Retiring the third law of black hole thermodynamics

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joint work with Christoph Kehle (ETH Zürich)
arXiv:2211.15742

Black hole thermodynamics

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

Black hole thermodynamics is a proposed close mathematical **analogy** between **classical** black hole dynamics and classical thermodynamics

Fundamentally, these are statements about Einstein gravity coupled to classical matter fields (and are already interesting in vacuum!)

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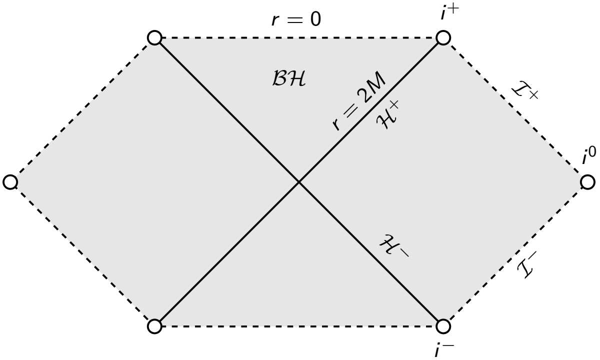
- Laws 0, 1, and 2 proved by Hawking, Carter, Bardeen–Carter–Hawking, Wald, ...

Refresher on Schwarzschild



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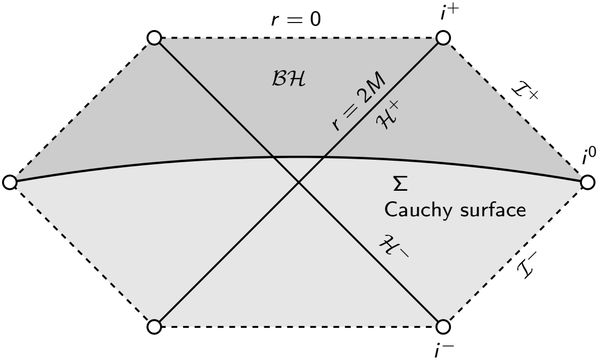

Maximally extended Schwarzschild is the unique maximal Cauchy development of the data induced on a spacelike hypersurface Σ as depicted here.

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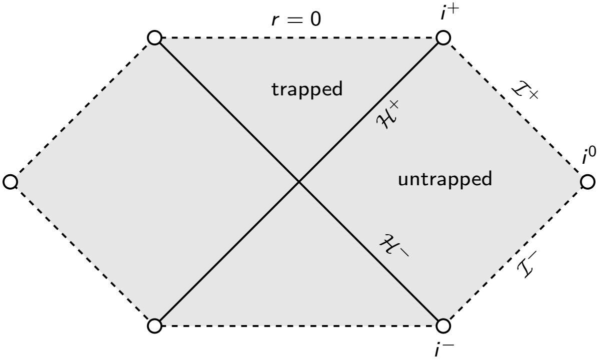
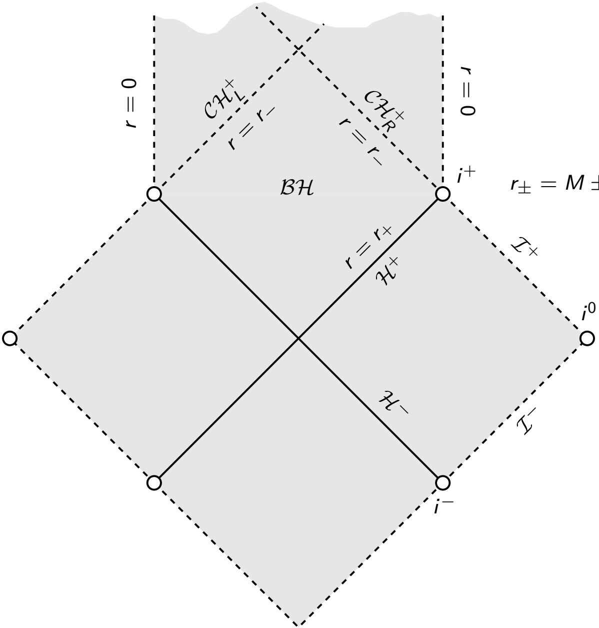


The black hole interior is foliated by **trapped surfaces** (both future null expansions negative)

Refresher on subextremal Reissner–Nordström: $0 < |e| < M$



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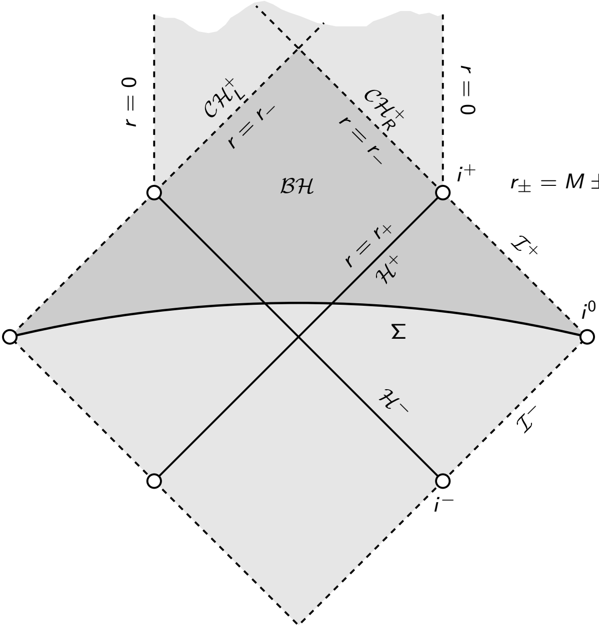

