Gaussian Elimination

$$\begin{array}{rrrcl} x_1 & -\,x_2 & + & 2x_3 & = 1 \\ -2x_1 & +\,2x_2 & - & 3x_3 & = -1 \\ -4x_1 & -\,x_2 & + & 2x_3 & = -3 \end{array}$$

$$\updownarrow$$

$$\begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$\updownarrow$$

$$\left(\begin{array}{rrrcr} \boxed{1} & -1 & 2 & | & 1 \\ \boxed{-2} & 2 & -3 & | & -1 \\ \boxed{-3} & -1 & 2 & | & -3 \end{array}\right)$$

- Find the first nonzero entry in each row (**pivot of the row**)
  - These are boxed
- Start with the pivot in the 1st row (assuming there is no pivot to the left of it)
- Eliminate nonzero entries below it
- Do this by adding or subtracting multiples of other equations to cancel out entries in other rows

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \left(\begin{array}{rrrcr} 1 & -1 & 2 & | & 1 \\ -2 & 2 & -3 & | & -1 \\ -3 & -1 & 2 & | & -3 \end{array}\right)$$

$$\begin{array}{l} (E1') = (E1) \\ (E2') = (E2) - (-2)(E1) \\ (E3') = (E3) - (-3)(E1) \end{array} \rightarrow \left(\begin{array}{rrrcr} \boxed{1} & -1 & 2 & | & 1 \\ 0 & 0 & \boxed{1} & | & 1 \\ 0 & \boxed{-4} & 8 & | & 0 \end{array}\right) = (A'|b')$$

$$\begin{array}{l} (E1'') = (E1') \\ (E2'') = (E3') \\ (E3'') = (E2') \end{array} \rightarrow \left(\begin{array}{rrrcr} \boxed{1} & -1 & 2 & | & 1 \\ 0 & \boxed{-4} & 8 & | & 0 \\ 0 & 0 & \boxed{1} & | & 1 \end{array}\right) = (\tilde{A}|\tilde{b})$$

This system is in **row echelon form**

- Can now be easily solved through back substitution

Using Matrices to Represent

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & -4 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Gauss-Jordan Elimination

Continuation where we make each pivot equal to 1 and then eliminate non-pivot rows

$$\left(\begin{array}{rrrcr} \boxed{1} & -1 & 2 & | & 1 \\ 0 & \boxed{1} & -2 & | & 0 \\ 0 & 0 & \boxed{1} & | & 1 \end{array}\right)$$

$$\begin{array}{l} (E1') = (E1) - 2(E3) \\ (E2') = (E2) + 2(E3) \\ (E3') = (E3) \end{array} \rightarrow \left(\begin{array}{rrrcr} 1 & -1 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{array}\right)$$

$$\begin{aligned}(E1'') &= (E1') + (E2) \\ (E2'') &= (E2') \\ (E3'') &= (E3')\end{aligned} \quad \rightarrow \begin{pmatrix} 1 & 0 & 0 & \mid & 1 \\ 0 & 1 & 0 & \mid & 2 \\ 0 & 0 & 1 & \mid & 1 \end{pmatrix}$$

Results in **reduced row echelon form** which is unique for each matrix