STATISTICS OF COHOMOLOGICAL AUTOMORPHIC REPRESENTATIONS ON UNITARY GROUPS VIA THE ENDOSCOPIC CLASSIFICATION

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ABSTRACT. Consider the family of automorphic representations on some unitary group with fixed (possibly non-tempered) cohomological representation π_0 at infinity and level dividing some finite upper bound. We compute statistics of this family as the level restriction goes to infinity. For unramified unitary groups and a large class of π_0, we are able to compute the exact leading term for both counts of representations and averages of Satake parameters. We get bounds on our error term similar to previous work by Shin-Templier that studied the case of discrete series at infinity.

This provides many corollaries: for example, we get new exact asymptotics on the growth of certain degrees of cohomology in certain towers of locally symmetric spaces, prove an averaged Sato-Tate equidistribution law for spectral families with specific non-tempered cohomological components at infinity, and extend bounds of Marshall and Shin to prove Sarnak-Xue density for cohomological representations at infinity on all unitary groups that don't have a $U(2,2)$ factor over infinity.

The main technical tool is an extension of an inductive argument that was originally developed by Taïbi to count unramified representations on Sp and SO and used the endoscopic classification of representations (which our case requires for non-quasisplit unitary groups).

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1. Introduction

1.1. Context.

1.1.1. *Statistics in general.* Let G be a reductive group over a number field F. Automorphic representations for G are very roughly irreducible subrepresentations of $L^2(G(F)\backslash G(\mathbb{A}))$ under right multiplication by $G(\mathbb{A})$. While this definition may seem unmotivated, automorphic representations encode information about many important applications—for example Galois representations through the Langlands program, so-called expanders used for computer algorithms through constructions akin to Ramanujan graphs, and explanations for geometric properties of locally symmetric spaces through automorphic decompositions of their cohomology.

Every automorphic representation π has a tensor product decomposition

$$\pi = {\bigotimes_v}' \pi_v$$

into unirreps π_v of G_v over places v of F. Applications usually depend on a key question: which combinations of π_v actually tensor together into an automorphic representation; i.e. which products are represented as functions in $L^2(G(F)\backslash G(\mathbb{A}))$?

This paper broadly focuses on an easier version of this key question—that of computing *statistics*. We will consider families $\mathcal{F}_i$ of automorphic representations satisfying local conditions on each π_v. Then we try to estimate as well as possible the asymptotics of the count of such representations weighted by some parameter of the representations when the local conditions "go to infinity" in some sense. For example, when $G = \mathrm{GL}_2$, this could correspond to looking at the "mth moment" of the Hecke eigenvalue at some place p averaged over weight-k, level-N holomorphic modular forms as $N \to \infty$. Problems of statistics are usually related to studying automorphic representations in spectral families in the precise sense of [SST16].

1.1.2. *Our specific case.* More specifically, we are interested in the specific case of restricting the component π_∞ at infinity to a specific representation—this can be thought of as fixing the "qualitative type" of an automorphic representation; for example, "holomorphic Siegel modular form of weight $\vec{v}$". We organize such questions by an informal ranking of the complexity of an automorphic representation based on the complexity of the component π_∞ [1].

The simplest automorphic representations correspond to the simplest components at infinity: the discrete series that can be realized as explicit subrepresentations of $L^2(G_\infty)$. Statistics of these discrete-at-infinity automorphic representations are well understood through Arthur's trace formula. Specifically, [Art89a] developed an explicit trace formula for studying them. The techniques of first and most importantly [ST16] and second [Fer07] as used in first author's thesis [Dal22] build on this explicit trace formula to provide good asymptotic estimates and error terms in the general discrete-at-infinity case (as far as the authors are aware, these are the strongest bounds known for the general case).

This paper goes beyond discrete series and studies automorphic representations with cohomological component at infinity—i.e. component at infinity that has some non-trivial $(\mathfrak{g}, K)$-cohomology. There is a critical new complication here:

[1] There are of course many powerful automorphic statistics results that don't fit into this story—as a good "most general" representative [FLM15] considers π_∞ contained in any subset of the unitary dual with finite, non-zero volume. We are not giving a full literature review here

non-discrete cohomological representations are always non-tempered. In particular, general cohomological automorphic representations can represent violations of the Ramanujan conjecture. They can therefore be very sparse in the automorphic spectrum and difficult to isolate.

Recent results suggest that known cases of the endoscopic classification—e.g. [Art13], [Mok15], and [KMSW14]—are a good way to study general cohomological automorphic representations. Work of Marshall, Shin and the second author have used it to provide good upper bounds for counts on unitary groups, see [Mar14], [MS19] and [GG21]. In specific simpler cases, explicit counts have even been computed: [CR15] and [Taï17] consider the case of level-1 representations on classical groups and [RSY22] develops techniques that apply to the case of low-level automorphic representations on Sp_4.

This work attempts to organize and synthesize the bounds of Marshal, Shin, and the second author together with an inductive analysis used in the work of Taïbi into a proposal for a general method to understand statistics of cohomological automorphic representations. While our proposed method in general depends on some wide-open and difficult problems in local representation theory, we are able to explicitly implement it in some specific cases on unitary groups. This gives a more general understanding of cohomological asymptotic statistics on unitary groups than any previous work.

We emphasize in particular that, in many cases, we are able to compute exact leading terms together with estimates on sub-leading terms. This gives us applications towards results like the Sarnak-Xue density conjecture, the growth of cohomology of locally symmetric spaces, and Sato-Tate equidistribution averaged over families of automorphic representations. We motivate some of these in more detail:

1.1.3. *Application: Growth of Cohomology.* A motivating problem for these statistical computation is that of the growth of cohomology in towers of arithmetic groups. Let Γ be an arithmetic lattice in $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$, and assume that Γ is both neat and cocompact for simplicity. Then the group cohomology of Γ (or equivalently, de Rham cohomlogy of the locally symmetric space $\Gamma\backslash G(F_\infty)/K_\infty$ for K_∞ a maximal compact subgroup) is computed in terms of automorphic forms via Matsushima's formula:

$$H^*(\Gamma, \mathbb{C}) = \bigoplus_{\pi_\infty \text{ cohomological}} m(\pi_\infty, \Gamma) H^*(\mathfrak{g}, K_\infty; \pi_\infty),$$

where $m(\pi_\infty, \Gamma) = \dim \mathrm{Hom}_{G_\infty}(\pi_\infty, L^2(\Gamma\backslash G_\infty))$. Beyond the fact that locally symmetric space provides a rich class of examples of manifolds, the cohomology $H^*(\Gamma, \mathbb{C})$ is also of interest because it carries an action of Hecke operators, making it a generalization of modular forms. The Hecke eigensystems that arise should correspond to Galois representations via the Langlands program.

Outside of some low-rank examples, dimensions of $H^*(\Gamma, \mathbb{C})$ are only known for specific lattices Γ, see for example [AGM08, GMY21]. A fruitful approach to studying the cohomology of Γ is doing so *in towers*: one fixes a sequence Γ_n of typically nested normal subgroups, and studies the asymptotics of $\dim H^*(\Gamma_n, \mathbb{C})$ as $n \to \infty$. Without giving a systematic survey of this problem, there have been multiple approaches: topological constructions, for example [ST22], non-abelian Iwasawa theory methods as in [CE09], and, beginning with the work of DeGeorge-Wallach [DW78], what can be referred to as spectral approaches.

As shown in [VZ84], cohomological discrete series representations only contribute to cohomology in degree $(1/2)\dim(G_\infty/K_\infty)$. Thus the exact asymptotics of [DW78], and later [Clo86] and [Sav89], show that the middle degree of cohomology grows like the volume of the corresponding symmetric space when G_∞ has discrete series, and that the growth of the lower degrees is slower. This leaves open the question of more precise upper bounds for other degrees of cohomology, and it seems (see [SX91, §1]) that this motivating question was at the heart of the discussion which led to the formulation of the Sarnak-Xue conjecture discussed in 1.1.5.

Progress on bounding growth of Betti numbers has been made in various directions, from the lower bounds of [Cos09, CM13, SK15, ST22] to the vanishing results of [CD06, Clo93] to extension to of the DeGeorge-Wallach results to more general sequences of lattices in [ABB$^+$17]. Starting with [SX91], there has also been progress on upper bounds: [CE09] have obtained a power saving for lattices in any group G_∞ admitting discrete series. Most influential for us is a series [CM13, Mar14, Mar16] of work of Simon Marshall and his collaborators, culminating in [MS19], in which Marshall-Shin give upper bounds for all degrees of cohomology for lattices in $U(N-1,1)$. Among other results in this article, we show that the bounds of Marshall-Shin are sharp in every other degree.

1.1.4. *Application: Averaged Sato-Tate.* Let π be an automorphic representation on a reductive group G. A Sato-Tate result for π is a statement that the Satake parameters for unramified components π_v are equidistributed over v according to a Sato-Tate law μ_π determined by some properties of π. This should be thought of as a generalized, automorphic-side analogue of the classical Sato-Tate conjecture (proved in [BLGHT11]) for the equidistribution over p of the coefficients a_p associated to point counts over $\mathbb{Z}/p$ of elliptic curves. See [ST16, §1.1] for a full introduction to the problem (in particular, [Ser94] state's a very general Galois-side version of the conjecture and [BLGG11], [BLGHT11] proves the conjecture for restrictions of scalars of GL_1 and GL_2).

Unfortunately, even stating what this Sato-Tate distribution for a single π should be depends on more-or-less the full conjecture of Langlands functoriality. Extremely roughly, π should correspond to another reductive group H_π and L-map $\varphi : {}^LH_\pi \hookrightarrow {}^LG$—the smallest such that the conjectural global L-parameter for π factors through φ. The law μ_π should then be thought of as pushforward of a μ_H determined by H from the space of Satake parameters of H to that of G. For general π on high-rank groups, Sato-Tate results therefore appear unapproachable. In fact, empirically measuring the Sato-Tate distribution for π is arguably one of the key currently accessible pieces of evidence for the existence of this conjectural H_π in the first place.

Following [ST16], we therefore instead try to study Sato-Tate laws averaged over some increasing sequence of families $\mathcal{F}_i$. Heuristically, representations $\pi \in \mathcal{F}$ can have many different H_π. However, for reasonable families, we expect the count of $\pi \in \mathcal{F}_i$ with $H_\pi = H_0$ to have growth rate proportional to the dimension of H_0. Therefore, we should expect most $\pi \in \mathcal{F}_i$ to have H_π be some maximum value $H^{\max}_{\mathcal{F}}$. In particular, if we look at Satake parameters π_v over all places v *and* all $\pi \in \mathcal{F}_i$, we should expect the distribution for $H_\pi = H^{\max}_{\mathcal{F}}$ to dominate.

Such "averaged Sato-Tate laws" end up being far easier to establish. For example, [ST16] studied families of automorphic representations on G with discrete series at infinity and showed they satisfied averaged Sato-Tate laws coming from G itself.

This corresponds to the heuristic that most π with discrete series at infinity should be "primitive", i.e. have $H_\pi = G$.

In section 12.3, we instead study families with certain "odd GSK-maxed" (defined in 11.2.11) cohomological components π_0 at infinity that are possibly non-tempered. In this case, the resulting averaged Sato-Tate laws are not those from G itself, but rather certain H_{π_0} that we explicitly compute from π_0. The pairs (φ, H_{π_0}) are not in general endoscopic embeddings, but instead compositions of endoscopic embeddings ${}^LH \hookrightarrow {}^LG$ and tensor-product maps analogous to $\mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{nm}$. This can be taken as evidence for a speculative interpretation that is nevertheless clearly suggested by the form of endoscopic classification: that the decomposition of an A-parameter in terms of cuspidal parameters is literally realizing the corresponding packet as a functorial transfer from a smaller, *non-endoscopic* group.

1.1.5. *Application: Sarnak-Xue Density.* The Sarnak-Xue density conjecture was motivated as a replacement for the naïve generalized Ramanujan conjecture—that if π is a cuspidal automorphic representation of reductive group G, all its local components π_v are tempered. This should be thought of as the generalization and translation to representation theoretic, adelic language of the classical Ramanujan conjecture bounding Hecke eigenvalues of the Δ function—see [Sar05] for a full introduction. Of course, the naïve Ramanujan was found not to be true even for split G due to counterexamples constructed in [HPS79].

Luckily, many desired applications don't require there to be no non-tempered representations: they only require there to be not too many. Sarnak and Xue in [SX91] conjectured a precise meaning for "too many": the exact, necessary upper bound on the asymptotic growth rate of representations π with component π_v non-tempered. They also proved it in some small-rank cases.

Their original conjecture was stated in terms of classical, real locally symmetric spaces and v an infinite place. We can modernize it slightly and focus it towards our context as follows: Let G/F be reductive and let U_i be a sequence of open compact subgroups of G^∞ decreasing to 1. Let $\Gamma_i = G(F) \cap U_i$ and choose a cohomological representation π_0 of G_∞. Then:

Conjecture 1.1.1 (Cohomological Sarnak-Xue)**.** *Let $m(\pi_0, \Gamma_i \backslash G_\infty)$ be the multiplicity of π_0 in $L^2(\Gamma_i \backslash G_\infty)$. Then for all $\epsilon > 0$,*

$$m(\pi_0, \Gamma_i \backslash G_\infty) \ll_\epsilon \mathrm{vol}(\Gamma_i \backslash G_\infty)^{\frac{2}{p(\pi_0)} + \epsilon}$$

where $p(\pi_0)$ is the infimum over p such that the (spherically finite) matrix coefficients of π_0 are in $L^p(G_\infty)$.

As alluded to in 1.1.3, it is known that if π_0 is discrete-series (i.e. $p(\pi_0) = 2$) then $m(\pi_0, \Gamma_i \backslash G_\infty) \sim \mathrm{vol}(\Gamma_i \backslash G_\infty)$. On the other hand, if π_0 is a character, (i.e. $p(\pi_0) = \infty$), then $m(\pi_0, \Gamma_i \backslash G_\infty) \sim 1$. This is therefore a claim that an asymptotically negligible fraction of automorphic representations have some non-tempered component π_0 at infinity and further that the quantitative strength of "asymptotically negligible" depends on $p(\pi_0)$ and is an interpolation between the cases $p = 2$ and $p = \infty$.

Many analytic applications are discussed in [GK20] including those from [EP22] relating to certain constructions (so-called golden gates and Ramunjuan complexes) used in computer science. There have been many recent breakthroughs proving the conjecture in specific cases: for example, [Blo22] proved a version of the conjecture at infinity for Maass forms on GL_n using the Kuznetsov formula and [FHMM20] proved many versions for Maass forms on products of $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{SL}_2\mathbb{C}$ using Arthur's trace formula.

Most importantly in our context is the work of Marshall and collaborators applying Arthur's classification to the problem at infinity for cohomological π_0 on unitary groups. The most general results are in [MS19] and prove the cohomological Sarnak-Xue conjecture for groups that are $U(N,1)$ at infinity. As one implication of this project, we are able to show that, when inputted into our general framework, Marshall-Shin's bounds extend to prove the Sarnak-Xue conjecture at infinity for cohomological π_0 on *all* unitary groups that don't have a $U(2,2)$ factor at infinity.

1.2. **Results.** To make this all precise, let E/F be a CM extension of number fields. Let G/F be a unitary group that splits over E, so that G has discrete series at infinity. We prove two main results.

First, assume E/F is unramified at all finite places. Let π_0 be a in certain class of good cohomological representations π_0 of G_∞ that satisfy a condition of being odd GSK-maxed (as in definition 11.2.10) and a technical parity condition from lemma 11.3.1. Theorem 11.4.1 then finds explicit constants $R(\pi_0)$ and $M(\pi_0) \neq 0$ such that for principal congruence subgroups U of G^∞,

$$\mathrm{vol}(U)^{R(\pi_0)/\dim G} L(U) \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^\infty)^U) = M(\pi_0) + O(\mathrm{vol}(U)^C)$$

for some correction factor $L(U)$ made precise in the theorem statement and some constant $C \geq 1/\dim G$. The sum should be thought of as the multiplicity of π_0 inside the automorphic quotient $L^2(G(F)\backslash G(\mathbb{A})/U)$ which has volume proportional to $\mathrm{vol}(U)^{-1}$.

The theorem also allows weighting the count by a Weyl-symmetric polynomial P in the Satake parameters s_{π_v} at a place v where U has hyperspecial component:

$$\mathrm{vol}(U)^{R(\pi_0)/\dim G} L(U) \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^\infty)^U) P(s_{\pi_v}) = M(\pi_0)M(P) + O(\mathrm{vol}(U)^C q_v^{A + B \deg P})$$

for some explicit constant $M(P)$ and inexplicit constants A, B.

The second result, theorem 11.4.2, applies for G arbitrary and π_0 arbitrary cohomological. It only provides upper bounds

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^\infty)^U) P(s_{\pi_v}) = O(\mathrm{vol}(U)^{R(\pi_0)} q_v^{A + B \deg P}).$$

When π_0 is discrete series, we recover the "trivial bound" of $R(\pi_0) = \dim G$. Otherwise, we get an improvement $R(\pi_0) < \dim G$.

Beyond some specific lower bounds on classical groups in [Cos09] extended to exact asymptotics in [CM13, cor. 1.3] on specific cohomological representations of symplectic groups, Theorem 11.4.1 seems to be the first exact asymptotic for counts of automorphic representations with non-tempered cohomological representations at infinity. As far as the authors are aware, it is the first with an estimate on the sub-leading term. It therefore gives many new corollaries:

1.2.1. *Corollary: Cohomology.* Our results give bounds for the growth of Betti numbers in towers of arithmetic manifolds. In this context, it is traditional and simplest, though by no means necessary, to fix a G_∞ which is isomorphic to $U(p,q)$, with $p+q=N$, at one infinite place and compact at all the others. The resulting $\Gamma(\mathfrak{n}_i) = G(F) \cap K(\mathfrak{n}_i)$ are then cocompact lattices in $U(p,q)$. Our results in this context imply two types of corollaries.

First, using the bounds of 11.4 and the explicit description of packets of cohomological representations in 11.2 and 11.1, we give an algorithm to compute upper bounds on the growth of the Betti numbers $h^k(\Gamma(\mathfrak{n}_i))$ and of the dimensions $h^{p,q}(\Gamma(\mathfrak{n}_i))$ of any piece of the Hodge decomposition. Though we don't expect the resulting upper bounds to be sharp in general, they should be so in many cases, and they always give a non-trivial power saving when compared to the volume of $X(\mathfrak{n}_i) = \Gamma(\mathfrak{n}_i)\backslash G_\infty/K_\infty$ in degrees strictly below $\frac{1}{2}\dim X(\mathfrak{n}_i)$. The algorithm is described in 12.2.2 and can be outlined as:

(1) Given a degree of cohomology or a Hodge weight, list all representations π_∞ for which the corresponding $(\mathfrak{g}, K)$-cohomology is nonvanishing using the parameterization of 11.1.
(2) For each representation π_∞, compute the shape $\Delta^{\max}(\pi_\infty)$ as in 11.2.2.
(3) Use Theorem 11.4.2 to give upper bounds

$$\dim \operatorname{Hom}(\pi_\infty, L^2(G(F)\backslash G(\mathbb{A})/K(\mathfrak{n}_i)K_\infty)) \ll |\mathfrak{n}_i|^{R(\Delta^{\max}(\pi_\infty))}.$$

(4) From Matsushima's formula and the growth $\gg |\mathfrak{n}_i|^{1-\epsilon}$ of components of $G(F)\backslash G(\mathbb{A})/K(\mathfrak{n}_i)$, deduce upper bounds for $h^k(\Gamma(\mathfrak{n}_i))$.

In practice, the combinatorics of computing the representations which contribute to cohomology in a given degree, as well as the corresponding $\Delta^{\max}(\pi_\infty)$ rapidly get complicated, but in some cases, the bounds can be expressed succinctly. For example, when $r = \min(p,q)$ is the smallest degree carrying the cohomology of a non-trivial representation, these non-trivial contributions appear in weights $(r,0)$ and $(0,r)$ we have

$$h^{r,0}(\Gamma(\mathfrak{n}_i)) + h^{0,r}(\Gamma(\mathfrak{n}_i)) \ll_\epsilon |\mathfrak{n}_i|^N + \epsilon.$$

Secondly, for some range of degrees, our exact asymptotics give lower bounds on a range of degrees of cohomology under the assumption that E/F is unramified. For example, using again $r = \min(p,q)$, we exhibit lattices with the property that for $1 \le j \le |p-q| - 1$ such that $j \not\equiv N \mod 2$, we show in Corollary 12.2.2

$$h^{kj,(r-k)j}(\Gamma(\mathfrak{n}_i)) \gg |\mathfrak{n}_i|^{Nj}, \quad 0 \le k \le r.$$

These results apply to a wider range of the degree when $U(p,q)$ is farther away from being quasisplit. In the extremal case where the noncompact part of G is $U(N-1,1)$, we deduce that in degrees j whose parity is opposite to that of N, the upper bounds of [MS19] are sharp.

1.2.2. *Corollary: Averaged Sato-Tate.* The error bound on our sub-leading term in Theorem 11.4.1 the same as in [ST16] so we can mimic their argument and prove an averaged Sato-Tate law. This is theorem 12.3.3.

More specifically, given an odd GSK-maxed representation π_0 on unramified unitary group G, we compute an unordered sequence of pairs $((T_i, d_i))_{1 \le i \le r}$ that's common to all elements of a set $\Delta^{\max}(\pi_0)$ understood by the algorithm at the end of §11.2.2. To this list of pairs we associate a group

$$H_{\pi_0} = (U_{E/F}(T_1) \times U_{E/F}(1)) \times \cdots \times (U_{E/F}(T_r) \times U_{E/F}(1))$$

and an L-embedding $\varphi : {}^L H_{\pi_0} \to {}^L G$ constructed in three stages: first we embed the second coordinate of each pair into ${}^L U_{E/F}(d_i)$ through the cocharacter corresponding to the parameter of the trivial representation. Then we take the tensor product embedding

$${}^L U_{E/F}(T_1) \times {}^L U_{E/F}(d_i) \to {}^L U_{E/F}(T_i d_i)$$

followed by the diagonal embedding

$$\prod_i {}^L U_{E/F}(T_i d_i) \hookrightarrow {}^L U_{E/F}(T_1 r_1 + \cdots + T_r d_r) = {}^L G.$$

The group ${}^L H_{\pi_0}$ has a canonical Sato-Tate measures on the space of Satake parameters for each splitting type θ of prime in E/F. We let $\mu_\theta^{\mathrm{ST}(\pi_0)}$ be the pushforward of this to G.

For each finite place v of the right splitting type and open compact $U \subseteq G^\infty$, we then define empirical measure:

$$\mu_{U,v}^{\pi_0} = \sum_{\substack{\pi \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim\left((\pi^\infty)^U\right) \delta_{s_{\pi_v}}$$

as a sum of delta-measure on the space of Satake parameters s_{π_v}. Then theorem 12.3.3 show that for certain sequences of U_i and v_i such that $\mathrm{vol}(U)^{-1}$ grows much faster than q_{v_i}, we have weak convergence,

$$C(\pi_0, U_i)^{-1} \mu_{U_i, v_i}^{\pi_0} \to \mu_\theta^{\mathrm{ST}(\pi_0)}$$

for appropriate scaling factor $C(\pi_0, U)$ as long as a parity condition from lemma 11.3.1 holds. This can be heuristically interpreted as evidence that most $\pi \in \mathcal{AR}_{\mathrm{disc}}(G)$ with $\pi_\infty = \pi_0$ are functorial transfers from H_{π_0} through φ.

1.2.3. *Corollary: Sarnak-Xue.* The bounds in theorem 11.4.2 are good enough to achieve the Sarnak-Xue bounds for cohomological component at infinity on all unitary groups without a $U(2,2)$ factor at infinity in theorem 12.4.7. It is very important to mention that this isn't strictly a new result since the local bounds proved in [MS19] were already good enough to achieve the Sarnak-Xue threshold. The only new work here is inputting them into a more general framework that applies beyond $U(N,1)$—absolutely no strengthening is needed.

Nevertheless, we do not expect these bounds to be optimal. Through some heuristics relating to GK-dimension, we conjecture an optimal $R_0(\pi)$ in section 9.6. We compare our bound R, the optimal bound R_0, the Sarnak-Xue threshold, and the trival bound $\dim G$ in many cases in table 12.4.1.

1.2.4. *Conditionality.* As a very important warning, our argument depends heavily on Mok's and Kaletha-Minguez-Shin-White's endoscopic classifications for unitary groups [Mok15] and [KMSW14]. The first depends on some references in Arthur's book [Art13] that are not yet publicly available. The second in addition pushes many technical details to a specific reference "KMSb" that is also not yet publicly available. All these missing references are expected to be completed soon.

1.3. **Summary of Argument.** We prove the result using the Arthur-Selberg trace formula (see [Art05] for a review). From the perspective of this project, it is an attempt to give an explicit "geometric side" formula for

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathrm{tr}_\pi f$$

for some compactly supported, smooth test function f on $G(\mathbb{A})$. There are main obstacles in applying it directly to our statistical problem:

- Given our chosen π_0, we would need to find a smooth compactly supported test function f_∞ such that $\mathrm{tr}_\pi f_\infty = \mathbf{1}_{\pi = \pi_0}$ to pick out only automorphic representation with component π_0 at infinity in this sum. This is the simplest way to understand
$$m(\pi_0, f^\infty) := \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \mathrm{tr}_{\pi^\infty}(f^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathrm{tr}_\pi(f_\infty f^\infty).$$
- The Arthur-Selberg trace formula is not very explicit in general.

Both these obstacles can be removed for the case of π_0 a discrete series. In this case, f_∞ can be chosen to be a pseudocoefficient of [CD90]. The paper [Art89a] (with some addenda in [Fer07]) then showed that the geometric side of Arthur's invariant trace formula $I^G_{\mathrm{disc}}(f_\infty f^\infty)$ simplifies to something explicit with f_∞ a pseudocoefficient. In [ST16] (with some addenda in [Dal22]), this explicit formula was understood well enough for the purposes of computing asymptotics with error terms as we desire.

As soon as we try to generalize to all cohomological π_0, finding f_∞ becomes a much larger issue—in fact, for non-tempered cohomological π_0, there is no such test function by the results of [CD90]. The endoscopic classification of [KMSW14] gives a way out: π_0 is contained some special finite sets of unirreps called A-packets Π_{ψ_∞} attached to A-parameters ψ_∞ at infinity. It turns out that we can find a pseudocoefficient associated to π_0 such that, while it doesn't isolate π_0 amongst all unirreps, it isolates it amongst unirreps that share an A-packet with it (lemma 3.4.3).

Next, the endoscopic classification also gives a decomposition

$$I^G_{\mathrm{disc}}(f_\infty f^\infty) = \sum_{\psi \in \Psi(G)} I^G_\psi(f_\infty f^\infty)$$

into pieces corresponding to global A-parameters ψ. The I_ψ should only involve traces against automorphic representations π such that $\pi_\infty \in \Pi_{\psi_\infty}$ where ψ_∞ is the associated local parameter at infinity. Understanding $m(\pi_0, f^\infty)$ is then reduced to finding an explicit, geometric expression for the part of the trace formula containing only those parts corresponding to A-parameters ψ_∞ with some given ψ_∞ at infinity such that $\pi_0 \in \Pi_{\psi_\infty}$.

We do something slightly different, defining instead a global invariant Δ called the *refined shape* for those A-parameters ψ that can have automorphic representations with cohomological factor at infinity. The Δ's will roughly have two key properties:

- Δ determines the component at infinity
- There is a an inductive method developed in [Taï17] to write
$$I^G_\Delta(f_\infty f^\infty) = \sum_{\psi \in \Delta} I^G_\psi(f_\infty f^\infty)$$

as a linear combination of terms of the form $I^H_{\mathrm{disc}}(f'_\infty (f')^\infty)$ with the f'_∞ pseudocoefficients on smaller groups H—i.e. these are terms that are already understood explicitly by [ST16] and [Dal22].

. Summing the inductive expressions for those Δ that correspond to ψ_∞ with $\pi_0 \in \Pi_{\psi_\infty}$, we would therefore get an explicit geometric formula for our desired

$$m(\pi_0, f^\infty) := \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \mathrm{tr}_{\pi^\infty}(f^\infty).$$

For technical reasons, we instead work with the analogous summand S_Δ of Arthur's stable S_{disc} (see [Art13, §3.1-3.3]). The two end up being more-or-less interchangeable for asymptotics by the "hyperendoscopy" techniques of [Fer07] as used in [Dal22].

Sections 2 and 3 give the necessary background material; §2 focused on the endoscopic classification and §3 on the real representation theory surrounding cohomological representations, their A-packets, and the pseudocoefficients. The definition of refined shape is made and understood in §4 while the inductive procedure understanding S_Δ is explained in §5. Going from S_Δ to $m(\pi_0, f^\infty)$ is the work of §10.

Unfortunately, serious technical obstacles intrude. The inductive procedure of §5 in general requires the construction and computation of certain transfers of f^∞: the conjectural "stable" and "Speh" transfers described in §6.4. We do not even attempt to understand these in general. Instead, throughout §6, we compute them in various trivial (i.e. already-known) cases while otherwise proving only inequalities. This is the main barrier towards generalizing theorem 11.4.1 to either E/F that are ramified somewhere or π_0 that aren't GSK-maxed. It is also the main barrier to generalizing our techniques to quasisplit symplectic and orthogonal groups.

Computing transfers isn't only technical obstruction: the sign $\epsilon_\psi(s_\psi)$ in the stable multiplicity formula 2.6.3 also confounds the inductive expansion of S_Δ. We are therefore only able to compute exact asymptotics for Δ such that this sign is always positive, only proving inequalities otherwise. This is the main barrier towards theorem 11.4.1 applying to all GSK-maxed π_0 instead of just the odd GSK-maxed ones.

Sections 7 and 8 make up the technical work of squeezing as much information about S_Δ as possible from this partial information about the inductive expansion. The key conclusions are propositions 7.3.3 on general Δ and 8.2.2 on a special class of Δ called odd GSK. A reformulation of the main technical trick of [GG21] plays a very important role as proposition 7.1.2. Finally, section 9 completes the full inductive analysis of S_Δ, concluding with the main technical results of the paper in Theorems 9.3.2 and 9.5.1. The later requires one final detail: a strengthening of 9.3.2 to corollary 9.4.4 using local bounds from [MS19] reformulated as lemmas 9.4.2 and 9.4.3. In §9.6, we highlight a heuristic for and possible strategy to prove conjectural optimal versions 9.6.3/9.6.4 of corollary 9.4.4.

All that remains is the previously mentioned work in §10 writing $m(\Delta, f^\infty)$ in terms of S_Δ and the computations in §11 using a parameterization of cohomological representations on unitary groups to get explicit numbers for $R(\pi_0)$ and $M(\pi_0)$. This produces our main results: exact asymptotic Theorem 11.4.1 and upper bound Theorem 11.4.2. The last section, 12, first computes more details for specific example π_0 and second applies the two main theorems to the problems of growth of cohomology, Sato-Tate equidistribution in families, and Sarnak-Xue density.

1.3.1. *Possible Extensions.* We specifically highlight the barriers towards generalizing our result further. The first is generalizing the computation of the stable transfer of indicators of congruence subgroups in lemma 6.1.2 to non-split places. This would of course allow us to extend theorems 11.4.1 and 11.4.2 to $\mathfrak{n}_i$ divisible by non-split primes. More importantly, it would allow extending the techniques here to the case of quasisplit symplectic and orthogonal G, where there are no "split places" v such that $G_v \cong \mathrm{GL}_n$.

Next, improving the bounds 6.1.3, 9.4.1, 9.4.2, and 9.4.3 on Speh transfers of indicators of congruence subgroups would allow tightening the bound in Theorem 11.4.2, possibly even to the conjectural optimal value 9.6.3. As explained in §9.6, it would help tremendously to have a good enough understanding of the local character expansions of Speh representations through the rich interplay of ideas involving generalized Whittaker models and A-parameters as studied in [MW87] and [JLZ22]. Beyond even this, in the dream case where exact formulas can be computed, Theorem 11.4.1 may even be extended to general π_0 instead of just those that are GSK-maxed.

Relatedly, 11.4.1 is restricted to π_0 that are odd GSK-maxed instead of generally GSK-maxed because of our inability to control signs $\epsilon_\psi(s_\psi)$ that appear in the stable multiplicity formula 2.6.3. Proving that these signs cause any cancellation at all in S_Δ for Δ where they are non-trivial would allow us to remove the "odd" restriction. See the discussion around conjecture 9.6.4.

Finally, even just proving the existence of Stable and Speh transfers for functions at ramified places as in §6.4 would allow extending Theorem 11.4.1 to unitary groups for ramified quadratic extensions E/F.

1.4. **How to Read.** Sections 2 and 3 are background material that can mostly be skipped by experts. Sections 4 and 5 are the conceptual heart of the paper and should be understood very well in full detail before moving on. Sections 6, 7, and 8 are technical details involved in implementing the strategy of §5. We recommend skipping them on a first read through and referring back depending on need/interest while reading §9, 10. The most important results from the technical sections to understand the statements of are propositions 7.3.3, 8.2.2, and corollary 8.1.2.

Finally, section 11.2 contains many pages of very involved but extremely elementary combinatorial arguments with the parameterization of cohomological representations on unitary groups. Unless very interested, we recommend reading enough to understand the definitions and statements while ignoring proofs. Subsection 12.4 on Sarnak-Xue density contains some sub-subsections that should be treated similarly.

Due to the length of the write-up and density of cross-references in later sections, we highly recommend reading this work electronically on a PDF reader that can handle intra-document hyperlinks and that has a back button.

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1.6. **Notation.**

1.6.1. *Global Variables:* As some notation used throughout:
Basics:

- $F/\mathbb{Q}$ a totally real number field.
- ∞ the set of infinite places of F.
- $\mathcal{O}_F$ is the ring of integers for F.
- q_v for finite place v is the residue field degree
- E/F an imaginary quadratic extension.
- Γ_F, the absolute Galois group of F
- $\Gamma(E/F) = \langle\sigma\rangle$, the Galois group of E over F.
- places of F will be denoted by v, with completion F_v.
- $\omega_{E/F}$ is the order-2 character associated to the quadratic extension E/F
- $\star_v$ is the local component at v of structure $\star$.
- $\star^S$ and $\star_S$ are components at S and away from S of structure $\star$ for S some set of places.
- "Irreps" are irreducible representations
- "Unirreps" are unitary irreducible representations.
- $v_1 \boxtimes v_2$ is the corresponding representation of $H_1 \times H_2$ if v_i is a representation of H_i.
- $[d]$ is the d-dimesional irrep of SL_2.
- $\mathbf{1}_X$ is the indicator function of set X
- $\bar{\mathbf{1}}_X$ is the indicator distribution on set X (indicator function normalized by volume to have integral 1)
- $\mathbf{1}_{x=y}$ is an indicator function if $x = y$.

Groups:

- $G_v = G(F_v)$ for v a place of F and G/F reductive
- $G_S, G^S = G(\mathbb{A}_S), G(\mathbb{A}^S)$ respectively for S a set of places of F and G/F reductive
- $\Omega_G, \Omega_{G,F}$ is the (geometric, F-rational) Weyl group of G
- ρ_G is the half-sum of positive roots of G.

- K_v^G is a chosen hyperspecial of G/F reductive at unramified place v.
- $\mathscr{H}^{\mathrm{ur}}(G_v)$ is the unramfied Hecke algebra for reductive G/F with respect to K_v^G
- $K_v^G(q_v^k)$ is the kth Moy-Prasad filtration group of K_v^G for G/F reductive and unramified at place v.
- $K^G(\mathfrak{n})$ for $\mathfrak{n}$ an ideal of $\mathcal{O}_F$ at which reductive G/F is unramified is the product of $K_v^G(q_v^k)$ that is the congruence subgroup corresponding to $\mathfrak{n}$.
- $\Pi^G_{\mathrm{disc}}(\lambda)$ is the discrete series L-packet at infinitesimal character λ of real group G
- φ_{π_d} is the pseudocoefficient of discrete series representation π_d
- EP_λ is the (endoscopically normalized) Euler-Poincaré function of infinitesimal character λ

Arthur's Classification:

- $G(N)$ is the GL_N-like group defined in §2.1
- $\widetilde{G}(N)$ is the GL_N-like twisted group defined in §2.1
- $U(N) := U_{E/F}(N)$ is the quasisplit unitary group as in §2.1
- $U(p,q)$ is the indefinite unitary group with signature (p,q) over $\mathbb{R}$.
- $\mathcal{E}_{\mathrm{ell}}(N), \mathcal{E}_{\mathrm{ell}}(G)$ are the elliptic endoscopic groups of $\widetilde{G}(N), G$ respectively as in §2.1.3
- $\mathcal{E}_{\mathrm{sim}}(N)$ are the simple endoscopic groups of $\widetilde{G}(N)$ as in §2.1.3
- f^H for f a test function on reductive G/F is a choice of endoscopic transfer to some $H \in \mathcal{E}_{\mathrm{ell}}(G)$
- f^N for f a test function on $G \in \mathcal{E}_{\mathrm{ell}}(N)$ is a test function on $\widetilde{G}(N)$ that transfers to f.
- $\psi = \oplus \tau_i[d_i]$ is an Arthur parameter with cuspidal building blocks τ_i as in §2.2.1
- $\Psi(N), \Psi(G)$ are the sets of parameters associated to $\widetilde{G}(N)$ and G respectively as in §2.2.1 and §2.2.4.
- The "Arthur SL_2" of a parameter is an unordered partition Q representing it's restriction to the Arthur SL_2.
- π_ψ is the automorphic representation of $G(N)$ corresponding to ψ as in §2.2.2
- $\tilde{\pi}_\psi$ is the extension of π_ψ to $\widetilde{G}(N)$ as in §2.2.3.
- φ_ψ is the L-parameter associated to A-parameter ψ as in equation (2.3.1)
- $\mathcal{S}_\psi, \mathcal{S}_{\psi_v}$ are Arthur's component groups associated to global or local parameters as in §2.4.1, 2.4.2
- $S^\natural_\psi, S^\natural_{\psi_v}$ are Kaletha's larger component groups associated to global or local parameters as in §2.4.1, 2.4.2
- s_ψ, s_{ψ_v} are the special elements identified in these component groups as in §2.4.1, 2.4.2.
- ϵ_ψ is the identified character on global $\mathcal{S}_\psi$ as §2.4.3.
- $\Pi_\psi(G), \Pi_{\psi_v}(G_v)$ are the A-packets associated to global or local A-parameters on group G, G_v.
- $\Pi_{\varphi_v}(G_v)$ is the local L-packet associated to L-parameter φ_v on G_v from §2.5.2
- $P\Pi_{\varphi_v}(G_v)$ is the local pseudo-L-packet associated to L-parameter φ_v on G_v from §2.5.2

- $\eta^{\psi_v}_{\pi_v}$ is the local character of $S^\natural_{\psi_v}$ associated to $\pi_v \in \Pi_{\psi_v}$
- η^ψ_π is the character of $\mathcal{S}_\psi$ associated to $\pi \in \Pi_\psi$
- $\mathrm{tr}_{\psi_v} := \mathrm{tr}^{G_v}_{\psi_v}$ is the stable packet trace for parameter ψ_v on group G_v

Shapes:

- $\Delta = (T_i, d_i, \lambda_i, \eta_i)$ is a refined shape as in §4.2
- $\psi \in \Delta$ means that A-parameter ψ has refined shape Δ
- $\Sigma_{\lambda,\eta}$ is the simple refined shape as in §4.2
- $\mathcal{S}_\Delta, s_\Delta$ are the component groups and special elements associated to refined shape Δ.
- ψ^Δ_∞ is the local component at infinity associated to refined shape Δ.
- $H(\Delta)$ is the $H \in \mathcal{E}_{\mathrm{ell}}(N)$ such that $\psi \in \Delta$ implies that $\psi \in \Psi(H)$.
- $\Delta(\pi_0)$ is the set of refined shapes Δ such that $\pi_0 \in \Pi_{\psi^\Delta_\infty}$.
- "GSK" and "Odd GSK" are conditions on shapes defined in 8.2.1.

