SATO-TATE EQUIDISTRIBUTION FOR FAMILIES OF AUTOMORPHIC REPRESENTATIONS THROUGH THE STABLE TRACE FORMULA

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Abstract. We extend the weight-aspect results of [ST16] to families of automorphic representations where the Archimedean component is restricted to a single discrete-series representation instead of an entire L-packet. We do this by using a so-called "hyperendoscopy" version of the stable trace formula from [Fer07]. The main technical difficulties are defining a version of hyperendoscopy that works for groups without simply connected derived subgroup and bounding the values of transfers of unramified functions. We also present an extension of the simple trace formula from [Art89] to non-cuspidal groups since it does not seem to appear elsewhere in the literature.

Contents

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1. Introduction

1.1. **Context.** This writeup generalizes work in [Shi12a] and [ST16] on equidistribution of local components of families of automorphic representations (see the summary next section). We roughly extend their weight-aspect to the case where the infinite component can be restricted to a single discrete series instead of an entire L-packet. Beyond the main result (theorem 9.1.1) the methods used should also be of interest. In particular, we do somewhat explicit computations with Arthur's stable trace formula and develop techniques to deal with some practical issues that thereby arise.

Generally, problems of statistics of families automorphic representations are interesting for a few potential reasons. First, when interpreted classically, such statistics are information on the spectra of lattices in locally symmetric spaces. Second, they give so-called globalization results such as [Art13, lem 6.2.2] through probabilistic method-style arguments. These allow the construction of automorphic forms satisfying desired local conditions. This is important since a very standard technique in studying local representations is to find a global rep with the local rep as a component and then using global methods to study the global rep: see for example the classification in [Art13] or the cohomology formula in [Shi12b]. Globalization results were the motivation for [Shi12a].

Next, certain bounds on automorphic representations—in particular the generalized Ramanujan conjecture and what it says about the sizes of Fourier coefficients—have various bizarre, unexpected implications. These include some striking ones outside of number theory such as the original construction of expander graphs. See [Sar05] for a review of this subject. As is common in analytic number theory, bounds on average in families instead of bounds on individual representations are often good enough for these applications. Conveniently enough, average bounds over families are also directly provided by studying statistics. This seems to be the original motivation for studying the problem in [ST16].

Specific to this work, we also hope that the practical methods developed here to compute with the stable trace formula could be useful in other places. Three of these to point out are:

- The version of the hyperendoscopy formula from [Fer07] that works when groups without simply connected derived subgroup appear in hyperendoscopy in section 4.
- The generalization of the simple trace formula in [Art89] to non-cuspidal groups with fixed central character datum in section 6.
- The computations and bounds on unramified transfers in sections 5.4 and 5.5.

We point out some relevant previous work: pseudocoefficients and their simplification of the trace formula were developed by Clozel and Delorme [CD90] and Arthur [Art89]. They were used to study statistics of families by Clozel [Clo86]. The exact families studied and the setup to study them are of course a small modification from [Shi12a] and [ST16]. The use of the stable trace formula is through the hyperendoscopy formula in Ferrari [Fer07] although the results of [Pen19] give a different potential strategy. The paper [KWY18] solves this problem for GSp_4 with far more explicit bounds through different methods. For a fuller history of this field of "limit multiplicity"-type problems, see the introduction to [FLM15].

Finally, the results here should be compared to [FLM15] and [FL18] by Finis, Mueller and Lapid. These use the non-invariant trace formula to develop similar though much more general results. In particular, they show Shin and Templier's level aspect with the Archimedean component restricted to any set of positive measure in the unitary dual. The result is dependent on some technical estimates on intertwining operators that are satisfied for GL_n and SL_n. A future work promises them for most other groups. In addition, those methods do not currently deal with the weight aspect or give error bounds though they could presumably be pushed to do both.

1.2. **Summary.**

1.2.1. *Shin-Templier's work.* Let G be a reductive group satisfying some technical conditions (described in section7.1). In [ST16] building off [Shi12a], Shin and Templier studied certain families of automorphic representations with level and weight restrictions:

$$\mathcal{F}_{U,\xi} = \{\pi \in \mathcal{AR}_{\mathrm{disc}}(G) : \pi_\infty \in \Pi_{\mathrm{disc}}(\xi), \dim(\pi^\infty)^U \geq 1\}$$

where $\mathcal{AR}_{\mathrm{disc}}(G)$ is set of the discrete automorphic representations of G, U is an open compact subgroup of $G(\mathbb{A}^{\infty,S_0})$ for some finite set of places S_0, ξ is a regular weight of $G_\mathbb{C}$ and $\Pi_{\mathrm{disc}}(\xi)$ is the discrete series L-packet corresponding to ξ. Pick another finite set of places $S \supseteq S_0$ and consider the empirical distribution

$$\mu_{\mathcal{F},S} = \sum_{\pi \in \mathcal{F}_{U,\xi}} a_\pi \delta_{\pi_S}$$

of S-components of $\pi \in \mathcal{F}$ weighted by

$$a_\pi = m_{\mathrm{disc}}(\pi) \dim(\pi^{S,\infty})^U.$$

Shin and Templier used Arthur's invariant trace formula to study the limits of these distributions under either increasing level ($U \to 1$) or increasing weight ($\xi \to \infty$). In both cases, they converged to the Plancherel measure. They furthermore provided bounds on how quickly the integrals $\mu_{\mathcal{F},S}(f)$ converge in the case where both f and the elements of $\mathcal{F}$ are unramified on S. The increasing weight aspect required that the center of G was trivial.

Their method was in a few broad steps:

(1) Realize the empirical distribution $\mu_{\mathcal{F},S}$ as the trace of a function with a special Archimedean component η_ξ against the discrete automorphic spectrum. Here, η_ξ is the Euler-Poincare function from [CD90].
(2) Since the Archimedean component is an Euler-Poincare function, Arthur's invariant trance formula reduces to the simple trace formula in [Art89] giving a reasonably tractable expression for this trace.
(3) Bound the appropriate terms and take a limit. This is most of the work.

The form of the error bound allowed proving so-called Sato-Tate equidistribution involving limits of $\mu_{\mathcal{F},v}$ for a single place v as v and ξ jointly go to infinity. They also provided some results on the statistics of low-level zeros of L-functions over the entire family.

1.2.2. *The extension.* Here, we extend the increasing weight result and bounds to smaller families where the Archimedean component is restricted to a single regular-weight discrete series representation π instead of averaged over the entire L-packet. We also remove the trivial center condition. The precise definition of the family we study is in section 7.1 and the final result is theorem 9.1.1. Here are the broad steps:

(1) Realize the empirical distribution $\mu_{\mathcal{F},S}$ as the trace of a function with a special Archimedean component φ_π against the discrete automorphic spectrum. The function φ_π is the pseudocoefficient from [CD90].
(2) Notice that pseudocoefficients have the same stable orbital integrals as Euler-Poincare functions
(3) Use the stable trace formula to write this trace as a linear combination of traces of functions with Euler-Poincare components at infinity on the smaller endoscopic groups
(4) Proceed as before to bound each term in the sum. Showing that enough technical conditions are satisfied and that the bounds are uniform enough that you are allowed to do so is most of the new work.
(5) Redo the computations showing the versions of Plancherel and Sato-Tate equidistribution that the new main term gives.

It is worth discussing step (3) in more detail. The key difficulty is that Arthur's simple trace formula only works when the Archimedean component is Euler-Poincare instead of a pseudocoefficient. However, the stable trace formula roughly gives the trace of a function as a linear combination of stable traces of transfers of the function on smaller endoscopic groups—we get an expansion of shape:

$$I^G(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} S^H(f^H).$$

Since pseudocoefficients have the same stable orbital integrals their corresponding Euler-Poincare functions, the f^H can wlog be chosen to have Euler-Poincare components at infinity. See section 5.1 for details on these transfers.

The most direct way to proceed is to then repeat the work in [Art89] on the stable distributions S^H instead of the invariant distribution I^G. We choose to instead use the hyperendoscopy formula from [Fer07] (see the remark at the beginning of section 4). It gives an expansion of shape

$$I^G(f) = \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} I^{\mathcal{H}}((f - f^*)^{\mathcal{H}})$$

where f^* is a function with the same stable orbital integrals as f and $\mathcal{HE}_{\mathrm{ell}}(G)$ is roughly the set of groups that can show up in sequence of iteratively choosing an endoscopic group starting from G. See section 4 for the full details. The distributions $I^{\mathcal{H}}$ can then be treated exactly as in [ST16] provided technical conditions still hold.

We also describe some of the complications in step (4). First, the distribution $I^G_{\mathrm{spec}}(f)$ isn't obviously the trace of f against the discrete automorphic spectrum like we want it to be. [Art89] shows this for Euler-Poincare at infinity and an unpublished lemma of Vogan (appearing here as lemma 6.3.1) is needed to extend to the pseudocoefficient case. Next, the groups appearing in $\mathcal{HE}_{\mathrm{ell}}(G)$ do not satisfy the technical simplifying conditions of [Art89]. We therefore need to slightly generalize the result, in particular to non-cuspidal groups. This is section 6. Thirdly, we need some bounds on endoscopic transfers of test functions so that Shin-Templier's orbital integral bounds apply. This takes some work in the non-Archimedean case and is sections 5.4 and 5.5.

For step (5), non-trivial center changes the main term in theorem 9.1.1 to something more complicated than originally in [ST16]. We therefore have to redo the computations for Sato-Tate and Plancherel equidistribution. This produces slightly different limiting measures that can be roughly thought of as Sato-Tate or Plancherel measure conditioned to be on a certain subset of $\widehat{G}_S$: representations with central character contained in a particular discrete set. This is section 10. We don't do the computation for low-lying zeros of L-functions due to complexity.

Finally, we save the level aspect computation for a future writeup. The main difficulty here is that as level gets larger, the test function f becomes more and more ramified adding more and more non-zero terms to the sum over $\mathcal{HE}_{\mathrm{ell}}(G)$. This necessitates proving much stronger uniformity of the bounds in [ST16, §8] over endoscopic groups.

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1.4. **Notational conventions.** Here are some notational conventions we will use throughout:
Basics

- F is a fixed number field
- G is a fixed reductive group over F
- $\mathbb{A}$ is $\mathbb{A}_F$ for shorthand
- $\mathbb{A}_\infty$, $\mathbb{A}^\infty$ are the at infinity and away from infinity parts of $\mathbb{A}$ respectively.
- W_E is the Weil group of local or global field E.
- $\mathcal{O}_E$ is the ring of integers of local field E
- k_E is the residue field of local field E.
- $\mathbf{1}_X$ is the indicator function for set X.
- $\widehat{H}$ is the reductive dual of reductive group H.
- $\widehat{S}$ is the unitary dual of abstract group S.
- $\widehat{S}^{\text{temp}}$ is the tempered part of $\widehat{S}$.
- $\widehat{f}$ is the Fourier transform of function f on abstract group S that should be clear from context
- $\bar{f}$ is the Fourier transform of f restricted to some subgroup of the center of S that should be clear from context

Reductive Groups

- Z_H is the center of abstract or reductive group H.
- $Z_H(G)$ is the centralizer of H inside G.
- A_H is the maximum split component in the center of reductive group H.
- H_∞ for group H over F is $(\mathrm{Res}^F_{\mathbb{Q}} H)(\mathbb{R}) = H(\mathbb{A}_\infty)$
- H_S for group H over F and finite set of places S of F is $H(\mathbb{A}_S)$. Use the standard conventions where an upper index means everything except S.
- $A_{H,\text{rat}}$ for group H over F is $A_{\mathrm{Res}^F_{\mathbb{Q}} H}(\mathbb{R})^0$ (the connected component is in the real topology)
- $A_{H,\infty} := A_{(\mathrm{Res}^F_{\mathbb{Q}} H)_{\mathbb{R}}}(\mathbb{R})^0$
- $H(\mathbb{A})^1 := H(\mathbb{A})/A_{H,\text{rat}}$
- $H^1_\infty := H_\infty/A_{H,\infty}$.
- H_γ is the centralizer of γ in H for H either an algebraic or abstract group.
- I^H_γ is the connected component of the identity in the centralizer of γ in H
- $\iota^H(\gamma)$ is the set of connected components of H_γ with an F-point.
- $[H]$, $[H]^{\text{ss}}$, $[H]^{\text{ell}}$ are the sets of (semisimple, elliptic) conjugacy classes in H
- $D^H(\gamma)$ is the Weyl discriminant for H
- K_S where S is a finite set of places of F is a chosen hyperspecial of $G(\mathbb{A}_S)$
- M usually represents some Levi subgroup
- P usually represents some parabolic subgroup
- $K_{S,H}$ for S some finite set of places usually represents some kind of maximal compact of $H(\mathbb{A}_S)$.

Lie Theory

- $\Phi^*(H), \Phi^+(H), \Phi^*_F(H), \Phi^+_F(H)$ are the sets of (positive, rational) roots of H.
- $\Phi_*(H), \Phi_+(H), \Phi_{*,F}(H), \Phi_{+,F}(H)$ are the sets of (positive, rational) co-roots of H.
- $\Delta^*(H), \Delta^*_F(H)$ are the sets of (rational) simple roots of H.
- $\Delta_*(H), \Delta_{*,F}(H)$ are the sets of (rational) simple roots of H.
- Ω_H is the Weyl group of $H_{\mathbb{C}}$ for H a reductive group.
- $\Omega_{H,E} = \Omega_E$ for H over F and E an extension of F is the subset of Ω_H generated by conjugating by elements of $H(E)$.

Volumes

- $\mu^{\text{tam}}, \mu^{\text{can}}, \mu^{EP}$ are the Tamagawa, Gross' canonical, or Euler-Poincaré measures on various groups
- $\bar{\mu}^\star$ is the quotient of measure $\mu^\star$ by something that should be clear from context
- $\tau(H)$ is the Tamagawa number of H
- $\tau'(H)$ is the modified Tamagawa number using the canonical measure $\mu^{\text{can},EP}$.

Endoscopy

- $(H, \mathcal{H}, s, \eta)$ is an endoscopic quadruple for G.
- $(\tilde{H}, \tilde{\eta})$ is a z-pair for $(H, \mathcal{H}, s, \eta)$
- (H_1, η_1) will also sometimes be used to represent a z-pair to keep diacritics from stacking too much.
- $\mathcal{E}_{\text{ell}}(H)$ is the set of elliptic endoscopic quadruples of reductive group H
- $\mathcal{H}\mathcal{E}_{\text{ell}}(H)$ is the set of elliptic hyperendoscopic paths of reductive group H
- $(\mathfrak{X}, \chi)$ is a central character datum on some reductive group
- $\mathcal{H}$ is further overloaded: when context is clear, it can also refer to either a hyperendoscopic path or the last group in the path.

Automorphic representations and the trace formula

- $\mathscr{H}(H, \chi) = \mathscr{H}(H, (\mathfrak{X}, \chi))$ is the space of compactly supported functions on $H(\mathbb{A})$ that transform according to character χ^{-1} on $\mathfrak{X} \subseteq Z_G(\mathbb{A})$.
- $\mathscr{H}(H_S, \chi_S)$ for S a finite set of places of F is compactly supported functions on $H(\mathbb{A}_S)$ similarly transforming according to χ_S^{-1}.
- $\mathscr{H}(H_S, K_S, \chi_S)$ if K_S is a product of hyperspecial subgroups and χ_S is unramified is the Hecke algebra of K_S-biinvariant elements of $\mathscr{H}(H_S, \chi)$.
- $\mathscr{H}(H_S, K_S, \chi_S)^{\leq \kappa}$ is the truncated Hecke algebra from section 5.3.
- $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \chi)$ for $(\mathfrak{X}, \chi)$ a central character datum is the unitary $G(\mathbb{A})$-rep of L^2-up-to-$\mathfrak{X}$ functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ transforming according to χ^{-1}.
- $L^2_{\text{disc}}(\cdot)$ is the discrete part of unitary representation $L^2(\cdot)$.
- $\mathcal{A}\mathcal{R}_{\text{disc}}(H, \chi)$ is the set of discrete automorphic representations on H with character χ on $A_{H,\infty}$.
- $O^H_\gamma(f)$ is the integral of f on the conjugacy orbit of γ. This can either be local or global; f can be a function on $H(\mathbb{A})$ or some $H(F_v)$.
- $I^{G,\chi}_{\text{spec}}, I^{G,\chi}_{\text{disc}}, I^{G,\chi}_{\text{geom}}$ are the distributions on G defined by Arthur's invariant trace formula depending on central character datum $(\mathfrak{X}, \chi)$.
- $S^{H,\chi}_{\text{spec}}, S^{H,\chi}_{\text{disc}}, S^{H,\chi}_{\text{geom}}$ are the distributions on H defined by Arthur's stable trace formula depending on central character datum $(\mathfrak{X}, \chi)$.

Rep theory

- $\pi(\lambda, w_0)$, $\pi(w_0(\lambda+\rho))$ are two different parametrizations for discrete series representations for λ a dominant weight.
- $\Pi_{\mathrm{disc}}(\lambda)$ is a discrete series L-packet where λ is a dominant weight.
- Θ_π is the Harish-Chandra character for representation π.
- ω_π is the central character of representation π.
- φ_π is the pseudocoefficient for discrete series representation π.
- η_λ is the Euler-Poincare function for the L-packet $\Pi_{\mathrm{disc}}(\lambda)$.

Families

- φ^∞ is a specific function defined in section 7.1.
- $\mathcal{F}$ is a specific family (as in [ST16]) of automorphic representations defined in section 7.1.
- $a_{\mathcal{F}}(\pi)$ are the coefficients defining $\mathcal{F}$.
- S_0, S_1, $U^{S\cup\infty}$, φ_{S_1}, f_{S_0} are data used to define φ^∞ and $\mathcal{F}$ as explained in section 7.1.
- $S_{\mathrm{bad},H}$ is the unknown finite set of bad places depending on reductive group H defined in [ST16, §B]
- L is the lattice $Z_G(F) \cap U^{S,\infty} \subseteq Z_{G_{S,\infty}}/A_{G,\mathrm{rat}}$.
- $E^{\mathrm{pl}}(\widehat{\varphi}|\omega)$ is the expectation defined in section 8.3.1
- $E^{\mathrm{pl}}(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$ is defined in proposition 8.3.5

1.4.1. *Dimensional Analysis.* A lot of the formulas here depend on choices of Haar measure. Since we are explicitly bounding terms, it is sometimes helpful to have notation for how they depend on these choices. For example, if we say that a value has dimension $[G][H]^{-1}$, then it is proportional to a choice of Haar measure on G and inversely proportional to a choice on H.

In any formula, dimensions on both sides need to match. In addition, any quantity with dimension needs to be normalized by a formula expressing it in terms of just dimensionless quantities and Haar measures—for example, the formulas defining traces of Hecke algebra elements, orbital integrals, or pseudocoefficients.

2. Trace Formula Background

2.1. **Invariant Trace Formula.** Let G be a connected reductive group over a number field F. Let $\mathbb{A} = \mathbb{A}_F$. Fix a central character χ on $A_{G,\mathrm{rat}}$. Let $\mathscr{H}(G,\chi)$ be the space of functions on $G(\mathbb{A})$ that are smooth and compactly supported when restricted to $G(\mathbb{A})^1$ and satisfy $f(ax) = \chi^{-1}(a)f(x)$ for all $a \in A_G(\mathbb{R})$.

Over a long series of papers that are summarized in [Art05] Arthur defines two equal distributions on $\mathscr{H}(G,\chi)$:

$$I^{G,\chi}_{\mathrm{geom}} = I^{G,\chi}_{\mathrm{spec}}.$$

Intuitively, one should think of I_{geom} as a sum of modified orbital integrals of f and I_{spec} as a sum of modified traces of f against components of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi)$. The exact definitions of these distributions are impractically complicated to use directly. However, enough useful special cases and abstract properties have been worked out—the most relevant being the simple trace formula in [Art89]. The χ will often be suppressed in notation.

Both sides have dimension $[G(\mathbb{A})^1]$. The individual terms in the expansions for both sides can have more complicated dimensions.

2.1.1. *Spectral side.* As a very rough description of the spectral side, Arthur defines components

$$I^G_{\mathrm{spec}} = I^G_{\mathrm{cts}} + \sum_{t \geq 0} I^G_{\mathrm{disc},t}.$$

$I_{\mathrm{disc},t}$ is 0 except for countably t and is much easier to evaluate. Expanding further,

$$I_{\mathrm{disc},t} = \sum_{M \in \mathscr{L}} \frac{|\Omega_M|}{|\Omega_G|} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|_{\mathfrak{a}^G_M}|^{-1} \operatorname{tr}(M_{P,t}(\omega)\mathcal{I}_{P,t}(f)).$$

To describe the most relevant terms, $\mathscr{L}$ is the set of Levi's of G containing a chosen minimal Levi, P is a chosen parabolic for M,$W(M)_{\mathrm{reg}}$ is a particular set of elements of a relative Weyl group (this and the Weyl group factor are a combinatorial term roughly parametrizing parabolics containing the Levi), and $M_{P,t}(\omega, \chi)$ is an intertwining operator between parabolic inductions through different parabolics containing M from the theory of Eisenstein series.

The last term is the most important for us. The χ induces a character on $A_{M,\mathrm{rat}}$ by pullback. Then $\mathcal{I}_P(\chi)$ is the representation of $G(\mathbb{A})$ produced from parabolically inducing $L^2_{\mathrm{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}), \chi)$. The term $\mathcal{I}_{P,t}$ is the subrepresentation of this with archimedean infinitesimal character having imaginary part of norm t. By lots of work, all these decompositions makes sense and the convolution operators $\mathcal{I}_{P,t}(f)$ for $f \in \mathscr{H}(G,\chi)$ are trace class.

There are well-known and simple sufficient conditions on f such that $I_{\mathrm{cts}}(f) = 0$:

Definition ([Art05, dfn 23.6])**.** If v is a place of F, $f \in \mathscr{H}(G(F_v))$ is *cuspidal* if for all Levi's M of G and π_v tempered representations of M

$$\operatorname{tr}_{\pi_v^G}(f) = 0.$$

Here π_v^G is (any) parabolic induction of π_v.

Note that this is an alternate definition to the original one from [Art88].

Theorem 2.1.1 ([Art88, thm 7.1])**.** *If f factors as $f_v \otimes f^v$ for some place v with f_v cuspidal, then $I_{\mathrm{cts}}(f) = 0$.*

2.1.2. *Geometric side.* The geometric side can be succinctly written as

$$I_{\mathrm{geom}}(f) = \sum_{M \in \mathscr{L}} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(\mathbb{Q})]_{M,S}} a^M(S,\gamma) I^G_M(\gamma, f).$$

Here S is a large enough set of places in particular including those at which f isn't the characteristic function of a hyperspecial and $[M(\mathbb{Q})]_{M,S}$ is the set of conjugacy classes under a complicated equivalence relation involving the away-from-S components of the unipotent parts. For γ semisimple,

$$a^M(S,\gamma) = |\iota^M(\gamma)|^{-1} \operatorname{vol}(I^M_\gamma(\mathbb{Q})\backslash I^M_\gamma(\mathbb{A})^1)$$

where $|\iota^M(\gamma)|$ is the number of connected components of M_γ that have an F-point. In general, there is no explicit description of $a^M(S,\gamma)$.

Next, I^G_M is a weighted orbital integral of the S-components of f. If $M = G$, it is simply the orbital integral at γ. If γ is semisimple, there is an explicit formula weighting the integral by a complicated combinatorial factor. Otherwise, it is only defined though some analytic continuations. The term I^G_M satisfies some splitting formulas ([Art05, 23.8] and [Art05, 23.9]) factoring it into local components in terms of traces of f against parabolic inductions. When f is cuspidal at some place, these splitting formulas of course then greatly simplify.

The a^M have dimension $[I^M_\gamma(\mathbb{A})^1]$ while the I^G_M have dimension $[G(\mathbb{A})^1][I^M_\gamma(\mathbb{A})^1]^{-1}$.

2.2. **The Simple Trace Formula.** Whenever $G(\mathbb{R})$ has discrete series, the trace formula can be simplified by setting the test function to have a special real component.

2.2.1. *Parametrizing discrete series.* The classification of discrete series is work of Harish-Chandra that can be found summarized in [Lab11, §III.5]. Let G be a reductive group over $\mathbb{R}$ with fixed elliptic maximal torus T (so it has discrete series). Let K be a maximal compact of $G(\mathbb{R})$ containing $T(\mathbb{R})$, B_K a Borel of $K_\mathbb{C}$ containing T, and B a Borel of $G_\mathbb{C}$. Let Ω_G be the Weyl group of $(G_\mathbb{C}, T_\mathbb{C})$ and $\Omega_{\mathbb{R},G}$ be the subgroup given by only conjugating by elements of $G(\mathbb{R})$.

The characters of $T(\mathbb{R})$ are contained in $T(\mathbb{C})$ so the root space of K is contained in G. Let ρ be half the sum of the positive roots of G. Finally, let $\Omega(B_K)$ be a particular set of coset representatives of $\Omega_{\mathbb{R},G}\backslash\Omega_G$: namely, w such that $w\lambda$ is B_K-dominant for any λ that is B-dominant.

The discrete series representations of G are parametrized by B-dominant weights $\lambda \in X^*(T)_\mathbb{C}$ and elements $w^* \in \Omega(B_K)$. Call the rep parameterized by λ and w_0 either $\pi(\lambda, w_0)$ or $\pi(w_0(\lambda+\rho))$. It is the unique rep with trace character

$$\Theta_{\pi(\lambda,w_0)} = (-1)^{1/2\dim(G(\mathbb{R})/KA_{G,\infty})}\frac{\sum_{w\in\Omega_K}\operatorname{sgn}(ww_0)e^{ww_0(\lambda+\rho)}}{\sum_{w\in\Omega_G}\operatorname{sgn}(w)e^{w\rho}}.$$

The infinitesimal character of $\pi(\lambda, w)$ is the same as that of V_λ, the finite dimensional representation with highest weight λ. Therefore the $\pi(\lambda, w)$ for a fixed λ are all in the same L-packet $\Pi_{\mathrm{disc}}(\lambda)$. We call $\pi(\lambda, w_0) = \pi(w_0(\lambda + \rho))$ regular if λ is.

2.2.2. *Pseudocoefficients and Euler-Poincare functions.* Given a discrete series representation π of a real reductive group $G(\mathbb{R})$ with character χ on $A_{G,\infty}$, Clozel and Delorme in [CD90] define a pseudocoefficient $\varphi_\pi \in C_c^\infty(\chi^{-1})$. The function φ_π is compactly supported and has the property that for irreducible representations ρ with character χ,

$$\operatorname{tr}_\rho(\varphi_\pi) = \begin{cases} 1 & \pi = \rho \\ 0 & \pi \neq \rho, \rho \text{ basic} \\ ? & \text{else} \end{cases}.$$

Here, a basic representation is a parabolic induction of a limit of discrete series. The non-basic case is much more complicated. Pseudocoefficients have dimension $[G(\mathbb{R})^1]^{-1}$.

If $\Pi_{\mathrm{disc}}(\lambda)$ is the discrete series L-packet for π, its useful to also consider Euler-Poincare functions:

$$\eta_\pi = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|}\sum_{\pi'\in\Pi_{\mathrm{disc}}(\lambda)}\varphi_{\pi'}.$$

Traces against Euler-Poincare functions can be interepreted as Euler characteristics of certain cohomologies for basic representations and therefore all representations by the Langlands classification. If λ is regular, these Euler characteristics can be shown to be 0 on non-tempered representations. Therefore, if λ is regular we get

$$\operatorname{tr}_\rho(\eta_\lambda) = \begin{cases} 1 & \pi \in \Pi_{\mathrm{disc}}(\lambda) \\ 0 & \text{else} \end{cases}$$

for all irreducible representations ρ (see sections 1 and 2 in [Art89]).

Note that both pseudocoefficients and Euler-Poincare functions are cuspidal since they have 0 trace against any non-discrete series basic representation and therefore against all parabolic inductions of tempered representations.

2.2.3. *Simple trace formula.* The simple trace formula is the main result of [Art89]. A more textbook exposition is in [Art05, §24]. We state it here. First, assume

- G is connected
- G is cuspidal over $\mathbb{Q}$: $\operatorname{Res}^F_{\mathbb{Q}} G/A_{G,\mathrm{rat}}$ has an $\mathbb{R}$-anisotropic maximal torus.

The last condition in particular gives that $G(\mathbb{R})$ has an elliptic maximal torus and therefore has discrete series mod center. In the case where $G(\mathbb{R})$ has discrete series mod center, cuspidal is equivalent to $A_{G,\mathrm{rat}} = A_{G,\infty}$: in other words, taking infinite place points of the maximum split torus in the center is the same as base changing to $\mathbb{R}$ and, looking at the maximal split torus in the center, and taking $\mathbb{R}$-points.

Consider a test function of the form $h = \eta_\xi \otimes h^\infty$ for regular weight ξ and $h \in \mathscr{H}(G(\mathbb{A}^\infty))$. Let χ be the character on $A_G(\mathbb{R})^0$ determined by ξ. Then

$$I_{\mathrm{spec}}(h) = I_{\mathrm{disc}}(h) = \sum_{\pi : \pi_\infty \in \Pi_{\mathrm{disc}}(\xi)} m_{\mathrm{disc}}(\pi) \operatorname{tr}_{\pi^\infty}(h^\infty) \tag{1}$$

where $m_{\mathrm{disc}}(\pi)$ is the multiplicity of π in $\mathcal{AR}_{\mathrm{disc}}(G,\chi)$. Let $\mathscr{L}$ be the set of Levi's containing a chosen minimal Levi of G. For each $M \in \mathscr{L}$, choose P_M a a parabolic for M. Then

$$\begin{aligned} I_{\mathrm{geom}}(h) = \sum_{M \in \mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|} \\ \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I^M_\gamma) |\iota^M(\gamma)|^{-1} \Phi_M(\gamma_\infty, \xi) O^M_\gamma(h^\infty_M) \end{aligned} \tag{2}$$

Here $\iota^M(\gamma)$ is the set of connected components of M_γ that have an F-point and

$$\chi(H) = (-1)^{q(H)} \operatorname{vol}(H(F)A^0_{H,\infty} \backslash H(\mathbb{A})) \operatorname{vol}(A^0_{H,\infty} \backslash \bar{H}_\infty)^{-1}$$

where $\bar{H}_\infty$ is an inner form of H_∞ such that $H_\infty/A_{H,\infty}$ has anisotropic center and $q(H) = 1/2 \dim(H_\infty/K_{H,\infty}A_{H,\infty})$ is the Kottwitz sign. Also

$$h^\infty_M(\gamma^\infty) = \delta_{P_M}(\gamma^\infty)^{1/2} \int_{K^\infty} \int_{N_M(\mathbb{A}^\infty)} h(k^{-1}\gamma^\infty n k) \, dn \, dk$$

where N_M is the unipotent group for P_M and K some chosen maximal compact. To make dimensions work out, the Haar measures choices should satisfy:

- The choices on I^M_γ, M, and in the orbital integral need to coincide
- The measure on $\bar{I}^M_\gamma$ comes from that on I^M_γ through them both coming from the same top form on $I^M_{\mathbb{C}}$.
- The choices on N_P, K, M, and G need to coincide according to the Iwasawa decomposition

Finally,

$$\Phi_M(\gamma_\infty, \xi) = \begin{cases} \left|\frac{D^G(\gamma_\infty)}{D^M(\gamma_\infty)}\right|^{1/2} \sum_{\pi \in \Pi^G_{\mathrm{disc}}(\xi)} \Theta_\pi(\gamma^\infty) & \gamma_\infty \text{ in an elliptic torus of } M \\ 0 & \text{else.} \end{cases}$$

As written, this is only defined on regular elements, but Arthur proves it extends to a function that is continuous on every elliptic torus.

As some notes for using this:

- Comparing character formulas computes that $\Phi_G(\gamma_\infty, \xi) = \operatorname{tr} \xi(\gamma_\infty)$ where ξ is overloaded to also denote the finite dimensional representation with highest weight ξ.
- If $M \neq G$, Φ_M cannot be evaluated through the standard Harish-Chandra character formula since it involves Θ_π's evaluated on tori that aren't elliptic in G. See [Art89, §4] for an algorithm to actually do so.
- The only M that contribute to the outer sum are those that are cuspidal over $\mathbb{Q}$. Arthur's original paper implicitly showed this for M cuspidal over $\mathbb{R}$ (for ease of reader, the full details are in section 6.4.1). The full result is actually missing from the original argument, but [GKM97] shows it using different methods.
- Because of the dimensions on η_ξ, both sides of this formula have dimension $[G^\infty]$. However, explicitly computing the $\chi(I^M_\gamma)$ terms still requires choosing Haar measures at ∞.

2.3. **Trace Formula with Central Character.** Stabilization requires a slightly different version of the trace formula where the fixed character χ is on a larger closed subgroup of $Z(\mathbb{A})$. There is a full theory in [Art02] taking quite a bit of work to describe. We summarize the relevant parts here.

