$, and holds everywhere because the ∂_{u_∞} and ∂_v derivatives of $v-u_\infty$ and the integral are equal. Writing (9.8) as

$$u_\infty-v=\frac{M^2}{\bar r_\infty-M}-2M\log(\bar r_\infty-M)+O(1), \tag{9.9}$$

we infer (9.7) by inspection. □

Let us consider the curve

$$\sigma_\beta\doteq\{(u_\infty,v)\in\mathcal{D}_\infty:u_\infty-v=u_\infty^\beta+C_\beta\}, \tag{9.10}$$

where the constant C_β is chosen so that $\sigma_\beta\subset\{\bar r_\infty\le\frac12\Lambda\}$. We also define the spacetime region

$$\mathfrak{t}_\beta\doteq\{(u_\infty,v)\in\mathcal{D}_\infty:v\le u_\infty-u_\infty^\beta-C_\beta\}. \tag{9.11}$$

Using (9.7), we observe that in the region $\mathfrak{t}_\beta$,

$$\bar r_\infty-M\lesssim(1+u_\infty)^{-\beta}\lesssim(1+v)^{-\beta}. \tag{9.12}$$

With these definitions, we have the following result:

Proposition 9.4. *Under the assumption* (9.6)*, for every $\beta\in(\frac23,1)$, there exists a $\beta'>0$ such that*

$$|u_\infty^2\partial_{u_\infty}\psi+M^2H_0[\phi]|\lesssim\varepsilon^3+\varepsilon u_\infty^{-\beta'} \tag{9.13}$$

in $\mathfrak{t}_\beta$, where $H_0[\phi]$ is the asymptotic Aretakis charge of the wave.

We first verify the proposition along $\underline{C}_{\mathrm{in}}$.

Lemma 9.5. *Let $\eta>0$. Under the assumption* (9.6)*, it holds that*

$$|u_\infty^2\partial_{u_\infty}\psi(u_\infty,0)+M^2H_0[\phi]|\lesssim\varepsilon^3+\varepsilon u_\infty^{-1+\eta} \tag{9.14}$$

for all $u_\infty>0$.

Proof. Using the chain rule and (8.30), we estimate

$$\partial_{\hat u}\big((\bar r_\infty-M)\circ\Phi_\infty(\hat u,0)\big)=\bar\nu_\infty\big(\Phi_\infty(\hat u,0)\big)\partial_{\hat u}\Phi(\hat u,0)\sim\nu_\infty\big(\Phi_\infty(\hat u,0)\big)\partial_{\hat u}\Phi(\hat u,0)=\hat\nu(\hat u,0)=-1.$$

Therefore, integrating backwards from $\hat u_{\mathcal{H}^+}$, we find that

$$(\bar r_\infty-M)\circ\Phi_\infty(\hat u,0)\sim\hat u_{\mathcal{H}^+}-\hat u$$

for $\hat u\in[0,\hat u_{\mathcal{H}^+}]$. By Taylor's theorem, the estimate (9.4), and the assumption (9.6), we have that

$$Y\psi(\hat u,0)=H_0[\phi]+O\big(\varepsilon(\bar r_\infty-M)\big)+O(\varepsilon^3)$$

for $\hat u\in[0,\hat u_{\mathcal{H}^+}]$. Of course, $H_0[\phi]=O(\varepsilon)$ as well.