Trace Formulas:

- $\mathcal{AR}_{\mathrm{disc}}(G)$ is the set of discrete automorphic representations of reductive G/F.
- I^G, S^G are Arthur's invariant and stable trace formulas for group G
- $I^G_{\mathrm{disc}}, S^G_{\mathrm{disc}}$ are their discrete parts
- R^G is the trace against $\mathcal{AR}_{\mathrm{disc}}(G)$
- I^G_ψ, S^G_ψ are the summands of $I^G_{\mathrm{disc}}, S^G_{\mathrm{disc}}$ associated to parameter ψ.
- I^G_Δ, S^G_Δ are the summands of $I^G_{\mathrm{disc}}, S^G_{\mathrm{disc}}$ associated to refined shape Δ.
- $\star^N$ is the version of any of the variants above associated to twisted group $\widetilde{G}(N)$.

Asymptotics:

- $|\mathfrak{n}_i|$ is the norm of ideal $\mathfrak{n}_i$ of $\mathcal{O}_F$.
- $\Gamma_{n_1,\dots,n_k}(\mathfrak{n}_i)$ is an Euler factor associated to ideal $\mathfrak{n}_i$ of $\mathcal{O}_F$ and list $n_1, \dots, n_k$ in §9.1.
- $\bar{R}(\Delta)$ is an upper bound on growth rate associated to refined shape Δ in theorem 9.3.2
- $R(\Delta)$ is a tighter upper bound on growth rate associated to refined shape Δ in corollary 9.4.4
- $R_0(\Delta)$ is a conjectural optimal growth rate associated to refined shape Δ in conjecture 9.6.3
- $L_\Delta(\mathfrak{n}_i)$ is an Euler factor associated to ideal $\mathfrak{n}_i$ of $\mathcal{O}_F$ and GSK shape Δ in theorem §9.5.1.
- $\tau'(G)$ is a modified Tamagawa number of reductive G/F as in [ST16, (9.5)].

Cohomological irreps of unitary groups

- $\mathcal{P}(N), \mathcal{P}(p,q), \mathcal{P}_1(p,q)$ are combinatorial parameterizing sets defined in 11.1.1
- β, δ are reduction maps between these parameterizing sets defined in (11.1.2)
- $\Delta^{\max}(\pi_0)$ is a subset of $\Delta(\pi_0)$ determined by 11.2.1.
- $R(\pi_0)$ is the common value of $R(\Delta)$ for $\Delta \in \Delta^{\max}(\pi_0)$
- $Q^{\max}(\pi_0)$ is the set of Arthur SL_2's of elements of $\Delta^{\max}(\pi_0)$ as in 11.2.4.
- $Q_{\mathrm{can}}(\pi_0)$ is a certain unordered partition assigned to cohomological representation π_0 in 11.2.8.

- "GSK-maxed", "odd GSK-maxed" are conditions on cohomological representations π_0 of unitary groups defined in 11.2.10, 11.2.11.

1.6.2. *Shorthand for Non-Factorizable Functions.* There is a technical annoyance that certain transfer maps from functions on a group G to a functions on a product of groups $H_1 \times H_2$ may not always have image in factorizable functions.

However, they will always land in linear combinations of factorizable functions. Therefore, we will use the notations like

$$\prod_i^{\oplus} f_i.$$

to represent the sum of the factored term over this linear combination.

At some points, we also for simplicity want to elide the fact that a transfer to a group like H^d may not be the same on each H-factor. Therefore we will use the even more abusive notation

$$(f_1)^{d\oplus}$$

to represent a sum over the factorizable pieces of the product over factors for each H in the H^d.

1.6.3. *Sequences.* Some objects we will consider will be indexed by finite sequences $n_1, \dots, n_k$. As shorthand, we will define the sequence

$$n_1^{(r_1)}, \dots, n_k^{(r_k)} := \overbrace{n_1, \dots, n_1}^{r_1}, \dots, \overbrace{n_k, \dots, n_k}^{r_k}.$$

Also, if L_1 and L_2 are sequences, "L_1, L_2" will represent their concatenation. We represent concatenation of multiple lists by a disjoint union symbol

$$\bigsqcup_i L_i.$$

Finally, if $P = (p_1, \dots, p_k)$ is an ordered partition of n and $a = (a_i)_i$ is a list of length n, the P-parts of a are defined by partitioning a in order according to P:

$$\overbrace{\xi_1 \dots, \xi_{n_1}}^{a_1^P}, \overbrace{\xi_{n_1+1} \dots, \xi_{n_1+n_2}}^{a_2^P}, \cdots, \overbrace{\xi_{N-n_k+1} \dots, \xi_N}^{a_k^P}.$$

2. A-parameters and the Classification

We attempt to summarize as concisely as possible the parts of endoscopic classification that are relevant to this project.

An Arthur/endoscopic classification for a group G is conceptually a "transfer" of two known facts about automorphic representations on GL_n—

- The classification of the discrete spectrum in [MW89],
- Local Langlands for local components.

—to a parameterization of automorphic representations of G and their local components.

We will focus on the example of Mok's and Kaletha-Minguez-Shin-White's versions from [Mok15] and [KMSW14] for quasisplit and general unitary groups respectively. Our summary will be in two pieces:

- A formalism of local and global parameters which encapsulates the known information on the GL_n side.

- A description of how the automorphic spectrum on G decomposes into pieces that correspond to each parameter together with a description of the structure of each of these pieces.

We will not go over background for endoscopy or the stable trace formula since sections 2.1 and 3.1-3 of [Art13] already give a good, relatively concise introduction with an eye towards the endoscopic classification.

2.1. **Groups Considered.** We begin needing to define certain groups and L-embeddings.

Fix a totally real number field F and totally complex quadratic extension E/F. For each $N > 0$, consider the group

$$G(N) = \mathrm{Res}_F^E \mathrm{GL}_{N,E}$$

Let θ_N be the automorphism of $G(N)$ in the outer class of conjugate inverse transpose that fixes the standard pinning. In particular, it is an involution. It can be written as $\theta_N(g) = \mathrm{Ad}(J_N)(\bar{g}^{-t})$ for a choice of J_N.

2.1.1. *Unitary Groups.* Let $U_{E/F}(N)/F$ be the reductive group

$$U_{E/F}(N, F) = \{g \in GL_N(E) : \theta(g) = g\} \tag{2.1.1}$$

It is a quasisplit unitary group and therefore a form of GL_N/F. We can choose a Borel and maximal torus (B, T) to be the upper triangular and diagonal θ-fixed matrices respectively

For any place v of F, we consider $U_{E/F}(N, F_v)$. When v is split in E, we have $U_{E/F}(N, F_v) \simeq GL_N(F_v)$. Otherwise, $U_{E/F}(N, F_v)$ is the unique quasisplit unitary group of rank N over F_v.

Finally, because of our choice of E/F, all inner forms of $U_{E/F}(N, F_\infty)$ will have discrete series.

2.1.2. *L-groups and embeddings.* All our groups split over E so in all our L-groups, the action of Γ_F factors through $\Gamma(E/F)$. We have

$${}^LG(N) = (GL_N(\mathbb{C}) \times GL_N(\mathbb{C})) \rtimes \Gamma_F$$

with σ swapping the two copies of $GL_N(\mathbb{C})$. We also have

$${}^LU_{E/F}(N) = GL_N(\mathbb{C}) \rtimes \Gamma_F,$$

with $\sigma(g) = \mathrm{Ad}(J_N)(g^{-t})$.

For $\kappa \in \pm 1$, we have L-embeddings

$$\xi_\kappa : {}^LU_{E/F}(N) \to {}^LG(N).$$

The explicit coordinates are not important to us and can be found in §2.1 of [Mok15].

2.1.3. *Endoscopic data.* We are interested in the twisted endoscopic groups of $\widetilde{G}(N) = G(N) \rtimes \theta_N$. As in §2.4.1 in [Mok15], these are parameterized as

$$\mathcal{E}_{\mathrm{ell}}(N) = \{U_{\kappa_1}(N_1) \times U_{\kappa_2}(N_2) : \kappa_i = \pm 1, N = N_1 + N_2, \kappa_1\kappa_2 = (-1)^{N-1}\}$$

with each $U_\pm(N_i)$ isomorphic to the quasisplit unitary group $U_{E/F}(N_i)$ and the κ_i determining the specific L-embedding ξ_κ.

Among these we highlight the simple endoscopic groups:

$$\mathcal{E}_{\mathrm{sim}}(N) := \{U_+(N), U_-(N)\}.$$

Note that $U_\pm$ are isomorphic as groups

We are also interested in the endoscopic groups of $G = U_{E/F}(N)$. As enumerated in [KMSW14, §1.1.1], these are parameterized as

$$\mathcal{E}_{\mathrm{ell}}(G) = \{U_{E/F}(N_1) \times U_{E/F}(N_2) : N = N_1 + N_2, N_1 \geq N_2\}.$$

We do not need the full information of the endoscopic triples involved. Beware that our $\mathcal{E}_{\mathrm{ell}}(G)$ is the $\overline{\mathcal{E}}_{\mathrm{ell}}(G)$ of [KMSW14].

2.1.4. *Inner Forms.* We will also consider extended pure inner forms of $U_{E/F}(N)$ as in [KMSW14, §0.3.3]. Since the general precise definition is complicated and not relevant to our computation, we simply recall their enumeration of possibilities for unitary groups.

Let $G \in \mathcal{E}_{\mathrm{ell}}(N)$. Then in the local case:

- If v is non-Archimedean and split in E, the extended pure inner forms of $G_v \cong \mathrm{GL}_{N,v}$ are of the form $\mathrm{Res}^{D_v}_{F_v} \mathrm{GL}_m$ for D_v a division algebra over F. They are associated invariant $a_v = N \cdot \mathrm{inv}(D_v)$.
- If v in non-Archimedean non-split[2] in E, the extended pure inner forms of $G_v \cong U_{E/F}(N)_v$ are:
  - $U_{E/F}(N)_v$ itself associated invariant $a_v = 0$
  - Another form associated to $a_v = 1$. If N is odd, this is isomorphic as a group to $U_{E/F}(N)_v$. If N is even, it is the unique non-quasisplit inner form of $U_{E/F}(N)_v$.
- If v is Archimedean real in F and inert in E, then the extended pure inner forms of $G_v \cong U_{\mathbb{C}/\mathbb{R}}(N)$ are the $U(p,q)$ for $p + q = N$ and associated invariant $a_v = N(N-1)/2 + q$. Note that $U(p,q) \neq U(q,p)$ as extended pure inner forms even though they are isomorphic as groups

Our choice of E/F never requires us to consider other types of Archimedean v.

A choice of local extended pure inner form and each v comes from a global extended pure inner form if and only if:

- Almost all a_v are trivial,
- The sum of the a_v is even.

Note that if we only care about inner forms as groups, the second condition is irrelevant for N odd: we can always switch an infinite-place $G_v = U(p,q)$ to $U(q,p)$, which is isomorphic as a group but has opposite a_v.

We are particularly interested in the isomorphism-as-groups classes of extended pure inner forms that are unramified at all finite places. This is only possible when E/F is unramified—for example

$$E/F = \mathbb{Q}[\sqrt{3}, i]/\mathbb{Q}[\sqrt{3}]$$

is CM and unramified at all finite places. Casework with respect to the parity of N then gives:

Lemma 2.1.1. *Assume E/F is unramified. Then every isomorphism-as-groups class of extended pure inner forms G of $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ that is unramified at all finite places has a representative that satisfies $G^\infty = (G^*)^\infty$ and is therefore determined by*

$$G_\infty = \prod_{v \in \infty} U(p_v, q_v).$$

---

[2]While the current as-of-this-comment draft of [KMSW14] only says inert, this seems to be a typo since the arguments they give work for ramified places as well.

Such a choice is valid if and only if

$$\frac{N(N-1)}{2}|\infty| + \sum_{v \in \infty} q_v$$

is even.

We will henceforth only consider these representatives to normalize local transfer factors later on so that they are consistent with the fundamental lemma at all finite places. Finally, since E/F is CM, we get the crucial property that all extended pure inner forms G of $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ satisfy that G_∞ has discrete series.

2.2. **Parameters: Definitions.**

2.2.1. *Definitions.*

Definition 2.2.1. A global A-parameter of rank N is a conjugate self-dual (through θ_N) formal expression

$$\psi = \tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k]$$

up to reordering the summands and where each τ_i is a cuspidal automorphic representation of $G(T_i)$ (equivalently, one of GL_{T_i}/E), $d_i \in \mathbb{Z}^+$, and $\sum_i T_i d_i = N$.

Definition 2.2.2. We say $\psi = \tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k]$ is

- *cuspidal* if $k = 1$ and $d_i = 1$,
- *simple* or *stable* if $k = 1$,
- *generic* if each $d_i = 1$,
- *elliptic* if each τ_i is conjugate self-dual by itself and the $\tau_i[d_i]$ are distinct.

Definition 2.2.3. Let $\Psi(N)$ be the set of elliptic parameters in $G(N)$.

2.2.2. *Representations.* As explained in [Art13, §1.3], the main result of [MW89] associates a unique discrete automorphic representation π_ψ of GL_n/E to each Arthur parameter ψ. First, for simple $\tau[d]$, consider the parabolic induction

$$\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_N(\mathbb{A})}(\tau|\det|^{(d-1)/2} \boxtimes \tau|\det|^{(d-3)/2} \boxtimes \cdots \boxtimes \tau|\det|^{-(d-1)/2})$$

where P is the parabolic associated to ordered partition $(\dim\tau, \dots, \dim\tau)$. We set $\pi_{\tau[d]}$ to be the unique Langlands quotient of this induction which exists and is unitary.

For general $\psi = \bigoplus_i \tau_i[d_i]$, we let

$$\pi_\psi := \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{GL}_N(\mathbb{A})}(\boxtimes_i \pi_{\tau_i[d_i]})$$

where P is the appropriate parabolic. This is always unitary and irreducible.

2.2.3. *Canonical Extensions to $\widetilde{G}(N)$.* Fix a Whittaker datum ω for $G(N)$ giving local Whittaker data ω_v on each $G_v(N)$. Then each π_ψ for $\psi \in \Psi(N)$ has a canonical extension $\tilde\pi_\psi := \tilde\pi_{\psi,\omega}$ to $\widetilde{G}(N)$ as explained in §2.2 of [Art13] or §3.2 of [Mok15]. We warn that this "choice of sign" is extremely important conceptually and not just a technicality to be ignored—it enters crucially into the computation of various sign characters in the works of Arthur and Mok and is the main difficulty in understanding conjecture 6.4.2.

The $\tilde\pi_\psi$ is a product of extensions $\tilde\pi_{\psi,v}$ of each $\pi_{\psi,v}$. By the Langlands classification, each $\pi_{\psi,v}$ is the Langlands quotient of an induction

$$\mathrm{Ind}_{P_v}^{G(N)_v}(\sigma_1|\det|^{r_i} \boxtimes \cdots \boxtimes \sigma_k|\det|^{r_k})$$

where each σ_i is tempered and therefore generic. We choose the θ-action on σ_i to be the one that acts as $+1$ instead of -1 on its one-dimensional space of Whittaker functionals with respect to ω_v.

Finally, since ψ is conjugate self-dual, we necessarily have that $r_j = -r_{k-j}$ and that σ_j and σ_{r-j} are conjugate-duals of each other. Therefore we can choose P to be fixed by θ and can define the action of θ on $\pi_{\psi,v}$ as coming from the induction of the actions on each σ_i.

2.2.4. *Assignment to Groups in $\mathcal{E}_{\mathrm{ell}}(N)$.* Every $\psi \in \Psi(N)$ can be assigned to a unique element of $\mathcal{E}_{\mathrm{ell}}(N)$ as in remark 2.4.6 of [Mok15]: If $\tau \in \Psi(N)$ is cuspidal, then Mok assigns it a parity δ. Then τ is assigned to $U_{\delta(-1)^{N-1}}(N)$. More generally, $\tau[d]$ is assigned to $U_\kappa(Nd)$ where

$$\kappa = (-1)^{(N-1)(d-1)}.$$

Finally, if

$$\psi = \bigoplus_i \tau_i[d_i]$$

with each $\tau_i \in \Psi(T_i)$, let N_O be the sum of $T_i d_i$ such that $\tau_i[d_i]$ are *orthogonal*: i.e. $\delta_i(-1)^{t_i+d_i} = 1$. Similarly, let N_S be defined similarly for the $\tau_i[d_i]$ that are the opposite: *symplectic*. The discussion after 2.4.6 in [Mok15] assigns ψ the group

$$U_{(-1)^{N_O-1}}(N_O) \times U_{(-1)^{N_S}}(N_S) \in \mathcal{E}_{\mathrm{ell}}(N_O + N_S).$$

Definition 2.2.4. For $G \in \mathcal{E}_{\mathrm{ell}}(N)$, let $\Psi(G)$ be the subset of $\Psi(N)$ assigned to G.

2.2.5. *As Morphisms.* We can interpret global parameters as morphisms into ${}^LG(N)$. This is a technical replacement for not having access to the conjectural global Langlands group and will be useful for discussing component groups later.

Given parameter

$$\psi = \bigoplus_i \tau_i[d_i] \in \Psi(N),$$

let $\tau_i \in \Psi(H_i) \subseteq \Psi(T_i)$ for H_i a group and T_i a number. Let μ_i be the endoscopic embedding

$$\tau_i' : {}^LH_i \hookrightarrow {}^LG(T_i).$$

Define the fiber product

$$\mathcal{L}_\psi := \prod_i ({}^LH_i \to W_F).$$

Then we define map

$$\psi' : \mathcal{L}_\psi \times \mathrm{SL}_2 \hookrightarrow {}^LG(N) : \psi' = \bigoplus_i \mu_i \boxtimes [d]$$

where $[d]$ is the d-dimensional irreducible representation of SL_2.

Note by construction that for the $G \in \mathcal{E}_{\mathrm{ell}}(N)$ such that $\psi \in \Psi(G)$, ψ' factors through ${}^LG$. Finally, we will say the Arthur SL_2 of parameter ψ to represent it's restriction to the SL_2-factor. This can be represented as an unordered partition encoding its decomposition into irreps.

2.3. **Parameters: Local Components.**

2.3.1. *Local Parameters.* We first define a formalism of local parameters. To deal with not knowing the Ramanujan conjecture, we will have to define two slightly different versions.

Definition 2.3.1. Let G/F be a reductive group. A local A-parameter for G at v, $\psi_v \in \Psi_v(G)$, is an L-morphism

$$\psi_v : L_{F_v} \times \mathrm{SL}_2 \to {}^L G$$

where

- L_{F_v} is the Weil group W_{F_v} if v is Archimedean and the Weil-Deligne group WD_{F_v} is v is non-Archimedean,
- $\psi_v|_{L_v}$ is a bounded L-parameter,

and up to conjugacy in $\widehat{G}$

A generalized local A-parameter, $\psi_v \in \Psi_v^+(G)$, is the same object without the boundedness condition.

We call L_{F_v} the local Langlands group and the SL_2 factor the Arthur SL_2. We can also define the unordered permutation Q representing the Arthur SL_2 of ψ_v similar to the global case.

Next, Every local parameter ψ has an associated L-parameter φ_ψ :

$$\varphi_\psi : L_f \to {}^L G : w \mapsto \psi \left(w, \begin{pmatrix} |w| & 0 \\ 0 & |w|^{-1} \end{pmatrix} \right) \tag{2.3.1}$$

Definition 2.3.2. Let $\Psi_v(N), \Psi_v^+(N)$ be the corresponding sets of conjugate self-dual local A-parameters for $\widetilde{G}(N)$.

We may also write local parameters as

$$\psi_v = \bigoplus_i \tau_i[d_i] := \bigoplus \tau_i \boxtimes [d],$$

where $[d]$ represents the d-dimensional representation of SL_2 and each τ_i is a representation of L_{F_v}.

As in section 2.2.4, we have decompositions

$$\Psi_v(N) = \bigsqcup_{G \in \mathcal{E}_{\mathrm{ell}}(N)} \Psi_v(G), \qquad \Psi_v^+(N) = \bigsqcup_{G \in \mathcal{E}_{\mathrm{ell}}(N)} \Psi_v^+(G)$$

determined by parities $\eta_{i,v}$ assigned to irreducible τ_i (see §2.2 in [Mok15]).

2.3.2. *Localization.* There is a localization map $\Psi(N) \to \Psi_v^+(N)$. Consider

$$\psi = \sum_i \tau_i[d_i] \in \Psi(N)$$

with each $\tau_i \in \Psi(T_i)$ cuspidal. By local Langlands, the component $\pi_{\tau_i,v}$ at v of π_{τ_i} corresponds to an L-parameter

$$\varphi_{i,v} : L_{F_v} \to {}^L G(N)_v.$$

Then we define local A-parameter

$$\psi_v := \bigoplus_i \varphi_{i,v} \boxtimes [d_i].$$

This ψ_v is currently only known to be an element of $\Psi^+(N)$. However, the Ramanujan conjecture would imply that each $\varphi_{i,v}$ is bounded since they come from local components of cuspidal automorphic representations. This would make $\psi_v \in \Psi(N)$.

Localization is consistent with the global picture: first, comparing with the construction of π_ψ shows that $(\pi_\psi)_v$ has L-parameter φ_{ψ_v}. In addition, corollary 2.4.11 in [Mok15] and the discussion afterwards shows that if $\psi \in \Psi(G)$ for $G \in \mathcal{E}_{\mathrm{ell}}(N)$, then ψ_v factors through ${}^LG$—in other words, the localization map restricts to

$$\Psi(G) \to \Psi_v^+(G)$$

for $G \in \mathcal{E}_{\mathrm{ell}}(N)$.

In addition, the discussion after corollary 2.4.11 explains how to produce localization maps

$$L_{F_v} \to \mathcal{L}_\psi \tag{2.3.2}$$

for any parameter $\psi \in \Psi(N)$ such that ψ_v is the pullback of ψ through the localization.

2.4. **Centralizer Subgroups and ϵ-Characters.** Now we start defining notions needed to understand the structure of the part of the automorphic spectrum of $G \in \mathcal{E}_{\mathrm{ell}}(N)$ corresponding to some ψ.

2.4.1. *Global Centralizers.* To each global parameter $\psi \in \Psi(G)$, Mok attaches a component group $\mathcal{S}_\psi$ defined as follows:

$$S_\psi(G) := Z_{\widehat{G}}(\operatorname{im}\psi'),$$
$$\mathcal{S}_\psi := \pi_0(S_\psi/Z(\widehat{G})^{\Gamma_F}).$$

In addition, [KMSW14] attaches a larger component group $S_\psi^\natural$. By the discussion around (1.3.6) there, for our special case of unitary groups we may use the formula:

$$S_\psi^\natural = \pi_0(S_\psi).$$

Also define

$$s_\psi := \psi'(1 \times -1) \in S_\psi^\natural.$$

These groups are explicitly computed in [KMSW14] around (1.3.6): if

$$\psi = \bigoplus_{i \in I} \tau_i[d_i]$$

with τ_i cuspidal, then the I^+ mentioned is all of I since ellipticity of ψ means that the τ_i all have multiplicity 1. Then there are canonical isomorphisms

$$S_\psi^\natural = (\mathbb{Z}/2)^I, \qquad \mathcal{S}_\psi = (\mathbb{Z}/2)^I/(\mathbb{Z}/2)^{\mathrm{diag}} \tag{2.4.1}$$

in which

$$s_\psi = \bigoplus_{\substack{i \in I \\ d_i \text{ odd}}} 1.$$

Note that s_ψ is trivial in $\mathcal{S}_\psi$ if the d_i all have the same parity.

2.4.2. *Local Centralizers.* Mok also defines local component groups:

$$S_{\psi_v}(G) := Z_{\widehat{G}_v}(\operatorname{im}\psi_v),$$
$$\mathcal{S}_{\psi_v} := \pi_0(S_{\psi_v}/Z(\widehat{G_v})^{\Gamma}_{F_v}).$$

We also similarly have an $S^{\natural}_{\psi_v}$. As explained at the end of §1.2.4 in [KMSW14], we may again use the formula

$$S^{\natural}_{\psi_v} := \pi_0(S_{\psi_v})$$

in our special case of unitary groups. We also define

$$s_{\psi_v} := \psi_v(1 \times -1) \in \mathcal{S}_{\psi_v}.$$

Just as in the global case, $\mathcal{S}_{\psi_v}$ and $S^{\natural}_{\psi_v}$ can be computed explicitly, though the lack of a corresponding "elliptic" condition makes this slightly more complicated—see the end of §1.2.4 in [KMSW14] again for details. We will only need to worry about local component groups explicitly for very specific parameters at ∞, so these details aren't relevant here.

Finally, the localization maps $L_{F_v} \to \mathcal{L}_\psi$ give localization maps $\mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$ and $S^{\natural}_\psi \to S^{\natural}_{\psi_v}$. Under these maps, we can identify s_{ψ_v} with s_ψ.

2.4.3. *ϵ characters.* Fix $H \in \mathcal{E}_{\mathrm{ell}}(N)$. The third structure we need to understand for global parameter $\psi \in \Psi(H)$ is a character ϵ_ψ on $\mathcal{S}_\psi$.

Definition 2.4.1. Let $\psi \in \Psi(H)$ and (ρ, V) a finite-dimensional representation of ${}^LH$. Then there is an action $\rho_\psi : S_\psi \times L_\psi \times \mathrm{SL}_2\mathbb{C}$ on V. Let ρ_ψ factor into irreducibles:

$$\sum_{i \in I} \sigma_i \otimes \gamma_i \otimes \delta_i.$$

Let I' be the set of indices such that

- γ_i is symplectic
- $\epsilon(1/2, \gamma_i) = -1$ (see Arthur for how to define this from local ϵ-factors).

Then we define

$$\epsilon^\rho_\psi : S_\psi \to \mathbb{C} : s \mapsto \prod_{i \in I'} \det(\sigma_i(s)).$$

Definition 2.4.2. Let $\epsilon_\psi := \epsilon^H_\psi$ be ϵ^ρ_ψ for ρ the adjoint representation of ${}^LH$ on $\operatorname{Lie}\widehat{H}$. Note that it factors through $\mathcal{S}_\psi$ itself.

The adjoint representation ρ on $\operatorname{Lie}(\hat{H})$ preserves the Killing form and is as such orthogonal, thus only the summands such that δ_i is symplectic, i.e. even dimensional, contribute non-trivially to ϵ_ψ. If $\hat{H} = GL_N$ and $\psi = \oplus_i \tau_i[d_i]$, then $\rho_\psi\mid_{SL_2\mathbb{C}} = \bigoplus_i \bigoplus_j \nu(d_i) \otimes \nu(d_j)$. A computation of the dimensions of the irreducible constituents of $\nu(d) \otimes \nu(d')$ then shows:

Lemma 2.4.3. *If $\psi = \oplus_i \tau_i[d_i]$ and all the d_i have the same parity, then $\epsilon_\psi \equiv 1$.*

2.5. **Main Theorems of the Classification.** Now we can state the two main theorems of the endoscopic classification

2.5.1. *Local Packets.* First, we state the existence of local A-packets. Let $G^* \in \mathcal{E}_{\mathrm{ell}}(G)$. We recall from (0.3.1) in [KMSW14] that every extended pure inner form G_v of G_v^* gets associated a character

$$\chi_{G_v} : Z(\widehat{G}_v)^\Gamma \to \mathbb{C}^\times.$$

This association is a bijection if v is non-Archimedean. Finally χ_{G^*} is trivial.

We define $\mathrm{Irr}(S_G^\natural, \chi)$ to be the set of trace characters of irreps of $S_G^\natural$ that pullback to χ through $Z(\widehat{G}_v)^\Gamma \to S_G^\natural$ (beware that this set can be empty—there's a condition of being "relevant" defined in [KMSW14, §0.4, 1.2] discussing when this happens).

Theorem 2.5.1 ([KMSW14, 1.6.1])**.** *Let $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ and $\psi_v \in \Psi(G_v^*)$. Fix a Whittaker datum on G_v^*. Then for each extended pure inner form G_v of G_v^*, there is an associated set $\Pi_{\psi_v}(G_v)$ of unitary representations of G_v together with a map*

$$\eta := \eta_{G_v} : \Pi_{\psi_v} \to \mathrm{Irr}(S^\natural_{\psi_v}, \chi_{G_v}) : \pi_v \mapsto \eta^{\psi_v}_{\pi_v}$$

These satisfy:

- *Assume ψ is generic. If v is non-Archimedean, then η is a bijection. If v is Archimedean, then the maps $\eta_{G'_v}$ for G'_v such that $\chi_{G'_v} = \chi_{G_v}$ jointly give a bijection from the disjoint union of the $\Pi_{\psi_v}(G'_v)$.*
- *The $\Pi_{\psi_v}(G_v)$ for generic ψ_v partition the set of tempered unirreps of G_v.*

We also make a definition:

Definition 2.5.2. Let ψ_v be a local A-parameter for $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$, G_v an extended pure inner form of G_v^*, and f_v be a test function on G_v. We define the stable packet trace:

$$\mathrm{tr}_{\psi_v}(f_v) := \mathrm{tr}^{G_v}_{\psi_v}(f_v) := \sum_{\pi_v \in \Pi_{\psi_v}(G_v)} \eta^{\psi_v}_{\pi_v}(s_{\psi_v}) \, \mathrm{tr}_\pi(f_v).$$

Theorem 2.5.3. *Let ψ_v, G_v as in the above definition. Then $\mathrm{tr}^{G_v}_{\psi_v}$ is a stable distribution on G_v.*

2.5.2. *The Associated L-packet.* Let $\psi_v : W_F \times \mathrm{SL}_2 \to {}^L G_v$ be a local A-parameter at v. Recall the associated L-parameter from (2.3.1).

This L-parameter further defines an L-packet of representations Π_{φ_ψ} as follows: there is a unique parabolic $P = MN$ of G such that φ_ψ factors through ${}^L M$ as a bounded L-parameter φ_ψ^M twisted by a character that is dual to character λ of M such that λ is P-dominant. Then

$$\Pi^G_{\varphi_\psi} := \{\text{Langlands quotient of}\, \mathrm{Ind}_P^G(\pi \otimes \lambda) : \pi \in \Pi^M_{\varphi_\psi^M}\}$$

Furthermore, M is a product of an arbitrary number of $G(N_i)$ together with a single $U_{E/F}(M)$. Therefore the image of $Z(\widehat{M})^\Gamma$ in $\pi_0(S_{\varphi_\psi})$ is the same as that of $Z(\widehat{G})^\Gamma$ implying that $\mathcal{S}_{\varphi_\psi} = \mathcal{S}_{\varphi_\psi^M}$. We can use this to assign characters

$$\text{Langlands quotient of}\, \mathrm{Ind}_P^G(\pi \otimes \lambda) \mapsto \eta_\pi^{\varphi_\psi^M}$$

See the discussion around (1.5.1) in [Art13] and the beginning of §2.5 in [Mok15] for more details.

It is also useful to sometimes consider the pseudo-L-packet:

$$P\Pi^G_{\varphi_\psi} := \{\mathrm{Ind}_P^G(\pi \otimes \lambda) : \pi \in \Pi^M_{\varphi_\psi^M}\}$$

made up of full induced representations. Finally, a similar construction can produce A-packets for general $\psi \in \Psi^+(G_v)$.

2.5.3. *Global Packets.* As before, let $G^* \in \mathcal{E}_{\text{ell}}(G)$ and G be an extended pure inner form of G^*. If $\psi \in \Psi(G^*)$ we define

$$\Pi_\psi^G := \Pi_\psi = \prod_v{}' \Pi_{\psi_v}^G$$

where the product is restricted so that $\eta_{\pi_v}^{\psi_v} = 1$ at almost all places.

We recall from (0.3.3) in [KMSW14] that

$$\prod_v \chi_{G_v} = 1,$$

so that for any $\pi \in \Pi_\psi$,

$$\eta_\pi^\psi := \prod_v \eta_{\pi_v}^{\psi_v}$$

is a character on $\mathcal{S}_\psi$.

Theorem 2.5.4 (Arthur's Multiplicity Formula, [KMSW14] thm. 1.7.1)**.** *We have that*

$$\mathcal{AR}_{\text{disc}}(G) = \bigoplus_{\psi \in \Psi(G^*)} \bigoplus_{\pi \in \Pi_\psi^G} m_\pi^\psi \pi$$

where

$$m_\pi^\psi = \langle \epsilon_\psi, \eta_\pi^\psi \rangle_{\mathcal{S}_\psi}$$

is the trace-character pairing.

2.6. **Trace Formula Decompositions.**

2.6.1. *Defintions.* Let I^G and S^G be Arthur's invariant and stable trace formulas for reductive G/F respectively and I^G_{disc} and S^G_{disc} their discrete parts. Also define distribution

$$R^G_{\text{disc}} := \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \operatorname{tr}_\pi$$

Arthur's classification gives various decompositions of these into smaller parts and identities between the parts.

Now let $G^* \in \mathcal{E}_{\text{ell}}(N)$ and G an extended pure inner form. For each $\psi \in \Psi(G^*)$, [KMSW14, §3.1,3.3] defines

$$I_\psi^G, S_\psi^G$$

as summands of I^G_{disc} and S^G_{disc} (In fact, these are defined for more than just elliptic parameters but we won't need the more general definitions).

It turns out (e.g. from the stable multiplicity formula 2.6.3 or an argument like [Art89a, (3.9)]) that for elliptic $\psi \in \Psi(G)$:

$$I_\psi^G := \sum_{\pi \in \Pi_\psi^G} m_\pi^\psi \operatorname{tr}_\pi$$

so that

$$R^G_{\text{disc}} = \sum_{\psi \in \Psi(G^*)} I_\psi^G. \tag{2.6.1}$$

Finally, recall the stabilization

$$I^G_{\mathrm{disc}}(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G,H) S^H_{\mathrm{disc}}(f^H)$$

for constants $\iota(G,H)$ and endoscopic transfers f^H. If

$$f = \prod_v f_v$$

then we can take

$$f = \prod_v f_v^{H_v}$$

for the $f_v^{H_v}$ defined up to non-canonical scalars that multiply to one and depend on choices of local transfer factors as in [Kal16a].

2.6.2. *Results.* The key point we will use is that the stabilization of I^G_{disc} in terms of S^H_{disc} for $H \in \mathcal{E}_{\mathrm{ell}}(G)$ descends to the level of I^G_ψ and S^G_ψ. First:

Proposition 2.6.1 ([KMSW14, §1.4]). *Let $\psi \in \Psi(G^*)$. Then there is a bijection from $\mathcal{S}_\psi$ taking*

$$s \mapsto (H(s), \psi^H(s))$$

with $H(s) \in \mathcal{E}_{\mathrm{ell}}(G)$, $\psi^H(s) \in \Psi(H)$ that pushes forward to ψ, and the pair taken up to conjugation by the the subset of $\mathrm{Out}(\widehat{H})$ produced by conjugation in $\widehat{G}$. In this bijection $1 \mapsto (G^*, \psi)$.

The analogous statement also holds for $\psi_v \in \Psi(G_v)$.

Then we have a local formula:

Theorem 2.6.2 (Local Character Relation, [KMSW14] theorem 1.6.1 (4)). *We have that for any test function f_v on G_v and $s \in \mathcal{S}_{\psi_v}$:*

$$\sum_{\pi \in \Pi_{\psi_v}} \eta^{\psi_v}_{\pi_v}(s' s_{\psi_v}) \operatorname{tr}_\pi(f_v) = \operatorname{tr}_{\psi^H(s)}(f_v^{H(s)})$$

Here, s' is a lift of s to $S^\natural_\psi$ that together with the chosen Whittaker datum on G_v determines the transfer factors for endoscopic transfer $f_v^{H(s)}$ (this is related to the difference we are ignoring between $\mathcal{E}_{\mathrm{ell}}(G)$ and $\overline{\mathcal{E}}_{\mathrm{ell}}(G)$ in [KMSW14]).

We also have a global formula:

Theorem 2.6.3 (Stable Multiplicity Formula [Mok15, thm. 5.1.2]). *Let $\psi \in \Psi(G)$. Then for all test functions f on G*

$$S^{G^*}_\psi(f) = |\mathcal{S}_\psi|^{-1} \epsilon_\psi(s_\psi) \operatorname{tr}^{G^*}_\psi(f).$$

Proof. We can ignore the σ term since we are restricting to elliptic ψ. □

We can use all the above to compute:

Theorem 2.6.4. *For any test function f on G*

$$I^G_\psi(f) = \sum_{(H,\psi_H)} \iota(G,H) S^H_{\psi^H}(f^H)$$

where (H,ψ_H) ranges over $H \in \mathcal{E}_{\mathrm{ell}}(G)$ and $\psi^H \in \Psi(H)$ that pushes forward to ψ (up to equivalence in H). Here, f^H is the endoscopic transfer and $\iota(G,H)$ is the constant that appears in the stabilization of the trace formula.

Furthermore, for any $s \in \mathcal{S}_\psi$

$$S^{H(s)}_{\psi^H(s)}(f^{H(s)}) = 2|\mathcal{S}_\psi|^{-1}\epsilon_\psi(ss_\psi)\sum_{\pi\in\Pi^G_\psi}\eta^\psi_\pi(ss_\psi)\,\mathrm{tr}_\pi(f).$$

Proof. The first follows from the exact definition of S_ψ and I_ψ in [KMSW14, §3.3]. The second can be computed from the stable multiplicity formula, the local character relation, an endoscopic sign identity ([Mok15, lem. 5.6.1]) that $\epsilon^H_{\psi^H}(s_{\psi^H}) = \epsilon^G_\psi(ss_\psi)$, and noting that $|\mathcal{S}_{\psi^H}|^{-1} = 2|\mathcal{S}_\psi|^{-1}$. □

3. AJ-packets and Pseudocoefficients

We need to recall some more background at the real place. First, we need to define cohomological representations on G_∞ and the A-packets that contain them. Second, we need some specific test function to eventually plug into the trace formula.

3.1. Cohomological Representations.

3.1.1. *Infinitesimal characters.* Recall that for a group G_∞ over $\mathbb{R}$, irreducible representations of G_∞ get associated infinitesimal characters $\lambda \in \Omega_{G_{\infty,\mathbb{C}}}\backslash \mathrm{Hom}(\mathfrak{t},\mathbb{C})$, where $\mathfrak{t}$ is a Cartan subalgebra of $G_{\infty,\mathbb{C}}$. This data is the same as a map from Weyl orbits of $X^*(\widehat{\mathfrak{t}}) = X_*(\mathfrak{t})$ to $\mathbb{C}$, which is further the same as a semisimple conjugacy class in $\widehat{\mathfrak{g}}$.

Definition 3.1.1. We say infinitesimal character λ is regular integral if it is that of a finite dimensional representation of $G_{\infty,\mathbb{C}}$.

If φ_∞ is a Langlands parameter for G_∞, then $\varphi|_{W_\mathbb{C}}$ is of the form $z \mapsto |z|^\mu (z/\bar{z})^\nu$ for cocharacters $\mu,\nu \in X_*(\widehat{\mathfrak{t}})$ on some Cartan $\widehat{\mathfrak{t}}$ of $\widehat{G}$. All $\pi \in \Pi^G_{\varphi_\infty}$ then have infinitesimal character $\mu+\nu$.

3.1.2. *Cohomological Representations.* Choose a maximal compact K of G_∞. Given a finite dimensional representation V_λ with highest weight λ, we can define the $(\mathfrak{g},K)$-cohomology groups $H^i(\mathfrak{g},K;\pi\otimes V_\lambda)$ for all unirreps π of G_∞ as in [BW00].

Definition 3.1.2. Unirrep π of G_∞ is called cohomological of weight λ if there is i such that $H^i(\mathfrak{g},K;\pi\otimes V_\lambda) \neq 0$.

Every unirrep that is cohomological of weight λ necessarily has infinitesimal character $\lambda + \rho_G$. This is regular integral by definitions. For any choice of real group G_∞ and finite-dimensional representation V_λ, there are only finitely many cohomological representations; an algorithm to construct them and compute their cohomology is given in [VZ84] where they are realized as "cohomologically induced " representations $A_\mathfrak{q}(\lambda)$. We will discuss this in more detail for the case of unitary groups in §11.1.

Finally, [SR99] proves that all unirreps with regular, integral infinitesimal character are cohomological. We therefore suggest using "regular integral infinitesimal character" as a simpler working definition of cohomological.