Definition. A central character datum on G is $(\mathfrak{X}, \chi)$ where

- $\mathfrak{X} \supseteq \mathbb{A}_{G,\infty}$ is closed inside $Z(\mathbb{A})$ such that $Z(F)\mathfrak{X}$ is also a closed subgroup
- $\chi : \mathfrak{X} \cap Z(F) \backslash \mathfrak{X} \to \mathbb{C}^\times$ is a continuous character

Furthermore, $\mathscr{H}(G, (\mathfrak{X}, \chi)) = \mathscr{H}(G, \chi)$ is the set of smooth functions f on $G(\mathbb{A})$ such that $f(gx) = \chi^{-1}(x) f(g)$ and f is compactly supported mod $\mathfrak{X}$.

Note. For our purposes here, it suffices to consider $\mathfrak{X}$ that are the product of the adelic points of some algebraic subtorus of Z multiplied by some abstract subgroup of $Z_{G_\infty}(\mathbb{R})$.

Fix central character data $(\mathfrak{X}, \chi)$. In [Art13, §3], Arthur defines $I_{\mathrm{disc},t,\chi}$ as a distribution on $\mathscr{H}(G, \chi)$:

$$(3)\quad I_{\mathrm{disc},t,\chi}(f) = \sum_{M \in \mathscr{L}} \frac{|\Omega_M|}{|\Omega_G|} \sum_{w \in W(M)_{\mathrm{reg}}} |\det(w-1)|_{\mathfrak{a}^G_M}|^{-1} \operatorname{tr}(M_{P,t}(\omega, \chi) \mathcal{I}_{P,t}(\chi, f))$$

This is a generalization of $I_{\mathrm{disc},t}$ and most of the terms are the same. The relevant part is how $\mathcal{I}_{P,t}$ changes. First, χ induces a character on $A_{M,\mathrm{rat}}\mathfrak{X}$ by pullback and therefore lets us define $L^2_{\mathrm{disc}}(M(\mathbb{Q}) \backslash M(\mathbb{A}), \chi)$ analogous to other L^2 spaces with character: as the discrete part of χ^{-1}-invariant, L^2-up-to-$\mathfrak{X}$ functions on $M(\mathbb{Q}) \backslash M(\mathbb{A})$ as an $M(\mathbb{A})$-representation. Then, $\mathcal{I}_{P,t}(\chi, f)$ can be defined analogously to $\mathcal{I}_{P,t}$ from the trace formula without central character. Decompositions and traces making sense in this context requires some extra work summarized on [Art13, pg 123]. The dimensions change to $[G(\mathbb{A})][\mathfrak{X}]^{-1}$.

For our work here, we only need to worry about the spectral side so we will not mention the geometric version.

3. Endoscopy and Stabilization Background

The standard reference for this material, [KS99], is written for the more general case of twisted endoscopy. It is therefore easier to follow the summary in [Kal16, §1.3]. The simpler summary in [Shi10, §2] for the simply connected derived subgroup case is also helpful. Finally, [Lab11] is a course-notes style writeup of this material and therefore more motivated albeit far less general.

For this section, allow F to be a local or global number field.

3.1. **Endoscopic groups.**

3.1.1. *Endoscopic quadruples.*

Definition ([KS99, pg 18])**.** An *endoscopic quadruple* for G is $(H, \mathcal{H}, s, \eta)$ with

- H a quasisplit connected reductive group over F
- $\mathcal{H}$ is a split extension of $\widehat{H}$ by W_F such that action of W_F on $\widehat{H}$ determined by the splitting is the same as the one coming from H.
- $s \in Z(\widehat{H})$ and semisimple in $\widehat{G}$.
- $\eta : \mathcal{H} \to {}^L G$ an L-embedding.

such that

(1) η restricts to an isomorphism $\widehat{H} \xrightarrow{\sim} \widehat{G}^0_{\eta(s)}$
(2) There is then a W_F-equivariant sequence

$$1 \to Z(\widehat{G}) \to Z(\widehat{H}) \to Z(\widehat{H})/Z(\widehat{G}) \to 0$$

which induces a map $(Z(\widehat{H})/Z(\widehat{G}))^{W_F} \to H^1(F, Z(\widehat{G}))$. Let $\mathfrak{K}(s, \eta)$ be elements of $Z(\widehat{H})/Z(\widehat{G})$ that map to something locally trivial under this. We require $s \in \mathfrak{K}(s, \eta)$

It is furthermore *elliptic* if

(3) $(Z(\widehat{H})^{W_F})^0 \subseteq Z(\widehat{G})$.

Definition. Two endoscopic quadruples $(H, \mathcal{H}, s, \eta), (H', \mathcal{H}', s', \eta')$ are isomorphic if there is an element $g \in \widehat{G}$ such that

(1) $\eta(\mathcal{H})$ and $\eta'(\mathcal{H}')$ are conjugate by g
(2) s and gsg^{-1} are equal in $Z(\widehat{H})/Z(\widehat{G})$.

Call the set of isomorphism classes of elliptic endoscopic quadruples $\mathcal{E}_{\mathrm{ell}}(G)$.

Note that the definition implicitly uses this fact which we state directly here to cite more easily later:

Lemma 3.1.1. *Let G be a reductive group over global or local field K, $(H, \mathcal{H}, \eta, s)$ an elliptic endoscopic quadruple. Then there is a map $Z(G) \hookrightarrow Z(H)$.*

Proof. See [KS99] pg. 53. ∎

3.1.2. *Endoscopic pairs.* Endoscopic quadruples actually contain a lot of redundant data. A more basic and easier to think about notion is the endoscopic pair defined in [Kot84, §7]:

Definition. An endoscopic pair for group G is (s, ρ) where

- s is a semisimple element of $\widehat{G}/Z(\widehat{G})$
- ρ is a map $W_F \to \mathrm{Out}(\widehat{H})$ where $\widehat{H} = \widehat{G}_s^0$.

satisfying

- $\rho(\sigma)$ for $\sigma \in W_F$ is conjugation by an element in the normalizer of $\widehat{H}$ in ${}^L G$ that projects to σ.
- Then, ρ induces a W_F-action on $Z(\widehat{G}_s^0)$ which fits into W_F-equivariant sequence
$$1 \to Z(\widehat{G}) \to Z(\widehat{H}) \to Z(\widehat{H})/Z(\widehat{G}) \to 0$$
which induces a map $(Z(\widehat{H})/Z(\widehat{G}))^{W_F} \to H^1(F, Z(\widehat{G}))$. Let $\mathfrak{K}(s, \rho)$ be elements of $(Z(\widehat{H})/Z(\widehat{G}))^{W_F}$ that map to something locally trivial under this. We require $s \in \mathfrak{K}(s, \rho)$

It is elliptic if $(Z(\widehat{H})^{W_F})^0 \subseteq Z(\widehat{G})$.

The ρ action can be further clarified: If $a \rtimes \gamma \in {}^L G$ and $(b, 1) \in \widehat{G} \subset {}^L G$,

$$\begin{aligned}(a \rtimes \gamma)(b \rtimes 1)(a \rtimes \gamma)^{-1} &= (a \rtimes \gamma)(b \rtimes 1)(\gamma^{-1}(a^{-1}) \rtimes \gamma^{-1}) \\ &= (a\gamma(b) \rtimes \gamma)(\gamma^{-1}(a^{-1}) \rtimes \gamma^{-1}) = (a\gamma(b)a^{-1} \rtimes 1)\end{aligned}$$

so if ρ is part of an endoscopic pair, any $\rho(\gamma)$ is of the form $b \mapsto a_\gamma \gamma_{\widehat{G}}(b) a_\gamma^{-1}$ for some $a_\gamma \in \widehat{G}$ where the γ action is as it is on $\widehat{G}$. The choices of a_γ are unique up to

$$a_\gamma \in \mathrm{Int}\, \widehat{H} \backslash \widehat{G} / Z_{\gamma \widehat{H}}(\widehat{G}) = \widehat{H}_{\mathrm{ad}} \backslash \widehat{G} / Z(\gamma \widehat{H}) = \widehat{H}_{\mathrm{ad}} \backslash \widehat{G} / Z(\widehat{H}) = \widehat{H} \backslash \widehat{G}$$

since $\gamma \widehat{H}$ is the centralizer of γs.

Definition. An isomorphism of endoscopic pairs (s, ρ) and (s', ρ') is an element $g \in \widehat{G}$ such that

- $\widehat{G}_s^0, \widehat{G}_{s'}^0$ and ρ, ρ' are g-conjugate.
- s, s' have the same image in $\mathfrak{K}(s, \rho)$.

As explained in [Kot84, pg 630-631], ρ determines a quasisplit group H from $\widehat{H}$ and therefore the (H, s, η) part of an endoscopic quadruple. Given H and G, we can define $\mathcal{H}$ as follows: $\widehat{H}$ embeds into both ${}^L H$ and ${}^L G$. let $\mathcal{H}$ be the $x \in {}^L G$ such that there exists $y \in {}^L H$ such that conjugation by x, y are the same on $\widehat{H}$ and x, y project to the same element of W_F. In terms of the a_γ from above, we can realize

$$\mathcal{H} = \bigcup_{\gamma \in W_F} \widehat{H} a_\gamma \rtimes \gamma$$

where we choose the representatives for a_γ that fix a pinning. Isomorphisms are also the same on each side so in summary

Lemma 3.1.2 ([Kot84, §7])**.** *The set of elliptic endoscopic pairs of G up to isomorphism are in bijection with $\mathcal{E}_{\mathrm{ell}}(G)$ where the bijection is as described above.*

3.1.3. *Motivation and the group $\mathfrak{K}$.* There are two motivations for this definition, either spectral or geometric. We briefly and very roughly describe the geometric explanation since it is somewhat relevant later. We ignore many, many Galois cohomology details. In increasing generality and detail, more information can be found in [Lab11, §III.3], [Kot86, §9], and [KS99, §6-7].

Let semisimple $\gamma \in G(E)$ be contained in maximal torus T. If γ is strongly regular, then we can write its stable orbit as $(T\backslash G)(E)$ and its orbit as $T(E)\backslash G(E)$. Therefore, the fibers of the map from $(T\backslash G)(E)$ onto

$$\mathfrak{D}(E, T\backslash G) = \ker(H^1(E,T) \to H^1(E,G))$$

are exactly the unstable conjugacy classes making up $(T\backslash G)(E)$. Let

$$\mathfrak{E}(E, T\backslash G) = \ker(H^1(E,T) \to H^1_{\mathrm{ab}}(E,G))$$

be the abelian group version of this and

$$\mathfrak{K}(E, T\backslash G) = \mathfrak{E}(E, T\backslash G)^\vee.$$

Elements $\kappa \in \mathfrak{K}$ are called endoscopic characters.

If v is a place of F and $\kappa \in \mathfrak{K}(F_v, T\backslash G)$, this allows the definition of twisted orbital integrals

$$O^\kappa_\gamma(f) = \int_{(T\backslash G)(F_v)} \kappa(g) f(g^{-1}\gamma g)\, dg$$

using the map $(T\backslash G)(F) \to \mathfrak{E}(F, T\backslash G)$.

We can also define adelic versions of these groups $\mathfrak{D}(\mathbb{A}, T\backslash G), \mathfrak{E}(\mathbb{A}, T\backslash G)$, and $\mathfrak{K}(\mathbb{A}, T\backslash G)$ using corresponding cohomology groups $H^1(\mathbb{A}, \cdot)$. If $\gamma \in G(\mathbb{A})$ is strongly regular, $\mathfrak{D}(\mathbb{A}, T\backslash G)$ parametrizes the γ' that have every component stably conjugate to γ. It is a restricted direct product of the $\mathfrak{D}(F_v, T\backslash G)$ by $\mathfrak{D}(\mathcal{O}_v, T\backslash G)$ which happens to be trivial. Define a measure on it by taking the product of the counting measures on $\mathfrak{D}(F_v, T\backslash G)$. Then for $\kappa \in \mathfrak{K}(F, T\backslash G)$ we can define global twisted orbital integral

$$O^\kappa_\gamma(f) = \sum_{e \in \mathfrak{D}(\mathbb{A}, T\backslash G)} \kappa(\mathrm{obs}(\gamma_e)) O_{\gamma_e}(f)$$

where γ_e is the conjugacy class corresponding to e with base point γ and obs is the obstruction defined in [Kot86] and [KS99].

Stabilization of the trace formula first produces sums of $O^\kappa_\gamma(f)$'s over triples of these (T, γ, κ) over F. The result ([KS99, lem 7.2.A]) shows that such triples are in bijection with quintuples $(H, \mathcal{H}, s, \eta, \gamma_H)$: endoscopic quadruples with a choice of strongly regular element $\gamma_H \in H$ up to appropriately defined equivalence. Through this equivalence, the group $\mathfrak{K}$ for T ends up being the same as the group κ defined above for (s, η) (see [KS99, pg 105-106]).

3.2. **z-Extensions.** Our next goal is to define transfers of functions. This naïvely needs an embedding ${}^L H \hookrightarrow {}^L G$, but in general ${}^L H \ncong \mathcal{H}$ so we don't have one. There are two possible strategies for dealing with this: the original in [LS87] is to take a nice enough central extension of G. This works for the standard endoscopy described here but not for the more general twisted endoscopy, so more modern sources prefer to take central extensions of H as described in [KS99]. As we will remark after proposition 3.2.2, these methods are more or less interchangeable in the standard endoscopy case.

We describe the second method in detail:

Definition. A *z-pair* $(\tilde{H}, \tilde{\eta})$ for endoscopic quadruple $(H, \mathcal{H}, s, \eta)$ is an extension $\tilde{H}$ by a central induced torus such that

(1) $\tilde{H}_{\text{der}}$ is simply connected (we call such an $\tilde{H}$ a *z-extension*).
(2) $\tilde{\eta} : \mathcal{H} \to {}^L\tilde{H}$ is an L-embedding that restricts to the map $\widehat{H} \to \widehat{\tilde{H}}$ dual to the projection $\tilde{H} \to H$.

By Lemma 2.2.A in [KS99], as long as (1) is satisfied, a valid η satisfying (2) always exists.

Lemma 3.2.1. *Let H be a reductive group that splits over K'. Then there exists a z-extension of H splitting over K'. Furthermore, the dimension of the extending torus is bounded by $[K' : \mathbb{Q}](\operatorname{rank}_{\text{ss}} H)$.*

Proof. We just go through the construction in [Lan79] or [MS82, pg 299] explicitly seeing how big things get at each step. Let T^{sc} be the maximal torus in the simply connected cover of H^{der}. Let $P = X_*(T)/X_*(T^{\text{sc}})$ as a Galois module. A z-extension would correspond to an extension of $X_*(T)$ making this quotient have no torsion. The torsion part has less than $\operatorname{rank}_{\text{ss}} G$ generators.

The argument starts with a lemma writing P as a quotient of Galois modules

$$0 \to M \to Q \to P \to 0$$

with M free over $Z[G]$ and Q free over P. The construction is [MS82, prop 3.1] and bounds $\operatorname{rank}_{\mathbb{Z}} M$ by $\dim K'$ times the number of generators of the torsion of $P/\mathbb{Z}$ which we can further bound by $(\dim K')(\operatorname{rank}_{\text{ss}} H)$.

Some work with reductive groups shows that M can be chosen to be the cocharacter space of the extending torus thereby finishing the argument. $\square$

In the case where G has simply connected derived subgroup, the Z-extension can be chosen to be trivial and $\mathcal{H} \simeq {}^L H$. In this case, an endoscopic triple (H, s, η) contains all the needed data.

3.2.1. *z-extensions and central character datum.* If $(\mathfrak{X}, \chi)$ is a central character datum for G, any $(H, \mathcal{H}, s, \eta)$ and $(\tilde{H}, \tilde{\eta})$ quadruple and extension determine a central character datum $(\mathfrak{X}_{\tilde{H}}, \chi_{\tilde{H}})$ on $\tilde{H}$. The central subgroup $\mathfrak{X}_{\tilde{H}}$ is produced from $\mathfrak{X}$ by first taking the image under the map $Z(G) \hookrightarrow Z(H)$ and then taking the preimage under $\tilde{H} \to H$.

To get $\chi_{\tilde{H}}$, pick a section c for $\mathcal{H} \to W_F$. Then if T is the extending torus defining $\tilde{H}$, the composition

$$W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\tilde{\eta}} {}^L\tilde{H} \to {}^L T$$

is an L-parameter for T. This determines a character $\lambda_{\tilde{\eta}}^{-1}$ on $T(F)$ if F is local or $T(F)\backslash T(\mathbb{A})$ if F is global through the Langlands correspondence for Tori. The inverse is to match our convention for defining Hecke algebras.

Through considerations of transfer factors (see section 3.3), $\lambda_{\tilde{\eta}}$ can be extended to the preimage of $Z(G)$ in $Z(\tilde{H})$. Therefore, we can set $\chi_{\tilde{H}}$ to be $\chi\lambda_{\tilde{\eta}}$ (where χ is defined on $\mathfrak{X}_H$ by pullback). We will discuss this and more properties of $\lambda_{\tilde{\eta}}$ when we discuss transfer. In particular, we will show that in the relevant cases, $\lambda_{\tilde{\eta},v}$ at a place v can be extended to a character on $\tilde{H}_v$.

3.2.2. *z-extensions don't change much.* There is a vague intuition that taking a z extension shouldn't change a groups endoscopy:

Proposition 3.2.2. *Let G be a group over F.*

(a) If G_1 is a central extension of G by induced torus T, then the (elliptic) endoscopic tuples for G are in bijection with those of G_1. This bijection takes a group H to a central extension H_1 by T.
(b) If H is an (elliptic) endoscopic group of G and H_1 is a central extension of H by induced torus T, then there is a central extension G_1 of G by T such that H_1 is an (elliptic) endoscopic group of G_1. Furthermore, the endoscopic tuples determining H and H_1 correspond under the bijection from (a).

Proof. Part (a):
The s: The map $\widehat{G} \to \widehat{G}_1$ gives a canonical W_F-equivariant isomorphism $\widehat{G}/Z(\widehat{G}) \to \widehat{G}_1/Z(\widehat{G}_1)$ so choices for s are the same. Given such an s, set $\widehat{H}_1 = (\widehat{G}_1)^0_s$. Then we have diagram

$$\begin{tikzcd}
\widehat{H} \arrow[r, hook] \arrow[d, hook] & \widehat{G} \arrow[d, hook] \\
\widehat{H}_1 \arrow[r, hook] \arrow[d, two heads] & \widehat{G}_1 \arrow[d, two heads] \\
\widehat{T} \arrow[r, "\sim"] & \widehat{T}
\end{tikzcd}$$

The ρ and H: This gives a canonical isomorphism $\widehat{H}_1\backslash\widehat{G}_1 \to \widehat{H}\backslash\widehat{G}$ so assignments $\gamma \to a_\gamma$ as in the comment after the definition of endoscopic pair are the same for G and G_1. There are two conditions for this assignment to give a valid ρ: The first is that $\gamma \mapsto \operatorname{Int} a_\gamma \circ \gamma$ is a homomorphism up to $\operatorname{Int}\widehat{H} = \operatorname{Int}\widehat{H}_1$. This condition is clearly the same with respect to either $\widehat{H}$ or $\widehat{H}_1$.

The second condition is that $\operatorname{Int} a_\gamma \circ \gamma$ needs to fix the appropriate group: $\widehat{H}$ or $\widehat{H}_1$. By construction, $\widehat{H} = \widehat{G} \cap \widehat{H}_1$. Therefore, since $\widehat{G}$ is W_F and $\operatorname{Int}\widehat{G}_1$-invariant, if such a map fixes $\widehat{H}_1$, it fixes $\widehat{H}$. For the other direction, since these are all complex groups and $\widehat{G} \supseteq (\widehat{G}_1)^{\mathrm{der}}$, all elements of $\widehat{G}_1$ can be written as zg for $z \in Z^0_{\widehat{G}_1}$ and $g \in \widehat{G}$. This is an element of $\widehat{H}_1$ iff $g \in \widehat{H}$. In total, $\widehat{H}_1 = Z^0_{\widehat{G}_1}\widehat{H}$ so we are done since $Z^0_{\widehat{G}_1}$ is fixed by W_F and Int_G. Therefore, this second condition is true for $\widehat{G}$ iff it is true for $\widehat{G}_1$.

Note that for any such ρ, the columns of the above diagram and the isomorphism between $\widehat{T}$'s are Γ-equivariant. Undoing this dual, this will give that H_1 is an extension of H by T.
The cohomology condition: In total, the possible pairs (s, ρ) ignoring the cohomology condition are the same for G and G_1. It remains to show that the cohomology condition holds with respect to G iff it does for G_1. We have W_F-equivariant diagram where the first two rows are exact sequences (note that the actions on $Z_{\widehat{G}}$

from ρ and $\widehat{G}$ coincide so the action here is according to ρ):

$$\begin{tikzcd}
1 \arrow[r] & Z_{\widehat{G}} \arrow[r] \arrow[d] & Z_{\widehat{H}} \arrow[r] \arrow[d] & Z_{\widehat{H}}/Z_{\widehat{G}} \arrow[r] \arrow[d, "\sim"] & 1 \\
1 \arrow[r] & Z_{\widehat{G}_1} \arrow[r] \arrow[d] & Z_{\widehat{H}_1} \arrow[r] \arrow[d] & Z_{\widehat{H}_1}/Z_{\widehat{G}_1} \arrow[r] & 1 \\
 & \widehat{T} \arrow[r, "\sim"] & \widehat{T} & &
\end{tikzcd}$$

This gives a corresponding diagram in cohomology:

$$\begin{tikzcd}
(Z_{\widehat{H}}/Z_{\widehat{G}})^\Gamma \arrow[r, "\varphi_1"] \arrow[d, "\sim"] & H^1(\Gamma, Z_{\widehat{G}}) \arrow[d, "\psi"] \\
(Z_{\widehat{H}_1}/Z_{\widehat{G}_1})^\Gamma \arrow[r, "\varphi_2"] & H^1(\Gamma, Z_{\widehat{G}_1})
\end{tikzcd}$$

Here $\Gamma \subseteq W_F$ is some local Galois group. The cohomology conditions for H and H_1 matching at Γ is equivalent to $\ker \varphi_1 = \ker \varphi_2$. To show this, consider sequence

$$\pi_0(\widehat{T}^\Gamma) \to H^1(\Gamma, Z_{\widehat{G}}) \xrightarrow{\psi} H^1(\Gamma, Z_{\widehat{G}_1}) \to H^1(\Gamma, \widehat{T}).$$

Since T is an induced torus, $\widehat{T}$ is a power of $\mathbb{G}_m$ with a Γ action by permuting coordinates. This gives first that $\widehat{T}^\Gamma$ is connected and second that $\widehat{T}$ is induced so $H^1(\Gamma, \widehat{T}) = 0$. Therefore ψ is an isomorphism and the cohomology conditions are equivalent at every place.

Ellipticity: The elliptic condition is that $(Z_{\widehat{H}_*}^{W_F})^0 \subseteq Z_{\widehat{G}_*}^{W_F}$. As before, $Z_{\widehat{H}_1} = Z_{\widehat{H}} Z_{\widehat{G}_1}$ and $Z_{\widehat{H}} \cap Z_{\widehat{G}_1} = Z_{\widehat{G}}$. Then we get sequence

$$1 \to Z_{\widehat{G}} \to Z_{\widehat{H}} \times Z_{\widehat{G}_1} \to Z_{\widehat{H}_1} \to 1$$

where the first map is the antidiagonal. This gives map in cohomology

$$Z_{\widehat{H}}^{W_F} \times Z_{\widehat{G}_1}^{W_F} \to Z_{\widehat{H}_1}^{W_F} \to H^1(W_F, Z_{\widehat{G}}) \to H^1(W_F, Z_{\widehat{G}_1}) \oplus H^1(W_F, Z_{\widehat{H}}).$$

From previous arguments, T being induced gives that the last map in injective into the first coordinate. Therefore the middle is 0 and the first is surjective. Therefore $Z_{\widehat{H}_1}^{W_F} = Z_{\widehat{H}}^{W_F} \times Z_{\widehat{G}_1}^{W_F} / Z_{\widehat{G}}^{W_F}$ and $(Z_{\widehat{H}_1}^{W_F})^0 \subseteq (Z_{\widehat{G}_1}^{W_F})^0 (Z_{\widehat{H}}^{W_F})^0 Z_{\widehat{G}}^{W_F}$. This gives that the elliptic condition on H implies that on H_1.

For the other direction, $Z_{\widehat{H}} = Z_{\widehat{H}_1} \cap \widehat{G}$ gives that $Z_{\widehat{H}}^{W_F} = Z_{\widehat{H}_1}^{W_F} \cap \widehat{G}^{W_F}$ which gives $(Z_{\widehat{H}}^{W_F})^0 \subseteq (Z_{\widehat{H}_1}^{W_F})^0 \cap \widehat{G}^{W_F}$. Assuming $(Z_{\widehat{H}_1}^{W_F})^0 \subseteq Z_{\widehat{G}_1}^{W_F}$ and further using that $Z_{\widehat{G}_1} \cap \widehat{G} = Z_{\widehat{G}}$ implies $Z_{\widehat{G}_1}^{W_F} \cap \widehat{G}^{W_F} = Z_{\widehat{G}}^{W_F}$ finally giving $(Z_{\widehat{H}}^{W_F})^0 \subseteq Z_{\widehat{G}}^{W_F}$.

Part (b):

We are given G, endoscopic group H, and extension H_1 by T. There is a map $Z_G \hookrightarrow Z_H$ (see [KS99] pg. 53) so we can pullback the extension Z_{H_1} to an extension Z_{G_1} of Z_G by T.

Set $G_1 = Z_{G_1} \times G_{\mathrm{der}}/Z_{G^{\mathrm{der}}}$ as an algebraic group where the $Z_{G^{\mathrm{der}}}$ is embedded antidiagonally. Then since $G = Z_G \times G_{\mathrm{der}}/Z_{G^{\mathrm{der}}}$, G_1 is an extension of G by T. If H comes from data (s, ρ), then through the construction of the bijection in (a), (s, ρ) gives data for H_1 and is elliptic iff (s, ρ) is. $\square$

Consider H an endoscopic group of G and H_1 is a z-extension so it has simply connected derived subgroup. Let $(H_1, \mathcal{H}_1, s, \eta)$ be the quadruple for G_1 produced by part (b). Then the map ${}^L H_1 \to \mathcal{H}_1$ is an isomorphism so we actually do have an embedding ${}^L H_1 \hookrightarrow {}^L G_1$. This is the z-extension construction described in [LS87].

3.3. **Transfer.** Consider quadruple $(H, \mathcal{H}, s, \eta)$ for G over local or global K and associated z-extension (H_1, η_1). There is a transfer map

$$\begin{aligned}\mathcal{T} : \{&\text{strongly } G\text{-regular semisimple conjugacy classes in } H(K)\} \\ &\to \{\text{strongly regular stable conjugacy classes } G(K)\} \cup \{*\}\end{aligned}$$

where the $*$ is a dummy variable to allow maps that aren't necessarily defined everywhere. We say that $\gamma_H \in H(K)$ is a norm of $\gamma_G \in G(K)$ if $\mathcal{T}$ takes the conjugacy class of γ_H to that of γ_G. Respectively, $\gamma_{H_1} \in H_1(K)$ is a norm of something if its projection to $H(F)$ is.

3.3.1. *Local Transfer.* Now, consider local F_v. If strongly G-regular γ_{H_1} is a norm of strongly regular γ_G, a transfer factor $\Delta(\gamma_{H_1}, \gamma_G) = \Delta_G^{H_1}(\gamma_{H_1}, \gamma_G)$ can be defined (this is the content of sections $4.1 - 5.1$ in [KS99]). The factor is non-canonical up to a uniform constant. We recall some useful properties from [KS99, §5.1]:

- $\Delta(\gamma_{H_1}, \gamma_G)$ is 0 unless γ_{H_1} is a norm of γ_G.
- $\Delta(\gamma_{H_1}, \gamma_G)$ is also constant over the stable conjugacy class of γ_{H_1}
- Let $Z_{G_1} = Z_{H_1} \times_{Z_H} Z_G$. There exists a character λ_{η_1} on $Z_{G_1}(F_v)$ such that if $(z_1, z) \in Z_{G_1}(F_v)$,
$$\Delta_G^{H_1}(z_1\gamma_{H_1}, z\gamma_G) = \lambda_{\eta_1}^{-1}(z_1, z)\Delta_G^{H_1}(\gamma_{H_1}, \gamma_G).$$
In fact, λ_{η_1} even extends to a character on $G_1(F_v)$ (see the construction on pg. 53 in [KS99] or pg. 55 in [LS87]).
- Let the quadruple $(H, \mathcal{H}, s, \eta, \gamma_{H_1})$ correspond to triple (T, γ_G, κ). Then γ_{H_1} is a norm of γ_G. If γ'_G is a stable conjugate of γ_G,
$$\kappa(\gamma'_G)\Delta(\gamma_{H_1}, \gamma_G) = \Delta(\gamma_{H_1}, \gamma'_G).$$

Fix central character datum $(\mathfrak{X}, \chi)$ for G. Let $f \in \mathscr{H}(G(F_v), \chi_v)$. We say that $f^{H_1} \in \mathscr{H}(H_1(F_v), \chi_{H_1,v})$ matches f if

$$SO_{\gamma_{H_1}}(f^{H_1}) = \sum_{\gamma_G} \Delta(\gamma_{H_1}, \gamma_G) O_{\gamma_G}(f)$$

for all strongly G-regular $\gamma_{H_1} \in H_1(F_v)$ where γ_G ranges over representatives of unstable conjugacy classes such that γ_{H_1} is a norm of γ_G. Note that the right-hand side is a twisted orbital integral multiplied by an appropriate constant.

Since γ_{H_1} and γ_G are strongly regular, If T is a maximal torus for G and Z the extending torus defining H_1 from H, the orbital integrals have dimension $[G(F_v)][T(F_v)]^{-1}$ and $[H_1(F_v)][T(F_v)]^{-1}[Z(F_v)]^{-1} = [H(F_v)][T(F_v)]^{-1}$. Therefore, f^{H_1} nees to have dimensions $[G(F_v)][H(F_v)]^{-1}$.

A big theorem is that such an f^H always exists. The Archimedean case is from Shelstad in [She82] while he non-Archimedean case was reduced to the fundamental lemma (which will be discussed later) by Waldspurger in [Wal97]. Call such an f^H a transfer of f.

3.3.2. *Global Transfer.* If F is global, then the endoscopic datum determine local endoscopic datum at each place v. The lets us define a global transfer factor $\Delta_{\mathbb{A}}(\gamma_{H_1}, \delta_G)$ as the product of all the local transfer factors. [KS99, cor 7.3.B] gives that all the choices defining the local factors can be made consistently giving a canonical choice of global factor.

If $f \in \mathscr{H}(G, \chi)$ factors into local factors at each place, then transferring each of the local factors gives a transfer f^H satisfying a similar identity. By the fundamental lemma, this is unramified almost everywhere and is therefore an element of $\mathscr{H}(H_1, \chi_{H_1})$.

After lots of cohomology work, f^H can be shown to satisfy a global identity

$$\mathrm{SO}_{\gamma_{H_1}}(f^H) = O^{\kappa}_{\gamma_G}(f).$$

When $(H, \mathcal{H}, s, \eta, \gamma_H)$ corresponds to (T, γ_G, κ). This is [KS99, lem 7.3.C].

3.3.3. *Characters from Transfer.* By the above, endoscopy always defines a character on $Z_{H_1}(F_v)$. However, for v non-Archimedean, this actually extends to a character on $H_1(F_v)$. We will need this to state some bounds on non-Archimedean transfers later.

Fix such a v and assume wlog G has simply connected derived subgroup (possibly by taking a z-extension and using proposition 3.2.2). Take the extension G_1 of G as in proposition 3.2.2(b). Then G_1^{der} is an isogenous cover of G^{der} so the two are equal. The map η determines a character λ_{η_1} on $Z_{G_1}(F_v) = Z_{H_1}(F_v) \times_{Z_H(F_v)} Z_G(F_v)$. Since this lifts to a character on $G_1(F_v)$, it is actually a character on $G_1(F_v)/G_1^{\mathrm{der}}(F_v)$. If F is local then $H^1(F_v, G_1^{\mathrm{der}}) = 0$ since G_1^{der} is semisimple and simply connected. Therefore this is a character on $(G_1)_{\mathrm{ab}}(F_v)$ so let it correspond to L-parameter $\alpha : W_{F_v} \hookrightarrow {}^L(G_1)_{\mathrm{ab}}$.

