$\text{proj}_V(x)$ is the unique vector v such that:

- It is the closest in V to x
- $x - v \perp V$

Projection Onto Vector

Suppose $Y = \text{span}\{y\}$ where $y \in \mathbb{R}^n \setminus \{0\}$

- To have v be the projection of x onto this, we have $v = cy$ and:

$$0 = y \cdot (x - v) = y \cdot (x - cy) = x \cdot y - c||y||^2$$

- Then: $v = cy = \left(\frac{x \cdot y}{|y|^2}\right) y$

This can alternatively be written as the projection matrix P_Y:

$$P_Y = \frac{1}{|y|^2} y y^t$$

Projection onto Matrix

Doing the same thing for projecting b onto the column space of a matrix A gives us:

- Our projection is $v = Ax$ for some x
- Being perpendicular to $V = C(A)$ means that it is in $N(A^t)$
  - $A^t(b - v) = 0 \implies x = (A^tA)^{-1}A^tb$
  - $v = Ax = A(A^tA)^{-1}A^tb$

Our projection matrix P_A is then:

$$P_A = A(A^tA)^{-1}A^tb$$

- Only holds if A^tA is invertible
  - If A has linearly independent columns (i.e. its columns are a basis for V$), then $A^tA will be invertible