Astron. Astrophys. 268, 326–349 (1993)

ASTRONOMY AND ASTROPHYSICS

# The orbits of the major satellites of Saturn

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Received June 15, accepted September 24, 1992

**Abstract.** Strugnell & Taylor's (1990) catalogue of observations of the major satellites of Saturn was augmented by several thousand additional observations. Theories of the motions of the satellites (excluding Hyperion) were fitted to these observations using a weighted least-squares procedure in order to derive improved values of the orbital elements and other parameters of the system, including the masses of the four inner satellites and the dynamical form factors $J_2$ and $J_4$ of Saturn. We find the following values:

Mass of Mimas = $(0.0646 \pm 0.0011) \times 10^{-6}$
Mass of Enceladus = $(0.213 \pm 0.046) \times 10^{-6}$
Mass of Tethys = $(1.076 \pm 0.018) \times 10^{-6}$
Mass of Dione = $(1.916 \pm 0.036) \times 10^{-6}$

Dynamical form factor $J_2 = 0.016478 \pm 0.000038$
Dynamical form factor $J_4 = -0.00110 \pm 0.00028$

Node of Saturn's equator plane = $168^{\circ}.8387 \pm 0^{\circ}.0035$
Inclination of Saturn's equator plane = $28^{\circ}.0653 \pm 0^{\circ}.0016$

where the masses of the satellites are expressed in units of Saturn's mass and the dynamical form factors have been calculated assuming the equatorial radius of Saturn to be 60000 kilometres. The position of the equator plane of Saturn is referred to the mean ecliptic and equinox of the epoch B1950. The masses of the satellites agree well with those determined by Dourneau (1987). The dynamical form factors agree well with those obtained from analysis of radio tracking data from the *Pioneer* and *Voyager* spacecraft.

We also find that inclusion of the long-period terms calculated by Vienne et al. (1991) in the mean longitude of Mimas causes a small but significant reduction in the coefficient of the secular acceleration which was introduced by Kozai (1957). This suggests that the discrepancy between observed and computed values of the mean longitude of Mimas may be due to the omission of terms of very long period, as proposed by Vienne et al. (1991)

**Key words:** Saturn – satellites – orbit – observations

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## 1. Introduction

In this paper, we analyse a large collection of observations of the positions of the major satellites of Saturn. The aim of the analysis is twofold. Firstly, we wish to determine improved values of the orbital elements, mean motions and secular rates of motion of the node and apse of each satellite, together with several other parameters such as the node and inclination of the equator plane of Saturn and the amplitude, phase and period of the Mimas-Tethys libration. From these quantities, we deduce the masses of the inner four satellites and the dynamical form factors of Saturn. Secondly, we wish to assess the quality of each of the sets of observations included in the analysis by comparing the distribution of the observed-minus-computed residuals of each set.

In this paper, we describe the method of analysis and compare the values obtained for the orbital elements and other parameters with those obtained by other authors. We also briefly present root-mean-square residuals for selected sets of observations. A full discussion and comparison of the residuals will be presented in a separate paper, hereafter referred to as Paper II.

In both papers, we compare our results with those of similar recent analyses by Kozai (1957), Garcia (1972), Chugunov (1983b), Dourneau (1987) and Taylor & Shen (1988). We have omitted Struve (1933) since Kozai analysed substantially the same set of observations as Struve using more refined theories. Hence Kozai's work supersedes that of Struve.

In Table 1 we give the time spans and types of the data analysed by previous authors. It should be noted that observations made before 1874 are of rather low quality. Micrometric measurements of the positions of the satellites were first made in 1846. Observations made before that date are estimates of the times of greatest elongation or conjunction of the satellites relative to Saturn and are of value only in improving the mean motions of the satellites.

Kozai (1957) analysed micrometric observations spanning the period 1846 to 1947 together with the lower-quality pre-1846 observations. He reduced the data to opposition mean elements. Garcia's (1972) analysis used only photographic observations from the Yale University southern station and the University of

**Table 1.** Analysis of observations of the satellites of Saturn by earlier authors

| Author | Years | Micrometric | Astrometric |
|---|---|---|---|
| Kozai (1957) | 1684-1947 | Yes | No |
| Garcia (1972) | 1926-1947 | No | Yes |
| Chugunov (1983) | 1682-1981 | Yes | Yes |
| Dourneau (1987) | 1886-1985 | Yes | Yes |
| Taylor & Shen (1988) | 1966-1983 | No | Yes |
| This paper | 1874-1989 | Yes | Yes |

Pittsburgh's Allegheny Observatory. These data span a period of 21 years. Chugunov (1983a) used new theories of the satellites (Chugunov 1983b) to analyse a large set of observations spanning 300 years. However, subsequent authors have suggested that the standard errors quoted by Chugunov are unrealistically small. Taylor & Shen (1988) analysed photographic observations of the satellites made between 1966 and 1983, including Pascu's unpublished observations. They were able to determine good values for the orbital elements of the eight major satellites (including Hyperion), but the short time span of the data prevented them from obtaining reliable values for the mean motions.

Dourneau (1987) analysed observations spanning the period 1886 to 1985 and was able to determine new values for the orbital elements, mean motions and secular rates. Dourneau's dataset included a new series of observations which he and his colleagues had made at the European Southern Observatory, Bordeaux, Hawaii and Pic du Midi. However, Dourneau gave equal weight to the modern astrometric data and to the older micrometric data. The analysis described in the current paper employs a weighted least-squares method in which different weights are assigned to the various sets of observations depending upon their quality. Furthermore, Dourneau used Sinclair's (1974) theory of the motion of Iapetus. As Rapaport (1978) and Harper et al. (1988) have shown, this omits several important periodic terms. In the current paper, we use the revised theory presented by Harper et al. (1988). Observations of Iapetus form a significant fraction of some sets and therefore an improved theory of the motion of Iapetus may be expected to lead to a significantly better fit to the observations. In addition, we have analysed a large number of observations which were not used by Dourneau. These include the extensive collection of observations by Pascu.

Our dataset consists of Strugnell & Taylor's (1990) catalogue augmented by more than 10000 further observations from several sources. It is probably the most complete collection of ground-based observations of Saturn's satellites currently available. It spans the period from 1874 to 1989 and should, when combined with the best current analytical theories of the satellites, provide much-improved values of the orbital elements, mean motions and secular rates of the satellites together with information on the long-period librations which are characteristic of the satellites of Saturn.

The additional data are in three groups: (1) Approximately 8000 micrometric observations made between 1890 and 1927 at the Yerkes, Lick and Leander-McCormick Observatories. We give full details of these observations in Paper II. (2) Over 2300 astrometric observations made by Pascu between 1974 and 1980 at the United States Naval Observatory. These data are unpublished, and requests for the data should be made to their author. (3) Approximately 1800 astrometric observations made by Veillet & Dourneau (1992) at the Mauna Kea, Pic du Midi and European Southern Observatories in period 1979-85.

The theories were fitted to the observations using a weighted least-squares procedure. Our analysis incorporates datasets of rather disparate quality. Before 1947, observations were predominantly visual micrometer measurements with a typical root-mean-square (RMS) residual of $0\rlap{.}''30$. After 1947, the observations are almost exclusively photographic astrometric measurements with a typical RMS residual of $0\rlap{.}''15$. We have assigned different weights to each of the series of observations which comprise the dataset; the weights were calculated on the basis of RMS residuals from a preliminary analysis where all observations were given equal weight.

The theories of the satellites are the same as those used by Taylor & Shen (1988). They are also the same as those used by Dourneau (1987) with the exception of the theory of Iapetus, as noted above. We summarise the theories in Sect. 3 and give them in full in the Appendix.

We have not used the new theories developed by Vienne & Duriez (Duriez & Vienne 1991; Vienne & Duriez 1991, 1992; Vienne 1991) because the form in which the theories are presented does not make it easy to compute partial derivatives of the positions of the satellites with respect to the parameters, which is a necessary part of the analysis of observations. The theories are presented as a combination of short-period terms calculated using standard general perturbation methods plus long-period and critical terms whose phases, frequencies and amplitudes have been determined empirically from Fourier analysis of numerical integrations of the averaged Lagrange planetary equations. The latter terms are the most significant part of the theories of most of the satellites, and are particularly important in determining the masses of the inner four satellites. Vienne (1991) gives purely numerical values for the phases, amplitudes and frequencies of the long-period and critical terms and hence we feel the theories of Vienne & Duriez are not yet sufficiently generalised for use in the work described in this paper.

It was decided to omit Hyperion from the analysis because the theory of the motion of Hyperion is not as precise as those of the other satellites. The motion of Hyperion is dominated by the 4:3 near commensurability of periods with Titan. The difficulties in constructing a theory for Hyperion are increased by the large and variable eccentricity of Hyperion (mean value about 0.105) and the large ratio (0.83) of the semi major axes of the two orbits. The relatively high eccentricity of Titan further complicates the problem by adding large oscillations in the eccentricity and apse longitude of Hyperion. In the work described in this paper, almost all observations are analysed in the form of the position of one satellite relative to another and therefore any inadequacies in the theory of the motion of Hyperion would reduce the accuracy of the solution for the elements of the other satellites. For this reason, all observations which involve Hyperion were excluded from the analysis.

Following the analysis of the observations, the parameters obtained from the least-squares procedure were used to determine the masses of the four inner satellites (Mimas, Enceladus, Tethys and Dione) and the dynamical form factors $J_2$ and $J_4$ which describe the gravitational field of Saturn. The masses were derived from the observed frequencies and amplitudes of the librations in the mean longitudes of the satellites. The dynamical form factors were determined from the secular motions of the nodes of the inner satellites.

## 2. The observations used in the analysis

The majority of observations used in this analysis are taken from the catalogue compiled by Strugnell & Taylor (1990). These observations fall into two categories. (1) Almost 35000 were made between 1874 and 1947 and are mostly visual micrometer measurements of the positions of satellites relative to the planet or to other satellites. (2) Over 16000 were made since 1966 and are mostly photographic observations, yielding the absolute astrometric Right Ascension and Declination of one or more satellites on each plate. The reader is referred to Strugnell & Taylor (1990) for full details of the data and references.

We have also included the series of observations made by Dr Dan Pascu at the United States Naval Observatory between 1974 and 1980. These are photographic astrometric observations. They have not been published and are not included in the catalogue of Strugnell & Taylor.

Taylor has continued to collect observations of the satellites of Saturn and to put them into the uniform format of the catalogue. He has added 8048 visual micrometric observations made at the Yerkes, Lick and Leander-McCormick Observatories between 1890 and 1927. These observations are included in the analysis described in this paper.

In addition, we have included approximately 1800 observations made by Veillet & Dourneau (1992) at the Mauna Kea, Pic du Midi and European Southern Observatories during the period 1979 to 1985. These are photographic astrometric observations. They are not included in the catalogue of Strugnell & Taylor.

When analysing photographic observations which give the absolute positions of the satellites with respect to an astrometric coordinate system (usually the mean equator and equinox at the epoch B1950.0), we have used the differencing method described in Taylor & Shen (1988). In this method, the absolute coordinates of pairs of satellites are differenced to form relative positions. All possible combinations of pairs of satellites are used in order to give equal weight to all the satellites which appear on a particular photographic plate. This method eliminates the effect of systematic errors in the astrometric positions of the satellites caused by zonal errors in the positions of the reference stars given in fundamental star catalogues. It also eliminates the effect of errors in the ephemeris of Saturn, as discussed by Taylor et al. (1991). Each difference is counted as one observation. The differences computed in this way are not independent, but this does not cause problems in the least-squares analysis process described in Sect. 4.

We have omitted several datasets as follows: (1) Observations of Titan and Iapetus made using the Carlsberg and Bordeaux automatic meridian circles. These observations have been analysed by Taylor et al. (1991) to determine corrections to the DE200 ephemeris of Saturn. The observations give the absolute Right Ascension and Declination of the satellites referred to the true equator and equinox of date, but at each transit only the position of one satellite was measured and so these observations can only be used in an analysis which seeks to determine corrections to Saturn's ephemeris position and systematic errors in star catalogue positions. (2) The small number of eclipse and conjunction observations made by Aksnes et al. (1984) and by Dourneau et al. (1982). These observations give very precise positions of pairs of satellites at a limited number of epochs during the opposition of 1980 when the Earth and Sun were close to Saturn's ring plane. (3) The images of Saturn and its satellites made by Voyager 1 and Voyager 2 during their encounters with Saturn. These also yield very precise positions but over a very short period of time. Unfortunately, they have not yet been published. (4) The series of observations made between 1787 and 1874 by various observers. Some of these data are estimated times of greatest elongation or conjunction of the satellites; others are micrometer observations but their accuracy is far lower than that of observers working after 1874. None of the pre-1874 datasets is large and the only advantage in including them might be to improve the mean motions slightly.

The position of Saturn and the nutation in longitude and obliquity were calculated from the DE200/LE200 solar system ephemeris (Standish 1991). This ephemeris yields coordinates referred to the mean equator and equinox of the epoch J2000, but the theories of the satellites gives coordinates referred to the mean equator and equinox of the epoch B1950. Since most previous authors quote orbital elements referred to the B1950 system, it was decided to adopt this system in the current paper. The astrometric coordinates of Saturn derived from the DE200/LE200 ephemeris were therefore processed to the B1950 system. The coefficients for this transformation were calculated using the expressions derived by Lieske (1979).

## 3. The theories used in the analysis

We have used the theories of Kozai (1957) for Mimas, Enceladus, Tethys and Dione. The theories developed by Sinclair (1977) were used for Rhea and Titan, and the theory of Harper et al. (1988) was used for Iapetus.