Next

Lemma 3.3.1. *Let G be a reductive group over F_v. Then $Z^0_{\widehat{G}} = \widehat{G_{\mathrm{ab}}}$ as groups with W_{F_v}-action.*

Proof. Let G have maximal torus T. As W_F-modules, $X_*(\widehat{G_{\mathrm{ab}}}) = X^*(G_{\mathrm{ab}}) = X^*(T)^\Omega$ and $X_*(Z^0_{\widehat{G}}) = X_*(\widehat{T})^\Omega = X^*(T)^\Omega$. This equality of cocharacters induces an equality of torii. □

Since $\widehat{H}_1$ is a connected centralizer in $\widehat{G}_1$, we get a map $Z^0_{\widehat{G}_1} \hookrightarrow Z^0_{\widehat{H}_1}$. Since H_1 is endoscopic, the map is Galois-equivariant so it extends to a map ${}^L(G_{1,\mathrm{ab}}) \to {}^L(H_{1,\mathrm{ab}})$. Therefore α can be pushed forward and determines a character λ'_{H_1} on H_1.

Note that λ_{H_1} and λ'_{H_1} are equal on $Z_{G_1}(F_v)$ since the correspond to the same parameter of $Z_{G_1}(F_v)$. This common value is the character λ_{η_1} from before determining which Hecke algebra transfers land in. The discussion here simply shows that it extends to a character on H_v.

3.3.4. *A trick for computing transfers with z-extensions.* Most formulas for transfers in the literature only apply in the case when ${}^L H \cong \mathcal{H}$. To use these in the general case, consider the same quadruple and z-extension as before with $T \hookrightarrow H_1$ the extending torus. proposition 3.2.2(b) lets us find G_1 such that $(H_1, \mathcal{H}_1, s, \eta)$ is an endoscopic quadruple for G_1 with ${}^L H_1 \cong \mathcal{H}_1$. Let $\pi : G_1 \to G$ be the projection.

The key property is that

$$\Delta_{G_1}^{H_1}(\gamma_1, \delta_1) = \Delta_G^{H_1}(\gamma_1, \delta)$$

whenever $\delta_1 \in G_1(F)$ projects to $\delta \in G(F)$ and γ_1 is a norm of δ_1 (see [LS87] pg. 55). Therefore, given $f \in \mathscr{H}(G(F), \chi)$, let

$$f_1(g) = f \circ \pi(g)$$

for $g \in G_1(F)$. If f_1 and ${f_1}^{H_1}$ match, then for all appropriate γ_1, δ

$$SO_{\gamma_1}(f_1^{H_1}) = \sum_{\delta_1} \Delta_{G_1}^{H_1}(\gamma_1, \delta_1) O_{\delta_1}(f_1) = \sum_{\delta_1} \Delta_G^{H_1}(\gamma_1, \pi(\delta_1)) O_{\pi(\delta_1)}(f)$$

which is the condition for f and ${f_1}^{H_1}$ matching. Therefore we can compute f^{H_1} by transferring f_1.

As a sanity check, note that γ_1 being a norm of δ_1 is true iff $z\gamma_1$ is a norm of $z\delta_1$ for all $z \in Z_{G_1}$. In particular, if $x = (z_1, z) \in Z_{G_1}$ then

$$\begin{aligned}\Delta_{G_1}^{H_1}(x\gamma_1, x\delta_1) = \Delta_{G_1}^{H_1}(z_1\gamma_1, x\delta_1) &= \Delta_G^{H_1}(z_1\gamma_1, z\pi(\delta_1)) \\ &= \lambda_{\eta_1}(x)^{-1} \Delta_G^{H_1}(\gamma_1, \pi(\delta_1)) = \lambda_{\eta_1}(x)^{-1} \Delta_{G_1}^{H_1}(\gamma_1, \delta_1).\end{aligned}$$

Therefore, the transfer factor transforms appropriately so that this transfer will be in $\mathscr{H}(H_1(F), \chi_H)$.

Beware that there is a small technical issue here. Theorems in the literature only give the existence of transfers of compactly supported functions. We get around this by finding a compactly supported function f' that averages to $f \circ \phi$ along the central character datum (see lemma 6.1.1 for example) and then transferring f'. We then average $(f')^{H_1}$ against the central character datum.

3.4. **Stabilization.** Using all the above and with much work, $I^{G,\chi}_{\mathrm{disc},t}(f)$ can be stabilized. In other words, it can be expanded as

$$I^G_{\mathrm{disc},t}(f) = \sum_{H \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, H) \widehat{S}^{\tilde{H}, \chi_{\tilde{H}}}_{\mathrm{disc},t}(f^{\tilde{H}})$$

for some choice of Z-extensions. Here $\widehat{S}^{\tilde{H}, \chi_{\tilde{H}}}_{\mathrm{disc},t}$ is a stable distribution on $\mathscr{H}(\tilde{H}, \chi_{\tilde{H}})$ depending only on $t, \tilde{H}$. We will not use any properties of S except that it is stable. There is no explicit construction of f^H in general, so its known properties will be cited as needed.

ι has an explicit formula. Recall there was a notion of automorphisms of quadruples $(H, \mathcal{H}, s, \eta)$ by elements $g \in \widehat{G}$. Let $\Lambda(H, \mathcal{H}, s, \eta)$ be the image of $\mathrm{Aut}(H, \mathcal{H}, s, \eta) \to \mathrm{Out}(\widehat{H})$. Then

$$\iota(G, H) = \Lambda(H, \mathcal{H}, s, \eta)^{-1} \tau(G) \tau(H)^{-1}$$

where τ is the Tamagawa number.

3.5. **Some Properties.**

3.5.1. *Endoscopy and root data.* The following is a summary of the relation between roots data of endoscopic groups and the original group:

Lemma 3.5.1. *Let G be a reductive group over global or local field K, $(H, \mathcal{H}, \eta, s)$ an elliptic endoscopic quadruple and $(\tilde{H}, \tilde{\eta})$ a z-extension. Then the following hold:*

1. *Let T_H be a maximal torus for $H_{\overline{K}}$. Then there is a maximal torus T of $G_{\overline{K}}$ and an isomorphism $T_H \to T$. The choice of T and the map are unique up to $G_{\overline{K}}$-conjugacy.*
2. *The positive (co)roots of (H, T_H) can be chosen to be a subset of those of (G, T) through $T_H \to T$.*
3. *For any root of α of (H, T_H), $s_\alpha \in \Omega(H)$ is the same as $s_\alpha \in \Omega(G)$ through the isomorphism $T_H \to T$*
4. *The positive roots of $(\tilde{H}, T_{\tilde{H}})$ can be chosen to be a subset of those of (G, T) through $T_{\tilde{H}} \to T_H \to T$.*
5. *The Weyl action on the roots of $(\tilde{H}, T_{\tilde{H}})$ restricts to that on (H, T_H) through $X^*(T_H) \hookrightarrow X^*(T_{\tilde{H}})$.*

Proof. (1)-(3) are done in [Kot86, §3.1] and [LS87, §1.3].

To deal with $\tilde{H}$, let the extension be $1 \to T \to \tilde{H} \to H \to 1$. Every maximal torus of $\tilde{H}$ is the preimage of one of H so $X^*(T_H)$ maps into the corresponding $X^*(T_{\tilde{H}})$. Since in the sequence

$$0 \to \operatorname{Lie} T \to \operatorname{Lie} \tilde{H} \to \operatorname{Lie} H \to 0,$$

$\operatorname{Lie} T$ maps into the center, the roots of $\tilde{H}$ have to be the images of those of H. Choose a Borel $\tilde{B}$ containing B_H to get containment of positive roots. The last statement on Weyl groups comes from $\Omega_{(H,T_H)} \cong N_H(T)/Z_H(T)$. □

Be careful that this lemma says nothing about the Galois actions on the roots. We will not need that information and getting it requires G to be quasisplit. Also beware that this does not give that the simple roots of H are a subset of the simple roots of G or that the coroots of $\tilde{H}$ are a subset of the coroots of G.

3.5.2. *Real endoscopic characters.* As another computational tool, the character κ has a nice form in the real case. If G is a real group, there is an isomorphism

$$\Omega_{\mathbb{C},G}/\Omega_{\mathbb{R},G} \to \mathfrak{D}(\mathbb{R}, T\backslash G).$$

An endoscopic character κ therefore be extended $\Omega_{\mathbb{C}}(G)$. [Lab11, §IV.1] gives that the extension is left-$\Omega_{\mathbb{C},H}$ invariant.

In addition, the composition

$$\Omega(B_K) \to \Omega_{\mathbb{C},G} \to \Omega_{\mathbb{C},G}/\Omega_{\mathbb{R},G}$$

is a bijection. This gives a bijection between any regular $\Pi_{\mathrm{disc}}(\xi)$ and $\mathfrak{D}(\mathbb{R}, T\backslash G)$ that depends on the choice of B_K.

This interpretation of κ will be used when computing transfers of pseudocoefficients.

4. The Hyperendoscopy Formula

Here we will describe Ferrari's hyperendoscopy formula with some modifications in the case where groups without simply connected derived subgroup appear in the hyperendoscopic paths. Using this formula may appear a little bizarres since it may seem more reasonable to try to directly mimic the work of [Art89] on the stable distributions $SO^H(f^H)$ like the main result [Pen19].

The advantage of using hyperendoscopy is that we can directly apply the work already done in [ST16] instead of proving slightly different bounds for the slightly different terms appearing in the stable trace formula. There are two disadvantages: first, it gives worse constants in bounds, but the constants were already not explicit due to the model theory bounds that go into them. Second, hyperendoscopy requires extending Shin-Templier's results to groups with fixed central character datum, but this is interesting in its own right. In addition, the hyperendsocopic formula itself may be a useful tool for studying future forms of the invariant trace formula that, unlike [Pen19], do not have a reasonable stabilization.

4.1. **Raw Formula.** Recalling the key trick from [Fer07], rearrange the stabilized trace formula:

$$\widehat{S}^{G^{\mathrm{qs}}}_{\mathrm{disc},t}(f^{G^{\mathrm{qs}}}) = I^G_{\mathrm{disc},t}(f) + \sum_{\substack{H\in\mathcal{E}_{\mathrm{ell}}(G)\\ H\neq G^{\mathrm{qs}}}} (-\iota(G,H))\widehat{S}^{\tilde{H}}_{\mathrm{disc},t}(f^{\tilde{H}})$$

where G^{qs} is the quasisplit form of G. We want to continue this expansion inductively to get a formula in terms of I_{disc} for the various groups. The result in [Fer07] uses endoscopic triples, seemingly assuming that if a group has simply connected derived subgroup, then so do all its endoscopic groups. This is not true as there can be SO_{2k} factors in endoscopic groups of Sp_{2n} (see [Wal10, §1.8]). Nevertheless, with a little more work, a formula more-or-less equivalent to Ferrari's can be derived.

Inductively substituting in the expansions for $\widehat{S}^{\tilde{H}}_{\mathrm{disc},t}(f^{\tilde{H}})$ since the $\tilde{H}$ are all quasisplit gives something like

$$\widehat{S}^{G^{\mathrm{qs}}}_{\mathrm{disc},t}(f^{G^{\mathrm{qs}}}) = I^G_{\mathrm{disc},t}(f) + \sum_{\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_\mathcal{H}}}_{\mathrm{disc},t}(f^{\mathcal{H}}).$$

Because of the non-canonical z-extensions, the notation defining the indexing set becomes somewhat painful. We will find a nicer set to index over later.

Definition. A consistent choice of length-1 raw endoscopic paths for G is a set $\mathcal{HE}^0_{\mathrm{ell}}(G)_1$ consisting of pairs (H,z) where H ranges over proper isomorphism classes in $\mathcal{E}_{\mathrm{ell}}(G)$ and z is a choice of z-pair for H.

Given a consistent choice of length-$(n-1)$ raw hyperendoscopic paths $\mathcal{HE}^0_{\mathrm{ell}}(G)_{n-1}$, a consistent choice of length-n hyperendoscopic paths is a set $\mathcal{HE}^0_{\mathrm{ell}}(G)_n$ consisting of tuples $(\mathcal{H},H,z)$ where $\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)_{n-1}$, H ranges over proper isomorphism classes in $\mathcal{H}_{\mathrm{ell}}(\mathcal{H})$ (overloading notation so that $\mathcal{H}$ also refers to the group in the last z-pair of $\mathcal{H}$), and z is a choice of z-pair for H.

A consistent choice of raw hyperendoscopic paths $\mathcal{HE}^0_{\mathrm{ell}}(G)$ is an (inductively-chosen) consistent choice of $\mathcal{HE}^0_{\mathrm{ell}}(G)_n$ for all $n>0$.

The sum is over a choice of $\mathcal{HE}^0_{\mathrm{ell}}(G)$. If $\mathcal{H}\in\mathcal{HE}^0_{\mathrm{ell}}(G)$, let $n_\mathcal{H}$ be its length. As shorthand, we will sometimes write

$$\mathcal{H} = (H_1, H_2, \cdots, H_{n_\mathcal{H}})$$

where H_n is the group in the z-pair for the nth step in the path. As further shorthand, $\mathcal{H}$ will sometimes be overloaded to refer to $H_{n_\mathcal{H}}$. For indexing purposes,

$H_0 = G$. Similarly define:

$$\iota(G,\mathcal{H}) = (-1)^{n_{\mathcal{H}}} \prod_{i=1}^{n_{\mathcal{H}}} \iota(H_{i-1}, H_i) \qquad f^{\mathcal{H}} = (\cdots (f^{H_1})^{H_2 \cdots})^{H_{n_{\mathcal{H}}}}.$$

Note that $f^{\mathcal{H}}$ is not canonical and the choice of $f^{\mathcal{H}}$ needs to be consistent with the choice of $f^{\mathcal{H}'}$ where $\mathcal{H}'$ is $\mathcal{H}$ truncated by removing the last step. Finally, a hyperendoscopic path $\mathcal{H}$ determines central character datum $(\mathfrak{X}_n, \chi_n)$ for each H_n.

This expansion of course only works if the paths are all finite. This holds:

Lemma 4.1.1. *Every element of $\mathcal{HE}^0_{\mathrm{ell}}(G)$ has $n_{\mathcal{H}} \le \mathrm{rank}_{\mathrm{ss}}\, G$*

Proof. Consider the quadruple $(H_i, \mathcal{H}, s_i, \eta_i)$ of H_{i-1}. The group $\widehat{H}_i$ is a centralizer of $s_i \in \widehat{H}_{i-1}$ that isn't $\widehat{H}_{i-1}$ since $H_{i-1} \neq H_i^{\mathrm{qs}}$. This has semisimple rank smaller than $\widehat{H}_{i-1}$ from which the result follows. □

The key point then is that

$$I^G_{\mathrm{disc},t}(f) + \sum_{\mathcal{H} \in \mathcal{HE}^0_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_{\mathcal{H}}}}_{\mathrm{disc},t}(f^{\mathcal{H}})$$

is a stable distribution in $f^{G^{\mathrm{qs}}}$. Finally, since G^{qs} corresponds to the trivial endoscopic character, if $f^{G^{\mathrm{qs}}} = f_1^{G^{\mathrm{qs}}}$, then f, f_1 have the same stable orbital integrals. Setting this equal for two such functions:

Proposition 4.1.2 ([Fer07, prop 3.4.3] corrected)**.** *Let f and f_1 be functions on $G(\mathbb{A})$ that have the same stable orbital integrals. Then*

$$I^G_{\mathrm{disc},t}(f) = I^G_{\mathrm{disc},t}(f_1) + \sum_{\mathcal{H} \in \mathcal{HE}^0_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{H_{n_{\mathcal{H}}}}_{\mathrm{disc},t}((f_1 - f)^{\mathcal{H}}).$$

4.2. **Simplifying Hyperendoscopic Paths.** To control which groups appear, it is nice to have an easier definition of hyperendoscopic path.

Definition. An endoscopic path for G is a sequence $(Q_1, \dots, Q_n)$ where $Q_1 \in \mathcal{E}_{\mathrm{ell}}(G)$ and $Q_i \in \mathcal{E}_{\mathrm{ell}}(H_{i-1})$ for $i > 1$ where H_i is the group in Q_{i-1} (note that if two endoscopic quadruples are isomorphic, then so are their groups).

We use the same notation for endoscopic paths as for raw endoscopic paths. The set of endoscopic paths for G will be called $\mathcal{HE}_{\mathrm{ell}}(G)$.

Definition. A z-pair path for an endoscopic path $(Q_1, \dots, Q_n)$ is a sequence of z-pairs $(\tilde{Q}_1, \dots, \tilde{Q}_n)$ where

- $\tilde{Q}_1 = (\tilde{H}_1, \tilde{\eta}_1)$ is a choice of z-pair for Q_1.
- For $i > 1$ assume we have already chosen $Q_1, \dots, Q_{i-1}$. We get a quadruple Q'_i for H_{i-1} through repeated applications of the bijection from lemma 3.2.2(a) down through the Q_i (it will be clear that H_{i-1} can be produced from the group in Q_{i-1} by a sequence of central extensions by induced torii). Then $\tilde{Q}_i = (\tilde{H}_i, \tilde{\eta}_i)$ should be a z-pair for Q'_i.

If $\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)$ with z-pair path $\tilde{\mathcal{H}}$, we will sometimes overload notation and use $\tilde{\mathcal{H}}$ to denote that last group $\tilde{H}_n$ in the path. If $(\mathfrak{X}, \chi)$ is a central character datum for G, we will also let $(\mathfrak{X}_{\tilde{\mathcal{H}}}, \chi_{\tilde{\mathcal{H}}})$ be the induced datum on $\tilde{\mathcal{H}}$. We can also define $\iota(G, \tilde{\mathcal{H}})$ and transfers $f^{\tilde{\mathcal{H}}}$ similarly.

As in the definition of raw hyperendoscopic paths, we can similarly inductively define a consistent choice of z-pair paths for all elements of $\mathcal{HE}_{\mathrm{ell}}(G)$.

Lemma 4.2.1. *Choose a consistent set of z-pair paths $\tilde{\mathcal{H}}$ for $\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)$. Then the set of combined data $\{[\mathcal{H}, \tilde{\mathcal{H}}] : \mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)\}$ concatenated properly form a consistent set of raw hyperendoscopic paths for G.*

Proof. We show this inductively on length. For length 1, this works by definition. For longer length, we use lemma 3.2.2(a): if we know this for length i and H_i is the ith group in $\mathcal{H}$, the corresponding H_i' in the corresponding raw endoscopic path has the same possible "next steps"—the elliptic quadruples of the two are in bijection. □

Finally

Lemma 4.2.2. *Let $\tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}'$ be two different z-extensions for hyperendoscopic path H. Let $f \in \mathscr{H}(G, \chi)$ for some central character datum $(\mathfrak{X}, \chi)$. Then the two terms $S^{\tilde{\mathcal{H}}}_{\chi_{\tilde{\mathcal{H}}}}(f^{\tilde{\mathcal{H}}})$ and $S^{\tilde{\mathcal{H}}'}_{\chi_{\tilde{\mathcal{H}}'}}(f^{\tilde{\mathcal{H}}'})$ are equal. In addition $\iota(G, \tilde{\mathcal{H}}) = \iota(G, \tilde{\mathcal{H}}')$*

Proof. First, let G be a group, H an endoscopic group, and $f \in \mathscr{H}(G, \chi)$ for some χ. Let $(\tilde{H}, \tilde{\eta})$ and $(\tilde{H}', \tilde{\eta}')$ be two z-pairs. Then part of the formalism of the stable trace formula gives that $S^{\tilde{H}}_{\chi_{\tilde{H}}}(f^{\tilde{H}}) = S^{\tilde{H}'}_{\chi_{\tilde{H}'}}(f^{\tilde{H}'})$. By definition, $\iota(G, \tilde{H}) = \iota(G, H) = \iota(G, \tilde{H}')$.

Second, if G_1 is a z-extension of G and f_1 the pullback of f to some $\mathscr{H}(G_1, \chi_1)$ where χ_1 is the pullback of χ, it induces extension H_0 of H according lemma 3.2.2(a). We can find a z-pair (H_1, η_1) of H such that H_1 is a z-extension of H_0. By a similar argument to section 3.3.4, $f^{H_1} = f_1^{H_1}$ and $\chi_{H_1} = (\chi_1)_{H_1}$. Therefore $S^{H_1}_{(\chi_1)_{H_1}}(f_1^{H_1}) = S^{H_1}_{\chi_{H_1}}(f^{H_1})$. Since Tamagawa measures are products of Tamagawa measures of factors, $\iota(G, H_1) = \iota(G, H) = \iota(G_1, H_1)$ by the explicit formula.

The result follows from an induction alternating on these two steps. □

Define $\iota(G, \mathcal{H})$ to be the common value of all the the $\iota(G, \tilde{\mathcal{H}})$. In total, we can choose whichever z-extensions we want and ignore the consistency condition:

Theorem 4.2.3 (The Hyperendoscopy Formula)**.** *Let f and f_1 be functions on $G(\mathbb{A})$ that have the same stable orbital integrals. Then*

$$I^G_{\mathrm{disc},t}(f) = I^G_{\mathrm{disc},t}(f_1) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G, \mathcal{H}) I^{\tilde{\mathcal{H}}}_{\mathrm{disc},t}((f_1 - f)^{\tilde{\mathcal{H}}})$$

where $\tilde{\mathcal{H}}$ is a choice of z-extension path for $\mathcal{H}$ and where we suppress the central character datum.

4.3. **Central Characters from Hyperendoscopy.** Let $\mathcal{H}$ be a hyperendoscopic path for G with z-extension $\tilde{\mathcal{H}}$ corresponding to sequence of groups and embeddings $(\tilde{H}_i, \eta_i)$. We can wlog assume that $H_0 = G$ has simply connected derived subgroup by taking further extensions. Then we can inductively define character on each $(H_i)_v$:

- χ_1 is the character λ_{η_1} on $(\tilde{H}_1)_v$ defined by η_1 as in section 3.3.3.
- Let χ_i' be the character on $(\tilde{H}_{i+1})_v$ coming from character χ_i on $(\tilde{H}_i)_v$ as in section 3.3.3. Let λ_{i+1} be the character on $(H_i)_v$ determined by η_{i+1}. Then set $\chi_{i+1} = \chi_i' \lambda_{i+1}$.

From all the previous discussion, we know χ_i are the characters such that given central character datum $(\mathfrak{X}, \chi)$ and $f \in \mathscr{H}(G, \chi)$, the successive transfers $f^{\tilde{H}_i}$ lie in $\mathscr{H}(G, (\mathfrak{X}_{\tilde{H}_i}, \chi\chi_i))$.

4.4. **Remarks on usage.** Some notes for using this:

- Beware that the transfers $(f_1 - f)^{\mathcal{H}}$ must be chosen explicitly, since the stable orbital integrals of $(f^{H_1})^{H_2}$ depend on the standard orbital integrals of f^{H_1}. Care should be taken in these choices since the ease of evaluating I_{disc} depends much on properties of f^{H_1} that are not determined by stable orbital integrals.
- As a sum of distributions, the sum over $\mathcal{E}_{\text{ell}}(H_i)$ can be infinite. However, for any particular f only finitely many terms are non-zero. Nevertheless, the number of such terms depends on the choices of $f^{\mathcal{H}}$ and can be arbitrarily large. Thankfully, if we choose the $f^{\mathcal{H}}$ so that they stay unramified outside of a finite set of places S, then there is a finite set of terms depending only on S that are non-zero. See lemma 5.6.1.
- If we can choose the f^H to be cuspidal, we don't need to worry that this formula is only in terms of I_{disc} instead of I_{spec}.
- If each of the H_i in path $\mathcal{H}$ are unramified, we can choose $\tilde{\mathcal{H}}$ to only have unramified groups since z-extensions can be chosen to have the same splitting field as the original group.

5. Lemmas on transfers

5.1. **Formulas for Archimedean Transfer.** This section will compute transfers of pseudocoeffcients. We take the Whittaker normalization of transfer factors as in [She10] and [Lab11]. Because pseudocoefficients already have the correct dimensions, we don't need to fix Haar measures.

Recall the parametrization of discrete series in section 2.2.1. Now let $(H_\infty, \mathcal{H}, \eta, s)$ be an endoscopic quadruple of G_∞. Fix an elliptic maximal torus T and let and κ the corresponding endoscopic character on Ω_G.

5.1.1. *Trivial z-Extension case.* We will first work out the formula for transfers in the case where $\mathcal{H} \cong {}^L H$ where we don't need a z-extension. To start,

Lemma 5.1.1. *Unless all elliptic tori G_∞ are transfers of elliptic torii of H_∞, transfers of pseudocoefficients can be taken to be* 0.

Proof. See lemma 3.2 in [She10] or the computation of κ-orbital integrals on page 186 of [Kot90]. □

Therefore, we can choose isomorphic maximal torii T_H and T of $H_{\mathbb{C}}$ and $G_{\mathbb{C}}$ respectively that are both elliptic over $\mathbb{R}$. The Weyl chambers of (H, T_H) are a coarser partition than those of (G, T) by lemma 3.5.1. Therefore, we can choose a positive Weyl chambers for H that contains a chosen one for G. Let B_H and B_G be the corresponding Borel subgorups. Let $\rho' = \rho_G - \rho_H$ be the half-sum of positive roots of G that aren't roots of H.

The transfer of pseudocoefficients is worked out in [Kot90, §7]. Special cases are worked out in terms of roots in [Lab11, §IV.3]. For full generality when ρ' isn't a character of T, we have to use a corrected transfer factor from [She82, pg 396] as worked out in [Fer07]. This involves an $\Omega(H)$-invariant $\mu^* = \mu^*_{G,H}$ such that

$\mu^* - \rho'$ is a character of T. The μ^* is determined by the exact chosen isomorphism ${}^L H \to \mathcal{H}$.

Proposition 5.1.2. *We can take*

$$(f_{\pi_G(\lambda)})^H = \sum_{\omega_* \in \Omega_*} \kappa(\omega_*)\epsilon(\omega^*\omega_0) f_{\pi_H(\omega_*\lambda - \mu^*)}$$

where $\omega_0^{-1}\lambda$ is B-dominant and Ω_* is the set of representatives w of $\Omega(H)\backslash\Omega(G)$ such that $w\lambda$ is B_H-dominant.

As a sanity check, note that if $A_{G,\infty} \in \mathfrak{X}$, then ellipticity forces $A_{H_1,\infty} \in \mathfrak{X}_H$.

[Fer07] explicitly computes the extension to hyperendoscopy: let $\Omega(G,H)$ be a set of representatives w of $\Omega(H)\backslash\Omega(G)$ such that $w\mu$ is B_H dominant for any μ that is B_G dominant. Reindexing $\omega_* = \omega_1\omega_0^{-1}$

$$(f_{\pi_G(\lambda)})^H = \sum_{\omega_1 \in \Omega(G,H)} \kappa(\omega_1\omega_0^{-1})\epsilon(\omega_1) f_{\pi_H(\omega_1\omega_0^{-1}\lambda - \mu^*)}.$$

Next, note that the Euler-Poincare function φ_λ has the same stable orbital integrals as the pseudocoefficient $f_{\pi_H(\lambda+\rho_H)}$. Let $\mu = \omega_0^{-1}\lambda - \rho_G$ so that $\pi_G(\lambda)$ becomes $\pi_G(\mu,\omega_0)$. Then

Corollary 5.1.3. *We can take*

$$(f_{\pi_G(\mu,\omega_0)})^H = \sum_{\omega_1 \in \Omega(G,H)} \kappa(\omega_1\omega_0^{-1})\epsilon(\omega_1)\varphi_{\omega_1(\mu+\rho_G)-\rho_H-\mu^*}.$$

Next, since κ is $\Omega_{\mathbb{R}}$-right invariant,

$$\sum_{\omega_0 \in \Omega(B_K)} \kappa(\omega_1\omega_0^{-1}) = \sum_{[\omega]\in\Omega_{\mathbb{R}}\backslash\Omega_{\mathbb{C}}} \kappa(\omega_1\omega^{-1}) = \sum_{[\omega]\in\Omega_{\mathbb{C}}/\Omega_{\mathbb{R}}} \kappa(\omega_1\omega) = \sum_{[\omega]\in\Omega_{\mathbb{C}}/\Omega_{\mathbb{R}}} \kappa(\omega)$$

where it doesn't matter which representatives ω we choose. Therefore, averaging over $\omega_0 \in \Omega(B_K)$,

Corollary 5.1.4. *We can take*

$$(\varphi_\mu)^H = \bar\kappa \sum_{\omega_1 \in \Omega(G,H)} \epsilon(\omega_1)\varphi_{\omega_1(\mu+\rho_G)-\rho_H-\mu^*}$$

where $\bar\kappa = \bar\kappa_{G,H}$ is the average value of κ over $\Omega_{\mathbb{C}}/\Omega_{\mathbb{R}}$.

5.1.2. *General case.* For $\mathcal{H} \ncong {}^L H$, we use the trick in section 3.3.4. Let $\varphi : (G_1)_\infty \to G_\infty$ be the surjection (since it is coming from a z-extension): if f is a function on G_∞, we choose $f^{H_1} = (f \circ \phi)^{H_1}$.

Given elliptic torii T_{G_1} and T_{H_1} as before, we can also get elliptic torus T_G by taking images under the z-extensions. The function $\varphi : (G_1)_\infty \to G_\infty$ gives a map $\phi^* : X^*(G_\infty, T_G) \hookrightarrow X^*((G_1)_\infty, T_{G_1})$. Then $f_\pi(\lambda) \circ \phi = f_{\pi(\phi^*\lambda)}$ so we can still use the above formulas in the general case as long as we treat λ as an element of $X^*(G_1, T_{G_1})$.

Note that the character λ_{H_1} shows up through the weight μ^*—each may be used to compute the other (not that we've explicitly described either here).

5.1.3. *Hyperendoscopic Transfers.* To simplify notation, for any weight μ of a group G, endoscopic group H, and $\omega \in \Omega(G,H)$ as before, let

$$T_{G,H}(\mu,\omega) = \omega(\mu+\rho_G) - \rho_H - \mu^*_{G,H}.$$

As in the previous section, we interpret μ as an character of G_1 corresponding to the chosen z-extension H_1.

For any hyperendoscopic path $\mathcal{H} = (H_i)_{0\le i\le n}$, let

$$\Omega(\mathcal{H}) = \prod_{i=1}^n \Omega(H_{i-1},H_i) \qquad \bar\kappa_{\mathcal{H}} = \prod_{i=2}^n \bar\kappa_{H_{i-1},H_i}.$$

For $\omega = (\omega_i)_{i\le i\le n} \in \Omega(\mathcal{H})$ let

$$\epsilon(\omega) = \prod_{i=1}^n \epsilon(\omega_i)$$

and let

$$T_{\mathcal{H}}(\mu,\omega) = T_{H_{n-1},H_n}(\cdots T_{G,H_1}(\mu_1,\omega_1)\cdots,\omega_n)$$

be the composition of all the T_{H_{i-1},H_i}. This gives

Proposition 5.1.5. *We can take*

$$(f_{\pi_G(\mu,\omega_0)})^{\mathcal{H}} = \bar\kappa_{\mathcal{H}} \sum_{\omega\in\Omega_{\mathcal{H}}} \kappa_{G,H_1}(\omega_1\omega_0^{-1})\epsilon(\mu)\varphi_{T_{\mathcal{H}}(\mu,\omega)}$$

with the terms defined as in the above paragraph.

Note that all the coefficients in the sum have norm 1 and define $\Xi_{\mu,\mathcal{H}}$ to be the set of $T_{\mathcal{H}}(\mu,\omega)$ for $\omega \in \Omega(\mathcal{H})$.

5.2. **Bounds on Archimedean Transfers.** Here are few lemmas on the terms that appear in the formula. For μ a weight of G define

- $m(\mu) = m_G(\mu) = \min_{\alpha\in\Phi^+(G)}\langle\alpha,\mu+\rho_G\rangle$
- $n(\mu) = n_G(\mu) = \min_{\alpha\in\Phi^+(G)}\langle\alpha,\mu\rangle$
- $\dim\mu = \dim_G(\mu)$ is the dimension of the finite dimensional representation with highest weight μ.