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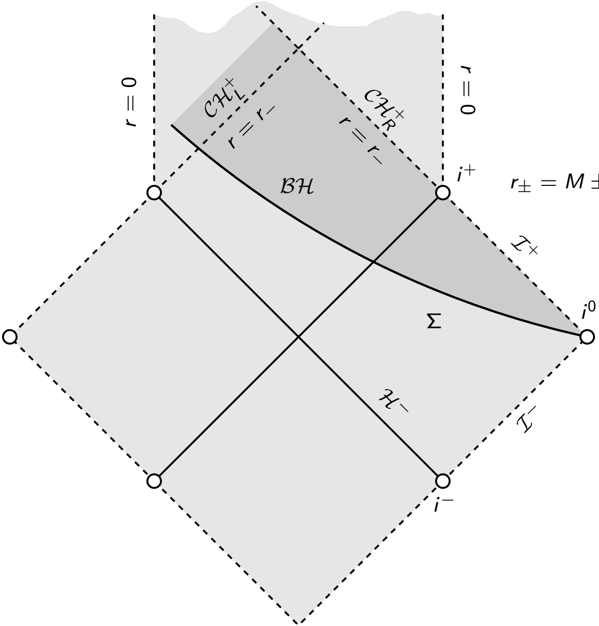

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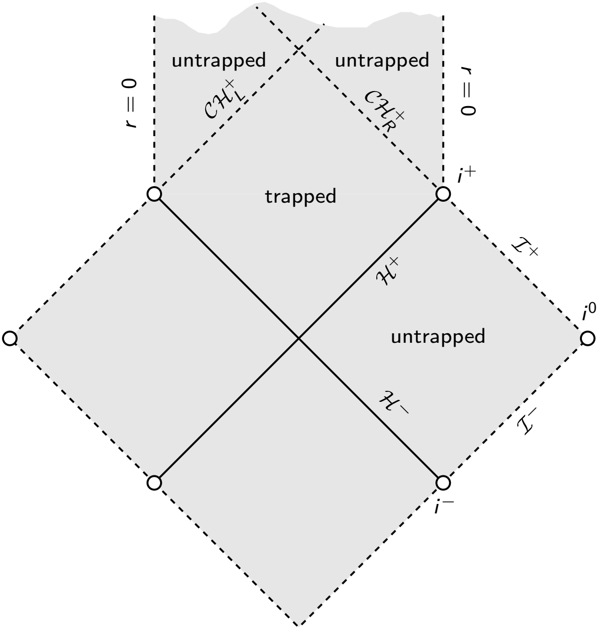

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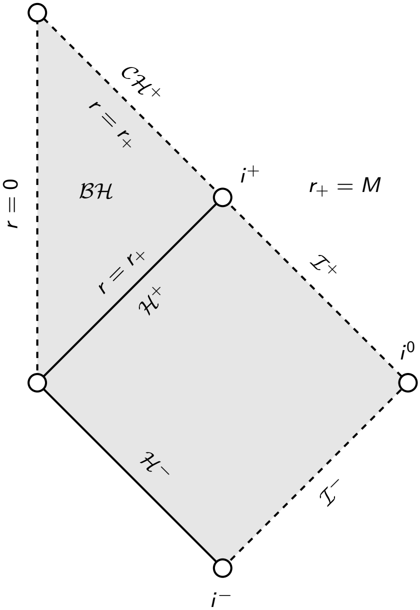

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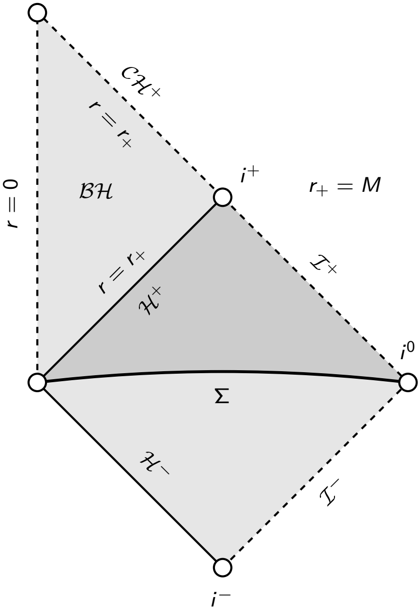

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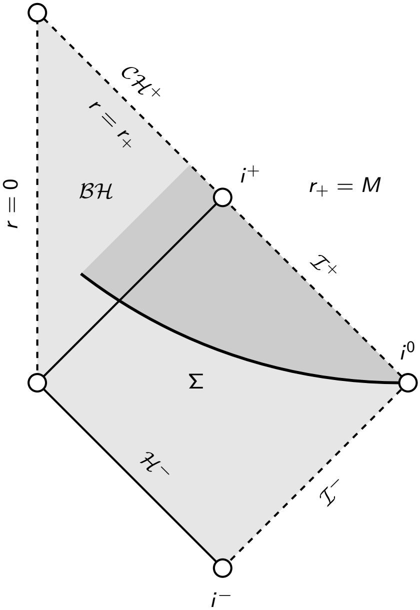

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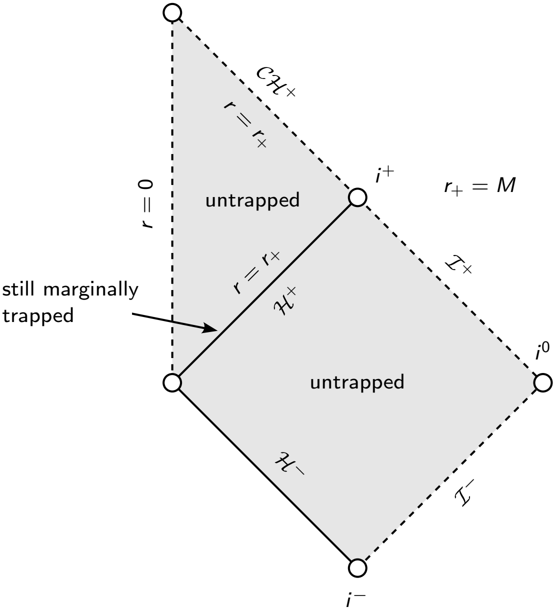

