

# Multi-Qubit Golden Gates

Rahul Dalal, Shai Evra, and Ori Parzanchevski

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## Abstract

Our goal in this paper is to construct optimal topological generators for compact unitary Lie groups, extending the works of [Sar15b],[PS18] on golden and super-golden gates to higher dimensions. To do so we consider a variant of the Sarnak–Xue Density Hypotheses [Sar90],[SX91] in the weight aspect for definite projective unitary groups and prove it using the endoscopic classification of automorphic representations.

Our main motivation is to construct efficient multi-qubit universal gate sets for quantum computers. For example, we find a set of universal gates that, for a given accuracy, can heuristically approximate arbitrary unitary operations on 2 qubits with  $\approx 10$  times fewer “expensive”  $T$ -type gates than the standard Clifford+ $T$  set. Our framework also covers the 2-qubit Clifford+CS gate set, well-known for being particularly friendly to fault-tolerant implementation. We thereby prove tight upper bounds on the required CS count for approximations (specifically, 4.8x fewer non-Clifford gates than Clifford+ $T$ ).

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# 1 Introduction

## 1.1 Background

Practical quantum computers must be able to approximate any unitary operator in  $U(2^n)$  acting on an  $n$ -qubit logical register. Because physical qubits are noisy, each logical qubit is expected to be encoded in many physical qubits using a quantum-error-correcting code (QECC) and then acted on by fault-tolerant implementations of logical gates—so the action on the physical register never allows an error on a single physical qubit to spread to many others; see [NC11, Chap 10] or the original threshold proofs [ABO97].

The simplest fault-tolerant logical gates are transversal: they apply independent physical operations to each physical qubit. The Eastin-Knill theorem [EK09] shows, however, that any fixed QECC admits only a finite transversal subgroup. A standard workaround is to supplement the transversal gates with a finite set  $T$  of “expensive” non-transversal gates implemented through more elaborate techniques such as the teleportation and magic-state distillation of [GC99, BK05]. Together, the transversal gates and  $T$  should generate a dense subset of  $U(2^n)$  such that arbitrary elements can be approximated well by products with a small count of factors from  $T$ .

Motivated by these considerations, Sarnak and Parzanchevski introduced the purely group-theoretic notions of golden gates and super-golden gates for  $PU(2)$  [Sar15b, PS18]. These are topological generating sets which possess optimal covering properties as well as an efficient algorithm for navigation and approximation (Definition 1.2.1); a super-golden gate set is further a finite group equipped with a distinguished set  $T$  that plays the role above. In [PS18] and [EP22], golden and super-golden gate sets were constructed for  $PU(2)$  and  $PU(3)$ , respectively. The main goal of this paper is to extend these constructions as far as possible: in particular to a multi-qubit setting of  $PU(2^b)$  for  $b = 2, 3$ .

The gate library most often used in the quantum-computing literature—the one-qubit Clifford+ $T$  set together with CNOT—works well for most known codes, but its  $T$ -count scales suboptimally. Super-golden gate sets can, in principle, approximate generic two-qubit

unitaries with asymptotically fewer costly  $T$ -type operations. One example we construct could theoretically save a multiplicative factor of 10; see Table §4.1 for comparisons of our new gate sets with those built from 1-qubit gates. Recent physical implementations of logical qubits built from QECC (e.g [AAAB<sup>+</sup>24, BEG<sup>+</sup>24]) make such constant-factor improvements practically relevant. As experimental progress pushes interest in more codes, potentially featuring more exotic transversal gates (e.g [KT23]) or more idiosyncratic optimization requirements, alternate hyper-efficient gate sets may become increasingly valuable.

Finally, the abstract problem of constructing golden (and super-golden) gate sets is mathematically rich, requiring sophisticated machinery from number theory. Even the  $PU(2)$  case required Ramanujan bounds for modular forms on quaternion algebras. The higher-rank problem draws on recent advances proving the Ramanujan conjecture for conjugate self-dual automorphic forms on  $GL_n$ , the fundamental lemma and endoscopic classification, explicit constructions of Arthur packets for classical groups, and trace-formula techniques for computing statistics of automorphic representations. We hope the present work highlights these surprising connections to active modern research areas.

## 1.2 Results

### 1.2.1 Golden Gates Abstractly

We begin by defining the notions of golden and super-golden gate sets for a general compact Lie group. Let  $L$  be a compact Lie group equipped with a probability Haar measure  $\mu = \mu_L$  and a bi-invariant metric  $d = d_L$ . For  $\varepsilon > 0$ ,  $x \in L$  and  $X \subseteq L$ , let  $B(x, \varepsilon)$  be the ball around  $x$  of volume  $\varepsilon$ , and let  $B(X, \varepsilon) := \bigcup_{x \in X} B(x, \varepsilon)$ . For  $S \subseteq L$  and  $\ell \in \mathbb{N}$ , let  $S^\ell \subset L$  (resp.  $S^{(\ell)} \subset L$ ) be the set of words with shortest representation of length at most (resp. precisely)  $\ell$  in  $S$ , and let  $\langle S \rangle$  (resp.  $\langle S \rangle_{sg}$ ) be the group (resp. semigroup) generated by  $S$ .

**Definition 1.2.1.** A finite subset  $S \subset L$  is said to be a *Golden Gate Set* if it satisfies the following conditions:

- (1) Covering: The covering rate of  $S^{(\ell)}$  in  $L$  is optimal up to a polylogarithmic factor; namely, there exists a fixed  $c \geq 1$ , such that

$$\mu\left(L \setminus B\left(S^{(\ell)}, \varepsilon_\ell\right)\right) \xrightarrow{\ell \rightarrow \infty} 0, \quad \varepsilon_\ell = \frac{(\log |S^{(\ell)}|)^c}{|S^{(\ell)}|}.$$

- (2) Growth: The size of  $S^{(\ell)}$  grows exponentially in  $\ell$ .
- (3) Navigation: There is an efficient algorithm such that, given  $g \in \langle S \rangle_{sg} \subset L$ , the algorithm writes  $g$  as a word of shortest possible length in  $S$ .
- (4) Approximation: There exists  $N \geq 1$  and a (heuristic, randomized) efficient algorithm such that given  $g \in L$ ,  $\varepsilon > 0$ , and  $\ell$  satisfying  $B(g, \varepsilon) \cap S^{(\ell)} \neq \emptyset$ , the algorithm outputs an element from  $B(g, \varepsilon) \cap S^{(\ell \cdot N)}$ .

A finite subgroup  $C \subseteq L$  and set of finite-order elements  $T \subseteq L$  are said to together form a *Super-Golden Gate Set* if the analogous four properties hold for  ${}^0S^{(\ell)}C$  where  ${}^0S = CTC^{-1}$ .

Following our main interest in quantum computation, in this paper, we shall only concern ourselves with the case where the compact Lie group is the group of unitary or projective unitary  $2^b \times 2^b$  matrices:

$$U(2^b) := \{g \in GL_{2^b}(\mathbb{C}) \mid g^*g = I\} \quad , \quad PU(2^b) := U(2^b)/\{c \cdot I \mid c \in U(1)\}.$$

Following [Sar15b, PS18, EP22], our constructions of golden and super-golden gate sets for  $PU(n)$  come from certain arithmetic groups of unitary matrices that we call (almost) “golden adelic groups” satisfying a “class-number-1” property:

**Theorem 1.2.2.** *Let  $n = 2^b = 4, 8$  and  $K'$  be a golden adelic group of a rank- $n$ , definite arithmetic unitary group that is almost golden (resp. almost super-golden) at some prime  $\mathfrak{p}$  as in Definition 4.1.1. Then there is a corresponding set  $S_{\mathfrak{p}}$  of golden (resp. super-golden) gates of  $PU(n)$ .*

### 1.2.2 Concrete Gate Sets

As some explicit examples:

**Theorem 1.2.3.** *Define the following Hermitian positive definite  $4 \times 4$  matrix,*

$$H_3 = 2 \cdot \begin{pmatrix} I_2 & A \\ -A & I_2 \end{pmatrix}, \quad A = \frac{\sqrt{-3}}{3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

and let  $B \in GL_4(\mathbb{C})$  be such that  $H = B^*B$ . For any prime  $p \neq 2$ , denote

$$S_p := \left\{ g \in M_4 \left( \mathbb{Z} \left[ \frac{1 - \sqrt{-3}}{2} \right] \right) \mid \begin{array}{l} g^* H_3 g = p' \cdot H_3, \ g \equiv I_4 \pmod{2} \\ g \text{ is not a scalar matrix} \end{array} \right\}$$

$$\text{where} \quad p' = \begin{cases} p & p \equiv 0, 1 \pmod{3} \\ p^2 & p \equiv 2 \pmod{3} \end{cases}.$$

Then the set,  $S'_p = \{BgB^{-1} \mid g \in S_p\}$ , is a golden gate set of  $PU(4)$ .

**Theorem 1.2.4.** *Let  $S_{\zeta_3}(4)$  be the group of  $4 \times 4$  monomial (i.e., generalized permutation) matrices with entries that are 3rd roots of unity. Let  $C_3$  be group with generators:*

$$C_3 := \left\langle S_{\zeta_3}(4), \quad \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix} \right\rangle \subseteq PU(4),$$

and let

$$T := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Then  $C_3$  is a finite group isomorphic to  $\text{PGSp}_4(\mathbb{F}_3)$  and together with  $T$  forms a super-golden gate set.

For compatibility with the current literature in quantum error correction, it is ideal for the finite group  $C$  to be the 2-qubit Clifford group. In addition, fault-tolerant implementations of  $T$  are usually done through the teleportation and magic-state distillation techniques of [GC99, BK05] which require  $T$  to be at a very low-level of the Clifford hierarchy (e.g., as described in [PRTC20]). One such 2-qubit  $T$  that has attracted interest in theoretical and experimental setups is the controlled-S, or CS, gate (see e.g., [GRT21], [Muk24], [FND+20], [XCY+20]).

This Clifford+CS gate set fits into our framework:

**Theorem 1.2.5.** *The controlled- $S$  or  $CS$  gate*

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & i \end{pmatrix} \subseteq PU(4)$$

*is at the 3rd level of the 2-qubit Clifford hierarchy and together with the 2-qubit Clifford group forms a super-golden gate set for  $PU(4)$ .*

Theorem 1.2.5 in particular gives an upper bound on the required CS count needed to approximate all but an asymptotically negligible set of unitaries that matches the worst-case lower bound of [GRT21].

Finally, the paper [MSG12] provides the only example of a golden adelic group for  $n = 2^b = 8$  and shows that no such groups exist for larger  $n$ . Various examples of golden and super-golden gates are described in more detail in Section 4.5. Theorems 1.2.4 and 1.2.5 come from Propositions 4.5.5 and 4.5.10 respectively together with the discussions afterwards. These also provide auxiliary structure needed for navigation in the gate sets.

We compare our new 2-qubit super-golden gates to previous 2-qubit gate sets in Section 4.6. In particular, the example in Theorem 1.2.5 can in theory approximate to a given accuracy with 4.8x fewer  $T$  gates than the current standard of three CNOT gates together with eight 1-qubit gates approximated by the 1-qubit Clifford+ $T$  set. The example of Theorem 1.2.4 needs  $\approx 10$ x fewer  $T$ -gates. We note however that our approximation algorithm as in Definition 1.2.1(4) has very unoptimized constant factors; the  $T$ -counts it produces are much worse than these theoretical optima.

### 1.2.3 Future Work

We see the results here as incomplete in three important ways. First, [MSG12] only classified all golden subgroups of unitary groups with rank  $n \geq 5$ . A full classification in the case  $n = 2^b = 4$  is crucial to fully optimize practical considerations for 2-qubit gate sets. In particular, we do not yet know the 2-qubit super-golden gate set that approximates to a given accuracy with the fewest number of  $T$ -gates.

Second, the algorithm we present for the approximation Definition 1.2.1(4) is very far from optimal. To properly realize the theoretical efficiency improvements from the gate sets of Theorems 1.2.4 and 1.2.5, it is crucial to improve the algorithm's constant-factor overhead to be closer to 1-qubit cases as in [BS23].

Finally, our working abstract framework for super-golden gates in Section 4.2.2 is not as general as possible and may be unnecessarily ruling out many interesting gate sets—see Remark 4.6.1. Furthermore, following [EP24], even gate sets coming from non-golden groups should be considered: the constant factors lost in the suboptimal covering bound may be made up for by a faster growth rate of the number of gates with a given  $T$ -count.

## 1.3 Techniques

The key property of golden adelic groups we use is that they determine lattices acting simply transitively on  $G_p$ -orbits in a Bruhat-Tits building. For example:

**Theorem 1.3.1.** *In the notations of Theorem 1.2.3, let  $p \neq 2$  and  $\Lambda_p$  be the group generated by  $S_p$  in  $PU(4, H)$ . Then  $\Lambda_p$  is a  $p$ -arithmetic subgroup and  $\Lambda_p$  acts simply transitively on the  $G_p$ -orbit of a hyperspecial vertex of a corresponding Bruhat-Tits building.*

In the notations of Theorem 1.2.5, let  $\Lambda_2$  be the group generated by  $CS$  and the 2-qubit Clifford group in  $PU(4)$ . Then  $\Lambda_2$  is conjugate to a  $(1+i)$ -arithmetic subgroup that acts transitively on the long edges of a corresponding Bruhat-Tits building.