3.2. **AJ-packets.** Both our multiplicity computations and subsequent cohomological applications will be phrased in terms of Adams-Johnson parameters, whose definition we now recall following [Kot90]; see also [AJ87, Art89b].

3.2.1. *AJ Parameters.* Denote the Weil group of $\mathbb{R}$ by $W_{\mathbb{R}} = W_{\mathbb{C}} \sqcup jW_{\mathbb{C}}$.

Definition 3.2.1. Let $\psi : W_{\mathbb{R}} \times SL_2(\mathbb{R}) \to {}^L G_\infty$ be an Arthur parameter, and denote by $\hat{L}$ the centralizer of $\psi(W_{\mathbb{C}})$ in $\widehat{G}$. Then ψ is an *Adams-Johnson* parameter if:

(i) $\psi(SL_2(\mathbb{C}))$ contains a principal unipotent element of $\widehat{L}$,
(ii) the identity component of $Z(\widehat{L})^{W_{\mathbb{R}}}$ is contained in $Z(\widehat{G})$,
(iii) the infinitesimal character of the parameter φ_ψ is regular.

In condition (ii), the action of $W_{\mathbb{R}}$ on $Z(\widehat{L})$ is defined as conjugation by $\psi(W_{\mathbb{R}}) \subset {}^L G$. Additionally, condition (i) implies that $\psi(W_{\mathbb{C}}) \subset Z(\widehat{L})$.

We recall a more explicit description of Adams-Johnson parameters, summarizing the discussion in [AMR18, §8]. Let $(\widehat{T}, \widehat{B}, X_\alpha)$ be $W_{\mathbb{R}}$-stable a pinning of $\widehat{G}$, so that ${}^L G_\infty$ is defined via this pinning. Then $\widehat{L}$ can be conjugated to be the Levi of a parabolic standard with respect to $\widehat{B}$. Assuming that G_∞ is quasisplit, we additionally make a choice of a Borel pair $(T_{\mathbb{C}}, B_{\mathbb{C}})$ for $G_{\infty,\mathbb{C}}$ such that T is a compact Cartan defined over $\mathbb{R}$. Via these splittings of G_∞ and $\widehat{G}$, $\widehat{L}$ is identified with the dual group of a Levi subgroup L of G, containing T. One can then construct an embedding $\xi_{L,G} : {}^L L \to {}^L G_\infty$ such that $\psi = \xi \circ \psi_L$ for ψ_L the A-parameter associated to a unitary one-dimensional representation of L; this is done in [Art89b, §5]. Moreover, it follows from [NP21, Theorem 5], that any parameter with regular integral infinitesimal character is an Adams-Johnson parameter (note that Nair-Prasad's self-dual Levi subgroups automatically satisfy condition (ii)).

Finally, if ψ is an AJ-parameter, let I_∞ be the set of blocks of the Levi ${}^L L$ constructed in this way. We then have a decomposition $\psi = \bigoplus_{i \in I_\infty} \psi_i$. All of the ψ_i are necessarily conjugate self-dual because they are pairwise distinct by regularity of the infinitesimal character. In particular, the multiplicity of each ψ_i is 1. As such, the computations of [KMSW14, §1.2.4] give a canonical isomorphism

$$S^\natural_{\psi_\infty} \simeq (\mathbb{Z}/2)^{I^+_\infty} = (\mathbb{Z}/2)^{I_\infty}. \tag{3.2.1}$$

Furthermore, the localization map $S^\natural_\psi \to S^\natural_{\psi_\infty}$ is the diagonal embedding induced by the surjection $I_\infty \to I$ and the canonical isomorphism (2.4.1).

3.2.2. *AJ Packets.* Adams-Johnson [AJ87] provide another construction of packets attached to the above parameters. Give AJ-parameter ψ, we get a pair (L, ω) of a Levi subgroup $L \subset G_\infty$ and ω the differential of a one-dimensional unitary representation of L: L is the dual Levi to $\widehat{L}$ and ω is the full infinitesimal character ψ minus ρ_L.

Let $\Omega(G,T)$, $\Omega(L,T)$, and $\Omega_{\mathbb{R}}(G,T)$ be the Weyl groups of $G_{\infty,\mathbb{C}}$ and $L_{\mathbb{C}}$, and G_∞ respectively. Then the elements of the packet $\Pi_\psi = \Pi(L,\omega)$ constructed by Adams-Johnson are in bijection with

$$\Sigma_L = \Omega(L,T) \backslash \Omega(G,T) / \Omega_{\mathbb{R}}(G,T).$$

For each $w \in \Sigma_L$, consider the inner form $L_w = w^{-1} L w$; its is the centralizer of an element $x \in i \operatorname{Lie}(T)$, itself giving to a so-called θ-stable parabolic subalgebra $\mathfrak{q}_w \subset \mathfrak{g}$. Similarly, let $\omega_w = w^{-1}\omega$. Then Adams-Johnson define

$$\Pi(L,\omega) = \{A_{\mathfrak{q}_w}(\omega_w); w \in \Sigma_L\},$$

where $A_{\mathfrak{q}_w}(\omega_w)$ is the cohomologically induced representation from [VZ84].

We will give more details on Adams-Johnson's packets of cohomological representations for unitary groups in §11.1, including a combinatorial parameterization of the representations in a packet, their relation with cohomology of locally symmetric spaces, and explicit formulas for the characters of S_ψ associated to each parameter.

3.2.3. *Compatibility of Descriptions.* We need to check that Adam's and Johnson's construction of packets matches that in [KMSW14]. We thank Jeffrey Adams for explaining this point to us.

By Theorem 4.18 in [AR22], Adams-Johnsons' packets are a special case of the ABV-packets defined in [ABV92]. These ABV-packets have further recently been shown by Arancibia-Mezo [ARM22] to agree with the packets built by Mok [Mok15] in the quasisplit case for any given parameter. Finally, the packets of [KMSW14] on non-quasisplit unitary groups are determined by trace identities comparing them to the quasisplit inner form. These trace identities are automatically satisfied by ABV-packets, so we also get that ABV-packets match the packets of [KMSW14].

In total, we may use the combinatorial description of Adams and Johnson to understand the structure of AJ-packets on all groups we are considering.

3.3. **Pseudocoefficients and Euler-Poincaré Functions.** We also recall the definitions of certain special test functions. Recall that a *standard module* of G_∞ is the full (possibly reducible) parabolic induction of discrete series or limit of discrete series representation on a Levi M.

If π_d is a discrete series representation of G_∞, the paper [CD90] constructs pseudocoefficients φ_{π_d} satisfying

$$\mathrm{tr}_\sigma(\varphi_{\pi_d}) = \mathbf{1}_{\sigma=\pi_d}$$

for all standard modules σ. We also define the Euler-Poincaré function

$$\mathrm{EP}_\lambda = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|} \sum_{\pi_d \in \Pi_{\mathrm{disc}}(\lambda)} \varphi_{\pi_d}$$

where $\Pi_{\mathrm{disc}}(\lambda)$ is the discrete series L-packet of infinitesimal character λ. Beware that our "endoscopic normalization" of Euler-Poincaré functions is different from the usual one in the literature to work better with endoscopic transfer.

3.4. **Trace Identities.** We also collect some useful computations about traces against AJ-packets.

3.4.1. *Character Formulas.* First, let $\psi : W_\mathbb{R} \times \mathrm{SL}_2 \to {}^L G_\infty$ be a parameter for an AJ-packet with infinitesimal character λ. We recall some combinatorial results about the character formulas of elements in Π_ψ.

Recall that the *character formula* for a representation π of G_∞ is its expansion in the Grothendieck group as a linear combination of standard modules.

Lemma 3.4.1. *Let π_d be a discrete series representation of G_∞ with infinitesimal character λ. Then there is a unique $\pi \in \Pi_\psi$ such that π_d appears in the character formula for π.*

Proof. This is a consequence of Lemma 8.8 in [AJ87] □

Next, let $\psi_{\mathrm{disc}} = \psi_{\mathrm{disc}}(\lambda)$ be the discrete parameter with infinitesimal character λ (This is the AJ-parameter with trivial Arthur-SL_2 and corresponds to a discrete series L-packet). We have an inclusion $S_\psi^\natural \subseteq S_{\psi_{\mathrm{disc}}}^\natural$.

Lemma 3.4.2. *Let $\pi \in \Pi_\psi$ and π_d any discrete series appearing in the character formula for π. Then $\eta^\psi_\pi = \eta^{\psi_{\mathrm{disc}}}_{\pi_d}|_{S^\natural_\psi}$.*

Proof. The values of η^ψ_π on the lifts s' of the endoscopic character identity 2.6.2 are determined by the realization of π as $A(w\lambda)$ and the values $\kappa(w)$ in theorem 2.21 of [AJ87]. It is therefore determined on all of $S^\natural_\psi$ since we also know that it also restricts to the character χ_{G_∞} of theorem §2.5.1. However, the character formula in theorem 8.2 of [AJ87] can be seen to show that π_d corresponds to the same w relative to a given choice of Whittaker datum. It therefore gets assigned the same values $\kappa(w)$.

See also the discussion on page 57 of [Taï17] summarizing parts of [Art89b, §5] for a more explicit computation of these characters in the quasisplit case. By a parenthetical note there, the argument should generalize to non-quasisplit groups through the methods of [Kal16b, §5.6]. □

3.4.2. *Identities.* Now we can prove our trace identities.

Lemma 3.4.3. *Let π_d be a discrete series of G_∞ with infinitesimal character λ.*

(1) *Let π_0 be a a representation of G_∞ such that π_d appears in its character formula with coefficient σ. Then for all parameters ψ such that $\pi_0 \in \Pi_\psi$: for all $\pi \in \Pi_\psi$,*

$$\operatorname{tr}_{\pi_\psi}(\varphi_{\pi_d}) = \begin{cases} \sigma & \pi = \pi_0 \\ 0 & \text{else} \end{cases}.$$

(2) *Let ψ be a parameter at infinity with infinitesimal character λ. Then for all $s \in §^\natural_\psi$*

$$\sum_{\pi_\infty \in \Pi_\psi} \eta^\psi_{\pi_\infty}(s) \operatorname{tr}_{\pi_\infty}(\varphi_{\pi_d}) = \eta^{\psi_{\mathrm{disc}}(\lambda)}_{\pi_d}(s_\psi s)$$

(where we implicitly use that $S^\natural_\psi \subseteq S^\natural_{\psi_{\mathrm{disc}}(\lambda)}$).

Proof. For (1), ψ then has infinitesimal character λ and is therefore necessarily an AJ-parameter. Therefore π_0 is then the unique $\pi_0 \in \Pi_\psi$ such that π_d appears in the character formula for π by 3.4.1.

For (2), choose π_0 to be the unique $\pi \in \Pi_\psi$ with π_d in its character formula. Let it appear with coefficient σ. By (1) the sum of traces is $\sigma\eta^\psi_{\pi_0}(s)$.

To compute σ, all discrete series appear in the character formula of the stable sum

$$\sum_{\pi \in \Pi_{\psi_0}} \eta^\psi_\pi(s_\psi)\pi \tag{3.4.1}$$

with multiplicity 1 by [AJ87, (8.10)]. Therefore, $\sigma = \eta^\psi_{\pi_0}(s_\psi)$. Lemma 3.4.2 finishes the argument. □

As an important special case:

Corollary 3.4.4. *Let π_d be a discrete series of G_∞ with infinitesimal character λ and let ψ be a parameter at infinity with infinitesimal character λ. Then*

$$\operatorname{tr}_\psi(\mathrm{EP}_\lambda) = \operatorname{tr}_\psi(\varphi_{\pi_d}) = 1.$$

4. Refined Shapes

We now come to the key definition of this paper: the invariant of a "refined shape" Δ attached to an an Arthur parameter ψ with appropriate infinitesimal character. We construct the refined shape invariant to satisfy:

- Δ determines the $H \in \mathcal{E}_{\mathrm{ell}}(N)$ attached to ψ.
- Δ determines the local factor ψ_∞.
- *Cuspidal* parameters ψ have refined shapes Δ such that there is a well-understood geometric-side expression for the trace against the Δ-part of the automorphic spectrum of H.
- Arthur's S_ψ can be understood well enough in terms of the S_{ψ_i} for ψ_i the cuspidal parameters that build up ψ. This understanding should be through processes/formulas that are uniform over all ψ with refined shape Δ. In particular, we need to understand the character ϵ_ψ and the local stable traces in this way.

These properties are in turn exactly the ones needed to run an inductive argument as in Section 5.

4.1. **Infinitesimal Characters and Central Characters.**

We need some preliminary details on infinitesimal and central characters.

4.1.1. *Infinitesimal characters in the classification.* Let $G \in \mathcal{E}_{\mathrm{sim}}(N)$. We have that $G_\infty = U_N^*(F_\infty)$ where the U^* represents the quasisplit inner form. Therefore the Lie algebra $\widehat{\mathfrak{g}}_\infty = \mathfrak{gl}_n(F_\infty \otimes_{\mathbb{R}} \mathbb{C})$, so consider an infinitesimal character of G as a semisimple matrix up to conjugacy, or other words an unordered sequence

$$\xi = (\xi_1, \xi_2, \dots, \xi_n)$$

with $\xi_i \in F_\infty \otimes_{\mathbb{R}} \mathbb{C}$. We can further expand out each ξ_i as a list of complex numbers $\xi_{i,v}$ for each place $v \in \infty$ of F (which is necessarily real).

It is also sometimes useful think of ξ as the generating function $\sum_j X^{\xi_j}$. In this way, if each $\tau_{i,\infty}$ has infinitesimal character $\xi^{(i)}$, we have infinitesimal character assignment

$$\left(\bigoplus_i \tau_i[d_i]\right)_\infty \mapsto \sum_i \xi^{(i)} \sum_{l=1}^{d_i} X^{\frac{d+1}{2}-l} \tag{4.1.1}$$

It can be seen from this that the character of $\tau[d]$ determines that of τ since the group ring $\mathbb{Z}[\mathbb{C}]$ is an integral domain.

Finally,

Definition 4.1.1. Call conjugacy class ξ *regular integral* if it corresponds to the infinitesimal character of a finite dimensional representation. Equivalently, for each $v \in \infty$ the $\xi_{i,v}$ are distinct values that are all integers if N is odd or all half integers if N is odd.

4.1.2. *Central characters in the classification.* Every parameter ψ has a central character η_ψ on $\mathbb{A}_E^\times$ that is conjugate self-dual. Through class field theory, we will also abuse notation and use η_ψ to denote the corresponding character of Γ_E.

As explained in [Mok15, 2.1], the assignments of parities to characters is simple:

$$\kappa_\eta = \begin{cases} 1 & \eta|_{\mathbb{A}_F^\times} = 1 \\ -1 & \eta|_{\mathbb{A}_F^\times} = \omega_{E/F} \end{cases}$$

where $\omega_{E/F}$ is the order-2 character associated to the quadratic extension E/F.

Then,

$$\eta_{\bigoplus_i \tau_i[d_i]} = \prod_i \eta_{\tau_i}^{d_i}. \tag{4.1.2}$$

4.2. **Definitions.** Our notion of refined shape is built off of the details of how Mok assigns a parameter to an element of $\mathcal{E}_{\mathrm{ell}}(N)$. First, recall the assigned parity η_i of each simple parameter τ_i in §2.2.4.

Lemma 4.2.1. *Let τ be an elliptic rank-N cuspidal parameter with regular integral infinitesimal character ξ at infinity and of parity $\eta = \pm 1$. Then there exists unique $H = H(\eta) \in \mathcal{E}_{\mathrm{sim}}(N)$ depending only on η such that τ is a parameter for $H(\eta)$.*

Furthermore, ξ is the infinitesimal character of a finite dimensional representation on H.

Proof. The first claim is a rephrasing of the material in §2.2.4. The second claim is by definition. □

Motivated by the above, we define:

Definition 4.2.2. An *refined shape* is a sequence

$$\Delta = (T_i, d_i, \xi_i, \eta_i)_i$$

up to permutation and where (T_i, d_i) are positive integers, ξ_i is an infinitesimal character of rank T_i, and $\eta_i = \pm 1$.

We say that $\psi \in \Delta$, or that ψ has refined global shape Δ if ψ is elliptic and $\psi = \bigoplus_i \tau_i[d_i]$ with each τ_i of rank T_i and such that each τ_i has infinitesimal character ξ_i at infinity and is of parity η_i.

We let $\Psi(\Delta) \subseteq \Psi(N)$ be the set of all elliptic, self-dual parameters on $G(N)$ of ∞-refined shape Δ.

Definition 4.2.3. Let ξ be an infinitesimal character of rank n and $\eta = \pm 1$. Then $\Sigma_{\xi,\kappa}$ is the refined shape $(n, 1, \xi, \eta)$.

Definition 4.2.4. The refined shape Δ is *integral* if its *total* infinitesimal character as determined by formula (4.1.1) is regular integral.

In particular, if Δ is integral, then each of the ξ_i must be regular integral, though this isn't sufficient.

4.3. **Properties.**

Proposition 4.3.1. *Let the refined shape Δ have rank N such that all the ξ_i are integral. Then there is a group $H(\Delta) \in \mathcal{E}_{\mathrm{ell}}(N)$ such that all $\psi \in \Delta$ are parameters for $H(\Delta)$.*

Proof. The assignment described in §2.2.4 only depends on T_i, d_i, and η_i. □

This gives us two quick corollaries:

Corollary 4.3.2. *Let $G \in \mathcal{E}_{\mathrm{ell}}(N)$. Then*

$$\Psi(G) = \bigsqcup_{\Delta : H(\Delta) = G} \Psi(\Delta).$$

Proof. This follows from the above since every parameter has a refined shape. □

Corollary 4.3.3. *Let ξ be an integral infinitesimal character of rank N. Then*

$$H(\Sigma_{\xi,\eta}) = U_{\eta(-1)^{N-1}}(N).$$

In particular, for every $G \in \mathcal{E}_{\mathrm{sim}}(N)$ and infinitesimal character ξ of a finite dimensional representation on G, there is η such that $H(\Sigma_{\xi,\eta}) = G$.

As two more facts, if Δ is a refined shape:

- All $\psi \in \Delta$ correspond to the same pairs $(S^{\natural}_{\psi}, s_{\psi})$ by formula (2.4.1). Call the common values $S^{\natural}_{\Delta}$ and s_{Δ}. We can similarly define common value $\mathcal{S}_{\Delta}$.
- All $\psi \in \Delta$ have the same Arthur SL_2 so we may speak of the Arthur SL_2 of Δ.
- The infinitesimal character at infinity and central character of $\psi \in \Delta$ are determined by (4.1.1) and (4.1.2).

Finally,

Lemma 4.3.4. *Let Δ be an integral refined shape. There exists AJ-parameter at infinity ψ^{Δ}_{∞} such that for all $\psi \in \Delta$, ψ_{∞} is conjugate to ψ^{Δ}_{∞}. Furthermore, the induced localization map $S^{\natural}_{\Delta} \to S^{\natural}_{\psi^{\Delta}_{\infty}}$ is also determined by Δ.*

Proof. The localization ψ_{∞} of any parameter $\psi \in \Delta$ is determined by Δ: for the restriction $\psi_{\infty} |_{SL_2} = \oplus_i \nu(d_i)^{T_i}$, this is immediate. Next, the infinitesimal character ξ_i prescribes $\psi_{\infty} |_{W_{\mathbb{R}}}$: let $\psi = \tau_i[d_i]$ such that each term has infinitesimal ξ_i. Then by a known case of the Ramanujan conjecture as explained in [MS19, Lem. 6.1], the parameter associated to $\tau_{i,\infty}$ is bounded with infinitesimal character ξ_i matching that of a finite dimensional representation, so it is uniquely determined. The parameter ψ_{∞} is determined from this data, following the constructions of [Mok15, §2.4].

By construction, the resulting ψ_{∞} has a regular integral infinitesimal character. Additionally, it maps $SL_2(\mathbb{C})$ to a principal SL_2 of the Levi $\hat{L} = \prod_i GL^{T_i}_{d_i} \subset \hat{G}$, with $\hat{L} = Z_{\hat{G}}(\psi_{\infty}(W_{\mathbb{C}}))$ following the assumption that the total infinitesimal character is regular. Then $\psi_{\infty}(W_{\mathbb{C}}) \subset Z(\hat{L})$, and since ψ_{∞} was built from parameters of discrete series, $\psi_{\infty} |_{W_{\mathbb{C}}}$ factors through $z \mapsto \frac{z}{\bar{z}}$ and $W_{\mathbb{R}}$ acts on $Z(\hat{L})$ by inversion. Thus the identity component of $Z(\hat{L})^{W_{\mathbb{R}}}$ is trivial, and ψ_{∞} is an Adams-Johnson parameter following Definition 3.2.1.

For the statement about localization maps, the map $I^{+}_{\infty} \to I^{+}$ described after formula (3.2.1) can be seen to depend only on Δ. This determines the localization map as determined there. $\square$

5. The Trace Formula with Fixed Shape

Let $G \in \mathcal{E}_{\mathrm{ell}}(N)$ (the case of $G \in \mathcal{E}_{\mathrm{sim}}(N)$ suffices for us). If Δ is a refined shape such that $H(\Delta) = G$, define

$$S^{G}_{\Delta} := \sum_{\psi \in \Delta} S^{G}_{\psi}.$$

This S^{G}_{Δ} is our main technical building block and understanding it is the key step for achieving all our applications.

For regular integral infinitesimal character λ, let EP_{λ} be the corresponding Euler-Poincaré function as in [CD90] or [Art89a] (this exists since G_{∞} has discrete series). The overarching goal for this section is:

Goal. *Let λ be the infinitesimal character of a finite dimensional representation on G. Understand $S^G_\Delta(\mathrm{EP}_\lambda f^\infty)$ as a linear combination of terms $I^H_{\mathrm{disc}}(\mathrm{EP}_{\lambda'}(f^\infty)')$ on other groups H.*

The terms $I^H_{\mathrm{disc}}(\mathrm{EP}_{\lambda'}(f^\infty)')$ are well-understood—they were given an explicit formula in [Art89a] that was studied in great detail and bounded in [ST16]. Achieving the goal would therefore give fine control over S_Δ. Note also that these terms depend only on the choice of transferred test function $\mathrm{EP}_{\lambda'}(f^\infty)'$ and make no other reference to the shape Δ.

Taïbi in [Taï17] found an explicit description in the level-1 case. This work is basically extending that method as much as possible to deeper level.

5.1. **Overall Strategy.** We build up $S^G_\Delta(\mathrm{EP}_\lambda f^\infty)$ from $I^H_{\mathrm{disc}}(\mathrm{EP}_{\lambda'}(f^\infty)')$ terms in stages:

(1) First, we switch from I to S to allow for application of various transfers in the classification: any $S^H_{\mathrm{disc}}(\mathrm{EP}_\lambda f^\infty)$ can be expanded as a linear combination of $I^{H'}_{\mathrm{disc}}(\mathrm{EP}_{\lambda'}(f^\infty)')$ terms through "hyperendscopy" as in [Fer07].
(2) Define a sum of traces on $\widetilde{G}(M)$ called $S^M(\lambda,\eta,f^\infty)$ satisfying that if $H = H(\Sigma_{\lambda,\eta})$, $S^M(\lambda,\eta,f^\infty) = S^H(\mathrm{EP}_\lambda(f^\infty)^H)$. Transferring to traces on $\widetilde{G}(M)$ allows to to induct by decomposing Δ down to its discrete constituents.
(3) Define subsums $S^N_\Delta(f^\infty)$ of $S^N(\lambda,\eta,f^\infty)$ corresponding to individual refined shapes. All the $S^N_\Delta(f^\infty)$ can be computed from terms $S^M(\lambda,\eta,(f^\infty)')$ by an induction in two steps:
   -
$$S^N_{\Sigma_{\lambda,\eta}}(f^\infty) = S^N(\lambda,\eta,f^\infty) - \sum_{\substack{H(\Delta)=H(\Sigma_{\lambda,\eta}) \\ \text{Inf. Char}(\Delta)=\lambda \\ \Delta\neq\Sigma_{\lambda,\eta}}} S^N_\Delta(f^\infty).$$
   - If $\Delta = (T_i, d_i, \lambda_i, \eta_i)_i$ with infinitesimal character λ, understand $S^N_\Delta(f^\infty)$ well enough in terms of the $S^{T_i}_{\Sigma_{\lambda_i,\eta_i}}$. This is the hardest part of the argument and the partial results needed for our applications will be postponed to sections 7 and 8.
     For certain bounds in this step, it will also be helpful to define a term $S^{|N|}_\Delta(f^\infty)$ that in essence removes signs coming from Arthur's ϵ_ψ characters.
(4) We finally need to transfer back to the classical group—$S^G_\Delta(\mathrm{EP}_\lambda f^\infty)$ can be written as $S^{N(G)}_\Delta(f^\infty_1)$ as long as we can find $(f^\infty_1)^G = f^\infty$.

In our actual argument, steps 2 and 3 won't be so clearly separated. We will use step 2's proposition 5.3.1 to switch freely between the perspective of traces on $G \in \mathcal{E}_{\mathrm{ell}}(N)$ and traces on $\widetilde{G}(N)$ as convenient while working the induction of step 3.

5.2. **Step 1: Understanding S^H.** The S^H terms can be understood through the hyperendoscopy formula of [Fer07]. We use notation from [Dal22], although the extension there isn't necessary since computation of the endoscopy of classical groups in [Wal10] show that we will never have to take a z-extension.

Theorem 5.2.1 (Ferrari's Hyperendoscopy Formula)**.** *Let $\mathcal{HE}_{\mathrm{ell}}(H)$ be the set of non-trivial elliptic hyperendoscopic paths of H as in* [Dal22, §4]. *Then,*

$$S^H(\mathrm{EP}_\lambda f^\infty) = I^H(\mathrm{EP}_\lambda f^\infty) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(H)} \iota(G, \mathcal{H}) I^{\mathcal{H}}(\mathrm{EP}_\lambda^{\mathcal{H}}(f^\infty)^{\mathcal{H}})$$

for constants $\iota(G, \mathcal{H})$ and transfers $\star^{\mathcal{H}}$ defined there.

Proof. See [Dal22, Thm. 4.2.3] and note that using Euler-Poincaré component at infinity lets us elide the distinction between trace formulas and their discrete parts. □

There is an explicit formula of Ferrari showing that the transfers EP_λ^H can be chosen to be linear combinations of Euler-Poincaré functions. See [Dal22, §5.1] for an English-language presentation though beware that there is a ρ-shift between the parametrization EP_λ used here and the parametrization η_λ used there.

5.3. **Step 2: Understanding S^M.** Fix infinitesimal character λ of rank M. Also fix global central character η. Using Proposition 4.3.1, define

$$\Psi(\lambda, \eta) = \bigcup_{\substack{\Delta : H(\Delta) = H(\Sigma_{\lambda,\eta}) \\ \text{Inf. Char}(\Delta) = \lambda}} \Psi(\Delta)$$

and define pieces of the spectral expansion:

$$\begin{aligned} S^M(\lambda, \eta, f^\infty) &:= \sum_{\Delta : H(\Delta) = H(\Sigma_{\lambda,\eta})} S^M_\Delta(f^\infty) \\ &:= \sum_{\psi \in \Psi(\lambda,\eta)} S^M_\psi(f^\infty) \\ &:= \sum_{\psi \in \Psi(\lambda,\eta)} \epsilon^H_\psi(s^H_\psi) m_\psi |S_\psi|^{-1} \operatorname{tr}_{\widetilde{\pi}^\infty_\psi}(f^\infty). \end{aligned} \tag{5.3.1}$$

Here, π_ψ is the automorphic representation of $\mathrm{GL}_M(\mathbb{A}_E)$ corresponding to ψ as in §2.2.2 and $\widetilde{\pi}_\psi$ is its extension to $\widetilde{G}(M)$ as in §2.2.3.

Proposition 5.3.1. *Let $H = H(\Sigma_{\lambda,\eta})$ which is necessarily in $\mathcal{E}_{\mathrm{sim}}(M)$. Then*

$$S^M(\lambda, \eta, f^\infty) = S^H(\mathrm{EP}_\lambda(f^\infty)^H).$$

Proof. Fix $\psi \in \tilde{\Psi}_2(H)$ with infinitesimal character λ. Then the stable multiplicity formula 2.6.3 shows that

$$S^H_\psi(\eta_\lambda(f^\infty)^H) = \epsilon_\psi(s_\psi)|S_\psi|^{-1} \left(\sum_{\pi_\infty \in \Pi_{\psi_\infty}} \eta^{\psi_\infty}_{\pi_\infty}(s_\psi) EP(\pi_\infty, \lambda) \right) \left(\sum_{\pi \in \Pi_{\psi^\infty}} \eta^{\psi^\infty}_{\pi^\infty}(s_\psi) \operatorname{tr}_{\pi^\infty}((f^\infty)^H) \right).$$

After recalling that S_ψ is a 2-group, lemma 3.4.3 shows that for each pseudocoefficient φ_{π_d} with infinitesimal character λ:

$$\sum_{\pi_\infty \in \Pi_{\psi_\infty}} \eta^{\psi_\infty}_{\pi_\infty}(s_\psi) \operatorname{tr}_{\pi_\infty}(\varphi_{\pi_d}) = 1.$$

Averaging over pseudocoefficients:

$$\sum_{\pi_\infty \in \Pi_{\psi_\infty}} \eta^{\psi_\infty}_{\pi_\infty}(s_\psi) EP(\pi_\infty, \lambda) = 1.$$

Furthermore, multiplying together the endoscopic character identity 2.6.2 over all finite places shows that:

$$\sum_{\pi \in \Pi_{\psi^\infty}} \eta_{\pi^\infty}(s_\psi) \operatorname{tr}_{\pi^\infty}((f^\infty)^H) = \operatorname{tr}_{\widetilde{\pi}^\infty_\psi}(f^\infty).$$

In total $S^H_\psi(\eta_\lambda(f^\infty)^H) = S^M_\psi(f^\infty)$.

Since $S_\psi(\eta_\lambda(f^\infty)^H) = 0$ for all ψ with infinitesimal character not equal to λ, summing over $\Psi(\lambda, \eta)$ and using Corollary 4.3.2 gives that

$$S^M(\lambda, \eta, f^\infty) = \sum_{\psi \in \Psi(H)} S^H_\psi(\mathrm{EP}_\lambda(f^\infty)^H).$$

By [Art89a, (3.9)] and equation (2.6.1), we know that

$$I^G(\mathrm{EP}_\lambda(f^\infty)^H) = R^G(\mathrm{EP}_\lambda(f^\infty)^H) = \sum_{\psi \in \Psi(H)} I^H_\psi(\mathrm{EP}_\lambda(f^\infty)^H).$$

By a hyperendoscopy argument using the expansion in theorem 2.6.4, the same sum expansion holds for S^G. Therefore, we can conclude that

$$S^M(\lambda, \eta, f^\infty) = S^H(\eta_\lambda(f^\infty)^H)$$

This finishes the argument. □

5.4. **Step 3: The Induction.** We give a heuristic overview to keep in mind for understanding step 3. All precise results will be postponed to Sections 7 and 8.

Recall the decomposition

$$S^N(\lambda, \eta, f^\infty) = \sum_{\Delta : H(\Delta) = H(\Sigma_{\lambda,\eta})} S^N_\Delta(f^\infty).$$

which we were using for the induction in step 3. We need to understand the second bullet point—reducing the individual S_Δ in terms of smaller groups.

For the sake of heuristic understanding, we will instead consider the simpler

$$S^{|H(\Delta)|}_\Delta(\eta_\lambda(f^\infty)^{H(\Delta)}) = S^{|N|}_\Delta(f^\infty) := \sum_{\psi \in \Delta} m_\psi |S_\psi|^{-1} \operatorname{tr}_{\widetilde{\pi}^\infty_\psi}(f^\infty).$$

without the ϵ-sign.

Consider $\psi = \tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k] \in \Delta$. Motivated by the way π_ψ is defined through parabolically inducing determinant twists of the τ_i, assume we could define a "generalized constant term" map

$$f^\infty \mapsto (f^\infty_i)_{\Delta,i}$$

such that for all $\psi \in \Delta$.

$$\operatorname{tr}_{\widetilde{\pi}^\infty_\psi}(f^\infty) = \prod_i \operatorname{tr}_{\widetilde{\pi}^\infty_{\tau_i}}(f^\infty_{\Delta,i}).$$

Because the infinitesimal character of Δ is regular and disallows repeated τ_i factors, the possible (elliptic) $\psi \in \Delta$ are exactly the $\tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k]$ for all choices of $\tau_i \in \Sigma^{T_i}_{\lambda_i,\eta_i}$. Therefore we get a heuristic factorization

$$S^{|N|}_{\Delta}(f^\infty) = C_\Delta \prod_i S^{|T_i|}_{\Sigma_{\lambda_i,\eta_i}}(f^\infty_{\Delta,i}), \tag{5.4.1}$$

where C_Δ is a constant depending on the various S_ψ, S_{τ_i} and m_ψ's that only depend on Δ.

Obviously, we are not allowed to just ignore the ϵ-sign and we do not have a actual definition of this generalized constant term. In fact, the definition of this generalized constant term would allow us to define the long-desired "stable transfer" between G and its endoscopic groups, so it is likely very difficult.

However, sections 7 and 8 will discuss enough partial results that an application to limit multiplicities at specifically split level can be completed.

5.5. **Step 4: Understanding S^G_Δ.** This step comes from a corollary to the arguments in step (2):

Corollary 5.5.1. *Choose refined shape Δ of rank N and let $G = H(\Delta)$ as in Proposition 4.3.1. Then for any test function f^∞ on $G(\mathbb{A}^\infty)$ with stable orbital integrals invariant under outer automorphisms of G, there is f^∞_1 on $\widetilde{\mathrm{GL}}_N(\mathbb{A}^\infty)$ such that $(f^\infty_1)^G = f^\infty$. Furthermore,*

$$S^G_\Delta(\eta_\lambda f^\infty) = S^N_\Delta(f^\infty_1).$$

Proof. The existence of f^∞_1 comes from [Mok15, Prop. 3.1.1(b)]. Then, arguing as in 5.3.1 gives $S^G_\psi(\eta_\lambda (f^\infty_1)^G) = S^N_\psi(f^\infty_1)$ for any $\psi \in \Delta$. Summing over all $\psi \in \Delta$ produces the result. $\square$

6. Lemmas on Local Transfers

Before understanding step 3 in detail, we need some local lemmas at non-Archimedean places. We will heavily use the shorthand from section 1.6.2 throughout. We will also use some new notation. If G_v is a group over F_v then

- $K_v := K^G_v$ is a hyperspecial for G_v when one exists
- $K_v(q^n_v)$ is the nth step in the Moy-Prasad filtration at K.

6.1. **Transfers at Split Places.** We first discuss some results that only hold at split places. In fact, our inability to extend lemmas 6.1.2 and 6.1.3 to non-split places is the main reason our final multiplicity result is restricted to split level.

Fix $G \in \mathcal{E}_{\mathrm{ell}}(N)$. Let v be a finite place of F which splits in E. Then $E_v := E \otimes_F F_v \simeq F_v \times F_v$, and σ permutes the two copies of F_v. Then $\widetilde{G}(N, F_v) \simeq GL_N(F_v) \times GL_N(F_v)$ and the action of θ_N on $\widetilde{G}(N, F_v)$ becomes

$$(g_1, g_2) \mapsto (\mathrm{Ad}(J_N) g_2^{-t}, \mathrm{Ad}(J_N) g_1^{-t}).$$

Then $G(F_v) \simeq GL_N(F_v)$ and is embedded in $\widetilde{G}(N, F_v)$ as the fixed points of θ. A possible isomorphism is $g \mapsto (g, \mathrm{Ad}(J_N) g^{-t})$.

For the reader's convenience, we now summarize some results that show Arthur's classification for $G(N)(F_v)$ as a element of $\mathcal{E}_{\mathrm{ell}}(N)$ matches Arthur's classification coming from the isomorphism $G \simeq \mathrm{GL}_N(F_v)$:

Lemma 6.1.1. *Let v be such that the global extension E/F splits at v and denote $E \otimes_F F_v$ by E_v. Let π_v be the irreducible conjugate self-dual representation of $G(N)(E_v) \simeq GL_N(F_v) \times GL_N(F_v)$ coming from Arthur parameter ψ_v. Then*

1. *π_v is of the form $\pi_v^0 \otimes (\pi_v^0)^\vee$ for an irrep π_v^0 of $\mathrm{GL}_N(F_v)$,*
2. *The canonical extension of π_v to $\widetilde{G}(N)$ as in* [Art13, §2.2] *has θ acting on $\pi_v^0 \otimes (\pi_v^0)^\vee$ through $x \otimes y \mapsto y \otimes x$,*
3. *For $\tilde{f}_v = (f_v^1, f_v^2) \in \mathcal{H}(G(N)(E_v) \rtimes \theta)$, we can choose transfer $\tilde{f}_v^G = f_v^1 \star {}^\theta f_v^2$ where we define ${}^\theta f_v(g) = f_v(\mathrm{Ad}(J_n) g^{-t})$.*
4. *For $\tilde{f}_v = (f_v^1, f_v^2) \in \mathcal{H}(G(N)(E_v) \rtimes \theta)$, $\mathrm{tr}_{\tilde{\pi}_v}(\tilde{f}_v) = \mathrm{tr}_{\pi_v^0}(f_v^1 \star {}^\theta f_v^2)$*
5. *$\mathrm{tr}_{\psi_v}(f_v) = \mathrm{tr}_{\pi_v^0}(f_v)$.*

Proof. (1) follows from the description of irreps of a product of groups and self-duality.

For (2), first assume π_v and therefore π_v^0 is tempered. A Whittaker functional on π_v is a product of a pair of functionals on π_v^0 and $(\pi_v^0)^\vee$. This product is preserved by the claimed Θ since the space Whittaker functionals is one dimensional. On the other hand, if π isn't tempered, the statement can be checked by the construction of Θ through parabolic induction. This gives the second statement in all cases.

For (3), this is a special case of corollary 1.1.6 in [KMSW14].

For (4), $\tilde{f}_v(x \otimes y) = (f_v^1 x \otimes {}^\theta f_v^2 y)$. By admissibility of π_v and smoothness of the f_v^i, we can compute the trace by choosing a basis for the finite-dimensional vector space $V = (\pi_v)^U$ for some open compact U. The result follows from computing the action on the standard induced bases for $\mathrm{Sym}^2 V \oplus \wedge^2 V = V \oplus V$.

Finally, (5) follows from (3), (4), and the endoscopic character relation after choosing f_v^1 and ${}^\theta f_v^2$ such that $f_v^1 \star {}^\theta f_v^2 = f_v$. □

As the two consequences:

Lemma 6.1.2. *Let v be a place that's split in E/F and $\psi_v = \psi_{1,v} \oplus \psi_{2,v}$ be an Arthur parameter for $U_{E/F}(N)(F_v) \simeq \mathrm{GL}_N(F_v)$. Then ψ factors through (the L-dual of) a Levi subgroup $M = \mathrm{GL}_{N_1} \times \mathrm{GL}_{N_2}(F_v)$ and*

$$\mathrm{tr}_{\psi_v} f = \coprod_{i=1,2}^{\oplus} \mathrm{tr}_{\psi_{i,v}} f_{M,i}$$

where $f_{M,1}$ and $f_{M,2}$ are the factors of the constant term map to M.

Proof. Then $\tilde{\pi}_{\psi_v}$ is a parabolic induction of $\tilde{\pi}_{\psi_{1,v}} \otimes \tilde{\pi}_{\psi_{2,v}}$. In the notation of lemma 6.1.1 (1), $\pi_{\psi_v}^0$ is therefore a parabolic induction of $\pi_{\psi_{1,v}}^0 \otimes \pi_{\psi_{2,v}}^0$. The result follows from 6.1.1 (5). □

Lemma 6.1.3. *Let v be a place that's split in E/F and $\psi_v = \psi_{i,v}[d]$ be an Arthur parameter for $U_{E/F}(N)(F_v) \simeq \mathrm{GL}_N(F_v)$. Then $\psi|_{W_F}$ factors through (the L-dual of) a Levi subgroup $M = {\mathrm{GL}_{N_1}}^d$.*

Furthermore, for all test functions f satisfying

- *f is supported on the kernel of $|\det|_v$,*
- *for all unirreps π_v of $\mathrm{GL}_N(F_v)$, $tr_{\pi_v}(f) \geq 0$,*

we have:

$$\mathrm{tr}_{\psi_v} f \leq (\mathrm{tr}_{\psi_{1,v}} f_{M,1})^{d\oplus}$$

where $f_{M,1}$ is the constant term to M restricted to the first factor.