The initial values of the elements, mean motions and secular rates of apses and nodes of the satellites were taken from Dourneau (1987). The libration parameters for Mimas, Enceladus, Tethys and Dione were also taken from this source.

All of the theories are referred to the fixed reference system of the mean ecliptic and equinox of B1950.0. The epoch of all of the theories has been standardised to JED 2426000.5, which is 1930 January $24^{\mathrm{d}}0^{\mathrm{h}}$ Ephemeris Time. This epoch was chosen because it lies approximately in the middle of the time span of the observations, and this minimises errors in the positions of the satellites due to errors in their mean motions.

We use subscripts as follows to denote the satellites: 1=Mimas, 2=Enceladus, 3=Dione, 4=Tethys, 5=Rhea, 6=Titan, 7=Hyperion, 8=Iapetus.

We summarise the theories below. The full theories are given in the appendix.

### *3.1. The theories of Mimas and Tethys*

The motion of Mimas and Tethys is characterised by two features:

(1) Rapid precession of the longitudes of the node and apse, mainly due to the oblateness of Saturn.

(2) A libration of large amplitude and long period in the mean longitudes of both satellites, caused by a 2:1 close commensurability in their mean motions. The libration argument is

$$\theta_{13} = 2\lambda_1 - 4\lambda_3 + \Omega_1 + \Omega_3 \tag{1}$$

where $\lambda$ denotes the mean longitude and $\Omega$ denotes the longitude of the ascending node. This angle oscillates about zero. The period of the libration is approximately 72 years and the amplitude of the principal libration term in the mean longitude is 43 degrees for Mimas and 2 degrees for Tethys. The theory of this libration was first studied by H. Struve and later by G. Struve who added libration terms with arguments $\psi$, $3\psi$ and $5\psi$ to the mean longitudes of both satellites, where $\psi$ is the libration argument defined by Eq. 5.

### *3.2. The theories of Enceladus and Dione*

The motion of Enceladus and Dione is characterised by similar features to those in the theories of Mimas and Tethys:

(1) The nodes and apses of the satellites precess rapidly due to the oblateness of Saturn.

(2) There is a small libration in the mean longitudes of the satellites caused by a 2:1 close commensurability in the mean motions. The libration argument is

$$\theta_{24}^{(1)} = 2\lambda_4 - \lambda_2 - \varpi_2 \tag{2}$$

which librates about zero. There is also a long-period term with the argument

$$\theta_{24}^{(2)} = 2\lambda_4 - \lambda_2 - \varpi_4 \tag{3}$$

which circulates. The angle $\varpi$ denotes the longitude of the apse. The periods of the two terms are 11 years and 3.8 years and the amplitudes in the mean longitudes are about 14′ for Enceladus and 1′.5 for Dione.

There are no other periodic perturbations included in the theories of Enceladus and Dione.

### *3.3. The theory of Rhea*

The motion of Rhea is dominated by a forced eccentricity due to the action of Titan. The longitude of the apse of Rhea oscillates about that of Titan with a period of 36 years.

In addition, the combined effect of the oblateness of Saturn and the action of Titan causes the orbit plane of Rhea to precess at a constant inclination to a fixed Laplacian plane. However, the orbital elements are referred to the ecliptic and therefore the precession introduces periodic variations in the mean longitude, inclination and longitudes of the node and apse.

### *3.4. The theory of Titan*

The orbit plane of Titan precesses at a constant inclination to a fixed Laplacian plane due to the combined secular effects of the Sun and the oblateness of Saturn. As in the case of Rhea, the choice of the ecliptic as the reference plane introduces apparent periodic perturbations into the mean longitude, inclination and longitudes of the node and apse.

There are also several significant periodic perturbations due to the action of the Sun. These affect all of the elements except for the semi major axis.

### *3.5. The theory of Iapetus*

The orbit plane of Iapetus precesses at a constant inclination to a fixed Laplacian plane due to the secular effects of the Sun, Titan and (to a lesser extent) the oblateness of Saturn. This is discussed in Harper (1987, 1988).

There are also periodic perturbations in all of the elements due to the action of Titan and of the Sun. The solar perturbations mainly affect the longitude of the node and the inclination, since the orbit of Iapetus has a high inclination to the orbit plane of Saturn. The perturbations by Titan principally affect the mean longitude.

Plana (1826) noted that the mean motions of Titan and Iapetus were approximately in the ratio 5:1 and suggested that certain perturbations in the mean longitude of Iapetus might be amplified because of the resulting small divisor. Rapaport (1978) calculated the contributions of a number of such terms; Harper et al. (1988) recalculated them following the method of Sinclair (1974) and showed that during the period covered by observations, they could be reduced to a single term with a slowly varying coefficient. The amplitude of the term is approximately 0.″07 as seen from the Earth at mean opposition distance, and it is therefore included in the theory of Iapetus

## 4. Method of analysis

### *4.1. The iterative least-squares process*

The observations were analysed using the theory of least-squares to minimise the sum of squares of the observed-minus-computed residuals. An equation of condition was formed from each observation. The right-hand-side of the equation of condition was the observed-minus-computed residual of the observation; the computed position was obtained using the theories described in the previous section. The coefficients in the left-hand-side of the equation were the partial derivatives of the observed position with respect to the unknown parameters.

Each equation involving a residual in Right Ascension was multiplied by the cosine of the computed Declination. Similarly, each equation in position angle was multiplied by the sine of the computed separation. This was done in order to ensure that the equations were directly comparable to corresponding equations in Declination or separation respectively.

All equations of condition whose right-hand-side was smaller (in absolute value) than a chosen rejection level were incorporated into the normal equations. The rejection level was chosen so that observations whose residuals were within $3\sigma$ would be included in the normal equations. In this context, $\sigma$ denotes the unweighted root-mean-square residual for all observations included in the normal equations. It was found that a rejection level of 1.″00 was satisfactory; it led to an unweighted root-mean-square residual of approximately 0.″27 in the final solution and 93 per cent of observations were included in the normal equations of this solution.

The least-squares procedure was used to calculate corrections to the orbital elements and other parameters of the system. These corrections were added to the elements and the least-squares procedure was started again. This iterative process was repeated until convergence was achieved. Convergence was assumed when the correction to each orbital element was no greater than one per cent of the standard error of that parameter. Using the elements and other parameters determined by Dourneau (1987) as starting values, convergence usually occurred after no more than six iterations.

### *4.2. Choice of weights*

Each data set within the catalogue of observations was assigned a weight which indicates the relative quality of the data within the set. As recent work has shown (Sinclair & Taylor 1985; Dourneau 1987; Harper et al. 1989; Taylor & Shen 1988), visual observations made during the period 1874-1947 are in general less precise than photographic observations made since 1966. There are exceptions – for example, the visual observations made by H Struve are equal in quality to some of the best modern photographic data. It was deemed necessary, therefore, to assign weights to each data set separately.

The work by Harper et al. (1989) also suggested that there may be variations in the quality of different observed quantities within each data set, particularly in the case of observations of position angle and separation. In order to allow for such variation, two weights were assigned to each data set: one for position angle or Right Ascension or $\cos\delta\Delta\alpha$ and the other for separation, Declination or $\Delta\delta$.

Every equation of condition was multiplied by the appropriate weight before it was incorporated into the normal equations. Weighted root-mean-square residuals were calculated for each data set. At the end of each iteration, these were compared with one another and with the overall standard error per unit weight to ensure that each data set had not been over- or under-weighted.

The weights for each data set were determined by carrying out an iterative least-squares process in which all the data sets were given equal weight. Weights were calculated in proportion to the inverse of the root-mean-square residual of the observations to which they were to be applied in subsequent analyses. This provided an objective and simple method of assigning weights to all of the data sets. The weights obtained by this method proved to be satisfactory and they were used in all later analyses.

The weights assigned to each dataset are given in full in Paper II. For the purpose of illustrating the range of weights, it may be noted that the weights assigned to the datasets of highest quality were approximately three times those assigned to those of the lowest quality.

## 5. Results

In this section, we present the orbital elements and other parameters of the satellite system that were determined by analysis of the observations. From these parameters, we determine the masses of the four inner satellites and the dynamical form factors $J_2$ and $J_4$ of Saturn. Our results are compared with the recent work of Dourneau (1987) and with Garcia (1972), Kozai (1957) and Taylor & Shen (1988).

The errors in the orbital elements and other parameters given in Table 2 are taken directly from the least-squares analysis where they were calculated by multiplying the standard error per unit weight by the square root of the appropriate diagonal element of the inverse of the normal matrix. They therefore represent the standard errors of the parameters calculated directly from the observations.

The quantities given in Tables 3 and 4, namely the masses and dynamical form factors, are derived from the elements and other parameters in Table 2. The standard errors given in Tables 3 and 4 were calculated using formal error analysis, as described by Topping (1972). This required knowledge of the partial derivatives of the masses and dynamical form factors with respect to the elements upon which they depend. Due to the complicated nature of these dependencies, the computer algebra system *Maple* (Char et al. 1991) was employed to calculate them analytically and to evaluate the derivatives. *Maple* was used extensively in the calculation of the results presented in Sects. 5.3 and 5.4.

We also present a summary of the root-mean-square residuals of several of the most important data sets. Full details of the residuals and an extensive discussion of the relative qualities of the data sets used in this analysis will be given in Paper II.

### *5.1. Keplerian elements and mean motions of the satellites*

The six orbital elements determined from the observations are given in Table 2a together with their standard errors. The values of the angular elements (mean longitude, longitude of the ascending node and longitude of the apse) correspond to the epoch JED 2426000.5 which is 1930 January $24^{\mathrm{d}}0^{\mathrm{h}}$ Ephemeris Time. The apse longitude of Enceladus is not determined because it is forced to obey the condition given by Eq. (2). For Enceladus, $e$ is the free eccentricity. For Rhea, $e$ and $\pi$ are the free eccentricity and apse. The reference planes for the inclinations and the node, apse and mean longitudes of the satellites are shown in Fig. 1. The mean motions of the satellites are given in Table 2b, where they are compared with the values obtained by Dourneau (1987).

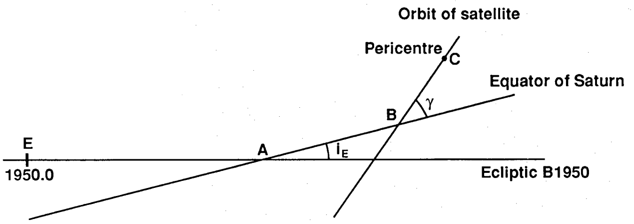


**Fig. 1a.** Reference system for the elements of Mimas, Enceladus, Tethys and Dione. E denotes the equinox of the epoch B1950. N = EA+AB; P=EA+AB+BC

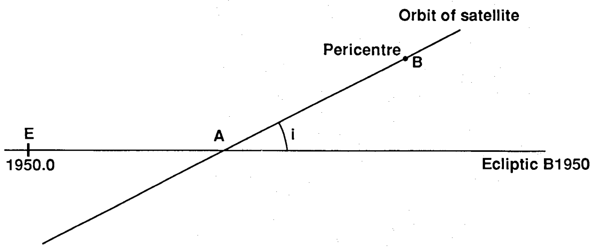


**Fig. 1b.** Reference system for the elements of Rhea, Titan and Iapetus. E denotes the equinox of the epoch B1950. N (or $\Omega$) = EA; $\varpi_0$ (or $\pi_0$) = EA+AB

In Fig. 2 we compare our determinations of the elements and the mean motions with those of other authors. These authors are: Taylor & Shen (1988), Dourneau (1987), Chugunov (1983), Garcia (1972) and Kozai (1957). We have not included Struve (1933) because Kozai analysed effectively the same set of observations as Struve using more refined theories for the inner satellites. Kozai's work thus supersedes that of Struve.

In the diagrams in Fig. 2, we have plotted the differences between our values of the elements and those of the other authors. For each of the six Keplerian elements, we have converted the difference and the standard errors into the corresponding difference in the position of the satellite in its orbit expressed in kilometres. The following scale factors were applied to each of the elements:

Semi major axis: no scaling
Mean longitude: multiply by $a$
Eccentricity: multiply by $a$
Longitude of the apse: multiply by $ae$
Inclination: multiply by $a$
Longitude of the node: multiply by $a \sin \gamma$

where $a$ denotes the semi major axis, $e$ the eccentricity and $\gamma$ the inclination ($i_0$ for Iapetus). The differences and standard errors in the mean motions have been converted to a distance by multiplying by the semi major axis, then multiplying by a time interval equal to 60 years; thus the diagrams comparing the mean motions indicate the error in the position of the satellite at the end points of the data due to the error in the mean motion. Taylor & Shen did not determine mean motions directly from their least-squares analysis of the observations; hence no mean motions from Taylor & Shen are given in Fig. 2.

The diagrams in Fig. 2 show the *maximum* differences between the positions of the satellites calculated from the elements determined in this paper and the positions calculated from the elements of the other authors, using the theories described in Sect. 3. Note that at mean opposition distance, $0\rlap{.}''1$ is equivalent to approximately 600 kilometres.