By (9.9), the identity $\bar\nu_\infty=-\bar r_\infty^{-2}(\bar r_\infty-M)^2$, and the Taylor expansion $\bar r_\infty^{-2}=M^{-2}+O(\bar r_\infty-M)$, we have

$$u_\infty^2\bar\nu_\infty=-M^2+O\big((\bar r_\infty-M)[1+\log(\bar r_\infty-M)]\big)$$

at $v=0$, and we may therefore compute

$$u_\infty^2\partial_{u_\infty}\psi=u_\infty^2\nu_\infty Y\psi=u_\infty^2\bar\nu_\infty\big(1+O(\varepsilon^3)\big)Y\psi=-M^2H_0[\phi]+O\big(\varepsilon(\bar r_\infty-M)[1+\log(\bar r_\infty-M)]\big)+O(\varepsilon^3)$$

at $v=0$. Using the estimate $x\log x\lesssim x^{1-\eta}$, we arrive at (9.14). □

Proof of Proposition 9.4. Integrating the wave equation (2.18) along the segment $\{u_\infty\}\times[0,v]\subset\mathfrak{t}_\beta$, we have

$$u_\infty^2\partial_{u_\infty}\psi(u_\infty,v)=u_\infty^2\partial_{u_\infty}\psi(u_\infty,0)+u_\infty^2\int_0^v\frac{2\kappa_\infty\nu_\infty\varkappa}{r}\psi\,dv'.$$

By (5.7), (5.41), (5.32), and (8.33), (8.30), (8.36), the second term on the right-hand side satisfies

$$\begin{aligned}\left|u_\infty^2\int_0^v\frac{2\kappa_\infty\nu_\infty\varkappa}{r}\psi\,dv'\right|&\lesssim u_\infty^2\int_0^v(\bar r_\infty-M)^3|\psi|\,dv'+u_\infty^2\int_0^v\varepsilon^2(1+v')^{-2+\delta}(\bar r_\infty-M)^2|\psi|\,dv'\\&\lesssim u_\infty^{-\beta'}\int_0^v u_\infty^{2+\beta'}(\bar r_\infty-M)^3|\psi|\,dv'+\int_0^v\varepsilon^2u_\infty^2(1+v')^{-2+\delta}(\bar r_\infty-M)^2|\psi|\,dv'.\end{aligned}\tag{9.15}$$

For the first term of the last expression we note that by (3.20) and (9.12), we have

$$u_\infty^{-\beta'}\int_0^v u_\infty^{2+\beta'}(\bar r_\infty-M)^3|\psi|\,dv'\lesssim\varepsilon u_\infty^{-\beta'}\int_0^v(1+v')^{-1+\delta/2-3\beta+2+\beta'}\,dv'\lesssim\varepsilon u_\infty^{-\beta'},$$

as $-1+\delta/2-3\beta+2+\beta'<-1$ for $\beta\in(2/3,1)$ and $\beta'>0$ sufficiently small.

To handle the second term on the right-hand side of (9.15), we break the region $\mathfrak{t}_\beta$ into two pieces. We split $\mathfrak{t}_\beta$ along the curve

$$\rho_\beta\doteq\{(u_\infty,v)\in\mathcal{D}_\infty:v=\tfrac12u_\infty-C'_\beta\},$$

where C'_β is chosen so that $\rho_\beta\cap\sigma_\beta=\emptyset$. Denote the region between ρ_β and $\mathcal{H}^+$ by $\mathfrak{t}^1_\beta$ and the region between ρ_β and σ_β by $\mathfrak{t}^2_\beta$, so that $\mathfrak{t}_\beta=\mathfrak{t}^1_\beta\cup\mathfrak{t}^2_\beta$. In $\mathfrak{t}^1_\beta$, it holds that $\bar r_\infty-M\lesssim(1+u_\infty)^{-1}$, which implies that

$$\int_0^v\varepsilon^2u_\infty^2(1+v')^{-2+\delta}(\bar r_\infty-M)^2|\psi|\mathbf{1}_{\mathfrak{t}^1_\beta}\,dv'\lesssim\varepsilon^3.$$

On the other hand, in $\mathfrak{t}^2_\beta$, it holds that $(1+v)^{-1}\lesssim(1+u_\infty)^{-1}$, so using (9.12) we estimate

$$\int_0^v\varepsilon^2u_\infty^2(1+v')^{-2+\delta}(\bar r_\infty-M)^2|\psi|\mathbf{1}_{\mathfrak{t}^2_\beta}\,dv'\lesssim\varepsilon^2u_\infty^{2-k-2\beta}\int_0^v(1+v')^{-3+k+3\delta/2}\,dv'\lesssim\varepsilon^3u_\infty^{-\beta'}$$

where k is chosen so that $-2+k+2\beta>\beta'$ and $-3+k+3\delta/2<-1$.