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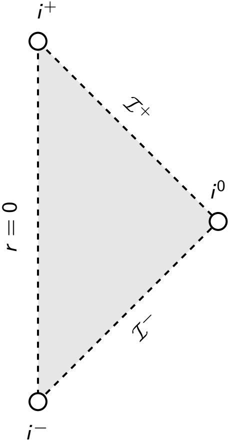

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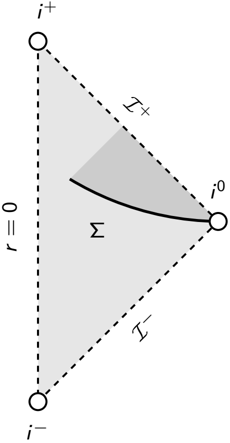

Surface gravity of Reissner–Nordström

- RN with mass M and charge e, $|e| \leq M$, has

$$\kappa = \frac{\sqrt{M^2 - e^2}}{(M + \sqrt{M^2 - e^2})^2}$$

- **Subextremal:** $\kappa > 0$
- **Extremal:** $\kappa = 0$

The third law

Original formulation of Bardeen–Carter–Hawking:

Extending the analogy even further one would postulate:

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

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   - This parenthetical remark is related to the "mistake" in Israel's paper and we will discuss it later

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Conjecture (The third law, BCH '73, Israel '86).
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More precisely, there exist regular solutions of the Einstein–Maxwell-charged scalar field (self-gravitating charged massless scalar field) system with the following behavior:

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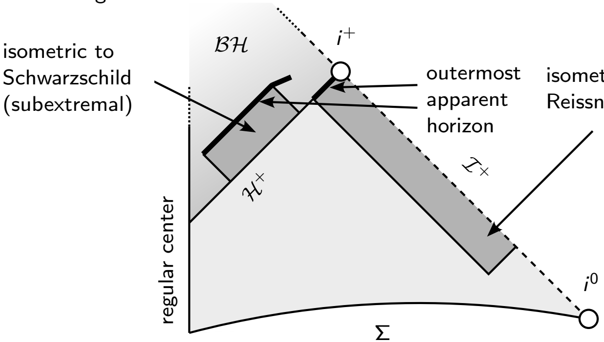

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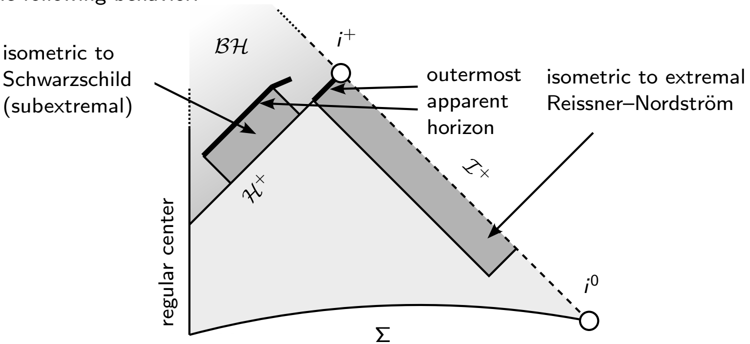

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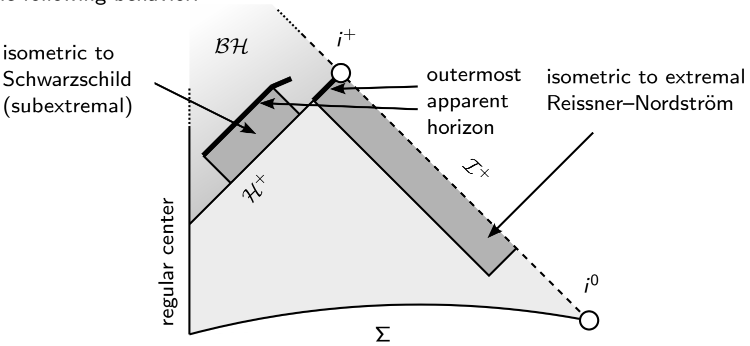

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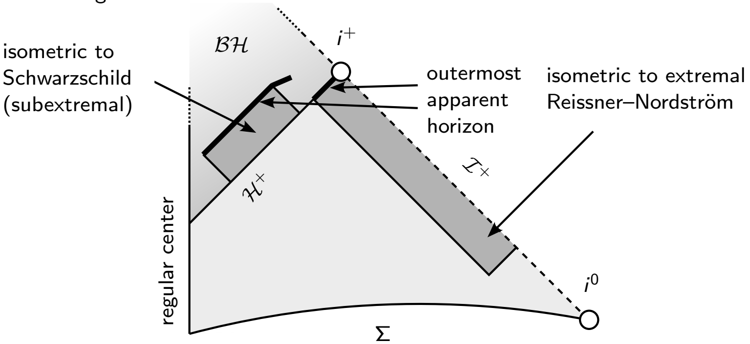