Variants of theorem 1.3.1, combined with the Ross-Selinger algorithm [RS15] (see Subsection 4.4), yields gate sets for  $PU(4)$  that satisfy the properties of growth, navigation and approximation (Theorem 4.4.1). The optimal covering property would follow if we could prove the naïve Ramanujan conjecture for the underlying algebraic group of  $\Lambda_p$  (this was the method of proof used in [PS18, EP22]). However, one can construct counterexamples for the naïve Ramanujan conjecture for  $n \geq 4$  (see Theorem 1.4 in [LSV05]).

To overcome this obstacle, we proceed according to the strategy suggested in [PS18] of replacing the naïve Ramanujan conjecture with a variant of the Sarnak-Xue Density Hypothesis ([Sar90, SX91]) in the weight aspect. We consider certain *automorphic families*  $\mathcal{F}$ : weighted subsets of the discrete automorphic spectrum  $\mathcal{AR}_{\text{disc}}(G)$  defined by assigning numbers  $\mathcal{F}(\pi)$  to each  $\pi$ . We also consider for any representation  $\pi_{\mathfrak{p}}$  of  $G_{\mathfrak{p}}$ , the matrix coefficient decay

$$\sigma(\pi_{\mathfrak{p}}) := \inf\{p : p \geq 2, \pi_{\mathfrak{p}} \text{ has matrix coefficients in } L_0^p(G_{\mathfrak{p}})\},$$

where  $L_0^p$  is defined by integration mod center on  $G_{\mathfrak{p}}/Z_{G_{\mathfrak{p}}}$  using that  $\pi_{\mathfrak{p}}$  has unitary central character. Then:

**Theorem 1.3.2.** *Let  $G$  be a definite,  $F$ -algebraic unitary group for some number field  $F$  and let  $\mathfrak{p}$  be a prime such that  $G(F_{\mathfrak{p}})$  is not compact. For the automorphic families  $\mathcal{F}_{\delta}$  for small  $\delta > 0$  defined in §7 roughly representing the decomposition over the automorphic spectrum of the indicator function of a ball of volume  $\delta$  in  $G_{\infty}$ , denote*

$$|\mathcal{F}_{\delta}| = \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_{\delta}(\pi), \quad |\mathcal{F}_{\delta}(\sigma, v)| = \sum_{\substack{\pi \in \mathcal{AR}_{\text{disc}}(G) \\ \sigma(\pi_{\mathfrak{p}}) \geq \sigma}} \mathcal{F}_{\delta}(\pi).$$

Then, for any  $\epsilon > 0$ , there exist  $c_{\epsilon} > 0$  such that for any  $\sigma \geq 2$  and small enough  $\delta$ :

$$|\mathcal{F}_{\delta}(\sigma, v)| \leq c_{\epsilon} |\mathcal{F}_{\delta}|^{\frac{2}{\sigma} - \epsilon} |\mathcal{F}_{\delta}(\infty, v)|^{1 - \frac{2}{\sigma} - \epsilon}$$

(where we note that both numbers under the exponents are  $\leq 1$ ).

Theorem 1.3.2 is proven through the heavy use of recent advances in the Langlands program, especially Arthur's work on the endoscopic classification of automorphic representations of classical groups.

## 1.4 Outline

This paper is organized as follows: In Sections 2 and 3, we collect facts about arithmetic unitary groups and Bruhat-Tits buildings, emphasizing the details particularly important to this application. In Section 4, we define the notions of golden and super-golden adelic groups, show how they give rise to gate sets that satisfy the last three properties of Definition 1.2.1, and give many examples.

Section 5 begins the second half of the paper: we first review material about automorphic representations, the Generalized Ramanujan Conjectures, endoscopic classifications, automorphic families, and the Sarnak-Xue density hypothesis. We also make a key definition of the shape of an automorphic representation  $\pi$ . In Section 6, we relate the shape of  $\pi$  to the local matrix coefficient decay  $\sigma(\pi_{\mathfrak{p}})$  at finite places  $v$ . We combine this with a bound on

counts of  $\pi$  with a given shape to prove Theorem 1.3.2 in Section 7. Finally, in Section 8, we use Theorem 1.3.2 to prove that gate sets coming from golden adelic groups satisfy the optimal covering property, thus proving Theorem 1.2.3.

A reader just interested in the quantum computing application could mostly just read parts of Sections 4.4-4.6, skip the most technical proofs, refer back to previous sections for notation and background, and ignore everything Section 5 or later. Conversely, a reader just interested in how the automorphic theory was inputted should start reading at Section 5.

## Conditionality

The proof of Theorem 1.3.2 depends heavily on Mok and Kaletha-Minguez-Shin-White’s endoscopic classifications for unitary groups [Mok15] and [KMSW14]. Both depend on the unpublished weighted twisted fundamental lemma. The second in addition pushes many technical details to a specific reference “KMSb” that is not yet publicly available.

We note that [AGI<sup>+</sup>24] recently resolved the dependence of [Mok15] and [KMSW14] on the unitary analogues of the unpublished references A25-27 in [Art13].

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## 1.5 Notation

### 1.5.1 Conventions

Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ . We say:

- $f \ll g$  if there is a constant  $C$  such that  $f(x) \leq Cg(x)$  for all large enough  $x$ ,
- $f \asymp g$  if both  $f \ll g$  and  $g \ll f$ ,
- $f \lesssim g$  if for any  $\epsilon > 0$ , there exists  $c_\epsilon > 0$  such that  $f(x) \leq c_\epsilon \cdot \max\{g(x)^{1+\epsilon}, g(x)^{1-\epsilon}\}$ ,
- $f \sim g$  if both  $f \lesssim g$  and  $g \lesssim f$ .

Given a number field  $F$ , denote its adeles by

$$\mathbb{A}_F := \prod'_v F_v,$$

i.e., the restricted product over localizations at all its places with respect to local integers. Given a set of places  $S$  of  $F$ , we use standard upper- and lower-indexing notation:

$$F_S := \prod_{v \in S}' F_v, \quad F^S := \prod_{v \notin S}' F_v.$$

If  $X$  is a variety over  $F$ , then also define  $X_S = X(F_S)$  and  $X^S = X(F^S)$ .



We use the standard notational conventions of [Art13], [Mok15], and [KMSW14] for Arthur parameters packets— $\psi$  is a parameter,  $\psi_v$  its localization to place  $v$ ,  $\Pi_{\psi_v}$  the local  $A$ -packet, etc.

Finally, if  $\pi_1, \pi_2$  are representations of groups  $G_1, G_2$  respectively, then  $\pi_1 \boxtimes \pi_2$  is the outer-product representation of  $G_1 \times G_2$ . If  $G$  is a group,  $Z_G$  is its center (either abstract or algebraic depending on context).

### 1.5.2 Reference List

We also use the following non-standard notations across many sections:

- $\mathcal{B}$  is usually the Bruhat-Tits building of the local group  $G$  being studied in the context,
- $\Sigma$  is a technical modification of the set of fundamental weights of  $G$  from §3.3.1/§3.3.2,
- $\Sigma_0$  is a subset of  $\Sigma$  as in §3.3.3 which can be *Weyl-complete* or *navigable*,
- $\Sigma^{\text{sc}}, \Sigma_0^{\text{sc}}$  are modifications of  $\Sigma, \Sigma_0$  from §3.4.1
- $\|\cdot\|_0, \|\cdot\|$  are the Cartan norm and modified Cartan norm of 3.3.1/3.3.6,
- $\|\cdot\|_{\Sigma_0}, \|\cdot\|_{\Sigma_0^{\text{sc}}}$  are the non-standard Cartan norm and decimated Cartan norm of 3.3.11 and §3.4.1 respectively,
- $S_\star, \tilde{S}_\star, {}^0S_\star, {}^0\tilde{S}_\star, C_\star, \tilde{C}_\star$  are gate sets and special subsets defined in 3.4.1. Tilde-versions resolve technical issues with non-trivial centers,
- $S_\star^{[\ell]}, \tilde{S}_\star^{[\ell]}$  are subsets of the group generated by a gate set in Proposition 3.4.2(2,3) that are slight modifications of  $S^{(\ell)}$ ,
- $T_\star, \tilde{T}_\star, {}^0T_\star, {}^0\tilde{T}_\star$  are subsets of a super-golden gate set defined in 4.2.4,
- $\tau$  is a subset of  $\mathcal{B}$  that can be (*decimated* or *non-standard*) *traversable* as in 4.2.3.

For the automorphic bounds in the second half:

- $\square$  is a shape of an automorphic representation as in 5.2.3,
- $G_F(\square)$  is a group associated to shape  $\square$  in (5.1)
- $L_\square^2$  is the space of automorphic representations with a given shape  $\square$  in 5.2.5
- $\mathbb{P}_\star$  is the orthogonal projection onto  $\star$  (e.g, as in 5.2.5)
- $\|\lambda\|, m(\lambda), \dim(\lambda), \dim_\square(\lambda)$  are various norms of infinitesimal characters from §5.3.2,
- $N_G, N_G^{\text{der}}, r_G, P_G$  are various dimensions associated to group  $G$  from §5.3.2
- $\lambda \in \square$  means infinitesimal character  $\lambda$  is compatible with shape  $\square$  as in 5.3.2
- $e(\square)$  is a dimension associated to shape  $\square$  from (5.4),
- $\mathcal{F}$  is usually an automorphic family as in 5.4.1,
- $\sigma_\square$  is a decay-bound associated to shape  $\square$  in 6.4.4,
- $f_{v_0}^{\epsilon, Z}$  is a test function defined in (7.4) that is almost the indicator function of a ball.



## 2 Arithmetic unitary groups

In this section, we collect some facts about arithmetic unitary groups and their class number.

### 2.1 Definitions

Let  $F$  be a totally real number field and let  $\mathcal{O}$  be its ring of integers. Let  $V$  be the set of places of  $F$  and let  $V_f$  and  $V_\infty$  be the subsets of finite and infinite places, respectively. For any  $v \in V$ , let  $F_v$  be the  $v$ -completion of  $F$  and, if  $\mathfrak{p} \in V_f$ , let  $\mathcal{O}_{\mathfrak{p}}$  be the ring of integers of  $F_{\mathfrak{p}}$  and  $q_{\mathfrak{p}}$  the residue field degree of  $F_{\mathfrak{p}}$ . For any  $S \subset V$ , let

$$\mathcal{O}[1/S] = F \bigcap_{\mathfrak{p} \in V_f \setminus S} \mathcal{O}_{\mathfrak{p}}$$

be the ring of  $S$ -integers of  $F$ . When  $S = \{\mathfrak{p}\} \cup V_\infty$ , we abbreviate  $\mathcal{O}[1/\mathfrak{p}] := \mathcal{O}[1/S]$ , called the ring of  $\mathfrak{p}$ -integers of  $F$ . Note that for  $S = V_\infty$ , we have  $\mathcal{O}[1/S] =: \mathcal{O}$ , the ring of integers of  $F$ .

Let  $E$  be a totally imaginary quadratic extension of  $F$  and let  $\mathcal{O}_E$  be its ring of integers. Let  $3 \leq n \in \mathbb{N}$  and let  $H \in GL_n(E)$  be a non-degenerate, Hermitian, totally positive-definite matrix. Assume for simplicity that  $H \in M_n(\mathcal{O}_E)$  and that  $\gcd(H_{ij}) \in \mathcal{O}_E^\times$ . Denote by  $U_n^{E,H}$  the unitary group scheme over  $\mathcal{O}$  with respect to  $E$  and  $H$  defined for any  $\mathcal{O}$ -algebra  $A$  by

$$U_n^{E,H}(A) := \{g \in GL_n(A \otimes_{\mathcal{O}} \mathcal{O}_E) : g^* H g = H\}.$$

Define the special unitary and projective unitary group schemes  $SU_n^{E,H}$  and  $PU_n^{E,H}$  over  $\mathcal{O}$  with respect to  $E$  and  $H$  by:

$$SU_n^{E,H} := \{g \in U_n^{E,H} : \det(g) = 1\}, \quad PU_n^{E,H} := U_n^{E,H}/U_1^E,$$

where  $U_1^E(A) := \{x \in (A \otimes_{\mathcal{O}} \mathcal{O}_E)^\times : \bar{x}x = 1\}$  is identified with the scalar matrices of  $U_n^{E,H}$ . Finally, define

$$GU_n := \{g \in GL_n(A \otimes_{\mathcal{O}} \mathcal{O}_E) : g^* H g = \alpha H, \alpha \in (A \otimes_{\mathcal{O}} \mathcal{O}_E)^\times\}.$$

We have equalities on points:  $PU_n(\mathbb{R}) = U_n(\mathbb{R})/U_1(\mathbb{R})$  and  $PU_n(S) = GU_n(S)/(S \otimes_{\mathcal{O}} \mathcal{O}_E)^\times$  for local and global fields  $S$ . However, this does not necessarily hold more generally.