By applying Lemma 9.5 with η chosen such that $-1+\eta<\beta'$, we conclude (9.13). □

We now use the previous proposition to obtain improved decay for ψ in $\mathfrak{t}_\beta$.

Proposition 9.6. *Under the assumptions of Proposition 9.4 and $\beta\in(\frac23,1-\delta-2\beta')$, it holds that*

$$\left|\psi(u_\infty,v)+M^2H_0[\phi]\left(v^{-1}-u_\infty^{-1}\right)\right|\lesssim\varepsilon^3|v^{-1}-u_\infty^{-1}|+\varepsilon(1+v)^{-1-\beta'}\tag{9.16}$$

in $\mathfrak{t}_\beta$.

Proof. Let $u_\beta=u_\beta(v)$ be defined by $(u_\beta,v)\in\sigma_\beta$. Then

$$|\psi(u_\beta,v)|=(\bar r_\infty-M)^{-1/2}(u_\beta,v)|(\bar r_\infty-M)^{1/2}\psi|(u_\beta,v)\lesssim\varepsilon(1+v)^{-3/2+\delta/2+\beta/2}\lesssim\varepsilon(1+v)^{-1-\beta'}\tag{9.17}$$

by (3.20), the fact that $(\bar r_\infty-M)^{-1}\sim1+v$ along σ_β, and the assumption on β. Using (9.13), we find for $(u_\infty,v)\in\mathfrak{t}_\beta$,

$$\psi(u_\infty,v)-\psi(u_\beta,v)=\int_{u_\beta}^{u_\infty}\partial_{u_\infty}\psi\,du'_\infty=-M^2H_0[\phi](u_\beta^{-1}-u_\infty^{-1})+O\big(\varepsilon^3(u_\beta^{-1}-u_\infty^{-1})+\varepsilon(u_\beta^{-1-\beta'}-u_\infty^{-1-\beta'})\big).$$

Now we note that

$$u_\beta^{-1}-u_\infty^{-1}=(u_\beta^{-1}-v^{-1})+(v^{-1}-u_\infty^{-1})=(v^{-1}-u_\infty^{-1})+O(v^{-2+\beta})$$

and $u_\beta+1\sim v+1$, and (9.16) readily follows. □

9.3 Sharpness of the horizon hierarchy

We will show that the range of p in the horizon hierarchy of Section 6.3 is sharp, resulting in failure of boundedness of the integrated non-degenerate energy near the horizon.

Proposition 9.7. *For any $\eta > 0$ and $C_2 > 0$ there exists a constant $0 < \varepsilon_{\eta,C_2} \leq \varepsilon_{\mathrm{stab}}$ such that the following holds. Let $\mathcal{S}_0(\alpha_\star) \in \mathfrak{M}_{\mathrm{stab}}$ with $0 < \mathfrak{D}[\mathcal{S}_0] \leq \varepsilon \leq \varepsilon_{\eta,C_2}$. Suppose also that $\mathring{\phi}$ satisfies the second order smallness condition* (9.6) *and that the asymptotic Aretakis charge is quantitatively nonvanishing in the sense that*

$$|H_0[\phi]| \geq \varepsilon^{2-\eta}. \tag{9.18}$$

Then for any $R > M$, it holds that

$$\iint_{\mathcal{D}_\infty \cap \{\bar{r}_\infty \leq R\}} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty}\, du_\infty dv = \infty. \tag{9.19}$$

Remark 9.8. Note that by the Morawetz estimate Proposition 6.4, the integral in (9.19), taken instead over $\mathcal{D}_\infty \cap \{R' \leq \bar{r}_\infty \leq R\}$ where $M < R' < R$, is finite.