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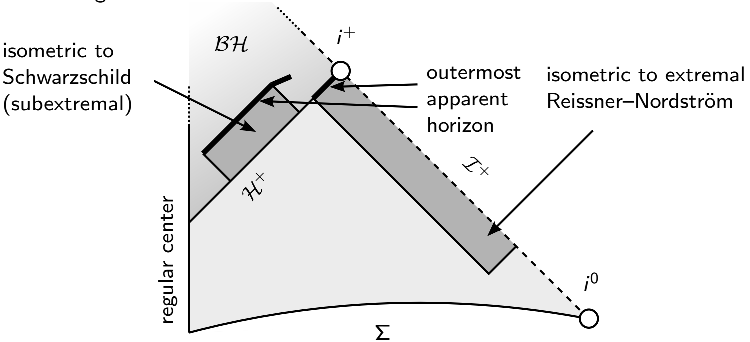

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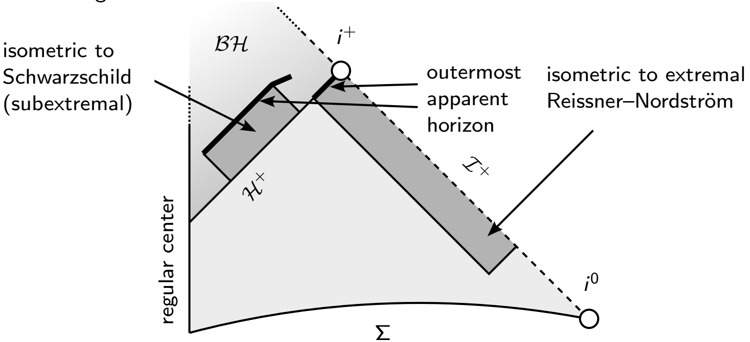

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But this is not true. The outermost apparent horizon can jump in smooth spacetimes.

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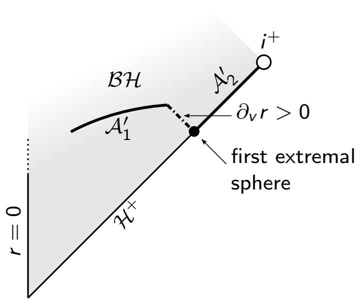
> Infractions can result from the absorption of infinitesimally thin, massive shells,[5] which force the apparent horizon to jump outward discontinuously

  But this is <u>not true</u>. The outermost apparent horizon can jump in smooth spacetimes.

- Under this implicit assumption, Israel concluded the third law to be true on the basis of Raychaudhuri's equation.

Israel's paper reinterpreted



Outermost apparent horizon becomes disconnected the instant the black hole becomes extremal!

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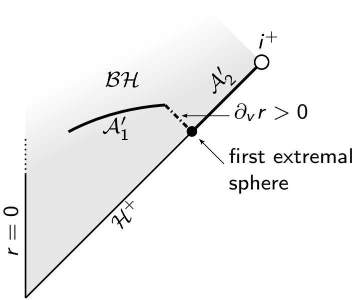

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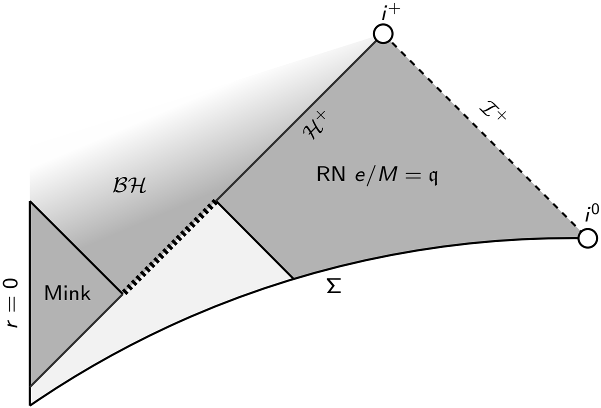
However, this is a feature, not a glitch!

Gravitational collapse for any charge to mass ratio

Theorem (Kehle–U.).

For any $\mathfrak{q} \in [-1, 1]$, there exist regular Cauchy data for the Einstein–Maxwell-charged scalar field system whose evolution has the following Penrose diagram:



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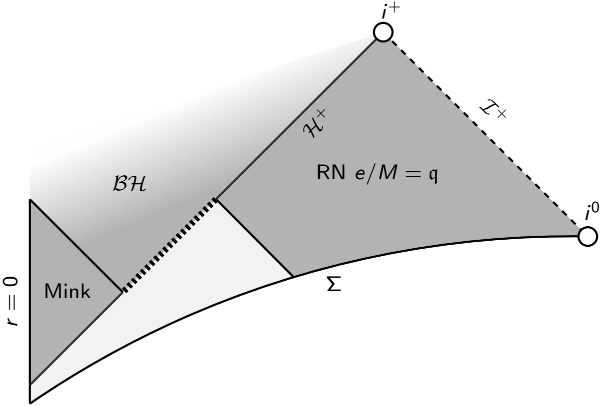

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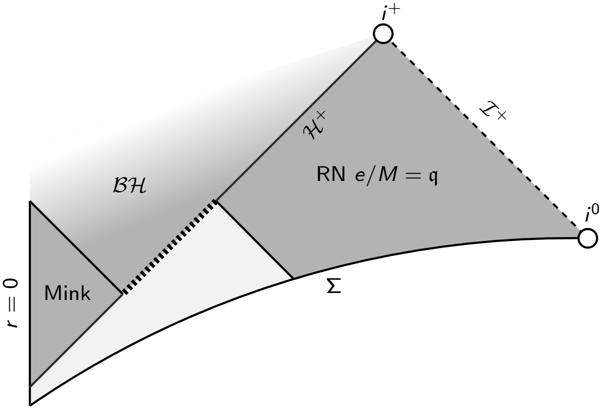