Lemma 5.2.1. *If μ is a weight of G and $\mathcal{H}$ as before, then for all $\mu' \in \Xi_{\mu,\mathcal{H}}$, $n_G(\mu') \ge n_{\mathcal{H}}(\mu)$. In particular, μ' is regular if μ is.*

Proof. In the situation where H is just an endoscopic group, consider arbitrary $\mu' = \omega(\mu+\rho_G) - \rho_H - \mu^*$ for appropriate $\omega \in \Omega_G$. Consider $\alpha \in \Phi^+(H)$. Since μ^* is invariant under $\Omega(H)$, $\langle\mu^*,\alpha\rangle = 0$ so

$$\langle\mu',\alpha\rangle = \langle\omega\mu,\alpha\rangle + \langle\omega\rho_G - \rho_H,\alpha\rangle.$$

Next, ρ_G is the sum of the fundamental weights so it is a regular weight. This implies that $\omega\rho_G$ is too. Therefore, for all $\beta \in \Phi^+(G)$, $\beta^\vee(\omega\rho_G) \in \mathbb{Z}\setminus\{0\}$. In particular, since $\omega\rho_G$ is B_H-dominant, for $\alpha \in \Phi^+(H)$, $\alpha^\vee(\omega\rho_G) \ge 1$. Computing, if α is in addition simple

$$\alpha^\vee(\omega\rho_G - \rho_H) \ge 1 - \alpha^\vee(\rho_H) = 0$$

so $\omega\rho_G - \rho_H$ is B_H-dominant. This gives

$$\langle\mu',\alpha\rangle \ge \langle\omega\mu,\alpha\rangle.$$

To finish this one-step case

$$n_H(\mu') = \min_{\alpha \in \Phi^+(H)} \langle \mu', \alpha \rangle \geq \min_{\alpha \in \Phi^+(H)} \langle \omega\mu, \alpha \rangle = \min_{\alpha \in \Phi^+(H)} \langle \mu, \omega^{-1}\alpha \rangle.$$

All the terms in the last two minimums have to be positive. However, μ is B_G-dominant so this means the $\omega^{-1}\alpha$ are all in $\Phi^+(G)$ giving

$$n_H(\mu') \geq \min_{\alpha \in \Phi^+(G)} \langle \mu, \alpha \rangle = n_G(\mu).$$

Finally, for an arbitrary endoscopic path, inductively continue this argument through each step. □

Lemma 5.2.2. *If μ is a weight of G and $\mathcal{H}$ as before, then for all $\mu' \in \Xi_{\mu,\mathcal{H}}$*

$$\frac{\dim_{\mathcal{H}}(\mu')}{\dim_G(\mu)} = O(m_G(\mu)^{-1})$$

with the implied constant only depending on G and $\mathcal{H}$.

Proof. This follows from the Weyl character formula. If H is just an endoscopic group, let $\mu' = \omega(\mu + \rho_G) - \rho_H - \mu^*$ for appropriate $\omega \in \Omega_G$. Using that μ^* pairs to zero with any root of H

$$\frac{\dim_H(\mu')}{\dim_G(\mu)} = \frac{\prod_{\alpha \in \Phi^+(G)} \langle \alpha, \rho_H \rangle}{\prod_{\alpha \in \Phi^+(H)} \langle \alpha, \rho_G \rangle} \frac{\prod_{\alpha \in \Phi^+(H)} (\langle \alpha, \omega\mu \rangle + \langle \alpha, \omega\rho_G \rangle)}{\prod_{\alpha \in \Phi^+(G)} (\langle \alpha, \mu \rangle + \langle \alpha, \rho_G \rangle)}.$$

The first fraction is a constant depending only on G and H. So are the second terms in the products in the second fraction. A priori, the $\langle \alpha, \omega\mu \rangle = \langle \omega^{-1}\alpha, \mu \rangle$ are a subset of the $\langle \pm\beta, \mu \rangle$ for $\beta \in \Phi^+(G)$. However, since they all have to be positive since $\omega\mu$ is B_H-dominant, they are actually a subset of the $\langle \beta, \mu \rangle$. Denote by A the subset of such β. Then

$$\frac{\dim_H(\mu')}{\dim_G(\mu)} = C \frac{\prod_{\alpha \in A} (\langle \alpha, \mu \rangle + O(1))}{\prod_{\alpha \in \Phi^+(G)} (\langle \alpha, \mu \rangle + O(1))} = O\left(\prod_{\alpha \in \Phi^+(G) \setminus A} \langle \alpha, \mu \rangle^{-1} \right)$$

using that the pairings are bounded below by a constant. Bounding the pairings again by $m_G(\mu)$ this is $O(m_G(\mu)^{|\Phi^+(H)| - |\Phi^+(G)|})$. Finally, since endoscopic groups have smaller rank, they don't have the same root data as the original group so this difference has to be negative.

Inducting on this argument for each step of the hyperendoscopic path $\mathcal{H}$ finishes the proof after a quick check that the $m_{\mathcal{H}_i}(\mu') = O(m_G(\mu))$. □

5.3. **Truncated Hecke algebras.** We now move on to the unramified finite places. Fix place v so that G_v is quasisplit and choose (B, T) a Borel and maximal torus defined over F_v. By G_v being quasisplit, all such choices are conjugate and T automatically contains a maximal split torus A. Furthermore, Ω_F can be identified with the fixed points Ω^{W_F} and therefore the Weyl group of the relative root system of rational roots in $X^*(A)$. Let K_v be a hyperspecial subgroup from a hyperspecial point in the apartment corresponding to A.

Eventually, we will evaluate $I_{\mathrm{geom}}(f)$ up to some error bounds which depend on how big the support of the finite part of f is. To precisely measure this size, we slightly modify the notion of truncated Hecke algebras as in [ST16, §2].

Recall then that the elements $\tau_\lambda^G = \mathbf{1}_{K_v \lambda(\varpi) K_v}$ for a chosen unformizer ϖ and $\lambda \in X_*(A)^+$ generate $\mathscr{H}r(G_v, K_v)$. Pick a basis $\mathcal{B}$ for the $X_*(A)$ and define norm:

$$\|\lambda\|_{\mathcal{B}} = \max_{\omega \in \Omega}(\text{biggest } \mathcal{B}\text{-coordinate of } \omega\lambda).$$

For $\lambda \in X_*(A)$. Define truncated Hecke algebra

$$\mathscr{H}(G, K)^{\leq \kappa, \mathcal{B}} = \langle \tau_\lambda^G : \|\lambda\|_{\mathcal{B}} \leq \kappa \rangle.$$

It turns out (see [ST16, §2]) that for any two $\mathcal{B}, \mathcal{B}'$, $\|\lambda\|_{\mathcal{B}} = \Theta(\|\lambda\|_{\mathcal{B}'})$. All the bounds we use will depend on κ only up to an unspecified constant. Therefore we can suppress the $\mathcal{B}$.

There is also a truncated Hecke algebra with central character data: choose an $(\mathfrak{X}, \chi)$ such that χ is unramified. In the case we care about, $\mathfrak{X}$ is a subtorus of Z_{G_v}. Let $A_{\mathfrak{X}}$ be its split part. Define

$$\mathscr{H}(G, K_v, \chi)^{\leq \kappa, \mathcal{B}} = \langle \tau_\lambda^G : \|\lambda + \zeta\|_{\mathcal{B}} \leq \kappa \text{ for some } \zeta \in X_*(A_{\mathfrak{X}}) \rangle \cap \mathscr{H}(G, K, \chi).$$

Note that for $x \in K_v \lambda(\varpi) K_v$ and $z \in \mathfrak{X}$, then there is $k \in K_v$ and $\zeta \in X_*(A_{\mathfrak{X}})$ such that $z = \zeta(\varpi) k$ implying $zx \in K_v(\lambda + \zeta)(\varpi) K_v$. Therefore this is a reasonable, non-empty intersection.

5.3.1. *A useful projection.* Working with the basis of τ_λ^G, it is sometimes useful to consider the following maps. First, there is a map $Q : \chi \mapsto \sum_{\omega \in \Omega_G} \omega\chi$ on $X_*(T)$. This sends every coroot of G to 0. Normalizing Q by $|\Omega_G|^{-1}$ gives a projection P on $X_*(T)_{\mathbb{Q}}$. Note that this projection is onto $X_*(Z(G))_{\mathbb{Q}}$ since Weyl-invariant cocharacters are the same as central cocharacters (they pair to zero with every root).

$X_*(A)$ embeds into $X_*(T)$ as the W_F invariants.

Lemma 5.3.1. *Let $\lambda \in X_*(A)$. Then $Q\lambda \in X_*(A)$.*

Proof. It suffices to show this for $P\lambda$. The map P is an orthogonal projection onto W_F-invariant $X_*(Z(G))_{\mathbb{Q}}$ with respect to a W_F-invariant inner product. Therefore it commutes with W_F and sends W_F invariants to W_F invariants. □

Therefore, we can consider Q, P as maps of $X_*(A), X_*(A)_{\mathbb{Q}}$ respectively. The kernel of P is the span of the roots of G so the kernel in $X_*(A)_{\mathbb{Q}}$ is V_F where V_F be the span of $\{\alpha^\vee | \alpha \in \Phi_F^*\}$ inside $X_*(A)_{\mathbb{Q}}$.

5.4. **Formulas for Unramified non-Archimedean Transfers.**

5.4.1. *The Fundamental Lemma.* The fundamental lemma allows computation of unramified non-Archimedean transfers (it is actually sufficient to show the existence of all non-Archimedean transfers). We will eventually use to to control which $\mathscr{H}(H_v, K_{H,v}, \chi_{H,v})^{\leq \kappa}$ transfers end up being in. Use the notation T, A, K analogous to the last section.

As explained in [ST16, §2.2], the Satake transform gives two isomorphisms

$$\varphi_G : \mathscr{H}(G_v, K_v) \to \mathscr{H}(A, A \cap K_v)^{\Omega_F} \to \mathbb{C}[X_*(A)]^{\Omega_F}.$$

We mention that this implies:

Lemma 5.4.1. *$\widehat{G}_v^{\mathrm{ur}}$ can be identified with $\Omega_F \backslash \widehat{A}$. The tempered part is $\Omega_F \backslash \widehat{A}_c$ where $\widehat{A}_c$ is the maximum compact torus on $\widehat{A}$.*

Proof. A result in representation theory of p-adic groups says that unramified representations of G_v are the same as characters of $\mathscr{H}(G_v, K_v)$ and therefore characters on $\mathbb{C}[X_*(A)]^{\Omega_F}$. These are the same as elements of $\Omega_F \backslash \widehat{A}$. Tempered representations need to correspond to tempered characters of $\mathscr{H}(G_v, K_v)$ which forces the element to be in $\widehat{A}_c$. □

There are more implications: let ${}^L G^{\mathrm{ur}}$ be defined like ${}^L G$ except that the semidirect product is only with $W_{F_v}^{\mathrm{ur}}$. Define $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$ to be the algebra of trace characters of representations of ${}^L G^{\mathrm{ur}}$ restricted to $(\widehat{G} \rtimes \mathrm{Frob})_{\mathrm{ss}}$. There is a third isomorphism

$$\mathcal{T} : \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \to \mathbb{C}[X_*(A)]^{\Omega_F}$$

that takes a representation π to a function on $\widehat{T}$ given by $a \mapsto \mathrm{tr}_\pi(a \rtimes \mathrm{Frob})$. This function can be shown to factor through $\widehat{A}$ (see [Bor79, prop 6.7]).

If we have a map $\eta : {}^L H^{\mathrm{ur}} \hookrightarrow {}^L G^{\mathrm{ur}}$, we get a pullback map $b_\eta : \mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})] \to \mathbb{C}[\mathrm{ch}({}^L H^{\mathrm{ur}})]$. We pick the Whittaker normalization for transfer factors and choose the measures μ^{can} on H_v and G_v that give K_v and $K_{H,v}$ volume 1.

Theorem 5.4.2 (Full Fundamental Lemma)**.** *Let G be an unramified reductive group over local field F_v. Let $(H, \mathcal{H}, \eta, s)$ be an elliptic endoscopic quadruple for G such that $\mathcal{H} \cong {}^L H$. Then, for $f \in \mathscr{H}(G, K)$ we can take*

$$f^H = \begin{cases} \varphi_H^{-1} \circ b_\eta \circ \varphi_G(f) & H \text{ unramified} \\ 0 & H \text{ ramified} \end{cases}.$$

Here we recall that if H_v and G_v are unramified, then the embedding $\mathcal{H} \hookrightarrow {}^L G$ descends to one $\mathcal{H}^{\mathrm{ur}} \hookrightarrow {}^L G^{\mathrm{ur}}$. In addition, H being unramified allows us to pick an $\eta : {}^L H \xrightarrow{\sim} \mathcal{H}$ that also descends to unramfied L-groups. The pullback b_η is defined through such an η.

Proof. The statements defining η come from the construction of $\mathcal{H}$ and the proof of 7.2A in [KS99].

The ramified H_v case is by [Kot86, §7.5]. Otherwise, it is reduced in [Hal95] to proving the result for just $\mathbf{1}_K$. This was further reduced to a fundamental lemma for Lie algebras in [Wal97] which was finally proven in [Ngô10]. [Hal95] removes a restriction on the size of the residue field of F_v. □

5.4.2. *Representations of ${}^L G^{\mathrm{ur}}$.* To compute with the fundamental lemma, we need to describe representations of ${}^L G^{\mathrm{ur}}$. As a start:

Lemma 5.4.3. *Representations π of ${}^L T^{\mathrm{ur}}$ are all of the following form: let λ be a character of $\widehat{T}$ up to W_F^{ur}-action and $\alpha \in \mathbb{C}^\times$. Then*

$$\chi_{\lambda,\alpha} = \bigoplus_{\gamma \in W_F / \operatorname{Stab} \lambda} V_{\gamma\lambda}$$

where each V_μ is a 1-dimensional space with a chosen generator v_μ on which $\widehat{T}$ acts through μ. If $\operatorname{Stab} \lambda$ is generated by $\mathrm{Frob}^{i(\lambda)}$, then $\mathrm{Frob}^{i(\lambda)}$ acts by $v_\lambda \mapsto \alpha v_\lambda$. Finally, $\mathrm{Frob}(v_\lambda) = \beta_\lambda v_{\mathrm{Frob}(\lambda)}$ for some constants β_λ. (Note that by scaling v_μ, wlog all the β_λ are 1 except one that is α).

Proof. Decompose π into eigenspaces V_μ for $\widehat{T}$. We can compute that, $\gamma V_\mu \subseteq V_{\gamma\mu}$ for $\gamma \in W_F^{\mathrm{ur}}$. Let γ_0 generate $\operatorname{Stab} \lambda$ for some non-empty V_λ. Then γ_0 acs as an element of $\mathrm{GL}(V_\lambda)$. Let v_λ be a chosen eigenvector of γ_0 with eigenvalue α. The vectors v_λ generates a $\pi_{\lambda,\alpha}$ inside π. $\square$

Beware that this parametrization depends on the splitting $W_F \hookrightarrow {}^LT$. Next

Proposition 5.4.4. *Representations $\pi_{\lambda,\alpha} = \pi^{{}^LG}_{\lambda,\alpha}$ of ${}^LG^{\mathrm{ur}}$ are parametrized by dominant-weight representations $\chi_{\lambda,\alpha}$ of ${}^LT^{\mathrm{ur}}$*

Proof. The is by [Kos61, pg 375-376] . We have that ${}^LT^{\mathrm{ur}}$ is the same as H^+ in the reference because the action of W_F fixes the Borel B used to define ${}^LG$. The construction is similar to that for connected complex Lie groups: $\pi_{\lambda,\alpha}$ forms a highest weight space on which the actions of the root subgroups of $\widehat{G}$ are determined. Together $\widehat{G}$ and ${}^LT^{\mathrm{ur}}$ generate ${}^LG^{\mathrm{ur}}$. $\square$

In fact, if $\pi^{\widehat{G}}_\mu$ is the representation corresponding to highest weight μ of $\widehat{G}$, then each of the $V_{\gamma\lambda} \subseteq V_{\lambda,\alpha}$ generates a copy of $\pi^{\widehat{G}}_{\gamma\lambda}$ under the action of $\widehat{G}$. The representation $\pi_{\lambda,\alpha}|_{\widehat{G}}$ therefore decomposes as a direct sum of the $\pi^{\widehat{G}}_{\gamma\lambda}$ and any $\gamma \in W_F$ sends $\pi^{\widehat{G}}_\mu$ to $\pi^{\widehat{G}}_{\gamma\mu}$. The exact description of this map in complicated but can be computed by the following trick: For any $\gamma \in \Gamma$, the μ coefficient of tr_π restricted to $\widehat{T} \rtimes \gamma$ is the trace of $1 \rtimes \gamma$ acting on the μ-weight space $V^{\lambda,\alpha}_\mu$ of $\pi_{\lambda,\alpha}$. This trace can be computed by Kostant's character formula [Kos61, thm 7.5].

As an easier way to think about this parametrization, let F_n be the splitting field for G. The groups $\mathrm{Gal}(F_n/F)$ and $\Omega_{\mathbb{C}}$ together generate a group C in automorphisms of the set of roots. Inside this, $\mathrm{Gal}(F_n/F)$ is the stabilizer of the positive Weyl chamber and $\Omega_{\mathbb{C}}$ acts simply on the Weyl chambers so $\mathrm{Gal}(F_n/F) \cap \Omega_{\mathbb{C}} = 1$. In addition, $\Omega_{\mathbb{C}}$ is normal since T is fixed by Galois. Therefore $C = \Omega_{\mathbb{C}} \rtimes \mathrm{Gal}(F_n/F)$. The λ parametrizing $\pi_{\lambda,\alpha}$ can be thought of as a C-orbit. This decomposes into $\Omega_{\mathbb{C}}$ orbits representing the constituent $\pi^{\widehat{G}}_{\gamma\lambda}$.

5.4.3. *Some Bases.* We also need to describe some bases of the various spaces.

If ϖ is a chosen uniformizer for $\mathcal{O}_F$ and $X_*(A)^+$ a chosen Weyl chamber, then the functions

$$\tau^G_\lambda = \mathbf{1}_{K\lambda(\varpi)K} \qquad \lambda \in X_*(A)^+$$

form a basis for $\mathscr{H}(G,K)$ (the corresponding double cosets partition G by the Cartan decomposition).

$\mathbb{C}[X_*(A)]^{\Omega_F}$ contains functions

$$\chi_\lambda = \frac{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_F(\sigma)\sigma(\lambda\cdot\rho)}{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_F(\sigma)\sigma(\rho)} \in \mathbb{C}[X_*(A)]^{\Omega_F}$$

for $\lambda \in X_*(A)^+$. We write the addition in $X_*(A)$ multiplicatively for clarity. Here, $\rho = \rho_F$ is the half-sum of the positive rational roots of $\widehat{G}$ which is the same as the half-sum of all positive roots since rational roots are sums over orbits of roots. We recall that Ω_F is the same as the Weyl group for the relative root system of rational roots of G by quasiplitness. The sgn_F here are -1 to the power the number of positive rational roots sent to negative roots (If the rational roots form a reduced root system, this is just the standard sgn on Ω_F).

If the relative root system is reduced, these are the standard characters from Weyl's character formula and are studied in [Kat82]. In the non-reduced case, these are the twisted characters from [CCH19, thm 1.4.1] or [Hai18, thm 7.9]. Either way, χ_λ for dominant weighs λ form a basis for $\mathbb{C}[X_*(A)]^{\Omega_F}$.

Finally,

Lemma 5.4.5.

$$\mathcal{T}(\pi_{\lambda,\alpha}) = \begin{cases} \alpha\chi_\lambda & \lambda \in X_*(A) \\ 0 & else \end{cases}.$$

Proof. This is just stated in the proof of [ST16] lemma 2.1. We give details here since there seems to be a minor mistake (that is irrelevant to all the work there and here) when λ isn't in $X_*(A)$. This is also proven as [CCH19, thm 1.4.1] and as [Hai18, thm 7.9] in a slightly different form.

We use Kostant's character formula [Kos61, thm 7.5]. Using the notation there, $a = t \rtimes \mathrm{Frob}$ for some $t \in \widehat{T}$ and W_a is the W_F^{ur} invariants in $\Omega_{\mathbb{C}}$ which is Ω_F. Also, let $\Phi_\sigma = \Phi^+_{\mathbb{C}} \cap \sigma(-\Phi^+_{\mathbb{C}})$ for $\sigma \in \Omega_{\mathbb{C}}$ where $\Phi^+_{\mathbb{C}}$ is the set of positive roots. Since Frob preserves a pinning, it acts by a permutation on some diagonal basis of $\bigoplus_{\phi\in\Phi_\sigma} \mathfrak{g}_{-\phi}$. Therefore, the determinant of the action of a is

$$\chi_1^\sigma(a) = \mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \prod_{\varphi\in\Phi_\sigma} \varphi^{-1}(t).$$

In addition $\chi_1^\delta(a)$ for δ the rep of ${}^L T$ parametrized by (λ, α) is $\alpha\lambda(t)$ if λ is fixed by Frob and 0 otherwise (the 0 otherwise case is what is missing in [ST16]). By a [LS87, pg 15], we can find representations of $\sigma \in W_a$ fixed by Frob so we get that $\chi_\sigma^\delta(a) = \alpha\sigma\lambda(t)$.

In total, the trace in the non-zero case is

$$\alpha \frac{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\,\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma})\sigma\lambda(t) \prod_{\varphi\in\Phi_\sigma} \varphi^{-1}(t)}{\sum_{\sigma\in\Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\,\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma}) \prod_{\varphi\in\Phi_\sigma} \varphi^{-1}(t)}$$
$$= \alpha \frac{\rho(t)^{-1} \sum_{\sigma\in\Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\,\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma})\sigma\lambda(t)\sigma\rho(t)}{\rho(t)^{-1} \sum_{\sigma\in\Omega_F} \mathrm{sgn}_{\mathbb{C}}(\sigma)\,\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma})\sigma\rho(t)}.$$

$\mathrm{sgn}_{\mathbb{C}}$ here is the sign character for $\Omega_{\mathbb{C}}$: the number of all positive roots sent to negative roots. This differs from the sgn_F in the formula for χ_λ by a factor of $\mathrm{sgn}(\mathrm{Frob}|_{\Phi_\sigma})$ through an argument breaking up Φ_σ into Frob-orbits and noting that each rational root is a sum over an orbit. Therefore we are done.

Note that the 0 case can be done more easily by thinking about the action in block matrix form with respect to the subspaces $\pi^{\widehat{G}}_{\gamma\lambda}$ and noticing that all diagonal blocks are 0. □

The key consequence of this is that the $\pi_{\lambda,1}$ for $\lambda \in X_*(A)$ form a basis for $\mathbb{C}[\mathrm{ch}({}^L G^{\mathrm{ur}})]$.

5.5. **Bounds on Unramified Transfers.**

5.5.1. *Trivial z-extension case.* As in the Archimedean case, we consider the trivial z-extension case first.

Recall the notation for various bases of spaces related to the Satake isomorphism. From [Gro98] and [Kat82] (again, see [Hai18, §7] or [CCH19, §1] for the non-split case), we can write

$$\varphi_G(\tau^G_\lambda) = \chi_\lambda + \sum_{\substack{\mu \in X^*(\widehat{A})^+ \\ 0 \leq \mu < \lambda}} b^G_\lambda(\mu)\chi^G_\mu$$

$$\varphi_H^{-1}(\chi^H_\nu) = q^{-\langle \nu, \rho_H \rangle}\tau^H_\nu + \sum_{\substack{\xi \in X^*(\widehat{A}_H)^+ \\ 0 \leq \xi < \nu}} q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi)\tau^H_\xi.$$

for some constants b and d. Here $\mu \leq \lambda$ means that there is some non-negative integer linear combination of roots $\alpha^\vee$ for $\alpha \in \Phi^*$ equal to $\lambda - \mu$.

Lemma 5.5.1. *$d^G_\lambda(\mu)$ and $q^{-\langle \lambda, \rho_H \rangle} b^G_\lambda(\mu)$ are polynomial in the norm $\|\mu\|$.*

Proof. First, lets show this for $d^G_\lambda(\mu)$. By above, we can ignore the $\lambda = \mu$ case. Otherwise, we apply [ST16, lem 2.2]. There is a small issue here: this lemma depends on the main result of [Kat82] which only works when the root system is reduced. Nevertheless, [Hai18, thm 7.10] and [CCH19, thm 1.9.1] provide an appropriate substitute in the non-reduced case.

[ST16, lem 2.2] bounds $d^G_\lambda(\mu)$ by $|\Omega_G|$ times the size of the set of tuples $(c_{\alpha^\vee})$ for α a positive root such that $\sum_{\alpha^\vee} c_{\alpha^\vee}\alpha^\vee = \mu - \lambda$ (since both μ and λ are in the positive Weyl chamber, the max in the lemma is achieved for the trivial element of the Weyl group). Looking at the coordinate of μ in the direction used to define positivity, every $\alpha^\vee$ is positive in this coordinate, so some weighted sum of the $c_{\alpha^\vee}$ is bounded. This implies that the number of tuples is only polynomial in this coordinate of μ. The result follows.

For $b^G_\lambda(\mu)$, note that the $q^{-\langle \beta, \rho_H \rangle} d^G_\alpha(\beta)$ for $\alpha, \beta \leq \lambda$ form an upper-triangular matrix with dimension polynomial in the size of λ. Then $b^G_\beta(\alpha)$ are coordinates of the inverse of this matrix. Making a change of variables, the $q^{-\langle \beta, \rho_H \rangle} b^G_\beta(\alpha)$ are the coordinates of the inverse of the matrix with coordinates $d^G_\alpha(\beta)$ so these are bounded by a polynomial in κ by solving through back substitution. □

It remains to understand the map b_η. This is computed exactly in terms of certain partition functions in [CCH19, §2.3] but we only need bounds so we do something slightly different and much simpler. For $\mu \in X_*(A)$ define coefficients $c_\mu(\nu)$ by

$$\pi^{\widehat{G}}_\mu|_{\widehat{H}} = \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \leq \nu \leq \mu}} c_\mu(\nu)\pi^{\widehat{H}}_\nu.$$

The $c_\nu(\mu)$ are in particular bounded by the dimension of $\pi^{\widehat{G}}_\mu$ so they are polynomial in the size of μ by the Weyl character formula.

Proposition 5.5.2. *As elements of $\mathbb{C}[\mathrm{ch}({}^L H^{\mathrm{ur}})]$*

$$b_\eta(\pi^{{}^L G}_{\mu,1}) = \bigoplus_{\substack{\nu \in X_*(A_H)^+ \\ 0 \leq \nu \leq \mu}} \alpha_\mu(\nu)c_\mu(\nu)\pi^{{}^L H}_{\nu,1}$$

where A_H is the maximal split torus of H contained in some maximal T_H contained in a rational Borel B_H and we consider $\mu \in X_*(T_H) = X_*(T)$ as dominant element by taking its Weyl-translate in the positive Weyl chamber.

For notational convenience, let $\Gamma = W^{\mathrm{ur}}_{F_v}$. There exists $t_\eta \in (Z^\Gamma_{\widehat{G}})^0$ depending only on η such that the constants $\alpha_\mu(\nu)$ satisfy two properties:

- $|\alpha_\mu(\nu)| \le |\nu(t_\eta)|$.
- *Let Y_G be the maximal split torus in Z_G^0. If $\zeta \in X_*(Y_G)$, then $\alpha_{\mu+\zeta}(\nu+\zeta) = \zeta(t_\eta)\alpha_\mu(\nu)$.*

Before starting the proof, note that all such T_H are isomorphic and that the map $X_*(T_H) \to X_*(T)$ is unique up to Weyl element. Therefore this is well defined.

Proof. Decomposition: To avoid confusion, $\Gamma_{\widehat{G}}$ is Γ acting on $\widehat{G}$ and visa versa for $\widehat{H}$ when it isn't clear from context. First,

$$b_\eta(\pi^{^LG}_{\mu,1})|_{\widehat{H}} = (\pi^{^LG}_{\mu,1}|_{\widehat{G}})|_{\widehat{H}} = \bigoplus_\gamma \pi^{\widehat{G}}_{\gamma\mu}|_{\widehat{H}} = \bigoplus_\gamma \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \le \nu \le \mu}} c_\mu(\nu)\pi^{\widehat{H}}_{\gamma_G\nu}$$

where the $\gamma\mu$ index the $\Gamma_{\widehat{G}}$-orbit of μ in $X_*(T)$. Note that $c_\mu(\nu)$ is constant on $\Gamma_{\widehat{G}}$ orbits and $\Omega_{\mathbb{C}}(\widehat{G})$ orbits.

The $\Gamma_{\widehat{H}}$-action is the composition of that of $\Gamma_{\widehat{G}}$ with conjugation by elements of $N_{\widehat{G}}(T)$ so since G is quasisplit, $\Gamma_{\widehat{H}}$ acts on $\widehat{T}_H$ through a subgroup W' with $\mathrm{Gal}(F_n/F) \subseteq W' \subseteq C_H \subseteq C_G$ (recall notation $C_G = \Gamma \rtimes \Omega_{\mathbb{C}}(\widehat{G})$). This implies that $c_\mu(\nu)$ is constant on $\Gamma_{\widehat{H}}$-orbits.

Therefore, the sum over such an orbit of the $c_\mu(\nu)\pi^{\widehat{H}}_\nu$ decomposes into $c_\mu(\nu)$ different $\pi^{^LH}_{\nu,\alpha_{i,\mu}}$ for possibly different $\alpha_{i,\mu}$. In total

$$b_\eta(\pi^{^LG}_{\mu,1}) = \bigoplus_{\substack{\nu \in X_*(T_H)^+ \\ 0 \le \nu \le \mu}} \bigoplus_{i=1}^{c_\mu(\nu)} \pi^{^LH}_{\nu,\alpha_{i,\mu}(\nu)} = \bigoplus_{\substack{\nu \in X_*(A_H)^+ \\ 0 \le \nu \le \mu}} \left(\sum_{i=1}^{c_\mu(\nu)} \alpha_{i,\mu}(\nu)\right) \pi^{^LH}_{\nu,1}$$

as elements of $\mathbb{C}[\mathrm{ch}(^LH^{\mathrm{ur}})]$ and for some $\alpha_{i,\mu}(\nu) \in \mathbb{C}^\times$. Let $\alpha_\mu(\nu)$ be the average of the $\alpha_{i,\mu}(\nu)$.

Properties of $\alpha_\mu(\nu)$: It remains to show the two properties of $\alpha_\mu(\nu)$. Since all (B,T)-pairs in $\widehat{G}$ are conjugate, wlog take an inner automorphism of ^LG so that $(\widehat{B}_H, \widehat{T}_H)$ is the pullback of $(\widehat{B}, \widehat{T})$. The map η determines a cocycle $c_\gamma \in C^1(\Gamma_{\widehat{G}}, \widehat{G})$ by $\eta(1 \rtimes \gamma) = c_\gamma \rtimes \gamma$. We then have that $\alpha_i(\nu)$ is the factor by which $c_{\mathrm{Frob}} \rtimes \mathrm{Frob}$ acts on the highest weight space V of the ith $\pi^{\widehat{H}}_\nu$.

There exists n such that the conjugation action of $(c_{\mathrm{Frob}} \rtimes \mathrm{Frob})^n$ on $X^*(\widehat{T})$ is trivial. Since this action also fixes a pinning of H, we must have

$$(c_{\mathrm{Frob}} \rtimes \mathrm{Frob})^n = z_0 \rtimes \mathrm{Frob}^n$$

for some $z_0 \in Z_{\widehat{H}}$. By the lemma below, we know $1 \rtimes \mathrm{Frob}$ acts trivially on V. Therefore, $\alpha_{i,\mu}(\nu)^n = \nu(z_0)$.

Next, note that the $\Gamma_{\widehat{H}}$-action is generated by conjugation by $c_{\mathrm{Frob}} \rtimes \mathrm{Frob}$. This fixes z_0 so $z_0 \in Z^\Gamma_{\widehat{H}}$. We can wlog make n bigger so that z_0 is trivial in the finite group $\pi_0(Z^\Gamma_{\widehat{H}})$—in other words, we may wlog assume $z_0 \in (Z^\Gamma_{\widehat{H}})^0$. Then by ellipticity of H, $z_0 \in (Z^\Gamma_{\widehat{G}})^0$. Since this a complex torus, there then exists $t_\eta \in Z^0_{\widehat{G}}$ such that $t_\eta^n = z_0$ so taking nth roots, $|\alpha_{i,\mu}(\nu)| = |\nu(t_\eta)|$. Summing over i then produces the bound on the $\alpha_\mu(\nu)$.

To get the central character transformation, $\zeta \in X_*(Y_G)$ iff it is a Γ_G and Ω_G-invariant element of $X_*(T) = X^*(\widehat{T})$. Such characters lift to Γ-invariant characters of $\widehat{G}$ and therefore characters on ${}^L G$. For such ζ, $\pi_{\mu+\zeta,1} = \zeta \otimes \pi_{\mu,1}$ so

$$b_\eta(\pi_{\mu+\zeta,1}) = b_\eta(\zeta) \otimes b_\eta(\pi_{\mu,1}) = \zeta(c_{\mathrm{Frob}})\zeta|_{\widehat{H}} b_\eta(\pi_{\mu,1}).$$

Since $c_\mu(\nu)$ is 0 unless μ and ν have the same central character and since $c_{\mu+\zeta}(\nu+\zeta) = c_\mu(\nu)$, this implies that $\alpha_{\mu+\zeta}(\nu+\zeta) = \zeta(c_{\mathrm{Frob}})\alpha_\mu(\nu)$. Therefore we are done if all the choices defining t_η above are such that t_η has the same image in $\widehat{G}_{\mathrm{ab}}$ as c_{Frob}. □

The lemma used in this proof follows:

Lemma 5.5.3. *Let V_ν for $\nu \in X_*(A)$ be a weight space for $\pi^{{}^L G}_{\mu,\alpha}$ for $\mu \in X_*(A)$. Then $1 \rtimes \mathrm{Frob}$ acts as multiplication by α on V_ν.*

Proof. For any $\gamma \in W_F^{\mathrm{ur}}$, the trace of γ acting on V_ν is the coefficient of ν in $\operatorname{tr} \pi^{{}^L G}_{\mu,\alpha}$ restricted to $\widehat{T} \rtimes \gamma$. Let n be the splitting degree of G. The same computation as lemma 5.4.5 gives that this is $\alpha^{ni+1} \dim V_\nu$ for any $\gamma = \mathrm{Frob}^{ni+1}$. The only representation of $W_F^{\mathrm{ur}} \cong \mathbb{Z}$ with these traces sends 1 to scaling by α. □

The element t_η defines a function χ_η^{-1} on G_v by $K\lambda(\varpi)K \mapsto \lambda(t_\eta)$ for $\lambda \in X_*(A)$. Since t_η is central, if Q is the map on $X_*(A)$ summing over Ω_G-orbits, this is constant fibers of Q. In particular, since products of basis elements $\tau_\lambda^G \in \mathscr{H}(G_v)^{\mathrm{ur}}$ are a linear combination of $\tau^G_{\lambda'}$ for λ' in a single fiber, χ is a character of G. This is the character that corresponds to t_η considered as a Weyl-orbit in $\widehat{A}$ through the Satake isomorphism.