Proof. This is a similar argument to 6.1.2 using that $\tilde{\pi}_{\psi_v}$ is a summand in the Grothendieck group of the parabolic induction of

$$\tilde{\pi}_{\psi_{1,v}}|\det|^{\frac{d-1}{2}} \otimes \cdots \otimes \tilde{\pi}_{\psi_{2,v}}|\det|^{\frac{1-d}{2}}$$

We may ignore the determinant factors by the support condition. The inequality comes from the positivity condition applied to traces against the other summands in the Grothendieck group expansion. ☐

6.2. **Transfers at Unramified Places.**

6.2.1. *Basic Transfer Result.* First we recall an "Arthur packet fundamental lemma" that was the key result making the strategy of Section 5 work in the level-1 case considered by [Taï17].

Lemma 6.2.1 (Fundamental Lemma for A-packets)**.** *Let v be a place that's unramified in E/F and*

$$\psi_v = \bigoplus_i \tau_{i,v}[d_i]$$

be an Arthur parameter for $U_{E/F}(N)(F_v)$. Then

$$\mathrm{tr}_{\psi_v} \mathbf{1}_{K_v} = \prod_i \mathrm{tr}_{\psi_{i,v}} \mathbf{1}_{K_{i,v}}$$

for appropriately chosen hyperspecial subgroups $K_{\star,v}$ in appropriate $U_{E/F}(N_\star)(F_v)$.

Proof. This follows from [Taï17, Lem. 4.1.1] since ψ_v is unramified if and only if each of the $\tau_{i,v}$ is. Then, just apply that for any π_v, $\mathrm{tr}_{\pi_v} \mathbf{1}_{K_{i,v}} = \mathbf{1}_{\pi_v \text{ unram.}}$.

The $K_{\star,v}$ are chosen as in the fundamental lemma according to a choice of Whittaker datum. ☐

We will eventually attempt to prove bounds in the style of [ST16], so we need a more general statement for any element of $\mathscr{H}^{\mathrm{ur}}(G_v)$.

6.2.2. *Truncated Hecke Algebras.* We begin by recalling the notion of a bounded Hecke algebra from [ST16]. The elements $\tau^G_\lambda = \mathbf{1}_{K_v\lambda(\varpi)K_v}$ for a chosen unformizer ϖ and $\lambda \in X_*(A)^+$ generate $\mathscr{H}^{\mathrm{ur}}(G_v)$. Pick a basis $\mathcal{B}$ for the $X_*(A)$ and define norm:

$$\|\lambda\|_{\mathcal{B}} = \max_{\omega\in\Omega}(\text{biggest } \mathcal{B}\text{-coordinate of } \omega\lambda).$$

For $\lambda \in X_*(A)$. Define truncated Hecke algebra

$$\mathscr{H}(G,K)^{\leq\kappa,\mathcal{B}} = \langle \tau^G_\lambda : \|\lambda\|_{\mathcal{B}} \leq \kappa\rangle.$$

It turns out (see [ST16, §2]) that for any two $\mathcal{B}, \mathcal{B}'$, $\|\lambda\|_{\mathcal{B}} = \Theta(\|\lambda\|_{\mathcal{B}'})$. All the bounds we use will depend on κ only up to an unspecified constant. Therefore we can suppress the $\mathcal{B}$.

6.2.3. *Basis of Characters.* Recall that the Satake transform gives an isomorphism

$$\mathscr{H}^{\mathrm{ur}}(G_v) \xrightarrow{\sim} \mathbb{C}[X_*(A)]^{\Omega_F}$$

where A is a maximally split maximal torus of G_v in good position with respect to K_v. The right side of this isomorphism has a basis χ_λ of trace characters of finite dimensional representations λ of the twisted group $\widehat{G} \rtimes \mathrm{Frob}_v$.

Any unramified parameter ψ_v determines an unramified L-parameter which determines a Satake parameter: a semisimple conjugacy class $s_{\psi_v} \in \widehat{G} \rtimes \mathrm{Frob}_v$. Because of [Taï17, Lem. 4.1.1], this satisfies that

$$\mathrm{tr}_{\psi_v} \chi_\lambda = \mathrm{tr}_\lambda(s_{\psi_v}). \tag{6.2.1}$$

See [ST16, §2.2] for more detail.

The consistency of unramified packets constructed by Arthur/Mok and the Satake isomorphism is implicit in the isolation of the ψ-part I_ψ of I_{disc}—see §3.3 in [Art13] for example. It depends on the full fundamental lemma for all spherical functions.

6.2.4. *General Unramified Transfer.* Let v be a place that's unramified in E/F and

$$\psi_v = \bigoplus_i \tau_{i,v}[d_i] \in \Delta = (t_i, d_i, \lambda_i, \eta_i)_i$$

be an Arthur parameter for $U_{E/F}(N)(F_v)$. Let each $\tau_{i,v}$ be a parameter for U_i. Then there is an associated embedding

$$\iota_\Delta : \mathbf{H} := {}^L H_v \times \prod_i {}^L\mathrm{GL}_{d_i} := \prod_i ({}^L U_i \times {}^L\mathrm{GL}_{d_i}) \hookrightarrow {}^L G_v. \tag{6.2.2}$$

Let ψ_n^I be the parameter of the trivial representation on GL_n. Then we can write the Langlands parameter φ_{ψ_v} corresponding to ψ_v as the pushforward of

$$\prod_i \tau_{i,v} \times \psi^I_{d_i}.$$

This gives map on Satake parameters

$$s_{\psi_v} = \mathcal{S}_\Delta((s_{\tau_{i,v}})_i) := \iota_\Delta \left(\prod_i s_{\tau_{i,v}} \times s^I_{d_i} \right) \tag{6.2.3}$$

where $s^I_{d_i}$ is the Satake parameter of the d_i-dimensional trivial representation.

Restricting the embedding to the $\star \rtimes \mathrm{Frob}$ cosets determines a map

$$\mathcal{T}_\Delta : \mathscr{H}^{\mathrm{ur}}(G_v) \to \mathscr{H}^{\mathrm{ur}}(H_v)$$

by pre-composing with $\mathcal{S}_\Delta$: $\mathcal{T}_\Delta(\chi_\lambda) = \chi_\lambda \circ \mathcal{S}_\Delta$. Equation (6.2.1) then gives:

Lemma 6.2.2. *With notation as above, let $f \in \mathscr{H}^{\mathrm{ur}}(G_v)$. Then*

$$\mathrm{tr}_{\psi_v}(f) = \prod_i \mathrm{tr}_{\tau_{i,v}}(\mathcal{T}_{\Delta,i} f),$$

where the $\mathcal{T}_{\Delta,i}$ are the factors at individual U_i.

We will often supress the Δ when it is clear from context. We need some control over what $\mathcal{T}_\Delta$ does:

Lemma 6.2.3. *Let $f \in \mathscr{H}^{\mathrm{ur}}(G_v)^{\leq \kappa}$ with $\|f\|_\infty \leq 1$. Then $\mathcal{T}_\Delta f \in \mathscr{H}^{\mathrm{ur}}(H_v)^{\leq \kappa}$ and $\|\mathcal{T}_\Delta f\|_\infty = O(q_v^{D\kappa} \kappa^E)$ for some constants D and E that only depend on G and Δ.*

Proof. This is a slightly more complicated version of the argument of Lemma 5.5.4 in [Dal22]. There is an additional step from bounding the trace of the Satake parameter of the trivial representation against the finite dimensional irreps of ${}^L\mathrm{GL}_{d_i}$ that appear as factors in restrictions from ${}^LG_v$ to $\mathbf{H}$. This can be seen to be $O(q_v^{D'\kappa})$ for some D' by the Weyl character formula. □

We also can similarly define a simpler map $\mathcal{T}_{\bar{\Delta}}$ such that

$$\mathrm{tr}_{\psi_v}(f) = \coprod_i \mathrm{tr}_{\tau_{i,v}[d_i]}(\mathcal{T}_{\bar{\Delta},i} f).$$

By the same arguments, this map also satisfies lemma 6.2.3. When we suppress Δ in notation, which version of $\mathcal{T}$ we are using should always be clear by the context of what the $\mathcal{T}_i$'s have image in. We note for intuition that since it acts on unramified functions, $\mathcal{T}_{\bar{\Delta}}$ can be thought of as hyperendscopic transfer as in theorem 5.2.1. See [Dal22, §5.5] for more details.

Finally, if Δ is a simple shape of the form $(1, d, \lambda, \eta)$, then for $\psi \in \Delta$ the possible s_{ψ_v} from formula (6.2.3) are all Satake parameters of characters. In addition, $H_v = U_1(F_v) = (G_v)^{\mathrm{ab}} = (G_{\mathrm{ab}})_v$ (note that $G_{v,\mathrm{der}}$ is semisimple and simply-connected and therefore has trivial cohomology). We can therefore compute

$$\mathcal{T}_\Delta(f_v)(h) = \int_{G_{v,\mathrm{der}}} f_v(hg)\, dg. \tag{6.2.4}$$

6.3. **Transfers at General Places: Inequalities.**

6.3.1. *Twisted Bernstein Components.* We recall some notions from [Rog88] on Bernstein components for twisted groups. Let G_v be the F_v points of a connected reductive group and let twisted group $\widetilde{G}_v = G_v \rtimes \theta$ for some automorphism θ of G_v. We assume there is a minimal parabolic P_0 and Levi factor M_0 that are θ-stable.

Let $\mathcal{L}(G_v)$ be the set of standard Levis of G_v with respect to M_0.

Definition 6.3.1. The twisted cuspidal supports for $\widetilde{G}_v$ are then pairs (M, σ) with $M \in \mathcal{L}(G_v)$ such that $(\theta M, \theta\sigma)$ is conjugate to (M, σ) in G_v.

Definition 6.3.2. We say that a θ-invariant irrep π of G_v has infinitesimal character (M, σ) if π is an irreducible subquotient of $\mathrm{Ind}_{MP_0}^{G_v} \sigma$.

Recall that every θ-invariant representation has an infinitesimal character that is a twisted cuspdial support.

Definition 6.3.3. If $\tilde{\pi}$ is an irrep of $\widetilde{G}_v$ with non-zero (twisted trace) character, then the infinitesimal character of $\tilde{\pi}$ is that of $\tilde{\pi}|_{G_v}$

Note that this restriction is necessarily θ-invariant if $\tilde{\pi}$ has non-zero character. Furthermore, all extensions of $\tilde{\pi}|_{G_v}$ differ by a root of unity of order dividing that of θ.

Definition 6.3.4. Let (M, σ) be a twisted cuspidal support. Its twisted Bernstein component is the set of irreps $\tilde{\pi}$ with non-zero character on $\widetilde{G}_v$ such that their infinitesimal characters are of the form $(M, \sigma\chi)$ for χ an unramified character of M (and $\sigma\chi$ θ-invariant).

The key point is that:

Lemma 6.3.5. *Let f_v be a compactly supported, smooth function on $\widetilde{G}_v$. Then $\pi \mapsto \operatorname{tr}_\pi f_v$ is supported on a finite number of twisted Bernstein components.*

Proof. This is part of the main result of [Rog88]. ☐

Now specialize to the case of $G = \tilde{G}(N)$.

Proposition 6.3.6. *Let $\mathfrak{s}$ be a twisted Bernstein component of $\widetilde{G}(N)_v$ and let $H \in \mathcal{E}_{\mathrm{ell}}(N)$. Then there is a finite list $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ of Bernstein components of H_v such that the Arthur packets Π_{ψ_v} for $\psi \in \Psi_v^+(H)$ with $\widetilde{\pi}_\psi \in \mathfrak{s}$ only contain representations in the $\mathfrak{s}_i$*

Proof. This follows from three facts: first, the compatibility of twisted endoscopic transfer of characters with Jacquet modules as in diagram (C.4) in [Xu17], second, the finiteness of A-packets, and third, the compatibility of transfer of characters with unramified character twist as in proposition 4.4 of [Oi21].

We thank Masao Oi for pointing this out to us. ☐

6.3.2. *Inequalities.* This lets us show:

Lemma 6.3.7. *Let v be a place of F and f_v be a test function on $U_{E/F}(N)(F_v)$. Let $dN_1 = N$. Then there exists test function φ_v on $U_{E/F}(N_1)(F_v)$ such that for all Arthur parameters $\psi_v = \psi_{1,v}[d]$ with $\psi_{\star,v}$ a parameter for $U_{E/F}(N_\star)(F_v)$ and $\psi_{1,v}$ cuspidal:*

$$|\operatorname{tr}_{\psi_v} f_v| \leq (\operatorname{tr}_{\psi_{1,v}} \varphi_v)^d.$$

Proof. First,

$$\operatorname{tr}_{\psi_v} f_v = \operatorname{tr}_{\tilde{\pi}_{\psi_v}} f_v^N$$

By Bernstein's admissibility theorem (as used in [ST16, prop. 9.6]) there is C (without loss of generality, $C > 1$) such that for all unirreps $\tilde{\pi}'$ of $\widetilde{G}(N)_v$,

$$|\operatorname{tr}_{\tilde{\pi}'} f_v^N| \leq C$$

Now consider the function from the unitary dual of $\widetilde{G}(N_1)_v$ to $\mathbb{C}$ given by

$$\Phi_0 : \tilde{\pi} \mapsto tr_{\tilde{\pi}[d]} f_v^N$$

This is supported on finitely many Bernstein components $\mathfrak{s}_1, \dots, \mathfrak{s}_i$ after noting that f_v^N is and that the cuspidal support of $\tilde{\pi}[d]$ determines that of $\tilde{\pi}$

Therefore, the representations $\pi \in \Pi_{\psi_{1,v}}$ for $\tilde{\pi}_{\pi_{1,v}} \in \mathfrak{s}_i$ lie in a finite set of Bernstein components by proposition 6.3.6. Since $U_{E/F}(N_1)$ is not twisted, it is easy to find a function φ_v on $U_{E/F}(N_1)$ so that $\operatorname{tr}_\star \varphi_v \geq C$ on each of these components (e.g. a scalar multiple of the indicator function of a small enough maximal compact depending on the Bushnell-Kutzko types associated to the Bernstein components).

Finally, since $\psi_{1,v}$ is cuspidal and therefore simple, $\operatorname{tr}_{\psi_{1,v}} \varphi_v$ is a sum of various $\operatorname{tr}_\pi \varphi_v$ and in particular larger. Therefore, this choice of φ_v suffices. ☐

Lemma 6.3.8. *Let v be a place of F and f_v be a test function on $U_{E/F}(N)(F_v)$. Let $N_1 + \dots + N_n = N$. Then there exists test functions f_i on $U_{E/F}(N_i)(F_v)$ such that for all Arthur parameters $\psi_v = \psi_{1,v} \oplus \dots \oplus \psi_{n,v}$ with each $\psi_{\star,v}$ a parameter for $U_{E/F}(N_\star)(F_v)$ and the $\psi_{i,v}$ simple:*

$$|\operatorname{tr}_{\psi_v} f_v| \leq \prod_i \operatorname{tr}_{\psi_{i,v}} f_{i,v}.$$

Proof. This is the same argument as lemma 6.3.7. ☐

6.4. **Transfers at General Places: Conjectural Equalities.** At the current moment, we do not have any exact equality results for transfers at general places. This is the main reason why our exact asymptotic result 11.4.1 is restricted to unitary groups for unramified E/F. We state the desired conjecture (which is probably extremely difficult):

Conjecture 6.4.1 (Full Transfer). *Let $\Delta = (T_i, d_i, \eta_i, \lambda_i)_i$ be a shape and φ_v a trace-positive test function on G_v. Then there are trace-positive functions $\varphi_{i,v}$ such that for all $\psi = \sum_i \psi_i \in \Delta$*

$$\mathrm{tr}_\psi\, \varphi_{i,v} = \prod_i \mathrm{tr}_{\psi_i}\, \varphi_{i,v}.$$

Except for the positivity statement, conjecture 6.4.1 would be implied by two local conjectures on $\widetilde{G}(N)_v$:

Conjecture 6.4.2 (Stable Transfer). *Let $\Delta = (\Delta_i)_i = (T_i, d_i, \eta_i, \lambda_i)_i$ be a shape of rank N and φ_v a test function on $\widetilde{G}(N)$. Then there are test functions $\varphi_{i,v}$ on each $\widetilde{G}(T_i d_i)$ such that for all choices $\psi_i \in \Delta_i$*

$$\mathrm{tr}_{\widetilde{\pi}(\psi)}\, \varphi_{i,v} = \prod_i \mathrm{tr}_{\widetilde{\pi}(\psi_i)}\, \varphi_{i,v}.$$

Note that $\pi(\psi)$ is just the parabolic induction of the product of the $\pi(\psi_i)$. The difficulty in constructing stable transfers is that the relation between $\widetilde{\pi}(\psi)$ and the $\widetilde{\pi}(\psi_i)$ is far more complicated because of the choice of extension in section 2.2.3.

The second necessary local conjecture is:

Conjecture 6.4.3 (Speh Transfer). *Let $\Delta = (T, d, \eta, \lambda)$ be a simple shape and φ_v a test function on $\widetilde{G}(Td)$. Then there is test function φ'_v on $\widetilde{G}(T)$ such that for all $\psi \in (T, 1, \eta, \lambda)$,*

$$\mathrm{tr}_{\widetilde{\pi}(\psi[d])}\, \varphi_v = \mathrm{tr}_{\widetilde{\pi}(\psi)}\, \varphi'_v.$$

For Speh transfer, even the relation between the untwisted representations $\pi(\psi[d])$ and $\pi(\psi)$ is difficult due to the reducibility of the relevant parabolic induction.

For this specific work, we will only desire conjecture 6.4.1 for odd GSK-shapes Δ as in definition 8.2.1.

7. Induction Step Details: General Shapes

For any refined shape Δ of rank N, recall

$$S^N_\Delta(f^\infty) := \sum_{\psi \in \Delta} S^N_\psi(f^\infty) = \sum_{\psi \in \Delta} \epsilon_\psi^{H(\Delta)}(s_\psi^{H(\Delta)}) m_\psi |S_\psi|^{-1}\, \mathrm{tr}_{\widetilde{\pi}(\psi)^\infty}(f^\infty).$$

Note that proposition 5.3.1 still works with subscripts of Δ added to both sides.

Also recall

$$S^{|H(\Delta)|}_\Delta(\eta_\lambda (f^\infty)^{H(\Delta)}) = S^{|N|}_\Delta(f^\infty) := \sum_{\psi \in \Delta} m_\psi |S_\psi|^{-1}\, \mathrm{tr}_{\widetilde{\pi}(\psi)^\infty}(f^\infty). \tag{7.0.1}$$

without the ϵ-sign.

The key point is that our eventual goal is an asymptotic formula for the trace formula with fixed shape, not anything exact. The induction process of step 3 writes this as a linear combination of terms for other shapes. Therefore, we will only need to solve two problems here:

- In this section, prove an upper bound (Theorem 7.3.3) for a general shape to show that terms for shapes with non-dominant contribution are negligible.
- In section 8, find an exact asymptotic for terms with the very special types of shapes that could be dominant in our application.

This section will heavily use our shorthand notations from 1.6.2 for non-factorizable functions.

7.1. **Preliminary Bound.** We first need a technical bound relating S^H to $S^{|H|}$ terms. This will rely on a key condition that we will need to assume henceforth for various test functions:

Definition 7.1.1. Let S be some finite or infinite set of places of F and f_S a function on G_S. We say that f_S is trace-positive on S if for all unirreps π_S of G_S that appear in an A-packet, $\mathrm{tr}_{\pi_S} f_S \geq 0$.

Proposition 7.1.2. *Let $H = H(\Delta) \in \mathcal{E}_{\mathrm{ell}}(N)$. Assume that f^∞ is trace-positive. Then there is an elliptic endoscopic group $H' = H_1 \times H_2$ of H such that for λ' any infinitesimal character at infinity for H' conjugate to λ over H:*

$$|S^N_\Delta((f^\infty)^N)| \leq C \coprod_{i=1,2} S^{H_i}_{\Delta_i}(\eta_{\lambda'_i}(f^\infty)^i)$$

Here we use shorthand: the data Δ, λ', and $(f^\infty)^{H'}$ all factor into components for H_1 and H_2 denoted with appropriate sub/superscripts. The C is a constant depending only on H and Δ.

The H' further satisfies that

$$S^{H_1}_{\Delta_1}(\eta_{\lambda'_1}(f^\infty)^1)S^{H_2}_{\Delta_2}(\eta_{\lambda'_2}(f^\infty)^2) = S^{|H_1|}_{\Delta_1}(\eta_{\lambda'_1}(f^\infty)^1)S^{|H_2|}_{\Delta_2}(\eta_{\lambda'_2}(f^\infty)^2).$$

Proof. Moving the absolute value within the sum from an intermediate computation in 5.3.1, we know that

$$|S^N_\Delta((f^\infty)^N)| \leq \sum_{\psi \in \Delta} m_\psi |\mathcal{S}_\psi|^{-1} \left(\sum_{\pi \in \Pi^H_{\psi^\infty}} \mathrm{tr}_{\pi^\infty}(f^\infty) \right).$$

Consider $\psi \in \Delta$. All such ψ have the same s_ψ and therefore the same endoscopic group H' corresponding to $(\psi, x = s_\psi)$. The tuples (H', ψ') that come from (ψ, x) form a $\widehat{H}$-conjugacy class intersected with the parameters of $\widehat{H}'$. They can therefore be specified by their infinitesimal character at ∞. In particular, the choice of λ' uniformly determines a unique choice of ψ' for each $\psi \in \Delta$.

Then, by the stable multiplicity formula, iterations of the computations of proposition 5.3.1, and the twisted character identity away from infinity:

$$\begin{aligned} S^{H'}_{\psi'}(\eta_{\lambda'}(f^\infty)') &= \epsilon^{H'}_{\psi'}(s_{\psi'}) m_{\psi'} |\mathcal{S}_{\psi'}|^{-1} \left(\sum_{\pi \in \Pi^H_{\psi^\infty}} \eta^{\psi^\infty}_{\pi^\infty}(s_\psi x)\, \mathrm{tr}_{\pi^\infty}(f^\infty) \right) \\ &= m_{\psi'} |\mathcal{S}_{\psi'}|^{-1} \left(\sum_{\pi \in \Pi^H_{\psi^\infty}} \mathrm{tr}_{\pi^\infty}(f^\infty) \right), \end{aligned}$$

where the second equality uses the endoscopic sign lemma $\epsilon^{H'}_{\psi'}(s_{\psi'}) = \epsilon^G_\psi(s_\psi x)$ and that $s_\psi x = s_\psi^2 = 1$. We also use f' as shorthand for the transfer to H'.

A similar computation applies to $S^{|H'|}$ so we also have

$$S^{H'}_{\psi'}(\eta_{\lambda'}(f^\infty)') = S^{|H'|}_{\psi'}(\eta_{\lambda'}(f^\infty)').$$

Summing over $\psi \in \Delta$ on one side and the corresponding ψ' uniquely determined by λ' then produces

$$|S^N_\Delta((f^\infty)^N)| \leq \frac{m_\psi |\mathcal{S}_\psi|^{-1}}{m_{\psi'}|\mathcal{S}_{\psi'}|^{-1}} S^{H'}_\Delta(\eta_{\lambda'}(f^\infty)')$$

where $S^{H'}_\Delta$ is defined by summing over parameters that pushforward to something of the right shape on H. Both results then follow from factoring the H' term into H_1 and H_2 terms in these last two expressions. □

7.2. **Reduction to Simple Shapes.** Now we discuss a specific test function. Choose $G \in \mathcal{E}_{\mathrm{ell}}(N)$ and pick finite sets of places $S = \infty \sqcup S_s \sqcup S_b$ where S_s is all split and S_b contains all the ramified places. We will look at test functions of the form:

$$f^\infty = \varphi_{S_b} f_{S_s} f^S$$

where K^S is a hyperspecial in G^S, φ_{S_b} and f_{S_s} are arbitrary, and $f^S \in \mathscr{H}^{\mathrm{ur}}(G^S)$

Consider refined shape $\Delta = (T_i, d_i, \lambda_i, \eta_i)_i$ and $\psi = \tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k] \in \Delta$. Iteratively applying lemma 6.1.2, we can realize

$$M = \prod_i \mathrm{GL}_{T_i d_i}(F_{S_s})$$

as a Levi of $G(F_{S_s}) \simeq \mathrm{GL}_N(F_{S_s})$ so that

$$\mathrm{tr}_{\psi_{S_s}} f_{S_s} = \prod_i \mathrm{tr}_{\tau_{i,S_s}[d_i]}(f_{S_s})_{M,i},$$

where $(\star)_{M,i}$ is the ith factor of the constant term map to M.

Lemma 6.2.2 gives that

$$\mathrm{tr}_{\psi^S} f^S = \prod_i \mathrm{tr}_{\tau_i^S[d_i]} \mathcal{T}_i f^S$$

and lemma 6.3.8 constructs φ_{i,S_b} so that

$$|\,\mathrm{tr}_{\psi_{S_s}} \varphi_{S_b}| \leq \prod_i \mathrm{tr}_{\tau_{i,S_b}[d_i]} \varphi_{i,S_b}.$$

Next, each $\epsilon_{\tau_i[d_i]}$ is trivial for each of the $\tau_i[d_i]$ since they are simple. Multiplying together the above trace identities,

$$\left| S^{|N|}_\psi((f_{S_s}\varphi_{S_b} f^S)^N) \right| \leq C_\Delta \left| \prod_i S^{T_i d_i}_{\tau_i[d_i]}((((f_{S_s})_{M,i}\varphi_{i,N}\mathcal{T}_i f^S)^{T_i d_i}) \right| \tag{7.2.1}$$

for some constant C_Δ that only depends on Δ.

By regularity of the infinitesimal character, none of the $(T_i, d_i, \lambda_i, \eta_i)$ appears with multiplicity more than 1. Therefore, summing over each factor $\tau_i[d_i] \in (T_i, d_i, \lambda_i, \eta_i)$ gives us a sum over all possible elliptic parameters of shape Δ:

Proposition 7.2.1. *Let notation be as in the above discussion and assume that f_{S_s} and f^S are trace-positive. Then*

$$S^{|N|}_\Delta((f_{S_s} f_{S_b} f^S)^N) = C_\Delta \coprod_i S^{T_i d_i}_{(T_i, d_i, \lambda_i, \eta_i)}((((f_{S_s})_{M,i} \varphi_{i,S_b} \mathcal{T}_i f^S)^{T_i d_i}).$$

for some constant C_Δ that only depends on Δ

Proof. First, $(f_{S_s})_{M,i}$ and $\mathcal{T}_i f^S$ satisfy the same positivity condition: positivity for $(f_{S_s})_{M,i}$ follows by "adjointness" of constant term and parabolic induction. That for $\mathcal{T}_i f^S$ comes from the "adjointness" between $\mathcal{T}_i$ and pushforward of Satake parameter. The $\varphi_{i,N}$ satisfy positivity by construction.

Since the $(T_i, d_i, \lambda_i, \eta_i)$ are simple, this guarantees that the terms in the right hand side of equation (7.2.1) are all positive since all the signs in the stable multiplicity formula are trivial. □

Doing the bookkeeping for what exactly the Δ_i are in proposition 7.1.2 finally produces

Corollary 7.2.2. *Let the notation be as in the above discussion and assume that f_{S_s}, φ_{S_b}, and f^S are all trace-positive. Then for some constant C_Δ only depending on Δ and some functions $\varphi'_{i,N}$:*

$$|S^N_\Delta((f_{S_s} f_{S_b} f^S)^N)| \leq C_\Delta \coprod_i S^{T_i d_i}_{(T_i, d_i, \lambda_i, \eta_i)}((((f_{S_s})_{M,i} \varphi'_{i,S_b} \mathcal{T}_i f^S)^{T_i d_i}).$$

Furthermore, $(f_{S_s})_{M,i}$, φ'_{i,S_b} and $\mathcal{T}_i f^S$ satisfy the same positivity condition.

Proof. Apply proposition 7.2.1 to the two terms on the right side of the inequality in proposition 7.1.2. As some details, we need to use the generalized fundamental lemma to transfer f^S to H'. The φ' are constructed by applying the above discussion to transfer to H' of f_N. Finally, the transfer of f_{S_s} to H' is given by taking constant terms since we are in a degenerate case of H'_{S_s} being a Levi subgroup.

The positivity condition follows for $(f_{S_s})_{M,i}$ by the "adjointness" of constant term and parabolic induction. That for $\mathcal{T}_i f^S$ comes from the "adjointness" between $\mathcal{T}_i$ and pushforward of Satake parameter. That for φ'_{i,S_b} is by construction. □

7.3. **Full Bound.** Now let $G \in \mathcal{E}_{\mathrm{ell}}(N)$ and $\Delta = (T, d, \lambda, \eta)$ a simple shape for G. As before, choose test function

$$f^\infty = f_{S_s} f_{S_b} f^S$$

on G where S_s is all split places, S_b contains all ramified places, and $f^S \in \mathscr{H}^{\mathrm{ur}}(G^S)$ Further assume:

- f_{S_s} is supported on the kernel of $|\det|_{S_s \sqcup S_b}$,
- f_{S_s} and f^S are trace-positive.

To start,

$$\begin{aligned} |S^M_\Delta((f_{S_s} f_N \mathbf{1}_{K^S})^M)| &\leq \sum_{\psi \in \Delta} |S^M_\psi((f_{S_s} f_N \mathbf{1}_{K^S})^M)| \\ &= \sum_{\psi \in \Delta} m_\psi |\mathcal{S}_\psi| (\mathrm{tr}_{\psi_{S_s}} f_{S_s})(\mathrm{tr}_{\psi^S} \mathbf{1}_{K^S})| \,\mathrm{tr}_{\psi_{S_b}} f_{S_b}|. \end{aligned}$$

For all $\psi = \tau[d]$ in the sum, now apply lemma 6.1.3 to get a Levi $M_{S_s} \simeq \mathrm{GL}_T{}^d$ of $\mathrm{GL}_M(F_{S_s})$ so that

$$\mathrm{tr}_{\psi_{S_s}} f_{S_s} \leq (\mathrm{tr}_{\tau_{S_s}} (f_{S_s})_{M,1})^{d\oplus}.$$

We also apply 6.3.7 to construct the functions φ'_{S_b} satisfying

$$|\,\mathrm{tr}_{\psi_{S_b}} \varphi_{S_b}| \leq (\mathrm{tr}_{\tau_{S_b}} \varphi'_{S_b})^{d\oplus}$$

since the τ_{S_b} are necessarily cuspidal. Note that the φ'_{S_b} so constructed is trace-positive. Applying lemma 6.2.2 then gives

$$|\,\mathrm{tr}_{\psi^\infty}(f_{S_s}\varphi_{S_b} f^S)| \leq (\mathrm{tr}_{\tau^\infty}((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S))^{d\oplus}. \tag{7.3.1}$$

Finally, summing over all $\tau[d] \in \Delta$ gives that:

Proposition 7.3.1. *With notation and conditions from the above discussion:*

$$|S^M_\Delta((f_{S_s}\varphi_{S_b} f^S)^M)| \leq C(S^T_{\Sigma_{\lambda,\eta}}((((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S))^T)^{d\oplus}$$

for some constant C that only depends on Δ. In addition, $(f_{S_s})_{M,1}$ and $\mathcal{T} f^S$ are trace positive and we can choose φ'_{S_b} to be so too.

Proof. Expanding out and using that $\Sigma_{\lambda,\eta}$ is simple:

$$\begin{aligned}(S^T_{\Sigma_{\lambda,\eta}}&((((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S))^T)^{d\oplus} \\ &= \sum_{\tau \in \Sigma_{\lambda,\eta}} m^d_\psi |\mathcal{S}_\psi|^d (\mathrm{tr}_{\tau^\infty}((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S))^{d\oplus} + \text{cross terms},\end{aligned}$$

where trace-positivity comes from the conditions in the above discussion and arguments similar to corollary 7.2.2. Since m_ψ and $|S_\psi|$ only depend on $\Sigma_{\lambda,\eta}$, equation (7.3.1) then proves the inequality since trace positivity gives that the cross terms are all positive. Note we use here that Δ is simple so there is no $\epsilon_\psi(s_\psi)$-term in the stable multiplicity formula. □

We will need a slight technical variation of this:

Proposition 7.3.2. *Fix notation and conditions as in proposition 7.3.1. Assume there is a constant B such that for all $\tau \in \Sigma_{\lambda,\eta}$:*

$$|\,\mathrm{tr}_{\tilde{\pi}^\infty_\tau}((((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S)^T)| \leq B$$

where we implicitly hide a sum over factorizable terms and a quantifier over all factors. Then

$$|S^M_\Delta((f_{S_s}\varphi_{S_b} f^S)^M)| \leq CB^{d-1} S^T_{\Sigma_{\lambda,\eta}}((((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S)^T)$$

for some constant C that only depends on Δ and we implicitly hide a sum over factorizable terms.

Proof. This is the same argument as 7.3.1 except we bound

$$\sum_{\tau \in \Sigma_{\lambda,\eta}} m^d_\psi |\mathcal{S}_\psi|^d (\mathrm{tr}_{\tau^\infty}((f_{S_s})_{M,1}\varphi'_{S_b}\mathcal{T} f^S))^{d\oplus}$$

directly instead of adding in cross terms. □

Our final bound for a general shape then becomes:

Proposition 7.3.3. *Let $\Delta = (T_i, d_i, \eta_i, \lambda_i)_i$ be a refined shape and $G = H(\Delta) \in \mathcal{E}_{\mathrm{ell}}(N)$. Let*

$$f = f_{S_s} \varphi_{s_b} f^S$$

be a test function on G^∞ where S_s is all split places, S_b contains all the ramified places, and $f^S \in \mathscr{H}^{\mathrm{ur}}(G^S)$. We further require

- *f_{S_s} is supported on the kernel of $|\det|_{S_s}$,*
- *f_{S_s}, φ_{S_b}, and f^S are trace-positive.*

Then there is a Levi subgroup

$$M \simeq \prod_i \mathrm{GL}_{T_i}(F_v)^{d_i}$$

of $G(F_{S_s}) \simeq \mathrm{GL}_N(F_{S_s})$ and functions φ'_{i,S_b} such that

$$|S^N_\Delta((f_{S_s} f_{S_b} f^S)^N)| \leq C_\Delta \prod_i (S^{T_i d_i}_{\Sigma_{\lambda_i,\eta_i}}((((f_{S_s})_{M,i} \varphi'_{i,S_b} \mathcal{T}_i f^S)^{T_i d_i}))^{d\oplus}.$$

Here $(f_{S_s})_{M,i}$ is the restriction of the constant term to M to the first $\mathrm{GL}_{T_i}(F_v)$ factor in $\mathrm{GL}_{T_i}(F_v)^{d_i}$, $\mathcal{T}_i$ is defined similarly, and C_Δ is a constant that only depends on Δ.

Finally, we may choose $\varphi'_{i,N}$ so that $(f_{S_s})_{M,i}$, $\varphi'_{i,N}$ and $\mathcal{T}_i f^S$ satisfy the same bulleted conditions.

Proof. The inequality comes from applying 7.2.2 and then 7.3.1. The first thing to check is that the necessary conditions for 7.3.1 still hold after applying 7.2.2. Positivity is guaranteed by the second implication of 7.2.2. Support follows from the integral formula for constant term.

The final φ', $(f_{S_s})_{M,i}$, and $\mathcal{T}_i f^s$ satisfy the bulleted conditions by last implications of 7.3.1. □

8. Induction Step Details: Particular Shapes

We compute more detailed information for certain particular kinds of shapes.

8.1. **Shapes of Characters.** Let $\Delta = (1, d, \lambda, \eta)$ be a shape for $H = H(\Delta)$. If $\psi \in \Delta$, then it can be seen by Adams and Johnson's combinatorial description of A-packets as in section 3.2.2 that Π_{ψ_∞} is a singleton containing a character ξ_∞ of H. Let $\mathrm{EP}_\xi = \mathrm{vol}(A_{G,\infty} \backslash G_\infty / G_{\mathrm{der},\infty})^{-1} \xi_\infty^{-1}$ be the corresponding Euler-Poincaré function (note that by our assumptions on E/F, H is cuspidal at ∞ so ξ_∞^{-1} is a valid test function).

Proposition 8.1.1.

$$S^H_\Delta(\mathrm{EP}_\xi f^\infty) = \sum_{\substack{\chi \in \mathcal{AC}_{\mathrm{disc}}(H) \\ \chi_\infty = \xi_\infty}} \mathrm{tr}_{\chi^\infty} f^\infty,$$

where $\mathcal{AC}_{\mathrm{disc}}(H)$ is the set of one-dimensional representations in the discrete automorphic spectrum of H.

Proof. If $\psi \in \Delta$, any $\pi \in \Pi_\psi$ has $\pi_\infty = \xi_\infty$ as explained above. This implies π is one-dimensional by a well-know result (see [KST16, lem. 6.2] for example). Furthermore, ψ is simple, so as distributions

$$S_\psi = \sum_{\pi \in \Pi_\psi} \mathrm{tr}_\pi = I_\psi.$$

Evaluating at our test function

$$S_\psi((\mathrm{EP}_\xi f^\infty) = I_\psi(\mathrm{EP}_\xi f^\infty) = \sum_{\pi \in \Pi_\psi} \mathrm{tr}_{\pi^\infty}(f^\infty).$$

On the other hand, any $\chi \in \mathcal{AC}_{\mathrm{disc}}(H)$ appears in some A-packet Π_ψ. If $\chi_\infty = \xi_\infty$, then ψ_∞ needs to have full Arthur-SL_2 and infinitesimal character λ_i, which forces $\pi \in \Delta$. Furthermore, we recall that characters appear with multiplicity at most one in the automorphic spectrum (realized as functions by evaluation). Therefore every such χ can only appear in one packet, since otherwise the above paragraph would show that they have multiplicity more than one. In total, the disjoint union over Π_ψ for $\psi \in \Delta$ is exactly the subset of $\mathcal{AC}_{\mathrm{disc}}(H)$ with infinite component ξ_∞.

Summing over $\psi \in \Delta$ then finishes the argument. □

Next, $\mathcal{AC}_{\mathrm{disc}}(H)$ are the characters of

$$\Xi(H) := H(F)^{\mathrm{ab}} \backslash H(\mathbb{A})^{\mathrm{ab}}$$

and we can write

$$S^H_\Delta(\mathrm{EP}_\xi f^\infty) = \frac{1}{\mathrm{vol}(H^{\mathrm{ab}}_\infty)} \sum_{\chi \in \Xi(H)^\vee} \widehat{\xi_\infty^{-1} f^{\infty,\mathrm{ab}}}(\chi)$$

where $f^{\infty,\mathrm{ab}}$ is the pushforward (by integration against $H(\mathbb{A}^\infty)_{\mathrm{der}}$) of f^∞ to $H(\mathbb{A}^\infty)^{\mathrm{ab}}$. Then:

Corollary 8.1.2. *Let $H = H(\Delta)$ for Δ be a shape of the form $(1, d, \lambda, \eta)$ corresponding to character ξ_∞ on H_∞. Then*

$$S^H_\Delta(\mathrm{EP}_\xi f^\infty) = \frac{\mathrm{vol}(H(F)^{\mathrm{ab}} \backslash H(\mathbb{A})^{\mathrm{ab}})}{\mathrm{vol}(H^{\mathrm{ab}}_\infty)} \sum_{\gamma \in H(F)^{\mathrm{ab}}} \xi_\infty^{-1} f^{\infty,\mathrm{ab}}(\gamma),$$

where $f^{\infty,\mathrm{ab}}$ is as immediately above.

Proof. Since $H(F) \backslash H(\mathbb{A})$ is compact for our specific H, we get that $H(F)^{\mathrm{ab}}$ is co-compact in $H(\mathbb{A})^{\mathrm{ab}}$. Since $H(F)^{\mathrm{ab}}$ is a subgroup of $H^{\mathrm{ab}}(F)$ and $H(\mathbb{A})^{\mathrm{ab}}$ is a subgroup of $H^{\mathrm{ab}}(\mathbb{A})$, discreteness of $H^{\mathrm{ab}}(F)$ in $H^{\mathrm{ab}}(\mathbb{A})$ gives discreteness of $H(F)^{\mathrm{ab}}$ in $H(\mathbb{A})^{\mathrm{ab}}$.