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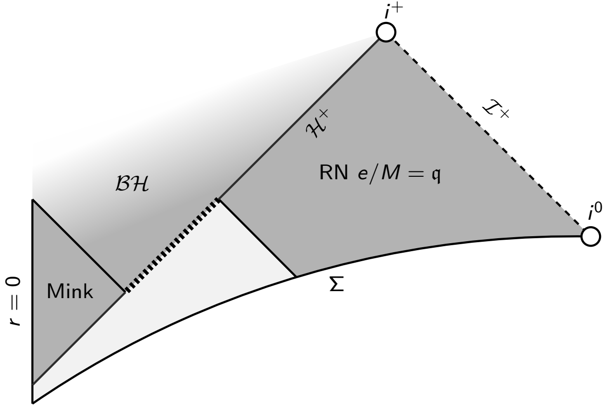
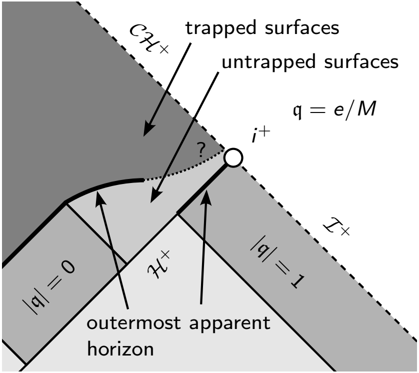


- From the point of view of our construction, the extremal case is exactly the same as the subextremal case!
- $|\mathfrak{q}| \to 1$ represents a regular limit
- $T = 0$ is not fundamentally different than $T > 0$ in these examples

Interior structure of third law violating solutions



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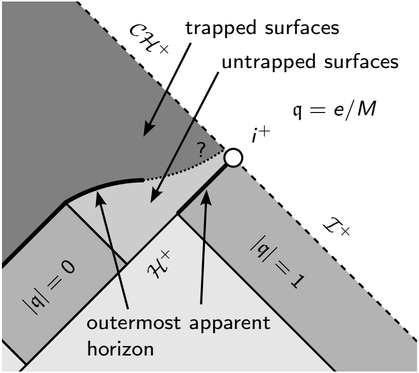

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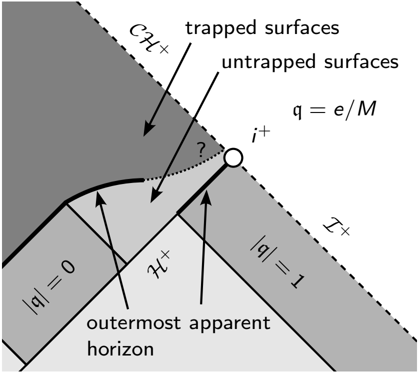
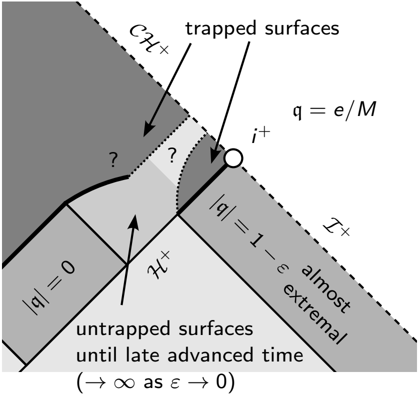


- Consistent with Israel's observation: the outermost apparent horizon becomes disconnected
- Trapped surfaces persist for all time and are found at any retarded time in the black hole interior

So where do the trapped surfaces go?

The geometry of a $|\mathfrak{q}| = 1 - \varepsilon$ example converges to a $|\mathfrak{q}| = 1$ example as $\varepsilon \to 0$.



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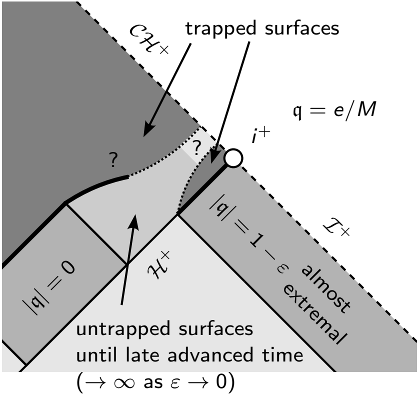

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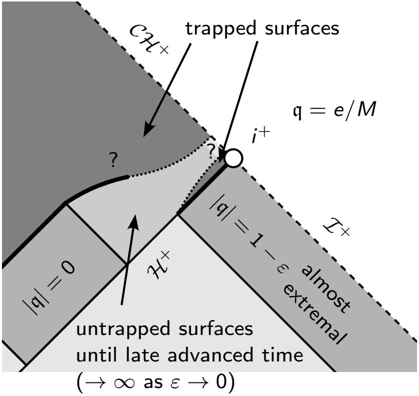

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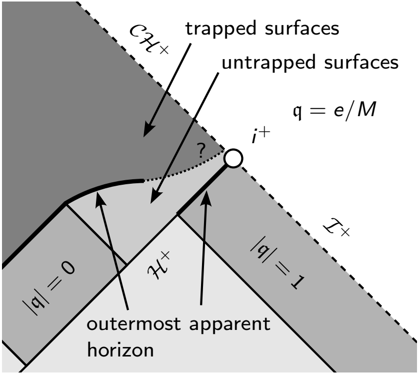

On the (non)genericity of our examples

The Third Law

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

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- However, such a reformulation would be completely trivial (and has nothing to do with $T = 0$)