Furthermore, the relation $\alpha_{\mu+\zeta}(\nu+\zeta) = \zeta(t_\eta)\alpha_\mu(\nu)$ forces χ_η to be the character associated to η through transfer factors as in section 3.3.3. This all finally gives that the character on H_v determined by $K_H\lambda(\varpi)K_H \mapsto \lambda(t_\eta)$ for $\lambda \in X_*(A_H)$ is the same as the one from transfer factors.

In summary, we get

$$(\tau_\lambda^G)^H = \delta_H^G(\lambda)\tau_\lambda^H + \sum_{\substack{\xi \in X^*(\widehat{A}_H) \\ 0 \le \xi < \lambda}} a_\lambda(\xi)\tau_\xi^H$$

where

$$a_\lambda(\xi) = \sum_{\substack{\mu \in X^*(\widehat{A}) \\ \nu \in X^*(\widehat{A}_H) \\ \xi \le_H \nu \le_H \mu \le_G \lambda}} \alpha_\mu(\nu) b_\lambda^G(\mu) c_\mu(\nu) q^{-\langle \xi, \rho_H \rangle} d_\nu^H(\xi)$$

setting terms of the form $*_\mu(\mu) = 1$ here for ease of indexing. We also know that the $\alpha_\mu(\nu)$ can be bounded in terms of the character on H_v determined by η. This finally allows us to compute:

Lemma 5.5.4. *Let G be a reductive group over a global field and $(H, \mathcal{H}, \eta, s)$ an endoscopic quadruple that has a trivial z-extension. Let S be a finite set of places v such that*

- *G_v, H_v are unramified*
- *$|k_v|$ doesn't divide $|\Omega(G)|$*

Let $\chi_{\eta,S}$ be the product of the characters $\chi_{\eta,v}$ on H_v for $v \in S$ determined by η.

If $f \in \mathscr{H}(G(F_S), K_S)^{\le \kappa}$ with $\|f\|_\infty \le 1$, we can take $f^H \in \mathscr{H}(H(F_S), K_S)^{\le \kappa}$ such that $\|\chi_{\eta,S} f_S^H\|_\infty = O(q_{S_1}^{E\kappa}\kappa^{C|S|})$ for a constants C, E independent of f_S and

q_S. In addition, E can be chosen uniformly over all G in endoscopic paths from a fixed G'.

Proof. Use the notation from the previous discussion. For $s \in S$, f_s is then a linear combination of some of τ^G_λ. If τ^G_λ has a τ^H_ξ component then $\lambda - \xi$ is in particular a non-negative sum of roots of G. The number of such λ polynomial in κ. Therefore, if f^H_s is written as a linear combination of τ^H_ξ, the coefficient for τ^H_ξ is bounded by a sum of polynomially many $a_\lambda(\xi)$. Furthermore, all these ξ are smaller than λ.

Moving to what we are actually bounding, if t_η is as in the previous discussion, the corresponding coefficient in $\chi^{-1}_{\eta,s} f^H_s$ is bounded by a sum of polynomially many $\xi(t_\eta)^{-1} a_\lambda(\xi)$. For all $\alpha_\mu(\nu)$ appearing in the sum defining $a_\lambda(\xi)$,

$$|\xi(t_\eta)^{-1}\alpha_\mu(\nu)| \leq |\xi(t_\eta)^{-1}\nu(t_\eta)| = 1$$

since ξ and ν have the same Ω_G-orbit sum. In particular, if we define

$$a'_\lambda(\xi) = \sum_{\substack{\mu \in X^*(\widehat{A}) \\ \nu \in X^*(\widehat{A}_H) \\ \xi \leq_H \nu \leq_H \mu \leq_G \lambda}} b^G_\lambda(\mu) c_\mu(\nu) q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi),$$

then $|\xi(t_\eta)^{-1} a_\lambda(\xi)| \leq |a'_\lambda(\xi)|$.

It remains to bound the polynomially many summands in $a'_\lambda(\xi)$. Bounding each of these terms, the $c_\mu(\nu)$ are polynomial in how big μ is. By lemma 5.5.1, the term

$$b^G_\lambda(\mu) q^{-\langle \xi, \rho_H \rangle} d^H_\nu(\xi)$$

is a polynomial in the size of λ times a factor of $q^{-\langle \xi, \rho_H \rangle + \langle \lambda, \rho_G \rangle}$. Therefore, we roughly bound the entire product, $a_\lambda(\xi)$, by a polynomial in κ times a factor of $q^{-\langle \lambda, \rho_G \rangle}$

Finally, note that $\langle \lambda, \rho_G \rangle \leq \mathrm{rank}_{\mathrm{ss}}(G)\kappa$. Taking the product of f^H_s over $s \in S$ and setting $E = \mathrm{rank}_{\mathrm{ss}}(G)$ gives the result. □

Note that this lemma can be inductively applied through a hyperendoscopic path letting χ at each step be the character defined from a hyperendoscopic path as in section 4.3.

5.5.2. *General case.* Starting as in the Archimedean case argument in section 5.1.2, consider z-pair (H_1, η_1) for H. The extension H_1 induces an extension G_1 such that H_1 is an endoscopic group for G_1 by proposition 3.2.2. If $\varphi : (G_1)_v \to G_v$ is the projection, we have that $f^{H_1} = (f \circ \varphi)^{H_1}$ for any H_1 on G (interpreted as before).

If H is ramified, then all κ-orbital integrals are still 0 so this transfer is 0.

If H is unramified, T can be pulled back to a maximal torus T_1 of G_1 and A can be pulled back to A_1. By lemma 3.2.1 the extending torus Z is wlog unramified so G_1 is too. As explained in [Kot86, §7], the reductive model of G corresponding to the chosen hyperspecial $K_{G,v}$ gives a reductive model of G_1 so we can find a hyperspecial $K_{G_1,v}$ that surjects onto $K_{G,v}$. The map φ induces $\varphi_* : X_*(A_1) \to X_*(A)$ so

$$\varphi(K_{G_1,v}\lambda(\varpi)K_{G_1,v}) = K_{G,v}\varphi_*\lambda(v)K_{G,v}.$$

Therefore,

$$\tau^G_\lambda \circ \varphi = \sum_{\lambda' \in \varphi_*^{-1}(\lambda)} \tau^{G_1}_{\lambda'}$$

and the transfer can be computed by the fundamental lemma.

We describe the transfer of τ^G_0 as an example computation:

Lemma 5.5.5. *Use the notation above. Then we can take*

$$(\tau_0^G)^{H_1} = \sum_{\lambda \in X_*(A_Z)} \chi_{\eta_1}(\lambda(\varpi))\tau_\lambda^{H_1}.$$

Here A_Z is the split part of the extending torus Z and χ_{η_1} is the character on Z_{G_1} determined by η_1.

Finally, we get an extension of lemma 5.5.4: that transfers from $\mathscr{H}(G_v, K_v, \chi)$ land in $\mathscr{H}(H_v^1, K_{H,v}, \chi\chi_{\eta_1})$ with the same bound.

5.6. Controlling Endoscopic Groups Appearing.

Lemma 5.6.1. *Let G be a reductive group over global field F that is cuspidal at infinity together with central character datum $(\mathfrak{X}, \chi)$ such that $\mathfrak{X}$ contains $A_{G,\infty}$. Let $f = \eta_\xi \otimes f^\infty$ be a function on $G(\mathbb{A})$ where η_ξ is some EP-function with central character matching χ. Let R be finite set of places containing those on which f^∞ or G are ramified. Then there are a finite number of elliptic endoscopic quadruples $(H, \mathcal{H}, \eta, s)$ up to equivalence for which $I_{\mathrm{disc}}(f^{H_1}) \neq 0$ for (all) z-extensions H_1. For each such H_1:*

- *H_1 is cuspidal at infinity and $\mathfrak{X}_{H_1}$ contains $A_{H_1,\infty}$.*
- *f^{H_1} is unramified outside of R and H_1 can be chosen to be.*
- *χ_{H_1} is unramified outisde of R*

Proof. If H_1 isn't cuspidal at infinity, then $I_{\mathrm{disc}}(g) = 0$ for any g with infinite part that is a EP function by the previous section. By corollary 5.1.4 and lemma 5.1.1, f^H is either a linear combination of such functions or 0. As before, we remark that $\mathfrak{X}_{H_1} \supseteq A_{H_1,\infty}$ due to ellipticity.

If H is ramified outside of R, then by the full fundamental lemma together with the trick to compute transfers on z-extensions, $f^{H_1} = 0$. Otherwise, by lemma 3.2.1, H_1 can be chosen to be unramified outside R so f^{H_1} is unramified outside of R by the full fundamental lemma again. The group H_1 being unramified outside of R further implies that χ_{H_1} is too.

Finiteness of the sum is implicit in the stabilization of the trace formula. Repeating the argument here, note that the roots of $H_{\overline{K}}$ are a subset of those of $G_{\overline{K}}$. Therefore, there are a finite number of possibilities for $H_{\overline{K}}$ and the splitting field of H has degree $\leq \Omega_G$. Since the splitting field is also unramified outside of R, there are a finite number of choices for it. This leaves only a finite number of choices for H.

To get finitely many quadruples it then suffices to show there are finitely many choices for $s \in (Z(\widehat{H})/Z(\widehat{G}))^{W_F}$. For this, $Z(H)^{W_F}/Z(G)^{W_F}$ is finite by ellipticity and $Z(H)^{W_F}$ having finitely many connected components. Therefore $(Z(\widehat{H})/Z(\widehat{G}))^{W_F}$ is finite by finiteness of a cohomology group. □

Note that this lemma can be inductively applied through a hyperendoscopic path.

6. Simple Trace Formula with Central Character

6.1. **Set-up.** To apply the hyperendoscopy formula, we will need two generalizations of the simple trace formula: first, allowing central characters and second, allowing pseudocoefficients at infinite places on the spectral side. We use a slightly convoluted and indirect argument to avoid having to go into too many technicalities of Arthur's distributions $I(f,\gamma)$ and $I(f,\pi)$:

Fix central character datum $(\mathfrak{X},\chi)$ and let χ_0 be the restriction of χ to $A_{G,\mathrm{rat}}$. We first define a variant of $I_{\mathrm{disc},\chi}$ to relate it to I_{geom,χ_0}. Let $\mathfrak{X}_F = \mathfrak{X}\cap Z(F)$. There is a map

$$\mathscr{H}(G,\chi_0)\to\mathscr{H}(G,\chi) : f(g)\mapsto \bar{f}_\chi(g) := \int_{\mathfrak{X}/A_{G,\mathrm{rat}}} f(gz)\chi(z)\,dz.$$

Lemma 6.1.1. *$f\mapsto \bar{f}_\chi$ is surjective.*

Proof. Let $h\in\mathscr{H}(G,\chi)$. There exists compact $U\subseteq G(\mathbb{A})/A_{G,\mathrm{rat}}$ such that $U\mathfrak{X}$ contains the support of h. Let c be a cutoff function: compactly supported, continuous, non-negative real valued, and positive on U. Then the function

$$m(g) = \int_{\mathfrak{X}/A_{G,\mathrm{rat}}} c(gz)\,dz$$

is continuous and non-zero on the support of h. If we take $f = m^{-1}ch$, then $\bar{f}_\chi = h$. □

We follow a strategy from [KSZ]. For any $\star\in\{\mathrm{geom},\mathrm{disc},\mathrm{spec}\}$, also define distributions on $\mathscr{H}(G,\chi_0)$:

$$I'_{\star,\chi}(f) = \frac{1}{\mathrm{vol}(\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_\mathbb{Q}\backslash\mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\star,\chi_0}(f_z)\,dz$$

where $f_z : g\mapsto f(gz)$. We of course have that

$$I'_{\mathrm{geom},\chi} = I'_{\mathrm{spec},\chi}.$$

In addition, if f is cuspidal, then so is f_z for any central z so

$$I'_{\mathrm{spec},\chi}(f) = I'_{\mathrm{disc},\chi}(f).$$

For our case, we can only consider central character datum where $A_{G,\infty}\subseteq\mathfrak{X}$. Fix $(\mathfrak{X},\chi)$ for the rest of this section and let χ_0 be the restriction of χ to $A_{G,\mathrm{rat}}$. The generalized simple trace formula can then be developed in three steps:

(1) Find a generalized pseudocoefficient φ so that $\bar{\varphi}_\chi$ is the pseudocoefficient φ_π and traces agains φ can be computed easily
(2) Compute $I'_{\mathrm{spec},\chi}(\varphi\otimes f^\infty)$ and show this equals $I_{\mathrm{spec},\chi}(\varphi_\pi\otimes\overline{(f^\infty)}_\chi)$. Both these are small modifications of Arthur's original spectral side argument together with an extra lemma of Vogan.
(3) Sum over φ to get a generalized Euler-Poincare function η. Evaluate $I_{\mathrm{geom},\chi_0}(\eta\otimes f^\infty)$ and average to get a formula for $I'_{\mathrm{geom},\chi}(\eta\otimes f^\infty)$.

To see how everything depends on Haar measures, φ will have dimension $[G_\infty/A_{G,\mathrm{rat}}]^{-1}$ and f^∞ will have dimension $[\mathfrak{X}^\infty]^{-1}$ so that both sides of our final formula will have dimension $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

6.2. **Generalized Pseudocoefficients.** We first need to define a version of truncated/generalized pseudocoefficients from [HL04, §1.9] in the real case. This actually can be done slightly more explicitly than the p-adic case. A lot of this section is probably implicit somewhere in [CD90].

For this section only, let $G = G(\mathbb{R})$ be a group over $\mathbb{R}$ with discrete series mod center. All other variables ($\mathfrak{a}$, A_G, etc.) will refer to real versions. There is a map

$$H_G : G(\mathbb{R}) \to \mathfrak{a}_*^G : \lambda(H_G(\gamma)) = \log|\lambda(\gamma)| \text{ for all } \lambda \in \mathfrak{a}_G^*.$$

It is well known that this maps $A^0 = A_G(\mathbb{R})^0$ isomorphically to a_*^G so since A^0 is central we get a splitting $G(\mathbb{R}) = G(\mathbb{R})^1 \times A^0$ where $G(\mathbb{R}^1)$ is the kernel of H_G.

Any character $\lambda \in (\mathfrak{a}_G^*)_\mathbb{C}$ of $\mathfrak{a}_*^G$ corresponds to the character $e^{\lambda(H_G(\gamma))}$ on A^0 and therefore G through this isomorphism. The unitary characters correspond to $\lambda \in i\mathfrak{a}_G^*$. Finally, if π is a representation of $G(\mathbb{R})$, let $\pi_\lambda = \pi \otimes e^{\lambda(H_G(\gamma))}$.

Let f be any compactly supported function on $\mathfrak{a}_*^G$ and π a discrete series representation. The main theorem [CD90] also allows us to construct a (again not-necessarily unique) compactly supported $\varphi_{\pi,f}$ such that for any unitary ρ

$$\operatorname{tr}_\rho(\varphi_{\pi,f}) = \begin{cases} \widehat{f}(\lambda) & \rho = \pi_\lambda \\ 0 & \rho \text{ basic}, \rho \neq \pi_\lambda \text{ for all } \lambda \in i\mathfrak{a}_G^* \\ ? & \text{else} \end{cases}.$$

Call such a $\varphi_{\pi,f}$ a generalized pseudocoefficient. For any character ω on A^0, we can define

$$\varphi_{\pi,f,\omega}(g) = \int_{A^0} \omega(a)\varphi_{\pi,f}(ag)\, da.$$

This is compactly supported mod center and transforms according to ω^{-1} on A^0. Therefore, if ρ has character ω on A^0, we can define

$$\begin{aligned} (4) \qquad \operatorname{tr}_\rho(\varphi_{\pi,f,\omega}) &= \int_{G/A^0} \varphi_{f,\pi,\omega}(g)\Theta_\rho(g)\, dg = \int_{G/A^0}\int_{A^0} \varphi_{\pi,f}(ag)\omega(a)\Theta_\rho(g)\, da\, dg \\ &= \int_{G/A^0}\int_{A^0} \varphi_{\pi,f}(ag)\Theta_\rho(ag)\, da\, dg = \int_G \varphi_{f,\pi}(g)\Theta_\rho(g)\, dg = \operatorname{tr}_\rho(\varphi_{\pi,f}). \end{aligned}$$

where Θ is the Harish-Chandra trace character. In particular, $\varphi_{\pi,f,\omega}$ appropriately scaled is a pseudocoefficient.

Averaging $\varphi_{\pi,f}$ over an L-packet $\Pi_{\mathrm{disc}}(\tau)$ for fixed f produces a generalized Euler-Poincare function $\eta_{\tau,f}$. Since the $\eta_{\tau,f,\omega}$ are averages of pseudocoefficients and therefore standard Euler-Poincare functions, the computation (4) gives that whenever τ is regular:

$$\operatorname{tr}_\rho(\eta_{\tau,f}) = \begin{cases} \widehat{f}(\lambda) & \rho = \pi_\lambda \text{ for } \pi \in \Pi_{\mathrm{disc}}(\tau), \lambda \in i\mathfrak{a}_G^* \\ 0 & \text{else} \end{cases}.$$

Generalized pseudocoefficients and Euler-Poincare functions are cuspidal for the same reason as the normal versions.

Finally, as a useful lemma relating our notion to the one in [HL04],

Lemma 6.2.1. *Let π be a discrete series representation with character $e^{\lambda(H_G(a))}$ on A^0 for $\lambda \in i\mathfrak{a}_G^*$. Let f on $\mathfrak{a}_*^G$ be compactly supported. Then we can make choices for φ_π and $\varphi_{\pi,f}$ such that $\varphi_{\pi,f} = f\varphi_\pi$.*

Proof. Make a preliminary choice for $\varphi_{\pi,f}$. Then $\widehat{f}(0)^{-1}\varphi_{\pi,f,\lambda}$ is a valid choice of φ_π We evaluate

$$\begin{aligned}\operatorname{tr}_\rho(f\varphi_{\pi,f,\lambda}) &= \int_G f(g)\varphi_{\pi,f,\lambda}(g)\Theta_\rho(g)\,dg \\ &= \int_{A^0}\int_{G/A_0} f(ag)\varphi_{\pi,f,\lambda}(ag)\Theta_\rho(ag)\,dg\,da \\ &= \int_{A^0} f(a)e^{(\mu-\lambda)(H_G(a))}\int_{G/A_0}\varphi_{\pi,f,\lambda}(g)\Theta_{\rho_{\lambda-\mu}}(g)\,dg\,da\end{aligned}$$

where we choose $\mu \in i\mathfrak{a}_G^*$ so that $e^{\mu(H_G(g))}$ is the central character of ρ on A^0. By previous properties, the inner integral therefore becomes $\operatorname{tr}_{\rho_{\lambda-\mu}}(\varphi_{\pi,f})$ and we get

$$\operatorname{tr}_\rho(f\varphi_{\pi,f,\lambda}) = \widehat{f}(\mu-\lambda)\operatorname{tr}_{\rho_{\lambda-\mu}}(\varphi_{\pi,f}).$$

Checking each of the three cases in its definition, $f\widehat{f}(0)^{-1}\varphi_{\pi,f,\lambda}$ is then a valid alternative choice for $\varphi_{\pi,f}$. □

A similar property also therefore holds for Euler-Poincare functions.

6.2.1. *A small modification.* Generalized pseudocoefficients are in $C_c^\infty(G_\infty)$. We instead want functions in some $C_c^\infty(G_\infty,\chi_0)$ so we make a small modification.

Return to the previous notation where G is a group over F. Let χ_0 be a character on $A_{G,\mathrm{rat}}$ and π_0 a representation of G_∞ consistent with χ_0. Let $\varphi_{\pi_0,f} = f\varphi_{\pi_0}$ be a generalized pseudocoefficient for π_0 and consider the partial average

$$\begin{aligned}\bar{\varphi}(g) &= \int_{A_{G,\mathrm{rat}}}\chi_0(a)f(ag)\varphi_{\pi_0}(ag)\,da \\ &= \int_{A_{G,\mathrm{rat}}}\chi_0(a)f(ag)\chi_0^{-1}(a)\varphi_{\pi_0}(g)\,da = \varphi_{\pi_0}(g)\int_{A_{G,\mathrm{rat}}} f(ag)\,da.\end{aligned}$$

This is an element of $C_c^\infty(G_\infty,\chi_0)$ and every function $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$ arises as an integral this way. Finally, by a similar computation to (4), this has the same traces against representations π consistent with χ_0 as $\varphi_{\pi_0,f}$.

Therefore, for any function $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$, we can construct analogues of generalized pseudocoefficients $\varphi_{\pi_0,f} = f\varphi_{\pi_0} \in C_c^\infty(G_\infty,\chi_0)$. For computations later, note that such f have Fourier transforms defined on any character of $A_{G,\infty}$ trivial on $A_{G,\mathrm{rat}}$. The same discussion carries over to Euler-Poincare functions. These are the functions we will actually be using.

We fix f to be dimensionless so these generalized pseudocoefficients have dimension $[G_\infty/A_{G,\mathrm{rat}}]^{-1}[A_{G,\infty}/A_{G,\mathrm{rat}}] = [G_\infty]^{-1}[A_{G,\infty}]$.

6.3. **Spectral side with central character.** To get a simple trace formula with central character, we need two spectral side computations: one for I'_{spec} and one for I_{spec}. Start with a lemma:

Lemma 6.3.1. *Let π_0 be a discrete automorphic representation of G_∞ with regular infinitesimal character ξ_0 and character χ_0 on $A_{G,\infty}$. Then for any real irrep ρ of G_∞ with character χ_0 on $A_{G,\infty}$, $\operatorname{tr}_\rho(\varphi_{\pi_0}) = \delta_{\pi_0}(\rho)$.*

Proof. We thank Vogan for this argument and note that all mistakes in this writeup are our own.

Assume not. Then in the Grothendieck group, ρ is a linear combination of basic representations with infinitesimal character ξ_0:

$$\rho = \sum_{\rho' \text{ basic}} m_\rho(\rho')\rho'.$$

Taking traces of both sides, $m_\rho(\pi_0) = \mathrm{tr}_\rho(\varphi_{\pi_0}) \neq 0$. Now, taking the trace against an EP-function η_{ξ_0}

$$\mathrm{tr}_\rho(\eta_{\xi_0}) = \frac{1}{|\Pi_{\mathrm{disc}}(\xi_0)|} \sum_{\rho' \in \Pi_{\mathrm{disc}}(\xi_0)} m_\rho(\rho')$$

where $\Pi_{\mathrm{disc}}(\xi_0)$ is the L-packet for ξ_0.

We now want to show that the $m_\rho(\rho')$ in this sum all have the same sign. The most direct way is to use the classification of all unitary representations with infinitesimal character of a discrete series from [SR99]. These are of the form of certain $A_{\mathfrak{q}}(\lambda)$ described in terms of Zuckerman functors. These have an explicit decomposition in the Grothendieck group through a version of Zuckerman's character formula proposition 9.4.16 in [Vog81]: λ is a character on Levi L_∞ so first get a character formula λ by twisting both sides of 9.4.16 for L_∞ by λ. Then cohomologically induce to get a character formula on G_∞. Alternatively, by Kazhdan-Lusztig theory, the $m_\rho(\rho')$ are Euler characteristics of stalks of certain perverse sheaves. By theorem 1.12 in [LV83] their cohomologies are either concentrated in even degree or odd degree. See the comments in the proof to corollary 4.6 in [Vir15] for example for why this applies to $\mathbb{C}$ in addition to $\overline{\mathbb{F}}_p$.

Therefore $\mathrm{tr}_\rho(\eta_{\xi_0}) \neq 0$. Since ξ_0 is regular, properties of Euler-Poincare functions imply then that ρ is in $\Pi_{\mathrm{disc}}(\xi_0)$. However, then the the trace against the pseudocoefficient has to be 0 unless $\rho \neq \pi_0$ which contradicts. □

Combining with computation (4) (note that twisting by a character doesn't change the regularity of the infinitesimal character) then gives:

Corollary 6.3.2. *Let π_0 be a discrete automorphic representation of G_∞ with regular infinitesimal character ξ_0. Let $f \in C_c^\infty(A_{G,\infty})$. Then for any real representation ρ of G_∞, $\mathrm{tr}_\rho(\varphi_{\pi_0,f}) = f(\rho,\pi_0)$ where*

$$f(\pi,\pi_0) = \begin{cases} \widehat{f}(\lambda) & \pi = \pi_\lambda \text{ for } \lambda \in i(a^*_{G_\infty})_{\mathbb{R}} \\ 0 & \text{else} \end{cases}.$$

A similar result holds for $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$.

This allows us to prove

Proposition 6.3.3. *Let π_0 be a discrete series representation of G_∞ with regular infinitesimal character ξ_0 and character χ_0 on $A_{G,\mathrm{rat}}$. Let $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Then:*

$$I^G_{\mathrm{spec}}(\varphi_{\pi_0,f} \otimes \varphi^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) f(\pi_\infty,\pi_0)\, \mathrm{tr}_{\pi^\infty}(\varphi^\infty)$$

where

$$f(\pi_\infty,\pi_0) = \begin{cases} \widehat{f}(\lambda) & \pi_\infty = (\pi_0)_\lambda \text{ for } \lambda \in i(a^*_{G_\infty})_{\mathbb{R}} \\ 0 & \text{else} \end{cases}.$$

Proof. This is simply a due-diligence check that none of the steps in the derivation of formula 3.5 in [Art89] break. First, $\varphi_{\pi_0,f}$ being cuspidal gives

$$\begin{aligned} I^G_{\mathrm{spec}}(\varphi) &= \sum_{t\geq 0} I^G_{\mathrm{disc},t}(\varphi) \\ &= \sum_{t\geq 0}\sum_{L\in\mathscr{L}(G)} \frac{|\Omega_M|}{|\Omega_G|} \\ &\qquad \sum_{s\in W^G(\mathfrak{a}_L)_{\mathrm{reg}}} |\det(s-1)|_{\mathfrak{a}_L/\mathfrak{a}_G}|^{-1}\,\mathrm{tr}(M_{Q|Q}(s,0)\rho_{Q,t}(0,(\varphi_{\pi_k}\varphi^\infty)^1)) \end{aligned}$$

using that G is connected. This uses a lot of the notation from [Art89]. In particular, $\mathscr{L}(G)$ is the set of Levi subgroups of G, Q is a parabolic for L, $M_{Q|Q}(s,0)$ is some intertwining operator, $\rho_{Q,t}$ is a sum of parabolically-induced representations from Q with Archimedean infinitesimal character having imaginary part of norm t, and $(\varphi_\pi\varphi^\infty)^1$ is the restriction of the function to $G(\mathbb{A})^1$

The full definition of the rest of the terms in the inner sum is unnecessary: the only detail Arthur uses is that when $Q\neq G$ it is a sum

$$\sum_{\pi\in\mathcal{AR}(G)} c_\pi\,\mathrm{tr}_\pi((\varphi_{\pi_0,f}\varphi^\infty)^1)$$

where the c_π vanish whenever the Archimedean infintesimal character of π is regular. However, a property of the pseudocoefficient $\varphi_{\pi_0,f}$ is that it is only supported on representations which have the same infinitesimal character as π_0 (similar to the the proof of [Clo86] lemma 1). This character has to be regular. Therefore the sum is 0.

For the leftover term, $Q=G$ so $L=G$ and $M_{Q|Q}(s,0)$ is trivial. This gives

$$I^G_{\mathrm{disc}}(\varphi) = \sum_{t\geq 0}\mathrm{tr}\,\rho_{G,t}(0,(\varphi_{\pi_0,f}\varphi^\infty)^1).$$

By its definition, $\rho_{G,t}(0)$ is all irreducible subrepresentations of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ with Archimedean infinitesimal character having imaginary part with norm t. By the restriction on infinitesimal characters that φ_{π_0} has support on, this sum is finite implying the operator $(\varphi_{\pi_0,f}\varphi^\infty)^1$ acting on L^2 is trace class. Finally, $(\varphi_{\pi_0,f}\varphi^\infty)^1$ acting on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ is the same operator as $\varphi_{\pi_0,f}\varphi^\infty$ acting on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\chi_0)$. Therefore, summing over the representations that are actually subrepresentations of L^2.

$$\begin{aligned} I^G_{\mathrm{disc}}(\varphi) &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi)\,\mathrm{tr}_\pi(\varphi_{\pi_0,f}\varphi^\infty) \\ &= \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi)\,\mathrm{tr}_{\pi_\infty}(\varphi_{\pi_0,f})\,\mathrm{tr}_{\pi^\infty}(\varphi^\infty). \end{aligned}$$

Corollary 6.3.2 gives that $\mathrm{tr}_{\pi_\infty}(\varphi_{\pi_0,f}) = f(\pi_\infty,\pi_0)$ finishing the argument. ☐

Next, let $\varphi=\varphi_{\pi_0,f}\otimes\varphi^\infty$. Then

$$I'_{\mathrm{spec},\chi}(\varphi) = \frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\mathrm{spec},\chi_0}(\varphi_z)\,dz.$$

Computing

$$I_{\mathrm{spec},\chi_0}(\varphi_z) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) \operatorname{tr}_\pi(\varphi_z)$$
$$= \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi_0)} m_{\mathrm{disc}}(\pi) \omega_\pi^{-1}(z) \operatorname{tr}_\pi(\varphi)$$

where ω_π is the central character of π. Substituting this in and factoring out the sum and constants from the integral gives

$$\int_{\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z)\omega_\pi^{-1}(z)\, dz = \begin{cases} \mathrm{vol}(\mathfrak{X}_{\mathbb{Q}} \backslash \mathfrak{X}/A_{G,\mathrm{rat}}) & \chi = \omega_\pi|_{\mathfrak{X}} \\ 0 & \text{else} \end{cases}.$$

Therefore a lot of terms in the sum go to 0. Finally, since π^∞ has central character χ^∞, it can be traced against functions in $\mathscr{H}(G^\infty, \chi^\infty)$. By definition

$$\operatorname{tr}_{\pi^\infty}(\varphi^\infty) = \operatorname{tr}_{\pi^\infty}(\overline{(\varphi^\infty)}_{\chi^\infty}).$$

Putting it all together,

Corollary 6.3.4. *Let π_0 be a discrete series representation of G_∞ with regular infinitesimal character ξ_0 and character χ_0 on $A_{G,\mathrm{rat}}$. Let $f \in C_c^\infty(A_{G,\infty}, \chi_0)$. Then:*

$$I'_{\mathrm{spec},\chi}(\varphi_{\pi_0,f} \otimes \varphi^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) f(\pi_\infty, \pi_0) \operatorname{tr}_{\pi^\infty}(\overline{(\varphi^\infty)}_{\chi^\infty})$$

(where we only sum over automorphic representations with the correct central character on all of $\mathfrak{X}$ instead of just $A_{G,\mathrm{rat}}$).

Finally, the same arguments as in Proposition 6.3.3 again work for the terms in equation (3) giving

$$I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \delta_{\pi_0,\pi_\infty} \operatorname{tr}_{\pi^\infty}(\varphi^\infty)$$

where we factor $\mathfrak{X}_\infty = \mathfrak{X}^1_\infty \times A_{G,\infty}$. Sanity checking dimensions here, we need

$$[G(\mathbb{A})][\mathfrak{X}]^{-1}[G_\infty]^{-1}[A_{G,\infty}] = [\mathfrak{X}_\infty/A_{G,\infty}]^{-1}[G^\infty][\mathfrak{X}^\infty]$$

which holds.