**Remark 2.1.1.** In our final construction, we will exclusively use  $U_n^{E,H}$  since it is the only case where we have access to the endoscopic classification and can prove optimal covering. However, the endoscopic classification is expected to be true for the other groups as well and these may be better suited to the construction of super-golden gates at non-split primes. We therefore consider more general groups in this section.

Let  $\text{disc}(H) \in F^\times/N_{E/F}(E^\times)$  be the discriminant:

$$\text{disc}(H) := (-1)^{n(n-1)/2} \det(\mathbf{H}) \tag{2.1}$$

for any matrix representation  $\mathbf{H}$  of  $H$  (this is well-defined up to norms). Since  $H$  is totally positive-definite, we recall a well-known fact (using the classification of Hermitian forms over  $p$ -adic fields from [Jac62]):

**Lemma 2.1.2.** *At each place  $v$ ,  $U_n^{E,H}(F_v)$  is isomorphic to*

- *The classical (i.e. complex-entry) compact unitary group of rank  $n$  if  $v|\infty$ ,*

- $\mathrm{GL}_n(F_{\mathfrak{p}})$  if  $v = \mathfrak{p}$  is split finite,
- The unique quasisplit unitary group for extension  $F_{\mathfrak{p}} \otimes_E E/F_{\mathfrak{p}}$  if  $v = \mathfrak{p}$  is non-split finite and either  $n$  is odd or  $\mathrm{disc}(H)$  is a norm from  $F_{\mathfrak{p}} \otimes_F E$  to  $F_{\mathfrak{p}}$ ,
- The unique non-quasisplit unitary group for extension  $F_{\mathfrak{p}} \otimes_F E/F_{\mathfrak{p}}$  if  $v = \mathfrak{p}$  is non-split finite,  $n$  is even, and  $\mathrm{disc}(H)$  is not a norm from  $F_{\mathfrak{p}} \otimes_F E$  to  $F_{\mathfrak{p}}$ .

## 2.2 Basic Structure

Let  $G$  be either  $U_n^{E,H}$ ,  $SU_n^{E,H}$ ,  $PU_n^{E,H}$ , or  $GU_n^{E,H}$ . For any  $v \in V$ , denote  $G_v := G(F_v)$ , and for any  $\mathfrak{p} \in V_f$ , denote  $K_{\mathfrak{p}} := G(\mathcal{O}_{\mathfrak{p}})$  and  $\Gamma_{\mathfrak{p}} := G(\mathcal{O}[1/\mathfrak{p}])$ . Call  $\Gamma_{\mathfrak{p}}$  the principal  $\mathfrak{p}$ -arithmetic subgroup of  $G$ ; any finite index subgroup of it is called a  $\mathfrak{p}$ -arithmetic subgroup. Denote  $G_{\infty} := \prod_{v \in V_{\infty}} G_v$  and  $\Gamma := G(\mathcal{O})$ . Call  $\Gamma$  the principal arithmetic subgroup of  $G$ ; any finite index subgroup of it is called an arithmetic subgroup.

**Lemma 2.2.1.** *The following hold for  $G \neq GU_n^{E,H}$ :*

- (1) Any arithmetic subgroup of  $G$  is finite.
- (2) Any  $\mathfrak{p}$ -arithmetic subgroup of  $G$  is a cocompact lattice of  $G_{\mathfrak{p}}$ .
- (3) Any  $\mathfrak{p}$ -arithmetic subgroup of  $G$  is a dense subgroup of  $G_{\infty}$ .

*Proof.* First note that, since  $H$  is totally positive-definite, we get that for any  $v \in V_{\infty}$ ,  $G_v \cong U(n)$ ,  $SU(n)$  or  $PU(n)$ , the unitary, special unitary, or projective unitary compact Lie groups; hence,  $G_{\infty}$  is a compact Lie group. By Borel–Harish-Chandra theory [BHC62], we get that any finite arithmetic subgroup of  $G$  is a cocompact lattice of  $G_{\infty}$  and any  $\mathfrak{p}$ -arithmetic subgroup of  $G$  is a cocompact lattice of  $G_{\infty} \times G_{\mathfrak{p}}$ .

Claim 1 follows from the fact that a discrete subgroup of a compact group is finite. Claim 2 follows from the fact that the projection of a cocompact lattice of  $G_{\infty} \times G_{\mathfrak{p}}$  onto the second component remains a cocompact lattice of  $G_{\mathfrak{p}}$  (since  $G_{\infty}$  is compact). Claim 3 follows from the fact that, since  $\mathrm{rank}_{F_{\mathfrak{p}}}(G_{\mathfrak{p}}) \geq \frac{n}{2} - 1 > 0$  (since  $n \geq 3$ ),  $G_{\mathfrak{p}}$  is non-compact and therefore projecting a cocompact lattice of  $G_{\infty} \times G_{\mathfrak{p}}$  to the first component is dense.  $\square$

Let  $\mathbb{A} := \prod'_{v \in V} F_v$  be the ring of adeles of  $F$ , let  $\widehat{\mathcal{O}} := \prod'_{\mathfrak{p} \in V_f} \mathcal{O}_{\mathfrak{p}}$ , let  $F_{\infty} = \prod_{v \in V_{\infty}} F_v$ , and consider  $F$  embedded diagonally in  $\mathbb{A}$ . Then  $\mathbb{A}$  is a locally compact ring,  $F$  (embedded diagonally) is a discrete subring,  $\widehat{\mathcal{O}}$  and  $F_{\infty}\widehat{\mathcal{O}}$  (embedded coordinate-wise) are compact and open subrings, respectively, and  $\mathbb{A} = F \cdot F_{\infty}\widehat{\mathcal{O}}$ .

**Lemma 2.2.2.** *Let  $E$  be of class number one. Then*

$$U_1^E(\mathbb{A}) = U_1^E(F) \cdot \prod_{v \in V_{\infty}} U_1^E(F_v) \prod_{\ell \in V_f} U_1^E(\mathcal{O}_{\ell}).$$

*Proof.* Let  $x = (x_v) \in U_1^E(\mathbb{A})$ , i.e.  $\overline{x_v}x_v = 1$  for any  $v$  and  $x_{\ell} \in U_1^E(\mathcal{O}_{\ell})$  for almost all finite places  $\ell$ . Note that if  $\ell$  is non-split in  $E$ , then  $U_1^E(F_{\ell}) = U_1^E(\mathcal{O}_{\ell})$ . If  $\ell = \ell_1\ell_2$  is split in  $E$ , then  $F_{\ell} \otimes_F E = E_{\ell_1} \times E_{\ell_2}$  so we can write  $x_{\ell} = (x_{\ell_1}, x_{\ell_2})$  for  $x_{\ell_i} \in E_{\ell_i}$ .

Let  $S$  be the finite set of split primes  $\ell$  such that  $|x_{\ell_1}|_{\ell_1} \neq 1$ . By the class-number-one hypothesis, there exists  $\alpha \in E$  such that  $|\alpha|_{\ell_1} = |x_{\ell_1}|_{\ell_1}$  and  $|\alpha|_{\ell_2} = 1$  for all  $\ell \in S$  and  $|\alpha|_{\ell_i} = 1$  for all  $\ell \notin S$ . Then  $\alpha/\bar{\alpha} \in U_1^E(F)$  and satisfies  $|\alpha_{\ell_i}| = |x_{\ell_i}|_{\ell_i}$  for  $i = 1, 2$  and all split  $\ell$ .

Putting it all together,  $x \in (\alpha/\bar{\alpha}) \prod_{v \in V_{\infty}} U_1^E(F_v) \prod_{\ell \in V_f} U_1^E(\mathcal{O}_{\ell})$ .  $\square$

**Lemma 2.2.3.** *Let  $G$  be either  $U_n^{E,H}$ ,  $SU_n^{E,H}$ ,  $PU_n^{E,H}$ , or  $GU_n^{E,H}$ . If  $G \neq SU_n^{E,H}$  assume that  $E$  is of class number one. Then for any prime  $\mathfrak{p}$ ,*

$$G(\mathbb{A}) = G(F) \cdot G_\infty G_{\mathfrak{p}} \prod_{\mathfrak{p} \neq \ell \in V_f} K_\ell.$$

*Proof.* If  $G = SU_n^{E,H}$ , then by the strong approximation property we get that  $G(F)G_{\mathfrak{p}}$  is dense in  $G(\mathbb{A})$ , so since  $K = G_\infty \prod_{\ell \in V_f} K_\ell$  is open, we get the claim. If  $G = U_n^{E,H}$ , then by the claim for  $SU_n^{E,H}$ , we get that  $G(F) \cdot G_\infty G_{\mathfrak{p}} \prod_{\mathfrak{p} \neq \ell \in V_f} K_\ell$  contains  $SU_n^{E,H}(\mathbb{A})$ . Then, since  $U_n = SU_n^{E,H} U_1^E$ , the claim follows from Lemma 2.2.2.

If  $G = GU_n^{E,H}$ , then we similarly use  $GU_n^{E,H} = SU_n^{E,H} \text{Res}_{\mathcal{O}_F}^{\mathcal{O}_E} \mathbb{G}_m$  and that  $E$  has class number one. Finally, the result for  $G = PU_n^{E,H}$  follows from that for  $GU_n^{E,H}$ : by Shapiro's lemma, we have surjections  $GU_n^{E/H}(S) \twoheadrightarrow PU_n^{E/H}(S)$  for  $S = \mathbb{A}, F_{\mathfrak{p}}, F$  under which  $PU_n^{E,H}(\mathcal{O}_{F_{\mathfrak{p}}})$  contains the image of  $GU_n^{E,H}(\mathcal{O}_{F_{\mathfrak{p}}})$ .  $\square$

**Lemma 2.2.4.** *Assume either  $G = SU_n^{E,H}$  or that  $E$  is of class number one. Then for any prime  $\mathfrak{p}$ , there is a bijective map from  $\Gamma_{\mathfrak{p}} \backslash G_{\mathfrak{p}} / K_{\mathfrak{p}}$  to  $G(F) \backslash G(\mathbb{A}) / K$ .*

*Proof.* The map from  $G_{\mathfrak{p}}$  to  $G(\mathbb{A})$ , defined by

$$g \mapsto (g_v)_{v \in V}, \quad g_v = \begin{cases} g & v = \mathfrak{p} \\ 1 & v \neq \mathfrak{p} \end{cases}$$

induces a well defined map from  $\Gamma_{\mathfrak{p}} \backslash G_{\mathfrak{p}} / K_{\mathfrak{p}}$  to  $G(F) \backslash G(\mathbb{A}) / K$ . This map is obviously injective, so the claim boils down to proving it is surjective, which follows from the strong approximation type result of Lemma 2.2.3.  $\square$

**Corollary 2.2.5.** *Assume either  $G = SU_n^{E,H}$  or that  $E$  is of class number one. The following are equivalent:*

- *There exists  $\mathfrak{p}$  such that  $G_{\mathfrak{p}} = \Gamma_{\mathfrak{p}} K_{\mathfrak{p}}$ .*
- *For any  $\mathfrak{p}$ ,  $G_{\mathfrak{p}} = \Gamma_{\mathfrak{p}} K_{\mathfrak{p}}$ .*

## 2.3 Class Numbers

Let  $G$  be either  $U_N^{E,H}$ ,  $SU_N^{E,H}$ , or  $PU_N^{E,H}$ . By [BHC62],  $G(F)$  is a cocompact lattice of  $G(\mathbb{A})$  and note that  $K := G_\infty G(\widehat{\mathcal{O}}) \leq G(\mathbb{A})$  is a compact open subgroup. Hence the double quotient space  $G(F) \backslash G(\mathbb{A}) / G_\infty G(\widehat{\mathcal{O}})$ , is finite.

**Definition 2.3.1.** Let  $G$  be either  $U_n^{E,H}$ ,  $SU_n^{E,H}$ , or  $PU_n^{E,H}$ . Define the class and  $\mathfrak{p}$ -class numbers of  $(G, K)$  to be

$$c(G) := |G(F) \backslash G(\mathbb{A}) / K|, \quad c_{\mathfrak{p}}(G) := |\Gamma_{\mathfrak{p}} \backslash G_{\mathfrak{p}} / K_{\mathfrak{p}}|.$$

We note that by Lemma 2.2.4 we get that  $c_{\mathfrak{p}}(G) = c(G)$ , for any prime  $\mathfrak{p}$ , and that by Corollary 2.2.5, if  $G_{\mathfrak{p}} = \Gamma_{\mathfrak{p}} K_{\mathfrak{p}}$  holds for one  $\mathfrak{p}$ , then it holds for all  $\mathfrak{p}$ .

When the chosen  $K$  satisfies a particular property of being “special maximal compact”<sup>(†)</sup> at all places  $v$  (see §3.2.1), the Mass formula of [GHY01] provides a good test if  $(G, K)$  has class number one.

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<sup>(†)</sup>beware that [GHY01] uses a non-standard definition of “maximal parahoric” that corresponds to the standard notion of special maximal compact. This is pointed out above (2.2) therein.