We find that our elements and their standard errors are in broad agreement with those obtained by Dourneau (1987). Our values of the elements are based upon a similar span of data to that used by Dourneau (1987) and we therefore expect to obtain similar values of the orbital elements. The mean longitudes determined by Taylor & Shen differ significantly from those of other authors. This discrepancy arises from the fact that they used values of the mean motions of the satellites taken from Kozai (1957), Garcia (1972) and Struve (1933) which were based upon observations whose mean epoch is around 1900. The mean epoch of the observations used by Taylor & Shen is about 1974, and since they did not solve for corrections to the mean motions, their solution compensated for the cumulative effect of errors in the mean motion by changing the mean longitude at epoch. Chugunov (1983b) obtained elements with standard errors that are an order of magnitude smaller than those of any other author. Except for the semi major axis, mean motion and secular rates of node and apse, Chugunov quotes standard errors which are of the order of 1 kilometre, corresponding to $0\rlap{.}''001$ or less at mean opposition distance. This is clearly a serious underestimate, since the root-mean-square residuals in the observed

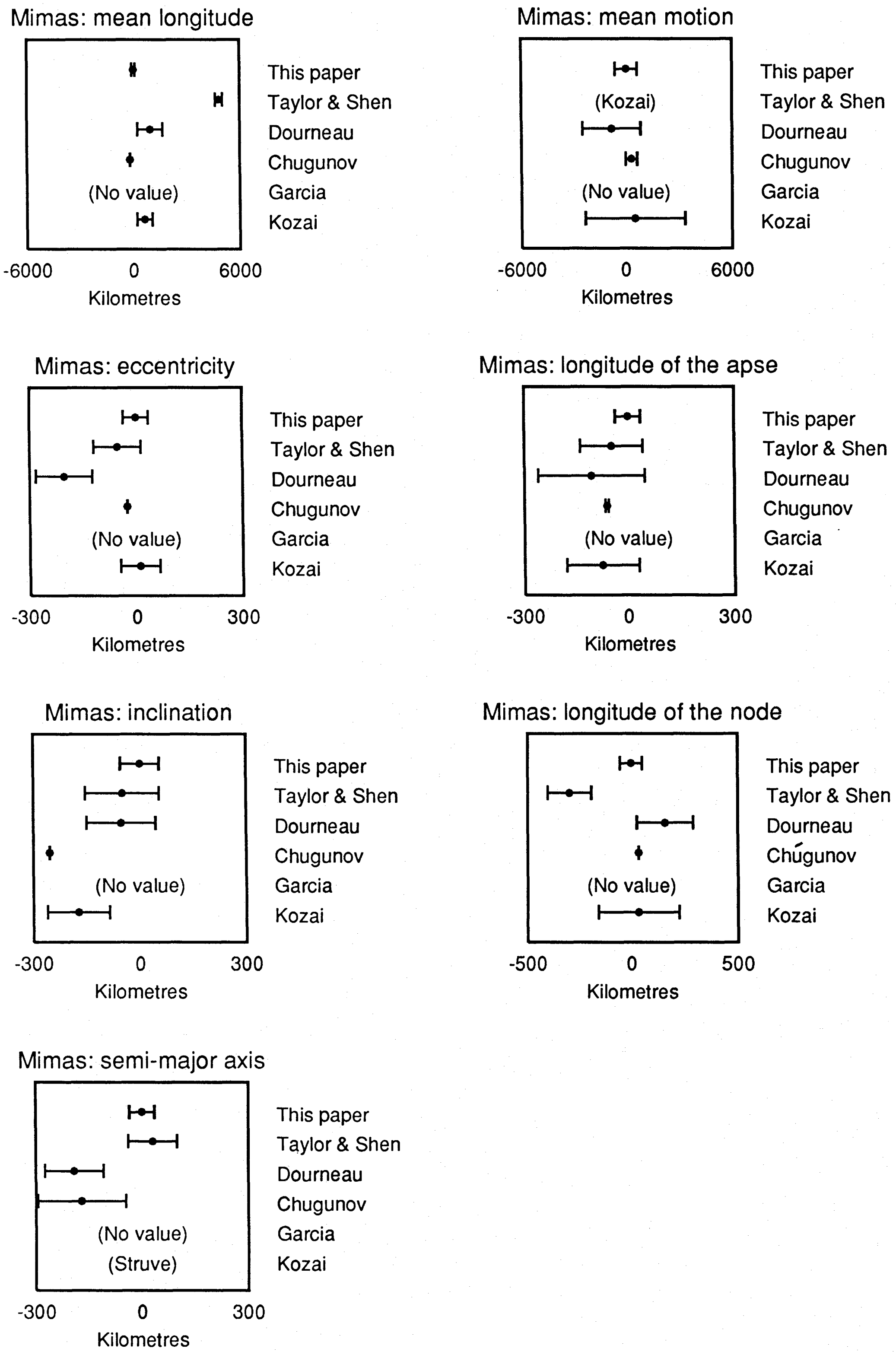


**Fig. 2a.** Comparison of the orbital elements and mean motion of Mimas determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411093.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Kozai" indicates that the author adopted the value determined by Kozai (1957). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

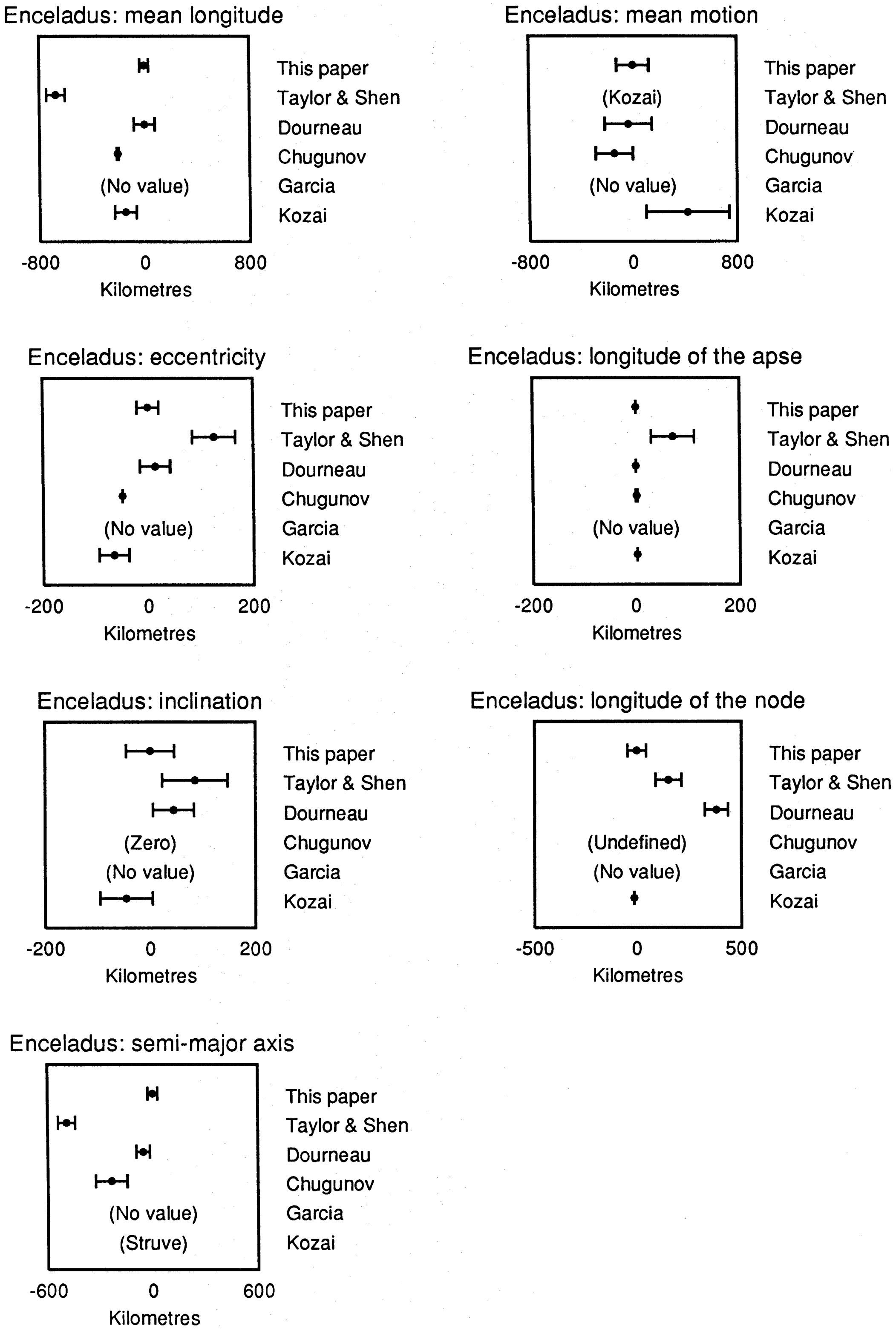


**Fig. 2b.** Comparison of the orbital elements and mean motion of Enceladus determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411093.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Kozai" indicates that the author adopted the value determined by Kozai (1957). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

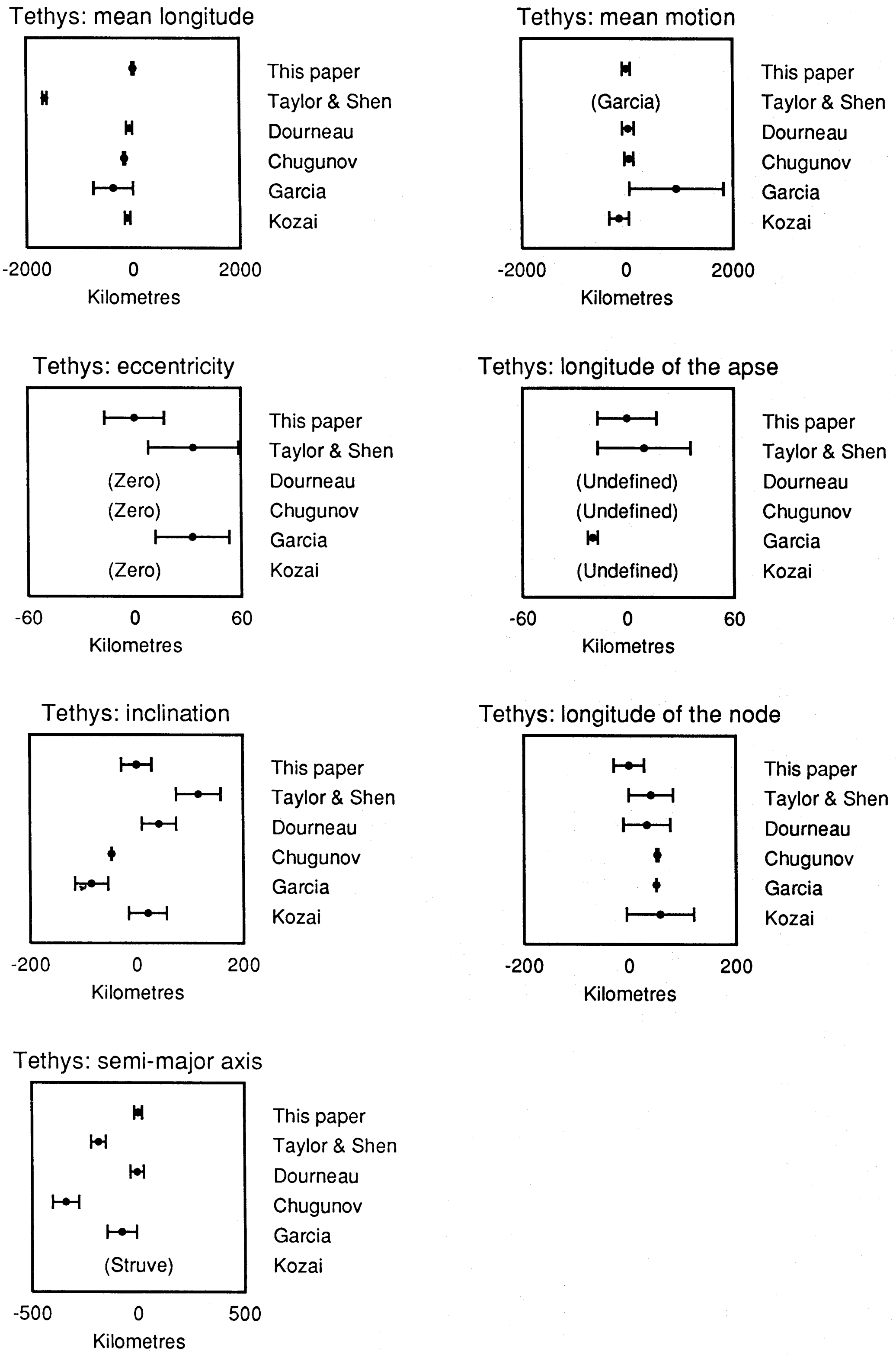


**Fig. 2c.** Comparison of the orbital elements and mean motion of Tethys determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411093.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Garcia" indicates that the author adopted the value determined by Garcia (1972). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