Putting everything together

Proposition 6.3.5. *Let π_0 be a discrete series representation of G_∞ with regular infinitesimal character ξ_0 matching character χ on $\mathfrak{X}$. Let $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$ and $\varphi^{\infty_1} \in \mathscr{H}(G^\infty, \chi^\infty)$ such that $\overline{(\varphi^{\infty_1})}_\chi = \varphi^\infty$. Then:*

$$\mathrm{vol}(\mathfrak{X}^1_\infty) I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi) \delta_{\pi_0,\pi_\infty} \operatorname{tr}_{\pi^\infty}(\varphi^\infty)$$
$$= \frac{1}{\widehat{f}(0)} I'_{\mathrm{spec},\chi}(\varphi_{\pi_0,f} \otimes \varphi^{\infty_1}).$$

The second equality uses that for any $\pi_\lambda \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)$, $\lambda = 0$. We fix φ^∞ to be dimensionless and normalize φ^{∞_1} by it. Therefore, the dimensions are all $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

6.4. **Geometric side with central character.**

6.4.1. *Vanishing of* $I_M^G(\gamma, \psi)$. Implicit in [Art89] is that the distributions $I_M^G(\gamma_\mathbb{R}, \psi)$ vanish when ψ is cuspidal and M isn't cuspidal over $\mathbb{R}$. For the ease of the reader, we mention the explanation in the summary [Art02, §24]: it is because then $\gamma_\mathbb{R}$ cannot be elliptic so the main result of [Art76] gives that it vanishes.

6.4.2. *Computation of* I_{geom,χ_0}. Next, we compute the geometric side. Let $\Pi_{\mathrm{disc}}(\lambda)$ be a regular discrete series L-packet for G_∞ consistent with χ and $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. We again try to mimic Arthur's arguments. Cuspidality of $\eta_{\lambda,f}$ and the splitting formulas reduce the geometric side to

$$I_{\mathrm{geom},\chi_0}(\eta_{\lambda,f} \otimes \varphi^\infty) = \sum_{M \in \mathscr{L}} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(\mathbb{Q})]_{M,S}} a^M(S,\gamma) I_M^G(\gamma_\mathbb{R}, \eta_{\lambda,f}) O_\gamma^M(\varphi_M^\infty).$$

Define for $\psi \in C_c^\infty(G_\infty, \chi)$

$$\varphi_M(\gamma_\mathbb{R}, \psi) = |D^M(\gamma)|^{-1/2} I_M^G(\gamma, \psi).$$

By the previous subsubsection, we can wlog set $\varphi_M(\gamma, \psi) = 0$ if M isn't cuspidal over $\mathbb{R}$

For L-packet $\Pi_{\mathrm{disc}}(\tau)$, and elliptic regular $\gamma \in M(\mathbb{R})$

$$\varphi_M(\gamma, \tau) = (-1)^{q(G)} |D_M^G|^{1/2} \sum_{\pi \in \Pi_{\mathrm{disc}}(\tau)} \Theta_\pi(\gamma).$$

Arthur shows that $\varphi_M(\gamma, \tau)$ can be extended by continuity to all elements in elliptic maximal tori. Define it to be 0 for other elements to extend it to all of $M(\mathbb{R})$; in particular, to non-semisimple elements.

Next, we need a defintion

Definition. Let χ be a character on $A_{G,\infty}$. A cuspidal function $\psi \in C_c^\infty(G_\infty, \chi)$ is *stable cuspidal* if its trace is supported on discrete series and constant on L-packets.

Note that Euler-Poincare functions are stable cuspidal. Part of the main result of [CD90] gives that Euler-Poincare functions are also K-finite.

As some notation for the next step, if H is a group over $\mathbb{R}$, let $\overline{H}$ be the compact form of H. Any Haar measure on H comes from a differential form on $H_\mathbb{C}$ and therefore induces a Haar measure on $\overline{H}$. Then:

Theorem 6.4.1 ([Art89, thm 5.1])**.** *Let χ be a character on $A_{G,\infty}$ and $\varphi \in C_c^\infty(G_\infty, \chi)$ be stable cuspidal and K-finite. Then for any $\gamma \in M(\mathbb{R})$*

$$\Phi_M(\gamma, \varphi) = (-1)^{\dim(A_M/A_G)} \nu(I_\gamma^M)^{-1} \sum_{\substack{\tau \in X^*(T)_\mathbb{C} \\ \tau \text{ matches } \chi}} \Phi_M(\gamma, \tau) \operatorname{tr}_{\tau^\vee}(\varphi) :$$

where $\nu(M_\gamma) = (-1)^{q(G)} \operatorname{vol}(\bar{I}_{\gamma,\infty}^M / A_{I_\gamma^M,\infty})$.

Since lemma 4 gives that wlog $\eta_{\lambda,f} = f\eta_\lambda$, we recall the following rephrasing of a fact used in deriving the invariant trace formula:

Lemma 6.4.2. *Let $f = f_1 \circ H_{G_\infty}$ be a function on $G/A_{G,\mathrm{rat}}$ where f_1 is a function on $C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Let φ be any function on $G(\mathbb{R})$ compactly supported mod center. Then for any $\gamma \in G(\mathbb{R})$ and Levi M*

$$I_M^G(\gamma, f\varphi) = f(\gamma) I_M^G(\varphi).$$

Proof. Remark 4 after theorems 23.2 and 23.3 in [Art05] gives that $I_M^G(\gamma, f\varphi)$ only depends on the values of $f\varphi$ on $g \in G(\mathbb{R})$ with the same image as γ under H_G. On this set f is constant so the result follows. $\square$

In particular, for any $\gamma \in G(\mathbb{R})$:

$$\Phi_M(\eta_{\lambda,f}, \gamma) = f(\gamma)\Phi_M(\eta_\lambda, \gamma) = (-1)^{\dim(A_M/A_G)} f(\lambda)\nu(M_\gamma)\Phi_M(\gamma, \lambda)$$

so following the computation in [Art89] section 6 gives:

Corollary 6.4.3. *Let λ_0 be weight consistent with χ_0 and $f \in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$. Then*

$$I_{\mathrm{geom},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \sum_{M \in \mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I_\gamma^M)|\iota^M(\gamma)|^{-1} f(\gamma)\Phi_M(\gamma, \lambda_0) O_\gamma^M(\varphi_M^\infty)$$

where

$$\chi(I_\gamma^M) = \frac{\mathrm{vol}(I_\gamma^M(F)\backslash I_\gamma^M(\mathbb{A})/A_{I_\gamma^M,\mathrm{rat}})}{\mathrm{vol}(\bar{I}_{\gamma,\infty}^M/A_{I_\gamma^M,\infty})}$$

and $\iota^M(\gamma)$ is the set of connected components of M_γ that have an F-point.

6.4.3. *Computation of $I'_{\mathrm{geom},\chi}$.* It remains to compute $I'_{\mathrm{geom},\chi}(\eta_{\lambda,f} \otimes \varphi^\infty)$ by averaging. To make the final formula more elegant, wlog assume λ_0 is consistent with χ. We have

$$I'_{\mathrm{geom},\chi}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z) I_{\mathrm{geom},\chi_0}((\varphi_{\pi_0,f} \otimes \varphi^\infty)_z)\, dz.$$

Wlog taking $\eta_{\lambda_0,f} = f\eta_{\lambda_0}$ by lemma 4:

$$(\eta_{\lambda_0,f} \otimes \varphi^\infty)_z = (\eta_{\lambda_0,f})_{z_\infty} \otimes \varphi_{z^\infty}^\infty = \omega_{\lambda_0}^{-1}(z_\infty)\eta_{\lambda_0,f_{z_\infty}} \otimes \varphi_{z^\infty}^\infty$$

where ω_{λ_0} is the central character associated to λ_0. Here, $\varphi_{\lambda_0,f_{z_a}}$ is still a generalized Euler-Poincare function so we substitute in corollary 6.4.3. The terms that change are

$$f(\gamma)\Phi_M(\gamma, \lambda_0) \mapsto \omega_{\lambda_0}^{-1}(z_\infty) f_{z_\infty}(\gamma)\Phi_M(\gamma, \lambda_0)$$

and

$$O_\gamma^M(\varphi_M^\infty) \mapsto O_\gamma^M((\varphi_{z^\infty}^\infty)_M).$$

By our simplifying assumptions, the $\omega_{\lambda_0}^{-1}(z_\infty)$ can be pulled out and partially cancelled against the $\chi(z)$. Finally, we use proposition 6.3.5:

$$\mathrm{vol}(\mathfrak{X}_\infty^1) I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\hat{f}(0)} I'_{\mathrm{spec},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \frac{1}{\hat{f}(0)} I'_{\mathrm{geom},\chi_0}(\eta_{\lambda_0,f} \otimes \varphi^\infty)$$

thereby getting the full formula we will use later:

Proposition 6.4.4. *Let $\Pi_{\mathrm{disc}}(\lambda_0)$ be a discrete series L-packet of G_∞ with regular infinitesimal character ξ_0 and central character χ on $\mathfrak{X}$, f a function pulled back* *through H_{G_∞} from $C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$, and $\varphi^{\infty_1} \in \mathscr{H}(G^\infty, \chi_0)$ such that $\bar{\alpha}_{\chi^\infty} = \varphi^\infty$. Then we have geometric expansion*

$$\begin{aligned}&\mathrm{vol}(\mathfrak{X}^1_\infty) I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \\ &\quad \frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) \sum_{M \in \mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|} \\ &\qquad \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} \chi(I^M_\gamma) |\iota^M(\gamma)|^{-1} f(z_\infty \gamma) \Phi_M(\gamma, \lambda_0) O^M_\gamma((\varphi^{\infty_1}_{z^\infty})_M)\, dz\end{aligned}$$

where

$$\chi(I^M_\gamma) = \frac{\mathrm{vol}(I^M_\gamma(F) \backslash I^M_\gamma(\mathbb{A})/A_{I^M_\gamma,\mathrm{rat}})}{\mathrm{vol}(\bar{I}^M_{\gamma,\infty}/A_{I^M_\gamma,\infty})}$$

and $\iota^M(\gamma)$ is the set of connected components of M_γ that have an F-point.

6.4.4. *Further Simplification.* Mimicking some simplifications from [KSZ], the integral can be evaluated to remove f and φ^1-dependence. This version of the formula and the method of its derivation are useful for some bounds later.

$\mathfrak{X}_F$ acts on $[M(F)]^{\mathrm{ss}}$ by multiplication. Let the set of orbits be $[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}$. For any γ, let $\mathrm{Stab}_{\mathfrak{X}}(\gamma)$ be the stabilizer of γ under this action. This is finite by using a faithful representation (which always induces a finite-to-one map on semisimple conjugacy classes) to reduce to the case $G = \mathrm{GL}_n$. Here conjugacy classes are just sets of eigenvalues and the $\mathfrak{X}$-action just scales each eigenvalue. Note also that since $\mathfrak{X}$ is central, ι and ν are constant on $\mathfrak{X}$-orbits.

We can therefore move the integral into the inner sum over γ and break it up as

$$\begin{aligned}&\sum_{\gamma \in [M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma) |\iota^M(\gamma)|^{-1} |\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \\ &\qquad \sum_{x \in \mathfrak{X}_F} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) f_{z_\infty}(x\gamma) \Phi_M(x\gamma, \lambda_0) O^M_\gamma((\varphi^{\infty_1}_{xz^\infty})_M)\, dz.\end{aligned}$$

Since χ is defined to be trivial on rational points, the innermost sum simplifies to

$$\begin{aligned}&\sum_{x \in \mathfrak{X}_F} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty x) f(z_\infty x \gamma) \omega^{-1}_{\lambda_0}(x) \Phi_M(\gamma, \lambda_0) O^M_\gamma((\varphi^{\infty_1}_{xz^\infty})_M)\, dz \\ &\qquad = \Phi_M(\gamma, \lambda_0) \left(\int_{\mathfrak{X}_\infty/A_{G,\mathrm{rat}}} f(z\gamma)\, dz \right) \left(\int_{\mathfrak{X}^\infty} \chi(z) O^M_\gamma((\varphi^{\infty_1}_z)_M)\, dz \right).\end{aligned}$$

Recalling

$$(\varphi^{\infty_1}_z)_M = \delta_{P_M}(\gamma^\infty)^{1/2} \int_{K^\infty} \int_{N_M(\mathbb{A}^\infty)} \varphi^{\infty_1}(k^{-1} \gamma^\infty z n k)\, dn\, dk,$$

a bunch of Fubini's steps gives that the non-Archimedean integral is $O^M_\gamma((\overline{(\varphi^{\infty_1})_\chi})_M) = O^M_\gamma((\varphi^\infty)_M)$ where we recall

$$\overline{\varphi_\chi}(g) = \int_{\mathfrak{X}^\infty} \varphi(gz) \chi(z)\, dz$$

for any φ.

For the Archimedean integral, let the $G_\infty = G^1_\infty \times A_{G,\infty}$ components of any g be $g_1 \times g_a$. Then $f(z\gamma) = f(z_a\gamma_a)$. This factorization gives a corresponding one $\mathfrak{X}^\infty/A_{G,\mathrm{rat}} = \mathfrak{X}^1_\infty \times A_{G,\infty}/A_{G,\mathrm{rat}}$. Then the integral becomes

$$\int_{\mathfrak{X}^1_\infty} \int_{A_{G,\infty}/A_{G,\mathrm{rat}}} f(z_a\gamma_a)\, dz_a\, dz_1 = \mathrm{vol}(\mathfrak{X}^1_\infty)\widehat{f}(0).$$

Putting it all together:

$$I'_{\mathrm{geom},\chi}(\eta_{\lambda_0,f} \otimes \varphi^\infty) = \frac{\mathrm{vol}(\mathfrak{X}^1_\infty)\widehat{f}(0)}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})} \sum_{M\in\mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|}$$
$$\sum_{\gamma\in[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\Phi_M(\gamma,\lambda_0)O^M_\gamma((\varphi^\infty)_M).$$

Using proposition 6.3.5 as before finally gives:

Proposition 6.4.5. *Let $\Pi_{\mathrm{disc}}(\lambda_0)$ be a discrete series L-packet of G_∞ with regular infinitesimal character ξ_0 and central character χ on $\mathfrak{X}$. Then we have geometric expansion*

$$I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})} \sum_{M\in\mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|}$$
$$\sum_{\gamma\in[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\Phi_M(\gamma,\lambda_0)O^M_\gamma((\varphi^\infty)_M).$$

The dimensions on both sides are $[G^\infty][\mathfrak{X}^\infty]^{-1}[\mathfrak{X}^1_\infty]^{-1} = [G^\infty][\mathfrak{X}/A_{G,\infty}]^{-1}$

6.5. **Irregular Discrete Series.** When λ_0 isn't regular, $\mathrm{tr}_{\pi_\infty}\eta_{\lambda_0}$ does not simply test if π_∞ is in a given L-packet. However, it can be interpreted as a cohomology as in [Art89, §2]. While we will not use this more general result, we state it here in case it is useful in other applications.

Even with irregular λ_0, we still have

$$I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi)\,\mathrm{tr}_{\pi_\infty}(\eta_{\lambda_0})\,\mathrm{tr}_{\pi^\infty}(\varphi^\infty).$$

The Euler-Poincaré function η_{λ_0} always satisfies $\mathrm{tr}_{\pi_\infty}(\eta_{\lambda_0}) = \chi_{\lambda_0}(\pi_\infty)$ where χ_{λ_0} is the Euler characteristic

$$\chi_{\lambda_0}(\pi_\infty) = \sum_q (-1)^q \dim H^q(\mathfrak{g}(\mathbb{R}), K_\infty, \pi_\infty \otimes \pi_{\lambda_0}).$$

H^q is the $(\mathfrak{g},K)$-cohomology: K_∞ is a maximal compact of G_∞ and π_{λ_0} is the finite dimensional complex rep with highest weight λ_0. The equality holds in general because it holds on basic representations which generate the Grothendieck group.

In particular, if we define the L^2-Lefschetz number

$$\mathscr{L}_{\lambda_0}(\varphi^\infty) = \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi)\chi_{\lambda_0}(\pi_\infty)\,\mathrm{tr}_{\pi^\infty}(\varphi^\infty),$$

we get

$$I_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)}\mathscr{L}_{\lambda_0}(\varphi^\infty).$$

Combining with proposition 6.4.5 gives formula:

Corollary 6.5.1. *Let π_0 be a discrete series representation of G_∞ with possibly irregular infinitesimal character ξ_0 matching character χ on $\mathfrak{X}$. Then for any φ^∞*

$$\mathscr{L}_{\lambda_0}(\varphi^\infty) = \frac{\mathrm{vol}(\mathfrak{X}^1_\infty)}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})} \sum_{M\in\mathscr{L}} (-1)^{\dim(A_M/A_G)} \frac{|\Omega_M|}{|\Omega_G|}$$
$$\sum_{\gamma\in[M(F)]^{\mathrm{ss}}_{\mathfrak{X}}} \chi(I^M_\gamma)|\iota^M(\gamma)|^{-1}|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1}\Phi_M(\gamma,\lambda_0)O^M_\gamma((\varphi^\infty)_M).$$

The dimensions on both sides are $[G^\infty][\mathfrak{X}^\infty]^{-1}$.

7. Trace Formula Computation Set-up

Now we can finally set up our main computation.

7.1. Conditions on G and Defining Families.

Let G be a reductive group over a number field F with discrete series at ∞. By instead looking at $\mathrm{Res}^F_{\mathbb{Q}} G$, we could wlog take $F = \mathbb{Q}$ since $\mathrm{Res}^F_{\mathbb{Q}} G(\mathbb{Q}) = G(F)$ and $\mathrm{Res}^F_{\mathbb{Q}} G(\mathbb{A}) = G(\mathbb{A}_F)$ as topological groups. Fix central character datum $(\mathfrak{X},\chi)$. Assume G is connected.

Let:

- π_0 be a real discrete series representation for G with regular infinitesimal character ξ_0 and character χ on $A_{G,\infty}$
- φ_{π_0} be its pseudocoefficient.
- S_0 be a finite set of places and choose $\varphi_{S_0} \in \mathscr{H}(G_{S_0},\chi_{S_0})$
- S_1 be another finite set of places disjoint from S_0 such that χ_{S_1} is unramfied.
- $S = S_0 \sqcup S_1$
- $U^{S,\infty} \subset G(\mathbb{A}^{S,\infty})$ open compact on which $\chi^{S,\infty}$ is trivial.

Define a family of automorphic representations $\mathcal{F}$ in $\mathcal{AR}_{\mathrm{disc}}(G,\chi)$ through discrete multiplicities

$$a_{\mathcal{F}}(\pi) = m_{\mathrm{disc}}(\pi)\delta_{\pi_0,\pi_\infty}\dim(\pi^{S,\infty})^{U^{S,\infty}}\frac{\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1})}{\mathrm{vol}(K_{S_1})}.$$

Note that the second-to-last term is just checking if π_{S_1} is unramified. The coefficient $a_{\mathcal{F}}(\pi)$ is dimensionless.

Define function

$$\mathbf{1}_{U^{S,\infty},\chi} = \mathrm{vol}(U^{S,\infty}\cap\mathfrak{X}^{S,\infty})^{-1}\overline{(\mathbf{1}_{U^{S,\infty}})}_\chi.$$

This is normalized so that $\mathbf{1}_{U^{S,\infty},\chi}(1) = 1$. For any test function $\varphi_{S_1} \in \mathcal{H}^{\mathrm{ur}}(G_{S_1},\chi_{S_1})$ let

$$\varphi = \varphi_{\pi_0,f,\varphi_{S_0}} = \varphi_{\pi_0}\otimes\varphi^\infty = \varphi_{\pi_0}\otimes\mathbf{1}_{U^{S,\infty},\chi}\otimes\varphi_{S_0}\otimes\varphi_{S_1}$$

where as before φ_π is the pseudocoefficient for π. Test function φ will momentarily be shown to pick out the family $a_{\mathcal{F}}$.

Intuitively, the test function is

- putting weight restrictions on the infinite place
- putting level restrictions on finite places away from S
- Forcing S_1 parts to be unramified
- Counting possible components at S according to to test function φ_S with φ_{S_1} unramified.

To make all the traces well-defined, we fix Haar measures on factors of $G(\mathbb{A}_F)$:

- use the normalization from [ST16, §6.6] of Gross' canonical measure from [Gro97] on G_s and the $\mathfrak{X}_s$.
- use Euler-Poincare measure on G_∞, $A_{G,\infty}$ and $\mathfrak{X}^1_\infty$.

This determines all appropriate Plancherel measures. We call the product measure $\mu^{\mathrm{can},EP}$ and the volume of the adelic quotient under it the modified Tamagawa number $\tau'(G)$.

7.2. **Spectral Side.** We can now directly compute the spectral expansion of $I_{\mathrm{spec},\chi}(\varphi)$:

Corollary 7.2.1. *Let π_0 be a discrete series representation of G with regular infinitesimal character ξ_0.*

$$I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = \bar{\mu}^{\mathrm{can}}(U^{S,\infty}_{\mathfrak{X}}) \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}}(\pi)\widehat{\varphi}_S(\pi)$$

where $U^{S,\infty}_{\mathfrak{X}} = U^{S,\infty}/\mathfrak{X}^{S,\infty} \cap U^{S,\infty}$.

Proof. By proposition 6.3.5 and using that $\mathrm{vol}(\mathfrak{X}^1_\infty) = 1$,

$$I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) = \frac{1}{\mathrm{vol}(\mathfrak{X}^1_\infty)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} m_{\mathrm{disc}}(\pi)\delta_{\pi_0,\pi_\infty} \operatorname{tr}_{\pi^\infty}(\varphi^\infty).$$

Factoring the finite trace into its S_0, S_1 and other components gives that

$$\operatorname{tr}_{\pi^\infty}(\varphi^\infty) = \widehat{\varphi}_{S_0}(\pi_{S_0}) \frac{\widehat{\mathbf{1}}_{K_{S_1}}(\pi_{S_1})}{\mathrm{vol}(K_{S_1})} \widehat{\varphi}_{S_1}(\pi)\mu^{\mathrm{can}}(U^{S,\infty}_{\mathfrak{X}}) \dim(\pi^{S,\infty})^{U^{S,\infty}}$$

so we are done. □

7.3. **Geomteric Side Outline.** We get a geometric expansion $I_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty)$ by using the hyperendoscopy formula (proposition 4.2.3). Since Euler-Poincare functions and pseudocoefficients have the same stable orbital integrals:

$$\begin{aligned} &I^G_{\mathrm{spec},\chi}(\varphi_{\pi_0} \otimes \varphi^\infty) \\ &\quad = I^G_{\mathrm{spec},\chi}(\eta_{\lambda_0} \otimes \varphi^\infty) + \sum_{\mathcal{H} \in \mathcal{HE}_{\mathrm{ell}}(G)} \iota(G,\mathcal{H}) I^{\mathcal{H}}_{\mathrm{spec},\chi_{\mathcal{H}}}((\eta_{\xi_0} - \varphi_{\pi_\infty})^{\mathcal{H}} \otimes (\varphi^\infty)^{\mathcal{H}}). \end{aligned}$$

Simplifying and bounding this takes a few steps:

(1) Notice that transfers $(\eta_{\xi_0} - \varphi_{\pi_\infty})^{\mathcal{H}}$ through hyperendoscopic paths can be chosen to be linear combinations of regular Euler-Poincare functions
(2) Substitute in proposition 6.4.4 for each hyperendoscopic group
(3) The result will have a main term consisting of central elements of G and an error term consisting of non-central elements, Levi terms, and terms from the hyperendoscopic groups.
(4) Use a Poisson summation argument to compute the main term.
(5) Bound the error term using bounds on non-Archimedean transfers and small generalizations of the results of [ST16].

For sanity checks later, note that both sides of our computation have dimension $[G^\infty][\mathfrak{X}/A_{G,\infty}]^{-1}$.

8. Geometric side details

We are eventually going to use the hyperendoscopic formula with f_1 of the form

$$f_1 = \eta_\xi \otimes \varphi^\infty.$$

All transfers appearing will have linear combinations of Euler-Poincare functions as infinite parts so we only need to analyze the geometric side with test functions of the form $\eta_\xi \otimes \varphi^\infty$. This is similar to what was done in [ST16].

8.1. **Original Bounds.** Recall notation and conditions from 7.1. We state the main bounds from [ST16] for reference. G determines a finite set of places $S_{\mathrm{bad},G}$ in a complicated, uncontrolled manner. We assume three conditions:

- S doesn't intersect $S_{\mathrm{bad},G}$
- G is cuspidal
- $\mathfrak{X}$ is trivial.

Then we get the following bounds:

Theorem 8.1.1 (Weight-aspect bound [ST16, thm 9.19])**.** *Consider the case where $Z(G) = 1$. Let $f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leq\kappa}$ such that $\|f_{S_1}\|_\infty \leq 1$. Let ξ be a dominant weight. Then*

$$\frac{1}{\tau'(G)\dim(\xi)\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{\varphi}_{S_0})} I_{\mathrm{spec}}(\eta_\xi \otimes \varphi^\infty) = \widehat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{f}_{S_1}) + O_{G,\varphi_{S_0}}(q_{S_1}^{A_{\mathrm{wt}}+B_{\mathrm{wt}}\kappa} m(\xi)^{-C_{\mathrm{wt}}})$$

for some constants $A_{\mathrm{wt}}, B_{\mathrm{wt}}, C_{\mathrm{wt}}$ depending only on G.

Theorem 8.1.2 (Level-aspect bound [ST16, thm 9.16])**.** *Consider the case where $U^{S,\infty}$ is a level subgroup $K^{S,\infty}(\mathfrak{n})$ for some ideal $\mathfrak{n}$ relatively prime to $S_{\mathrm{bad},G}$. Let $f_{S_1} \in \mathscr{H}^{\mathrm{ur}}(G(F_{S_1}))^{\leq\kappa}$ such that $\|f_{S_1}\|_\infty \leq 1$. Let ξ be a dominant weight. Then, if $\mathbb{N}(\mathfrak{n})$ is large enough,*

$$\frac{1}{\tau'(G)\dim(\xi)\widehat{\mu}^{\mathrm{pl}}_{S_0}(\widehat{\varphi}_{S_0})} I_{\mathrm{spec}}(\eta_\xi \otimes \varphi^\infty) = \widehat{\mu}^{\mathrm{pl}}_{S_1}(\widehat{f}_{S_1}) + O_{G,\varphi_{S_0}}(q_{S_1}^{A_{\mathrm{lv}}+B_{\mathrm{lv}}\kappa} \mathbb{N}(\mathfrak{n})^{-C_{\mathrm{lv}}})$$

for some constants $A_{\mathrm{lv}}, B_{\mathrm{lv}}, C_{\mathrm{lv}}$ depending only on G.

For clarity later, we emphasize that the implied constants in the big O depend on G and φ_{S_0}. As noted in errata on the authors' website, there is a mistake in [ST16, §7] so the alternate argument in [ST16, B] must be used for the orbital integral bounds that go into the results. This alternate argument does not provide any control on the constants or S_{bad}.

8.1.1. *Clarifying a minor detail.* As another note, there is a small detail assumed in the bound for $a_{\gamma,M}$ used in proving the weight aspect bound: corollary 6.16 used to bound the L function in the formula for $\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\mathrm{rat}})$ only applies to groups with anisotropic center. However 6.17 uses it for centralizers of elements and these can have arbitrary center. We can use to following lemma to get an alternate bound for $\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\mathrm{rat}})$ in general in terms of the bound for groups with anisotropic center:

Lemma 8.1.3. *Let G be a connected reductive group over F and $G' = G/A_G$. Then*

$$\begin{aligned}&\bar{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A})/A_{G,\infty}) \\ &\qquad = \bar{\mu}^{\mathrm{can},EP}(G'(F)\backslash G'(\mathbb{A}))\bar{\mu}^{\mathrm{can},EP}(A_G(F)\backslash A_G(\mathbb{A})/A_{A_G,\mathrm{rat}}).\end{aligned}$$

Note that the factor $\mu^{\mathrm{can},EP}(A_G(F)\backslash A_G(\mathbb{A})/A_{A_G,\mathrm{rat}})$ is a constant depending only on the field F and the dimension of A_G.

Proof. If G is quasisplit at finite v, there is a special model $\underline{G}$ over F_v. Then $\underline{G}(\mathcal{O}_v)\cap A_G(F_v)$ is a maximal (a bigger subgroup times $G(\mathcal{O}_v)$ is otherwise a bigger compact) connected compact subgroup and therefore a corresponds to a model $\underline{A}_G$ consistent with the inclusion. Consider the quotient model $\underline{G/A_G}$. By Lang's theorem, $\underline{G'}(k_v)=\underline{G}(k_v)/\underline{A_G}(k_v)$ so by Hensel's lemma and smoothness of quotient maps by smooth subgroups, $\underline{G/A_G}(\mathcal{O}_v)=\underline{G}(\mathcal{O}_v)/\underline{A_G}(\mathcal{O}_v)$. By Hilbert 90, $G'(F_v)=G(F_v)/A_G(F_v)$ for any local F_v. This gives that $G'(\mathbb{A})=G(\mathbb{A})/A_G(\mathbb{A})$ implying $G'(\mathbb{A})^1=G'(\mathbb{A})=G(\mathbb{A})^1/A_G(\mathbb{A})^1$.

Using $G'(F)=G(F)/A_G(F)$, we then get an isomorphism of topological spaces

$$G(F)\backslash G(\mathbb{A})^1\cong G'(F)\backslash G'(\mathbb{A})\times A_G(F)\backslash A_G(\mathbb{A})^1.$$

Next, $\mu^{\mathrm{can},EP}$ on $G'(\mathbb{A})$ and $G(\mathbb{A})$ induces a measure μ_A on $A_G(\mathbb{A})$. By the above factorization, it suffices to show that this equals $\mu_A^{\mathrm{can},EP}$ place by place. At the infinite place, they are the same by definition (see [ST16, §6.5]).

If G is quasisplit at finite v, then μ^{can} is characterized by giving any special subgroup volume 1. As before, $\underline{G/A_G}(\mathcal{O}_v)=\underline{G}(\mathcal{O}_v)/\underline{A}_G(\mathcal{O}_v)$. In particular, $\underline{G/A_G}(\mathcal{O}_v)$ also needs to be maximal connected so it is special. Since these are all special subgroups, this forces $\mu_A=\mu_A^{\mathrm{can}}$ at v.

If G isn't quasisplit at v, then μ^{can} is determined by transfer of a top-form $\omega_{G^{\mathrm{qs}}}$ from G^{qs} (since the normalization factor Λ in [ST16] depends only on the motive for G which depends only on the quasisplit form of G). The isomorphism $G_{\overline{k}}\xrightarrow{\sim}G^{\mathrm{qs}}_{\overline{k}}$ carries $(A_G)_{\overline{k}}$ to $(A_{G^{\mathrm{qs}}})_{\overline{k}}$ since centers are identified between inner forms. This means that $G'^{\mathrm{qs}}=G^{\mathrm{qs}}/A_{G^{\mathrm{qs}}}$ through the isomorphism over $\overline{k}$. By the previous paragraph, the defining top-forms for G'_{qs} and $A_{G^{\mathrm{qs}}}$ wedge together to that of G^{qs}. Therefore, this same property holds for G and A_G which is what we want. □

This argument is implicit but not clearly summarized in later sections of the paper.

8.2. **New bounds Set-up.** For our use, we will need a generalization of these bounds that works when $Z(G)\neq 1$ and when G isn't necessarily cuspidal. We will also need the big O, $S_{\mathrm{bad},H}$, and the constants A,B,C to be uniform over all groups H appearing in hyperendoscopic paths of G. The final statement requires some notation and will be at the end of this section.

Let ξ be a dominant weight and choose central character datum $(\mathfrak{X},\chi)$ where $A_{G,\infty}\subseteq\mathfrak{X}$ and χ is consistent with ξ. Let χ_0 be its restriction to $A_{G,\mathrm{rat}}$. We start similar to [Shi12a, thm 4.11] and [ST16, thm 9.19], instead trying to apply proposition 6.4.4. This requires making some choices:

- A cutoff function $f\in C_c^\infty(A_{G,\infty}/A_{G,\mathrm{rat}})$
- A $\varphi^{\infty_1}\in\mathscr{H}(G^\infty,\chi_0)$ such that $\overline{(\varphi^{\infty_1})}_\chi=\varphi^\infty$.
- Lots of Haar measures: fix them to be $\mu^{\mathrm{can}\times EP}$ whenever necessary.

We need to bound the term for all endoscopic groups. Considering all the previous lemmas on transfers, we are interested in the case where:

- φ and χ are unramified outside of S_0 and ∞.
- χ extends to a character on G_v.

- $(\varphi^{S,\infty})^1$ can be chosen to be $\mathrm{vol}(\mathfrak{X}^{S,\infty} \cap U^{S,\infty})^{-1}\mathbf{1}_{U^{S,\infty}}$. For endoscopic groups we will wlog expand S_0 so that $U^{S,\infty} = K^{S,\infty}$. Then this follows from the computation of transfers in section 5.5.2.
- $\varphi_s \in \mathscr{H}(G_s, K_s, \chi_s)^{\leq \kappa}$ and $\|\chi_s \varphi_s\|_\infty \leq 1$ for all $s \in S_1$.