Therefore Poisson summation gives the result. □

Our $H \in \mathcal{E}_{\mathrm{ell}}(N)$ is necessarily isomorphic as a reductive group to $U(N)$ so we will usually have $H^{\mathrm{ab}} = U(1)$ in the above.

8.2. **Odd GSK Shapes.** We will eventually focus on shapes that are similar to the Saito-Kurokawa case (d) at the end of [Art04]:

Definition 8.2.1. We say a shape $\Delta = (T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k}$ is *GSK* or *generalized Saito-Kurokawa* if

- $d_1 = 1$ and for all $i > 1$, $T_i = 1$,
- The d_i are distinct integers.

We furthermore say it is *odd GSK* if

- The d_i are all odd.

We will be able to get exact asymptotic bounds for such S_Δ since these are exactly the ones where the only Speh representations that appear are characters. Understanding more general shapes would require understanding Speh representations on GL_n better.

We recall the setup of sections 7.2 and 7.3: choose $G \in \mathcal{E}_{\mathrm{ell}}(N)$ and pick finite sets of places $S = \infty \sqcup S_s \sqcup S_b$ where S_s is all split and S_b contains all the ramified places. We will look at test functions of the form:

$$f^\infty = \varphi_{S_b} f_{S_s} f^S$$

where K^S is a hyperspecial in G^S, φ_{S_b} and f_{S_s} are arbitrary, and $f^S \in \mathscr{H}^{\mathrm{ur}}(G^S)$. We will further assume that the pair (Δ, φ_{S_b}) satisfies the stable transfer conjecture 6.4.1.

Consider arbitrary $\psi = \tau_1[d_1] \oplus \cdots \oplus \tau_k[d_k] \in \Delta$. Then, for all such ψ, lemma 6.1.2 gives Levi

$$M = \prod_i \mathrm{GL}_{T_i d_i}(F_{S_s})$$

of $G(F_{S_s}) \simeq \mathrm{GL}_N(F_{S_s})$ so that

$$\mathrm{tr}_{\psi_{S_s}} f_{S_s} = \bigoplus\!\!\!\!\!\!\prod_i \mathrm{tr}_{\tau_{i,S_s}[d_i]}(f_{S_s})_{M,i},$$

and lemma 6.2.2 gives that

$$\mathrm{tr}_{\psi^S} f^S = \bigoplus\!\!\!\!\!\!\prod_i \mathrm{tr}_{\tau_i^S[d_i]} \mathcal{T}_i f^S.$$

Note that the constant terms and $\mathcal{T}_i f^S$ satisfy the positivity condition by various "adjointesses" of trace and various transfers.

Finally, for all such ψ, lemma our assumption that φ_{S_b} satisfies conjecture 6.4.1 gives φ_{i,S_b} such that

$$\mathrm{tr}_{\psi_{S_b}} \varphi_{S_b} = \bigoplus\!\!\!\!\!\!\prod_i \mathrm{tr}_{\tau_{i,S_s}[d_i]} \varphi_{i,S_b}$$

where the φ_{i,S_b} are trace-positive.

This produces:

Proposition 8.2.2. *With notation and conditions as above (in particular, that the pair (Δ, φ_{S_b}) satisfies conjecture 6.4.1),*

$$S^H_\Delta(\mathrm{EP}_\lambda \varphi_{S_b} f_{S_s} f^S) = 2^{-k+1} \bigoplus\!\!\!\!\!\!\prod_i S^{H_i}_{(1,d,\lambda_i,\eta_i)}(\mathrm{EP}_\lambda (f_{S_s})_{M,i} \varphi_{i,S_b} \mathcal{T}_i f^S),$$

where $H_i = H(1, d, \lambda_i, \eta_i)$.

Proof. As in section 7.2, we multiply the together and sum the trace equalities above. Note that Δ having all odd d_i gives that $S^H_\Delta = S^{|H|}_\Delta$. Furthermore, since we are working with unitary groups $m_\psi = 1$ always. Finally, for $\psi \in \Delta$ as above, since the $\tau_i[d_i]$ are simple,

$$|\mathcal{S}_\psi|^{-1} \prod_i |\mathcal{S}_{\tau_i[d_i]}| = 2^{-k+1}.$$

This computes all the terms in the stable multiplicty formula. $\square$

9. Level-Aspect Asymptotics

9.1. **Setup.** In this section, we use the strategy outlined above to compute some asymptotics of $S^G_\Delta(\eta_\lambda f^\infty)$ for a specific sequence of f^∞.

Fix N and $G \in \mathcal{E}_{\mathrm{sim}}(N)$ such that $G = H(\Sigma_{\lambda,\eta})$. We use the setup from [ST16]. First, as some notation,

- $K := K^G_R$ is a choice of hyperspecial subgroup of G at a set of unramified places R. Similarly define $K^{G,R}$.
- $\dim \lambda$ for λ an integral regular infinitesimal character is the dimension of the associated finite-dimensional representation on $\mathrm{GL}_n\mathbb{C}$.
- $K(\mathfrak{n}) := K^G_R(\mathfrak{n})$ for $\mathfrak{n}$ and ideal of $\mathcal{O}_F$ supported on a set of unramified places R is the $\mathfrak{n}$th congruence subgroup of K^G_R:
$$K^G_R(\mathfrak{n}) = \prod_{v|R} K^G_v(q_v^{v(\mathfrak{n})}),$$
where $K^G_v(q_v^{v(\mathfrak{n})})$ is the corresponding step in the Moy-Prasad filtration. Define $K^{G,R}(\mathfrak{n})$ similarly.
- A choice of measure on G_∞ induces one on the compact form G^c_∞ in the standard way by relating to top forms on $G_{\infty,\mathbb{C}}$.

Then, fix:

- A finite set of finite places S_0 including all those where E/F is ramified.
- An arbitrary test function φ_{S_0} at S_0
- A finite set of finite places S_1 disjoint from S_0.
- An $f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(G_{S_1})^{\le\kappa}$ for some κ
- A sequence of ideals $\mathfrak{n}_i$ of F relatively prime to $2, 3, S_0, S_1$ that are all split in E and such that $|\mathfrak{n}_i| \to \infty$.
- As notation, let S be the union of ∞, S_0, and S_1.

Define test function:
$$f_i^\infty = \varphi_{S_0} f_{S_1} \bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$$
where $\bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$ denotes the indicator function normalized by volume to have total integral 1.

We will also need some constants related to the $\mathfrak{n}_i$:

- The norm
$$|\mathfrak{n}_i| := \prod_{v|\mathfrak{n}_i} q_v^{v(\mathfrak{n}_i)}$$
- Euler factors: for $n, n_i \in \mathbb{Z}^+$
$$\Gamma_n(\mathfrak{n}_i) := \prod_{v|\mathfrak{n}_i} (1 - q_v^{-1})(1 - q_v^{-2}) \cdots (1 - q_v^{-n}),$$
$$\Gamma_{-n}(\mathfrak{n}_i) := \prod_{v|\mathfrak{n}_i} (1 + q_v^{-1})(1 + q_v^{-2}) \cdots (1 + q_v^{-n}),$$
$$\Gamma_{\pm n_1, \dots, \pm n_k}(\mathfrak{n}_i) := \Gamma_{\pm n_1}(\mathfrak{n}_i) \cdots \Gamma_{\pm n_k}(\mathfrak{n}_i).$$

Our initial input will be the level-aspect bounds of [ST16] that were proven through a very detailed analysis of the geometric side of Arthur's discrete-at-infinity trace formula from [Art89a].

Theorem 9.1.1 (Special case of [ST16, Thm. 9.16])**.** *With notation defined as above, there are constants A, B, C, D, E with $C \geq 1$ and depending only on G such that whenever $|\mathfrak{n}_i| \geq D q_{S_1}^{E\kappa}$,*

$$|\mathfrak{n}_i|^{-\dim G}\Gamma_N(\mathfrak{n}_i)^{-1} I^G(\mathrm{EP}_\lambda f_i^\infty) = \Lambda + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}),$$

where we define mass:

$$\Lambda = \Lambda(G, f, \varphi) = \varphi_{S_0}(1) f_{S_1}(1) \frac{\dim \lambda}{|\Pi_{\mathrm{disc}}(\lambda)|} \frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A}_F))}{\mathrm{vol}(K^S)\,\mathrm{vol}(G_\infty^c)}.$$

Proof. Since only split primes divide $\mathfrak{n}_i$, we can assume

$$[K : K(\mathfrak{n}_i)] = |\mathfrak{n}_i|^{\dim G}\Gamma_N(\mathfrak{n}_i)$$

by a standard formula for the sizes of the $\mathrm{GL}_n(\mathcal{O}_v/q_v^{v(\mathfrak{n}_i)}\mathcal{O}_v)$. □

When $S_0 = S_1 = \emptyset$, the quotient of volumes is the modified Tamagawa number computed in [ST16] corollary 6.14:

$$\frac{\mathrm{vol}(G(F)\backslash G(\mathbb{A}_F))}{\mathrm{vol}(K^\infty)\,\mathrm{vol}(G_\infty^c)} = \tau'(G) = 2^{-(N-1)\deg F}\tau(G) L(\mathrm{Mot}_G)|\Omega_G||\Omega^c_{G_\infty}|^{-1} \tag{9.1.1}$$

where $\tau(G)$ is the Tamagawa number, $L(\mathrm{Mot}_G)$ is the L-value of the motive from [Gro97], and $\Omega^c_{G_\infty}$ is the Weyl group of the maximal compact at ∞.

Since the A, B, C, D, E aren't very explicit, we will allow them to change throughout the following argument.

9.2. **Bounds on Stable Trace.** To extend the input bound to S^G, we next need to recall a standard formula:

Lemma 9.2.1. *Let G be a reductive group over F, $\mathfrak{n}_i$ and ideal relatively prime to all places where G is ramified, and M a Levi component of parabolic subgroup P. Then we have identity of indicator functions normalized by volume:*

$$(\bar{\mathbf{1}}_{K^G(\mathfrak{n}_i)})_M = I(\mathfrak{n}_i)\bar{\mathbf{1}}_{K^M(\mathfrak{n}_i)}$$

where

$$I(\mathfrak{n}_i) = [K : K \cap K(\mathfrak{n}_i)P].$$

Furthermore, if $G = \mathrm{GL}_N$ then

$$I(\mathfrak{n}_i) = (1 + O(|\mathfrak{n}_i|^{-2}))|\mathfrak{n}_i|^{\dim G/P}\Gamma_{-1}(\mathfrak{n}_i)^{\sigma(M)}$$

where

$$\sigma(M) = \mathrm{rank}_{\mathrm{ss}}\, G - \mathrm{rank}_{\mathrm{ss}}\, M.$$

Proof. This is well known—see for example the proof of Lemma 5.2 in [MS19]. □

Note for intuition later that $\dim G/P = 1/2(\dim G - \dim M)$. Using this:

Proposition 9.2.2. *For the f_i^∞ and Λ as in section 9.1, there are constants A, B, C, D, E depending only on G with $C \geq 1$ such that whenever $|\mathfrak{n}_i| \geq D q_{S_1}^{E\kappa}$,*

$$|\mathfrak{n}_i|^{-\dim G}\Gamma_N(\mathfrak{n}_i)^{-1} S^G(\mathrm{EP}_\lambda f_i^\infty) = \Lambda + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}).$$

Proof. We apply theorem 5.2.1 and apply Theorem 9.1.1 to each term. This argument is a much less general and much simpler version of the main result of [Dal22] so we present it tersely.

We simply need to show that all the non-I^G summands can be put into the error term:

- In this case of unitary groups, there are a finite number of such terms.
- We can ignore dependence on φ_{S_0}.
- Lemma 5.5.4 in [Dal22] allows us to bound the value at 1 and support of the $f^{\mathcal{H}}_{S_1}$.
- Lemma 6.1.2 lets us iteratively use lemma 9.2.1 to bound the $\bar{1}^{\mathcal{H}}_{K^{G,S}(\mathfrak{n}_i)}$.
- Averaging corollary 5.1.6 in [Dal22] shows that the transfer of the EP-function is a linear combination of a number of EP-functions uniformly bounded over $\mathcal{H}$.

Putting all this together, 9.1.1 shows that the sum of the non-I^G terms are upper-bounded by the claimed error as long as we extremize A, B, C, D, E appropriately over all hyperendoscopic groups. □

9.3. **The Induction.** Now we induct to bound the limit multiplicities restricted to specific shape. Their are two pieces to this argument—first, as a consequence of proposition 5.3.1,

$$S^G_{\Sigma_{\lambda,\eta}}(\eta_\lambda f_i^\infty) = S^G(\mathrm{EP}_\lambda f_i^\infty) - \sum_{\substack{H(\Delta)=H(\Sigma_{\lambda,\eta}) \\ \text{inf. char}(\Delta)=\lambda \\ \Delta\neq\Sigma_{\lambda,\eta}}} S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty). \tag{9.3.1}$$

Second, the bound in proposition 7.3.3 lets us show that non-Σ terms S^G_Δ have limit multiplicities controlled by the groups they are lifts from:

We start with a technical trick that allows us to get the trace-positivity conditions needed to apply the results of the previous section.

Lemma 9.3.1. *Let f_v be a test function on G_v. Then there are trace-positive functions $f_1, \dots, f_k$ such that*

$$f_v = \lambda_1 f_1 + \cdots + \lambda_k f_k.$$

Furthermore, if $f_v \in \mathscr{H}^{\mathrm{ur}}(G_v)^{\leq\kappa}$, then so are the f_i and

$$\sum_i |\lambda_i| \leq C, \qquad \sum_i \|f_i\|_\infty \leq C\|f_v\|_\infty$$

for some uniform constant C.

Proof. The first statement works in great generality by the Dixmier-Malliavin decomposition theorem as used [Sau97, Lem. 3.5].

More concretely, we can without loss of generality assume $f_v^* = f_v$ by symmetrizing:

$$f_v = \left(\frac{f_v + f_v^*}{2}\right) - i\left(\frac{f_v - f_v^*}{2i}\right).$$

By smoothness, there is open compact subgroup U such that $f_v = f_v \star \bar{\mathbf{1}}_U$. Then

$$f_v = \frac{1}{4}(f_v + \bar{\mathbf{1}}_U) \star (f_v + \bar{\mathbf{1}}_U) - \frac{1}{4}(f_v - \bar{\mathbf{1}}_U) \star (f_v - \bar{\mathbf{1}}_U).$$

This is a linear combination of functions of the form $g \star g^*$, which are necessarily trace-positive.

Using $U = K_v$ shows the required bounds for the second statement. □

Theorem 9.3.2. *Fix the f_i^∞, $G \in \mathcal{E}_{\mathrm{ell}}(N)$, and Λ as in section 9.1. Then there are constants A, B, C, D, E with $C \geq 1$ such that whenever $|\mathfrak{n}_i| \geq Dq_{S_1}^{E\kappa}$,*

$$|\mathfrak{n}_i|^{-\dim G}\Gamma_N(\mathfrak{n}_i)^{-1} S^G_{\Sigma_{\lambda,\eta}}(\mathrm{EP}_\lambda f_i^\infty) = \Lambda + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa})$$

and for $\Delta \neq \Sigma_{\lambda,\eta}$,

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = O(|\mathfrak{n}_i|^{\bar R(\Delta)} q_{S_1}^{A+B\kappa})$$

where

$$\bar R((T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k}) = \frac{1}{2}\left(N^2 + \sum_i T_i^2 d_i\right). \tag{9.3.2}$$

Proof. We induct on the N such that $G \in \mathcal{E}_{\mathrm{ell}}(N)$. For $N = 1$, $\Sigma_{\lambda,\eta}$ is the only possible shape so (by extreme overkill) this follows from Theorem 9.1.1.

For the inductive step, we first argue that it suffices to show the second statement for $\Delta \neq \Sigma_{\lambda,\eta}$ if we take D and E to be the maximum values over that for all smaller shapes appearing. Then we can use proposition 7.3.3 to get that all the terms in the sum are lower order in $|\mathfrak{n}_i|$ than the bound from proposition 9.2.2.

Therefore let $\Delta = (T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k} \neq \Sigma_{\lambda,\eta}$ be a rank N shape. First, apply lemma 9.3.1 on each of the finitely many unramified factors of φ_{i,S_0} and $\mathcal{T}_i f_{S_i}$ to write

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = \sum_j \lambda_j S^G_\Delta(\mathrm{EP}_\lambda f_{i,j}^\infty)$$

where the number of terms and sum of the $|\lambda_j|$ are both $O(C^{S_1}) = O(q_{S_1})$ and each $f^\infty_{i,j}$ is trace-positive.

We then apply proposition 7.3.3 with S_s being the primes that divide $\mathfrak{n}_i$ and $S_b = S_0$. Over S_s, let M be the Levi from that theorem and P the parabolic the constant term is defined through. Then (in the notation from the theorem statement),

$$\begin{aligned} |S^G_\Delta(\mathrm{EP}_\lambda f^\infty_{i,j})| &\le C_\Delta \coprod^{\oplus} \left(S^{T_i d_i}_{\Sigma_{\lambda_i,\eta_i}}(((f_{S_s})_{M,i} \varphi'_{i,S_0} \mathcal{T}_i f_{S_1} \mathbf{1}_{K^S})^{T_i d_i})\right)^{d\oplus} \\ &= C_\Delta I(\mathfrak{n}_i) \coprod^{\oplus} \left(S^{H_i}_{\Sigma_{\lambda_i,\eta_i}}(\mathrm{EP}_{\lambda_i} \bar{\mathbf{1}}_{K^{H_i}(\mathfrak{n}_i)} \varphi'_{i,S_0} \mathcal{T}_i f_{S_1})\right)^{d\oplus} \end{aligned} \tag{9.3.3}$$

where each $H_i = H(\Sigma^{T_i}_{\lambda_i,\eta_i})$ and the second equality uses lemma 9.2.1 together with factoring $\mathrm{vol}(K^M(\mathfrak{n}_i))$ over places in S_s.

From lemma 6.2.3, we know that $\mathcal{T} f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(M_{S_1})^{\le \kappa}$ and

$$\|\mathcal{T} f_{S_1}\|_\infty = \|f_{S_1}\|_\infty O(q_v^{\kappa F} \kappa^G).$$

Therefore, we can use the inductive hypothesis on the $\Sigma_{\lambda_i,\eta_i}$, summing the asymptotics of the Λ and the error to get that each term in the product is:

$$O\left(|\mathfrak{n}_i|^{\dim H_i} \Gamma_{T_i^{(d_i)}}(\mathfrak{n}_i) q_{S_1}^{A'+B'\kappa}\right)$$

for some constants A', B' depending on Δ. Note that the positivity condition is guaranteed by the last implication of proposition 7.3.3. Summing over j, this finally produces that

$$|S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty)| = O\left(I(\mathfrak{n}_i)\Gamma_{T_1^{(d_1)},\dots,T_k^{(d_k)}}(\mathfrak{n}_i)|\mathfrak{n}_i|^{\dim M} q_{S_1}^{A'+B'\kappa}\right),$$

ignoring all dependence on the φ'.

Recalling that $G_{S_s} \simeq \mathrm{GL}_{N,S_s}$, we can use the asymptotic for I and get:

$$I(\mathfrak{n}_i)|\mathfrak{n}_i|^{\dim M} = O\left(|\mathfrak{n}_i|^{\dim P} \Gamma_{-1}(\mathfrak{n}_i)^{\sigma(M)}\right).$$

We finally note that $\Gamma_{T_1^{d_1},\dots,T_k^{d_k}}(\mathfrak{n}_i)\Gamma_{-1}(\mathfrak{n}_i)^{\sigma(M)} \leq 1$ and that

$$\dim P = \frac{1}{2}\left(N^2 + \sum_i T_i^2 d_i\right).$$

This finishes the bound for Δ and therefore the induction. □

We emphasize that the value $\bar{R}(\Delta)$ should be thought of as an approximate upper-bound to some growth rate attached to Δ. It is the dimension of the parabolic attached to partition

$$(T_1^{(d_1)}, \dots, T_k^{(d_k)})$$

and upper bounds the true growth rate. It is also maximized at $\bar{R}(\Sigma_{\lambda,\eta}) = \dim G$ where it is exact.

9.4. **Improving the Bound.** We will actually need a slightly tighter bound $R(\Delta)$. This will come from improving our bounds for blocks $(T_i, d_i, \lambda_i, \eta_i)$ with small T_i instead of directly applying proposition 7.3.3. This argument is nothing more than a rephrasing to fit our current context of the key technical trick that makes the bounds in [MS19] work.

9.4.1. *Terms with* $T_i = 1$*.* We will actually prove a stronger exact formula for terms with $T_i = 1$

Lemma 9.4.1. *We can bound the terms for summands* $(T_i, d_i, \lambda_i, \eta_i)$ *with* $T_i = 1$ *implicit in an intermediate step of equation* (9.3.3)*:*

$$\begin{aligned} &S^{H_i'}_{(1,d_i,\lambda_i,\eta_i)}(EP_{\lambda_i'}\bar{\mathbf{1}}_{K^{H_i'}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}) \\ &\qquad = |\mathfrak{n}_i|\Gamma_1(\mathfrak{n}_i)\frac{\operatorname{vol}(H_i'(F)^{\mathrm{ab}}\backslash H_i'(\mathbb{A})^{\mathrm{ab}})}{\operatorname{vol}(K^{(H_i')^{\mathrm{ab}},S})\operatorname{vol}((H'_{i,\infty})^{\mathrm{ab}})}\int_{H'_{i,\mathrm{der},S_1,S_0}} \mathcal{T}_i f_{S_1}\varphi_{i,S_0}(h)\,dh \\ &\qquad = O(|\mathfrak{n}_i|\Gamma_1(\mathfrak{n}_i) q_{S_1}^{A+B\kappa}) \end{aligned}$$

as long as $|\mathfrak{n}_i| > D q_{S_1}^{E\kappa}$*.*

Proof. By corollary 8.1.2,

$$\begin{aligned} &S^{H_i'}_{(1,d_i,\lambda_i,\eta_i)}(EP_{\lambda_i'}\bar{\mathbf{1}}_{K^{H_i'}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}) \\ &= \frac{\operatorname{vol}(H_i'(F)^{\mathrm{ab}}\backslash H_i'(\mathbb{A})^{\mathrm{ab}})}{\operatorname{vol}((H'_{i,\infty})^{\mathrm{ab}})}\sum_{\gamma \in H_i'(F)^{\mathrm{ab}}} (\bar{\mathbf{1}}_{K^{H_i'}(\mathfrak{n}_i)})^{\mathrm{ab}}(\gamma)(\mathcal{T}_i f_{S_1})^{\mathrm{ab}}(\gamma)\varphi^{\mathrm{ab}}_{i,S_0}(\gamma)\xi_i^{-1}(\gamma) \end{aligned}$$

where ξ_i is the character of H_i' associated to λ_i'.

Since at places dividing $\mathfrak{n}_i$ we know H_i is a general linear group, we can compute

$$(\bar{\mathbf{1}}_{K^{H_i'}(\mathfrak{n}_i)})^{\mathrm{ab}} = |\mathfrak{n}_i|\Gamma_1(\mathfrak{n}_i)\bar{\mathbf{1}}_{K^{(H_i')^{\mathrm{ab}}}(\mathfrak{n}_i)}.$$

using standard formulas for $|\mathrm{SL}_n(\mathcal{O}_F/\mathfrak{p}_v^n)|$. Next, $H_i'(F)^{\mathrm{ab}} \subseteq (H')_i^{\mathrm{ab}}(F)$ so a trivial case of lemma 8.4 [ST16] applied to $(H_i')^{\mathrm{ab}}$ gives that there are D_i and E_i such that whenever $|\mathfrak{n}_i| \geq D_i q_{S_1}^{E_i\kappa}$, all terms in the sum vanish except for $\gamma = 1$. This finishes the argument. □

9.4.2. *Terms with $T_i = 2$.*

Lemma 9.4.2. *We can bound the terms for summands $(T_i, d_i, \lambda_i, \eta_i)$ with $T_i = 2$ implicit in an intermediate step of equation* (9.3.3)*:*

$$S^{H'_i}_{(2,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i}\bar{\mathbf{1}}_{K^{H'_i}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}) \\ = O(|\mathfrak{n}_i|^{2d_i(d_1-1)+d_i+3}\Gamma_{-1^{(2d-2)},2}(\mathfrak{n}_i)q_{S_1}^{A+B\kappa}).$$

Proof. We use proposition 7.3.2 instead of proposition 7.3.1. First, we bound $\mathrm{tr}_{\pi_{S_0}}(\varphi_{i,S_0})$ by some constant through Bernstein admissibility as in lemma 6.3.7. Next, the Satake eigenvalues of unirreps of GL_N always have their $|\log_{q_v}(\cdot)|$ bounded by $(N-1)/2$ by the main result of [Tad86] (this is the value achieved by the trivial representation). Therefore, arguments as in [Dal22, §5.5] show that

$$\mathrm{tr}_{\pi_{S_1}}(\mathcal{T}_i f_{S_1}) = O(q_{S_1}^{A+B\kappa})$$

for all unirreps π_{S_1} of GL_2 at S_1. (the number of factorizable summands of $\mathcal{T} f_{S_1}$ also bounded similarly).

Finally for any irrep π^S

$$\mathrm{tr}_{\pi^S}\ \bar{\mathbf{1}}_{K^{\mathrm{GL}_2}(\mathfrak{n}_i)} \leq \Gamma_{-1}(\mathfrak{n}_i)|\mathfrak{n}_i|$$

as in the proof of lemma 5.2 in [MS19]. Applying 7.3.2 by putting it all together (and changing A, B):

$$S^{H'_i}_{(2,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i}\bar{\mathbf{1}}_{K^{H'_i}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}) \\ = O(|\mathfrak{n}_i|^{d-1}\Gamma_{-1}(\mathfrak{n}_i)^{d_i-1}q_{S_1}^{A+B\kappa}) \\ \times |\mathfrak{n}_i|^{2d_i(d_i-1)}\Gamma_{-1}(\mathfrak{n}_i)^{d_i-1}S^{H_i}_{\Sigma_{\lambda_i,\eta_i}}(EP_{\lambda_i}\bar{\mathbf{1}}_{K^{H_i}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}).$$

Where the factors on the second line outside the big-O come from taking the constant term of $\bar{\mathbf{1}}_{K^H(\mathfrak{n}_i)}$. Substituting in theorem 9.3.2 produces the result noting that $H_i = H(2,1,\lambda_i,\eta_i) \in \mathcal{E}_{\mathrm{ell}}(2)$. $\square$

9.4.3. *Terms with $T_i = 3$.*

Lemma 9.4.3. *We can bound the terms for summands $(T_i, d_i, \lambda_i, \eta_i)$ with $T_i = 3$ implicit in an intermediate step of equation* (9.3.3)*:*

$$S^{H'_i}_{(3,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i}\bar{\mathbf{1}}_{K^{H'_i}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}) \\ = O_\epsilon(|\mathfrak{n}_i|^{\frac{9}{2}d_i(d_i-1)+(4+\epsilon)d_i+5}\Gamma_{-1^{(d)},3}(\mathfrak{n}_i)q_{S_1}^{A+B\kappa}).$$

for all $\epsilon > 0$.

Proof. This is the same argument as lemma 9.4.2 except we use

$$\mathrm{tr}_{\pi^S}\ \bar{\mathbf{1}}_{K^{\mathrm{GL}_3}(\mathfrak{n}_i)} \leq C(\epsilon)|\mathfrak{n}_i|^{4+\epsilon}$$

from corollary 9.2 in [MS19]. Our $C(\epsilon)$ here is the product of Marshall-Shin's $C(\epsilon, q_v)$ for $q_v \leq q(\epsilon)$. $\square$

9.4.4. *The full bound.* Applying the previous results with $T_i = 1, 2, 3$ instead of directly applying proposition 7.3.3:

Corollary 9.4.4. *The bound for*

$$\Delta = (T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k} \neq \Sigma_{\lambda,\eta}$$

in theorem 9.3.2 may be tightened to

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = O(|\mathfrak{n}_i|^{R(\Delta)} q_{S_1}^{A+B\kappa})$$

under all the same conditions and where

$$(9.4.1) \quad R(\Delta) = \bar{R}(\Delta) - \sum_{i:T_i=1} \left(\frac{1}{2}(d_i^2 + d_i) - 1\right) - \sum_{i:T_i=2} (4d_i - (d_i+3)) - \sum_{\substack{i:T_i=3 \\ d_i>1}} \left(9d_i - ((4+10^{-100})d_i + 5)\right).$$

Proof. For each summand of Δ with $T_i = 1$, let $H'_i = H((1, d_i, \lambda_i, \eta_i))$. The the method of proof of theorem 9.3.2 implicitly bounds terms

$$S^{H'_i}_{(1,d_i,\lambda_i,\eta_i)}(\mathrm{EP}_{\lambda'_i} \bar{\mathbf{1}}_{K^{H'_i}(\mathfrak{n}_i)} \varphi'_{i,S_0} \mathcal{T}_i f_{S_1}) = O(|\mathfrak{n}_i|^{\frac{1}{2}(d_i^2+d_i)} \Gamma_{-1^{(d_i-1)},1^{(d_i)}}(\mathfrak{n}_i) q_{S_1}^{A+B\kappa})$$

We instead use lemma 9.4.1 to get

$$S^{H'_i}_{(1,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i} f^\infty) = O(|\mathfrak{n}_i| \Gamma_1(\mathfrak{n}_i) q_{S_1}^{A+B\kappa})$$

instead.

For summands of Δ with $T_i = 2$, we similarly use lemma 9.4.2 to replace

$$O(|\mathfrak{n}_i|^{2(d_i^2+d_i)} \Gamma_{-1^{(d_i-1)},2^{(d_i)}}(\mathfrak{n}_i) q_{S_1}^{A+B\kappa}) \mapsto O(|\mathfrak{n}_i|^{2d_i(d_i-1)+d_i+3} \Gamma_{-1^{(2d_i-2)},2}(\mathfrak{n}_i) q_{S_1}^{A+B\kappa})$$

For $T_i = 3$, we use 9.4.3 to replace

$$O(|\mathfrak{n}_i|^{\frac{9}{2}(d_i^2+d_i)} \Gamma_{-1^{(d_i-1)},3^{(d_i)}}(\mathfrak{n}_i) q_{S_1}^{A+B\kappa}) \mapsto O_\epsilon(|\mathfrak{n}_i|^{\frac{9}{2}d_i(d_i-1)+(4+10^{-100})d_i+5} \Gamma_{-1^{(d)},3}(\mathfrak{n}_i) q_{S_1}^{A+B\kappa}).$$

Substituting in these stronger bounds produces the result.

Here, we were not very careful with the φ_{S_0} terms since we are not making claims about how the error term depends on them. □

Remark. The $R(\Delta)$ is a better upper bound of the true growth rate than $\bar{R}(\Delta)$. It can be thought of as making three modifications to the dimension count of the parabolic:

- When $T_i = 1$, replace the dimension of the Borel in the GL_{d_i}-block corresponding to that summand with 1
- When $T_i = 2$, replace the dimension of the Levi $\mathrm{GL}_2^{d_i}$ in the GL_{2d_i}-block corresponding to that summand with $d_i + 3$,
- When $T_i = 3$, replace the dimension of the Levi $\mathrm{GL}_3^{d_i}$ in the GL_{3d_i}-block corresponding to that summand with $(4+\epsilon)d_i + 5$.

The next section describe a case where $R(\Delta)$ is the optimal growth rate.

9.5. **Odd GSK Shapes.** Now that we understand $S_{\Sigma_{\lambda,\eta}}$, we can compute the limiting asymptotics for odd GSK shapes. Keep the same setup as section 9.1. Additionally consider shape $\Delta = (\Delta_i)_{1 \le i \le k} = (T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k}$ such that $H(\Delta) = G$. Let $H_i = H(\Delta_i)$ and $\lambda'_\star$ the total infinitesimal character of $\Delta_\star$ as in (4.1.1).

We will be able to now get that $R(\Delta)$ is the exact growth rate for S^G_Δ.

Theorem 9.5.1. *Fix the f_i^∞ and $G \in \mathcal{E}_{\mathrm{ell}}(N)$ as in section 9.1. Assume Δ is odd GSK as in definition 8.2.1 and that the pair (Δ, φ_{S_0}) satisfies conjecture 6.4.1.*

Then there are constants A, B, C, D, E with $C \ge 1$ depending only on G and Δ such that whenever $|\mathfrak{n}_i| \ge D q_{S_1}^{E\kappa}$,

$$\begin{aligned}|\mathfrak{n}_i|^{-R(\Delta)}\Gamma_{L(\Delta)}(\mathfrak{n}_i)^{-1} S^G_\Delta(\mathrm{EP}_{\lambda'} f_i^\infty) \\ = 2^{-k+1}\Lambda(H_1, \mathcal{T}_1 f_{S_1}, \varphi_{1,S_0}) \times \coprod_{i \ge 2} \Lambda^{\mathrm{ab}}(H_i, \mathcal{T}_i f_{S_1}, \varphi_{i,S_0}) + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}),\end{aligned}$$

with growth rate from corollary 9.4.4

$$\begin{aligned}R(\Delta) &= \dim H_1 + (k-1) + \frac{1}{2}\left(\dim G - \sum_i \dim H_i,\right) \\ &= \bar{R}(\Delta) - \sum_{i>1}\left(\frac{1}{2}(d_i^2 + d_i) - 1\right),\end{aligned}$$

indexing list

$$L(\Delta) = T_1, 1^{(k-1)}, -1^{(k-1)},$$

and masses

$$\Lambda(H_1, \mathcal{T}_1 f_{S_1}, \varphi_{1,S_0}) = \varphi_{1,S_0}(1)(\mathcal{T}_1 f_{S_1})(1)\frac{\dim \lambda_1}{|\Pi_{\mathrm{disc}}(\lambda_1)|}\frac{\mathrm{vol}(H_1(F)\backslash H_1(\mathbb{A}_F))}{\mathrm{vol}(K^S_{H_1})\,\mathrm{vol}(H^c_{1,\infty})},$$

$$\Lambda^{\mathrm{ab}}(H_i, \mathcal{T}_i f_{S_1}, \varphi_{i,S_0}) = \frac{\mathrm{vol}(H_i(F)^{\mathrm{ab}}\backslash H_i(\mathbb{A})^{\mathrm{ab}})}{\mathrm{vol}(K^S_{H_i^{\mathrm{ab}}})\,\mathrm{vol}((H_{i,\infty})^{\mathrm{ab}})}\int_{H_{i,\mathrm{der},S_0,S_1}} \mathcal{T}_i f_{S_1}\varphi_{i,S_0}(h)\,dh.$$

(The $\mathcal{T}_i f_{S_1}$ and φ_{i,S_0} are defined as in lemmas 6.2.2 and conjecture 6.4.1.)

Proof. We first apply proposition 8.2.2 with $S_b = S_0$ and S_s the places dividing the $\mathfrak{n}_i$: whenever $|\mathfrak{n}_i|$ is big enough,

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = 2^{-k+1} I(\mathfrak{n}_i) \coprod_i S^{H_i}_{(T_i,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i}\bar{\mathbf{1}}_{K^{H_i}(\mathfrak{n}_i)}\varphi_{i,S_0}\mathcal{T}_i f_{S_1}),$$

where we applied lemma 9.2.1 to compute constant terms.

For $i = 1$, $d_i = 1$ so we can apply theorem 9.3.2 and get (for each summand in the $\coprod$)

$$\begin{aligned}S^{H_i}_{(T_1,1,\lambda_1,\eta_1)}(EP_{\lambda'_1}\bar{\mathbf{1}}_{K^{H_1}(\mathfrak{n}_i)}\varphi_{1,S_0}\mathcal{T}_1 f_{S_1}) \\ = |\mathfrak{n}_i|^{\dim H_1}\Gamma_{T_1}(\mathfrak{n}_i)\Lambda(H_1, \mathcal{T}_1 f_{S_1}, \varphi_{1,S_0}) + O(|n_i|^{\dim H_1 - C} q_{S_1}^{A+B\kappa}).\end{aligned}$$

For $i > 1$, $T_i = 1$, so we can apply lemma 9.4.1 and get:

$$\begin{aligned}S^{H_i}_{(1,d_i,\lambda_i,\eta_i)}(EP_{\lambda'_i} f^\infty) \\ = |\mathfrak{n}_i|\Gamma_1(\mathfrak{n}_i)\frac{\mathrm{vol}(H_i(F)^{\mathrm{ab}}\backslash H_i(\mathbb{A})^{\mathrm{ab}})}{\mathrm{vol}(K^{H_i^{\mathrm{ab}},S})\,\mathrm{vol}((H_{i,\infty})^{\mathrm{ab}})}\int_{H_{i,\mathrm{der},S_1,S_0}} \mathcal{T}_i f_{S_1}\varphi_{i,S_0}(h)\,dh.\end{aligned}$$

The result follows from multiplying and summing over factorizable summands in the ∐ and taking the maximum over the various A's through E's above. We use the second part of lemma 9.2.1 to estimate $I(\mathfrak{n}_i)$ and note that the $\dim G/P$ there is $1/2(\dim G - \dim M)$. □

Note. We write the scaling factor in the theorem statement as it is to emphasize the exact growth in $\mathfrak{n}_i$. It comes from the more conceptual formula:

$$|\mathfrak{n}_i|^{-R(\Delta))}\Gamma_{L(\Delta)}(\mathfrak{n}_i) = \left(|\mathfrak{n}_i|^{\frac{1}{2}(\dim G - \sum_i \dim H_i)}\Gamma_{-1}(\mathfrak{n}_1)^{k-1}\right)\left(\prod_i [K^{\mathrm{GL}_{T_i}} : K^{\mathrm{GL}_{T_i}}(\mathfrak{n}_i)]\right).$$

where the first factor come from the parabolic descent of the functions $\mathbf{1}_{K(\mathfrak{n}_i)}$ and the second from plugging in the expected scaling factors on the groups that the cuspidal pieces that make up Δ come from.

Example. Consider the case when $S_1 = S_0 = \emptyset$. Then this reduces to

$$|\mathfrak{n}_i|^{-R(\Delta))}\Gamma_{L(\Delta)}(\mathfrak{n}_i)S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = 2^{-k+1}\frac{\dim\lambda_1}{|\Pi_{\mathrm{disc}}(\lambda_1)|}\frac{\mathrm{vol}(H(F)\backslash H(\mathbb{A}_F))}{\mathrm{vol}(K_H^\infty)\,\mathrm{vol}(H_\infty^c)} + O(|\mathfrak{n}_i|^{-C}q_{S_1}^{A+B\kappa})$$

where

$$H = H_1 \times \prod_{i>1} H_i^{\mathrm{ab}}.$$

9.6. **General Shapes: Conjectural Optimal Bound.** Considerations of a concept called GK-dimension give us a heuristic for what the optimal growth rate bound should be for any Δ.

First, consider π_v a representation of some p-adic group G_v. The Harish-Chandra-Howe local character expansion gives an expression

$$\Theta_{\pi_v}(\exp g) = \sum_{O\in N} c_O(\pi)\widehat{\mu}_O(g)$$

where N is the set of nilpotent orbits of G_v acting on $\mathrm{Lie}\,G$, $\mu_O(G)$ is the Fourier transform of the δ-measure on the orbit O, $c_O(\pi)$ are constants, and $g \in \mathrm{Lie}\,G$ is in small enough open compact at the identity.

Definition 9.6.1. With the notation as above, let the GK-dimension

$$d_{GK}(\pi_v) := \frac{1}{2}\max\{\dim O : c_O(\pi) \neq 0\}.$$

We can then compute

Lemma 9.6.2. *Assume G_v is unramified. Then*

$$\dim\left(\pi_v^{K^G(q_v^n)}\right) = \mathrm{tr}_{\pi_v}\,\bar{\mathbf{1}}_{K^G(q_v^n)} \asymp q_v^{nd_{GK}(n)}$$

Moeglin and Walspurger in [MW87] associate to each $O \in N$ a particular "degenerate Whittaker model" W_O. The prove that the maximal O such that $c_O(\pi) \neq 0$ are exactly the same as the maximal O such that $\mathrm{Hom}(\pi_v, W_O) \neq 0$.