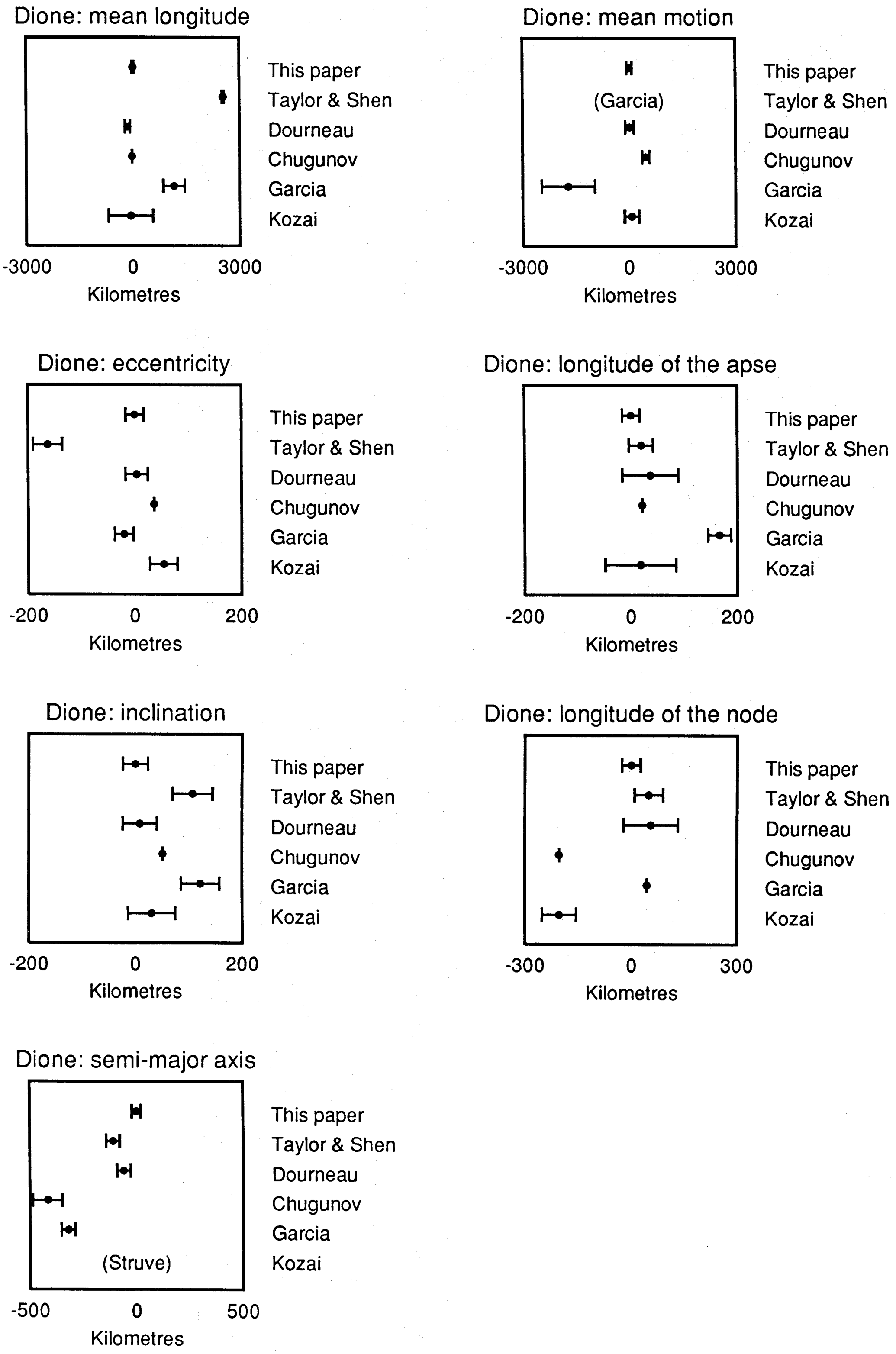


**Fig. 2d.** Comparison of the orbital elements and mean motion of Dione determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411093.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Garcia" indicates that the author adopted the value determined by Garcia (1972). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

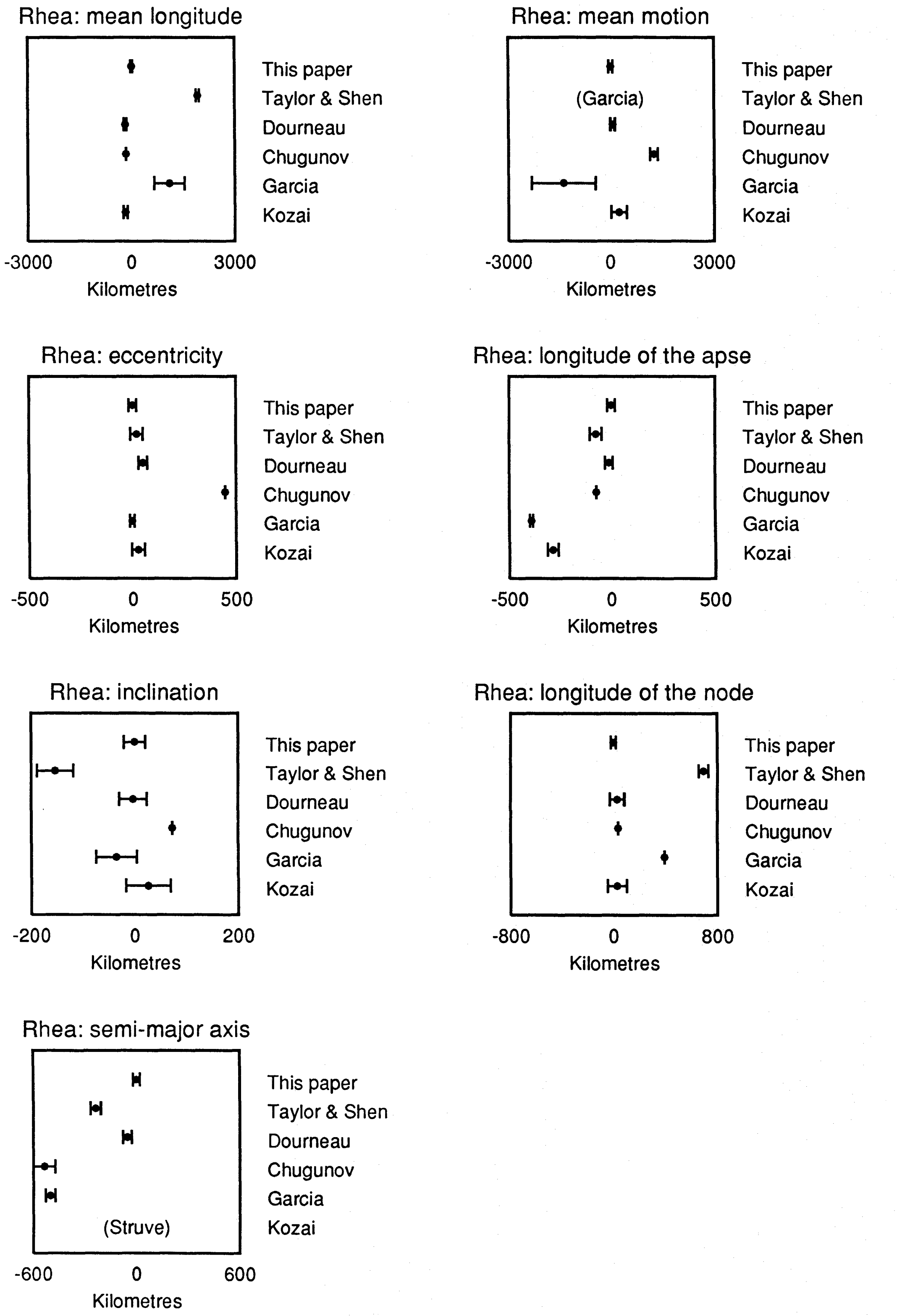


**Fig. 2e.** Comparison of the orbital elements and mean motion of Rhea determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411368.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Garcia" indicates that the author adopted the value determined by Garcia (1972). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

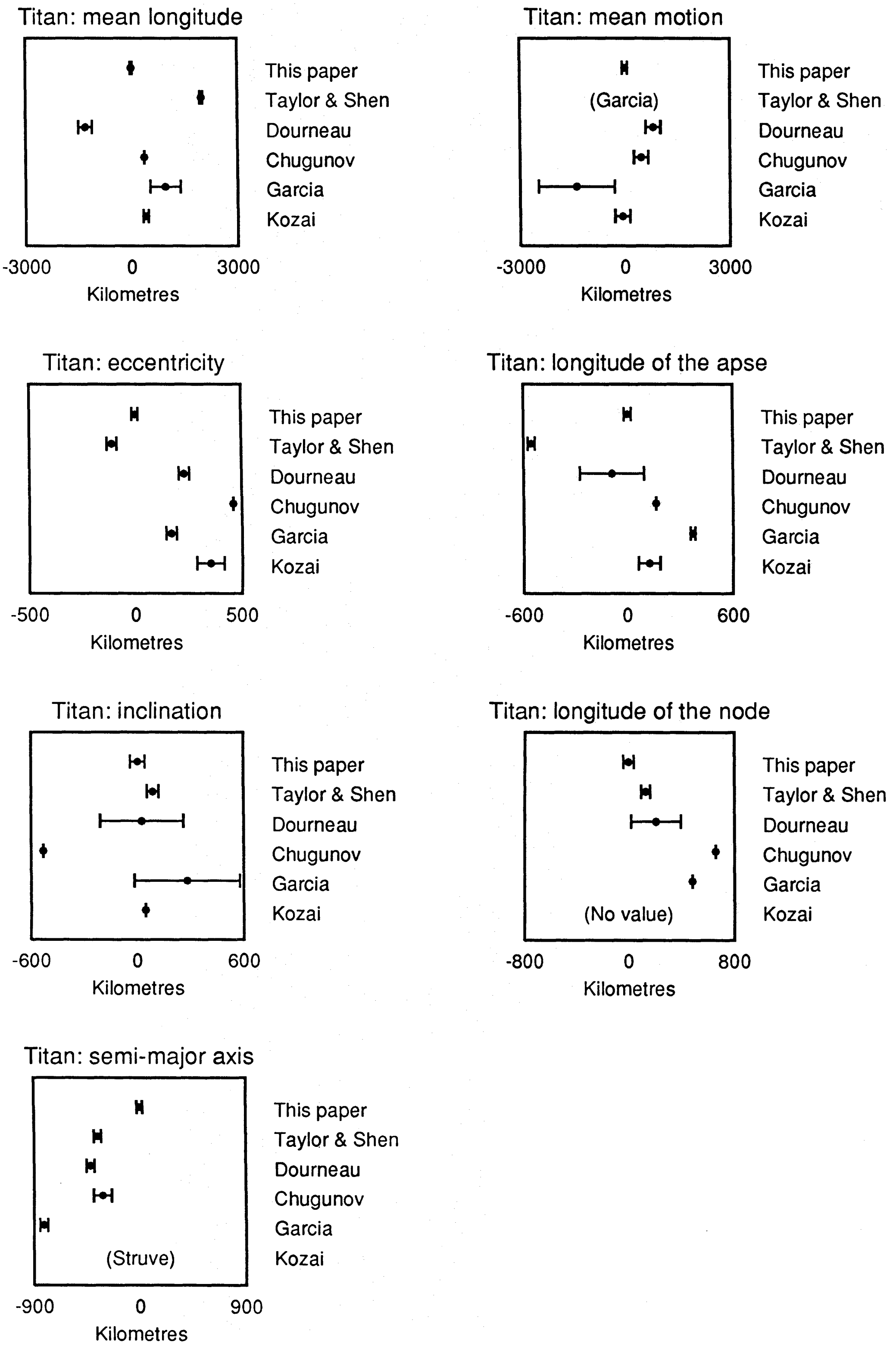


**Fig. 2f.** Comparison of the orbital elements and mean motion of Titan determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2411368.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Garcia" indicates that the author adopted the value determined by Garcia (1972). Note that the standard errors quoted by Chugunov are underestimated; see Sect. 5.1

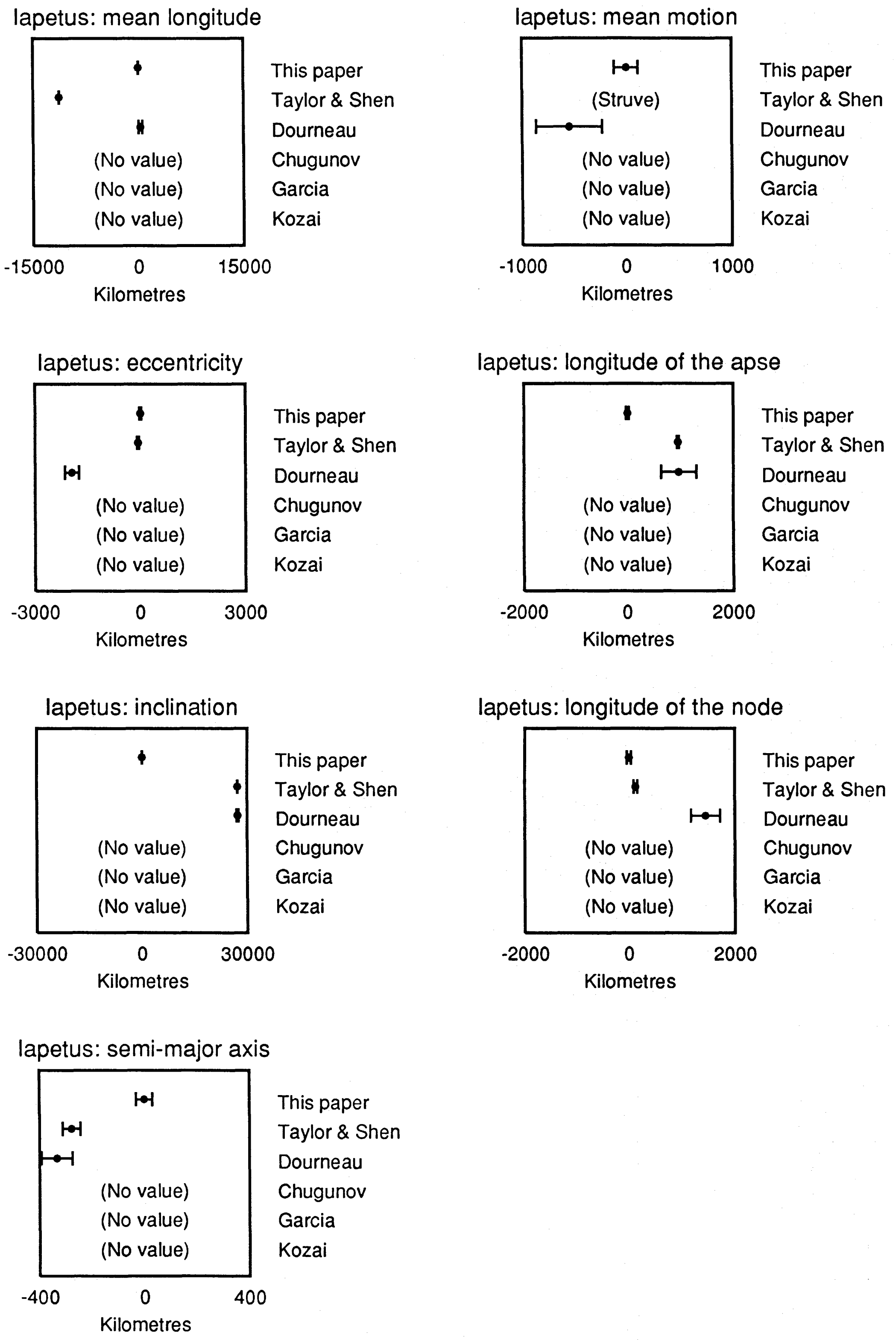


**Fig. 2g.** Comparison of the orbital elements and mean motion of Iapetus determined in this paper with those of other authors. The mean longitude and the longitudes of node and apse determined in this paper have been transformed to the epoch JED 2409786.0 which is the epoch used by all previous authors. Error bars denote the standard error. Values are plotted as the difference between the cited author's value and that determined in this analysis. The label "No value" indicates that the author did not determine the element; "Zero" indicates that the author assumed a value of zero; "Undefined" indicates that the element is undefined; "Struve" indicates that the author adopted the value determined by Struve (1933).

