Null Space

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The **null space** of a $m \times n$ matrix A is the space $N(A)$ of all vectors x such that

$$Ax = 0$$

Proposition:

Invertible row operations do not change the null space of a matrix A

Proof:

- For GA where G is invertible, we have $GAx = 0$ implies $Ax = G^{-1}0 = 0$
- Additionally $Ax = 0$ implies $GAx = 0$, so bijection

Computing Null Space of RREF Matrix

$$R = \begin{pmatrix} \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & \boxed{1} & 2 \end{pmatrix}$$

- Pivot variables are those boxed
- Each other column represents a "free" variable
  - Selecting values for the free variables determines the pivot variables
  - Dimension of null space = number of free variables

Setting this to zero:

$$\begin{gathered} x_1 + 3x_2 + 7x_4 = 0 \\ x_3 + 2x_4 = 0 \\ \downarrow \\ x_1 = -3x_2 - 7x_4 \\ x_3 = -2x_4 \end{gathered}$$

$$N(R) = \left\{ \begin{pmatrix} -3x_2 - 7x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\} = \left\{ x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} = \operatorname{span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} \Rightarrow \text{2 dimensional subspace of } \mathbb{R}^4$$

General Method

- Find reduced row echelon form R
- Null space is span of $v_1, v_2, \dots, v_l$
  - $l = n - \#\text{ of pivot variables}$
  - Find v_i by:
    - Set i$th free variable coefficient to $1
    - Set all other free variables to 0
    - Let pivot variables be determined by the respective equations (since thier values are governed by free variables)