For any CM field  $E/F$  and any Hermitian form  $H \in M_n(E)$ , denote by  $\text{disc}(E)$  the discriminant of  $E$  and  $\text{disc}(H)$  that of  $H$  as in (2.1). Denote by  $\text{Ram}(E)$  the set of primes that divide  $\text{disc}(E)$  and  $\text{Ram}(H)$  the sets of primes  $v$  at which  $\text{disc}(H) \notin N_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}(E_{\mathfrak{q}}^{\times})$ . Denote by  $\zeta_F(s)$  and  $L(s, \chi_{E/F})$  the (analytic continuations of the) Dedekind zeta function of  $F$  and the Dirichlet  $L$ -function of  $\chi_{E/F}$ , the quadratic Dirichlet character associated to  $E/F$  by class field theory.

**Definition 2.3.2.** Let  $G = U_n^{E,H}$ . Define the set of ramified primes of  $G$  to be

$$\text{Ram}(G) := \text{Ram}(E) \bigcup \text{Ram}(H),$$

define the  $\lambda$ -constant of  $G$  to be

$$\lambda(G) := \prod_{\ell \in \text{Ram}(G)} \lambda_{\ell}$$

with

$$\lambda_{\ell} = \begin{cases} 1/2 & 2 \nmid n, \ell \in \text{Ram}(E) \\ (q_{\ell}^n + 1)/(q_{\ell} + 1) & 2 \nmid n, \ell \notin \text{Ram}(E) \\ 1 & 2 \mid n, \ell \notin \text{Ram}(H) \\ (q_{\ell}^n - 1)/(q_{\ell} + 1) & 2 \mid n, \ell \notin \text{Ram}(E), \ell \in \text{Ram}(H) \\ (1/2)(q_{\ell}^n - 1)/(q_{\ell}^{n/2} + 1) & 2 \mid n, \ell \in \text{Ram}(E), \ell \in \text{Ram}(H), K_{\ell} \text{ extraspecial} \\ (1/2)(q_{\ell}^n - 1)/(q_{\ell} + 1) & 2 \mid n, \ell \in \text{Ram}(E), \ell \in \text{Ram}(H), K_{\ell} \text{ not extraspecial} \end{cases},$$

define the  $L$ -special value of  $G$  to be

$$\begin{aligned} L(G) &:= 2^{-n[F:\mathbb{Q}]+1} \cdot \prod_{r=1}^n L(1-r, \chi_{E/F}^r) \\ &= 2^{-n[F:\mathbb{Q}]+1} \cdot \prod_{r=1}^{\lfloor n/2 \rfloor} \zeta_F(1-2r) \cdot \prod_{r=1}^{\lfloor (n+1)/2 \rfloor} L(2-2r, \chi_{E/F}), \end{aligned}$$

and finally, define the mass constant of  $G$  to be

$$R(G) := L(G) \cdot \lambda(G).$$

Then:

**Lemma 2.3.3.** *Let  $G = U_n^{E,H}$  and assume that  $G(\mathcal{O}_{\mathfrak{p}})$  is special maximal compact at all finite places  $\mathfrak{p}$ . Then*

$$c(G) = 1 \quad \Leftrightarrow \quad |G(\mathcal{O})|^{-1} = R(G).$$

*Proof.* Let  $\mu_{\text{tam}}$  be the Tamagawa measure of  $G(\mathbb{A})$  and denote the mass of  $G$  by

$$\text{Mass}(G) := \frac{\mu_{\text{tam}}(G(F) \backslash G(\mathbb{A}))}{\mu_{\text{tam}}(K)}.$$

On the one hand, Propositions 4.4 and 4.5 in [GHY01], yield

$$\text{Mass}(G) = R(G).$$

Note that Tables 1 and 2 in [GHY01] do not cover the cases of  $\ell$  ramified,  $K_\ell$  extraspecial, and  $G_\ell$  either odd or even non-quasisplit. These cases can be filled in by [GHY01, (2.12)]. For the first case, the reductive quotient of the special fiber of the corresponding parahoric integral model is  $\mathrm{SO}_n$  since  $G_\ell$  corresponds to type  $C\text{-}BC_{(n-1)/2}$  in the tables of [Tit79]. This has the same number of points as  $\mathrm{Sp}_{n-1}$ . In the second case, the reductive quotient is  ${}^2\mathrm{SO}_n$  coming from type  ${}^2B\text{-}C_{n/2}$ . In both cases, the parahoric is index two in the maximal compact using results from [HR10] (see §3 for more details) and formulas for point counts over finite fields can be found in [CCN<sup>+</sup>85, §2].

On the other hand, by Siegel mass formula, the mass of  $G$  is equal the sum over the representatives of the genus of  $G$  weighted by the reciprocal of the size of the associated arithmetic finite group:

$$\mathrm{Mass}(G) = \sum_{g \in G(F) \backslash G(\mathbb{A})/K} |G(F) \cap gKg^{-1}|^{-1} = |G(\mathcal{O})|^{-1} + \sum_{1 \neq g \in G(F) \backslash G(\mathbb{A})/K} \dots$$

Since each member in the sum is positive rational number, comparing both estimates shows that  $|G(\mathcal{O})|^{-1} = R(G)$  if and only if  $c(G) = 1$ .  $\square$

**Remark 2.3.4.** There are only finitely many values of  $n$ ,  $\mathrm{disc}(E)$ , and  $\mathrm{disc}(H)$  such that the definite unitary group  $G = U_N^{E,H}$  has class number one (see [BP89]). In fact, Mohammadi and Salehi-Golsefidy in [MSG12] showed that  $n = 8$  is the threshold for definite unitary groups of class number one; namely, for any  $n > 8$ , there are no class number one definite unitary groups of the form  $G = U_N^{E,H}$  and for any  $4 < n \leq 8$ , there exists class number one definite unitary groups of the form  $G = U_N^{E,H}$ .

### 3 Bruhat-Tits buildings

This subsection summarizes facts about the theory of Bruhat–Tits buildings of unitary and general linear groups that we will need. The standard reference summarizing the theory is [Tit79]. A modernized textbook treatment of the full details can be found in [KP23].

#### 3.1 Description

Let  $\mathbb{F}$  be a non-Archimedean local field,  $\mathcal{O}_{\mathbb{F}} \subset \mathbb{F}$  its ring of integers,  $\varpi_{\mathbb{F}} \in \mathcal{O}_{\mathbb{F}}$  a choice uniformizer and  $q_{\mathbb{F}} = |\mathcal{O}_{\mathbb{F}}/\varpi_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}|$  its residue degree (e.g.  $\mathbb{F} = \mathbb{Q}_p$ ,  $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}_p$ ,  $\varpi_{\mathbb{F}} = p$  and  $q_{\mathbb{F}} = p$ ). Let  $G$  be the group of  $\mathbb{F}$ -rational points of an  $\mathbb{F}$ -almost-simple, connected reductive group  $\mathcal{G}$ . Then Bruhat–Tits theory constructs a pure, simplicial, infinite, locally-finite, contractible complex  $\mathcal{B} := \mathcal{B}(G)$  called the Bruhat-Tits building of  $G$  such that  $G$  acts simplicially on  $\mathcal{B}(G)$  and transitively on its maximal faces (called chambers). The dimension of  $\mathcal{B}(G)$  is equal to the semisimple rank  $r = \mathrm{rank}_{\mathbb{F}}(G^{\mathrm{der}})$ .

To define nice stabilizers, we also sometimes consider the enlarged building  $\tilde{\mathcal{B}}(G)$  which is  $\mathcal{B}(G)$  times a real vector space coming from the split part of the center of  $G$ . If  $\tau \subseteq \mathcal{B}(G)$ , we may consider it as  $\tilde{\tau} \subseteq \tilde{\mathcal{B}}(G)$ . Then the stabilizer  $G_{\tilde{\tau}} = G_{\tau} \cap G^1$ , where  $G^1$  is the subgroup of  $G$  on which all characters take values of valuation 0, i.e.  $G^1 = \bigcap_{\chi} \ker(\mathrm{val} \circ \chi)$ , where  $\chi$  runs over all characters from  $G$  to  $\mathbb{F}^{\times}$  and  $\mathrm{val}: \mathbb{F}^{\times} \rightarrow \mathbb{Z}$  is the valuation map.

The building is equipped with a natural type function from its 0-simplices to  $[r] := \{0, 1, \dots, r\}$ , which is a bijection on the vertices of each chamber. We say the type of a  $k$ -simplex is the set consisting of the  $k$  types of its vertices. If  $G = \mathbf{G}(\mathbb{F})$  for  $\mathbf{G}$  semisimple and simply connected, then the  $H$ -orbit of a simplex of type  $\tau$  is the set of all simplices of type  $\tau$ . In general, it might be bigger.

Bruhat-Tits theory highlights some vertices as being special or hyperspecial. Hyperspecial vertices exist if and only if  $G$  is unramified: i.e,  $G = \mathbf{G}(\mathbb{F})$ , where  $\mathbf{G}$  is quasi-split over  $\mathbb{F}$  and splits over an unramified extension of  $\mathbb{F}$ . In this case, for hyperspecial  $v_0$ , there is known to be a reductive model  $\mathbf{G}$  of  $G$  over  $\mathcal{O}_{\mathbb{F}}$  such that  $K_G = \mathbf{G}(\mathcal{O}_{\mathbb{F}})$  is the stabilizer of  $v_0$ . Special vertices always exist and similarly have stabilizer coming from a model with special fiber having the “largest possible” reductive quotient in some sense. We fix a choice of special  $v_0$  that is hyperspecial if  $G$  is unramified and without loss of generality assume to be of type 0.

Lemma 2.1.2 enumerates our cases of interest. We describe  $\mathcal{B}$  in these cases when  $\mathbb{F}$  has residue characteristic different from 2; see [KP23, §15] for the full details:

**Example 3.1.1.** Let  $G = PGL_n(\mathbb{F})$  be the projective general linear group of  $\mathbb{F}$ -points. The Bruhat-Tits building  $\mathcal{B} = \mathcal{B}(PGL_n(\mathbb{F}))$  is the  $(n-1)$ -dimensional simplicial complex, whose vertices are homotetic classes  $[L] = \{\alpha L \mid \alpha \in \mathbb{F}^\times\}$  of  $\mathcal{O}_{\mathbb{F}}$ -lattices  $L$  in  $\mathbb{F}^n$  and a collection of vertices  $\sigma = \{v_0, \dots, v_k\}$ , forms a face in  $\mathcal{B}$  if there exists representatives  $L_i \in v_i$ ,  $0 \leq i \leq k$  such that  $L_0 \subsetneq \dots \subsetneq L_k \subsetneq \varpi_{\mathbb{F}}^{-1} L_0$ .

The group  $PGL_n(\mathbb{F})$  acts on the homotetic classes of lattices by matrix multiplication and this action extends to the entire complex. The subgroup  $PGL_n(\mathcal{O}_{\mathbb{F}})$  is the stabilizer of the hyperspecial vertex  $v_0 = [\mathcal{O}_{\mathbb{F}}^n]$ . The group  $PGL_n(\mathbb{F})$  acts transitively on the vertices of the building, hence all vertices are hyperspecial. The degree of (i.e. the number of chambers containing) any  $(n-2)$ -dimensional face is  $q_{\mathbb{F}} + 1$ .

For example,  $\mathcal{B}(PGL_2(\mathbb{F}))$  is the infinite  $(q_{\mathbb{F}} + 1)$ -regular tree, and  $\mathcal{B}(PGL_3(\mathbb{F}))$  is the infinite 2-dimensional simplicial complex, all of whose edges are of degree  $q_{\mathbb{F}} + 1$  and all of whose vertices are contained in  $2(q_{\mathbb{F}}^2 + q_{\mathbb{F}} + 1)$  edges.

**Example 3.1.2.** Let  $\mathbb{E}$  be a tame quadratic extension of  $\mathbb{F}$  and let  $c \mapsto \bar{c}$  be the non-trivial element in  $\text{Gal}(\mathbb{E}/\mathbb{F})$ . Let  $G = PU_n(\mathbb{F})$  be the corresponding quasisplit projective unitary group:

$$PU_n(\mathbb{F}) = \{g \in GL_n(\mathbb{E}) : g^* J g = J\} / \{c I_n : \bar{c} c = 1, c \in \mathbb{E}^\times\},$$

where  $g^* = \bar{g}^t$  and  $J = (\delta_{i, n+1-j})_{i,j}$  is the anti-diagonal Hermitian form. Let  $\sharp$  be the involution on  $PGL_n(\mathbb{E})$  defined by  $g^\sharp = J(g^*)^{-1} J$ . Note that  $PU_n(\mathbb{F})$  is the subgroup of  $\sharp$ -fixed elements of  $PGL_n(\mathbb{E})$ :

$$PU_n(\mathbb{F}) = PGL_n(\mathbb{E})^\sharp := \{g \in PGL_n(\mathbb{E}) : g^\sharp = g\}.$$

Similarly, define the order 2 automorphism  $\sharp$  on  $\mathcal{B}(PGL_n(\mathbb{E}))$  as follows: on the  $\mathcal{O}_{\mathbb{E}}$ -lattices  $L$  of  $\mathbb{E}^n$ , define

$$L^\sharp := \{v \in \mathbb{E}^n \mid v J \bar{u} \in \mathcal{O}_{\mathbb{E}}, \forall u \in L\},$$

on the homothetic class of  $\mathcal{O}_{\mathbb{E}}$ -lattices (i.e. the vertices)  $v = [L]$ , define  $v^\sharp = [L^\sharp]$ , and on the faces  $\sigma = \{v_0, \dots, v_k\}$ , define  $\sigma^\sharp = \{v_0^\sharp, \dots, v_k^\sharp\}$ .