Proof of Proposition 9.7. We repeat the main calculation for the horizon hierarchy, Lemma 6.18, with $p = 3$. Let

$$\mathcal{A}_{u_f} \doteq \mathcal{D}_\infty \cap \{r \leq \Lambda\} \cap \{u_\infty \leq u_f\},$$

where we will let $u_f \to \infty$ and v_0 will be chosen later.

Integrating the expression

$$\iint_{\mathcal{A}_{u_f}} \partial_v \left((\bar{r} - M)^{-1} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty} \right) du_\infty dv$$

by parts and using the wave equation (2.18), we find that

$$\begin{aligned} &- \int_{\underline{C}_0 \cap \mathcal{A}_{u_f}} (\bar{r}_\infty - M)^{-1} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty}\, du_\infty + \int_{\Gamma \cap \mathcal{A}_{u_f}} n^v_\Gamma (\bar{r}_\infty - M)^{-1} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty}\, ds \\ &\qquad = - \iint_{\mathcal{A}_{u_f}} \left(\frac{2M + \bar{r}_\infty}{\bar{r}_\infty^3} \right) \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty}\, du_\infty dv - \iint_{\mathcal{A}_{u_f}} \frac{4\kappa_\infty \nu_\infty}{r \bar{\nu}_\infty} (\bar{r}_\infty - M)^{-1} \varkappa \psi \partial_{u_\infty}\psi\, du_\infty dv. \end{aligned} \tag{9.20}$$

The term along Γ is $\lesssim \varepsilon^2$ using (6.27) and (6.30). For ε sufficiently small using (9.14) with $\eta = 1 - \beta'$ and the condition (9.18), we have

$$\left(u_\infty^2 \partial_{u_\infty}\psi(u_\infty, 0) \right)^2 \gtrsim |H_0[\phi]|^2 - O(\varepsilon^2 u_\infty^{-\beta'})$$

Therefore, by (9.7), it holds that

$$\begin{aligned} \int_{\underline{C}_0 \cap \mathcal{A}_{u_f}} (\bar{r}_\infty - M)^{-1} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty}\, du_\infty &\gtrsim \int_{\underline{C}_0 \cap \mathcal{A}_{u_f}} (\bar{r}_\infty - M)^{-3} (\partial_{u_\infty}\psi)^2\, du_\infty \gtrsim \int_{\underline{C}_0 \cap \mathcal{A}_{u_f}} u_\infty^3 (\partial_{u_\infty}\psi)^2\, du_\infty \\ &\gtrsim \int_{\underline{C}_0 \cap \mathcal{A}_{u_f}} u_\infty^{-1} \left(|H_0[\phi]|^2 - O(\varepsilon^2 u_\infty^{-\beta'}) \right) du_\infty \gtrsim |H_0[\phi]|^2 \log(u_f) - \varepsilon^2, \end{aligned} \tag{9.21}$$

where we have used that $u_\infty^{-1-\beta'}$ is integrable.

We now estimate the mixed term on the right-hand side of (9.20). We use the geometric estimate (8.36) and break up the region of integration using $\mathfrak{t}_\beta$ to estimate

$$\begin{aligned} \left| \iint_{\mathcal{A}_{u_f}} \frac{4\kappa_\infty \nu_\infty}{r \bar{\nu}_\infty} (\bar{r}_\infty - M)^{-1} \varkappa \psi \partial_{u_\infty}\psi\, du_\infty dv \right| &\lesssim \iint_{\mathcal{A}_{u_f}} |\psi||\partial_{u_\infty}\psi|\, du_\infty dv \\ &+ \left(\iint_{\mathcal{A}_{u_f} \setminus \mathfrak{t}_\beta} + \iint_{\mathcal{A}_{u_f} \cap \mathfrak{t}_\beta} \right) \varepsilon^2 (\bar{r}_\infty - M)^{-1} \tau^{-2+\delta} |\psi||\partial_{u_\infty}\psi|\, du_\infty dv \doteq \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{9.22}$$