Now specialize to $G_v = \mathrm{GL}_n(F_v)$. For π_v a tempered representation of some $\mathrm{GL}_d(F_v)$ define $\pi_v[d]$ to be the Langlands quotient of the parabolic induction

$$\mathrm{Ind}_P^{G_v}(\pi_v|\det|^{(d-1)/2} \boxtimes \pi_v|\det|^{(d-3)/2} \boxtimes \cdots \boxtimes \pi_v|\det|^{-(d-1)/2})$$

This is like a local component of the Speh representation from the construction of $\pi_{\tau[d]}$ §2.2.2.

Any tempered π_v on $\mathrm{GL}_t(F_v)$ is generic and therefore satisfies

$$GK(\pi_v) = \frac{1}{2}t(t-1).$$

On the other hand, Mitra in [Mit20] computes that the maximal $O \in N$ such that $\mathrm{Hom}(\pi_v[d], W_0) \neq 0$ is the principle nilpotent orbit in the Levi $M \subseteq \mathrm{GL}_{td}$ missing the simple roots indexed by $\{t, 2t, \dots, (d-1)t\}$. This is the O that corresponds to partition $(t^{(d)})$ in Jordan normal form. We can therefore compute

$$GK(\pi_v[d]) = \frac{1}{2}(t^2d^2 - td^2) = \frac{1}{2}d^2t(t-1)$$

and get

$$\dim\left(\pi_v[d]^{K(q_v^n)}\right) \asymp q_v^{\frac{1}{2}t(t-1)(d^2-1)n} \dim\left(\pi_v^{K(q_v^n)}\right) \tag{9.6.1}$$

By the Ramanujan conjecture, for any simple parameter $\psi[d]$, we expect all the ψ_v to correspond to tempered representations on the GL side (this is in fact known for our case, see lemma 6.1 in [MS19]). Therefore we can try to use the heuristic (9.6.1) instead of lemmas 9.4.1, 9.4.2,and 9.4.3 in theorem 9.4.4 and get:

Conjecture 9.6.3. *The bound for*

$$\Delta = (T_i, d_i, \lambda_i, \eta_i)_{1 \le i \le k} \neq \Sigma_{\lambda,\eta}$$

in theorem 9.3.2 may be tightened to

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = O(|\mathfrak{n}_i|^{R_0(\Delta)} q_{S_1}^{A+B\kappa})$$

under all the same conditions and where

$$\begin{aligned} R_0(\Delta) &:= \frac{1}{2}\left(N^2 - \sum_i T_i^2 d_i^2\right) + \sum_i \left(T_i^2 + \frac{1}{2}T_i(T_i-1)(d_i^2-1)\right) \\ &= \bar{R}(\Delta) - \sum_i \left(\frac{1}{2}T_i^2 d_i(d_i+1) - \left(T_i^2 + \frac{1}{2}T_i(T_i-1)(d_i^2-1)\right)\right). \end{aligned}$$

We think of $R_0(\Delta)$ again as making a modification to the dimension count of the the parabolic that gives $\bar{R}(\Delta)$: replace the dimension of the parabolic corresponding to partition $(d_i^{(t_i)})$ in the $\mathrm{GL}_{t_i d_i}$-block on the diagonal with $T_1^2 + \frac{1}{2}T_i(T_i-1)(d_i^2-1)$.

The main obstacle in proving conjecture 9.6.3 is proving that the asymptotic (9.6.1) is uniform enough in π_v. This appears to require proving upper bounds that are uniform over tempered π_v on coefficients $c_0(\pi_v)$ for all O. In particular, we need to understand c_O for *non-maximal* O in the wavefront set of π_v so the techniques of [MW87] don't apply.

Of course, conjecture 9.6.3 can only be exact for $S^{|G|}_\Delta$ instead of S^G_Δ. In the case where all the d_i have the same parity, $s_\psi = 1$ for all $\psi \in \Delta$ so these are the same. Otherwise, the terms S_ψ for different $\psi \in \Delta$ are attached to a varying sign $\epsilon_\psi(s_\psi)$. If we naïvely assume some non-trivial cancellation, we get the following:

Conjecture 9.6.4. *Recall the setup and conditions for theorems 9.3.2, 9.4.4, and conjecture 9.6.3. Then, if all the d_i have the same parity:*

$$C_{\epsilon,1}|\mathfrak{n}_i|^{R_0(\Delta)-\epsilon} \le S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) \le C_{\epsilon,2}|\mathfrak{n}_i|^{R_0(\Delta)+\epsilon}$$

for all $\epsilon > 0$ *and some constants* $C_{\epsilon,1}, C_{\epsilon,2}$ *such that* $C_{\epsilon,2} = O_\epsilon(q_{S_1}^{A+B\kappa})$.

If the d_i*'s have different parities, then*

$$S^G_\Delta(\mathrm{EP}_\lambda f_i^\infty) = o(|\mathfrak{n}_i|^{R_0(\Delta)} q_{S_1}^{A+B\kappa}).$$

10. Application to Limit Multiplicities

Let G be a pure inner form of some $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ and fix a cohomological representation π_0 of G_∞. Fix f^∞ that is nowhere negative and unramified outside of a finite set of finite places S. In this section, we want to compute

$$m^G(\pi_0, f^\infty) := \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} m_\pi \mathbf{1}_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty).$$

in terms of the form of $S^H_{\Delta'}(\eta_\lambda(f^\infty)')$ for each H in some $\mathcal{E}_{\mathrm{ell}}(N)$.

10.1. **Preliminaries.** Let the ψ^i_∞ be the Arthur parameters at infinity such that $\pi_0 \in \Pi^G_{\psi^i_\infty}$ (we recall the packet is now defined as in Theorem 1.6.1 of [KMSW14]). This is a finite (possibly empty) list since it is a subset of the set of Arthur parameters with a particular infinitesimal character. It can be seen that for each ψ^i_∞, there is a finite number of choices of Δ such that ψ^i_∞ is the unique infinite component of global $\psi \in \Delta$ as in Lemma 4.3.4—intuitively, these are parameterized by ways to group together simple factors of ψ^i_∞ that share the same Arthur SL_2.

In total, we get a finite (possibly empty) set of refined shapes $\Delta(\pi_0)$ such that

$$\pi_0 \otimes \pi^\infty \in \Pi_\psi \implies \psi \in \Delta \text{ for some } \Delta \in \Delta(\pi_0). \tag{10.1.1}$$

The spectral decomposition then produces:

$$m^G(\pi_0, f^\infty) = \sum_{\Delta \in \Delta(\pi_0)} \sum_{\psi \in \Delta} \sum_{\pi \in \Pi_\psi} m^\psi_\pi \mathbf{1}_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty). \tag{10.1.2}$$

Specializing to a single $\Delta \in \Delta(\pi_0)$, consider for any test function, f_∞ on G_∞

$$I^G_\Delta(f_\infty f^\infty) := \sum_{\psi \in \Delta} I^G_\psi(f_\infty f^\infty) = \sum_{\psi \in \Delta} \sum_{\pi \in \Pi_\psi} m^\psi_\pi \operatorname{tr}_{\pi_\infty}(f_\infty) \operatorname{tr}_{\pi^\infty}(f^\infty). \tag{10.1.3}$$

Here the I^G_ψ are defined analogously to the quasisplit case using Theorem 1.7.1 of [KMSW14].

Applying Lemma 3.4.3 part 1 to equation (10.1.3) and comparing to equation (10.1.2) then gives:

Corollary 10.1.1. *Let* π_d *be a discrete series representation appearing in the character formula for* π_0 *with sign* σ*. Then:*

$$m^G(\pi_0, f^\infty) = \sigma \sum_{\Delta \in \Delta(\pi_0)} I^G_\Delta(\varphi_{\pi_d} f^\infty).$$

10.2. **Stabilization.** Now we can finish by stabilizing to compute the I^G_Δ. We start with a more conceptual version:

Let $H \in \mathcal{E}_{\mathrm{ell}}(G)$. Then $H = H_1 \times H_2$ for H_i quasisplit unitary groups. Therefore, we can abuse notation and use each H_i to also denote a representative in some $\mathcal{E}_{\mathrm{ell}}(N)$ that is isomorphic as an algebraic group. Any refined shape Δ on H corresponds to a finite and possibly empty set of shapes $\Delta_1 \times \Delta_2$ on $H_1 \times H_2$ that push forward to Δ. Stabilization of each I^G_ψ (through [KMSW14] equation (3.3.2)) gives:

Proposition 10.2.1. *Let π_d be a discrete series representation appearing in the character formula for π_0 with sign σ. Then:*

$$m^G(\pi_0, f^\infty) = \sigma \sum_{\Delta \in \Delta(\pi_0)} \sum_{H_1 \times H_2 \in \mathcal{E}_{\mathrm{ell}}(G)} \sum_{\Delta_1 \times \Delta_2} \iota(G, H_1 \times H_2) \coprod_{i=1,2} S^{H_i}_{\Delta_i}((\varphi_{\pi_d})^{H_i}(f^\infty)^{H_i}).$$

where the $\star^{H_i}$ terms represent the corresponding factors of transfers to H.

Each $(\varphi_{\pi_d})^{H_i}$ can be chosen to be a linear combination of EP-functions by standard formulas for transfers of pseudocoefficients. To be explicit about this, let some pair $(H(s), \Delta'(s))$ that pushes forward to Δ correspond to $s \in \mathcal{S}_\Delta$. Then,

$$\mathrm{tr}_{\psi^{\Delta'}_\infty}(\varphi^H_{\pi_d}) = \eta^{\psi^\Delta_\infty}_{\pi_0}(s' s_\Delta) = \sigma \eta^{\psi^\Delta_\infty}_{\pi_0}(s')$$

by the endoscopic character identity 2.6.2 where s' is the lift of s to $S^\natural_\Delta$ therein. For the second equality, we use $\eta^{\psi^\Delta_\infty}_{\pi_0}(s_\Delta) = \sigma$ as in the proof of part 2 of lemma 3.4.3. If Δ' corresponds to total infinitesimal character λ, then this gives

$$\mathrm{tr}_{\psi^{\Delta'}_\infty}(\varphi^H_{\pi_d}) = \sigma \eta^{\psi^\Delta_\infty}_{\pi_0}(s')\, \mathrm{tr}_{\psi^\Delta_\infty}(\mathrm{EP}_\lambda).$$

since the EP-function trace is just 1. Changing notation a bit produces:

Corollary 10.2.2. *Let $G \in \mathcal{E}_{\mathrm{ell}}(N)$ and fix a cohomological representation π_0 of $G(\mathbb{R})$. Then*

$$\begin{aligned} m^G(\pi_0, f^\infty) &= \sum_{\Delta \in \Delta(\pi_0)} m^G(\pi_0, \Delta, f^\infty) \\ &:= \sum_{\Delta \in \Delta(\pi_0)} \sum_{s \in \mathcal{S}_\Delta} \iota(G, H(s)) \eta^{\psi^\Delta_\infty}_{\pi_0}(s') \sum_{\Delta_1 \times \Delta_2} \coprod_{i=1,2} S^{H_i(s)}_{\Delta_i}(\mathrm{EP}_{\lambda_i}(f^\infty)^{H_i}). \end{aligned}$$

where $(H(s), \Delta'(s))$ is the pair of group and shape corresponding to $s \in \mathcal{S}_\Delta$ which is lifted to $s' \in S^\natural_\Delta$ as in theorem 2.6.2, $\Delta_1 \times \Delta_2$ ranges over refined shapes on $H_1(s) \times H_2(s)$ such that the pair $(H(s), \Delta_1 \times \Delta_2)$ is equivalent to $(H(s), \Delta'(s))$, and λ_i is the total infinitesimal character of Δ_i.

Note. The sum over $\Delta_1 \times \Delta_2$ will be a singleton unless $H_1 \cong H_2$ in which case there will be two terms that differ by transposing the factors.

Note. We briefly discuss the relation between this and the formulas in [Lab11] for transfers of pseudocoefficients. For simplicity, assume we are in a case where we never have $H_1 \cong H_2$.

Then, for each Δ there is only ever one possible choice $\Delta_1 \times \Delta_2$ due to the fixing of infinitesimal characters at infinity in shapes. The sum over EP-functions in Labesse's formulas then comes from the sum over $\Delta(\pi_0)$—these then correspond to different $\Delta_1 \times \Delta_2$ with non-conjugate infinitesimal characters.

Next, while different $\Delta \in \Delta(\pi_0)$ may have the same $\psi_\infty(\Delta)$ or even $\mathcal{S}_\Delta$, the embeddings $\mathcal{S}_\Delta \hookrightarrow \mathcal{S}_{\psi^\Delta_\infty}$ will differ. This accounts for unexpected differences in signs of coefficients of EP-functions in Labesse's formulas.

10.3. **Transfer Factors.** We eventually want to apply proposition 10.2.2 to f^∞ as in section 9.1. In the most important case where $S_0 = \emptyset$, we will need to compute explicit endoscopic transfers at all places so we need to choose explicit local transfer factors that are consistent globally.

First, pick a global Whittaker datum ω on the quasisplit form G^* of G. Since $S_0 = \emptyset$, we necessarily need that G is unramified at all finite places. Therefore $G^*_v = G_v$ for all finite v and G^*_v is in particular also unramified for all v. This implies that G^* can be defined over $\mathcal{O}_F$ so we can choose ω so that the the induced local data ω_v are unramified/admissible everywhere as in [Hal93, §7]. This allows us to use the fundamental lemma for each G_v.

Next, [Kal18, §4.4] shows that the choice of ω also gives us compatible local transfer factors on G itself (we note that G has simply connected derived subgroup to make the extra term in theorem 4.4.1 disappear). At finite places G_v, the local factors stay the same as for the G^*_v.

10.4. **Limit Multiplicities.** As a preliminary/example computation, we work out what the summand $m^G(\pi_0, \Delta, f_\infty)$ from proposition 10.2.2 is for

$$\Delta = (T_1, 1, \lambda_1, \eta_1), (1, d_2, \lambda_2, \eta_2)$$

with d_2 odd so that Δ is odd GSK. In our eventual application to unitary groups, this will be the dominant term in the sum.

Then, $\mathcal{S}_\Delta \cong \mathbb{Z}/2$ and the non-identity element s corresponds to

$$H(s) = H_1 \times H_2 = U(T_1) \times U(d_2).$$

If $T_1 \neq d_2$, there is a unique choice

$$\Delta_1 \times \Delta_2 = (T_1, 1, \lambda_1, \eta_1) \times (1, d_2, \lambda_2, \eta_2)$$

and $\iota(G, H(s)) = 1/2$. If $T_2 = d_2$, there are two choices for $\Delta_1 \times \Delta_2$ that correspond to the exact same product of S-terms and $\iota(G, H(s)) = 1/4$. Either way, 10.2.2 reduces to

$$\begin{aligned}(10.4.1)\quad m^G(\pi_0, \Delta, f_\infty) &= S^{G^*}_\Delta(\mathrm{EP}_\lambda(f^\infty)^{G^*}) \\ &+ \frac{1}{2}\eta^{\psi^\Delta_\infty}_{\pi_0}(s') S^{U(T_1)}_{(T_1,1,\lambda_1,\eta_2)}(\mathrm{EP}_{\lambda_1}(f^\infty)^{U(T_1)}) \times S^{U(d_2)}_{(1,d_2,\lambda_2,\eta_2)}(\mathrm{EP}_{\lambda_2}(f^\infty)^{U(d_2)})\end{aligned}$$

where G^* is the quasisplit form of G and where the product implicitly includes a sum over factorizable summands of the transfer to $H(s)$.

Now assume

$$f^\infty = \varphi_{S_0} f_{S_1} \bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$$

is of the form in section 9.1. Furthermore, assume that he chosen transfers of φ_{S_0} satisfy conjecture 6.4.1 for Δ.

Then, by theorem 9.5.1, the first summand has main term (ignoring non-factorizability as usual):

$$\frac{1}{2}|\mathfrak{n}_i|^{\frac{1}{2}(N^2+T_1^2-d_2^2)+1}\Gamma_{T_1,-1,1}(\mathfrak{n}_i)\frac{\dim\lambda_1}{|\Pi_{\mathrm{disc}}(\lambda_1)|}\frac{\mathrm{vol}(H'(F)\backslash H'(\mathbb{A}_f))}{\mathrm{vol}(K^S_{H'})\,\mathrm{vol}((H'_\infty)^c)}$$
$$\varphi_{1,S_0}(1) f^{H_1}_{S_1}(1)\int_{H_{2,\mathrm{der},S_0,S_1}} f^{H_2}_{S_1}\varphi_{2,S_0}(h)\,dh$$

where $H' = H_1 \times H_2^{\mathrm{ab}}$. We use here that being unramified makes $\mathcal{T}_i f_{S_1} = f^{H_i}_{S_1}$

Next, considerations as in lemma 6.1.2 give that the transfer of the $\mathbf{1}_{K^{G,S}(\mathfrak{n}_i)}$ term is a constant term to a Levi so lemma 9.2.1 gives

$$\bar{\mathbf{1}}^H_{K^{G,S}(\mathfrak{n}_i)} = I(\mathfrak{n}_i)\bar{\mathbf{1}}_{K^{H_1,S}(\mathfrak{n}_i)} \times \bar{\mathbf{1}}_{K^{H_2,S}(\mathfrak{n}_i)}.$$

We can therefore use lemma 9.2.1 to get that the second summand has main term (ignoring factorizability issues):

$$\frac{1}{2}|\mathfrak{n}_i|^{\frac{1}{2}(N^2-T_1^2-d_2^2)}\Gamma_{-1}(\mathfrak{n}_i)\eta_{\pi_0}^{\psi_\infty^\Delta}(s')$$
$$S^{H_1}_{(T_1,1,\lambda_1,\eta_2)}(\mathrm{EP}_{\lambda_1}\varphi^{H_1}_{S_0} f^{H_1}_{S_1}\bar{\mathbf{1}}_{K^{H_1,S}(\mathfrak{n}_i)}) \times S^{H_2}_{(1,d_2,\lambda_2,\eta_2)}(\mathrm{EP}_{\lambda_2}\varphi^{H_2}_{S_0} f^{H_2}_{S_1}\bar{\mathbf{1}}_{K^{H_2,S}(\mathfrak{n}_i)}).$$

Theorem 9.3.2 gives that the first factor has main term

$$|\mathfrak{n}_i|^{T_1^2}\Gamma_{T_i}(\mathfrak{n}_i)\varphi^{H_1}_{S_0}(1) f^{H_1}_{S_1}(1)\frac{\mathrm{vol}(H_1(F)\backslash H_1(\mathbb{A}_f))}{\mathrm{vol}(K^S_{H_1})\,\mathrm{vol}(H^c_{1,\infty})}$$

and proposition 8.1.2 gives that second factor is eventually

$$|\mathfrak{n}_i|\Gamma_1(\mathfrak{n}_i)\frac{\mathrm{vol}(H_2^{\mathrm{ab}}(F)\backslash H_2^{\mathrm{ab}}(\mathbb{A}_f))}{\mathrm{vol}(K^S_{H_2^{\mathrm{ab}}})\,\mathrm{vol}(H^{\mathrm{ab}}_{2,\infty})}\int_{H_{2,\mathrm{der},S_0,S_1}} f^{H_2}_{S_1}\varphi^{H_2}_{S_0}(h)\,dh$$

after using an argument as in corollary 9.4.4 to remove all the terms in the sum except for 1. After multiplying everything together, this shows that the summands for G and $H_1 \times H_2$ in (10.4.1) have the exact same asymptotic dependence on $\mathfrak{n}_i$.

When $S_0 = \emptyset$, we can drop the dependence on conjecture 6.4.1 and put everything together reasonably cleanly: for some A, B, C, D, E with $C \geq 1$, as long as $|\mathfrak{n}_i| \geq Dq_{S_1}^{E\kappa}$:

$$(10.4.2)\quad |\mathfrak{n}_i|^{-\frac{1}{2}(N^2+T_1^2-d_2^2)-1}\Gamma_{T_1,-1,1}(\mathfrak{n}_i)^{-1}m^G(\pi_0,\Delta,f_{S_1}\bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)})$$
$$= \mathbf{1}_{\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta)=1}\frac{\dim\lambda_1}{|\Pi_{\mathrm{disc}}(\lambda_1)|}\frac{\mathrm{vol}(H'(F)\backslash H'(\mathbb{A}_f))}{\mathrm{vol}(K^S_{H'})\,\mathrm{vol}((H'_\infty)^c)}\left(f^{H_1}_{S_1}(1)\int_{H_{2,\mathrm{der},S_1}} f^{H_2}_{S_1}(h)\,dh\right)$$
$$+ O(|\mathfrak{n}_i|^{-C}q_{S_1}^{A+B\kappa}).$$

If $S_0 = \emptyset$, then G is unramified at all finite places so we necessarily have that $\eta_{\pi_0}^{\psi_\infty^\Delta}$ factors through $\mathcal{S}_\Delta$ because of the conditions on the χ_{G_v} in theorem 2.5.2 to glue together to a global group as in lemma 2.1.1.

The case when $\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta)$ isn't trivial can actually be understood more simply. If $S_0 = \emptyset$, then each factor f_v of the test function is either on $\mathrm{GL}_{N,v}$ or unramified. Therefore, it traces to 0 against all $\pi_v \in \Pi_{\psi_v}$ such that $\eta_{\pi_v}^{\psi_v} \neq 1$. By the multiplicity formula 2.5.4, this implies that

$$m_\pi^\psi\,\mathrm{tr}_{\pi^\infty}(f^\infty) = 0$$

whenever $\eta_{\pi_\infty}^{\psi_\infty} \neq 1$. In total,

$$\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta) \neq 1 \implies m^G(\pi_0,\Delta,f^\infty) = 0.$$

A slightly more complicated generalization of this computation gives:

Theorem 10.4.1. *Let G be an extended pure inner form of $G \in \mathcal{E}_{\mathrm{ell}}(N)$ that is unramified at all finite places and let π_0 be a cohomological representation of G_∞. Choose odd GSK $\Delta = (T_i, d_i, \lambda_i, \eta_i)_{1\leq i\leq k} \in \Delta(\pi_0)$ and let $f = f_{S_1}\bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$ be as in section 9.1 with $S_0 = \emptyset$.*

Then, if $\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta) \neq 1$

$$m^G(\pi_0, \Delta, f^\infty) = 0.$$

Otherwise, there are A, B, C, D, E *with* $C \geq 1$ *such that as long as* $|\mathfrak{n}_i| \geq D q_{S_1}^{E\kappa}$*:*

$$\begin{aligned}
&|\mathfrak{n}_i|^{R(\Delta)} \Gamma_{L(\Delta)}(\mathfrak{n}_i)^{-1} m^G(\pi_0, \Delta, f^\infty) \\
&\quad = \frac{\dim \lambda_1}{|\Pi_{\mathrm{disc}}(\lambda_1)|} \frac{\mathrm{vol}(H'(F)\backslash H'(\mathbb{A}_f))}{\mathrm{vol}(K_{H'}^S)\,\mathrm{vol}((H'_\infty)^c)} \left(f_{S_1}^{H_1}(1) \coprod_{i>1}^{\oplus} \int_{H_{i,\mathrm{der},S_1}} f_{S_1}^{H_i}(h)\, dh \right) \\
&\qquad\qquad + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa})
\end{aligned}$$

where $H_i = H(T_i, d_i, \lambda_i, \eta_i)$*,* $H' = H_1 \times \prod_{i>1} H_i^{\mathrm{ab}}$*, and* $R(\Delta)$ *is as in corollary 9.4.4.*

For our upper bound we can allow $S_0 \neq \emptyset$:

Theorem 10.4.2. *Let* G *be an extended pure inner form of* $G \in \mathcal{E}_{\mathrm{ell}}(N)$ *that may or may not be unramified and let* π_0 *be a cohomological representation of* G_∞*. Choose arbitrary* $\Delta \in \Delta(\pi_0)$ *and let* $f^\infty = \varphi_{S_0} f_{S_1} \bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$ *be as in section 9.1.*

Then there are A, B, D, E *such that as long as* $|\mathfrak{n}_i| \geq D q_{S_1}^{E\kappa}$

$$m^G(\pi_0, \Delta, f^\infty) = O(|\mathfrak{n}_i|^{R(\Delta)} q_{S_1}^{A+B\kappa})$$

where $R(\Delta)$ *is as in corollary 9.4.4.*

Proof. Apply proposition 10.2.2 and then 9.4.4 to each term. □

Note that Theorem 10.4.2 does *not* depend on conjecture 6.4.1 since corollary 9.4.4 doesn't.

11. Explicit Computations on Unitary Groups

In this section, we recall the explicit combinatorial parameterization of cohomological representations of $U(p, q)$, their A-packets and Adams-Johnson parameters, following [MR19], see also [VZ84, Tra01, BC05]. This allows us to compute the sets $\Delta(\pi_0)$ and work out explicit limit multiplicity statements from proposition 10.2.1 together with the bounds in theorems 10.4.1 and 10.4.2.

11.1. Cohomological Representations of $U(p, q)$.

11.1.1. *Setup:* We recall some general facts about the construction of cohomological representations. Let G be a reductive group with Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition corresponding to a choice of maximal compact subgroup K, with Cartan involution ι[3]. Assume that G has a compact Cartan subgroup T with Lie algebra $\mathfrak{t} \subset \mathfrak{k}$. Let $\Delta(\mathfrak{t}, \mathfrak{g})$ be the root system for $\mathfrak{t}$ in $\mathfrak{g}$.

In [VZ84], Vogan-Zuckerman introduce the notion of a ι-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, henceforth referred to as VZ subalgebras. To construct such a $\mathfrak{q}$, choose $x \in i\mathfrak{t}_0$ and define $\lambda_0 \in \mathfrak{t}^*$ by $[x, \mathfrak{g}^\alpha] = \langle \lambda_0, \alpha \rangle \mathfrak{g}^\alpha$ for $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$. Then

$$\mathfrak{q} := \mathfrak{l} \oplus \mathfrak{u}, \quad \mathfrak{l} := \mathfrak{t} \oplus \bigoplus_{\langle \lambda_0, \alpha \rangle = 0} \mathfrak{g}^\alpha, \quad \mathfrak{u} := \oplus \bigoplus_{\langle \lambda_0, \alpha \rangle \geq 0} \mathfrak{g}^\alpha. \tag{11.1.1}$$

Let $L = Z_G(\lambda_0)$ be the Levi subgroup of G with lie algebra $\mathfrak{l}$. Let $\Delta(\mathfrak{t}, \mathfrak{u})$ be the roots of $\mathfrak{t}$ in $\mathfrak{u}$, and let $\lambda : \mathfrak{l} \to \mathbb{C}$ be a character such that

[3] In lieu of the traditional notation θ, which we reserve for the involution defining our unitary group.

(i) λ is the differential of a one-dimensional representation of L, and
(ii) $\langle \alpha, \lambda \rangle \geq 0$ for $\alpha \in \Delta(\mathfrak{t}, \mathfrak{u})$.

Then Vogan-Zuckerman define a representation $A_{\mathfrak{q}}(\lambda)$, and show that if F_λ is the irreducible finite-dimensional G-representation of highest weight λ, then $A_{\mathfrak{q}}(\lambda)$ and F_λ have the same infinitesimal character $\lambda + \rho_G$ and

$$H^*(\mathfrak{g}, K; A_{\mathfrak{q}}(\lambda) \otimes F_\lambda) \neq 0.$$

Moreover, any representation with nontrivial $(\mathfrak{g}, K)$-cohomology is isomorphic to $A_{\mathfrak{q}}(\lambda)$ for some pair $(\mathfrak{q}, \lambda)$.

11.1.2. *The Parameterization.* The parameterization of cohomological representations of $U(p, q)$ will be given in terms of the following combinatorial data.

Definition 11.1.1. For a pair of nonnegative integers p, q with $p + q = N$, let:

- $\mathcal{P}(N)$ to be the set of *ordered* partitions of N, i.e. tuples $(N_1, ..., N_r)$ where r is arbitrary, each N_i is positive, and $\sum_i N_i = N$.
- $\mathcal{P}(p, q)$ be the set of ordered bipartitions of (p, q), i.e. the set of tuples of pairs $((p_1, q_1), ..., (p_r, q_r))$ where the p_i, q_i are a nonnegative integers with $\max p_i, q_i > 0$ and $\sum_i p_i = p$, $\sum_i q_i = q$.
- $\mathcal{P}_1(p, q) \subset \mathcal{P}(p, q)$ be the subset consisting of expressions $((p_1, q_1), ..., (p_r, q_r))$ where if $p_i q_i = 0$, then $\max(p_i, q_i) = 1$.

There are natural surjective maps

$$\beta : \mathcal{P}(p, q) \to \mathcal{P}(N), \quad ((p_1, q_1), ..., (p_r, q_r)) \mapsto (p_1 + q_1, ..., p_r + q_r), \tag{11.1.2}$$

$$\gamma : \mathcal{P}(p, q) \to \mathcal{P}_1(p, q) \tag{11.1.3}$$

where γ replaces any term of the form $(n, 0)$ (resp. $(0, m)$) by n copies of $(1, 0)$ (resp. m copies of $(0, 1)$.)

The bipartitions $B \in \mathcal{P}(p, q)$ parameterize VZ subalgebras following [Tra01, §1-3], who proves:

Proposition 11.1.2. *Let $G = U(p, q)$.*

(i) *The K-conjugacy classes of VZ subalgebras of $\mathfrak{g}$ are in bijection with $\mathcal{P}(p, q)$.*
(ii) *Let $\mathfrak{q}$, $\mathfrak{q}'$ be VZ subalgebras corresponding to $B_{\mathfrak{q}}$, $B_{\mathfrak{q}'} \in \mathcal{P}(p, q)$. Then $A_{\mathfrak{q}}(0) \simeq A_{\mathfrak{q}'}(0)$ if and only if $\gamma(B_{\mathfrak{q}}) = \gamma(B_{\mathfrak{q}'})$.*

We write the Levi subgroup L_B associated to $B \in \mathcal{P}(p, q)$ as

$$L_B = U(p_1, q_1) \times ... \times U(p_r, q_r).$$

To realize the bijection, embed $K = U(p, 0) \times U(0, q)$ in $U(p, q)$, write

$$\mathfrak{t}_0 = \mathfrak{t}_0 \cap \mathfrak{u}_0(p, 0) \oplus \mathfrak{t}_0 \cap \mathfrak{u}_0(0, q) \simeq \mathbb{R}^p \times \mathbb{R}^q,$$

and associate to $B = ((p_1, q_1), ..., (p_r, q_r))$ the element ix_B for

$$x_B = (\overbrace{r, ..., r}^{p_1}, \overbrace{r-1, ..., r-1}^{p_2}, ..., \overbrace{1, ..., 1}^{p_r}, \overbrace{r, ..., r}^{q_1}, ..., \overbrace{1, ..., 1}^{q_r}) \in \mathfrak{t}_0 \subset \mathfrak{k}_0.$$

Let λ be an infinitesimal character. Recall the notion of P-parts of λ from §1.6.3.

Definition 11.1.3. We say that a regular integral infinitesimal character

$$\lambda = \xi_1 > \cdots > \xi_n$$

is adapted to partition P if the P-parts of λ are all of the form

$$X^r \sum_{i=1}^{n} X^{(n-2i+1)/2}$$

for some integer or half integer r and some integer n.

For example the infinitesimal character ρ_G of the trivial representation is adapted to all partitions of N. The following is also deduced from [Tra01]:

Proposition 11.1.4. *Let $G = U(p,q)$. The cohomological representations with regular integral infinitesimal character λ are all of the form $A_{\mathfrak{q}}(\lambda - \rho_G)$ with $\mathfrak{q}$ corresponding to bipartition $B \in \mathcal{P}(p,q)$ such that λ is adapted to $\beta(B)$. In particular, they are in bijection with the bipartitions $B \in \mathcal{P}_1(p,q)$ with λ adapted to $\beta(B)$.*

To compute the cohomology, associated to the representations, one makes a choice of complex structure on $\mathfrak{p}$ i.e. on the quotient G/K. To do this, fix a Shimura datum for G to be the conjugacy class of

$$h_K : \mathbb{S} \to U(p,q), \quad h_K(z) = \left(\frac{z}{\bar{z}} I_p, I_q\right) \in U(p,0) \times U(0,q) \subset U(p,q).$$

This induces a decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathrm{Ad}(h(z))$ acts on $\mathfrak{p}^+$ by $z/\bar{z}$ and on $\mathfrak{p}^-$ by its inverse.

Lemma 11.1.5. *Let $G = U(p,q)$ and a choice of Shimura datum as above. Let F_λ be a finite-dimensional representation with highest weight λ. Let $B = ((p_1,q_1),...,(p_r,q_r)) \in \mathcal{P}(p,q)$ be a partition such that $\lambda + \rho_G$ is adapted to $\beta(B)$, and let*

$$R = pq - \sum_i p_i q_i, \quad R^+ = \sum_{i<j} p_i q_j, \quad R^- = \sum_{i>j} p_i q_j.$$

Then

(i) the smallest value of i such that $H^i(\mathfrak{g}, K; A_{\mathfrak{q}_B}(\lambda) \otimes F_\lambda) \neq 0$ is $i = R$;
(ii) $H^j(\mathfrak{g}, K; A_{\mathfrak{q}_B}(\lambda) \otimes F_\lambda) \neq 0$ if and only if $j = R + 2p$ for $0 \leq p \leq \sum_i p_i q_i$;
(iii) the Hodge weights of $A_{\mathfrak{q}_B}(\lambda)$ in degree $R + 2p$ are $(R^+ + p, R^- + p)$.

Proof. This follows from [VZ84, Thm. 3.3, 5.4, 6.19]: the first nonzero degree of cohomology of $A_{\mathfrak{q}}(\lambda)$ is $R = \dim \mathfrak{u} \cap \mathfrak{p} = \dim \mathfrak{p} - \dim \mathfrak{l} \cap \mathfrak{p} = pq - \sum_i p_i q_i$. More precisely, the cohomology of $A_{\mathfrak{q}_B}(\lambda)$ in degree $\dim \mathfrak{u} \cap \mathfrak{p}$ appears in weight $(R^+, R^-) = (\mathfrak{u} \cap \mathfrak{p}^+, \mathfrak{u} \cap \mathfrak{p}^-)$. All other weights (a,b) for which $A_{\mathfrak{q}}(\lambda)$ has cohomology are of the form $(R^+ + p, R^- + p)$. One computes R^+ and R^- from x_B and the choice of Shimura datum, see e.g. [BC05, §5] for more details. □

11.1.3. *Packets of cohomological representations.* Recall form §3.2 that any parameter associated to a packet of cohomological representations is an Adams-Johnson parameter. In particular, there is a Levi subgroup

$$\hat{L} \simeq GL_{n_1} \times ... \times GL_{n_r} \subset \hat{G}$$

such that $\psi(W_{\mathbb{C}} \times I) \subset Z(\hat{L})$, and such that $\psi(1 \times SL_2)$ is a principal SL_2 in $\hat{L}$. Thus we have

$$\psi \mid_{W_{\mathbb{C}} \times SL_2} = \bigoplus_{i=1}^{r} \chi_{t_i} \otimes [n_i]$$

where $[n_i]$ is the irreducible n_i-dimensional representation of $SL_2(\mathbb{C})$ and

$$\chi_{t_i}(z) = \left(\frac{z}{\bar{z}}\right)^{t_i/2}, \quad z \in W_{\mathbb{C}} \simeq \mathbb{C}^\times.$$

In this case, the total infinitesimal character of ψ is

$$\lambda_\psi = \sum_{i=1}^{r} X^{\frac{t_i}{2}} \sum_{j=1}^{n_i} X^{\frac{n_i-2j+1}{2}}.$$

Since we assume that λ_ψ is the infinitesimal character of a finite-dimensional representation, we have for each i that $t_i - n_i \equiv N \mod 2$, i.e. we are in the good parity case in the sense of [MR19]. To a good parity parameter, Mœglin-Renard attach the ordered partition $P = (n_1, ..., n_r) \in \mathcal{P}(N)$: specifically, the unordered multiset of n_i come from restricting to the Arthur-SL_2 and the ordering is such that the infinitesimal character of the summand $\chi_{t_i} \otimes [n_i]$ of ψ is the ith P-part of λ_ψ (in particular, P is adapted to λ_ψ). They also show that the corresponding packet is

$$\Pi_\psi = \{A_{\mathfrak{q}_B}(\lambda_B) \mid \beta(B) = P\},$$

where $\lambda_B = \boxtimes_i \det^{(t_i+a_i-N)/2-a_{<i}}$ for $a_i = \sum_{j<i} a_j$. In short, the representations in the packet π_ψ are in bijection with the bipartitions $((p_1, q_1), ..., (p_r, q_r)) \in \mathcal{P}(p, q)$ such that $p_i + q_i = n_i$.

The article [MR19] does more. First they fix a Whittaker datum for a choice of quasisplit form G^* of G (see [MR19, Remarque 4.5]) which without loss of generality we can have match the one from section 10.3. Then, they write down explicitly the character $\eta_\pi : S_\psi^\natural \to \pm 1$ attached by Arthur to each representation in the packet Π_ψ, in terms of the bipartition B such that $\pi = \pi_B$. These constructions are recalled in §11.3, where they are used.

11.2. **Understanding $\Delta(\pi_0)$ and $R(\Delta)$.** Now, fix an extended pure inner form G of $G^* \in \mathcal{E}_{\mathrm{sim}}(N)$. Let

$$\pi_0 = \bigotimes_{v \in \infty} \pi_{0,v}$$

be a cohomological representation of

$$G_\infty = \prod_{v \in \infty} U(p_v, q_v).$$

We want to understand the set $\Delta(\pi_0)$ introduced in §10.1: specifically, for each $\Delta \in \Delta(\pi_0)$, we want to compute the invariant $R(\Delta)$ from theorem 9.4.4 and the set of shapes that realize it:

Definition 11.2.1. Let $\Delta^{\max}(\pi_0)$ be the set of $\Delta \in \Delta(\pi_0)$ with maximal $R(\Delta)$.

Definition 11.2.2. Let $R(\pi_0)$ be the common value of $R(\Delta)$ for $\Delta \in \Delta^{\max}(\pi_0)$.

11.2.1. *Ignoring η_i.* First, since each $\Delta = (T_i, d_i, \lambda_i, \eta_i)_i \in \Delta(\pi_0)$ satisfies $H(\Delta) = G^*$, the η_i are completely determined by T_i and d_i according to section 2.2.4. Therefore, their information is redundant so we will ignore it in this section.

11.2.2. *Arthur* SL_2*'s.* We first try to understand Δ according to their Arthur SL_2. These N-dimensional representations of SL_2 correspond to unordered partitions of N via their decomposition into irreducibles. It is easy to see that

Lemma 11.2.3. *Among the shapes Δ with Arthur SL_2 given by unordered partition*

$$Q = (a_1^{(r_1)}, \dots, a_k^{(r_k)})$$

with a_i distinct, $R(\Delta)$ (defined as in theorem 9.4.4) is maximized for ones of the form $(r_i, a_i, \lambda_i, \eta_i)_{1 \le i \le k}$, (i.e. shapes such that each distinct integer a_j appears once). Denote by $R(Q)$ this maximized value of $R(\Delta)$.

Furthermore, for cohomological representations π_0 of G_∞, the sets $\Delta(\pi_0)$ satisfy that if there is $\Delta \in \Delta(\pi_0)$ whose Arthur SL_2 matches Q, then there is also $\Delta' \in \Delta(\pi_0)$ with the maximizing form above.

Proof. Constructing any other shape with the same Arthur SL_2 restriction would require splitting up some of the $(r_i, a_i, \lambda_i, \eta_i)$ into smaller blocks which would decrease $R(\Delta)$.

For the second part, we may always merge blocks in Δ with the same d_i by concatenating their infinitesimal characters and leave ψ^∞_Δ unchanged by the construction in lemma 4.3.4. □

Therefore, to understand the values of $R(\Delta)$, it suffices to understand the possible Arthur SL_2's for $\Delta \in \Delta(\pi_0)$:

Definition 11.2.4. Let $Q^{\max}(\pi_0)$ be the unordered partitions representing the Arthur SL_2's of $\Delta \in \Delta(\pi_0)$ with maximal $R(\Delta)$. Equivalently by lemma 11.2.3, it is the set of Arthur SL_2's of $\Delta \in \Delta^{\max}(\pi_0)$.