We choose a specific φ_s^1 for $s \in S_1$ according to the following lemma.

Lemma 8.2.1. *Pick unramified character datum $(\mathfrak{X}_v, \chi_v)$ such that χ_v extends to a character on G. Let $\varphi_v \in \mathscr{H}(G_v, K_v, \chi_v)^{\leq \kappa}$ such that $\|\chi_v \varphi_v\|_\infty \leq 1$. Fix the canonical measure on $\mathfrak{X}_v$ so that $\mathrm{vol}(K \cap \mathfrak{X}_v) = 1$. Then there exists $\varphi_v^1 \in \mathscr{H}(G_v, K_v)^{\leq \kappa}$ such that $\overline{(\varphi_v^1)}_{\chi_v} = \varphi_v$ and $\|\chi_v \varphi_v^1\|_\infty \leq 1$.*

Proof. Let
$$\varphi_v = \sum_{\lambda \in X_*(A)} a_\lambda \tau_\lambda.$$
Let $A_{\mathfrak{X}_v}$ be the split part of $\mathfrak{X}_v$. Then for any $\zeta \in X_*(A_{\mathfrak{X}_v})$, $a_{\lambda+\zeta} = \chi(\zeta(\varpi))^{-1} a_\lambda$. For each λ such that $a_\lambda \neq 0$, there is a representative λ' of its class $[\lambda] \in X_*(A)/X_*(A_{\mathfrak{X}_v})$ such that $\|\lambda'\| \leq \kappa$. Let Λ be the set of all these chosen representatives. Then
$$\varphi_v^1 = \varphi_v = \sum_{\lambda \in \Lambda} a_\lambda \tau_\lambda$$
satisfies $\overline{(\varphi_v^1)}_{\chi_v} = \varphi_v$ The L^∞ bound on φ_v gives that $|\chi_v(\lambda(\varpi)) a_\lambda| = 1$ implying the needed bound on φ_v^1. □

Note. There is a small technicality there. The original χ_v chosen on G_v may not necessarily extend to G_v. However, section 4.3 still gives that $\chi_{\mathcal{H},v}$ on any $\mathcal{H}_v$ is a character λ that extends to $\mathcal{H}_v$ times χ_v. Since $Z_{G^{\mathrm{der}}}$ is finite, χ_v can be factored as a unitary character times a character on G_v so since the bounds here are only up to absolute value, this doesn't matter.

Beginning the computation:
$$\frac{1}{\tau'(G)\dim(\xi)} I_{\mathrm{spec},\chi}(\eta_\xi \otimes \varphi^\infty) = \frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty)$$
$$\sum_{M \in \mathscr{L}} \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} a_{M,\gamma} |\iota^M(\gamma)|^{-1} f(z_\infty \gamma) \frac{\Phi_M(\gamma,\xi)}{\dim \xi} O_\gamma^M((\varphi_{z^\infty}^{\infty_1})_M)\, dz.$$
Here
$$a_{M,\gamma} = \tau'(G)^{-1} \frac{|\Omega_M|}{|\Omega_G|} \frac{\overline{\mu}^{\mathrm{can},EP}(I_\gamma^M(F) \backslash I_\gamma^M(\mathbb{A}_F)/A_{I_\gamma^M,\mathbb{Q}})}{\overline{\mu}^{EP}(\bar{I}_{\gamma,\infty}^M / A_{I_\gamma^M,\infty})}.$$

This double sum breaks into three pieces: $M = G$ and $\gamma \in Z(G)$, $M = G$ otherwise, and $M \neq G$. for $M = G$, $\Phi_M(\gamma, \xi) = \mathrm{tr}\,\xi(\gamma_\infty)$. For central γ, the centralizer is everything so $|\iota^G(\gamma)| = 1$. In addition, the measure on the quotient is just counting measure on a point so $O_\gamma^M(\varphi_{z^\infty}^{\infty_1}) = \varphi^{\infty_1}(z^\infty \gamma)$. Finally,
$$a_{G,\gamma} = \tau'(G)^{-1} \frac{\overline{\mu}^{\mathrm{can},EP}(G(F)\backslash G(\mathbb{A}_F)/A_{G,\mathrm{rat}})}{\overline{\mu}^{EP}(\bar{G}_\infty / A_{G,\infty})} = \overline{\mu}^{EP}(\bar{G}_\infty/A_{G,\infty})^{-1} = 1$$
since existence of a discrete series requires that the last group is compact and therefore has EP-measure 1.This leaves us with
$$\frac{1}{\widehat{f}(0)} \frac{1}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}})} \int_{\mathfrak{X}_F \backslash \mathfrak{X}/A_{G,\mathrm{rat}}} \chi(z^\infty) \sum_{\gamma \in Z_G(F)} \varphi^{\infty_1}(\gamma) f(z\gamma) \frac{\mathrm{tr}\,\xi(z_\infty \gamma)}{\dim \xi}\, dz.$$

Next, note that by a Fourier inversion formula

$$\frac{\operatorname{tr}\xi(\gamma)}{\dim\xi} = \omega_\xi^{-1}(\gamma) = \omega_\xi(z_\infty)\omega_\xi^{-1}(z_\infty\gamma) = \omega_\xi(z_\infty)\eta_\xi(z_\infty\gamma)\eta_\xi(1)^{-1}$$

where ω_ξ is the central character for ξ. Therefore, the term inside the sum is simply $\omega_\xi(z_\infty)f(z\gamma)\varphi^1(z\gamma)$ where $\varphi^1 = \eta_\xi\varphi^{\infty_1}$.

Combining the $\omega_\xi(z_\infty)$ factor with the χ, we get a a main term

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)}\frac{1}{\operatorname{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z)\sum_{\gamma\in Z_G(F)}f(z_\infty\gamma)\varphi^1(z\gamma)\,dz \tag{5}$$

The leftovers form an error term

$$\begin{aligned}\frac{1}{\widehat{f}(0)}&\frac{1}{\operatorname{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z^\infty)\\ &\left(\sum_{\substack{\gamma\in[G(F)]^{\mathrm{ss}}\\ \gamma\notin Z(G)}}a_{G,\gamma}|\iota^G(\gamma)|^{-1}f(z_\infty\gamma)\frac{\operatorname{tr}\xi(\gamma_\infty)}{\dim\xi}O^G_{z^\infty\gamma}(\varphi^{\infty_1})\right.\\ &\left.+\sum_{\substack{M\in\mathscr{L}\\ M\neq G}}\sum_{\gamma\in[M(F)]^{\mathrm{ss}}}a_{M,\gamma}|\iota^M(\gamma)|^{-1}f(z_\infty\gamma)\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_{z^\infty\gamma}(\varphi_M^{\infty_1})\right)dz\end{aligned} \tag{6}$$

We compute these separately since they require pretty different ideas to understand.

8.3. **The Main Term.**

8.3.1. *Central Fourier transforms.* This section uses material on Fourier analysis on non-abelian group. See [Fol16] chapter 7 for a good reference. That p-adic reductive groups are type I is a classic result from [Ber74].

The main term initially simplifies in terms of the Fourier transform $\bar{f}_S$ of f_S with respect to $Z(G_S)$. To actually get a reasonable interpretation, we need to relate $\bar{f}_S$ to $\widehat{f}_S$. Therefore, for this subsection only, redefine $G = G_S$, $Z = (Z_G)_S$ and consider arbitrary $f \in \mathcal{H}(G)$. Note that the following results probably hold for general type I unimodular groups with an appropriate modification of $\mathcal{H}(G)$ to a more complicated function space; the case of p-adic groups just makes the analytic issues a lot nicer.

There is a map from $P : \widehat{G} \to \widehat{Z}_G$ taking π to its central character ω_π.

Lemma 8.3.1. *P is measurable with respect to the usual sigma algebras on $\widehat{G}$ and $\widehat{Z}$.*

Proof. Fix a Hilbert space H_i of each dimension and consider the set Π of irreducible unitary representations of G on some H_i. Consider the functions on Π defined by $\pi \mapsto \langle\pi(g)v, w\rangle$ for $g \in G$ and v, w in the appropriate Hilbert space. Since G is type I, the σ-algebra on $\widehat{G}$ is the quotient of the smallest one on Π that makes these functions continuous. An analogous statement holds for $\widehat{Z}$.

Then, since central elements act by central characters, the functions defined by $z \in Z$ on $\widehat{G}$ are exactly the pullbacks by P of the analogous functions on $\widehat{Z}$. $\square$

Denote the Fourier transform of $f|_{Z_G}$ by $\bar{f}$.

Lemma 8.3.2. *For any functions* $\varphi \in \mathcal{H}(\widehat{Z})$ *and* $f \in \mathcal{H}(G)$

$$\int_{\widehat{Z}} \varphi \bar{f}\, d\mu^{\mathrm{pl}} = \int_{\widehat{G}} (\varphi \circ P) \widehat{f}\, d\mu^{\mathrm{pl}}.$$

Proof. Using both Fourier inversion theorems, for any $z \in Z$

$$\int_{\widehat{Z}} \omega(z) \bar{f}(\omega)\, d\omega = f(z) = \int_{\widehat{G}} \omega_\pi(z) \widehat{f}(\pi)\, d\pi.$$

For a general φ

$$\begin{aligned}
\int_{\widehat{G}} \varphi(\omega_\pi) \widehat{f}(\pi)\, d\pi &= \int_{\widehat{G}} \int_Z \bar{\varphi}(z) \omega_\pi^{-1}(z) \widehat{f}(\pi)\, dz\, d\pi \\
&= \int_Z \bar{\varphi}(z) \int_{\widehat{G}} \omega_\pi^{-1}(z) \widehat{f}(\pi)\, d\pi\, dz \\
&= \int_Z \bar{\varphi}(z) \int_{\widehat{Z}} \omega^{-1}(z) \bar{f}(\omega)\, d\omega\, dz \\
&= \int_{\widehat{Z}} \int_Z \bar{\varphi}(z) \omega^{-1}(z) \bar{f}(\omega)\, dz\, d\omega = \int_{\widehat{Z}} \varphi(\omega) \bar{f}(\omega)\, d\omega
\end{aligned}$$

so we are done. □

Intuitively, we can therefore think of $\bar{f}(\omega)$ as an average of $\widehat{f}$ over representations with central character ω. To make this notion precise, push $\widehat{f}\, d\mu^{\mathrm{pl}}$ forward to a measure $\mu_{\widehat{f}}$ on $\widehat{Z}_G$.

Lemma 8.3.3. $\mu_{\widehat{f}}$ *is absolutely continuous with respect to Haar measure on* $\widehat{Z}_G$*.*

Proof. Let $X \subset \widehat{Z}$ have measure 0. By σ-finiteness, outer regularity, and continuity of $\bar{f}$, for any $\epsilon > 0$, X is contained in a union X_ϵ of countably many compact open sets such that $\int_{X_\epsilon} \bar{f} d\mu^{\mathrm{pl}} < \epsilon$. Then

$$\mu_{\widehat{f}}(X) \le \mu_{\widehat{f}}(X_\epsilon) = \int_{\widehat{G}} \mathbf{1}_{P^{-1}(X_\epsilon)} \widehat{f}\, d\mu^{\mathrm{pl}} = \int_{\widehat{Z}} \mathbf{1}_{X_\epsilon} \bar{f}\, d\mu^{\mathrm{pl}} < \epsilon.$$

Since this is true for every $\epsilon > 0$, $\mu_{\widehat{f}}(X) = 0$ and we are done. □

Therefore we can define:

Definition. The conditional Plancherel expectation is the Radon-Nikodym derivative

$$E^{\mathrm{pl}}(\widehat{f}|\omega) := \frac{d\mu_{\widehat{f}}}{d\mu^{\mathrm{pl}}_{Z_G}}(\omega).$$

This is defined up to a set of measure 0. However, note that the measures $E^{\mathrm{pl}}(\widehat{f}|\omega)\, d\mu^{\mathrm{pl}}$ and $\bar{f} d\mu^{\mathrm{pl}}$ are the same on $\widehat{Z}$ so:

Corollary 8.3.4. $E^{\mathrm{pl}}(\widehat{f}|\omega)$ *can be taken to be continuous. If so* $E^{\mathrm{pl}}(\widehat{f}|\omega) = \bar{f}(\omega)$*.*

We borrow the notation of conditional expectation from probability theory to emphasize first, the same definition in terms of Radon-Nikodym derivatives and second, the analogous intuition as an average over the measure-zero set of representations with central character ω. Beware that under this analogy, E^{pl} is an unnormalized expectation since $E^{\mathrm{pl}}(\widehat{f}|\omega) = \bar{f}$ and the operation $f \mapsto \bar{f}$ multiplies in a factor of $[Z]$ to the dimensions of f.

8.3.2. *Main term computation.*

Proposition 8.3.5. *The main term* (5) *simplifies to*

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)}\sum_{\omega_S\in\widehat{Z}_{S,L,\xi,\chi}} E^{\mathrm{pl}}(\widehat{\varphi}_S|\omega_S)$$

where $Z'_{S,\infty}=Z_{G_{S,\infty}}/A_{G,\mathrm{rat}}$, $L=Z_G(F)\cap U^{S,\infty}$, *and* $\widehat{Z}_{S,L,\xi,\chi}$ *is the set of* $\omega_S\in\widehat{Z}_S$ *such that* $\omega_S|_L=\omega_\xi|_L$ *and* $\omega_S|_{\mathfrak{X}_S}=\chi_S$. *The normalizing factors are*

- $\mu=\mu_{Z'_\infty}/\mu^{EP}_{Z'_\infty}$ *where* $\mu_{Z'_\infty}$ *is the measure chosen on* Z'_∞ *to compute the other terms*
- X *is the finite group* $\mathfrak{X}^{S,\infty}/\mathfrak{X}^{S,\infty}\cap\overline{Z_G(F)}Z_{U^{S,\infty}}$ *where the closure is taken in* $Z^{S,\infty}$.

For shorthand, we denote this sum $E(\widehat{\varphi}_S|\omega_\xi,L,\chi_S)$.

Proof. Start with (5):

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)}\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z)\sum_{\gamma\in Z_G(F)}f(z_\infty\gamma)\varphi^1(z\gamma)\,dz.$$

$Z_G(F)$ is cocompact and discrete inside $Z^1=Z_G(\mathbb{A})/A_{G,\mathrm{rat}}$. Then by Poisson summation, the inner sum becomes

$$\frac{1}{\mathrm{vol}(Z/Z_G(F))}\sum_{\substack{\omega\in\widehat{Z^1}\\ \omega(Z_G(F))=1}}\omega^{-1}(z)\overline{f\varphi^1}(\omega)$$

since if $\varphi_z:x\mapsto\varphi(zx)$, then $\bar{\varphi}_z(\omega)=\omega^{-1}(z)\bar{\varphi}(\omega)$. Integrating over z, all terms with $\omega\neq\chi$ vanish so (5) becomes

$$\frac{1}{\widehat{f}(0)}\frac{1}{\mathrm{vol}(Z^1/Z_G(F))}\sum_{\substack{\omega\in\widehat{Z^1}\\ \omega(Z_G(F))=1\\ \omega|_{\mathfrak{X}}=\chi}}\overline{f\varphi}(\omega).$$

Here we use that φ^∞ has Fourier transforms on any ω^∞ in the sum and $\overline{\varphi^\infty}=\overline{\varphi^{\infty 1}}$ on these characters. We next break this up into local components to make it more interpretable. First,

$$\bar{\varphi}(\omega)=\overline{f\eta_\xi}(\omega_\infty)\bar{\varphi}_S(\omega_S)\bar{\varphi}^{S,\infty}(\omega^{S,\infty})$$

after choosing Haar measures on the components of Z^1. Let ω_ξ be the central character associated to ξ. For any ψ compactly supported on $Z'_\infty=Z_{G,\infty}/A_{G,\mathrm{rat}}$, by lemma 8.3.2 applied to $G_\infty/A_{G,\mathrm{rat}}$,

$$\int_{\widehat{Z'_\infty}}\psi(\omega)\overline{f\eta_\xi}(\omega)\,d\omega^{\mathrm{pl}}=\int_{(G_\infty/A_{G,\mathrm{rat}})^\vee}\psi(\omega_\pi)\widehat{f\eta_\xi}(\pi)\,d\pi^{\mathrm{pl}}=$$
$$\int_{\widehat{A}}\int_{\widehat{G^1_\infty}}\psi(\omega\omega_\pi)\widehat{f\eta_\xi}(\pi\otimes\omega)\,d\pi\,d\omega=\mathrm{vol}_{\widehat{G^1_\infty}}(\Pi_{\mathrm{disc}}(\xi))\int_{\widehat{A}}\psi(\omega_\xi\omega)\widehat{f}(\omega)\,d\omega^{\mathrm{pl}}$$

where $A=A_{G,\infty}/A_{G,\mathrm{rat}}$. Picking measures to factor the integral, the measure chosen on Z'_∞ induces Plancherel measure $\widehat{Z}'_\infty$ which restricts to a measure on $\widehat{A}$ lying discretely inside. This corresponds to the quotient measure on A coming from setting $\mathrm{vol}(Z^1_\infty)=1$. If we had EP-measure on Z'_∞, this would therefore induce EP-measure on G^1_∞ so let $\mu=\mu_{Z'_\infty}/\mu^{EP}_{Z'_\infty}$.

By Fourier inversion, this finally becomes

$$\int_{\widehat{Z'_\infty}} \psi(\omega)\overline{f\eta_\xi}(\omega)\, d\omega^{\mathrm{pl}} = \mu\eta_\xi(1)\int_{\widehat{Z'_\infty}} \psi(\omega_\xi\omega)\mathbf{1}_{\widehat{A}}(\omega)\widehat{f}(\omega)\, d\omega^{\mathrm{pl}}$$

so we get

$$\overline{f\varphi_\infty}(\omega) = \mu\eta_\xi(1)\delta_{\omega|_{Z'_\infty}=\omega_\xi|_{Z'_\infty}}\widehat{f}(\omega\omega_\xi^{-1}).$$

In our case $A_{G,\infty} \subseteq \mathfrak{X}_\infty$ so for $\omega|_{\mathfrak{X}_\infty} = \omega_\xi|_{\mathfrak{X}_\infty}$, this simplifies to

$$\overline{f\varphi_\infty}(\omega) = \mu\eta_\xi(1)\delta_{\omega_\infty=\omega_\xi}\widehat{f}(0).$$

Next, let $Z_{U^{S,\infty}} = U^{S,\infty}\cap Z^1$. Since it is an integral over a subgroup

$$\bar{\varphi}^{S,\infty}(\omega^{S,\infty}) = \begin{cases}\mathrm{vol}(Z_{U^{S,\infty}}) & \omega^{S,\infty}|_{Z_{U^{S,\infty}}} = 1\\ 0 & \text{else}\end{cases}.$$

In total, the terms that don't vanish are

$$\frac{\mu\,\mathrm{vol}(Z_{U^{S,\infty}})}{\mathrm{vol}(Z^1/Z_G(F))}\bar{\varphi}_S(\omega_S)$$

for every character ω satisfying

(1) $\omega(Z_G(F)) = 1$
(2) $\omega|_{\mathfrak{X}} = \chi$.
(3) $\omega_\infty = \omega_\xi$
(4) $\omega^{S,\infty}(Z_{U^{S,\infty}}) = 1$

We try to characterize such ω. Consider $\omega = \omega_\infty\omega_S\omega^{S,\infty}$. Let $L = Z_G(F)\cap U^{S,\infty}$. These conditions require that $\omega_S\omega_\infty = 1$ on L and that $\omega_S\chi_S^{-1} = 1$ on $\mathfrak{X}_S$. Given ω_S satisfying this, the conditions determine $\omega^{S,\infty} = \omega_S^{-1}\omega_\infty^{-1}$ on $Z_G(F)$. Since the determined $\omega^{S,\infty}$ is trivial on $Z_G(F)\cap U^{S,\infty}$ it extends to a continuous character on $\overline{Z_G(F)} \subseteq Z^{S,\infty}$. The character $\omega^{S,\infty}$ is also determined on $U^{S,\infty}$ and $\mathfrak{X}^{S,\infty}$ so in total, the possible choices of $\omega^{S,\infty}$ are those that restrict to a particular value on $E^{S,\infty} = \overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}$.

Since quotient maps of groups are open, $Z_{U^{S,\infty}}$ is open mod $\overline{Z_G(F)}$. Therefore, since $Z^{S,\infty}/\overline{Z_G(F)}$ is compact, $Z^{S,\infty}/E^{S,\infty}$ is a finite group. Therefore the number of choices is in bijection with $Z^{S,\infty}/E^{S,\infty}$.

By comparing $U^{S,\infty}$ times a fundamental domain for $Z^1/Z_G(F)$ to a fundamental domain for $Z^1_{S,\infty}/L$, we get

$$\frac{\mathrm{vol}(Z_{U^{S,\infty}})}{\mathrm{vol}(Z^1/Z_G(F))} = \frac{1}{\mathrm{vol}(Z'_{S,\infty}/L)|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|}.$$

Therefore, pulling out just the non-zero terms in the sum gives

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)}\sum_{\substack{\omega_S\in\widehat{Z}_S\\ \omega_S\omega_\xi^{-1}(L)=1\\ \omega_S\chi_S^{-1}(\mathfrak{X}_S)=1}}\bar{\varphi}_S(\omega_S)$$

where

$$|X|^{-1} = \frac{|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}|}{|Z^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|} = |\overline{Z_G(F)}Z_{U^{S,\infty}}\mathfrak{X}^{S,\infty}/\overline{Z_G(F)}Z_{U^{S,\infty}}|^{-1}.$$

An application of lemma 8.3.2 to $G_S/\mathfrak{X}_S$ then finishes the argument. $\square$

The formula here is complicated and requires some discussion. First, ω_ξ determines a character on L consistent with χ_S. Therefore, ω_ξ and χ_S together determine a character λ on $L\mathfrak{X}_S$. The term $E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$ can be thought of as some sort of normalized average of $\hat{\varphi}_S$ along representations with central character extending λ.

Note that if Z_G is compact and $\mathfrak{X} = A_{G,\mathrm{rat}} = 1$, we can choose a measure so that $\mu(Z_v) = 1$ for all v. This gives $\mu = 1$ so

$$\frac{1}{|X|}\frac{\mu}{\mathrm{vol}(Z_{S,\infty}/L)} = \frac{1}{\mu(Z_{S,\infty}/L)} = |L| = |Z_G(F) \cap U^{S,\infty}|$$

and $\widehat{Z}_S$ has the counting measure. Therefore, $E^{\mathrm{pl}}(\widehat{f}|\omega)$ is the literal integral of $f\,d\mu^{\mathrm{pl}}$ over representations with character ω. This is in line with the result in [KST16].

This computation can be compared to the very short argument at the beginning of [FL18, §2]. Reconciling notation, Θ in that paper is the same as L here and S there is $S \cup \infty$ here. Our argument is much longer since we are factoring out the infinite part of $\mu_{\Theta,S}$ requiring a sum over a complicated set of ω_S instead of just a term for $E^{\mathrm{pl}}(\varphi_{S,\infty}|1)$. In addition, issues involving $\mathfrak{X}$ appear.

8.3.3. *Main term bound.* It will also be useful to have a very rough bound on the magnitude of this main term.

Proposition 8.3.6. *Let $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq\kappa}$ such that $|\chi_{S_1}(x)\varphi_{S_1}(x)| \leq 1$ for all x. Then for some constant C depending only on G, the main term* (5) *is $O_{\varphi_{S_0}}(q_{S_1}^{C\log\kappa})$ where the implied constant is independent of φ_{S_1} and ξ. .*

Proof. Start with the expression (5):

$$\frac{1}{\widehat{f}(0)\eta_\xi(1)}\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\int_{\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}}}\chi(z)\sum_{\gamma\in Z_G(F)} f(z_\infty\gamma)\varphi^1(z\gamma)\,dz.$$

Here it is actually convenient to evaluate the integral, giving the central terms in 6.4.5:

$$\frac{1}{\mathrm{vol}(\mathfrak{X}_F\backslash\mathfrak{X}/A_{G,\mathrm{rat}})}\sum_{\gamma\in[Z_G(F)]^{\mathrm{ss}}_{\mathfrak{X}}}\omega_\xi^{-1}(\gamma)\varphi(\gamma).$$

The sum becomes

$$\sum_{\gamma\in[Z_G(F)]^{\mathrm{ss}}_{\mathfrak{X}}}\chi_{S_1}(\gamma)\varphi_{S_1}(\gamma)\chi_{S_0}(\gamma)\varphi_{S_0}(\gamma)\varphi^{S,\infty}.$$

By construction, $\varphi^1_{S_1}$ and $(\varphi^1)^{S,\infty}$ intersect every $\mathfrak{X}$-class in $Z_G(F)$ that φ_{S_1} does. Pick a $\varphi^1_{S_0}$ with the same property. Finally let $U_\infty \subset Z_\infty$ be such that every point with non-zero summand can be translated into it. We will choose specific U_∞ later.

We may then instead bound

$$\sum_{\gamma\in[L]^{\mathrm{ss}}_{\mathfrak{X}}}\mathbf{1}_{U_\infty}\chi_{S_1}(\gamma)\varphi^1_{S_1}(\gamma)\chi_{S_0}(\gamma)\varphi^1_{S_0}(\gamma)$$

where $L = Z_G(F) \cap U^{S,\infty}$. We will do this by first bounding the number of terms in this sum by the size of $L \cap U_\infty \operatorname{Supp}\varphi_S$.

If K_s are the chosen maximal compacts, for each $s \in S_1$, $\varphi^1_s \in \mathscr{H}(G_s, K_s)^{\leq\kappa}$ so φ^1_s is a linear combination of indicator functions $\mathbf{1}_{K_s\lambda(\omega)K_s}$ for a number of possible ω that is polynomial in κ. Therefore, for some constant C, φ_{S_1} is supported on a union of $O(\kappa^{C|S_1|})$ double cosets of K_{S_1}. Since $\varphi^1_{S_0}$ is compactly supported, this gives that φ^1_S is supported on a union of $O_{\varphi^1_{S_0}}(\kappa^{C|S_1|})$ double cosets of K_S. Note that $\kappa^{C|S_1|} \leq \kappa^{C \log q_{S_1}} = q_{S_1}^{C \log \kappa}$.

Let $Z_{K_S} = Z_S \cap K_S$ be the maximal compact for abelian Z_S. Consider double coset $D = K_S \alpha K_S$. If $D \cap Z_S \neq \emptyset$, wlog let α be in the intersection. Then $D = \alpha K_S$ and $D \cap Z_S$ is a union of cosets of Z_{K_S} in Z_S. Consider two of these cosets xZ_{K_S} and yZ_{K_S}. Then there exists $k \in K_S$ such that $x = ky \implies k = xy^{-1} \implies k \in Z_S$. Therefore $x \in Z_{K_S}$ and the two cosets are equal. In total, $D \cap Z_S$ is either empty or a coset of K_S. This finally implies that $\operatorname{Supp} \varphi \cap Z_S$ is contained in a union of $O_{\varphi^1_{S_0}}(q_{S_1}^{C \log \kappa})$ cosets of Z_{K_S}.

To continue, we need to choose a particular U_∞. First, Z_∞ factors as $A_{G,\infty}/A_{G,\mathrm{rat}}$ times a compact real torus Z_c. Let U'_∞ be some subset of $A_{G,\infty}/A_{G,\mathrm{rat}}$ and choose f to be the pullback of the characteristic function of U'_∞ through H_{G_∞} (we aren't technically allowed to do this due to the smoothness restriction but we can take a close enough approximation in L^1). Then f has support on $U_\infty = U'_\infty \times Z_c$.

Let $c_1 = |L \cap Z_{K_S} U_\infty|$ and assume for now that this is finite. If coset $C = \alpha_S Z_{K_S} U_\infty$ contains an element of L, the wlog let this element be α_S. Multiplying by α_S^{-1} bijects $L \cap C$ to $L \cap Z_{K_S} U_\infty$ so $|L \cap C| = c_1$. Counting all possible cosets, $|L \cap \operatorname{Supp}(f\varphi_S)| = O_{\varphi_{S_0}}(c_1 q_{S_1}^{C \log \kappa})$. By similar argument, $|L \cap \operatorname{Supp}(f\varphi_S)_z| = O_{\varphi_{S_0}}(c_z q_{S_1}^{C \log \kappa})$ where $c_z = |L \cap Z_{K_S} z_\infty^{-1} U_\infty|$.

It remains to bound

$$c_z = |Z_G(F) \cap Z_{K_S} Z_{U^{S,\infty}} z_\infty^{-1} U_\infty| \leq |Z_G(F) \cap Z_{K_S} Z_{K^{S,\infty}} z_\infty^{-1} U_\infty|$$

where $K^{S,\infty}$ is the maximal compact (since $Z^{S,\infty}$ is abelian). This is finite since $Z_G(F)$ is discrete inside $Z/A_{G,\mathrm{rat}}$. Then, $Z_G(F) \cap Z_{K_S} Z_{K^{S,\infty}}$ is a co-compact lattice inside Z_∞. It is still so when projecting down to $A_{G,\infty}/A_{G,\mathbb{Q}}$. Choose U'_∞ to be a fundamental domain for this lattice. Then $c_z = 1$ for all z and $\widehat{f}(0) = \mathrm{vol}(U'_\infty)$ which depends only on G.

Finally, the terms in the sum all have norm 1 up to the factor $\chi_{S_0}\varphi^1_{S_0}$ that depends on φ_{S_0}. Therefore the sum is $O_{\varphi_{S_0}}(q_{S_1}^{C \log \kappa})$ for all z. The factor in front depends only on $(G, \mathfrak{X})$ so the entire term is $O_{\varphi_{S_0}, G}(q_{S_1}^{C \log \kappa})$. □

8.4. **The Error Term.** We need to do a few things to bound the error term. First, the orbital integral bounds used only apply to elements in $\mathscr{H}(H_v, K_{H,v})^{\leq \kappa'}$ so we need to extend them to spaces like $\mathscr{H}(H_v, K_{H,v}, \chi)^{\leq \kappa}$.

Second, a given group has infinitely many endoscopic groups. Unfortunately, the alternate proof of orbital integral bounds in [ST16, §B] which gives no control over constants and S_{bad}. Therefore, it is useful to have some result that allows the use of the same constants and S_{bad} for all groups.

.

Finally, we need to do another due-diligence check that one, all the lemmas used in the proofs of theorems 8.1.1 and 8.1.2 still hold over to the non-trivial center case and two, all the constants from those lemmas can also be uniformly bounded over all hyperendoscopic groups that contribute a non-zero term. This in particular uses the correction to [ST16, cor 6.17].

8.4.1. *Uniform bounds for orbital integrals.* The model-theoretic method for bounding orbital integrals gives the following

Theorem 8.4.1 ([ST16, thm B.2]). *Let ξ be the root datum for an unramified group over some non-Archimedean local field (so the Galois action is determined by the Frobenius action). Choose a norm of the form $\|\cdot\|'_{\mathcal{B}}$ on $X_*(A)$. Then there exist T, a_ξ, b_ξ depending only on $(\xi, \|\cdot\|')$ such that for all non-Archimedean local fields F (including ones of positive characteristic) with residue field degree $q \geq T$ the following holds:*

Let G^F be the unramified group over F with root datum ξ, K a hyperspecial of G^F, A a maximal split torus, and ϖ a uniformizer for F. Then for all $\lambda \in X_*(A)$ with $\|\lambda\|' \leq \kappa$ and semisimple $\gamma \in G^F(F)$:

$$|O_\gamma(\tau_\lambda^{G^F})| \leq q^{a_\xi \kappa + b_\xi} D^{G^F}(\gamma)^{-1/2}$$

where as before, $\tau_\lambda^{G^F} = \mathbf{1}_{K\lambda(\varpi)K}$.

By the following and lemma, we can choose a_ξ, b_ξ, T uniformly over all H appearing in an endoscopic path of G and places v where H is unramified:

Lemma 8.4.2. *Let H be a group appearing in a hyperendoscopic path for G, M_H a Levi of H, v a place where H is unramified, and ξ the unramified root data for $(M_H)_v$. Then ξ is an element of a finite set depending only on G.*

Proof. The (co)root spaces of M_H are isomorphic to those of G and the (co)roots of M_H are a subset of those of G so there are only finitely many possibilities for the root system of M_H (without Galois action) since its rank is bounded by a finite number through iteratively applying lemma 3.2.1. Then, there are only finitely many ways for Frobenius to map into the automorphisms of this root systems. □

This bound is extremely rough—in any application one should use properties of the exact group being studied to describe the set more explicitly.