By (9.7), we have $(\bar{r}_\infty - M)^{-1} \lesssim u - v \le u$ in $\mathcal{A}_{u_f} \cap \{\bar{r}_\infty \le \frac{1}{2}\Lambda\}$ and by definition of σ_β, $(\bar{r}_\infty - M)^{-1} \lesssim 1 + v$ in $\mathcal{A}_{u_f} \setminus \mathfrak{t}_\beta$. It follows that $\mathrm{II} \lesssim \varepsilon^2 \mathrm{I}$. Using the bulk terms in the energy estimate Proposition 6.14 with $p = 5/2$ (after passing to $\tau_f \to \infty$, refer also to the proof of Theorem 1 in Section 8.3.5), we have $\mathrm{I} \lesssim \underline{\mathcal{E}}^\infty_{5/2}(1) \lesssim \varepsilon^2$. By Proposition 9.6, we have $|\psi| \lesssim \varepsilon u_\infty^{-1} + \varepsilon v^{-1}$ in $\mathfrak{t}_\beta$ and by Proposition 9.4, we have $|u_\infty^2 \partial_{u_\infty} \psi| \lesssim \varepsilon$ in $\mathfrak{t}_\beta$. Therefore, we estimate

$$\begin{aligned}\mathrm{III} &\lesssim \iint_{\mathcal{A}_{u_f} \cap \mathfrak{t}_\beta^1} \varepsilon^2 u_\infty^{-1} v^{-2+\delta} |\psi| |u_\infty^2 \partial_{u_\infty} \psi| \, du_\infty dv \\ &\lesssim \varepsilon^4 + \varepsilon^4 \iint_{\{1 \le u_\infty \le u_f\} \cap \{v \ge 1\}} \big(u_\infty^{-2} v^{-2+\delta} + u_\infty^{-1} v^{-3+\delta}\big) du_\infty dv \lesssim \varepsilon^4 (1 + \log(u_f)). \end{aligned} \tag{9.23}$$

Combining (9.20), (9.21), (9.22), and (9.23), we find

$$\iint_{\mathcal{A}_{u_f}} \frac{(\partial_{u_\infty}\psi)^2}{-\bar{\nu}_\infty} \, du_\infty dv \gtrsim \big(|H_0[\phi]|^2 - \varepsilon^4\big) \log(u_f) - \varepsilon^2 \gtrsim |H_0[\phi]|^2 \log(u_f) - \varepsilon^2$$

for ε sufficiently small. Letting $u_f \to \infty$ completes the proof. □

Remark 9.9. There is an interesting difference between the proof of Proposition 9.7 and the corresponding fact in the uncoupled case. In our case, the mixed bulk term on the right-hand side of (9.20) is in general *not* bounded as $u_f \to \infty$, while it *is* bounded in the uncoupled case. This is because the dynamical redshift factor $\varkappa$ does not in general exactly cancel out the singular factor of $(\bar{r}_\infty - M)^{-1}$. Though the growth rate is logarithmic, just as with the first term on the left-hand side of (9.20), it comes with an additional smallness factor which allows the proof to go through. The cost is that we can only prove Proposition 9.7 under the "quantitative nonvanishing" assumption (9.18) on the asymptotic Aretakis charge $H_0[\phi]$. In the uncoupled case, where $H_0[\phi]$ is constant, it suffices to assume that $H_0[\phi] \neq 0$.

On the other hand, in the uncoupled case on a *subextremal* Reissner–Nordström background, the same bulk term is also infinite. It should be noted though, in the uncoupled subextremal case, that it is infinite in a way that cancels out the other unbounded term in (9.20) (the first term on the left-hand side). As a result, in the uncoupled subextremal case, there is no contradiction with the fact that the spacetime integral (9.19) is finite due to the redshift effect [DR09]. See [AAG17, Section 4.6] for more details.