**Table 2a.** Orbital elements of the satellites determined in this analysis. The epoch in all cases is JED 2426000.5. The elements of the inner four satellites are given in the reference frame of Fig. 1a; the elements of the outer three satellites are given in the reference frame of Fig. 1b

| Parameter | | | Units |
|---|---|---|---|
| *Mimas* | | | |
| $a_0$ | 0.001 241 51 | ± 0.000 000 24 | A.U. |
| $L_0$ | 230.489 | ± 0.027 | degrees |
| $e_0$ | 0.020 14 | ± 0.000 19 | |
| $P_0$ | 266.73 | ± 0.55 | degrees |
| $\gamma_0$ | 1.585 | ± 0.017 | degrees |
| $N_0$ | 272.85 | ± 0.57 | degrees |
| *Enceladus* | | | |
| $a_0$ | 0.001 592 63 | ± 0.000 000 18 | A.U. |
| $L_0$ | 76.1250 | ± 0.0081 | degrees |
| $e_0$ | 0.004 795 | ± 0.000 086 | |
| $P_0$ | 307.3348 | — | |
| $\gamma_0$ | 0.016 | ± 0.010 | degrees |
| $N_0$ | 310 | ± 40 | degrees |
| *Tethys* | | | |
| $a_0$ | 0.001 971 95 | ± 0.000 000 12 | A.U. |
| $L_0$ | 194.4419 | ± 0.0042 | degrees |
| $e_0$ | 0.000 100 | ± 0.000 051 | |
| $P_0$ | 56 | ± 24 | degrees |
| $\gamma_0$ | 1.0895 | ± 0.0056 | degrees |
| $N_0$ | 42.75 | ± 0.29 | degrees |
| *Dione* | | | |
| $a_0$ | 0.002 524 86 | ± 0.000 000 14 | A.U. |
| $L_0$ | 191.7299 | ± 0.0037 | degrees |
| $e_0$ | 0.002 147 | ± 0.000 044 | |
| $P_0$ | 353.0 | ± 1.2 | degrees |
| $\gamma_0$ | 0.0126 | ± 0.0036 | degrees |
| $N_0$ | 38 | ± 18 | degrees |
| *Rhea* | | | |
| $a_0$ | 0.003 525 59 | ± 0.000 000 12 | A.U. |
| $\lambda_0$ | 338.6372 | ± 0.0022 | degrees |
| $e_0$ | 0.000 172 | ± 0.000 035 | |
| $\pi_0$ | 42 | ± 11 | degrees |
| $\gamma_0$ | 0.3472 | ± 0.0022 | degrees |
| $N_0$ | 294.00 | ± 0.36 | degrees |
| *Titan* | | | |
| $a_0$ | 0.008 170 06 | ± 0.000 000 16 | A.U. |
| $\lambda_0$ | 138.8328 | ± 0.0014 | degrees |
| $e_0$ | 0.028 905 | ± 0.000 012 | |
| $\varpi_0$ | 297.278 | ± 0.030 | degrees |
| $\gamma_0$ | 0.2949 | ± 0.0019 | degrees |
| $N_0$ | 19.56 | ± 0.38 | degrees |

positions of the satellites are no smaller than 0.″05 even for the best data sets.

**Table 2a.** (continued)

| Parameter | | | Units |
|---|---|---|---|
| *Iapetus* | | | |
| $a_0$ | 0.023 811 70 | ± 0.000 000 21 | A.U. |
| $\lambda_0$ | 216.99743 | ± 0.00068 | degrees |
| $e_0$ | 0.028 836 7 | ± 0.000 004 9 | |
| $\varpi_0$ | 357.824 | ± 0.016 | degrees |
| $i_0$ | 18.02066 | ± 0.00049 | degrees |
| $\Omega_0$ | 141.4750 | ± 0.0022 | degrees |

Our standard errors in the elements and other parameters given in Table 2 are slightly smaller, in most cases, than those given by Dourneau. This arises from a combination of three factors. Firstly, we have analysed a larger number of observations than Dourneau, including Pascu's collection which is of very high quality. Secondly, we have weighted each dataset according to its overall quality, and our solution is therefore not degraded by poor datasets to the same degree as Dourneau's solution. Of these two effects, the former is the more significant; the standard errors obtained in our preliminary unweighted solution were also smaller than those of Dourneau, and do not differ greatly from those in our final weighted solution. Thirdly, Dourneau included observations of Hyperion in his analysis and as a consequence, his values of some elements may be degraded by the lower precision of the Hyperion theory.

The eccentricities and inclinations of the orbits of all the satellites are small, with the exception of Iapetus, and therefore errors in these elements will have a relatively small effect upon the observed positions of the satellites. This is reflected in the scale of the plots of eccentricity, apse longitude, inclination and node longitude in Figs. 2a to 2f, which are generally an order of magnitude smaller than the scales of the plots of mean longitude and mean motion. Only in the case of Iapetus, whose inclination to the reference plane (the ecliptic) is large, are the differences comparable to those in the mean longitude, and this is also partly due to the large semi major axis of Iapetus.

The differences and standard errors in the mean longitude at epoch are similar in size to those in the corresponding mean motion. Since the mean motion diagrams have been scaled to indicate the cumulative effect of differences in the mean motion over a period of 60 years, which is half the length of the data set, we expect this to be the case. In fact, the values of the mean motions agree very well with most other authors. Garcia's values of the mean motions are somewhat discrepant and have larger standard errors. This is due to the very short time span of his data set.

The overall scale of the diagrams ranges from 60 kilometres (eccentricity of Tethys) to 30000 kilometres (inclination of Iapetus). As we noted above, the mean longitude plots give a truer indication of uncertainties in the positions of the satellites than the plots of other elements. If we consider only the mean longitude plots, then the scale ranges from 800 to 20000 kilometres. The typical value for most of the satellites is around 3000

**Table 2b.** Mean motions, secular node and apse rates, libration parameters and the position of the equator plane of Saturn determined in this analysis. The node and inclination of the equator plane of Saturn are referred to the ecliptic and equinox of B1950. Values from Dourneau (1987) are also given for comparison

| Parameter | This paper | | Dourneau | | Units |
|---|---|---|---|---|---|
| *Equator plane of Saturn* | | | | | |
| $\Omega_e$ | 168.8387 | ±0.0035 | 168.8112 | ±0.0089 | degrees |
| $i_e$ | 28.0653 | ±0.0016 | 28.0817 | ±0.0035 | degrees |
| *Mimas* | | | | | |
| $n$ | 381.994 508 7 | ±0.000 005 3 | 381.994 497 | ±0.000 014 | degrees/day |
| $\dot{N}$ | −365.063 | ±0.016 | −365.072 | ±0.029 | degrees/year |
| $\dot{P}$ | 365.532 | ±0.019 | 365.549 | ±0.061 | degrees/year |
| *Enceladus* | | | | | |
| $n$ | 262.731 900 58 | ±0.000 000 82 | 262.731 900 2 | ±0.000 001 2 | degrees/day |
| $\dot{N}$ | −151.43 | ±0.77 | −151.95 | ±0.66 | degrees/year |
| *Tethys* | | | | | |
| $n$ | 190.697 911 96 | ±0.000 000 42 | 190.697 912 26 | ±0.000 000 60 | degrees/day |
| $\dot{N}$ | −72.2351 | ±0.0060 | −72.2441 | ±0.0080 | degrees/year |
| $\dot{P}$ | 70.03 | ±0.79 | — | — | degrees/year |
| *Dione* | | | | | |
| $n$ | 131.534 931 86 | ±0.000 000 30 | 131.534 931 93 | ±0.000 000 49 | degrees/day |
| $\dot{N}$ | −30.30 | ±0.48 | −30.27 | ±0.77 | degrees/year |
| $\dot{P}$ | 30.887 | ±0.029 | 30.820 | ±0.054 | degrees/year |
| *Rhea* | | | | | |
| $n$ | 79.690 046 87 | ±0.000 000 17 | 79.690 047 20 | ±0.000 000 20 | degrees/day |
| *Titan* | | | | | |
| $n$ | 22.576 976 82 | ±0.000 000 09 | 22.576 978 55 | ±0.000 000 28 | degrees/day |
| $\dot{\varpi}$ | 0.51273 | ±0.00077 | 0.51187 | ±0.0035 | degrees/year |
| *Iapetus* | | | | | |
| $n$ | 4.537 951 65 | ±0.000 000 04 | 4.537 951 25 | ±0.000 000 14 | degrees/day |
| $\dot{\Omega}$ | −3.7119 | ±0.0049 | −3.919 | ±0.015 | degrees/century |
| $\dot{\varpi}$ | 12.285 | ±0.037 | 11.71 | ±0.20 | degrees/century |
| *Mimas-Tethys libration* | | | | | |
| $A_1$ | −43.635 | ±0.038 | −43.57 | ±0.13 | degrees |
| $x_{13}$ | 0.09539 | ±0.00028 | 0.09470 | ±0.00074 | |
| $\nu_{13}$ | 5.0866 | ±0.0033 | 5.095 | ±0.014 | degrees/year |
| $\tau_0$ | 1866.261 | ±0.045 | 1866.39 | ±0.021 | years |
| *Enceladus-Dione libration* | | | | | |
| $p_2$ | 0.297 | ±0.013 | 0.257 | ±0.017 | degrees |
| $p_4$ | −0.0262 | ±0.0053 | −0.0215 | ±0.0070 | degrees |
| $\mu_{24}$ | 314.3 | ±2.0 | 318.6 | ±5.6 | degrees |
| $\nu_{24}$ | 32.567 | ±0.069 | 32.39 | ±0.13 | degrees/year |

kilometres. This represents the range of values determined by authors from Kozai (1957) to the present analysis. The standard errors are somewhat smaller; a typical value is around 300 kilometres which corresponds to 0.″05 as seen from mean opposition distance. The root-mean-square residuals from the best data sets used in this analysis correspond to approximately 600 kilometres; hence, generally speaking, it is not possible to increase the accuracy with which the positions of the satellites can be predicted simply by adding further terms to the theories of the satellites, because the limiting factor is the precision of the observations. The use of charge-coupled detector (CCD) devices for ground-based astrometric imaging of satellites of the outer planets may provide slightly more accurate positions than photographic methods, but improvements of an order of magnitude or better will only be achieved by spacecraft such as *Cassini* which are intended to approach Saturn to a distance of less than 1 astronomical unit.

### 5.2. Libration parameters and masses of the inner satellites

The theory of the libration was developed by Struve (1898) and improved by Struve (1933).

The libration in the mean longitude of Mimas can be written as

$$\delta\lambda_1 = A_1 \sin\psi + B_1 \sin 3\psi + C_1 \sin 5\psi, \tag{4}$$

where the libration argument $\psi$ is given by

$$\psi = \nu_{13}(t - \tau_0), \tag{5}$$

where the time $t$ is expressed in years.

It can be shown that the corresponding libration in the mean longitude of Tethys may be written as

$$\delta\lambda_3 = -\frac{x_{13}}{2}\delta\lambda_1, \tag{6}$$

where

$$x_{13} = \frac{a_3}{a_1}\frac{m_1}{m_3}, \tag{7}$$

in which $m_1$ and $m_3$ denote the masses of Mimas and Tethys respectively.

We have determined the values of $A_1$ and $x_{13}$ from the observations of Mimas and Tethys. Fixed values of $B_1$ and $C_1$ were used; they were taken to be $-0\overset{\circ}{.}72$ and $-0\overset{\circ}{.}02144$ respectively. These are the values obtained by Dourneau (1987). We have also determined the frequency $\nu_{13}$ and the phase $\tau_0$ of the libration. The values of these parameters are presented in Table 2b with their standard errors. The values obtained by Dourneau (1987) are also given for comparison.