By the construction in lemma 4.3.4, the possible Arthur SL_2's for $\Delta \in \Delta(\pi_0)$ are the possible Arthur SL_2's for ψ_∞ with $\pi \in \Pi_{\psi_\infty}$. We can therefore enumerate them by our classification of cohomological representations.

Fix a place v and let $\pi_{0,v}$ correspond to $B_v = (p_{i,v}, q_{i,v})_i \in \mathcal{P}_1(p_v, q_v)$ and infinitesimal character λ_v. We next study $\Delta(\pi_{0,v})$: the union of $\Delta(\pi_0')$ over all π_0' with $\pi_{0,v}' = \pi_{0,v}$. Recall from (11.1.2) that $\beta(B_v)$ is the ordered partition of N associated to a bipartition of (p_v, q_v). We define some combinatorial objects:

- $\beta_+(B)$ is the unordered subpartition of $\beta(B)$ corresponding to parts with size bigger than 1.
- $Q_p(\pi_0)$ is the unordered partition $(n_j)_{j \in J}$ where the j correspond to runs of consecutive (p_i, q_i) of the form $(1, 0)$ such that the corresponding piece of λ is of the form
$$X^r \sum_{i=1}^{n_j} X^{(n-2i+1)/2}$$
so that n_j is the length of the run.
- $Q_q(\pi_0)$ is similarly defined for parts of the form $(0, 1)$

Next, if $Q_1 = (n_i^1)_{i \in I}$ and $Q_2 = (n_j^2)_{j \in J}$ are two unordered partitions, we say that Q_2 refines Q_1 if there is a map $J \to I$ such that the sum of n_j^2 over the fiber at i is n_i^1.

Lemma 11.2.5. *The possible Arthur* SL_2*'s for $\Delta \in \Delta(\pi_{0,v})$ correspond exactly to unordered partitions*

$$(X, Y, \beta_+(B_v))$$

where X refines $Q_p(\pi_{0,v})$ and Y refines $Q_q(\pi_{0,v})$

Proof. Lemma 11.1.5 tells us the $P \in \mathcal{P}(p+q)$ that correspond to ψ_∞ with $\pi_0 \in \Pi_{\psi_\infty}$. These are produced by merging runs of consecutive 1's in $\beta(B_v)$ that correspond to parts in B_v all of the form $(1,0)$ or all of the form $(0,1)$. Furthermore, we require that the coarsened partition thereby produced is still adapted to λ_v. These conditions together show that P is an ordering of something of the form $(X, Y, \beta_+(B_v))$.

Next, consider

$$P = (n_1, \dots, n_k)$$

of claimed form. We will show that there is $\Delta \in \Delta(\pi_{0,v})$ with ψ_v^Δ corresponding to P. Let I_1 be subset of indices such that $n_i = 1$ and I_+ its complement. Let λ_v have P-parts $\lambda_1^P, \dots, \lambda_l^P$. Since λ_v is adapted to P, for each $i \in I_+$, there exists $\lambda_{i,v}$ such that shape $(1, n_i, \lambda_{i,v})$ has total infinitesimal character λ_i^P at v. Next, let λ'_v be the concatenation of λ_i^P for $i \in I_1$. Finally, choose the other components for $w \neq v$ of λ_i and λ' arbitrarily. Consider

$$\Delta = (|I_1|, 1, \lambda', \eta), ((1, n_i, \lambda_i, \eta_i))_{i \in I_+}$$

Then by the constructions in §11.1.3,ψ_v^Δ corresponds to P. Note that $\Delta \in \Delta(\pi_0)$ by the form of P. □

Next, we define

$$\beta_+(\pi_0) := \bigcup_{v \in \infty} \beta_+(B_v)$$

where the union is interpreted as of non-disjoint multisets (i.e, the union contains an element with multiplicity exactly equal to the maximum of its multiplicities in the multisets that the union is over).

Lemma 11.2.6. *The possible Arthur* SL_2*'s for $\Delta \in \Delta(\pi_0)$ correspond exactly to unordered Q partitions that, for each $v \in \infty$, can be written in the form:*

$$Q = (X_v, Y_v, \beta_+(B_v))$$

where X_v refines $Q_p(\pi_{0,v})$ and Y_v refines $Q_q(\pi_{0,v})$

In particular:

- *All Arthur* SL_2*'s for $\Delta \in \Delta(\pi_0)$ contain $\beta_+(\pi_0)$ as a subpartition*
- *if $\Delta(\pi_0)$ isn't empty, there is $\Delta \in \Delta(\pi_0)$ with Arthur* SL_2

$$(1, \dots, 1, \beta_+(\pi_0))$$

Proof. That the Arthur SL_2's must be contained in this set is an elementary combinatorial extension of the argument in 11.2.5.

Existence of a $\Delta = (T_i, d_i, \lambda_i)_i$ with a particular SL_2 follows from fixing each component of the λ_i as in the argument of lemma 11.2.5 instead of just the v-component.

Finally, the two bullet points are also elementary combinatorial properties of the set of such simultaneous $(X_v, Y_v, \beta_+(B_v))$. □

Note. We can summarize this as a three step algorithm for computing $\Delta^{\max}(\pi_0)$:

1. Find the possible Arthur SL_2's for $\Delta \in \Delta(\pi_0)$ by lemma 11.2.6.
2. Compute $R(Q)$ for each of these Arthur SL_2's to compute $Q^{\max}(\pi_0)$.
3. $\Delta^{\max}(\pi_0)$ is then partitioned into non-empty parts corresponding to the $Q \in Q^{\max}(\pi_0)$. Each part can be determined by lemma 11.2.3.

It turns out that the second step becomes much easier for GSK-shapes. Showing this will make up the remainder of our combinatorial work.

11.2.3. *The Key Bound.* Next, we need an elementary combinatorial bound that is basically a reformulation of lemma 7.1 in [MS19]. Recall the introduction of the numerical invariants $\bar{R}(\Delta)$ in (9.3.2), and $R(\Delta)$ in (9.4.1); $R(Q)$ is by definition the the maximum of $R(\Delta)$ over the Δ such that ψ_∞^Δ corresponds to Q. Part of the complexity of this argument is an artifact of only being able to prove the suboptimal bound $R(\Delta)$ from corollary 9.4.4 instead of the optimal $R_0(\Delta)$ from conjecture 9.6.3.

Lemma 11.2.7. *Let Q_0 be a unordered partition that has distinct parts and no parts of size* 1*. Then the maximum value of $R(Q)$ over all unordered partitions Q of N that have subpartition Q_0 is achieved by*

$$Q_{\text{can}} = (1^{(r)}, Q_0).$$

Furthermore, if either $r \neq 2$ or Q_0 has no parts of size 2*, this is the unique such Q that achieves this maximum.*

Proof. Let

$$Q_{\text{can}} = (1, \dots, 1, Q_0) = (1^{(r)}, (a_i^{(r_i)})_i)$$

for a_i distinct and $r_i = 1$. The other possible Q containing Q_0 are produced by decreasing the number of 1's and increasing one of the r_i's. We will therefore show that $R(Q)$ decreases if we increase any of the r_i.

Recall from Remark 9.4.4 that $R(Q)$ is obtained by starting from $\bar{R}(Q)$, equal to the dimension of a certain parabolic block matrix and then replacing the dimensions of certain blocks on the diagonal with modified counts. Along the diagonal, the blocks are indexed by the number 1 and the distinct a_i. Changing an r_i changes the three blocks of the parabolic: the two on the diagonal associated to 1 and a_i and one corresponding to this pair above the diagonal. In particular, we can treat changes in each r_i independently. Furthermore, changing an r_i from 0 (i.e. creating a new block) can be easily seen to decrease $R(Q)$.

In general, going from $r_i = 1$ to $r_i = k$ changes the modified summand associated to these three blocks from:

$$\begin{aligned} &r^2 + ra_i + 1 \mapsto \\ &\qquad (r - (k-1)a_i)^2 + ka_i(r - (k-1)a_i) + (\text{modified count for } a_i \text{ with } r_i = k). \end{aligned}$$

Expanding out, the change in $R(Q)$ is

$$(\text{modified count for } a_i \text{ with } r_i = k) - a_i(a_i + r)(k-1) - 1$$

where the modified counts are the counts for the part of formula for $R(Q)$ associated to the a_i-block on the diagonal. The modified count defining $R(Q)$ is defined differently for $r_i = 1, 2, 3$ versus everything else. Therefore we have to look at cases:

- If an r_i is increased to 2, the modified count is

  $$2a_i(a_i - 1) + (a_i + 3)$$

  making the total difference

  $$2a_i(a_i - 1) + (a_i + 3) - a_i(a_i + r) - 1$$

Using $r \geq a_i$, this is bounded above by

$$-a_i + 2$$

and is always negative unless $r = a_i = 2$.

- If an r_i is increased to 3, the modified count is

$$\frac{9}{2}a_i(a_i - 1) + ((4 + \epsilon)a_i + 5)$$

making the total difference

$$\frac{9}{2}a_i(a_i - 1) + ((4 + \epsilon)a_i + 5) - 2a_i(a_i + r) - 1.$$

Using $r \geq 2a_i$, this is bounded above by

$$-\frac{3}{2}a_i^2 - \left(\frac{1}{2} - \epsilon\right) a_i + 4$$

which is always negative for in particular $\epsilon < 1/2$ since $a_i \geq 2$

- If an r_i is increased to $k \geq 4$, then the change in modified counts is

$$\frac{k^2}{2}a_i(a_i + 1)$$

making the total difference

$$\frac{k^2}{2}a_i(a_i + 1) - a_i(a_i + r)(k - 1) - 1.$$

Using $r \geq (k - 1)a_i$, this is bounded above by

$$-\left(\frac{k^2}{2} - k\right) a_i^2 + \frac{k^2}{2}a_i - 1$$

and using $a_i \geq 2$ and $k^2/2 - k > 0$ gives an upper bound by

$$-\left(\frac{k^2}{2} - 2k\right) a_i - 1$$

which is always negative when $k \geq 4$.

In total, if we increase any of the r_i, then $R(Q)$ decreases. □

11.2.4. *Summary of combinatorial work.* To conclude:

Definition 11.2.8. Let π_0 factor into places $\pi_{0,v}$ for each v Let each $\pi_{0,v}$ correspond to $B_v = (p_{i,v}, q_{i,v})_v \in \mathcal{P}_1(p_v, q_v)$. Let $\beta_+(B_v)$ be the unordered subpartiton of pieces of size greater than 1 in $\beta(B_v)$. Let $\beta_+(\pi_0)$ be the union of all the $\beta_+(B_v)$ as multisets. Finally, define

$$Q_{\mathrm{can}}(\pi_0) := (1, \ldots, 1, \beta_+(\pi_0))$$

as an unordered partition of N (if it exists).

Then:

Proposition 11.2.9. *Let $Q_{\mathrm{can}}(\pi_0)$ be of the form*

$$Q_{\mathrm{can}}(\pi_0) = (1^{(r)}, a_1, \ldots, a_k)$$

for the a_i distinct (i.e. $\beta_+(\pi_0)$ has distinct parts). Further assume that either $r \neq 2$ or there is no i such that $a_i = 2$. Then $\Delta^{\max}(\pi_0)$ consists of all shapes of the form

$$(r, 1, \lambda), (1, a_i, \lambda_i)_{1 \leq i \leq k}$$

in $\Delta(\pi_0)$ and $\Delta^{\max}(\pi_0) \neq \emptyset$ provided that $\Delta(\pi_0) \neq \emptyset$.

Proof. This is the result of applying algorithm 11.2.2 keeping lemma 11.2.7 in mind for step (2) to get that $Q^{\max}(\pi_0) = \{Q_{\mathrm{can}}(\pi_0)\}$. □

We warn that proposition 11.2.9 doesn't necessarily hold if $\beta_+(\pi_0)$ has repeated elements. For example, consider $\beta_+(\pi_0) = (2, 2, 2, 2)$. Then

$$R(2^{(4)}, 1, 1) = 67$$

while

$$R(2^{(5)}) = 74.$$

In fact, this is a counterexample to even the analogous statement with the conjectural optimal bound R_0. Whether any $\Delta \in \Delta(\pi_0)$ can have Arthur SL_2 given by $(2^{(5)})$ depends on what exactly the $Q_p(\pi_{0,v})$ and $Q_q(\pi_{0,v})$ are. Therefore, a general description of $Q^{\max}(\pi_0)$ is much more complicated.

11.2.5. *GSK-maxed representations.* Now we restrict to the special class of representations we can study:

Definition 11.2.10. Let π_0 be a cohomological representation of some $U(p, q)$ corresponding to bipartition $(p_i, q_i)_i \in \mathcal{P}_1(p + q)$. We say π_0 is GSK-maxed if the only value of $p_i + q_i$ that appears with multiplicity is 1.

We say π_0 is odd GSK-maxed if in addition all the $p_i + q_i$ are odd.

Definition 11.2.11. Let π_0 factor into $\pi_{0,v}$ that each corresponding to bipartition $(p_{i,v}, q_{i,v})$. Then we say that π_0 is GSK-maxed if $\Delta(\pi_0) \neq \emptyset$ and the only number that appears with multiplicity among the $p_{i,v} + q_{i,v}$ is 1.

We say π_0 is odd GSK-maxed if in addition all the $p_{i,v} + q_{i,v}$ are odd.

Equivalently, $\beta_+(\pi_0)$ is a partition into (odd) distinct parts.

These definitions are of course justified by corollary:

Corollary 11.2.12. *Fix an extended pure inner form G of $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ and let π_0 be a cohomological representation of G_∞ that is (odd) GSK-maxed. Further assume that if $\beta_+(\pi_0) = (1^{(r)}, a_1, \dots, a_k)$, then either $r \neq 2$ or none of the $a_i = 2$ (this is automatically satisfied if π_0 is odd GSK-maxed).*

Then all elements of $\Delta^{\max}(\pi_0)$ are (odd) GSK.

Proof. This follows from proposition 11.2.9. □

We warn that in general, $\Delta^{\max}(\pi_0)$ isn't a singleton The different possibilities differ by different assignments of infinitesimal characters $\lambda_{i,v}$ to each block (T_i, d_i).

Example. Consider F with two infinite places v, w and $G_v \cong G_w \cong U(6, 1)$. Let

$$\pi_{0,v} = (1, 1), (1, 0)^{(5)},$$
$$\pi_{0,w} = (2, 1), (1, 0)^{(4)}$$

at the infinitesimal character of the trivial representation:

$$\lambda = (3, 2, 1, 0, -1, -2, -3).$$

If $\Delta^{\max}(\pi_0) \neq \emptyset$, any $\Delta \in \Delta^{\max}(\pi_0)$ is of the form

$$\Delta = (2, 1, (\lambda_v^1, \lambda_w^1)), (1, 2, (\lambda_v^2, \lambda_w^2)), (1, 3, (\lambda_v^3, \lambda_w^3)),$$

and the unordered partition $Q^{\max}(\pi_0)$ is $(3, 2, 1, 1)$. We are forced to choose

$$\lambda_v^2 = (3, 2), \qquad \lambda_w^3 = (3, 2, 1).$$

However, we still need to pick λ_v^3 and λ_w^2. This will correspond to a choice of ordering of $Q^{\max}(\pi_0)$ at each of v and w.

There are three choices

$$\lambda_v^3 = (1,0,-1), (0,-1,-2), \text{ or } (-1,-2,-3)$$

corresponding to three choices

$$\lambda_v^1 = (-2,-3), (1,-3), \text{ or } (1,0)$$

and three orderings of $Q^{\max}(\pi_0)$:

$$(2,3,1,1), (2,1,3,1), \text{ or } (2,1,1,3).$$

Similarly, there are three choices

$$\lambda_w^2 = (0,-1), (-1,-2), \text{ or } (-2,-3)$$

Corresponding to three choices

$$\lambda_w^1 = (-2,-3), (0,-3), \text{ or } (0,-1)$$

and three orderings of $Q^{\max}(\pi_0)$:

$$(3,2,1,1), (3,1,2,1), \text{ or } (3,1,1,2).$$

Thus in total $\Delta^{\max}(\pi_0)$ can contains up to nine elements. Note that the different possibilities for λ_v^1 and λ_w^2 aren't necessarily even character twists of each other.

Example. As an simple example to keep in mind where this difficulty doesn't appear, assume

$$G_\infty = U(N-1,1)^r$$

and that $\pi_0 \cong \pi_{0,v}^r$ is diagonal. There is only one non-1 entry in any element of $\mathcal{P}_1(N-1,1)$ so we necessarily have that π_0 is GSK-maxed. Furthermore, inspection of the argument in lemmas 11.2.5 and 11.2.6 shows that since $\beta_+(\pi_0) = \beta_+(B_v)$ for all v and is a singleton, there is exactly one way to assign infinitesimal characters $\lambda_{i,v}$ to the blocks (T_i, d_i). This implies that $\Delta^{\max}(\pi_0)$ is a singleton.

11.3. **Characters on the Component Group.** Let $P = (a_1, \dots, a_k) \in \mathcal{P}(N)$ correspond to an A-parameter ψ_∞ at infinity for some $U(p,q)$ considered as an extended pure inner form. Recall the group $S^\natural_{\psi_\infty}$ from [KMSW14] mentioned in §2.4.2. As mentioned after [MR19, (1.3)], equation (3.2.1) reduces to a canonical isomorphism

$$S^\natural_{\psi_\infty} = \bigoplus_{1\le i\le k} \mathbb{Z}/2 = \langle \epsilon_i \rangle_{1\le i\le k}$$

where each index i corresponds to part a_i of P. In addition, the subgroup

$$\mathbb{Z}(\widehat{G})^\Gamma = \mathbb{Z}/2 = \left\langle \sum_{i=1}^k \epsilon_i \right\rangle$$

is embedded diagonally.

We explain how to attach characters of $S^\natural_{\psi_\infty}$ to $\pi \in \Pi_{\psi_\infty}$, following [MR19]. Let

$$\pi_0 = (p_i, q_i)_i \in \beta^{-1}(P) \subseteq \mathcal{P}(p,q)$$

and define $a_{<i} = \sum_{j=1}^{i-1} a_i$. Then Mœglin-Renard compute [MR19, (1.3)]

$$\eta_{\pi_0}^{\psi_\infty}(\epsilon_i) = (-1)^{p_i a_{<i} + q_i(a_{<i}+1) + a_i(a_i-1)/2}$$

We can simplify this to

$$\eta_{\pi_0}^{\psi_\infty}(\epsilon_i) = (-1)^{a_i a_{<i} + q_i + \chi_4(a_i)} \tag{11.3.1}$$

where

$$\chi_4(a_i) := \begin{cases} 0 & a_i \equiv 0,1 \pmod 4 \\ 1 & a_i \equiv 2,3 \pmod 4 \end{cases}$$

If the a_i are all odd, this further simplifies to

$$\eta_{\pi_0}^{\psi_\infty}(\epsilon_i) = (-1)^{(i-1)+q_i+\chi_4(a_i)}. \tag{11.3.2}$$

Finally, if $\psi_\infty = \psi_\infty^\Delta$ for Δ of the type in lemma 11.2.3, then

$$\mathcal{S}_\Delta^\natural = \left\langle s_d := \sum_{i : a_i = d} \epsilon_i \right\rangle_{d \in \mathbb{Z}^+}$$

is a subgroup of $S_{\psi_\infty}^\natural$. As a result, we can characterize representations such that $\eta_{\pi_0}^{\psi_\infty^\Delta}(S_\Delta^\natural) = 1$; this will be used to give asymptotics for multiplicities of individual representations.

Lemma 11.3.1. *Let $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ and G an extended pure inner form of G^*. Let $\pi_0 = \bigotimes_v \pi_v$ be an odd GSK-maxed representation of $G_\infty = \prod_v U(p_v, q_v)$.*

Let $Q^{\max}(\pi_0) = (1^{(r)}, d_1, \dots, d_k)$ with d_i odd and distinct. The $\Delta \in \Delta^{\max}(\pi_0)$ each determine orderings

$$P_v^\Delta := (a_{1,v}, \dots, a_{k+r,v})$$

of $Q^{\max}(\pi_0)$ for each v such that $\psi_v^\Delta = P_v^\Delta$. Let

$$\pi_{0,v} = (p_{i,v}^\Delta, q_{i,v}^\Delta)_i \in \beta^{-1}(P_v^\Delta) \subseteq \mathcal{P}(p_v, q_v).$$

and for each d_j, let $i_v^\Delta(d_j)$ be the index i such that $d_j = a_{i,v}$.

Then, for $\Delta \in \Delta^{\max}(\pi_0)$, $\eta_{\pi_0}^{\psi_\infty^\Delta}(S_\Delta^\natural) = 1$ if and only if

$$t_j := \sum_{v \in \infty} (i_v^\Delta(d_j) - 1 + q_{i_v^\Delta(d_j),v}^\Delta + \chi_4(d_j))$$

is even for each $1 \le j \le k$ and G_∞ satisfies the parity conditions from 2.1.1.

Proof. The condition on t_j comes from checking that $\eta_{\pi_0}^{\psi_\infty^\Delta}(s_d) = 1$ for $d > 1$. The second comes from checking that $\eta_{\pi_0}^{\psi_\infty^\Delta}$ factors through $\mathcal{S}_\Delta$. □

Beware that it is very important to keep track of whether $G_v = U(p,q)$ or $U(q,p)$ as an extended pure inner form to compute the characters $\eta_{\pi_0}^{\psi_\infty}$.

11.4. **Limit Multiplicities.** Now that we understand $\Delta(\pi_0)$, we can compute our main result: limit multiplicities for certain π_0.

Theorem 11.4.1. *Let G be a pure inner form of $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$. Choose*

$$f^\infty = f_{S_1} \bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$$

as in section 9.1 with $S_0 = \emptyset$ (in particular, G and therefore E/F is unramified at all finite places and as in lemma 2.1.1).

Pick a cohomological representation π_0 of G_∞ that is odd GSK-maxed and for

$$\Delta = (T_1, 1, \lambda_1, \eta_1), (1, d_i, \lambda_i, \eta_i)_{2 \le i \le k} \in \Delta^{\max}(\pi_0),$$

let $\lambda_1(\Delta) = \lambda_1$. *Let* T_1, d_i, R *and* L *be the common values of* T_1, d_i, $R(\Delta)$ *and* $L(\Delta)$ *over* $\Delta^{\max}(\pi_0)$*:*

Then there are A, B, C, D, E *with* $C \geq 1$ *such that as long as* $|\mathfrak{n}_i| \geq Dq_{S_1}^{E\kappa}$*:*

$$|\mathfrak{n}_i|^{-R}\Gamma_L(\mathfrak{n}_i)^{-1} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty)$$

$$= \frac{\mathrm{vol}(H'(F)\backslash H'(\mathbb{A}_f))}{\mathrm{vol}(K_{H'}^S)} \left(\sum_{\Delta \in \Delta^{\max}(\pi_0)} \mathbf{1}_{\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta) = 1} \frac{\dim \lambda_1(\Delta)}{|\Pi_{\mathrm{disc}}(\lambda_1(\Delta))|} \right.$$

$$\left. \times \left(f_{S_1}^{H_1}(1) \coprod_{i>1} \int_{H_{i,\mathrm{der},S_1}} f_{S_1}^{H_i}(h)\, dh \right) + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}), \right.$$

where $H_i = H(T_i, d_i, \lambda_i, \eta_i)$ *and* $H' = H_1 \times \prod_{i>1} H_i^{\mathrm{ab}}$ *which are both constant over* $\Delta \in \Delta^{\max}(\pi_0)$

Finally, we recall that the condition on $\eta_{\pi_0}^{\psi_\infty^\Delta}$ *can be checked as in lemma 11.3.1, that*

$$R = (k-1) + \frac{1}{2}\left(N^2 + T_1^2 - \sum_{i \geq 2} d_i^2, \right),$$

and that

$$L = T_1, 1^{(k-1)}, -1^{(k-1)}.$$

Proof. Apply proposition 10.2.2. Then corollary 11.2.12 allows us to apply theorem 10.4.1 to compute the main terms $m(\pi_0, \Delta, f^\infty)$ for $\Delta \in \Delta^{\max}(\pi_0)$. We compute $R = R(\pi_0)$ by corollary 11.2.9. Theorem 10.4.2 bounds all the terms $m(\pi_0, \Delta', f^\infty)$ for $\Delta' \notin \Delta^{\max}(\pi_0)$. □

We also have much more general upper bound:

Theorem 11.4.2. *Let* G *be an extended pure inner form of* $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$*. Choose*

$$f^\infty = \varphi_{S_0} f_{S_1} \bar{\mathbf{1}}_{K^{G,S}(\mathfrak{n}_i)}$$

as in section 9.1 with f_{S_1} *and* φ_{S_0} *is arbitrary.*

Pick a cohomological representation π_0 *of* G_∞*. Then if* $\Delta(\pi_0) = \emptyset$

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} = 0.$$

Then there are A, B, D, E *such that as long as* $|\mathfrak{n}_i| \geq Dq_{S_1}^{E\kappa}$*:*

$$\sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \operatorname{tr}_{\pi^\infty}(f^\infty) = O(|\mathfrak{n}_i|^{R(\pi_0)} q_{S_1}^{A+B\kappa}).$$

Proof. Apply proposition 10.2.2 and theorem 10.4.2. □

This upper bound 11.4.2 applies to any CM-extension E/F, any extended pure inner form G, and any φ_{S_0}. If we accept conjecture 9.6.3, the exponent can of course be improved to an analogous $R_0(\pi_0)$. Furthermore, in the sum from theorem 10.2.1, at least one of the $\Delta_1 \times \Delta_2$ for each $\Delta \in \Delta^{\max}(\pi_0)$ satisfies that the d_i assigned to each Δ_i all have the same parity. Therefore, if we accept the even stronger conjecture 9.6.4 and don't have an obstruction from the multiplicity formula (like the $\eta_{\pi_0}^\Delta$ condition from Theorem 11.4.2), we should expect $R(\pi_0)$ to be optimal.

12. Examples and Corollaries

We give examples and corollaries of the main theorems 11.4.1 and 11.4.2. This includes our applications to Sato-Tate equidistribution in familes, the Sarnak-Xue density hypothesis, and studying the cohomology of locally symmetric spaces. Before jumping into things, we link to all our various growth rates for reader's convenience:

Let π_0 be on a rank-N group. Each $\star(\pi_0)$ is a maximum of $\star(\Delta) = \star((T_i, d_i, \lambda_i, \eta_i)_i)$ for $\Delta \in \Delta(\pi_0)$. Then we have:

The first-pass, rough growth rate from proposition 9.3.2:

$$\bar{R}(\pi_0) = \max_{\Delta \in \Delta(\pi_0)} \frac{1}{2}\left(N^2 + \sum_i T_i^2 d_i\right).$$

This will not be used much in this section.

The provable growth rate from proposition 9.4.4:

$$R(\pi_0) = \max_{\Delta \in \Delta(\pi_0)} \bar{R}(\Delta) - \sum_{i:T_i=1}\left(\frac{1}{2}(d_i^2+d_i)-1\right) - \sum_{i:T_i=2}(4d_i-(d_i+3)) - \sum_{\substack{i:T_i=3\\ d_i>1}}\left(9d_i - ((4+10^{-100})d_i+5)\right). \tag{12.0.1}$$

This will be used in all theorem statements.

The conjectural growth rate from conjecture: 9.6.3:

$$R_0(\pi_0) = \max_{\Delta \in \Delta(\pi_0)} \frac{1}{2}\left(N^2 - \sum_i T_i^2 d_i\right) + \sum_i \left(T_i^2 + \frac{1}{2}T_i(T_i-1)(d_i^2-1)\right)$$

We will compare to this for certain discussions.

12.1. Examples.

First, we work out what theorem 11.4.1 says in some simpler case. To discuss infinitesimal characters, we let $\lambda_1, \dots, \lambda_{n-1}$ be the fundamental weights of GL_n

$$\lambda_i = (1^{(i)}, 0^{(N-i)})$$

in the standard basis of $X^*(T)$ corresponding to the entries of a diagonal matrix. Define λ_0, λ_n similarly for indexing purposes, though these aren't fundamental weights. Two weights are character twists of each other if they differ by a multiple of λ_n and as usual

$$\rho_n = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right)$$

is the half-sum of positive roots.

12.1.1. *Example 1: parallel case for $U(N-1,1)$.*

For the simplest example with non-tempered representations at infinity, assume:

- $\deg F/\mathbb{Q} = d$ is even,
- $G_\infty \cong U(N-1,1)^d$ as is allowed by §2.1,
- $\pi_\infty \cong \pi_0^d$ with $\pi_0 = ((1,0)^{(r)}, (k-1,1), (1,0)^{(N-k-r)})$ for $k > 1$ odd and $r + k \leq N$,
- $S_1 = \emptyset$.

Then we can check that $\Delta^{\max}(\pi_0)$ is a singleton

$$(N-k, 1, (\lambda_{1,v})_v, \eta_1), (1, k, (\lambda_{2,v})_v, \eta_2),$$

with $\lambda_{1,v}$ a character twist of $k\lambda_r + \rho_{n-k}$ on GL_{n-k} and $\lambda_{2,v}$ the infinitesimal character of a 1-dimensional irrep. We can also recall that

$$|\Pi^{U(N-1,1)}_{\mathrm{disc}}(\lambda_{2,v})| = N.$$

Finally, $\eta^{\psi_\infty}_{\pi_\infty} = (\eta^{\psi_v}_{\pi_0})^d$ is an even power of some $\eta^{\psi_v}_{\pi_0}$ so it is trivial.

If $\pi_{k\lambda_r}$ is the finite dimensional representation of $\mathrm{GL}_{N-k}\mathbb{C}$ with highest weight $k\lambda_r$, We can then compute

$$\begin{aligned}(12.1.1)\quad |\mathfrak{n}_i|^{-(N(N-k)+1)} L_{k,1,-1}(\mathfrak{n}_i)^{-1} \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim((\pi^\infty)^{K(\mathfrak{n}_i)}) \\ = \frac{1}{N^d} \dim(\pi_{k\lambda_r})^d \tau'(U_{E/F}(N-k) \times U(1)) + O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}).\end{aligned}$$

as an asymptotic count of automorphic *forms* of level $\mathfrak{n}_i$ corresponding to π_∞. Recall that τ' is the modified Tamagawa number from (9.1.1).

Note that the "masses" (i.e. relative abundances in the automorphic spectrum)

$$\frac{1}{N^d} \dim(\pi_{k\lambda_r})^d$$

depend on r even though the corresponding π have the same infinitesimal character and come from the same Levi in the Langlands classification. How the infinitesimal character divides up between the blocks of this Levi also matters.

More specifically, $\dim(\pi_{k\lambda_r})$ is largest for r close to $(N-k)/2$ and decreases to 1 towards the extreme cases of $r = 0$ or $N-k$. In cohomological terms, all the representations considered contribute to cohomology in degree $d(N-k)$. The representations with Hodge weights closer to $\left(\frac{d}{2}(N-k), \frac{d}{2}(N-k)\right)$ are attached to larger masses, whereas the ones whose weights are closer to $(0, d(N-k))$ and $(d(N-k), 0)$ are rarer.

12.1.2. *Example 2.* As a slight complication, now assume:

- $\deg F/\mathbb{Q} = d$ is odd,
- $N \not\equiv 0 \pmod 4$,
- $$G_\infty \cong \begin{cases} U(N-1,1)^d & N \equiv 2,3 \pmod 4 \\ U(1,N-1)^d & N \equiv 1 \pmod 4 \end{cases}$$
  to satisfy the conditions in §2.1,
- $\pi_\infty = \pi_0^d$ with
  $$\pi_0 = \begin{cases} ((1,0)^{(r)}, (k-1,1), (1,0)^{(N-k-r)}) & N \equiv 2,3 \pmod 4 \\ ((0,1)^{(r)}, (1,k-1), (0,1)^{(N-k-r)}) & N \equiv 1 \pmod 4 \end{cases}$$
  for $k > 1$ odd and $r + k \leq N$,
- $S_1 = \emptyset$.

This situation is the same as the first example except that we no longer necessarily have $\eta^{\Delta}_{\pi_\infty} = 1$.

Using the test from lemma 11.3.1, $\eta^\Delta_{\pi_\infty} = 1$ if and only if

$$r + \begin{cases} 1 & N \equiv 2,3 \pmod 4 \\ N-1 & N \equiv 1 \pmod 4 \end{cases} + \begin{cases} 1 & k \equiv 0,1 \pmod 4 \\ 0 & k \equiv 2,3 \pmod 4 \end{cases} \equiv 0 \pmod 2.$$

is even. We write this condition as

$$r + 1 + \chi_4(N)) + \chi_4(k) \equiv 0 \pmod 2. \tag{12.1.2}$$

in terms of the quadratic character χ_4 from lemma 11.3.1 and think of it as a parity condition on r.

If this equation (12.1.2) holds, then then we have the same result as (12.1.1). Otherwise, we have that

(12.1.3)

$$|\mathfrak{n}_i|^{-(N(N-k)+1)} L_{k,1,-1}(\mathfrak{n}_i)^{-1} \sum_{\substack{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) \\ \pi_\infty = \pi_0}} \dim((\pi^\infty)^{K(\mathfrak{n}_i)}) = O(|\mathfrak{n}_i|^{-C} q_{S_1}^{A+B\kappa}).$$

There are two consequences. First, we only have exact asymptotics in the case where r satisfies the parity condition (12.1.2). Second and surprisingly, even the asymptotic growth rate of counts of forms in level can be different for representations coming from the same Levi in the Langlands classification.

This is basically an obstruction from the multiplicity formula. In such examples, all automorphic representations counted have only unramified local components at non-split, non-Archimedean places. In particular, they all corresponded to trivial character on the component group. Furthermore, all those representations came from parameters ψ with $\epsilon_\psi = 1$. In total, the multiplicity formula requires $\eta^\psi_{\pi_0} = 1$ for packet ψ to contribute to the multiplicity of π_0 at infinity.

Even more surprisingly, growth rates can be different for different members of the same L-packet in cases beyond $U(N,1)$. If $\pi = ((p_i, q_i))_i$ on $U(p,q)$, then it can be seen from the description in §6 of [VZ84] that other members of its (pseudo- and therefore true) L-packet can be produced by reversing some number of pairs (p_i, q_i) such that we remain in $\mathcal{B}(p,q)$. For an L-packet like

$$\{((2,1),(0,1)),((1,2),(1,0))\}$$

on $U(2,2)$, only one member can satisfy the parity condition from lemma 11.3.1. This is starkly different from the discrete-at-infinity case in [Dal22] and caused by our dominant contribution to growth rates coming from shapes Δ with non-trivial $\mathcal{S}_\Delta$.

12.1.3. *Example 3.* We will also consider an example where there is only one non-compact place. Assume:

- $\deg F/\mathbb{Q} = d$ with a fixed place $v_0 \in \infty$
- $G_\infty \simeq U(p,q) \times U(N,0)^{d-1}$ where
$$q \equiv \begin{cases} 0 \pmod 2 & d \text{ even or } N \equiv 0,1 \pmod 4 \\ 1 \pmod 2 & d \text{ odd and } N \equiv 2,3 \pmod 4 \end{cases}$$
to satisfy the conditions of §2.1. The $U(p,q)$ factor is at v_0.
- $\pi_\infty = \pi_0 \times \mathbf{1}^{d-1}$ where π_0 is odd GSK-maxed with the infinitesimal character of the trivial representations and $\mathbf{1}$ is the trivial representation. We parameterize
$$\pi_0 = (p_i, q_i)_i, \qquad n_i = p_i + q_i.$$

- $S_1 = \emptyset$.

We need some more combinatorial parameters

- d_j for $1 \le i \le k$ are the distinct non-1 values among the n_i,
- If $d_j = n_{i(d_j)}$, let $r_j = i(d_j) - \#\{i(d_{j'}) < i(d_j)\}$
- $M = N - \sum_j d_j$.

The $\Delta \in \Delta^{\max}(\pi_\infty)$ are then all of the form

$$(M, 1, (\lambda_v)_v, \eta), (1, d_j, (\lambda_{j,v})_j, \eta_j)_j$$

where all the $\lambda_{j,v}$ are infinitesimal characters of 1-dimensional irreps and λ_{v_0} is a character twist of

$$\rho_M + \sum_j d_j \lambda_{r_j}.$$

The possible choices for each λ_v with $v \neq v_0$ are in bijection with reorderings $(n_{v,i})_i$ of $(n_i)_i$. We can define $i_v(d_j)$ and $r_{v,j}$ from $(n_{v,i})_i$ similar to the original definitions or $i(d_j)$ and r_j. Then λ_v is equal to a character twist of

$$\rho_M + \sum_j d_j \lambda_{r_{v,j}}.$$

The condition on the character from lemma 11.3.1 then reduces to

$$i(d_j) + \sum_{v \neq v_0} i_v(d_j) \equiv d(\chi_4(d_j) - 1) + q_{i(d_j)} \pmod 2 \tag{12.1.4}$$

for all $1 \le j \le k$. There is always a set of reorderings $(n_{v,i})_i$ that satisfy (12.1.4): the only way there couldn't be is if the d_j are all the n_i and a parity condition from summing (12.1.4) over all j fails (this boils down to a combinatorial puzzle about filling in a $(\#\{n_i\} - 1) \times k$ grid of squares black or white such that each row has $\lfloor (\#\{n_i\} - 1)/2 \rfloor$ black squares and the first $d - 1$ columns have a fixed parity of black squares). However, the parity condition on q makes sure the condition from summing always holds

Therefore, this example is analogous to example 1 with a much more complicated factor replacing the $N^{-d} \dim(\pi_{r\lambda})^d$.

12.2. **Growth of Cohomology.** In this section, we give applications to computing the cohomology of arithmetic lattices in G_∞. Recall that for a cocompact lattice Γ in a Lie group G_∞ with maximal compact K_∞ and Lie algebra $\mathfrak{g}$, Matsushima computed the cohomology of Γ with coefficients in the finite-dimensional representation F of G using representations of G_∞:

$$H^*(\Gamma, F) = \sum_{\pi \in G_\infty^\vee} m(\pi, \Gamma) H^*(\mathfrak{g}, K_\infty; \pi \otimes F^*). \tag{12.2.1}$$

Here $G_\infty^\vee$ is the unitary dual of G_∞, the integer $m(\pi, \Gamma)$ is the multiplicity of π in $L^2(\Gamma \backslash G_\infty)$, and $H^*(\mathfrak{g}, K_\infty)$ is the $(\mathfrak{g}, K)$ *cohomology* of π with coefficients in F, see [BW00]. We say π is *cohomological with coefficients in* F if $H^*(\mathfrak{g}, K_\infty; \pi \otimes F^*)$ is nontrivial.

As an application, we can give upper and lower bounds for the growth of cohomology of lattices in $U(p, q)$. Thus we will restrict our groups G so that $G_\infty = U(p, q) \times U(N, 0)^a \times U(0, N)^b$. Following Lemma 2.1.1, such G can exist for all values of N, though possibly not over all unramified extensions E/F. We will abuse notation and say that a degree i of cohomology appears in an A-packet Π_ψ

if Π_ψ contains a cohomological representation with nonvanishing cohomology in degree i.