References

[Ang16] Y. Angelopoulos. "Global spherically symmetric solutions of non-linear wave equations with null condition on extremal Reissner-Nordström spacetimes". *IMRN* 11 (2016), pp. 3279–3355.

[AAG17] Y. Angelopoulos, S. Aretakis, and D. Gajic. "The trapping effect on degenerate horizons". *Ann. Henri Poincaré* 18.5 (2017), pp. 1593–1633.

[AAG18a] Y. Angelopoulos, S. Aretakis, and D. Gajic. "A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes". *Ann. PDE* 4.2 (2018), Paper No. 15, 120.

[AAG18b] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Asymptotic blow-up for a class of semilinear wave equations on extremal Reissner-Nordström spacetimes" (2018). arXiv: `arXiv:1612.01562v2`.

[AAG18c] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes". *Adv. Math.* 323 (2018), pp. 529–621.

[AAG20a] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Late-time asymptotics for the wave equation on extremal Reissner-Nordström backgrounds". *Adv. Math.* 375 (2020), pp. 107363, 139.

[AAG20b] Y. Angelopoulos, S. Aretakis, and D. Gajic. "Nonlinear scalar perturbations of extremal Reissner-Nordström spacetimes". *Ann. PDE* 6.2 (2020), Paper No. 12, 124.

[Ape22] M. A. Apetroaie. "Instability of gravitational and electromagnetic perturbations of extremal Reissner-Nordström spacetime" (2022). arXiv: `2211.09182`.

[Are10] S. Aretakis. "The Wave Equation on Extreme Reissner-Nordström Black Hole Spacetimes: Stability and Instability Results" (2010). arXiv: 1006.0283.

[Are11a] S. Aretakis. "Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations I". *Comm. Math. Phys.* 307.1 (2011), pp. 17–63.

[Are11b] S. Aretakis. "Stability and instability of extreme Reissner-Nordström black hole spacetimes for linear scalar perturbations II". *Ann. Henri Poincaré* 12.8 (2011), pp. 1491–1538.

[Are12] S. Aretakis. "Decay of axisymmetric solutions of the wave equation on extreme Kerr backgrounds". *J. Funct. Anal.* 263.9 (2012), pp. 2770–2831.

[Are15] S. Aretakis. "Horizon instability of extremal black holes". *Adv. Theor. Math. Phys.* 19.3 (2015), pp. 507–530.

[AKS23] S. Aretakis, G. Khanna, and S. Sabharwal. "An observational signature for extremal black holes" (2023). arXiv: 2307.03963.

[Blu08] P. Blue. "Decay of the Maxwell field on the Schwarzschild manifold". *J. Hyperbolic Differ. Equ.* 5.4 (2008), pp. 807–856.

[CGZ16] M. Casals, S. Gralla, and P. Zimmerman. "Horizon instability of extremal Kerr black holes: Nonaxisymmetric modes and enhanced growth rate". *Phys. Rev. D* 94.6 (2016).

[CG69] Y. Choquet-Bruhat and R. Geroch. "Global aspects of the Cauchy problem in general relativity". *Comm. Math. Phys.* 14 (1969), pp. 329–335.

[Chr99] D. Christodoulou. "On the global initial value problem and the issue of singularities". *Classical and Quantum Gravity* 16.12A (1999), A23.

[Chr00] D. Christodoulou. *The action principle and partial differential equations.* Vol. 146. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2000, pp. viii+319.

[CT84] W. Couch and R. Torrence. "Conformal invariance under spatial inversion of extreme Reissner-Nordström black holes". *General relativity and gravitation* 16 (1984), pp. 789–792.

[Daf03] M. Dafermos. "Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations". *Ann. of Math. (2)* 158.3 (2003), pp. 875–928.