8.4.2. *Error term bound for weight aspect.* We can now show

Proposition 8.4.3. *Assume that $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq \kappa}$ with $\|\chi_{S_1}\varphi_{S_1}\|_\infty \leq 1$. Consider error term (6) for any group H unramified on S_1 and appearing in an endoscopic path of G with induced central character datum $(\mathfrak{X}, \chi)$ such that $A_{H,\infty} \subseteq \mathfrak{X}$ and χ is unramified on S_1. It is $O_{\varphi_0,H}(q_{S_1}^{A_{\mathrm{wt},H}+B_{\mathrm{wt},H}\kappa} m(\xi)^{-C_{\mathrm{wt},H}})$ for some constants A, B, C as long as S_1 contains no fields with residue degree less than some M_G uniform over all H.*

Proof. Let S_{bad} be from proposition 8.4.1. This is then a due-diligence check that all the steps in [ST16, thm 9.19] still hold. We start by evaluating the integral in (6) getting term

$$\frac{\mathrm{vol}(\mathfrak{X}^1_\infty)}{\mathrm{vol}(\mathfrak{X}_F \backslash \mathfrak{X}/A_{H,\mathrm{rat}})} \left(\sum_{\substack{\gamma \in [H(F)]^{\mathrm{ss}} \\ \gamma \notin Z(H)}} a_{H,\gamma} |\iota^H(\gamma)|^{-1} |\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \frac{\operatorname{tr}\xi(\gamma_\infty)}{\dim \xi} O_\gamma^M(\varphi_M^\infty) \right.$$
$$\left. + \sum_{\substack{M \in \mathscr{L}_H \\ M \neq H}} \sum_{\gamma \in [M(F)]^{\mathrm{ss}}} a_{M,\gamma} |\iota^H(\gamma)|^{-1} |\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \frac{\Phi_M(\gamma, \xi)}{\dim \xi} O_\gamma^M((\varphi^\infty)_M) \right).$$

Wlog, expand S_0 so that φ is the characteristic function of a hyperspecial $K^{S,\infty}$ away from $S \cup \infty$ and that S_{bad} is contained in S_0. If a conjugacy class intersects the support of φ_{S_1}, then we can scale it by an element of $\mathfrak{X}_S$ so that it intersects the support of $\varphi^1_{S_1}$. The same holds for $\varphi^{S,\infty}$ which has support $K^{S,\infty}$. Choose φ_{S_0} and φ_S similarly and let their supports after taking constant terms to M be $U_{S_0,M}$ and $U_{\infty,M}$. We can then replace terms in the sum through the rule

$$\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_\gamma((\varphi^\infty)_M) \mapsto \mathbf{1}_{U_{\infty,M}}\frac{\Phi_M(\gamma,\xi)}{\dim\xi}O^M_\gamma((\varphi^\infty)^1_M).$$

Let $U_{S_1,M} = \operatorname{Supp}\mathscr{H}^{\mathrm{ur}}(M_{S_1})^{\le\kappa}$. Let Y_M be the set of semisimple rational conjugacy classes intersecting the set $U_{S_1,M}U_{S_0,M}U_{\infty,M}K^{S,\infty}_M$. The number of terms in the sum is less than or equal $|Y_M|$.

We check that each of factors can be bounded as in the proof of [ST16, thm 9.19]. The finite set of places $S_{M,\gamma}$ disjoint from S can be defined in the same way. Then:

- [ST16, cor 6.17] still applies to $a_{M,\gamma}$, (see the missing step lemma 8.1.3 for why this works for general center).
- The bound in [ST16, lem 6.11] still applies to the $\Phi_M(\gamma,\xi)$ terms. There is an extra factor of $\chi^{-1}_\infty(\gamma_\infty)$.
- A version of [ST16, thm A.1] modified to work on functions with central character still applies to bound $O^M_\gamma(\varphi_{S_0,M})$. There is an extra factor of $\chi^{-1}_{S_0}(\gamma_{S_0})$.
- Proposition 8.4.1 still bounds $O^M_\gamma(\varphi_{S_1,M})$. There is an extra factor of $\chi^{-1}_{S_1}(\gamma_{S_1})$.
- Proposition 8.4.1 still gives the same bound for $O^M_\gamma(\varphi_{v,M})$ for $v \in S_{M,\gamma}$ since $M_H \le M_G$.There is again an extra factor of χ.
- [ST16, lem 2.18] and [ST16, lem 2.21] still provide a bound on the D^M terms since we can still construct the embedding from [ST16, prop 8.1].
- $|Y_M|$ can still be bounded bound by [ST16, cor 8.10] (this also applies to groups with general center).
- $|\operatorname{Stab}_{\mathfrak{X}}(\gamma)|^{-1} \le 1$

Since χ is trivial at rational elements, all the χ_v terms cancel. Therefore, the entire term can similarly be bounded by

$$O(q_{S_1}^{A_{\mathrm{wt},H}+B_{\mathrm{wt},H}\kappa}m(\xi)^{-c_{\mathrm{wt},H}})$$

folding in the constant that only depends on H and $\mathfrak{X}$. □

This very weak uniformity is all we will need for the weight aspect.

9. Final Computation

9.1. **Weight aspsect.** Assume the previous conditions on $(G,\mathfrak{X},\chi)$ from section 7.1. Let π_k be a sequence of discrete series representations of $G(\mathbb{R})$ such that their corresponding finite-dimensional representations ξ_k have regular weights $m(\xi_k) \to \infty$. Let S_1 be disjoint from $S_{\mathrm{bad},G}$: the set of places with residue degree less than the uniform M_G from proposition 8.4.3. Choose constant S_0, φ_{S_0} and $U^{S,\infty}$ to define a sequence of families $\mathcal{F}_k$ for each ξ_k.

Theorem 9.1.1. *There are constants $A'_{G,\mathrm{wt}}$ and $B'_{G,\mathrm{wt}}$ such that for any φ_{S_0} and $\varphi_{S_1} \in \mathscr{H}(G_{S_1}, K_{S_1}, \chi_{S_1})^{\leq \kappa}$,*

$$\frac{\bar{\mu}^{\mathrm{can}}(U^{S,\infty}_{\mathfrak{X}})}{\tau'(G)\dim(\xi_k)} \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_S(\pi)$$
$$= E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S) + O(q_{S_1}^{A'_{\mathrm{wt}}+B'_{\mathrm{wt}}\kappa} m(\xi_k)^{-1})$$

(using notation from corollary 7.2.1 and theorem 8.3.5). The constants in the error depend on $(G, \mathfrak{X}, \chi)$, φ_{S_0}, and $U^{S,\infty}$.

Proof. For $\|\varphi_{S_1}\chi_{S_1}\|_\infty \leq 1$, let

$$\varphi_k = \varphi_{\pi_k} \otimes \mathbf{1}_{U^{S,\infty},\chi} \otimes \varphi_{S_1} \otimes \varphi_{S_0}$$

as in section 7.1. Let $\varphi^1_k = \eta_{\xi_k} \otimes \varphi^\infty_k$. Then φ_k and φ^1_k are unramified outside of S. Let $\mathcal{A}$ be the set of hyperendoscopic tuples that contribute a non-zero value to the hyperendoscopy formula as in lemma 5.6.1.

Then using the hyperendoscopy formula

$$\frac{1}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k)$$
$$= \frac{1}{\tau'(G)\dim(\xi_k)} \left(I^G_{\mathrm{disc}}(\varphi^1_k) + \sum_{\mathcal{H} \in \mathcal{A}} \iota(G, \mathcal{H}) I^{H_{n_\mathcal{H}}}_{\mathrm{disc}}((\varphi^1_k - \varphi_k)^{\mathcal{H}}) \right).$$

We choose arbitrary transfers of φ_0. Choose $(\mathbf{1}_{K^{S,\infty}_G})^{\mathcal{H}}$ according to lemma 5.5.5 since by lemma 5.6.1, $\mathcal{H}$ stays unramified away from S, ∞. Let $\Pi_{\mathrm{disc}}(\xi_k)$ be the L-packet containing π_k and let its size be X_k. Then

$$(\varphi^1_k - \varphi_k)^\infty = \varphi^\infty_k \qquad (\varphi^1_k - \varphi_k)_\infty = \frac{1}{X_k} \sum_{\pi_k \neq \pi \in \Pi_{\mathrm{disc}}(\xi_k)} \varphi_\pi - \frac{X_k - 1}{X_k} \varphi_{\pi_k}.$$

By proposition 5.1.5, we can choose the infinite part transfer to be a linear combination of EP-functions

$$\sum_{\xi \in \Xi_{\xi_k,\mathcal{H}}} c_\xi \eta_\xi$$

for some constants

$$|c_\xi| \leq (X_k - 1)\frac{1}{X_k} + \frac{X_k - 1}{X_k} \leq 2.$$

Now, checking some conditions:

- All groups in the hyperndoscopic paths are and unramified on S_1 and cuspidal at infinity with $\mathfrak{X}_\mathcal{H} \supseteq A_{\mathcal{H},\infty}$ by lemma 5.6.1
- Let $\chi_\mathcal{H}$ be the character determined by $\mathcal{H}$ as in section 4.3. The transfer $\chi_{\mathcal{H},S_1}\varphi^{\mathcal{H}}_{S_1}$ can be chosen to be in $\mathscr{H}(G_{S_1}, K_{S_1}, \chi_{\mathcal{H},S_1})^{\leq\kappa}$ and have L^∞-norm bounded by some $q_{S_1}^{E_\mathcal{H}\kappa}\kappa^{|S_1|}$ by repeated application of lemma 5.5.4. We can apply this due to the above.
- The ξ are regular by lemma 5.2.1
- Wlog enlarge S_0 so that $U^{S,\infty} = K^{S,\infty}$. Then $\mathbf{1}^{\mathcal{H}}_{U^{S,\infty}}$ is still the indicator function of an open compact subgroup averaged over $\chi^{S,\infty}_{\mathcal{H}}$.

We can therefore apply the main term bound in proposition 8.3.6 and the error term bound in propostion 8.4.3 to each term in the sum and get

$$I^{\mathcal{H}}_{\mathrm{disc}}((\varphi^1_k-\varphi_k)^{\mathcal{H}}) = I^{\mathcal{H}}_{\mathrm{spec}}((\varphi^1_k-\varphi_k)^{\mathcal{H}}) = \sum_{\xi\in\Xi_{\xi_k,\mathcal{H}}} U_{\xi,\mathcal{H}}\dim(\xi) O_{\varphi_0^{\mathcal{H}},U^{S,\infty},\mathcal{H}}(q_{S_1}^{(A_{\mathrm{wt},\mathcal{H}}+E_{\mathcal{H}}+\epsilon)\kappa+B_{\mathrm{wt},\mathcal{H}}})$$

for some constant $U_{\xi,\mathcal{H}}$. We use here that $O(\kappa^{C|S_1|})O(q_{S_1}^{(A+E)\kappa+B}) = O(q_{S_1}^{(A+E+\epsilon)\kappa+B})$.

By the computation in 8.3.5 and the error term bound 8.4.3,

$$\frac{1}{\tau'(G)\dim(\xi_k)} I^G_{\mathrm{disc}}(\varphi^1_k) = E + O(q_{S_1}^{A_{\mathrm{wt},G}+B_{\mathrm{wt},G}\kappa_1} m(\xi_k)^{-C_{\mathrm{wt},G}})$$

where we shorthand $E = E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S)$. Multiplying through,

$$\frac{1}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k) = E + \sum_{\mathcal{H}\in\mathcal{A}}\sum_{\xi\in\Xi_{\xi_k,\mathcal{H}}} W_{\xi,\mathcal{H}}\frac{\dim(\xi)}{\dim(\xi_k)} O_{\mathcal{H},\varphi_0^{\mathcal{H}}}(q_{S_1}^{(A_{\mathrm{wt},\mathcal{H}}+E_{\mathcal{H}}+\epsilon)\kappa+B_{\mathrm{wt},\mathcal{H}}}) + O(q_{S_1}^{A_{\mathrm{wt},G}\kappa+B_{\mathrm{wt},G}} m(\xi_k)^{-C_{\mathrm{wt}}})$$

where

$$W_{\xi,\mathcal{H}} = \iota(G,\mathcal{H})c_\xi\frac{\tau'(H)}{\tau'(G)}.$$

The $W_{\xi,\mathcal{H}}$ here are independent of k and q_{S_1}. Finally, by lemma 5.2.2 the ratio of dimensions is $O(m(\xi_k)^{-1})$.

In total, the inner sum has $|\Omega_{\mathcal{H}}|$ elements so the entire double sum has finite size independent of S_1 and ξ. Therefore, it can be bounded to

$$\frac{1}{\tau'(G)\dim(\xi_k)} I_{\mathrm{disc}}(\varphi_k) = E + O(q_{S_1}^{A'_{\mathrm{wt}}+B'_{\mathrm{wt}}\kappa} m(\xi_k)^{-1}).$$

where $A'_{\mathrm{wt}}, B'_{\mathrm{wt}}$ are anything bigger than the maxima over all groups appearing in $\mathcal{A}$ (Note that C_{wt} can be chosen to be ≥ 1). Finally, plug in corollary 7.2.1. □

10. Corollaries

Theorem 9.1.1 can be substituted in for [ST16]'s 9.19 to most of the same corollaries. We leave the result on zeros of L-functions for the future because the computations are complicated—the term β_v^{pl} gets replaced by something far more complicated in the case with central character.

Recall the notation from last section and for brevity define

$$\mu_{\mathcal{F}_k}(\widehat{\varphi}_S) = \frac{\bar{\mu}^{\mathrm{can}}(U_{\mathfrak{X}}^{S,\infty})}{\tau'(G)\dim(\xi_k)}\sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_S(\pi)$$

for any $\widehat{\varphi}_S$ on $\widehat{G}_S$. Theorem 9.1.1 computes this when $\varphi_S\in\mathscr{H}(G_S)$ and φ_{S_1} is unramified.

10.1. **Plancherel Equidistribution.** First, we get a version of [ST16, cor 9.22] using a similar Sauvageot density argument. We phrase things as in [FL18]. Restrict to the case where all the ξ_k have the same central character ω_ξ and S_1 is trivial. Let $\Theta = L\mathfrak{X}_S$ and let ψ be the character on Θ induced by ω_ξ and χ_S. Let $\widehat{G}_{S,\psi} \subseteq \widehat{G}_S$ be all representations with central character extending ψ. We can define a measure μ^{pl}_ψ on $\widehat{G}_{S,\psi}$ by $\mu^{\mathrm{pl}}_\psi(f) = E(f^*|\omega_\xi, L, \chi_S)$ where f^* is a continuous extension of f to $\widehat{G}_S$.

When ψ is trivial, $\mu^{\mathrm{pl}}_\psi = \mu_{\Theta,\mathrm{pl}}$ from [FL18] up to some constant. The lemma in the middle of the proof of [FL18, thm 2.1] extends to our case of non-trivial ψ and Θ a general subgroup of Z_{G_S}.

Lemma 10.1.1. *Let $\epsilon > 0$:*

(1) For any bounded $A \subseteq \widehat{G}_S \setminus \widehat{G}_S^{\mathrm{temp}}$, there exists $h \in \mathscr{H}(G_S)$ such that $\widehat{h} \geq 0$ on $\widehat{G}_S$, $\widehat{h} \geq 1$ on A, and $\mu^{\mathrm{pl}}_\psi(\widehat{h}) \leq \epsilon$

(2) For any Riemann integrable function $\widehat{f}$ on $\widehat{G}^{\mathrm{temp}}_{S,\psi}$, there exist $h_1, h_2 \in \mathscr{H}(G_S)$ such that $|\widehat{f} - \widehat{h}_1| \leq \widehat{h}_2$ on $\widehat{G}_{S,\psi}$ and $\mu^{\mathrm{pl}}_\psi(\widehat{h}_2) \leq \epsilon$.

Proof. We try to mimic the argument in [FL18, thm 2.1]. Let $\Theta_f = \Theta \cap Z_{G_{\mathrm{der}}}(F_S)$ and $\overline{\Theta} = \Theta/\Theta_f$. Then Θ_f is finite. In addition, if we denote by $X(\cdot)$ taking complex-valued characters, the map $X(G_S) \to X(Z_{G,S}/Z_{G_{\mathrm{der}}}(F_S)) \to X(\overline{\Theta})$ is surjective. Choose a set-theoretic section s of this map.

We can ignore normalization constants by wlog changing ϵ. Then this result for Θ trivial follows from the main result of [Sau97]. If $\overline{\Theta}$ is trivial, then the various $\widehat{G}_{S,\psi}$ are positive-measure clopen subsets of $\widehat{G}_S$ so we can use the h_i for either A or the extension of f by 0 on $\widehat{G}_S$.

For the general case, given f on $\widehat{G}_{S,\psi}$ define F on $\widehat{G}_{S,\psi|_{\Theta_f}}$ by $F(\pi) = f(\pi \otimes s(\omega_\pi^{-1}\psi))$. Choose H_1, H_2 satisfying the condition for F. For any finite subset $T_0 \subseteq X(\overline{\Theta})$, the averages

$$h_i = \frac{1}{|T_0|} \sum_{\lambda \in T_0} s(\lambda) H_i$$

satisfy $|\widehat{f} - \widehat{h}_i| \leq \widehat{h}_2$ (each individual term in the sum does) so we simply need to find an T_0 such that $\mu^{\mathrm{pl}}_\psi(h_2) \leq \epsilon$.

Up to some constants

$$\mu^{\mathrm{pl}}_\psi(h_2) = \int_\Theta \psi(z) h_2(z)\, dz = \frac{1}{|T_0|} \sum_{\lambda \in T_0} \overline{H}_2(\lambda\psi) \qquad \mu^{\mathrm{pl}}_{\psi|_{\Theta_f}}(H_2) = \sum_{z \in \Theta_f} \psi(z) H_2(z)$$

by variations of the arguments in section 8.3.1. Choose a sparse enough lattice M so that $\sup H_2 \cap M = \{1\}$. Then up to constants,

$$\epsilon > \mu^{\mathrm{pl}}_{\psi|_{\Theta_f}}(H_2) = \sum_{z \in \Theta_f M} \psi(z) H_2(z) = \sum_{\lambda \in T} \overline{H}_2(\lambda\psi)$$

where T is some subset of $X(\overline{\Theta})$. The last step was Poisson summation on Θ. Therefore, since the last sum converges, choosing T_0 to be a large enough finite subset of T suffices.

The argument for subsets A is the same averaging trick—in place of the function F, we use set $A' = \{\pi \otimes \lambda : \pi \in A, \lambda \in X(\overline{\Theta})\}$. $\square$

The same "3ϵ"-argument as [ST16, cor 9.22] then gives:

Corollary 10.1.2 (Plancherel equidistribution up to central character)**.** *Recall the conditions and notation from the above discussion. Then*

(1) *For any bounded* $A \subseteq \widehat{G}_S \setminus \widehat{G}_S^{\mathrm{temp}}$

$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\mathbf{1}_A) = 0.$$

(2) *For any Riemann integrable* $\widehat{f}$ *on* $\widehat{G}_{S,\psi}^{\mathrm{temp}}$,

$$\lim_{k\to\infty} \mu_{\mathcal{F}_k}(\widehat{f}) = \mu_\psi^{\mathrm{pl}}(\widehat{f}).$$

Beware that part (1) does not give a Ramanujan conjecture at S on average; it cannot count that the total number of π in $\mathcal{F}$ with non-tempered π_S is $O(m(\xi_k))^{-1}$ since A needs to be bounded. It is nevertheless somewhat close.

10.2. **Sato-Tate Equidistribution.** For this section we need to slightly modify our notation. Allow S_1 to be infinite and define modified measure

$$\mu^{\natural}_{\mathcal{F}_k,v}(\widehat{\varphi}_v) = \frac{\bar{\mu}^{\mathrm{can}}(U_{\mathfrak{X}}^{S,\infty})}{\tau'(G)\dim(\xi_k)} \sum_{\pi\in\mathcal{AR}_{\mathrm{disc}}(G,\chi)} a_{\mathcal{F}_k}(\pi)\widehat{\varphi}_{S_0}(\pi_{S_0})\widehat{\varphi}_v(\pi_v)$$

for any $v \in S_1$. Then $\mu^{\natural}_{k,v}(\widehat{\varphi}_v)$ can still be picked out by a test function φ of the form we have been considering by setting $\varphi_w = \mathbf{1}_{K_w}$ for all $w \in S_1 \setminus v$.

10.2.1. *Sato-Tate measures.* We recall the definition of the Sato-Tate measure from [ST16, §3,§5]. Recall the Satake isomorphism $\mathscr{H}(G_v, K_v) \to \mathbb{C}[X_*(A)]^{\Omega_F}$ in the notation of section 5.4.1 and how it identifies $\widehat{G}_v^{\mathrm{ur,temp}}$ with $\Omega_{F_v}\backslash\widehat{A}_c$.

We can find a maximal compact $\widehat{K}$ of $\widehat{G}$ invariant under Frob_v. Then since G_v is unramified, $\Omega_{F_v}\backslash\widehat{A}_c$ can be identified with the $\widehat{G}$ classes in $\widehat{K} \rtimes \mathrm{Frob}_v \subseteq {}^LG$ and also $\widehat{T}_{c,v} = \Omega_{F_v}\backslash\widehat{T}_c/(\mathrm{Frob}_v - \mathrm{id})\widehat{T}_c$ (see [ST16, lem 3.2]).

In general, let G split over F_1 and let $\Gamma_1 = \mathrm{Gal}(F_1/F)$. Given $\Theta \in \Gamma_1$, define

$$\widehat{T}_{c,\Theta} = \Omega_G^\Theta\backslash\widehat{T}_c/(\Theta - \mathrm{id})\widehat{T}_c.$$

Given $\tau \in \Gamma_1$, $t \mapsto \tau t$ canonically identifies $T_{c,\Theta}$ with $T_{c,\tau\Theta\tau^{-1}}$. All these identifications are consistent with each other so $T_{c,\Theta}$ depends only on the Γ_1-conjugacy class of Θ. Note then that $\widehat{T}_{c,\mathrm{Frob}_v} = \widehat{T}_{c,v}$ since G_v is quasisplit.

Choose the Haar measure on $\widehat{K}$ with total volume 1. This induces a quotient measure on the set of conjugacy classes in $\widehat{K} \rtimes \Theta$ and therefore on $\widehat{T}_{c,\Theta}$. Call this μ_Θ^{ST}. Finally, let $\mathcal{V}_F(\Theta)$ be the set of places v such that F_1 is unramified at v and Frob_v is in the conjugacy class of Θ. For such a v, we get a measure $\mu_v^{\mathrm{pl,ur}}$ from the identification $T_{c,\Theta}$ with $\widehat{G}_v^{\mathrm{ur,temp}}$. Normalize this to also have total volume 1.

Proposition 10.2.1 ([ST16, prop 5.3])**.** *For any* $\Theta \in [\Gamma_1]$, *let* $v \to \infty$ *in* $\mathcal{V}_F(\Theta)$. *Then there is weak convergence* $\mu_v^{\mathrm{pl,ur}} \to \mu_\Theta^{\mathrm{ST}}$.

Proof. by the explicit formulas [ST16, prop 3.3] and [ST16, lem 5.2] □

10.2.2. *Central character issues.* Recall all the notation from proposition 8.3.5. Our result is in terms of $E(\widehat{\varphi}|\omega_\xi, L, \chi_S)$ instead of $\mu_v^{\mathrm{pl,ur}}$ so we need to define an alternate Sato-Tate measure in terms of this. First, we need to understand $E_v^{\mathrm{pl,ur}}$ better.

There is a central character map $T_{c,\Theta} \to \widehat{Z}_{G_v}$. This lets us define $E^{\mathrm{ST},\Theta}(\widehat{\varphi}|\omega)$ for any $\widehat{\varphi}$ on $T_{c,\Theta}$ similar to $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega)$ from section 8.3.1. Now Langlands for torii gives that $\widehat{Z}_{G_v}$ is the set of L-parameters $\varphi : W_{F_v}^{\mathrm{ur}} \hookrightarrow {}^L(Z_G)_{F_v}^{\mathrm{ur}}$. If $\mathrm{Frob}_v, \mathrm{Frob}_w$ are conjugate in Γ_1, we can identify the set of these parameters and therefore $\widehat{Z}_{G_v}$ and $\widehat{Z}_{G_w}$. For $v \in \mathcal{V}_F(\Theta)$, call this common set $\widehat{Z}_\Theta$. Note that these identifications commute with the identifications of $\widehat{T}_{c,v}$ and the map taking central characters.

Lemma 10.2.2. *Fix a common measure on $\widehat{Z}_\Theta$. Choose $\widehat{\varphi}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega) \to E^{\mathrm{ST},\Theta}(\widehat{\varphi}|\omega)$ pointwise for $\omega \in \widehat{Z}_\Theta$*

Proof. The previous result gives weak convergence $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}_\Theta|\omega) \to E^{\mathrm{ST},\Theta}(\widehat{\varphi}_\Theta|\omega)$ in $L^2(\widehat{Z}_\Theta)$. By the formula [ST16, prop 3.3], the $E_v^{\mathrm{pl,ur}}(\widehat{\varphi}|\omega)$ are equicontinuous so this implies pointwise convergence. □

To understand the more complicated $E(\widehat{\varphi}|\omega_\xi, L, \chi_S)$, we now have to parametrize $Z_{S,\xi,L,\chi}$ in terms of local components. Assume $\omega_S = \omega_{S_1}\omega_{S_0} \in Z_{S,\xi,L,\chi}$: i.e. $\omega_S\omega_\xi = 1$ on L and $\omega_S|_{\mathfrak{X}_S} = \chi_S$. Assume also that ω_{S_1} is unramified. Let $L_0 = L \cap K_{S_1}$. It is a cocompact lattice in Z_{S_0}. Then we always have that $\omega_{S_0}\omega_\xi = 1$ on L_0 and that $\omega_{S_0}|_{\mathfrak{X}_{S_0}} = \chi_{S_0}$.

Given such ω_{S_0}, it forces $\omega_{S_1} = \omega_{S_0}^{-1}\omega_\xi^{-1}$ on L. The determined ω_{S_1} is trivial on $L \cap K_{S_1}$ and therefore extends to a continuous character on $\overline{L} \subseteq Z_{S_1}$. Therefore the possible choices for ω_{S_1} are those that restrict to $\omega_{S_0}^{-1}\omega_\xi^{-1}$ on L, χ_{S_1} on $\mathfrak{X}_{S_1}$, and are unramified.

Let E_{S_1} be the group $\overline{L}K_{S_1}\mathfrak{X}_{S_1}$. Since Z_{S_1}/E_{S_1} is finite, there are finitely many choices for ω_{S_1} and we can factor

$$\sum_{\substack{\omega_S \in \widehat{Z}_S \\ \omega_S\omega_\xi(L)=1 \\ \omega_S|_{\mathfrak{X}_S}=\chi_S}} E^{\mathrm{pl}}(\widehat{\varphi}_S|\omega_S) = \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}}=\chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\substack{\omega_S \in \widehat{Z}_{S_1}^{\mathrm{ur}} \\ \omega_{S_1}\omega_{S_0}\omega_\xi(L)=1 \\ \omega_{S_1}|_{\mathfrak{X}_{S_1}}=\chi_{S_1}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_1}|\omega_{S_1}).$$

To compute $\mu_{k,v}^\natural$, we consider $\varphi_{S_1} = \mathbf{1}_{K_{S_1\setminus v}}\varphi_v$ so

$$E^{\mathrm{pl}}(\widehat{\varphi}_{S_1}|\omega_{S_1}) = E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v)\prod_{w\in S_1\setminus v}\mathrm{vol}(Z_w \cap K_w) = \frac{\mathrm{vol}(Z_{S_1}\cap K_{S_1})}{\mathrm{vol}(Z_v \cap K_v)}E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v).$$

Let the set of summands for the second sum be $\widehat{Z}_{v,\omega_{S_0},\chi_v} \subseteq \widehat{Z}_v$ and let $\omega_S \in \widehat{Z}_{v,\omega_{S_0},\chi_v}$. The possible ω_v components are those satisfying two conditions: $\omega_v\omega_{S_0}\omega_\xi$ extends continuously to $\overline{L} \subseteq Z_{S\setminus v}$, and $\omega_v|_{\mathfrak{X}_v} = \chi_v$. The first condition is equivalent to ω_v being the F_v-component of a character ω on $Z_G(\mathbb{A})/Z_G(F)$ trivial on $U^{S,\infty}$ that also has $F_{S_0,\infty}$-component $\omega_{S_0}\omega_\xi$.

Next, by global Langlands for torii, this is equivalent to its parameter $\psi_{\omega_v} : W_{F_v} \to {}^L(Z_G)_{F_v}$ being a restriction of a global parameter $\psi_\omega : W_F \to {}^LZ_G$ satisfying certain conditions. However, if Frob_w is conjugate to Frob_w, then $\psi_\omega|_{W_{F_w}}$ is the transport of $\psi_\omega|_{W_{F_v}}$ through the identification before. In particular, if we identify all the $\widehat{Z}_v$ for $v \in \mathcal{V}_F(\Theta) \cap S_1$, $\widehat{Z}_{v,\omega_{S_0},\chi_v}$ depends on v only through Θ. Call the common value $\widehat{Z}_{\Theta,\omega_{S_0},\chi_v} \subseteq \widehat{Z}_\Theta$.

In total, if we set $\varphi_{S_1} = \mathbf{1}_{K_{S_1 \setminus v}} \varphi_v$ for some $v \in \mathcal{V}_F(\Theta) \cap S_1$,

$$E(\widehat{\varphi}_S|\omega_\xi, L, \chi_S) = \frac{1}{|X|} \frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)} \frac{\mathrm{vol}(Z_{S_1} \cap K_{S_1})}{\mathrm{vol}(Z_v \cap K_v)} \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}} = \chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\omega_v \in \widehat{Z}_{\Theta,\omega_{S_0},\chi_v}} E^{\mathrm{pl}}(\widehat{\varphi}_v|\omega_v).$$

This allows us to define an $E_{\mathrm{ST},\Theta}(\widehat{\varphi}_v|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0})$ analogously:

$$E_{\mathrm{ST},\Theta}(\widehat{\varphi}_S|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0}) = \frac{1}{|X|} \frac{\mu}{\mathrm{vol}(Z'_{S,\infty}/L)} \frac{\mathrm{vol}(Z_{S_1} \cap K_{S_1})}{\mathrm{vol}(Z_v \cap K_v)} \sum_{\substack{\omega_{S_0} \in \widehat{Z}_{S_0} \\ \omega_{S_0}\omega_\xi(L_0)=1 \\ \omega_{S_0}|_{\mathfrak{X}_{S_0}} = \chi_{S_0}}} E^{\mathrm{pl}}(\widehat{\varphi}_{S_0}|\omega_{S_0}) \sum_{\omega_v \in \widehat{Z}_{\Theta,\omega_{S_0},\chi_v}} E^{\mathrm{ST},\Theta}(\widehat{\varphi}_v|\omega_v).$$

Then we get

Proposition 10.2.3. *Choose a sequence $v \to \infty$ in $\mathcal{V}_F(\Theta) \cap S_1$ such that the characters χ_v all correspond in $\widehat{\mathfrak{X}}_\Theta$. Choose $\widehat{\varphi}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then*

$$E(\widehat{\mathbf{1}}_{K_{S_1 \setminus v}} \widehat{\varphi}_\Theta \widehat{\varphi}_{S_0}|\omega_\xi, L, \chi_S,) \to E_{\mathrm{ST},\Theta}(\widehat{\varphi}_\Theta|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0}).$$

Proof. Use the above formula for E_{ST} and E together with the previous lemma. We can compute both sides by fixing a common measure on $\widehat{Z}_\Theta$ which makes $\mathrm{vol}(Z_v \cap K_v)$ constant on $v \in \mathcal{V}_F(\Theta)$. □

This is a replacement for [ST16, prop 5.3] in our case.

10.2.3. *Final Statment.* Arguing as in [ST16, thm 9.26], we finally get

Corollary 10.2.4 (Sato-Tate equidistribution up to central character)**.** *Choose a sequence $v_j \to \infty$ in $\mathcal{V}_F(\Theta) \cap S_1$ such that the characters χ_v all correspond in $\widehat{\mathfrak{X}}_\Theta$. Choose a Riemann integrable function $\widehat{f}_\Theta$ on $\widehat{T}_{c,\Theta}$. Then*

$$\lim_{(j,k)\to\infty} \mu^\natural_{\mathcal{F}_k, v_j}(\widehat{f}_\Theta) = E_{\mathrm{ST},\Theta}(\widehat{f}_\Theta|\omega_\xi, L, \chi_S, \widehat{\varphi}_{S_0})$$

where the limit is over any sequence of pairs (j,k) such that $q_{v_j}^N m(\xi_k)^{-1} \to 0$ for all integers N.

This can be thought of as sort of a "diagonal" equidistribution as opposed to the "vertical" Plancherel equidistribution involving $\lim_{k\to\infty} \mu^\natural_{\mathcal{F}_k, v_j}(\widehat{f}_\Theta)$ or the conjectural "pure horizontal" Sato-Tate equidistribution involving $\lim_{j\to\infty} \mu^\natural_{\mathcal{F}_k, v_j}(\widehat{f}_\Theta)$.

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