Then the Bruhat-Tits building of  $PU_n(\mathbb{F})$  is the subcomplex of  $\sharp$ -fixed faces of the Bruhat-Tits building of  $PGL_n(\mathbb{E})$ : i.e,

$$\mathcal{B}(PU_n(\mathbb{F})) = \mathcal{B}(PGL_n(\mathbb{E}))^\sharp := \{\sigma \in \mathcal{B}(PGL_n(\mathbb{E})) : \sigma^\sharp = \sigma\}.$$

The group  $PU_n(\mathbb{F}) = PGL_n(\mathbb{E})^\sharp$  acts naturally on  $\mathcal{B}(PU_n(\mathbb{F})) = \mathcal{B}(PGL_n(\mathbb{E}))^\sharp$ . The vertices of the buildings of  $\mathcal{B}(PU_n(\mathbb{F}))$  are either  $\sharp$ -fixed vertices, which are the special vertices (hyperspecial if  $\mathbb{E}/\mathbb{F}$  is unramified), or edges whose endpoints are swapped by  $\sharp$ , which are non-special.

Note that when  $\mathbb{E}/\mathbb{F}$  is unramified,  $v_0 = [\mathcal{O}_{\mathbb{E}}^n] \in \mathcal{B}(PGL_n(\mathbb{E}))^\sharp$  is a hyperspecial vertex and its stabilizer in  $PU_n(\mathbb{F})$  is  $PU_n(\mathcal{O}_{\mathbb{F}})$ , the subgroup of elements with coefficients in  $\mathcal{O}_{\mathbb{F}}$ .

The dimension of  $\mathcal{B}(PU_n(\mathbb{F}))$ , which is equal to the  $\mathbb{F}$ -rank  $PU_n(\mathbb{F})$ , is  $\lfloor \frac{n}{2} \rfloor$ . When  $\mathbb{E}/\mathbb{F}$  is ramified, a special vertex can instead be given by  $v_0 = [\mathcal{O}_{\mathbb{E}}^{[n/2]} \oplus (\mathfrak{D}_{\mathbb{E}/\mathbb{F}}^{-1})^{[n/2]}]$ , where  $\mathfrak{D}_{\mathbb{E}/\mathbb{F}}$  is the different ideal.

For example, when  $\mathbb{E}/\mathbb{F}$  is unramified,  $\mathcal{B}(PU_3(\mathbb{F}))$  is the infinite  $(q_{\mathbb{F}}^3 + 1, q_{\mathbb{F}} + 1)$ -biregular tree. All vertices are special and those of degree  $q_{\mathbb{F}}^3 + 1$  are also hyperspecial. When  $\mathbb{E}/\mathbb{F}$  is ramified,  $\mathcal{B}(PU_3(\mathbb{F}))$  is an infinite  $(q_{\mathbb{F}} + 1)$ -regular tree all of whose vertices are special with every other vertex satisfying a stronger property of being extraspecial.

**Example 3.1.3.** Now, consider  $\mathbb{E}/\mathbb{F}$  tame quadratic,  $n$  even, and  $G = PU'_n$  the non-quasisplit unitary group instead preserving  $J \text{diag}(1, \dots, 1, a)$  where  $a \notin N_{\mathbb{E}/\mathbb{F}}(\mathbb{E}^\times)$ . Then we still have for an analogously defined  $\sharp'$  that

$$\mathcal{B}(PU'_n(\mathbb{F})) = \mathcal{B}(PGL_n(\mathbb{E}))^{\sharp'}.$$

However, this now has dimension  $(n - 2)/2$  instead of  $n/2$  and a slightly different structure.

For example, if  $\mathbb{E}/\mathbb{F}$  is unramified, then  $\mathcal{B}(PU'_4(\mathbb{F}))$  is an infinite  $(q_{\mathbb{F}}^3 + 1)$ -regular tree with all vertices extraspecial. If on the other hand  $\mathbb{E}/\mathbb{F}$  is ramified, then  $\mathcal{B}(PU'_4(\mathbb{F}))$  is an infinite  $(q_{\mathbb{F}}^2 + 1, q_{\mathbb{F}} + 1)$ -biregular tree. The degree- $(q_{\mathbb{F}}^2 + 1)$  vertices are extraspecial.

**Example 3.1.4.** For  $G = GL_n$ , the building  $\mathcal{B}$  is that same as that of  $PGL_n$ . Since the center intersect a maximally split torus is in this case connected, the action of  $GL_n$  on  $\mathcal{B}$  factors through  $PGL_n$ .

**Example 3.1.5.** For  $G = U_n$ , the building  $\mathcal{B}$  is the same as that of  $PU_n$ . However, when  $n$  is even, the natural map  $U_n(\mathbb{F}) = GL_n(\mathbb{E})^{\sharp}$  to  $PU_n(\mathbb{F})$  is not necessarily a surjection so the action might be smaller.

**Remark 3.1.6.** In the unramified case, for  $G = PGL_n(\mathbb{F})$ ,  $G = PU_n(\mathbb{F})$ , or  $G = U_{2n+1}(\mathbb{F})$ , the group acts transitively on the hyperspecial vertices of  $\mathcal{B}(G)$ . However this is not true in general.

**Remark 3.1.7.** When  $\mathbb{E}/\mathbb{F}$  is wildly ramified in 3.1.2 (or 3.1.3), we can only guarantee that

$$\mathcal{B}(PU_n(\mathbb{F})) \subseteq \mathcal{B}(PGL_n(\mathbb{E}))^{\sharp}.$$

The fixed points contain extra pieces called “barbs”, each one branching away from  $\mathcal{B}(PU_n(\mathbb{F}))$  a finite distance.

Nevertheless, the tree description and valencies of vertices for the low-rank examples holds in all cases and, by the end of [Tit79, §2.4], can be read off from the integers  $d(v)$  attached to each vertex in the tables at the end of [Tit79] (Table 3.1 gives a quick reference for our cases).

## 3.2 Vertices, Chambers, and Compact Open Subgroups

### 3.2.1 Special Maximal Compacts

All stabilizers in  $G$  of vertices in  $\mathcal{B}$  are maximal compact since every compact subgroup fixes a point of  $\mathcal{B}$ . Certain vertices of  $\mathcal{B}$  are classified as special, extraspecial, or hyperspecial (see, e.g. [KP23, Def 1.3.39, §7.11]) and have stabilizers with particularly nice abstract behavior:

**Definition 3.2.1.** A compact open  $K \leq G$  is called special (resp. extraspecial, hyperspecial) maximal compact if it is the stabilizer of a special (resp. extraspecial, hyperspecial) vertex in the (enlarged) building  $\tilde{\mathcal{B}}$ .



When  $G$  is semisimple and simply connected, these are the same as the more standard notion of special parahorics (this is not always the case—see [HR08] and [HR10, §8]). Hyperspecial maximal compact subgroups are also always special parahorics. In our particular case of  $G = U^{E,H}$ :

**Lemma 3.2.2.** *Let  $G = U_n^{E,H}(F_{\mathfrak{p}})$  for some choices of  $E, H, \mathfrak{p}$  and let  $K \subseteq G$  be a special parahoric subgroup corresponding to vertex  $x_0 \in \tilde{\mathcal{B}}(G)$ . Then except for the cases when  $E_{\mathfrak{q}}/F_{\mathfrak{p}}$  is ramified and either  $n$  is odd or  $U_n^{E,H}$  is non-quasisplit,  $K = \text{Stab}_G(x_0)$  and is special maximal compact.*

*Proof.* These are the cases with connected special fiber in the tables of [GHY01, §3]. This can also be seen by computing that the  $\Lambda_M$  from [HR10, Thm 1.0.1] has no torsion.  $\square$

**Remark 3.2.3.** Explicit Hermitian forms  $H_{\mathfrak{p}}$  such that  $U^{E_{\mathfrak{q}}, H_{\mathfrak{p}}}(\mathcal{O}_{\mathfrak{p}})$  is special maximal compact can be found in [GHY01, §3]. For example, when  $n$  is even and  $G_{\mathfrak{p}}$  is quasisplit, we may choose

$$H_{\mathfrak{p}} = \begin{pmatrix} & \mathbf{I}_{n/2} \\ \mathbf{I}_{n/2} & \end{pmatrix} \text{ or } \begin{pmatrix} & a_{E_{\mathfrak{q}}} \mathbf{I}_{n/2} \\ \bar{a}_{E_{\mathfrak{q}}} \mathbf{I}_{n/2} & \end{pmatrix}$$

respectively for  $\mathfrak{p}$  unramified or  $\mathfrak{p}$  ramified with  $a_{E_{\mathfrak{q}}}$  a generator of the different ideal  $\mathfrak{D}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}$ . When  $G_{\mathfrak{p}}$  isn't quasisplit and  $E_{\mathfrak{p}}/F_{\mathfrak{q}}$  is unramified, we can similarly pick

$$H_{\mathfrak{p}} = \begin{pmatrix} & & \mathbf{I}_{(n-2)/2} \\ & 1 & 0 \\ & 0 & -\pi_{E_{\mathfrak{p}}} \\ \mathbf{I}_{(n-2)/2} & & \end{pmatrix}$$

for uniformizer  $\pi_{E_{\mathfrak{p}}}$  of  $F_{\mathfrak{p}}$ . When  $E_{\mathfrak{p}}/F_{\mathfrak{q}}$  is ramified, then we can pick either

$$\begin{pmatrix} & & a_{E_{\mathfrak{q}}} \mathbf{I}_{(n-2)/2} \\ & 1 & 0 \\ & 0 & -\beta \\ \bar{a}_{E_{\mathfrak{q}}} \mathbf{I}_{(n-2)/2} & & \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{I}_{n-1} & \\ & \beta \end{pmatrix}$$

for the non-extraspecial or extraspecial case respectively and where  $\beta$  is an element of  $\mathcal{O}_{\mathfrak{p}}$  that isn't a norm from  $E_{\mathfrak{q}}$ . The extraspecial case is not described in [GHY01] and can be seen by computing the points over the residue field.

These examples work even for  $\mathfrak{p}|2$ . Global Hermitian forms  $H$  that localize to these specific  $H_{\mathfrak{p}}$  can be constructed as in [MSG12, §11.3.2].

We finally note:

**Lemma 3.2.4.** *Let  $G = U^{E,H}$  and  $\mathfrak{p} \notin \text{Ram}(E)$  such that  $H$  has matrix representation  $\mathbf{H} \in \text{GL}_n(\mathcal{O}_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} E)$ . Then  $U^{E,H}(\mathcal{O}_{\mathfrak{p}}) \subseteq U^{E,H}(F_{\mathfrak{p}})$  is hyperspecial.*

*Proof.* Then  $U^{E,H}$  is a smooth group scheme over  $\mathbb{Z}_{\mathfrak{p}}$  with reductive special fiber.  $\square$

### 3.2.2 The Chamber

Since  $G$  acts transitively on the chambers of  $\mathcal{B}$ , to understand the conjugacy classes of maximal parahorics, it suffices to understand the vertices of these chambers and their  $G$ -orbits.

In our case of  $G = U^{E,H}$ , Table 3.1 describes the structure of the chamber in each of the possibilities in lemma 2.1.2. Much of the information is from [KP23]: the types can be

looked up in table 6.4.1 and §10.7b,c, extra special vertices are discussed in remark 7.11.9, and the intersection of the action of  $G$  with the automorphisms of a chamber can be found by computing the image of  $\pi_1(G)_I^{\text{Frob}}$  in  $\pi_1(G_{\text{ad}})_I^{\text{Frob}}$  (using Corollary 11.7.5 to interpret  $\pi_1(G_{\text{ad}})_I^{\text{Frob}}$  as the setwise stabilizer of a chamber in  $G_{\text{ad}}(\mathbb{F})$  mod its pointwise stabilizer).

We note that there is a unique extraspecial (hyperspecial if the group is unramified) orbit in all cases except when  $v = \mathfrak{p}$  is inert unramified and  $n$  is even.

$n$	Q-split?	$v$	Label		$G$ -action	Orbits	
			[KP23]	[Tit79]		xs	s
		split	$A_{n-1}$	$A_{n-1}$	transitive	1	0
even	yes	ur.	$C_{n/2}$	${}^2A'_{n-1}$	trivial	2	0
		ram.	$B_{n/2}^\vee$	$B-C_{n/2}$	involutive	1	0
	no	ur.	$(C_{(n-2)/2}^\vee, C_{(n-2)/2})$	${}^2A''_{n-1}$	trivial	2	0
		ram.	$BC_{(n-2)/2}$	${}^2B-C_{n/2}$	trivial	1	1
odd	yes	ur.	$(BC_{(n-1)/2}, C_{(n-1)/2})$	${}^2A'_{n-1}$	trivial	1	1
		ram.	$BC_{(n-1)/2}$	$C-BC_{(n-1)/2}$	trivial	1	1

**Note:** Labels in [KP23] are in tables 1.3.3-4, here  $C_1 = C_1^\vee = B_1 = B_1^\vee := A_1$ ,  $B_2 := C_2$ ,  $B_2^\vee := C_2^\vee$ . For labels in [Tit79],  $B-C_2 := C-B_2$  and  ${}^2B-C_2 := {}^2C-B_2$ . xs is extraspecial, s is special but not xs,  $\text{xs} \implies \text{hyperspecial}$  if  $G$  is unramified.