The third law and supercharging

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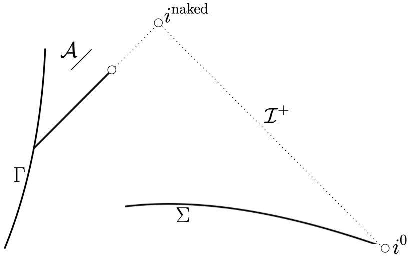

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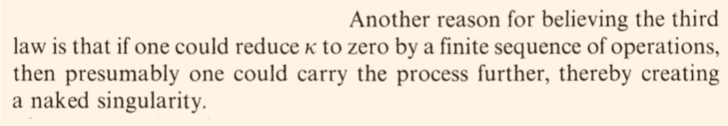
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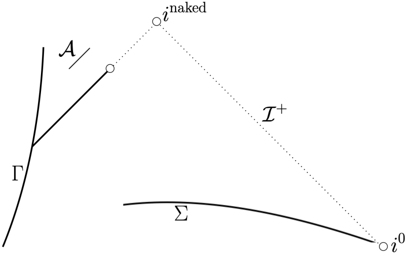

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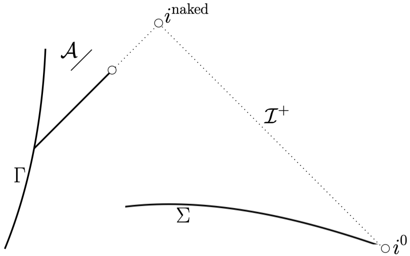

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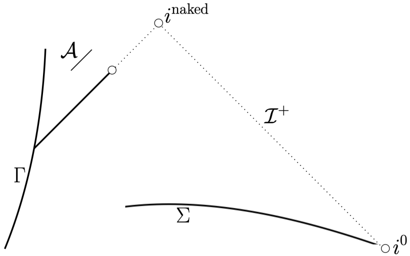

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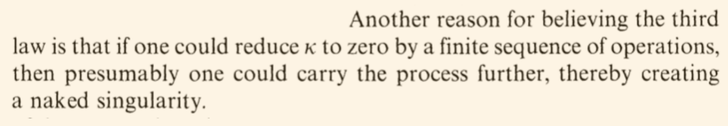
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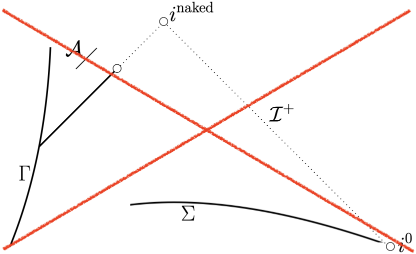

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The black hole region $\mathcal{BH}$ and event horizon $\mathcal{H}^+$ of a spacetime are **teleological** notions—they can't be located *without knowing the entire future of the spacetime*.

Prototype: Minkowski to Schwarzschild gluing

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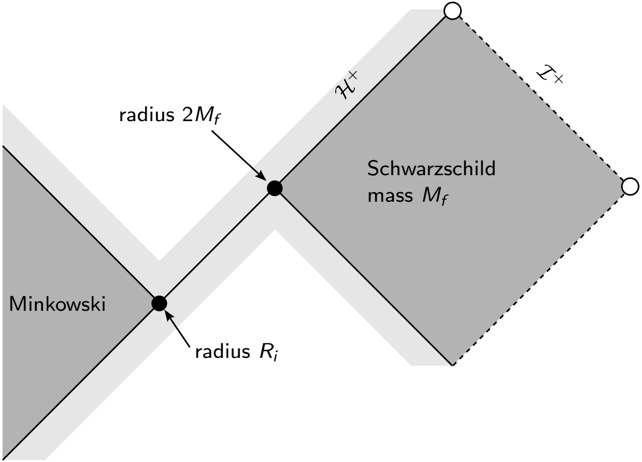

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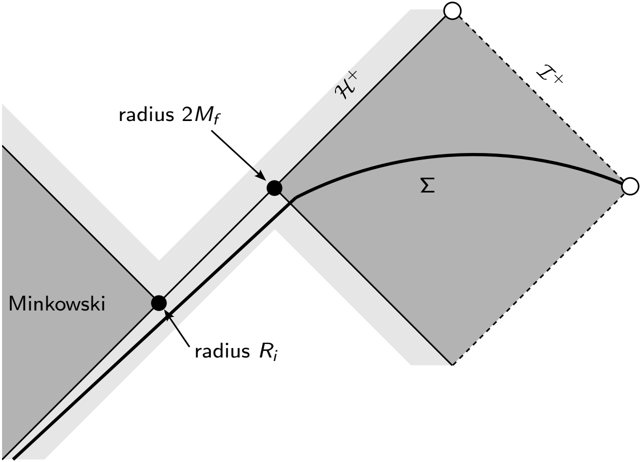

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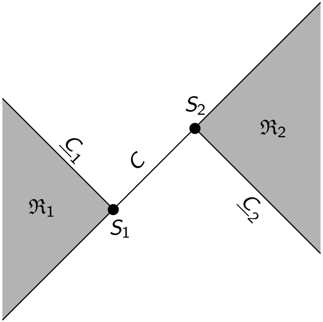
Characteristic/null gluing for the linear wave equation



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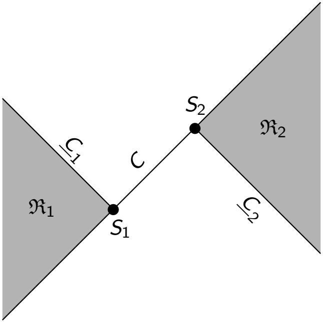