[Daf05a] M. Dafermos. "On naked singularities and the collapse of self-gravitating Higgs fields". *Advances in Theoretical and Mathematical Physics* 9.4 (2005), pp. 575–591.

[Daf05b] M. Dafermos. "Spherically symmetric spacetimes with a trapped surface". *Classical and Quantum Gravity* 22.11 (2005), p. 2221.

[Daf05c] M. Dafermos. "The interior of charged black holes and the problem of uniqueness in general relativity". *Comm. Pure Appl. Math.* 58.4 (2005), pp. 445–504.

[Daf24] M. Dafermos. "The stability problem for extremal black holes". In: *Black Holes Inside and Out 2024: visions for the future of black hole physics.* Niels Bohr Institute. 2024. arXiv: 2410.14414.

[DHR] M. Dafermos, G. Holzegel, and I. Rodnianski. "The linear stability of the Schwarzschild solution to gravitational perturbations". *Acta Math.* 222.1 (2019), pp. 1–214.

[DHRT] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor. "The non-linear stability of the Schwarzschild family of black holes" (2021). arXiv: 2104.08222.

[DHRT22] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor. "Quasilinear wave equations on asymptotically flat spacetimes with applications to Kerr black holes" (2022). arXiv: 2212.14093.

[DR05] M. Dafermos and I. Rodnianski. "A proof of Price's law for the collapse of a self-gravitating scalar field". *Invent. Math.* 162.2 (2005), pp. 381–457.

[DR09] M. Dafermos and I. Rodnianski. "The red-shift effect and radiation decay on black hole spacetimes". *Communications on Pure and Applied Mathematics* 62.7 (2009), pp. 859–919.

[DR10] M. Dafermos and I. Rodnianski. "A new physical-space approach to decay for the wave equation with applications to black hole spacetimes". *XVIth ICMP.* World Scientific. 2010, pp. 421–432.

[DR13] M. Dafermos and I. Rodnianski. “Lectures on black holes and linear waves”. *Clay Math. Proc* 17 (2013), pp. 97–205.

[Gaj17a] D. Gajic. “Linear waves in the interior of extremal black holes I”. *Comm. Math. Phys.* 353.2 (2017), pp. 717–770.

[Gaj17b] D. Gajic. “Linear waves in the interior of extremal black holes II”. *Ann. Henri Poincaré* 18.12 (2017), pp. 4005–4081.

[Gaj23] D. Gajic. “Azimuthal instabilities on extremal Kerr” (2023). arXiv: 2302.06636.

[GL19] D. Gajic and J. Luk. “The interior of dynamical extremal black holes in spherical symmetry”. *Pure Appl. Anal.* 1.2 (2019), pp. 263–326.

[Gau24] O. Gautam. “Decay and mass inflation for the spherically symmetric Einstein–Maxwell-scalar field system”. *Preprint* (2024).

[Gio20a] E. Giorgi. “The linear stability of Reissner-Nordström spacetime for small charge”. *Ann. PDE* 6.2 (2020), Paper No. 8, 145.

[Gio20b] E. Giorgi. “The linear stability of Reissner-Nordström spacetime: the full subextremal range $|Q| < M$”. *Comm. Math. Phys.* 380.3 (2020), pp. 1313–1360.

[Gio22] E. Giorgi. “Electromagnetic-gravitational perturbations of Kerr–Newman spacetime: The Teukolsky and Regge–Wheeler equations”. *Journal of Hyperbolic Differential Equations* 19.01 (2022), pp. 1–139.

[Gio23] E. Giorgi. “Boundedness and Decay for the Teukolsky System in Kerr-Newman Spacetime I: The Case $|a|, |Q| \ll M$” (2023). arXiv: 2311.07408.

[Gio24] E. Giorgi. “The Carter tensor and the physical-space analysis in perturbations of Kerr–Newman spacetime”. *Journal of Differential Geometry* 127.1 (2024), pp. 277–371.