12.2.1. *Lower Bounds.* We give a sample result of the kind of lower bounds on growth of cohomology produced by theorem 11.4.1. First, as a direct consequence of Lemmas 11.1.5 and of Section 11.1.3, we find:

Lemma 12.2.1. *Let $G_v = U(p,q)$ with $p+q = N$ and $\min(p,q) = r$. Let $1 < d \leq N$ be odd.*

(i) *Let $Q \in \mathcal{P}(N)$ be an ordered partition with one entry equal to d and all others equal to 1. Then the lowest degree of cohomology appearing in the packet $\Pi_{\psi_Q,v}$ is*

$$i = i(d,N,r) = \begin{cases} r(N-r) - \frac{d^2-1}{4} & r \geq \frac{d-1}{2} \\ r(N-d) & r \leq \frac{d-1}{2} \end{cases}.$$

(ii) *If $r \leq \frac{d-1}{2}$, the degree i is achieved by a unique representation $\pi_i \in \Pi_{\psi_Q,v}$. If*

$$Q = (\overbrace{1,...,1}^{s}, d, 1, ..., 1) \in \mathcal{P}(N),$$

then the only Hodge weights of π_i in degree $i = r(N-d)$ are

$$(a,b) = \begin{cases} (rs, r(N-d-s)) & G = U(N-r,r) \\ (r(N-d-s), rs) & G = U(r, N-r). \end{cases}$$

From this we can deduce the following:

Corollary 12.2.2. *Assume that E/F is unramified at all finite places, and that there exists a pure extended inner form G of G^* which is unramified at all finite places, isomorphic to $U(p,q)$ at one finite place, and compact at all the others. Let $\Gamma(\mathfrak{n}_i) = G(F) \cap K^G(\mathfrak{n}_i)$ be sequence of lattices in $U(p,q)$ of level $|\mathfrak{n}_i| \to \infty$, only divisible by primes that split in E. Let $p+q = N$ with $r = \min(p,q)$. Let $j \not\equiv N$* mod 2 *be such that $j \leq |p-q| - 1$. Then*

$$\dim H^{rj}(\Gamma(\mathfrak{n}_i), \mathbb{C}) \gg |\mathfrak{n}_i|^{Nj}.$$

The exact same bounds hold when $H^{rj}(\Gamma(\mathfrak{n}_i), \mathbb{C})$ is replaced by $H^{rk, r(j-k)}(\Gamma(\mathfrak{n}_i), \mathbb{C})$ for $0 \leq k \leq j$.

Proof. Let v_0 be the infinite place at which G is not compact. By Lemma 12.2.1, it suffices to give lower bounds on multiplicities for the representation $\pi_{rj} = \pi_{rj,v_0} \otimes \mathbf{1}^{[F:\mathbb{Q}]-1}$. Since π_{rj} is odd GSK-maxed by the congruence condition on j, Theorem 11.4.1 gives exact multiplicities with $R = Nj + 1$ provided that

$$\eta_{\pi_{rj}}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta) \equiv 1.$$

Thus we need to check that there exists at least one shape Δ such that the restriction to $\mathcal{S}_\Delta$ of the character attached to π_{rj} is trivial. Note that $\mathcal{S}_\Delta \simeq \mathbb{Z}/2\mathbb{Z}$: in the notation of 11.3, we have

$$\mathcal{S}_\Delta^\natural = \left\langle \sum_{i:a_i=1} \epsilon_i, \epsilon_{i^\Delta(d)} \right\rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2, \quad \mathcal{S}_\Delta = \mathcal{S}_\Delta^\natural / \left\langle \sum_i \epsilon_i \right\rangle \simeq \mathbb{Z}/2\mathbb{Z}. \tag{12.2.2}$$

Thus it suffices to find a shape Δ such that $\eta_{\pi_{rj}}^{\psi_\infty^\Delta}$ is trivial on the non-identity element.

Note that S_Δ is embedded in $\prod_v S_{\psi_v}^\natural$ according to the ordered partitions Q_v associated to each ψ_v for $v \mid \infty$. accordingly we have

$$\eta_{\pi_{rj}}^{\psi_\infty^\Delta} = \prod_{v|\infty} \eta_{\pi_{rj}}^{\psi_v^\Delta}.$$

Recall that each Q_v is a reordering of $(N-j, 1, ..., 1)$. Starting with any shape Δ such that $\psi_{v_0}^\Delta = \psi_{Q_{v_0}}$, we can change the value of $\eta_{\pi_{rj}}^{\psi_\infty^\Delta}$ on the nontrivial element of $\mathcal{S}_\Delta$ by picking $v \neq v_0$, and changing the ordering of Q_v by swapping the $N-j$ with one of the 1's. Since G_v is compact, this will result in a new shape Δ' such with the same $\psi_{v_0}^{\Delta'} = \psi_{Q_{v_0}}$ and such that the packets at the compact place contain only the trivial representation. But following the formulas from 11.3, the character $\prod_{v|\infty} \eta_{\pi_{rj}}^{\psi_v^{\Delta'}}$ will be trivial on $\mathcal{S}_\Delta$ if $\prod_{v|\infty} \eta_{\pi_{rj}}^{\psi_v^\Delta}$ was not, and vice versa. Thus at least one of Δ and Δ' contributes asymptotically $|\mathfrak{n}_j|^{Nj+1}$ to $\dim H^{rj}(Y(\mathfrak{n}_i), \mathbb{C})$. The argument made no use of the precise ordering in Q_{v_0}, so it holds for the representations or any Hodge weights in Lemma 12.2.1. This shows the result for the disconnected locally symmetric spaces $Y(\mathfrak{n}_i) = G(F)\backslash G(\mathbb{A})/K(\mathfrak{n}_i)$.

Let G^{der} be the derived subgroup of G, and $T = G/G^{\mathrm{der}}$ with quotient map ν. Since G^{der} is a special unitary group, it is simply connected, and by [Del71, §2], the connected components of $Y(\mathfrak{n}_i)$ are in bijection with the points of $T(\mathbb{A})/T(F)\nu(K_\infty \times K(\mathfrak{n}_i))$. Moreover, if $\mathfrak{n}_i$ is large enough, we have following [Del79, §2.7] that $H^*(Y(\mathfrak{n}), \mathbb{C}) = \mathrm{Ind}_1^{T(\mathbb{A})/T(F)\nu(K_\infty \times K(\mathfrak{n}_i))} H^*(\Gamma(\mathfrak{n}_i), \mathbb{C})$. Since $|T(\mathbb{A})/T(F)\nu(K_\infty \times K(\mathfrak{n}_i))| \gg |\mathfrak{n}_i|^{1-\epsilon}$, we conclude. $\square$

Recall that Marshall-Shin [MS19] showed that $\dim H^d(\Gamma(\mathfrak{n}_i), \mathbb{C}) \ll_\epsilon |\mathfrak{n}_i|^{Nd+\epsilon}$ for lattices $\Gamma(n_i)$ in $U(N-1,1)$ and conjectured that they were sharp.

Corollary 12.2.3. *If $d \not\equiv N \mod 2$, then the upper bounds obtained by Marshall-Shin are sharp.*

12.2.2. *Upper Bounds.* As an example of the upper bounds, fix:

- a CM quadratic extension E/F,
- an extended pure inner form G of $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$ that is isomorphic to $U(p,q)$ at one infinite place v_0 and compact at all other infinite places.
- a finite set of finite places S at which G is split
- an open compact $U^{S,\infty}$ away from S and ∞.
- $\mathfrak{n}_i$ a sequence of ideals supported over S such that $|\mathfrak{n}_i| \to \infty$.
- $\Gamma_i = G(F) \cap U^{S,\infty} K_S^G(\mathfrak{n}_i)$ a sequence of lattices

As before, we are interested in $H^{a,b}(\Gamma_i, \mathbb{C})$. To compute this:

(1) We can use lemma 11.1.5 to enumerate all the $\pi_0 \in \mathcal{B}_1(p,q)$ that contribute to $H^{a,b}$.
(2) We use the algorithm at the end of §11.2.2 and the formula (12.0.1) to compute all the $R(\pi_0)$. Let the maximum value be $R(a,b)$.

Then

Proposition 12.2.4. *For all $\epsilon > 0$,*

$$\dim H^{a,b}(\Gamma_i, \mathbb{C}) \ll_\epsilon |\mathfrak{n}_i|^{R(a,b)-1+\epsilon}.$$

Proof. This follows from applying theorem 11.4.2 to each of the π_0 contributing, applying Matushima's formula, and then using the bounds on the number of connected components to reduce to a single connected component of the adelic quotient. □

Example. If the noncompact factor of G is isomorphic to $U(p,q)$ and let $r = \min(p,q)$. Then r is the lowest degree of cohomology that is not guaranteed to vanish for local reasons. There are only two nontrivial representations contributing with cohomology in degree r, and they have weights $(r,0)$ and $(0,r)$ respectively. Then one computes that $R(0,r) = R(r,0) = p+q$.

12.3. **Sato-Tate Equidistribution in Families.** We can prove an averaged Sato-Tate result similar to theorem 9.26 in [ST16] since our main theorem 11.4.1 has error bounds of the same strength in f_{S_1}.

We consider families of automorphic representations with infinite component equal to an odd GSK-maxed π_0. Their Satake parameters will not equidistribute with respect to the Sato-Tate measure on G, but rather with respect to the pushforward of the Sato-Tate measure on a smaller group related to the $\Delta \in \Delta^{\max}(\pi_0)$. Otherwise, this section will follow [ST16] extremely closely.

12.3.1. *Sato-Tate Measures.* First we need to recall the definition of Sato-Tate measures from [ST16, §3,5] (the full details can be found there). Choose place v of F and unramified reductive group G_v over F_v. Let $A \subseteq T$ be a maximally split torus of G and maximal torus containing it. Let A_c, T_c be their maximal compact subgroups. By looking at Satake parameters, we get a parameterization of the tempered, unramified dual of G_v:

$$\widehat{G}_v^{\mathrm{ur,temp}} \simeq \Omega_{F_v} \backslash \widehat{A}_c \simeq \Omega_{F_v} \backslash \widehat{T}_c / (\mathrm{id} - \mathrm{Frob}_v) \widehat{T}_c.$$

This space is also the same as $\widehat{G}$-conjugacy classes in $\widehat{K}_{\mathrm{Frob}_v} \rtimes \mathrm{Frob}_v$ where $\widehat{K}_{\mathrm{Frob}_v}$ is the maximal Frob_v-invariant compact subgroup of $\widehat{G}$.

Of course, not every v has the same Frobenius. To deal with this, if G splits over F_1, let $\Gamma = \mathrm{Gal}(F_1/F)$. Then for each $\theta \in \Gamma$, let

$$\widehat{T}_{c,\theta} := \Omega_{F_v} \backslash \widehat{T}_c / (\mathrm{id} - \theta) \widehat{T}_c.$$

For $\gamma \in \Gamma$, $t \mapsto \gamma t$ canonically identifies $T_{c,\theta}$ with $T_{c,\gamma\theta\gamma^{-1}}$ so $\widehat{T}_{c,\theta}$ can be considered to only depend on the conjugacy class of θ. This is therefore a uniform description of $\widehat{G}_v^{\mathrm{ur,temp}}$ whenever $\mathrm{Frob}_v = \theta$.

Definition 12.3.1. The Sato-Tate measure $\mu_\theta^{\mathrm{ST}} := \mu_\theta^{\mathrm{ST}}(G)$ on $\widehat{T}_{c,\Theta}$ is the quotient under $\widehat{G}$-conjugation of the measure on $\widehat{K}_\theta \rtimes \theta$ with total volume 1.

This should be thought of as the "most canonical" possible measure to put on $T_{c,\Theta}$.

Now, let $\mathcal{V}_F(\theta)$ be the set of places v of F such that $\mathrm{Frob}_v = \theta$. For $v \in \mathcal{V}_F(\theta)$, the Plancherel measure on $\widehat{G}_v^{\mathrm{ur,temp}}$ (normalized so that a maximal compact of G_v has volume 1) gives another measure $\mu_v^{\mathrm{pl,ur}} := \mu^{\mathrm{pl,ur}}(G_v)$ on $\widehat{T}_{c,\theta}$

Lemma 12.3.2. *Let sequence $v \in \mathcal{V}_F(\theta)$ such that $q_v \to \infty$. Then there is weak convergence $\mu_v^{\mathrm{pl,ur}} \to \mu_\theta^{\mathrm{ST}}$.*

Proof. by the explicit formulas [ST16, prop 3.3] and [ST16, lem 5.2] □

12.3.2. *Equidistribution.* We can now state and prove the equidistribution result. Fix G an unramified extended pure inner form of some $G^* \in \mathcal{E}_{\mathrm{ell}}(N)$. Note that at all finite places, G splits over E. We fix

- Odd GSK-maxed cohomological representation π_0 of G_∞,
- $\theta \in \mathrm{Gal}(E/F)$,
- sequence $v_i \in \mathcal{V}_F(\theta)$ (i.e. either all split or all non-split),
- ideals $\mathfrak{n}_i$ of $\mathcal{O}_F$ relatively prime to v_i.

The different $\Delta = (T_i, d_i, \lambda_i, \eta_i) \in \Delta^{\max}(\pi_0)$ differ only in their λ_i-coordinates and therefore correspond to the same map

$$\mathcal{S}_\Delta : \widehat{H}_v^{\mathrm{ur,temp}} \hookrightarrow \widehat{G}_v^{\mathrm{ur}}$$

as in formula (6.2.3) and for some common group H_v as in (6.2.2). For $\theta \in \mathrm{Gal}(E/F)$, define pushforward

$$\mu_\theta^{\mathrm{ST}(\pi_0)} := \mu_\theta^{\mathrm{ST}(\pi_0)}(G) := (\mathcal{S}_\Delta)_*(\mu_\theta^{\mathrm{ST}}(H))$$

Beware that this is a measure on the full unramified dual $\widehat{G}_v^{\mathrm{ur}}$ for $v \in \mathcal{V}_F(\theta)$ instead of just the tempered part.

Finally, for each i, define empirical distribution on $\widehat{G}_\theta^{\mathrm{ur}}$:

$$\mu_{\mathfrak{n}_i, v_i}^{\pi_0} := \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^\infty)^{K^G(\mathfrak{n}_i)}) \delta(s_{\pi_{v_i}}).$$

Here, $\delta(s_{\pi_{v_i}})$ is the delta-measure at Satake parameter $s_{\pi_{v_i}}$.

Then

Theorem 12.3.3 (Sato-Tate Equidistribution in Families)**.** *Recall the notation for constants in the statement of main theorem 11.4.1. Assume that $|\mathfrak{n}_i|$ grows faster than any power of q_{v_i}. Then for all continuous $\widehat{f}$ on $\widehat{G}_\theta^{\mathrm{ur}}$,*

$$|\mathfrak{n}_i|^{-R} \Gamma_L(\mathfrak{n}_i)^{-1} \mu_{\mathfrak{n}_i, v_i}^{\pi_0}(\widehat{f}) \to C(\pi_0) \mu_\theta^{\mathrm{ST}(\pi_0)}(\widehat{f})$$

as $i \to \infty$ and where normalizing constant

$$C(\pi_0) = \frac{\mathrm{vol}(H'(F) \backslash H'(\mathbb{A}_f))}{\mathrm{vol}(K_{H'}^S)} \sum_{\Delta \in \Delta^{\max}(\pi_0)} \mathbf{1}_{\eta_{\pi_0}^{\psi_\infty^\Delta}(\mathcal{S}_\Delta) = 1} \frac{\dim \lambda_1(\Delta)}{|\Pi_{\mathrm{disc}}(\lambda_1(\Delta))|}$$

Proof. It suffices to check this for non-negative $\widehat{f}$. By the Weierstrass approximation argument in remark of [ST16], it is then further sufficient to show that

$$|\mathfrak{n}_i|^{-R} \Gamma_L(\mathfrak{n}_i)^{-1} \mu_{\mathfrak{n}_i, v_i}^{\pi_0}(\widehat{f}_{v_i}) \to C(\pi_0) \mu_\theta^{\mathrm{ST}(\pi_0)}(\widehat{f}_{v_i})$$

just for trace-positive $f_{v_i} \in \mathscr{H}^{\mathrm{ur}}(G_{v_i})$ (note that this Hecke algebra is constant on $\mathcal{V}_F(\theta)$). We do this by applying theorem 11.4.1 with $S_1 = \{v_i\}$. Note that the growth condition on $|\mathfrak{n}_i|$ shows that we will eventually have $|\mathfrak{n}_i| \geq D q_{v_1}^{E\kappa(f_{v_i})}$.

After using formula (6.2.4) and the Plancherel theorem to get

$$\begin{aligned} f_{S_1}^{H_1}(1) \prod_{i>1} \int_{H_{i,\mathrm{der},S_1}} f_{S_1}^{H_i}(h)\, dh &= \mathcal{T}_\Delta f_{S_1}(1) \\ &= \mu_v^{\mathrm{pl,ur}}(H'_v)(\widehat{\mathcal{T}_\Delta f_{S_1}}) = \mu_v^{\mathrm{pl,ur}}(H'_v)(\widehat{f}_{S_1} \circ \mathcal{S}_\Delta), \end{aligned}$$

the argument follows exactly as that for 9.26 in [ST16]. $\square$

We repeat an interpretation from the introduction: recall that as part of the conjectures surrounding Langlands functoriality, every automorphic representation π on some G/F should correspond to a group H_π/F that is the smallest group it is a functorial transfer from. The Satake parameters s_{π_v} for v ranging over a particular $\mathcal{V}_F(\theta)$ are then expected to equidistribute according to a Sato-Tate distribution coming from H.

At the current time, actually finding H appears out-of-reach. However, in reasonable families of automorphic representations, most π should correspond to some fixed "maximal" H that is computable. Therefore, if we look at Satake parameters over the entire family, we can hopefully prove an equidistribution-on-average result towards the Sato-Tate measure for this maximal H.

This is conceptually what is happening here: most automorphic representations with π_0 at infinity come from group $H_{\pi_0} = H'$. Therefore the s_{π_v} ranging over both v and a reasonable family of such π should equidistribute according to Sato-Tate measures from H'. Unlike previous cases built off of [ST16], we are in a more complicated situation where this maximal H_{π_0} isn't actually G itself.

12.4. **Sarnak-Xue Conjecture.** Theorem 11.4.2 has applications towards the Sarnak-Xue conjecture of [SX91] on unitary groups. This conjecture is stated in terms of classical symmetric spaces instead of adelic quotients. Consider reductive G/F, open compact $U \subseteq G^\infty$, and π_0 a unirrep of G_∞. Let $\Gamma(U) = U \cap G(F)$ and let

$$m(\pi_0, \Gamma(U)) := \dim \operatorname{Hom}(\pi_0, L^2(\Gamma(U)\backslash G_\infty))$$

Note that $\Gamma(U)\backslash G_\infty$ is a connected component of the adelic quotient

$$Y(U) := G(F)\backslash G(\mathbb{A})/U.$$

Then:

Conjecture 12.4.1 (Sarnak-Xue density hypothesis)**.** *Let U_i be an sequence of open compacts of G^∞ decreasing to the identity. Then for all unirreps π_0 of G_∞:*

$$m(\pi_0, \Gamma(U_i)) \ll_\epsilon \operatorname{vol}(\Gamma(U_i)\backslash G_\infty)^{\frac{2}{p(\pi_0)}+\epsilon},$$

where $p(\pi_0)$ is the infimum over p such that the K-finite matrix coefficients of π_0 are in $L^p(G_\infty)$.

In this section, we will study this conjecture for the case of unitary groups and U_i only decreasing through deepening principle-congruence components at split places. We will prove for it all π_0 except for those that have a single particular representation on $U(2,2)$ as a factor.

It is important to mention that the work [MS19] already had strong enough bounds to achieve the Sarnak-Xue threshold. We did not need to improve these in any way, only use them as input (through lemmas 9.4.2 and 9.4.3) for our more general framework.

12.4.1. *Computing $p(\pi)$.* Before we can check this for unitary groups, we first need extend the computations of [GG21] to find a way to understand $p(\pi)$ in terms of our parameterization of cohomological representations.

First, given (possibly ordered) bipartition $Q = ((p_1, q_1), \dots, (p_r, q_r))$, let

$$\Xi(B) = (\chi_j(B))_j$$

be the concatenated list

$$\bigsqcup_{i:m_i\neq 0}(n_i-1,n_i-3,\dots,n_i-2m_i+1)$$

reordered to be decreasing and where $m_i=\min\{p_i,q_i\}$ and $n_i=p_i+q_i$. For indexing purposes, we define $\chi_j(B)=0$ for j out of bounds. Also define

$$\sigma_j(B)=\sum_{k\leq j}\chi_k(B).$$

for all j.

Proposition 12.4.2. *Let $G=U(p,q)$ and π_0 be the cohomological representation of G associated to $B=((p_1,q_1),...,(p_r,q_r))$. Then*

$$\frac{2}{p(\pi_0)}\geq 1-\max_i\left\{\frac{\sigma_i(B)}{i(N-i)}\right\}$$

Proof. First, we recall how to obtain, from results in [Kna01, §§7-8], a formula to compute $p(\pi)$, when $\pi=J(S,\sigma,\nu)$ is a Langlands quotient. To describe such a quotient we need

- S_0, a minimal parabolic of G, with Langlands decomposition $S_0=M_0A_0N_0$, whose respective subgroups have Lie algebras $\mathfrak{m}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$,
- $\alpha_1,...,\alpha_{\dim\mathfrak{a}_0}$ the simple roots of $\mathfrak{a}_0$ in $\mathfrak{g}$, and $\omega_1,...,\omega_{\dim\mathfrak{a}_0}$ the basis of $\mathfrak{a}_0$ dual to the α_i.
- ρ_0 the corresponding half-sum of positive roots of $\mathfrak{a}_0$ in $\mathfrak{g}$.
- $S=MAN$, a parabolic subgroup of G stantdard with respect to S_0, with Lie algebras $\mathfrak{m}$, $\mathfrak{a}$, and $\mathfrak{n}$,
- a discrete series representation σ of M,
- a weight $\nu\in\mathfrak{a}^*$ such that $\langle\nu,\alpha\rangle>0$ for all roots α of $\mathfrak{a}$ in $\mathfrak{n}$.

Then the parabolic induction $I(S,\sigma,\nu)$ has a unique Langlands quotient $J(S,\sigma,\nu)$. We have a direct sum decomposition

$$\mathfrak{a}_0=\mathfrak{a}\oplus\mathfrak{a}_M,$$

where $\mathfrak{a}_M$ is the Lie algebra of the maximal split torus of M. Define $\nu_0\in\mathfrak{a}_0^*$ by extending ν by zero on $\mathfrak{a}_M$. Proposition 5.13 of [GG21] then deduces from [Kna01, §§7-8] the inequality

$$p(J(S,\sigma,\nu))\leq\inf\left\{p\geq 2\mid p>\frac{2\langle\rho_0,\omega_j\rangle}{\langle\rho_0-\nu_0,\omega_j\rangle}\text{ for all }\omega_j\right\}. \tag{12.4.1}$$

We now compute the appropriate pairings for π corresponding to $B=((p_1,q_1),...,(p_r,q_r))$, by first writing it as a Langlands quotient following [VZ84, §6]. Begin with the Levi subgroup

$$L=U(p_1,q_1)\times...\times U(p_r,q_r)$$

associated to π_0. Let K the maximal compact of G and T a maximal torus, and let $(K\cap L)A_LN_L$ the Iwasawa decomposition of L. Define ν_L to be the the half-sum of positive roots of A_L acting on N_L, and let $Z=\{\alpha\text{ a root of }\mathfrak{a}_L\text{ in }\mathfrak{g}\mid\langle\alpha,\nu_L\rangle=0\}$, and let

$$A=\cap_{\alpha\in\mathbb{Z}}\ker\alpha\subseteq A_L.$$

Then M is defined to be the centralizer of A in G. Define also M_L to be the centralizer of A_L in G, and let $S_L=M_LA_LN_L$ be any parabolic subgroup of G with respect to which ν_L is dominant. Then by construction, there is a unique parabolic subgroup S containing with Levi MA and containing S_L. Let $\nu = \nu_L \mid_A$. Then following [VZ84, Theorem 6.16], there exists a discrete series representation σ of M such that $\pi_0 = J(S, \sigma, \nu)$.

We write this in coordinates: for this we choose a posteriori a minimal parabolic subgroup S_0 for which S is standard. Define numbers $m_\star = \min\{p_\star, q_\star\}$ and $n_\star = p_\star + q_\star$. Each $U(p_\star, q_\star)$ has minimal parabolic corresponding to partition

$$(1^{(m_\star)}, n_\star - 2m_\star, 1^{(m_\star)})$$

and maximal split torus isomorphic to $\mathbb{R}^{m_\star}$. Choose coordinates for each group's $X^*(T)$ where T is a maximal torus in the standard way:

$$X^*(T) = \mathbb{Z}\langle e_1, \dots, e_{n_\star}\rangle$$

with simple roots $\alpha_i = e_i - e_{i+1}$. We can therefore realize as a vector:

$$\rho_0 = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{n-2m+1}{2}, 0^{(n-2m)}, \frac{-n+2m-1}{2}, \dots, \frac{-n+3}{2}, \frac{-n+1}{2}\right)$$

in $X^*(T)$. In $X^*(A_0)$, this becomes

$$\rho_0 = \rho_{p,q} = (n-1, n-3, \dots, n-2m+1).$$

Similarly, $\nu_L \in X^*(A_L)$ is the concatenation of sequences

$$\bigsqcup_{k=1}^{r} \rho_{p_i,q_i}. \tag{12.4.2}$$

modified to be in decreasing order to reflect the choice of S_0 with respect to which which S_L is standard. The subtorus $A \subset A_L$ is then chosen so that $\langle \mathrm{Re}(\nu_L), \alpha\rangle > 0$ for all simple roots α of A. We have further direct sum decompositions

$$A_0 = A' \oplus A_L = A' \oplus A'' \oplus A$$

where A' is the maximal split torus of L and $A' \oplus A''$ is that of M. The extension of ν by 0 to A_L is then just ν_L again. Let ν_0 be the common extension by 0 to A_0, obtained in coordinates by adding a string of zeros to (12.4.2). Following (12.4.1), we have

$$\frac{2}{p(\pi_0)} \geq 1 - \max_i \left\{ \frac{\langle \nu_0, \omega_i\rangle}{\langle \rho_0, \omega_i\rangle} \right\}$$

By symmetry of the ν_0 and ρ_0, the maximum value is achieved for some $i \leq m$. In this case, we check that $2\langle \rho_0, \omega_i\rangle = i(N-i)$ and $2\langle \nu_0, \omega_i\rangle = \sigma_i(B)$. $\square$

12.4.2. *Some Combinatorial Lemmas.* We next need some more involved but elementary combinatorial bounds, this time for the $\sigma_i(B)$. Once again, the complexity of this section is entirely from needing to use the suboptimal bound $R(\Delta)$ from corollary 9.4.4 instead of the conjectural optimal bound $R_0(\Delta)$ from 9.6.3.

First, for (possibly ordered) partition Q, define $\sigma_i(Q) = \sigma_i(B)$ for the $B = ((p_i, q_i))_i \in \beta^{-1}(Q)$ such that $|p_i - q_i| \leq 1$ so that for all i, $\sigma_i(B) \leq \sigma_i(\beta(B))$. Next

Lemma 12.4.3. *Let $d < N \in \mathbb{Z}^+$.*

(1) If

$$Q_d := \left(d^{(\lfloor N/d \rfloor)}, N - d\lfloor N/d \rfloor\right),$$

then for all $Q = (n_1, \dots, n_r)$ a partition of N with each part of size $\leq d$, $\sigma_i(Q) \leq \sigma_i(Q_d)$ for all i.

(2) *Asssume $N \geq 2d$. If*

$$Q'_d := \begin{cases} \left(d^{(\lfloor N/d \rfloor)-1}, d-1, N - d\lfloor N/d \rfloor + 1\right) & N \not\equiv -1 \pmod d \\ \left(d^{(\lfloor N/d \rfloor)-1}, d-1, d-1, 1\right) & N \equiv -1 \pmod d \end{cases}$$

then for all $Q = (n_1, \dots, n_r)$ a partition of N with each part of size $\leq d$ and at most $\lfloor n/d \rfloor - 1$ parts of size d (i.e. not equal to Q_d), $\sigma_i(Q) \leq \sigma_i(Q'_d)$ for all i

Proof. For the first claim, choose such $Q = (n_1, \dots, n_r) \neq Q_d$ wlog in decreasing order. Then $n_r, n_{r-1} < d$ so $Q' = (n_1, \dots, n_{r-2}, n_{r-1} + 1, n_r - 1)$ also satisfies the conditions. Switching any pair (a, b) with $a \geq b$ to $(a+r, b-r)$ in $\Xi(Q)$ will increase or keep equal every $\sigma_i(Q)$. With this in mind, $\Xi(Q')$ differs from $\Xi(Q)$ by removing the numbers

$$(n_r - 1, n_r - 3, \dots,), (n_{r-1} - 1, n_{r-1} - 3, \dots,)$$

and adding in the numbers

$$(n_r - 2, n_r - 4, \dots,), (n_{r-1}, n_{r-1} - 2, \dots,).$$

Since $n_{r-1} \geq n_r$, this is a sequence of pair-switches as above together with some strict increases of coordinates. Therefore $\sigma_i(Q') \geq \sigma_i(Q)$ for all i.

Repeating this process until producing Q_d proves the first part. The second part follows by similar argument. □

Lemma 12.4.4. *With notation from lemma 12.4.3:*

$$\max_i \left\{ \frac{\sigma_i(Q_d)}{i(N-i)} \right\} = \frac{d-1}{N - \lfloor N/d \rfloor}.$$

Furthermore, if $N \geq 2d$,

$$\max_i \left\{ \frac{\sigma_i(Q'_d)}{i(N-i)} \right\} = \frac{d-1}{N - \lfloor N/d \rfloor + 1}.$$

Proof. by computer check □

This gives our final results:

Proposition 12.4.5. *Let $B \in \mathcal{P}_1(p, q)$ with $p+q = N$ and let π_0 be the cohomological representation of $U(p, q)$ corresponding to β. Let $Q \in Q^{\max}(B)$ and assume $\beta(B) \neq (2, 2)$. Then*

$$(N^2 - 1)\left(1 - \max_i \left\{ \frac{\sigma_i(B)}{i(N-i)} \right\}\right) \geq R(Q) - 1$$

with equality only if Q has a single element. (Recall the definition of $R(Q)$ from corollary 9.4.4.)

Proof. It suffices to prove the bound with

$$\max_i \left\{ \frac{\sigma_i(\beta(B))}{i(N-i)} \right\} \geq \max_i \left\{ \frac{\sigma_i(B)}{i(N-i)} \right\}$$

instead. Let d be the maximal element of $\beta(B)$. First, if $N < 2d$, then $\beta(B) = (d, (a_i)_i)$ for $\sum_i a_i < d$. Then by lemma 12.4.3,

$$\max_i \left\{ \frac{\sigma_i(\beta(B))}{i(N-i)} \right\} \leq \frac{d-1}{N-1}.$$

Furthermore, by lemma 11.2.5, Q has a part of size d so by lemma 11.2.7,

$$R(Q) \leq R(d, 1^{(N-d)}) = N(N-d)+1$$

The result then follows from

$$1 - \frac{d-1}{N-1} = \frac{N-d}{N-1} > \frac{N(N-d)}{N^2-1}.$$

Therefore assume $N \geq 2d$. If $\beta(B) \neq Q_d$, then by lemmas 12.4.3 and 12.4.4, we have that

$$\max_i \left\{ \frac{\sigma_i(\beta(B))}{i(N-i)} \right\} \leq \max_i \left\{ \frac{\sigma_i(\beta(Q_d'))}{i(N-i)} \right\} = \frac{d-1}{N - \lfloor N/d \rfloor + 1}.$$

Using again that $R(Q) \leq N(N-d)+1$, the result then follows since

$$1 - \frac{d-1}{N - \lfloor N/d \rfloor + 1} > \frac{N(N-d)}{N^2-1}$$

always. If on the other hand $\beta(B) = Q_d$, then

$$\max_i \left\{ \frac{\sigma_i(\beta(B))}{i(N-i)} \right\} = \frac{d-1}{N - \lfloor N/d \rfloor}.$$

In addition, by lemma 11.2.5, $Q = Q_d$ so

$$R(\beta(B)) \leq \bar{R}(Q) = \frac{1}{2}\left(N^2 + \left\lfloor \frac{N}{d} \right\rfloor^2 d + N - d \left\lfloor \frac{N}{d} \right\rfloor \right).$$

By a computer check, the desired inequality

$$1 - \frac{d-1}{N - \lfloor N/d \rfloor} > \frac{1}{N^2-1}\left(\frac{1}{2}\left(N^2 + \left\lfloor \frac{N}{d} \right\rfloor^2 d + N - d \left\lfloor \frac{N}{d} \right\rfloor \right) - 1 \right)$$

is true except for the cases

$$\beta(B) = (d,d), \quad \beta(B) = (d,d,1), \quad \beta(B) = (2,2,2).$$

All these cases except $(2,2)$ can be checked by using the tighter bound $R(Q)$ instead of $\bar{R}(Q)$. ∎

Extending to all $F/\mathbb{Q}$:

Corollary 12.4.6. *Let G be an extended pure inner form of some $G^* \in \mathcal{E}_{\mathrm{sim}}(N)$ and let $\pi_0 = \prod_v \pi_v$ be a cohomological representation of G_∞ such that $\Delta^{\max}(\pi_0) \neq \emptyset$ and where each π_v corresponds to bipartition $B_v \in \mathcal{P}_1(p_v, q_v)$.*

Then, for all B_v such that $\beta(B_v) \neq (2,2)$,

$$(N^2-1)\left(1 - \max_i \left\{ \frac{\sigma_i(B_v)}{i(N-i)} \right\}\right) \geq R(\pi_0) - 1$$

with equality only if π_∞ is a character.

Proof. For all v, $R(\pi_v) \geq R(\pi_0)$ since it is a maximum over a larger set by lemma 11.2.6. The result then follows from proposition 12.4.5. ∎

12.4.3. *Sarnak-Xue Density.* We can now return to the original setup and specialize to unitary groups. Let G be an extended pure inner form of some $G^* \in \mathcal{E}_{\mathrm{sim}}(N)$. We choose

- Cohomological representation $\pi_0 = \prod_v \pi_v$ of G_∞,
- Finite set of places S_0 containing all places where G is ramified,
- Sequence of ideals $\mathfrak{n}_i \to \infty$ relatively prime to S_0,
- Open compact $U_{S_0} \subseteq G_{S_0}$.

We then define
$$U_i = U_{S_0} K^G(\mathfrak{n}_i)$$
using the principle congruence subgroups associated to $\mathfrak{n}_i$.

Theorem 12.4.7 (Split-level Sarnak-Xue Density for Unitary Groups)**.** *With setup as above, assume*

- $\mathfrak{n}_i$ *is only divisible by split places of* F*.*
- *If* $N = 4$*: for each* v *with* $G_v = U(2,2)$*,* $\pi_v \neq ((1,1),(1,1))$*.*

Then
$$m(\pi_0, \Gamma(U_i)) \ll_\epsilon \mathrm{vol}(\Gamma(U_i)\backslash G_\infty)^{\frac{2}{p(\pi)}+\epsilon}.$$
(The ϵ *may be removed if* π_0 *isn't a character).*

Proof. As in §1.1 of [MS19], $Y(U_i)$ contains $\gg_\epsilon |\mathfrak{n}_i|^{1-\epsilon}$ copies of $\Gamma(U_i)\backslash G_\infty$ and that
$$\mathrm{vol}(\Gamma(U_i)\backslash G_\infty) \gg_\epsilon |\mathfrak{n}_i|^{N^2-1+\epsilon}.$$

The bound on connected components gives us that
$$\begin{aligned} m(\pi_0, \Gamma(U)) &\ll_\epsilon |\mathfrak{n}_i|^{-1+\epsilon} \dim \mathrm{Hom}(\pi_0, L^2(Y(U_i))) \\ &= |\mathfrak{n}_i|^{-1+\epsilon} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G)} \mathbf{1}_{\pi_\infty = \pi_0} \dim((\pi^\infty)^{U_i}). \end{aligned}$$

We can bound the sum by Theorem 11.4.2 with $S_1 = \emptyset$ and $\varphi_{S_0} = \bar{\mathbf{1}}_{U_{S_0}}$ to get
$$m(\pi_0, \Gamma(U)) \ll_\epsilon |\mathfrak{n}_i|^{R(\pi_0)-1+\epsilon}$$

Next proposition 12.4.2 computes
$$\frac{2}{p(\pi)} = \min_v \frac{2}{p(\pi_v)} \geq \min_v \left(1 - \max_i \left\{ \frac{\sigma_i(B_v)}{i(N-i)} \right\}\right),$$
where each π_v corresponds to bipartition B_v and with a strict inequality if π_0 isn't a character. The result follows from applying the bound in corollary 12.4.6 and the volume estimate. □

Of course either by varying φ_{S_0} beyond just the indicator function of U_{S_0} or by using non-trivial S_1, we can prove similar statements for very general weighted counts of representations. These don't have as clean a statement in terms of the classical symmetric spaces $\Gamma(U_i)\backslash G_\infty$ however.

The leftover $\pi_0 = ((1,1),(1,1))$ is likely an artifact of $R(\Delta)$ not being optimal. However, provably improving it enough seems to be hard—see [Mar16] (the conjectural optimal bound $R_0(\Delta)$ of conjecture 9.6.3 would of course be enough).

12.4.4. *Examples.* We compute some small cases where π_0 on a group of rank N is the same value π_v at all infinite places v so that $R(\pi_0) = R(\pi_v)$. Let $\pi_v = B \in \mathcal{P}_1(p_v, q_v)$. Let Q be the unordered partition corresponding to $\beta(B)$. If π_v is GSK-maxed, let

$$Q = (d_1, \dots, d_r, 1^{(k)})$$

with $d_1 > d_2 > \cdots > d_r > 1$. Then we can compute

$$R(\pi_v) - 1 = R_0(\pi_v) - 1 = \frac{1}{2}\left(N^2 + k^2 - \sum_i d_i^2\right) + (r-1)$$

so we get

$$m(\pi_0, \Gamma(U_i)) \ll_\epsilon |\mathfrak{n}_i|^{\frac{1}{2}\left(N^2+k^2-\sum_i d_i^2\right)+(r-1)+\epsilon}$$

This is conjecturally the exact exponent and provably so in the odd GSK-case for unramified E/F and U_i as in theorem 11.4.1. We also have

$$\frac{2}{p(\pi_0)} \geq \frac{N-d}{N-1}$$

so the Sarnak-Xue bound asks for $m(\pi_0, \Gamma(U_i))$ to be asymptotically less than

$$\operatorname{vol}(\Gamma(U_i)\backslash G_\infty)^{\frac{2}{p(\pi_0)}} \gg_\epsilon |\mathfrak{n}_i|^{(N+1)(N-d)-\epsilon}.$$

which is always true. If $r = 1$, we save a factor of $|\mathfrak{n}_i|^{N-d}$ over the Sarnak-Xue bound. If we keep n and k fixed but increase r, the saving is even larger.

When π_v isn't GSK-maxed, the formulas are much more complicated and $R(\pi_v) \neq R_0(\pi_v)$. Table 12.4.1 lists some values based on $Q = \beta(\pi_v)$. For each Q, we list the maximum possible values values of $R(\pi_0) - 1$ and $R_0(\pi_0) - 1$ which in our setup only depends on Q. These are the the provable and conjectural exponents on $|\mathfrak{n}_i|$ in the growth rate of $m(\pi_0, \Gamma(U_i))$ respectively. We also list the target exponent from the Sarnak-Xue density bound and the "trivial bound" growth rate $N^2 - 1$ when π_0 is discrete series. Finally, we italicize cases where a failure of lemma 11.2.7 occurs due to Q not being GSK.

Our $R(Q)$ beats target growth rate in every case except the bolded number when $Q = (2,2)$. The improvement is often large, though not in some cases like $(3,3), (4,4)$ and $(2,2,2,2)$. The conjectural optimum $R_0(Q)$ is usually much smaller still.

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TABLE 12.4.1. Comparison of Growth Rates

Arthur-SL_2 Q	provable: max $R(\pi_0)-1$	conjectural: max $R_0(\pi_0)-1$	SX goal: $2(N^2-1)p(\pi_v)^{-1}$	trivial: N^2-1
(2,2)	**8**	6	**7.5**	15
(2,2,1)	13	11	16	24
(2,2,2)	21	17	23.33	35
(2,2,1,1)	*21*	18	26.25	35
(3,3)	17	11	17.5	35
(2,2,2,1)	28	24	36	48
(3,3,1)	24	18	28.8	48
(3,2,2)	21	19	32	48
(2,2,2,2)	47	33	47.25	63
(2,2,2,1,1)	*47*	33	50.4	63
(4,4)	30	18	31.5	63
(3,3,3)	43	32	53.33	80
(3,2,2,2)	40	36	60	80
(5,5)	47	27	49.5	99
(2,2,2,2,2)	74	54	79.2	99
(2,2,2,2,1,1)	*74*	*54*	82.5	99

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