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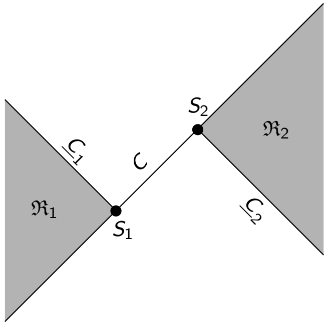

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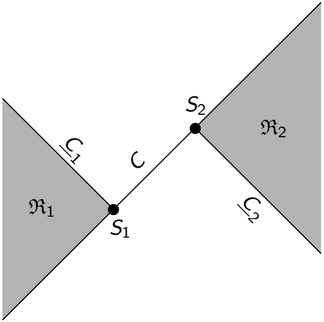

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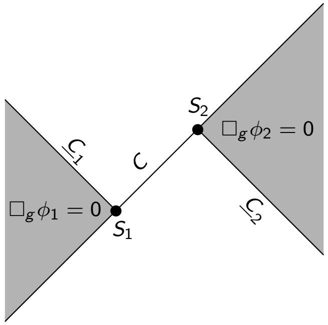

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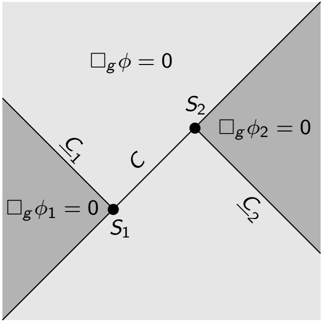

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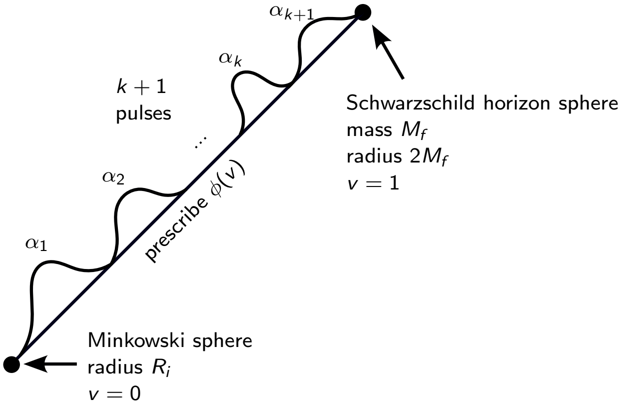
For any $k \in \mathbb{N}$ and $0 < R_i < 2M_f$, the Minkowski sphere of radius R_i can be characteristically glued to the Schwarzschild event horizon sphere with mass M_f to order C^k within the Einstein-scalar field model in spherical symmetry.



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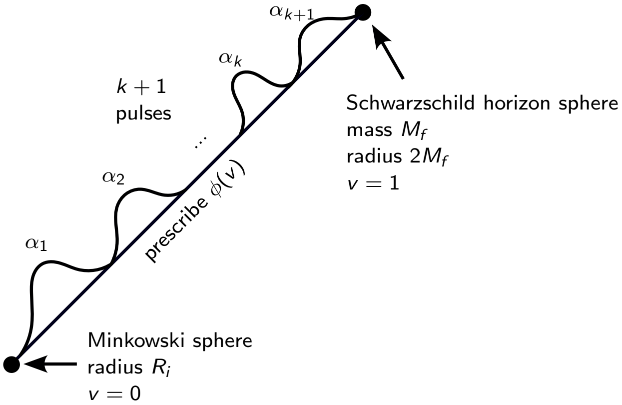

Pulse amplitudes α_j are chosen using the Borsuk–Ulam theorem, making use of the symmetry $\phi \mapsto -\phi$ of the equations.

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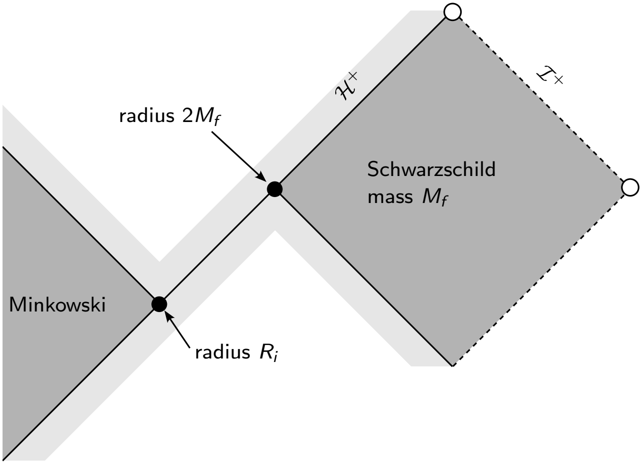

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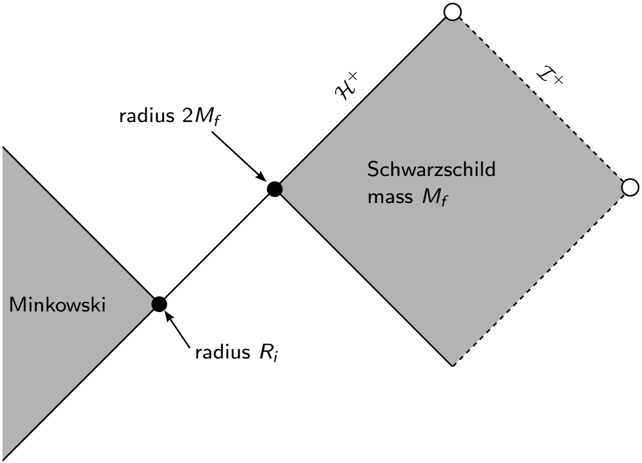

