Complementary Subspaces

Subspaces $U, V \in \mathbb{R}^n$ are complementary if $U \cap V = \{0\}$ and $U + V = \mathbb{R}^n$

- $U + V = \mathbb{R}^n$ if any $x \in \mathbb{R}^n$ can be written as $u + v$

Proposition: Unique Decomposition

Suppose $U, V \in \mathbb{R}^n$ are complementary. Then any $x \in \mathbb{R}^n$ has a unique decomposition $x = u + v$.

Proof:

- Suppose we had another decomposition $x = u' + v'$
- Then $u - u' = v - v'$ but $u - u' \in U$ and $v - v' = V$
  - The only thing in the intersection of U and V is 0, so the decompositions are the same

Proposition:

The vectors u, v that are the decomposition of x are the projections of x on U and V respectively

Proof Strategy

- Show that v and w are the closest points to x in V and W

Fundamental Theorem of Linear Algebra

$N(A)$ and $C(A^t)$ are orthogonal complements