The mass of Tethys can be determined from the frequency of the libration, following the method described by Struve (1933). Struve gives

$$\left(\frac{2K}{\pi}\right)^2 \left(\frac{\nu_{13}}{n_1}\right)^2 = 12 m_3 \gamma_1 \gamma_3 \alpha^2 b_{3/2}^{(3)} (1 + x_{13}) \left(1 + \frac{\dot{\Omega}_1}{2n_1}\right), \tag{8}$$

where $\alpha$ is the ratio of the semi major axes in the sense $a_1/a_3$, $b_{3/2}^{(3)}$ is a Laplace coefficient,

$$\frac{2K}{\pi} = 1 + 4\sum_{s=1}^{\infty} \frac{q^s}{1 + q^{2s}}, \tag{9}$$

and $q$ is found from

$$\frac{\sqrt{q}}{1+q} = -\left(\frac{1}{4}\frac{1 + x_{13}}{1 - \dot{\Omega}_1/2n_1}\right) A_1, \tag{10}$$

in which the coefficient $A_1$ must be expressed in radians. $K$ is a complete elliptic integral of the first kind; Eq. (9) gives its power series expansion in terms of Jacobi's parameter $q$.

The mass of Mimas can then be calculated from the mass of Tethys, the ratio of the amplitudes of the librations and the ratio of the semi major axes, using Eq. (7). This gives

$$m_1 = \frac{a_1}{a_3} x_{13} m_3 \,. \tag{11}$$

The masses obtained by this method are given in Table 3, where they are compared with the values determined by Dourneau (1987) and Kozai (1957) from ground-based observations and by Tyler et al. (1982) and Campbell & Anderson (1989) from *Pioneer* and *Voyager* radio tracking data.

The libration in the mean longitude of Enceladus can be written as

$$\delta\lambda_2 = p_2 \sin(\nu_{24} t + \mu_{24}) + q_2 \sin(2\lambda_4 - \lambda_2 - \varpi_4) \,, \tag{12}$$

where the time $t$ is expressed in years from the epoch of the theory. The libration in the mean longitude of Dione can be expressed similarly in the form

$$\delta\lambda_4 = p_4 \sin(\nu_{24} t + \mu_{24}) + q_4 \sin(2\lambda_4 - \lambda_2 - \varpi_4) \,, \tag{13}$$

and from the theory of the libration (Struve 1933) we expect to find that

$$\frac{p_4}{p_2} = \frac{q_4}{q_2} = -\frac{1}{2}\frac{a_4}{a_2}\frac{m_2}{m_4} \,, \tag{14}$$

in which $m_2$ and $m_4$ denote the masses of Enceladus and Dione respectively. When calculating the argument of the long-period terms, only the linear parts of $\lambda_2$ and $\lambda_4$ are used.

We have determined the amplitudes $p_2$ and $p_4$, phase $\mu_{24}$ and frequency $\nu_{24}$ of the libration by analysis of observations of Enceladus and Dione. The values are presented in Table 2b with their standard errors. We also give the values obtained by Dourneau (1987) for comparison. We have adopted Dourneau's values for the amplitudes $q_2$ and $q_4$ of the long-period terms.

The mass of Dione can be determined from the frequency $\nu_{24}$ of the libration term in the mean longitude of the two satellites. We have

$$\left(\frac{\nu_{24}}{n_2}\right)^2 = 3\left(1 + 8x_{24}\left(\frac{n_4}{n_2}\right)^2\right)\alpha A e_2 m_4 + \left(\frac{\alpha A}{e_2}\right)^2 m_4^2 \,, \tag{15}$$

where

$$x_{24} = -\frac{p_4}{p_2} \,, \tag{16}$$

and

$$A = \frac{1}{2}(4 + \alpha D_\alpha) b_{1/2}^{(2)} \,, \tag{17}$$

in which $\alpha$ is the ratio of the semi major axes $a_2/a_4$, $b_{1/2}^{(2)}$ is a Laplace coefficient and $D_\alpha$ denotes differentiation with respect to $\alpha$. This is a quadratic equation in $m_4$ and we solve to find the positive root. Note that the forced eccentricity of Enceladus, $e_2$, appears in this equation and therefore the mass of Dione determined from Eq. 15 depends upon the value determined for $e_2$.

**Table 3.** Mass of the four inner satellites. Units are $10^{-6}$ times the mass of Saturn. Campbell & Anderson's (1989) value of the mass of Saturn has been used to express the masses given by Tyler et al. (1982) in units of the mass of Saturn. The mass of Mimas given by Tyler et al. was not obtained directly, but rather by multiplying the mass of Tethys by Kozai's value of the mass ratio

| Satellite | Mimas | Enceladus | Tethys | Dione |
|---|---|---|---|---|
| *From ground-based observations* | | | | |
| This paper | $0.0646 \pm 0.0011$ | $0.213 \pm 0.046$ | $1.076 \pm 0.018$ | $1.916 \pm 0.036$ |
| Dourneau (1987) | $0.0648 \pm 0.0021$ | $0.206 \pm 0.055$ | $1.088 \pm 0.031$ | $1.954 \pm 0.058$ |
| Kozai (1957, 1976) | $0.0659 \pm 0.0015$ | $0.134 \pm 0.041$ | $1.095 \pm 0.022$ | $1.850 \pm 0.040$ |
| *From spacecraft radio-tracking* | | | | |
| Campbell & Anderson (1989) | | | $1.19 \pm 0.26$ | |
| Tyler et al. (1982) | $0.0800 \pm 0.0095$ | | $1.33 \pm 0.16$ | |

Then the mass of Enceladus can be deduced from the mass of Dione and from the ratio of the amplitudes of the librations and the ratio of the semi major axes, using Eq. (11). We find

$$m_2 = 2\frac{a_2}{a_4}x_{24}m_4 \,. \tag{18}$$

The masses obtained by this method are given in Table 3, where they are compared with the values determined by Dourneau (1987) and Kozai (1976).

Our values of the masses of the four inner satellites agree well with those of Dourneau (1987) and Kozai (1957, 1976). The agreement is closer between our values and those of Dourneau (maximum difference = 1.1 standard errors) than between our values and those of Kozai (maximum difference = 1.8 standard errors). The discrepancy is especially notable in the masses of Enceladus and Dione. Kozai's values are based upon a 260-year span of data, but the observations made before 1874 are small in number and low in precision. Consequently, Kozai's values of the frequencies and amplitudes of the librations have larger standard errors than those given in Table 2b which were determined from only 100 years span of higher-quality data.

Tyler et al. (1982) and Campbell & Anderson (1989) determined the mass of Tethys from analysis of radio tracking data from the *Voyager 2* spacecraft. Both values differ significantly from those obtained by analysis of ground-based observations and have much larger standard errors. Campbell & Anderson note that the Doppler tracking of the spacecraft during its close approach to Tethys was dependent upon the stability of the on-board crystal oscillator which may not have been thoroughly radiation-hardened during the spacecraft's encounter with Jupiter. They conclude that the mass of Tethys may be more reliably determined from ground-based observations than from the *Voyager 2* fly-by.

### *5.3. Secular node and apse rates and Saturn's dynamical form factors*

The longitudes of the node and apse of each satellite precess due to the oblateness of Saturn and the secular effect of the Sun and the other satellites. The secular rates of motion of the nodes and apses of the satellites were determined from the observations. Their values are presented in Table 2b with the corresponding standard errors. The values obtained by Dourneau (1987) are also given for comparison. Note that the apse motion of Enceladus was not determined since the apse longitude of Enceladus is forced to obey the constraint of the libration described by Eq. (1). The node motion of Titan was not determined because it is taken to be equal to the apse motion of the satellite but opposite in sign.

The dynamical form factors $J_2$ and $J_4$ of Saturn can be determined from the secular motions of the nodes of the satellites. The inner four satellites alone were chosen for this calculation since their node rates are rather large (between 30 and 365 degrees per year) and may be expected to yield a better determination of the dynamical form factors than those of the outer satellites. Furthermore, the outer satellites are more affected by the secular contributions of the Sun and other satellites than the inner satellites. The effect of the Sun is proportional to the square of the satellite's orbital period (Eq. 20), hence the effect of the Sun upon Iapetus is nearly a thousand times greater than its effect upon Mimas. Comparison of Eq. 21 with Eq. 22 shows that an interior satellite has a greater effect (for a given mass and ratio of semi major axes) than an exterior satellite due to the extra factor $\alpha$ in Eq. 22.

The secular motion of the longitude of the node may be written as the sum of several terms, in the form

$$\frac{\dot{\Omega}}{n} = -\Gamma_{\mathrm{S}} - \sum_i \Gamma_{\mathrm{I}}^{(i)} - \sum_j \Gamma_{\mathrm{E}}^{(j)} - \Gamma_{\mathrm{J}} \,, \tag{19}$$

where $\Gamma_{\mathrm{S}}$ denotes the secular effect of the Sun, $\Gamma_{\mathrm{I}}^{(i)}$ the effect of the $i^{\text{th}}$ internal satellite, $\Gamma_{\mathrm{E}}^{(j)}$ the effect of the $j^{\text{th}}$ external satellite and $\Gamma_{\mathrm{J}}$ the effect of the oblateness of Saturn. These terms may be written as follows:

$$\Gamma_{\mathrm{S}} = \frac{3}{4}\left(\frac{n_{\mathrm{s}}}{n}\right)^2 \,, \tag{20}$$

where $n_{\mathrm{s}}$ is the mean motion of Saturn in its orbit around the Sun and $n$ is the mean motion of the satellite,

$$\Gamma_{\mathrm{I}}^{(i)} = \frac{1}{4}\mu_i\alpha_i b_{3/2}^{(1)}(\alpha_i) \,, \tag{21}$$

**Table 4a.** Secular node rates of the four inner satellites ($\times 10^8$). The contributions from $J_2$ and $J_4$ are calculated from the average values of these parameters from spacecraft radio-tracking

| Satellite | $-\dot{\Omega}/n$ | Contributions from | | | |
|---|---|---|---|---|---|
| | | Sun+satellites | $J_2$ | $J_2^2$ | $J_4$ |
| Mimas | 261 650 ± 11 | 214 ± 10 | 257 949 | −998 | 3815 |
| Enceladus | 156 760 ± 802 | 623 ± 6 | 156 749 | −369 | 1409 |
| Tethys | 103 708 ± 9 | 916 ± 20 | 102 245 | −157 | 599 |
| Dione | 63 062 ± 999 | 1427 ± 29 | 62 368 | − 58 | 223 |

where $\mu_i$ is the mass ratio of the $i^{\text{th}}$ satellite to Saturn, $\alpha_i = a_i/a$ and $b_{3/2}^{(1)}$ is a Laplace coefficient,

$$\Gamma_{\mathrm{E}}^{(j)} = \frac{1}{4}\mu_j \alpha_j^2 b_{3/2}^{(1)}(\alpha_j) , \tag{22}$$

where $\alpha_j = a/a_j$, and

$$\Gamma_{\mathrm{J}} = \frac{3}{2}\left(\frac{a_{\mathrm{e}}}{a}\right)^2 J_2 - \frac{27}{8}\left(\frac{a_{\mathrm{e}}}{a}\right)^4 J_2^2 - \frac{15}{4}\left(\frac{a_{\mathrm{e}}}{a}\right)^4 J_4 , \tag{23}$$

where $a_{\mathrm{e}}$ is the equatorial radius of Saturn. The contribution from the rings is negligible when compared to the standard errors in the observed values of the node rates, and it has been omitted.

The Solar and satellite contributions to the node rate of each of the four inner satellites were determined from the orbital elements given in Table 2. The masses of the inner four satellites were those obtained during this analysis. The mass of Rhea was taken from Tyler et al. (1982), the mass of Titan from Sinclair & Taylor (1985) and the mass of Iapetus from Campbell & Anderson (1989). The mean motion of Saturn was assumed to be $n_{\mathrm{s}} = 0\overset{\circ}{.}0334594$ per day. The mass of Hyperion was assumed to be zero. The observed node rates of the inner four satellites and the contributions from the various secular perturbations are shown in Table 4a together with their standard errors.

Adopting a value of 60 000 km for the equatorial radius of Saturn, the values of $J_2$ and $J_4$ were calculated from the node rates of the inner four satellites. The inclinations of the orbits of Enceladus and Dione are small (of the order of 0°.01) and hence the longitudes of the nodes of these satellites are not well determined. This is reflected in the standard errors of $-\dot{\Omega}/n$ for Enceladus and Dione given in Table 4a. The inclinations of Mimas and Tethys are much larger (greater than 1°) and their nodes are much better defined, so we might expect to obtain more precise values of $J_2$ and $J_4$ from analysis of the node rates of Mimas and Tethys alone. Evaluation of Eq. 19 for Mimas and Tethys yields two simultaneous equations in $J_2$ and $J_4$. These equations are non-linear due to the second-order term in $J_2$ in Eq. 23. However, they can easily be solved using *Maple* to yields values for $J_2$ and $J_4$. These are presented in Table 4b together with the values obtained by Kozai (1957) and Garcia (1972). Dourneau (1987) did not determine the dynamical form factors of Saturn. However, we have derived values using the orbital elements and node rates obtained by Dourneau and these are given in Table 4b where they are labelled "Dourneau". We also give values obtained from analysis of radio tracking data from the *Pioneer 11*, *Voyager 1* and *Voyager 2* spacecraft.