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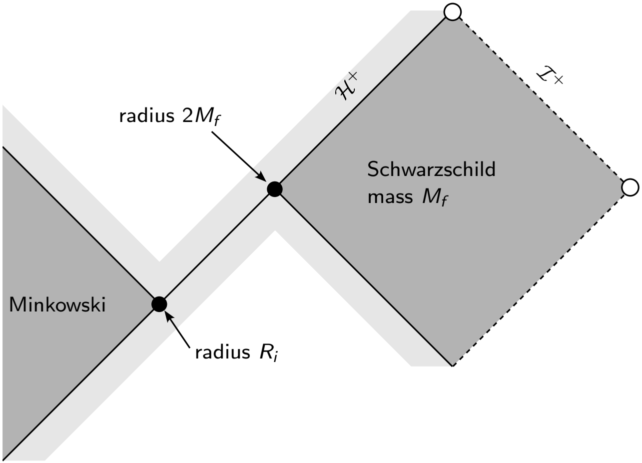
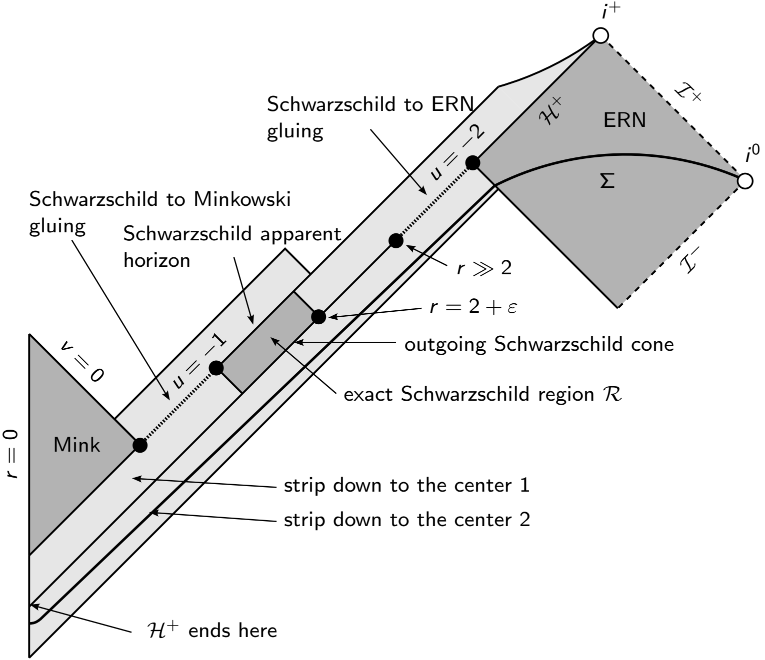


Disproof of the third law



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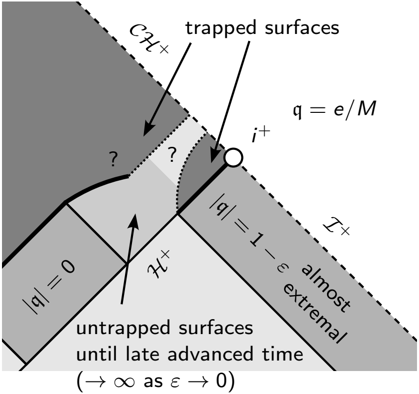

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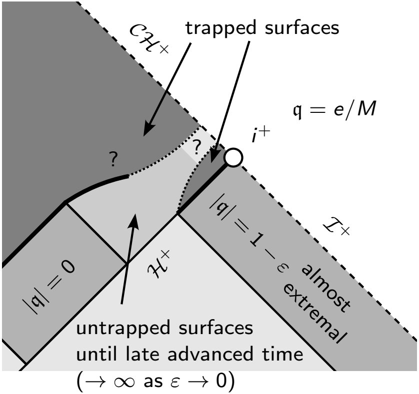

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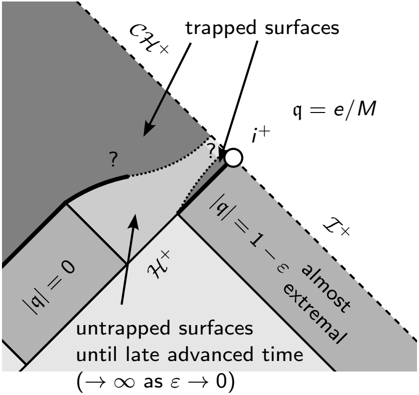

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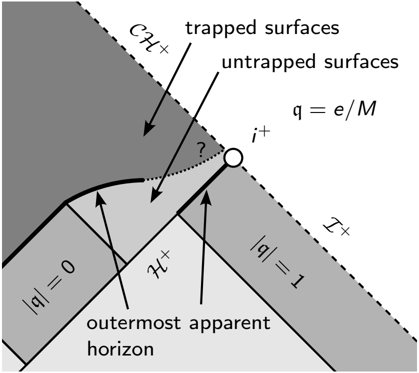

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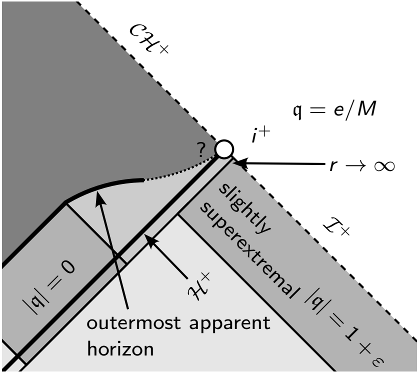

Critical behavior: We expect that there are smooth 1-parameter families of solutions passing through extremality where the event horizon "jumps" as soon as the exterior becomes superextremal. There is **no naked singularity.**

The vacuum case

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Thank you!