[GW24] E. Giorgi and J. Wan. “Boundedness and Decay for the Teukolsky System in Kerr-Newman Spacetime II: The Case $|a| \ll M$, $|Q| < M$ in Axial Symmetry” (2024). arXiv: 2407.10750.

[HMVR24] G. Holzegel, G. Mavrogiannis, and R. Velozo Ruiz. “A note on integrated local energy decay estimates for spherically symmetric black hole spacetimes” (2024). arXiv: 2403.02533.

[Kád24a] I. Kádár. “A scattering theory construction of dynamical solitons in 3d” (2024). arXiv: 2403.13891.

[Kád24b] I. Kádár. “Construction of multi-soliton solutions for the energy critical wave equation in dimension 3” (2024). arXiv: 2409.05267.

[KU22] C. Kehle and R. Unger. “Gravitational collapse to extremal black holes and the third law of black hole thermodynamics”. To appear in *J. Eur. Math. Soc. (JEMS)* (2022).

[KU24] C. Kehle and R. Unger. “Extremal black hole formation as a critical phenomenon” (2024). arXiv: 2402.10190.

[Kod80] H. Kodama. “Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation”. *Prog. Theoretical Phys.* 63.4 (1980), pp. 1217–1228.

[Kom13] J. Kommemi. “The global structure of spherically symmetric charged scalar field spacetimes”. *Comm. Math. Phys.* 323.1 (2013), pp. 35–106.

[LMRT13] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi. “On the horizon instability of an extreme Reissner-Nordström black hole”. *J. High Energy Phys.* 3 (2013), 035, front matter+43.

[LR12] J. Lucietti and H. S. Reall. “Gravitational instability of an extreme Kerr black hole”. *Physical Review D—Particles, Fields, Gravitation, and Cosmology* 86.10 (2012), p. 104030.

[Luk12] J. Luk. “On the local existence for the characteristic initial value problem in general relativity”. *Int. Math. Res. Not. IMRN* 20 (2012), pp. 4625–4678.

[LO19a] J. Luk and S.-J. Oh. “Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. The interior of the black hole region”. *Ann. of Math. (2)* 190.1 (2019), pp. 1–111.

[LO19b] J. Luk and S.-J. Oh. "Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data II: the exterior of the black hole region". *Ann. PDE* 5.1 (2019), Paper No. 6, 194.

[LOS23] J. Luk, S.-J. Oh, and Y. Shlapentokh-Rothman. "A scattering theory approach to Cauchy horizon instability and applications to mass inflation". *Ann. Henri Poincaré* 24.2 (2023), pp. 363–411.

[MRT13] K. Murata, H. S. Reall, and N. Tanahashi. "What happens at the horizon (s) of an extreme black hole?" *Classical Quantum Gravity* 30.23 (2013), p. 235007.

[Nor18] G. Nordström. "On the energy of the gravitation field in Einstein's theory". *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B* 20 (1918), pp. 1238–1245.

[Pas19] F. Pasqualotto. "The spin ±1 Teukolsky equations and the Maxwell system on Schwarzschild". *Ann. Henri Poincaré* 20.4 (2019), pp. 1263–1323.

[Pen68] R. Penrose. "Structure of space-time". *Battelle Rencontres.* 1968, pp. 121–235.

[Rei16] H. Reissner. "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie". *Annalen der Physik* 355.9 (1916), pp. 106–120.

[Sbi16] J. Sbierski. "On the existence of a maximal Cauchy development for the Einstein equations: a dezornification". *Ann. Henri Poincaré* 17.2 (2016), pp. 301–329.

[Sch16] K. Schwarzschild. "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* (1916), pp. 189–196.

[TdC20] R. Teixeira da Costa. "Mode stability for the Teukolsky equation on extremal and subextremal Kerr spacetimes". *Comm. Math. Phys.* 378.1 (2020), pp. 705–781.