Our value of $J_2$ determined from the node rates of Mimas and Tethys agrees well with those from Dourneau's elements. This is to be expected since both sets are based upon a similar span of data. It also agrees with that of Kozai. Garcia's values of the dynamical form factors are rather larger than any of the others in Table 4b and we are inclined to suspect that his standard errors are underestimated because they are based upon only 21 years of data. In particular, Garcia's value of $J_4$ is much larger than any other value.

The value of $J_2$ obtained in this paper also agrees well with those determined by Null et al. (1981), Nicholson & Porco (1988) and Campbell & Taylor (1989) from analysis of spacecraft radio tracking data. The high degree of consistency between the three sets of coefficients based upon spacecraft data leads us to believe that these values are to be preferred over those determined by secular node and apse rates from ground-based observations. Nevertheless, they provide a useful confirmation of solutions from ground-based observations.

A further solution was sought using the node rates of Mimas, Enceladus, Tethys and Dione. Evaluation of Eq. 19 for the four satellites yields four simultaneous non-linear equations in $J_2$ and $J_4$. Each equation is of the form

$$p_i J_2 + q_i J_2^2 + r_i J_4 = s_i , \tag{24}$$

where $p_i$, $q_i$, $r_i$ and $s_i$ are numerical coefficients. These must be solved by the method of least squares to determine the values of $J_2$ and $J_4$ which minimise the function

$$\Lambda = \sum_{i=1}^{4} \left(p_i J_2 + q_i J_2^2 + r_i J_4 - s_i\right)^2 . \tag{25}$$

However, when *Maple* was used to draw a three-dimensional plot of $\Lambda$ in the vicinity of the values of $J_2$ and $J_4$ given in Table 4b, it was evident that the function did not have a well-defined local minimum corresponding to a unique solution. Analysis of the behaviour of the function shows that it has a "valley-shaped" minimum corresponding to points in the $J_2$-$J_4$ plane which satisfy the equation

$$J_2 - 0.2128 J_4 - 0.016546 = 0 . \tag{26}$$

**Table 4b.** Dynamical form factors of Saturn. The equatorial radius of Saturn is assumed to be 60000 km. The values labelled "Dourneau" are calculated from the node rates determined by Dourneau (1987)

| Author | $J_2$ | $J_4$ |
|---|---|---|
| *From ground-based observations* | | |
| This paper | $0.016\,478 \pm 0.000\,038$ | $-0.001\,10 \pm 0.000\,28$ |
| "Dourneau" | $0.016\,506 \pm 0.000\,055$ | $-0.000\,86 \pm 0.000\,41$ |
| Kozai (1957) | $0.016\,496 \pm 0.000\,011$ | $-0.000\,897 \pm 0.000\,043$ |
| Garcia (1972) | $0.016\,951 \pm 0.000\,008$ | $-0.003\,4 \pm 0.000\,1$ |
| *From spacecraft radio-tracking* | | |
| Null et al. (1981) | $0.016\,479 \pm 0.000\,018$ | $-0.000\,937 \pm 0.000\,038$ |
| Nicholson & Porco (1988) | $0.016\,477 \pm 0.000\,018$ | $-0.000\,930 \pm 0.000\,062$ |
| Campbell & Anderson (1989) | $0.016\,478 \pm 0.000\,010$ | $-0.000\,935 \pm 0.000\,041$ |

This is not a dynamical constraint such as that described by Nicholson & Porco (1988). Rather, it is a statistical correlation between the two parameters which indicates that it is not possible to determine independent values of $J_2$ and $J_4$ from the data.

The apse rate of Mimas is well-determined due to the large eccentricity of Mimas (approximately 0.02) and it might be thought that this observed quantity could also be used to determine the dynamical form factors. However, the expression for the apse rate is almost identical to that for the node rate, with the opposite sign. All contributions have the same coefficients except for the $J_2^2$ term where the numerical coefficient is $-9/8$ instead of $+27/8$. This leads to the small difference between the numerical values of the node rate and apse rate of Mimas given in Table 2b. However, the difference is very small and will not affect the value of $J_2$ determined from the node rates alone.

### 5.4. The secular acceleration of Mimas

Kozai (1957) noted that the residuals in the mean longitude of Mimas obtained from opposition means showed a trend that suggested a secular acceleration term. He determined a coefficient for this term, $-2\overset{\circ}{.}78 \pm 1\overset{\circ}{.}76/\text{century}^2$, but the standard error was large compared to the value of the coefficient itself. Kozai was therefore unable to confirm the existence of the secular acceleration. Dourneau re-determined the coefficient using a longer span of data and found a coefficient of $-2\overset{\circ}{.}05 \pm 0\overset{\circ}{.}55/\text{century}^2$. However, there is no theoretical basis for the inclusion of a secular acceleration term, and the results presented in Sects. 5.1-5.4 are based upon a theory of Mimas that does not include a secular acceleration in the mean longitude.

In order to determine whether inclusion of a secular acceleration term improves the fit between theory and observation, we repeated the least-squares analysis described in Sect. 4 using a theory of Mimas which uses the following expression for the mean longitude:

$$\lambda = L_0 + n \cdot d + sT^2 + \delta\lambda \,. \tag{27}$$

The last term, $\delta\lambda$, represents the 72-year libration. The coefficient $s$ was regarded as an unknown parameter whose value was to be determined from the observations. A value $-2\overset{\circ}{.}74 \pm 0\overset{\circ}{.}64/\text{century}^2$ was obtained. This agrees with Dourneau's value and is close to that obtained by Kozai. However, the standard error is still large. Inclusion of the secular acceleration term increased the mean longitude at epoch from $230\overset{\circ}{.}674 \pm 0\overset{\circ}{.}027$ to $230\overset{\circ}{.}912 \pm 0\overset{\circ}{.}061$ which corresponds to $9\sigma$. The frequency of the libration, $\nu_{13}$, also changed by approximately $1.6\sigma$. Other orbital elements and parameters changed by less than a standard error. The overall fit of the theories to observations was not changed significantly.

Recently, Vienne et al. (1991) have investigated long-period terms in the mean longitude of Mimas and have suggested that the residuals noted by Kozai are due to periodic terms involving the apse of Tethys. Previous authors have neglected such terms in the theory of Mimas. We repeated the least-squares fitting process using a theory of Mimas which includes the terms given by Vienne et al. (1991). We find that the value of the secular acceleration coefficient becomes $-1\overset{\circ}{.}63 \pm 0\overset{\circ}{.}86/\text{century}^2$ when these terms are included. This corresponds to a change of 2 standard errors in the previous value and may therefore indicate that part of the discrepancy in the mean longitude of Mimas can be explained by the omission of such terms.

### 5.5. Observed-minus-computed residuals of the data sets

A total of 43279 observations were compared with the corresponding positions computed from the theories described in Sect. 3. Of these, 2809 were rejected because their observed-minus-computed residual was greater than $1\overset{''}{.}0$, leaving 40470 observations which contributed to the normal equations. In Table 5 we give the root-mean-square residuals of several important data sets which were included in this analysis. We defer a more complete discussion of the comparative quality of the datasets to Paper II.

**Table 5.** Root-mean-square residuals of several major data sets used in this analysis. "Ref" denotes the reference number of the data set given in Strugnell & Taylor (1990). "Datum type 1" denotes observations of Right Ascension ($\Delta\alpha$), $\cos\delta\Delta\alpha$ or position angle. "Datum type 2" denotes observations of Declination ($\Delta\delta$) or separation

| Ref. | Observer | Datum type 1 | | | Datum type 2 | | |
|---|---|---|---|---|---|---|---|
| | | $N_{used}$ | $N_{total}$ | RMS | $N_{used}$ | $N_{total}$ | RMS |
| 1 | USNO (1877-1887) | 533 | 650 | 0."40 | 543 | 619 | 0."40 |
| 3 | USNO (1911) | 1838 | 1962 | 0."26 | 1881 | 1949 | 0."29 |
| 4 | USNO (1929) | 1196 | 1207 | 0."19 | 1195 | 1198 | 0."18 |
| 5 | USNO (1954) | 1256 | 1301 | 0."22 | 1261 | 1301 | 0."23 |
| 6 | Struve (1933) | 1360 | 1462 | 0."40 | 1434 | 1443 | 0."20 |
| 9 | Struve (1898) | 906 | 907 | 0."14 | 872 | 872 | 0."15 |
| 10 | Alden & O'Connell (1928) | 297 | 297 | 0."14 | 291 | 291 | 0."11 |
| 18 | Kisseleva et al. (1977) | 174 | 184 | 0."17 | 182 | 184 | 0."16 |
| 26 | Sinclair (1974,1977) | 430 | 434 | 0."13 | 430 | 434 | 0."12 |
| 31 | Pascu (1982) | 1163 | 1173 | 0."14 | 1164 | 1173 | 0."12 |
| 33 | Tolbin (1985) | 363 | 363 | 0."12 | 361 | 363 | 0."16 |
| 46 | Dourneau et al. (1989) | 473 | 474 | 0."32 | 474 | 474 | 0."25 |
| 47 | Veillet & Dourneau (1992) | 148 | 148 | 0."23 | 148 | 148 | 0."13 |
| 48 | Veillet & Dourneau (1992) | 698 | 698 | 0."15 | 698 | 698 | 0."11 |

## 6. Conclusions

We have determined the Keplerian elements and other orbital parameters of the major satellites of Saturn. Our values agree well with those of other authors, notably Dourneau (1987) who analysed observations covering a similar time span. Our standard errors are slightly smaller than those of Dourneau; this can be attributed to the fact that our data set is somewhat larger than that used by Dourneau and includes Pascu's (1982) large collection of high-quality observations. In addition, we have weighted the observations according to their source. This means that our solution is not degraded by datasets of poor quality.

The standard errors in the elements (particularly the mean longitude at the epoch) and the range of values determined by other authors both indicate that analysis of ground-based observations is limited mainly by the precision of the observations themselves, not by the current theories of the motions of the satellites. In particular, the standard error in the mean longitude at epoch is of the order of 300 kilometres.

We have determined the masses of the four inner satellites by analysis of the frequencies and amplitudes of the resonances which characterise their motions. Our values agree closely with those of Dourneau (1987). We have also determined the dynamical form factors $J_2$ and $J_4$ of Saturn. Our values are in good agreement with those derived in this paper from Dourneau's (1987) elements. However, the standard error in $J_4$ is rather large and the value may not be reliable. Furthermore, it was found that the values of $J_2$ and $J_4$ were closely correlated when determined from the node rates of the four inner satellites.

We have analysed observations of Mimas using a theory which includes the long-period terms calculated by Vienne et al. (1991). The inclusion of these terms makes a small but significant difference to the coefficient of the secular acceleration in the mean longitude whose existence was first suggested by Kozai (1957). It seems likely that terms involving the eccentricity of Tethys should be included in the mean longitude of Mimas in order to reduce the observed discrepancy between observation and theory.

The Royal Greenwich Observatory, Queen Mary and Westfield College and the University of Liverpool have an ongoing observing program of the major satellites of the outer planets. This program was inaugurated in 1987 and uses a CCD detector on the 1m Jacobus Kapteyn Telescope at the Observatorio del Roque de los Muchachos on the island of La Palma. The observations are currently being reduced and will be incorporated with those used in this paper to further improve the orbits of Saturn's satellites.

*Acknowledgements.* We would like to thank the referee, Dr. M. Rapaport, for his helpful comments on the manuscript. We are grateful to Dr Dan Pascu and Dr Gérard Dourneau for permission to use their extensive unpublished collections of observations of the satellites of Saturn. We are also grateful to Jim Message, Carl Murray and Andrew Sinclair for their helpful comments on earlier drafts of this paper. This work was supported by the Science and Engineering Research Council.

## Appendix A: Theories of the satellites

Note on notation: we use the following symbols to denote the time since the epoch of the theories (JED 2426000.5), which is 1930 January $24^{d}0^{h}$ Ephemeris Time.

$$d = \text{JED} - 2426000.5,$$
$$t = d/365.25,$$
$$T = d/36525,$$

so that $d$ is measured in days, $t$ in Julian years and $T$ in Julian centuries. Additional time arguments appear in the theories of several of the satellites and are explained where they are used.