Table 3.1: Vertices in Chamber for  $G = U_n^{E,H}(F_{\mathfrak{p}})$

### 3.3 Cartan Invariants

Now we define a key invariant related to a notion of “distance” between points in the building.

#### 3.3.1 Unramified Case

Assume first  $G$  is unramified and  $K_G$  is the stabilizer of the hyperspecial vertex  $v_0$  of  $\tilde{\mathcal{B}}(G)$ . Let  $A_G \cong (\mathbb{F}^\times)^r$  be a maximally split torus of  $G$  and let  $X_+(A_G)$  be a positive Weyl chamber in the cocharacter lattice  $X_*(A_G)$  and  $\Phi^*$  the corresponding set of simple roots. Then the following Cartan decomposition holds:

$$G = K_G \cdot X_+(A_G) \cdot K_G := \bigsqcup_{a \in X_+(A_G)} K_G a(\varpi_{\mathbb{F}}) K_G. \quad (3.1)$$

For any  $h \in G$ , define  $a_h \in X_+(A_G)$  to be the unique element such that  $h \in K_G a(h) K_G$ .

To resolve a technicality when  $G$  has non-anisotropic center, let

$$\bar{A}_G := A_G / Z_G^{\text{spl}} := A_G / (A_G \cap Z_G)^0,$$

where  $Z_G^{\text{spl}}$  is the maximal split torus in  $Z_G$ . Define  $\bar{a}_h$  to be the image of  $a_h$  in this quotient. There is a dominance ordering on  $X_+(\bar{A}_G)$  given by

$$a \preceq b \iff b - a \in X_+(\bar{A}_G).$$

Let  $\Sigma$  be the set of minimal non-zero elements of  $(X_+(\bar{A}_G), \preceq)$ . Then  $X_+(\bar{A}_G)$  is exactly the non-negative integer linear combinations of elements of  $\Sigma$ . In general,  $|\Sigma|$  might be larger

than the semisimple rank  $r_{ss}(G)$  (e.g.  $G = \mathrm{SL}_n$ ,  $n \geq 3$ ). However, when they are equal,  $X_+(\bar{A}_G) = \mathbb{Z}_{\geq 0}^\Sigma$  as semigroups. In this case, index a dual basis to  $\Sigma$  by elements  $\alpha' \in \Phi_{mod}^*$  that we will call modified simple roots. These are the same as the (non-multipliable) simple roots  $\alpha \in \Phi^*$  when  $Z_G \cap A_G$  is connected. Otherwise, they are scalings of the  $\alpha$ .

We can now make the key definition that will let us construct gate sets.

**Definition 3.3.1.** Define the Cartan norm on  $G$  to be:

$$\|\cdot\|_0 := \|\cdot\|_{G,0}: G \rightarrow \mathbb{N}_0 \quad , \quad \|h\|_{G,0} = \|a_h\|_{G,0} = \sum_{\alpha \in \Phi^*} n_\alpha |\alpha(a_h)|.$$

where  $n_\alpha$  is the coefficient of  $\alpha$  in the root  $\alpha_0$  added to produce the affine root system for  $G$  from its spherical root system at a special point (note that  $\|g\|_0 = \alpha_0(a_g)$  on the positive chamber).

In the case where  $X_+(\bar{A}_G) = \mathbb{Z}_{\geq 0}^\Sigma$ , define the modified Cartan norm to be:

$$\|\cdot\| := \|\cdot\|_G: G \rightarrow \mathbb{N}_0 \quad , \quad \|h\|_G = \|a_h\|_G = \sum_{\alpha' \in \Phi_{mod}^*} |\alpha'(a_h)|.$$

Given two points  $v_1, v_0 \in \mathcal{B}$ , we can also define norms of  $\|v_0 - v_1\|$ , where  $v_0 - v_1$  is always interpreted as an element of  $X_+(\bar{A}_G)$  in a common apartment. Note also that

$$\Sigma = \{\alpha \in X_+(\bar{A}_G) : \|\alpha\| = 1\}. \quad (3.2)$$

To justify the name norm:

**Lemma 3.3.2.** For all  $x, y \in G$ :

- (1) The Cartan norm satisfies  $\|xy\|_0 \leq \|x\|_0 + \|y\|_0$ ,
- (2) If  $G$  has all simple factors of type  $A, B$ , or  $C$ , then the modified Cartan norm satisfies  $\|xy\| \leq \|x\| + \|y\|$ .

*Proof.* By the  $W$ -metric space interpretation of the building, we have that in the positivity ordering  $\leq$  (as opposed to the dominance ordering  $\preceq$ ),  $a_{xy} \leq a_x + a_y$ . For (1), it then suffices to show that  $\alpha_0(a_x + a_y - a_{xy}) \geq 0$ , which follows since  $\alpha_0$  is dominant.

For (2), the result similarly follows if  $\sum_i \alpha'_i$  is dominant, which is true under the conditions on  $G$ .  $\square$

The Cartan norm  $\|a\|_0$  has a clean interpretation in terms of  $\mathcal{B}$ . Let  $\mathrm{dist}$  be the graph metric on the vertices in the 1-skeleton of  $\mathcal{B}$ .

**Lemma 3.3.3.** For any  $h \in H$ , its Cartan norm  $n = \|h\|_{G,0}$  satisfies the following:

$$\mathrm{dist}(h.v_0, v_0) = \|h\|_0.$$

*Proof.* Note that for any  $h \in G$  and any  $k_1, k_2 \in K_G$ , we get

$$\mathrm{dist}(k_1 h k_2.v_0, v_0) = \mathrm{dist}(h.(k_2.v_0), k_1^{-1}.v_0) = \mathrm{dist}(h.v_0, v_0).$$

Hence by the Cartan decomposition, it suffice to assume that  $h = a \in A_G$ . The split torus  $A_G$  acts by translation on the apartment of  $\mathcal{B}$  corresponding to it, which we assume to  $v_0$ . As a fact about buildings, one chamber of this apartment is the convex hull of 0 and the  $\lambda_i/n_i$  where  $\lambda_i$  are the (non-divisible) fundamental coweights corresponding to the  $\alpha_i$ .

Therefore,  $\mathrm{dist}(h.v_0, v_0)$  is the sum of the coordinates of  $h.v_0 - v_0$  in the basis of  $\lambda_i/n_i$ . This is exactly  $\sum_{1 \leq i \leq s} n_i |\alpha_i(h)|$ .  $\square$

Similarly, the modified Cartan norm is a weighted graph distance whenever it can be defined.

**Example 3.3.4.** Let  $G = PGL_n(\mathbb{F})$  or  $GL_N(\mathbb{F})$  and  $K_G = PGL_n(\mathcal{O}_{\mathbb{F}})$  or  $GL_N(\mathcal{O}_{\mathbb{F}})$ , respectively. Then  $\bar{A}_G$  is a quotient of the group of diagonal matrices  $\text{diag}(x_1, \dots, x_n)$  where  $x_i \in \mathbb{F}^\times$ .

The relative root system is the absolute root system which is type- $A_{n-1}$ , so

$$X_+(\bar{A}_G) = \{(m_1, \dots, m_n) \in \mathbb{Z}^n / \langle (1, \dots, 1) \rangle \mid m_1 \geq \dots \geq m_n\}.$$

The Cartan norm is the same as the modified Cartan norm and is defined on  $X_+(A_G)$  by

$$\|(m_1, \dots, m_n)\| = \|(m_1, \dots, m_n)\|_0 = \sum_{1 \leq i \leq n-1} (m_i - m_{i+1}) = m_1 - m_n.$$

The set  $\Sigma$  is the set of standard fundamental weights.

**Example 3.3.5.** Let  $G$  be quasisplit  $U_n(\mathbb{F})$  with respect to the unramified quadratic extension  $\mathbb{E}$  and standard antidiagonal Hermitian form  $\mathbf{J}$ . Let  $K_G = U_n(\mathcal{O}_{\mathbb{F}})$  be the hyperspecial. Then we can choose a maximal torus consisting of elements  $\text{diag}(x_1, \dots, x_n)$  with  $x \in \mathbb{E}^\times$  and  $x_i \overline{x_{n-i+1}} = 1$ . Inside this,  $A_G = \bar{A}_G$  is those elements with  $x_i \in \mathbb{F}^\times$ . Then,

$$X_*(A_G) = \{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_i = -m_{n-i+1}\}.$$

The spherical relative root system is the restriction of the roots of  $G_{\bar{\mathbb{F}}} \cong GL_N$  to  $A_G$  which is type- $C_{\lfloor n/2 \rfloor}$  if  $n$  is even and type- $BC_{\lfloor n/2 \rfloor}$  when  $n$  is odd. This extends to affine root systems  $C_{\lfloor n/2 \rfloor}$  and  $(BC_{\lfloor n/2 \rfloor}, C_{\lfloor n/2 \rfloor})$  respectively. Either way, the non-multipliable part is spherical  $C_{\lfloor n/2 \rfloor}$  extending to affine  $C_{\lfloor n/2 \rfloor}$  so

$$X_+(A_G) = \{(m_1, \dots, m_n) \in X_*(A_G) \mid m_i \in \mathbb{Z}, m_1 \geq \dots \geq m_{\lfloor n/2 \rfloor} \geq 0\}$$

and the Cartan norm is defined on  $X_+(A_H)$  by

$$\|(m_1, \dots, m_n)\|_0 = \left( \sum_{i=1}^{\lfloor n/2 \rfloor - 1} 2 \cdot (m_i - m_{i+1}) \right) + 1 \cdot (2m_{\lfloor n/2 \rfloor}) = 2m_1.$$

The set  $\Sigma$  is the all elements of the form  $(1, \dots, 1, 0, \dots, 0, -1, \dots, -1) \in X_*(A)$  which is of the correct size to define modified Cartan norm:

$$\|(m_1, \dots, m_n)\| = \left( \sum_{i=1}^{\lfloor n/2 \rfloor - 1} (m_i - m_{i+1}) \right) + m_{\lfloor n/2 \rfloor} = m_1.$$

Then  $\Sigma$  is exactly the elements of modified Cartan norm 1.

In this specific case, we get a clean replacement for lemma 3.3.3: the modified Cartan norm is exactly half the graph distance.

### 3.3.2 Ramified or Non-Quasisplit Cases

Now assume  $G$  is general and let  $A_G$  be a maximally split torus corresponding to apartment  $\mathcal{A} \subseteq \mathcal{B}(G)$  with dominant chamber  $\mathcal{A}_+$ . Then  $M = Z_G(A)$  is a minimal Levi subgroup. Let  $\bar{A}_G = A_G/Z_G^{\text{spl}}$  as before.

Define  $X_1(A_G)$  to be the subgroup of translations in the action of  $N_G(A)$  on the enlarged apartment  $\bar{\mathcal{A}} \simeq X_*(A_G) \otimes_{\mathbb{Z}} \mathbb{R}$  (this is the same as the  $\Lambda$  defined on [Car79, pg 140]). Note

that  $X_*(A_G) \subseteq X_1(A_G) \subseteq \tilde{\mathcal{A}}$  and is a free group of the same rank. There is also therefore an action of the Weyl group  $\Omega(G, A_G)$  on  $X_1(A_G)$  and  $\mathcal{A}_+$  determines a dominant subset  $X_{1+}(A_G) \subseteq X_1(A_G)$  and dominance ordering  $\preceq$ .

Let  $\tilde{K}$  be the stabilizer of special vertex  $x_0$  in  $\mathcal{A}$ . In many cases,  $\tilde{K}$  is a the special parahoric for  $x_0$  by lemma 3.2.2. Then [Car79, pg 140] gives a Cartan decomposition

$$G = \bigsqcup_{a \in X_{1+}(A_G)} \tilde{K} n_a \tilde{K},$$

where  $n_a$  is a choice of element of  $N_G(A_G)$  that acts as translation by  $a$  on  $\mathcal{A}$ .

To compute  $X_1(A_G)$ , we may use [HR10, Thm 1.0.1]: there is a Kottwitz map  $M \rightarrow X^*(Z(\widehat{M}))_I^{\text{Frob}}$ , where  $\star_I^{\text{Frob}}$  denotes the Frobenius invariants of the inertia coinvariants and  $\widehat{M}$  is the complex dual group of the algebraic group underlying  $M$ . This induces an isomorphism through  $n_\alpha$ :

$$X_1(A_G) \simeq X^*(Z(\widehat{M}))_I^{\text{Frob}} / \text{torsion}.$$

In other words, all  $n_\alpha$  can be chosen to be in  $M$  and the elements of  $M$  acting trivially on  $\mathcal{A}$  are the preimage of torsion under the Kottwitz map. We also have the translation group  $X_*(A_G) \subseteq X_1(A_G)$  embedded through  $X_*(A_G) = X^*(\widehat{A}_G) = X^*(\widehat{A}_G)_I^{\text{Frob}}$  and functoriality of the Kottwitz homomorphism.

Let  $X_1(\bar{A}_G)$  be the image of  $X_1(A_G)$  under the map  $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ . Given  $g \in G$ , we can define  $a_g$  to be the  $a \in X_{1+}(A)$  such that  $\tilde{K} g \tilde{K} = \tilde{K} n_{a_g} \tilde{K}$  and  $\bar{a}_g$  to be the image of  $a_g$  in  $X_1(\bar{A}_G)$ . Finally, as before, define  $\Sigma$  to be the set of minimal elements of  $(X_{1+}(\bar{A}_G), \preceq)$ . This allows us to define modified Cartan norms  $\|\cdot\|$  analogously to Definition 3.3.1.

To satisfy lemma 3.3.3, the Cartan norm  $\|\cdot\|_0$  needs to be defined differently since the chamber for ramified groups is smaller (see e.g [KP23, Def 6.3.4] describing the affine roots of  $G$ ). As before, let  $\alpha_0$  be the (non-multipliable) root added to the relative spherical root system of  $G$  to produce its affine root system. The root subgroup  $U_{\alpha_0}$  has a filtration with jumps at  $(1/e)\mathbb{Z} + y$  for some  $e \in \mathbb{Z}^+$  and  $y \in \mathbb{Q}$ .

**Definition 3.3.6.** Define the Cartan norm on possibly ramified  $G$  to be

$$\|\cdot\|_0 := \|\cdot\|_{G,0} : G \rightarrow \mathbb{N}_0 \quad , \quad \|h\|_{G_0} = \|a_h\|_{G,0} = e \sum_{\alpha \in \Phi^*} n_\alpha |\alpha(a_h)|,$$

similarly to definition 3.3.1.

Note that the analogue of the triangle inequality 3.3.2 holds in these ramified cases too.

**Example 3.3.7.** Let  $G = U_n(\mathbb{F})$  for  $n$  even and  $U_n$  the quasisplit unitary group with respect to ramified extension  $\mathbb{E}/\mathbb{F}$  and standard anti-diagonal Hermitian form  $\mathbf{J}$  with alternating signs.

Then, we may choose  $A_G$  as in example 3.3.5 and  $M$  is the diagonal maximal torus  $T$  containing it. We then normalize

$$X^*(Z(\widehat{M})) = X^*(\widehat{T}) = X_*(T) \cong \mathbb{Z}^n$$

using the isomorphism  $T \simeq (\text{Res}_{\mathbb{F}}^{\mathbb{E}} \mathbb{G}_m)^{n/2}$  by looking at the first  $n/2$  coordinates. The action of  $I$  is through a  $\mathbb{Z}/2$  that swaps coordinate  $2i+1$  with  $2i+2$  and that of Frob is trivial. Then

$$X_1(A_G) = X_*(T)_I = 1/2 X^*(T)^I = 1/2 X_*(A_G).$$

Therefore, fix coordinates

$$X_*(A_G) = \{(m_1, \dots, m_{n/2}) \in \mathbb{Z}^n\} \subseteq \{(m_1, \dots, m_{n/2}) \in 1/2\mathbb{Z}^n\} = X_1(A_G)$$

Next  $G$  has spherical relative root system  $C_{n/2}$  embedded in  $X_*(A_G)$  as in Example 3.3.5. However, this now extends to affine root system  $B_{n/2}^\vee$  and the added root has jumps in its root group filtration at half-integers. Therefore, the positive chambers are defined by

$$m_1 \geq \dots \geq m_{n/2} \geq 0$$

and the Cartan norm on the positive chamber can be given by

$$\begin{aligned} \|(m_1, \dots, m_n)\|_0 &= 2 \left[ 1 \cdot (m_1 - m_2) + \left( \sum_{i=2}^{n/2-1} 2 \cdot (m_i - m_{i+1}) \right) + 1 \cdot (2m_{n/2}) \right] \\ &= 2m_1 + 2m_2. \end{aligned}$$

On the other hand, since  $X_{1+}(A_G) = 1/2X_+(A_G)$  the modified Cartan norm is exactly twice that of Example 3.3.5:

$$\|(m_1, \dots, m_n)\| = 2m_1.$$

This can now only be interpreted as a weighted graph distance in  $\mathcal{B}$ .

**Example 3.3.8.** Now consider  $G = U'_n(\mathbb{F})$  for  $n$  even and  $U'_n$  the non-quasisplit unitary group with respect to an unramified  $\mathbb{E}/\mathbb{F}$  and Hermitian form

$$\mathbf{J}' = \begin{pmatrix} & & \mathbf{J} \\ & 1 & 0 \\ 0 & -\beta & \\ \mathbf{J} & & \end{pmatrix},$$

where  $J$  is the antidiagonal matrix with alternating signs and  $\beta$  is not a norm from  $\mathbb{E}^\times$  to  $\mathbb{F}^\times$ . Then  $A_G$  is elements  $\text{diag}(x_1, \dots, x_n)$  with  $x_i \in \mathbb{F}^\times$  such that  $x_{n/2}, x_{n/2+1} = 1$  and  $x_i = x_{n-i+1}^{-1}$  for  $1 \leq i \leq n/2 - 1$  making  $M \cong (\text{Res}_{\mathbb{F}}^{\mathbb{E}} \mathbb{G}_m)^{n/2-1} \times U'_2$ .

Denote the first factor by  $T$ . Then since  $\text{GL}_2$  has simply connected derived subgroup,

$$X^*(Z(\widehat{M})) = X^*(\widehat{T}) \times X^*(Z(\widehat{U'_2})) = X_*(T) \times \mathbb{Z} = \{(m_1, \dots, m_n, x) \in \mathbb{Z}^n\}$$

on which  $I$  acts trivially and Frob acts through a  $\mathbb{Z}/2$  that swaps coordinate  $2i + 1$  with  $2i + 2$  and takes  $x \mapsto -x$ . This computes

$$X_1(A_G) = X_*(T)^{\text{Frob}} \times 1 = X_*(A_G)$$

so fix coordinates

$$X_*(A_G) = \{(m_1, \dots, m_{n/2-1}) \in \mathbb{Z}^n\} = X_1(A_G).$$

Next,  $G$  has spherical root system  $BC_{n/2-1}$  embedded in  $X_*(A_G)$  by restricting the roots of  $G_{\overline{\mathbb{F}}} \cong \text{GL}_N$  to  $A_G$ . This extends to affine root system  $(C_{n/2-1}^\vee, C_{n/2-1})$ . Since the non-reduced piece is the spherical  $C_{n/2-1}$  extended to affine  $C_{n/2-1}$ , this produces the same dominant chambers

$$m_1 \geq \dots \geq m_{n/2-1} \geq 0,$$

Cartan norm

$$\|(m_1, \dots, m_{n/2-1})\|_0 = 2m_1,$$

and modified Cartan norm

$$\|(m_1, \dots, m_{n/2-1})\| = m_1$$

as the case of quasisplit  $U_{n/2-1}$  in example 3.3.5. In particular, the modified Cartan norm is half the graph distance.

**Example 3.3.9.** Consider the same  $U'_n(\mathbb{F})$  as Example 3.3.8 except now with  $\mathbb{E}/\mathbb{F}$  ramified. Then

$$X^*(Z(\widehat{M})) = \{(m_1, \dots, m_n, x) \in \mathbb{Z}^n\}$$

as in 3.3.8 except the roles of  $I$  and Frob in the action are reversed. This computes

$$X_1(A_G) = (X_*(T)_I \times \mathbb{Z}/2)_{\text{non-torsion}} = 1/2 X_*(A_G).$$

In particular, since there was torsion to remove,  $\widetilde{K}$  is no longer a special parahoric subgroup. We fix coordinates

$$X_*(A_G) = \{(m_1, \dots, m_{n/2-1}) \in \mathbb{Z}^n\} \subseteq \{(m_1, \dots, m_{n/2-1}) \in 1/2\mathbb{Z}^n\} = X_1(A_G).$$

Next,  $G$  still has spherical root system  $BC_{n/2-1}$  as in 3.3.8. However, this now extends to affine root system  $BC_{n/2-1}$  and the added root has jumps in its root group filtration at half-integers. Therefore, the positive chambers are defined by

$$m_1 \geq \dots \geq m_{n/2-1} \geq 0$$

and the Cartan norm on the positive chamber can be given by

$$\|(m_1, \dots, m_{n/2-1})\|_0 = 2 \cdot \left[ \left( \sum_{i=1}^{n/2-2} 2 \cdot (m_i - m_{i+1}) \right) + 1 \cdot (2m_{n/2-1}) \right] = 4m_1.$$

The modified Cartan norm can be computed exactly as in 3.3.7:

$$\|(m_1, \dots, m_{n/2-1})\| = 2m_1,$$

but this can now be interpreted simply as half the graph distance.

### 3.3.3 Non-Standard Cases

The analysis of super golden gates may require one more complication. Define  $X_1(A_G)$ ,  $X_{1+}(A_G)$ , and  $\Sigma$  as above. Assume that  $|\Sigma| = \text{rank}_{\text{ss}}(G)$  and also define the  $\alpha' \in \Phi_{\text{mod}}^*$  as above.

**Definition 3.3.10.** A subset  $\Sigma_0 \subseteq \Sigma$  is called *Weyl-complete* if:

- every element of  $\Sigma$  can be written as a sum of Weyl-conjugates of elements of  $\Sigma_0$ ,
- every element of  $-\Sigma_0$  is a Weyl-conjugate of an element of  $\Sigma_0$ .

**Definition 3.3.11.** Given Weyl-complete subset  $\Sigma_0 \subsetneq \Sigma$ , define the *non-standard Cartan norm*

$$\|\cdot\|_{\Sigma_0} : G \rightarrow \mathbb{N}_0 \quad , \quad \|h\|_{\Sigma_0} = \|a_h\|_{\Sigma_0} = \sum_{\alpha' \in \Phi_{\text{mod}}^*} w_{\Sigma_0}(\alpha') |\alpha'(a_h)|,$$

where if  $\alpha'$  is dual to  $\lambda \in \Sigma$ ,  $w_{\Sigma_0}(\alpha') := w_{\Sigma_0}(\lambda)$  is the minimum number of Weyl translates of elements of  $\Sigma_0$  needed to sum to  $\lambda$ .

If  $\Sigma_0 = \Sigma$ , we let  $\|\cdot\|_{\Sigma_0} := \|\cdot\|$  be the modified Cartan norm.



**Definition 3.3.12.** Call a Weyl-complete subset  $\Sigma_0 \subseteq \Sigma$  *navigable* if

$$\sum_{\alpha' \in \Phi_{mod}^*} w_{\Sigma_0}(\alpha') \alpha'$$

is dominant.

**Lemma 3.3.13.** *If  $\Sigma_0$  is navigable and Weyl-complete, then for all  $x, y \in G_{\mathfrak{p}}$ ,  $\|xy\|_{\Sigma_0} \leq \|x\|_{\Sigma_0} + \|y\|_{\Sigma_0}$ .*

*Proof.* This is the same argument as lemma 3.3.2.  $\square$

**Example 3.3.14.** Consider the example of quasiplit unramified unitary groups, recall the coordinates from Example 3.3.5 and fix  $n = 4$ . Then  $\Sigma_0 = \{(1, 0, 0, -1)\} \subseteq \Sigma$  is Weyl-complete and produces non-standard Cartan norm:

$$\|(m_1, \dots, m_4)\|_{\Sigma_0} = 1 \cdot (m_1 - m_2) + 2 \cdot m_2 = m_1 + m_2.$$

In particular,  $\Sigma_0$  is also navigable.

**Example 3.3.15.** Consider the example of quasiplit ramified unitary groups, recall the coordinates from Example 3.3.7, and fix  $n = 4$ . Then  $\Sigma_0 = \{(1/2, 0)\} \subseteq \Sigma$  is Weyl-complete and produces non-standard Cartan norm:

$$\|(m_1, m_2)\|_{\Sigma_0} = 1 \cdot 2(m_1 - m_2) + 2 \cdot 2m_2 = 2m_1 + 2m_2.$$

In particular,  $\Sigma_0$  is also navigable.

### 3.4 Discrete Actions and General Gate Sets

Our construction of gate sets will have to do with an analysis of discrete subgroups acting on  $\mathcal{B}$  whose action is transitive on certain special vertices.

Fix  $K \leq G$  a special maximal compact (hyperspecial in the unramified case) and a positive Weyl chamber. Let  $\Sigma$  be the set of minimal non-zero elements of  $(X_{1+}(\bar{A}_G), \preceq)$ . Assume that  $|\Sigma| = \text{rank}_{\text{ss}} G$  so we can define a modified Cartan norm  $\|\cdot\|$ .