*The theory of Mimas*

$$\begin{aligned} a &= a_0, \\ \lambda &= L_0 + n \cdot d + \delta L, \\ e &= e_0, \\ P &= P_0 + \dot{P}t, \\ \gamma &= \gamma_0, \\ N &= N_0 + \dot{N}t, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \delta L &= A_1 \sin\psi - 0^\circ\!.72 \sin 3\psi - 0^\circ\!.02144 \sin 5\psi, \\ \psi &= \nu_{13} \cdot (\tau - \tau_0), \\ \tau &= 1950.0 + (\text{JED} - 2433282.423)/365.2422. \end{aligned} \tag{2}$$

The free parameters are:

$$\{a_0, L_0, e_0, P_0, \gamma_0, N_0, n, \dot{P}, \dot{N}, A_1, \nu_{13}, \tau_0\}.$$

*The theory of Enceladus*

$$\begin{aligned} a &= a_0, \\ \lambda &= L_0 + n \cdot d + \delta L, \\ e &= e_0, \\ P &= 2\lambda_4 - \lambda_2, \\ \gamma &= \gamma_0, \\ N &= N_0 + \dot{N}t, \end{aligned} \tag{3}$$

where

$$\delta L = p_2 \sin(\nu_{24}t + \mu_{24}) + 12'\!.53 \sin(2\lambda_4 - \lambda_2 - \varpi_4). \tag{4}$$

Only the linear parts of $\lambda_2$ and $\lambda_4$ are used when calculating $P$ and the argument of the long-period term. The free parameters are:

$$\{a_0, L_0, e_0, \gamma_0, N_0, n, \dot{N}, p_2, \nu_{24}, \mu_{24}\}.$$

*The theory of Tethys*

$$\begin{aligned} a &= a_0, \\ \lambda &= L_0 + n \cdot d + \delta L, \\ e &= e_0, \\ P &= P_0 + \dot{P}t, \\ \gamma &= \gamma_0, \\ N &= N_0 + \dot{N}t, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \delta L &= -x_{13} \cdot (A_1 \sin\psi - 0^\circ\!.72 \sin 3\psi - 0^\circ\!.02144 \sin 5\psi), \\ \psi &= \nu_{13} \cdot (\tau - \tau_0), \\ \tau &= 1950.0 + (\text{JED} - 2433282.423)/365.2422. \end{aligned} \tag{6}$$

The free parameters are:

$$\{a_0, L_0, e_0, \gamma_0, N_0, n, \dot{N}, x_{13}, A_1, \nu_{13}, \tau_0\}.$$

*The theory of Dione*

$$\begin{aligned} a &= a_0, \\ \lambda &= L_0 + n \cdot d + \delta L, \\ e &= e_0, \\ P &= P_0 + \dot{P}t, \\ \gamma &= \gamma_0, \\ N &= N_0 + \dot{N}t, \end{aligned} \tag{7}$$

where

$$\delta L = p_4 \sin(\nu_{24}t + \mu_{24}) - 1'\!.04 \sin(2\lambda_4 - \lambda_2 - \varpi_4). \tag{8}$$

Only the linear parts of $\lambda_2$ and $\lambda_4$ are used when calculating the argument of the long-period term. The free parameters are:

$$\{a_0, L_0, e_0, P_0, \gamma_0, N_0, n, \dot{P}, \dot{N}, p_4, \nu_{24}, \mu_{24}\}.$$

*The theory of Rhea*

$$\begin{aligned} a &= a_0 \\ \lambda &= \lambda_0 + n \cdot d + \kappa \sin\gamma_0 \tan\frac{i_e}{2} \sin N, \\ e \sin\varpi &= e_0 \sin\pi + 0.00100 \sin\varpi_6, \\ e \cos\varpi &= e_0 \cos\pi + 0.00100 \cos\varpi_6, \\ i &= i_e - 0^\circ\!.04550 + \kappa \sin\gamma_0 \cos N + 0^\circ\!.02007 \cos N_6, \\ \Omega &= \Omega_e - 0^\circ\!.007792 \\ &\quad + (\kappa \sin\gamma_0 \sin N + 0^\circ\!.02007 \sin N_6)/\sin i_e, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \pi &= \pi_0 + 10^\circ\!.057t, \\ N &= N_0 - 10^\circ\!.057t, \\ \varpi_6 &= (\varpi_0)_6 + \dot{\varpi}_6 t, \\ N_6 &= (N_0)_6 + \dot{N}_6 t. \end{aligned} \tag{10}$$

Note that $\kappa = 57^\circ\!.29578$ is a conversion factor from radians to degrees, and $\varpi_6$ and $N_6$ are the osculating apse and node longitudes of Titan. The free parameters are:

$$\{a_0, \lambda_0, e_0, \pi_0, \gamma_0, N_0, n\}.$$

*The theory of Titan*

$$a = a_0, \tag{11}$$

$$\lambda = \lambda_0 + n \cdot d + \kappa \left\{ \sin\gamma_0 \tan\frac{i_e}{2} \sin N - 0.0001757 \sin \ell_s \right.$$
$$\left. -0.0002151 \sin 2L_s + 0.0000567 \sin(2L_s + \Psi) \right\},$$
$$e = e_0 - 0.0001841 \cos 2g + 0.0000731 \cos 2(L_s - g),$$
$$\varpi = \varpi_a + \kappa \left\{0.0063044 \sin 2g + 0.0025027 \sin 2(L_s - g)\right\},$$
$$i = i_a + 0.0002320\kappa \cos(2L_s + \Psi),$$
$$\Omega = \Omega_a + 0.0005034\kappa \sin(2L_s + \Psi),$$

where approximate values of the apse, inclination and node of Titan are given by

$$\begin{aligned} i_a &= i_e - 0.^\circ 6204 + \kappa \sin\gamma_0 \cos N, \\ \Omega_a &= \Omega_e - 0.^\circ 1418 + \kappa \sin\gamma_0 \sin N / \sin i_e, \\ \varpi_a &= \varpi_0 + \dot\varpi t, \end{aligned} \tag{12}$$

the angle $N$ is given by

$$N = N_0 + \dot N t, \tag{13}$$

where it is assumed that $\dot N \equiv -\dot\varpi$, and the auxiliary angles for solar perturbations (as defined in Sinclair 1977) are

$$\begin{aligned} \ell_s &= 175.^\circ 4762 + 1221.^\circ 5515 T_s - 0.^\circ 0005 T_s^2, \\ \lambda_s &= 267.^\circ 2635 + 1222.^\circ 1136 T_s, \\ i_s &= 2.^\circ 489139 + 0.^\circ 002435 T_s - 0.^\circ 000034 T_s^2, \\ \Omega_s &= 113.^\circ 349952 - 0.^\circ 259679 T_s - 0.^\circ 000038 T_s^2, \end{aligned} \tag{14}$$

in which the time argument is given by

$$T_s = (\mathrm{JED} - 2415020.0)/36525.$$

The other auxiliary angles $\Psi$ and $g$ are defined by

$$\begin{aligned} \cos\Gamma &= \cos i_s \cos i_a + \sin i_s \sin i_a \cos(\Omega_a - \Omega_s), \\ \sin\Gamma \sin\Psi &= \sin i_s \sin(\Omega_a - \Omega_s), \\ \sin\Gamma \cos\Psi &= \cos i_s \sin i_a - \sin i_s \cos i_a \cos(\Omega_a - \Omega_s), \\ \sin\Gamma \sin\Theta' &= \sin i_a \sin(\Omega_a - \Omega_s), \\ \sin\Gamma \cos\Theta' &= -\sin i_s \cos i_a + \cos i_s \sin i_a \cos(\Omega_a - \Omega_s), \\ \Theta &= \Theta' + \Omega_s, \\ L_s &= \lambda_s - (\Theta - \Omega_s) - \Omega_s, \\ g &= \varpi_a - \Omega_a - \Psi. \end{aligned} \tag{15}$$

The factor $\kappa = 57.^\circ 29578$ is a conversion factor from radians to degrees.

The free parameters are:

$$\{a_0, L_0, e_0, \varpi_0, \gamma_0, N_0, n, \dot\varpi\}.$$

*The theory of Iapetus*

This theory is a further (unpublished) revision of the theory of Sinclair (1974) undertaken by Harper, Taylor and Sinclair.

$$\begin{aligned} a &= a_0 + \delta a, \\ \lambda &= \lambda_0 + n \cdot d + \delta\lambda, \\ e &= e_0 + \dot e T + \delta e, \\ \varpi &= \varpi_0 + \dot\varpi \cdot T + \delta\varpi, \\ i &= i_0 + i_a T + i_b T^2 + i_c T^3 + \delta i, \\ \Omega &= \Omega_0 + \dot\Omega \cdot T + \Omega_b T^2 + \Omega_c T^3 + \delta\Omega. \end{aligned} \tag{16}$$

The polynomial terms in the eccentricity, inclination and node represent Sinclair's approximate expressions for the periodic perturbations of very long period, in which the coefficients have the following values:

$$\begin{aligned} \dot e &= 0.001156, \\ i_a &= -1.^\circ 0125, \\ i_b &= -0.^\circ 0648, \\ i_c &= +0.^\circ 0054, \\ \Omega_b &= +0.^\circ 127, \\ \Omega_c &= +0.^\circ 008. \end{aligned} \tag{17}$$

The periodic perturbations in the elements are due to the effect of Titan and the Sun, and they are given by

$$\begin{aligned} 10^5 \delta a / a_0 = {} & 7.87 \cos(2\ell + 2g - 2\ell_s - 2g_s) \\ & + 98.79 \cos(\ell + g_1 - \ell_T - g_T), \end{aligned} \tag{18}$$

$$\begin{aligned} \delta\lambda = {} & -0.^\circ 04299 \sin(\ell + g_1 - l_T - g_T) \\ & - 0.^\circ 00356 \sin(5\ell - \ell_T + 5g_1 - g_T) \\ & - 0.^\circ 00087 \sin(5\ell - \ell_T + 5g_1 - 3g_T) \\ & + 0.^\circ 00519 \sin(5\ell - \ell_T + 4g_1 - 2g_T) \\ & - 0.^\circ 00794 \sin(5\ell - \ell_T + 3g_1 - g_T) \\ & - 0.^\circ 00789 \sin(2\ell + 2g - 2\ell_s - 2g_s) \\ & - 0.^\circ 06312 \sin\ell_s - 0.^\circ 00295 \sin 2\ell_s \\ & - 0.^\circ 02231 \sin(2\ell_s + 2g_s) \\ & + 0.^\circ 00650 \sin(2\ell_s + 2g_s + \theta), \end{aligned} \tag{19}$$

$$\begin{aligned} 10^5 \delta e = {} & -140.97 \cos(g_1 - g_T) + 24.08 \cos\ell \\ & + 37.33 \cos(2\ell_s + 2g_s - 2g) \\ & + 0.50 \cos(3\ell_s + 2g_s - 2g) \\ & + 11.80 \cos(\ell + 2g - 2\ell_s - 2g_s) \\ & + 28.49 \cos(2\ell + g_1 - l_T - g_T) \\ & + 61.90 \cos(\ell_T + g_T - g_1), \end{aligned} \tag{20}$$

$$\begin{aligned} e\,\delta\varpi = {} & 0.^\circ 08077 \sin(g_1 - g_T) \\ & + 0.^\circ 02139 \sin(2\ell_s + 2g_s - 2g) \\ & + 0.^\circ 00028 \sin(3\ell_s + 2g_s - 2g) \\ & - 0.^\circ 00676 \sin(\ell + 2g - 2\ell_s - 2g_s) \\ & + 0.^\circ 01380 \sin\ell \\ & + 0.^\circ 01632 \sin(2\ell + g_1 - \ell_T - g_T) \\ & + 0.^\circ 03547 \sin(\ell_T + g_T - g_1), \end{aligned} \tag{21}$$

$$
\begin{aligned}
\delta i = {} & 0\overset{\circ}{.}00106 \cos \ell_s \\
& - 0\overset{\circ}{.}00242 \cos(\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}04204 \cos(2\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00565 \cos(3\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00057 \cos(4\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00035 \cos(2\ell_s + 2g_s + 2g + \theta) \\
& + 0\overset{\circ}{.}00235 \cos(\ell + g_1 + \ell_T + g_T + \phi) \\
& + 0\overset{\circ}{.}00360 \cos(\ell + g_1 - \ell_T - g_T - \phi),
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\sin i\, \delta\Omega = {} & -0\overset{\circ}{.}01449 \sin \ell_s - 0\overset{\circ}{.}00060 \sin 2\ell_s \\
& + 0\overset{\circ}{.}00242 \sin(\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}04204 \sin(2\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00565 \sin(3\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00057 \sin(4\ell_s + 2g_s + \theta) \\
& + 0\overset{\circ}{.}00035 \sin(2\ell_s + 2g_s + 2g + \theta) \\
& + 0\overset{\circ}{.}00235 \sin(\ell + g_1 + \ell_T + g_T + \phi) \\
& + 0\overset{\circ}{.}00358 \sin(\ell + g_1 - \ell_T - g_T - \phi),
\end{aligned} \tag{23}
$$

The arguments for the periodic perturbations are

$$
\begin{aligned}
\lambda_s &= 267\overset{\circ}{.}263 + 1221\overset{\circ}{.}114 T_s, \\
\varpi_s &= 91\overset{\circ}{.}796 + 0\overset{\circ}{.}562 T_s, \\
\theta &= 4\overset{\circ}{.}367 - 0\overset{\circ}{.}195 T_s, \\
\Theta &= 146\overset{\circ}{.}819 - 3\overset{\circ}{.}918 T_s, \\
\lambda_T &= 261\overset{\circ}{.}319 + 22\overset{\circ}{.}576974(\mathrm{JED} - 2411368.0), \\
\varpi_T &= 277\overset{\circ}{.}102 + 0\overset{\circ}{.}001389(\mathrm{JED} - 2411368.0), \\
\phi &= 60\overset{\circ}{.}470 + 1\overset{\circ}{.}521 T_s, \\
\Phi &= 205\overset{\circ}{.}055 - 2\overset{\circ}{.}091 T_s,
\end{aligned} \tag{24}
$$

in which the time argument $T_s$ is given by

$$T_s = (\mathrm{JED} - 2415020.0)/36525\,. \tag{25}$$

From these, the auxiliary angles are computed as follows:

$$
\begin{aligned}
\ell &= \lambda - \varpi, \\
g &= \varpi - \Omega - \theta, \\
g_1 &= \varpi - \Omega - \phi, \\
\ell_s &= \lambda_s - \varpi_s, \\
g_s &= \varpi_s - \Theta, \\
\ell_T &= \lambda_T - \varpi_T, \\
g_T &= \varpi_T - \Phi.
\end{aligned} \tag{26}
$$

The free parameters are

$$\{a_0, \lambda_0, e_0, \varpi_0, i_0, \Omega_0, n, \dot{\varpi}, \dot{\Omega}\}.$$

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