Let  $\Lambda \leq G$  be a discrete subgroup. As some technicalities when  $Z_G$  isn't anisotropic, let  $\bar{\Lambda} = \Lambda / (\Lambda \cap Z_G^{\text{spl}})$  and note that  $\gamma \mapsto \bar{a}_\gamma$  is well-defined on  $\bar{\Lambda}$  and the action of  $\Lambda$  on  $\mathcal{B}$  factors through  $\bar{\Lambda}$ . Also, for each  $\alpha \in \Sigma$ , choose once and for all a lift  $\tilde{\alpha} \in X_1(A_G)$  and let  $\tilde{\Sigma}$  be the set of lifts. Since under our assumptions,  $X_{1+}(A_G) = \mathbb{Z}_{\geq 0}^\Sigma$ , we can extend this to well-defined map

$$X_{1+}(\bar{A}_G) \rightarrow X_{1+}(A_G) : \alpha \mapsto \tilde{\alpha}.$$

**Definition 3.4.1.** Let  $\Lambda \leq G$  be a discrete subgroup such that  $G = \Lambda \cdot K$ . Define the corresponding Gate set

$$S_\Lambda := \{\gamma \in \bar{\Lambda} : \bar{a}_\gamma \in \Sigma\}.$$

The corresponding lifted Gate set is

$$\tilde{S}_\Lambda = \{\gamma \in \Lambda : a_\gamma \in \tilde{\Sigma}\}.$$

In addition, define:

$$C_\Lambda = \bar{\Lambda} \cap K / (K \cap Z_G^{\text{spl}}), \quad \tilde{C}_\Lambda = \Lambda \cap K.$$

Finally, denote by  ${}^0S_\Lambda$  and  ${}^0\tilde{S}_\Lambda$  a choice of coset representatives for  $S_\Lambda/C_\Lambda$  and  $\tilde{S}_\Lambda/\tilde{C}_\Lambda$  respectively.

Note that quotienting by  $\Lambda \cap Z_G^{\text{spl}}$  gives a bijection between these choices of coset representatives. We denote the inverse as:

$${}^0S_\Lambda \xrightarrow{\sim} {}^0\tilde{S}_\Lambda : s \mapsto \tilde{s}.$$

This necessarily satisfies  $\tilde{\tilde{a}}_{s_i} = a_{\tilde{s}_i}$ .

**Proposition 3.4.2.** *In the notation of definition 3.4.1,*

- (1) *All  $\gamma \in \bar{\Lambda}$  are of the form  $s_1 \cdots s_k c$  for  $c \in C_\Lambda$  and  $s_i \in {}^0S_\Lambda$ .*
- (2) *Let  $S_\Lambda^{[\ell]} := {}^0S_\Lambda^{(\ell)} C_\Lambda$  be  $\gamma \in \bar{\Lambda}$  for which the minimum possible  $k$  in an expansion as above is exactly  $\ell$ . Then,*

$$S_\Lambda^{[\ell]} = (S_\Lambda \cup C_\Lambda)^{(\ell)} = \{\gamma \in \bar{\Lambda} : \|\bar{a}_\gamma\| = \ell\},$$

*where  $\|\bar{a}_\gamma\|$  is the modified Cartan norm.*

- (3) *Let  $\tilde{S}_\Lambda^{[\ell]} := {}^0\tilde{S}_\Lambda^{(\ell)} \tilde{C}_\Lambda = \{\tilde{s}_1 \cdots \tilde{s}_\ell \tilde{c} : s_1 \cdots s_\ell \in S_\Lambda^{[\ell]}, \tilde{c} \in \tilde{C}_\Lambda\}$ . Then similarly,*

$$\tilde{S}_\Lambda^{[\ell]} = (\tilde{S}_\Lambda \cup \tilde{C}_\Lambda)^{(\ell)} = \{\gamma \in \Lambda : a_\gamma = \tilde{\alpha} \text{ for some } \alpha \in X_{1+}(\bar{A}_G) \text{ with } \|\alpha\| = \ell\}.$$

*Proof.* Let  $\gamma \in \bar{\Lambda}$  and pick a common apartment  $\mathcal{A}$  of  $v_0$  and  $\gamma v_0$ . Choose an embedding  $X_1(\bar{A}_G) \hookrightarrow \mathcal{A}$  by setting  $v_0 = 0$  and orienting it in some way such that  $\gamma v_0$  corresponds to the point  $v_1 = \bar{a}_\gamma \in X_{1+}(\bar{A}_H)$ .

If  $v_1 \neq 0$ , then there is  $0 \preceq v'_1 \in X_*(\bar{A}_H)$  such that  $v_1 - v'_1 \in \Sigma$ . Since  $G = \Lambda \cdot K_H$ , there is  $\gamma' \in \bar{\Lambda}$  such that  $v_1 = \gamma'.v_0$ . Then,

$$\bar{a}_{\gamma(\gamma')^{-1}} = |\bar{a}_{(\gamma')^{-1}} - \bar{a}_{\gamma^{-1}}| = \bar{a}_\gamma - \bar{a}_{\gamma'} \in \Sigma,$$

where the absolute value denotes the Weyl conjugate in  $X_{1+}(\bar{A}_G)$ . In total, we have produced  $s \in S_\Lambda$  and  $\gamma' \in \bar{\Lambda}$  such that  $\gamma = s\gamma'$  and  $\bar{a}_{\gamma'} \prec \bar{a}_\gamma$ . We may without loss of generality choose  $s \in {}^0S_\Lambda$ .

Inductively reducing  $\bar{a}_{\gamma'}$  further, we can find  $s_1, \dots, s_k \in {}^0S_\Lambda$  and  $\gamma' \in \bar{\Lambda}$  such that  $\gamma = s_1 \cdots s_k \gamma'$  and  $\bar{a}_{\gamma'} = 0$ . Since  $\bar{a}_x = 0$  if and only if  $x \in K_H Z_G^{\text{spl}}$ , this implies that  $\gamma' \in C_\Lambda$  which proves (1).

For (2), the induction for (1) and equation (3.2) also gives that the claimed set is contained in  $S_\Lambda^{(\ell)}$ . Containment the other way follows from the triangle inequality 3.3.2 that  $\|\bar{a}_{xy}\| \leq \|\bar{a}_x\| + \|\bar{a}_y\|$ .

For (3), we repeat the argument for (1) and (2). Note that the inequality-squeezing for (2) forces that if  $\gamma = s_1 \cdots s_\ell \in S_\Lambda^{[\ell]}$ , then  $\bar{a}_\gamma = \bar{a}_{s_1} + \cdots + \bar{a}_{s_\ell}$ . Therefore,  $a_\gamma = a_{\tilde{s}_1} + \cdots + a_{\tilde{s}_\ell} = \tilde{\tilde{a}}_\gamma$ .  $\square$

**Remark 3.4.3.** It is more intuitive to think of

$$s_1 \cdots s_k c = c' s_1 \cdots s_k$$

for  $c'$  in the stabilizer of  $(s_1 \cdots s_k)\tau$ .

**Remark 3.4.4.** Given Weyl-complete  $\Sigma_0 \subseteq \Sigma$  as in 3.3.10, we may define analogous

$$S_{\Lambda, \Sigma_0} := \{\gamma \in \bar{\Lambda} : \bar{a}_\gamma \in \Sigma_0\},$$

etc. If  $\Sigma_0$  is furthermore navigable as in 3.3.12, then the natural variant of Proposition 3.4.2 then with analogous statements like:

$$S_{\Lambda, \Sigma_0}^{[\ell]} = (S_{\Lambda, \Sigma_0} \cup C_\Lambda)^{(\ell)} = \{\gamma \in \bar{\Lambda} : \|\bar{a}_\gamma\|_{\Sigma_0} = \ell\}$$

in terms of the non-standard Cartan norm from 3.3.11. Note that the argument for navigation needs the alternate triangle inequality 3.3.13.

### 3.4.1 Decimation

We can sometimes use one more trick to sparsify the gate sets. Define through the Kottwitz map

$$G_0 := \ker(G \rightarrow G_{\text{ad}} \rightarrow \pi_1(G_{\text{ad}})_I^{\text{Frob}}).$$

Then  $G_0$  is the elements of  $G$  that fix pointwise every chamber of  $\mathcal{B}(G)$  that they stabilize. This gives that for every point  $v_0 \in \mathcal{B}(G)$ ,  $G_0 v_0 = G^{\text{sc}} v_0$ , where  $G^{\text{sc}}$  is the simply connected cover of the derived subgroup  $G^{\text{der}}$ ; stated otherwise:

$$G_0 = \{g \in G : a_g \in X_{1+}(A_{G^{\text{sc}}}) \subseteq X_{1+}(A_G)\}$$

Therefore define  $\Sigma^{\text{sc}}$  and  $\|\cdot\|_{\Sigma_0^{\text{sc}}}$  analogously to before except with respect to  $X_{1+}(A_{G^{\text{sc}}})$ .

Next, note that if  $\Lambda K = G$  and we define  $\Lambda^{\text{sc}}, K_0$  by intersecting with  $G_0$ , we have that  $\Lambda^{\text{sc}} K_0 = G_0$ . By lemma 3.2.2, note also that  $K_0 = K$  in all cases when  $G_0 \neq G$ .

**Definition 3.4.5.** If  $\Lambda K = G$ , then define decimated gate sets analogously to Definition 3.4.1: i.e

$$S_{\Lambda^{\text{sc}}} = \{\gamma \in \bar{\Lambda}^{\text{sc}} : \bar{a}_\gamma \in \Sigma^{\text{sc}}\}, \text{ etc.}$$

Given Weyl-complete  $\Sigma_0^{\text{sc}} \subseteq \Sigma^{\text{sc}}$ , we may also define non-standard decimated gate sets  $S_{\Lambda^{\text{sc}}, \Sigma_0^{\text{sc}}}$  and in remark 3.4.4.

**Corollary 3.4.6.** For Weyl-complete and navigable  $\Sigma_0^{\text{sc}}$ , the natural variant of Proposition 3.4.2 holds for standard and non-standard decimated gate sets: i.e.

$$S_{\Lambda^{\text{sc}}, \Sigma_0^{\text{sc}}}^{[\ell]} = (S_{\Lambda^{\text{sc}}, \Sigma_0^{\text{sc}}} \cup C_{\Lambda^{\text{sc}}})^{(\ell)} = \{\gamma \in \bar{\Lambda}^{\text{sc}} : \|\bar{a}_\gamma\|_{\Sigma_0^{\text{sc}}} = \ell\}, \text{ etc.}$$

**Example 3.4.7.** In the even, quasisplit, ramified case of Example 3.3.7 (and the coordinates there),  $X_1(A_{G^{\text{sc}}})$  is the  $(m_1, \dots, m_{n/2}) \in 1/2\mathbb{Z}^n$  such that  $m_1$  is an integer. Then  $\Sigma^{\text{sc}}$  is  $(1, 0, \dots, 0)$  together with all elements of  $i$ -many  $1/2$ 's followed by  $(n/2 - i)$ -many 0's for  $1 \leq i \leq n/2$ . Then

$$\|(m_1, \dots, m_n)\|_{\Sigma^{\text{sc}}} = (m_1 - m_2) + \left( \sum_{i=2}^{n/2} 2(m_i - m_{i+1}) \right) + 2m_{n/2} = m_1 + m_2$$

so  $\Sigma^{\text{sc}}$  is navigable.

Fix  $n = 4$ , and also consider  $\Sigma_0^{\text{sc}} = (1/2, 1/2)$ . Then  $w_{\Sigma_0^{\text{sc}}}((1, 0)) = 2$ , giving non-standard modified Cartan norm

$$\|(m_1, m_2)\|_{\Sigma_0^{\text{sc}}} = 2m_1,$$

In particular,  $\Sigma_0^{\text{sc}}$  is navigable.

## 4 Golden Adelic Subgroups and Gate Sets

In this section we introduce the notions of golden and super-golden adelic groups, use them to construct gate sets of  $U(n)$ , and prove that these gate sets satisfy the first three properties of Definition 1.2.1: growth, navigation and approximation (Subsection 4.4).

We end this section by giving examples of golden and super-golden groups for  $n = 4$  (Subsection 4.5), noting that all previous constructions comes from such golden and super-golden groups ([Sar15b, PS18] for  $n = 2$  and [EP22] for  $n = 3$ ), and discussing practical considerations for their use in quantum computing (Subsection 4.6).

#### 4.1 Definition and First Properties

Let  $G = U_n^{E,H}$ ,  $SU_n^{E,H}$  or  $PU_n^{E,H}$ . Let  $K' = \prod_\ell K'_\ell \leq G(\widehat{\mathcal{O}})$  be a finite index subgroup.

**Definition 4.1.1.** The adelic subgroup,  $K' \leq G(\widehat{\mathcal{O}})$ , is said to be *golden* if

- (1)  $G(\mathbb{A}) = G(F) \cdot G_\infty K'$ .
- (2)  $G(F) \cap K' = \{1\}$

We say it is *golden at  $\mathfrak{p}$*  if  $K'_\mathfrak{p}$  is the stabilizer of a special vertex in the (enlarged) building  $\widetilde{\mathcal{B}}$  (in, this case,  $K'_\mathfrak{p} = K_\mathfrak{p}$  necessarily. By lemma 3.2.2,  $K'_\mathfrak{p}$  is also often special parahoric).

We say it is  *$\tau$ -super-golden* at  $\mathfrak{p}$  if  $K'_\mathfrak{p}$  is the stabilizer of a facet  $\tau \subseteq \widetilde{\mathcal{B}}$  of dimension  $> 0$ .

Finally, if only the weaker condition  $G(F) \cap K' \subseteq Z_G(F)$  holds instead of (2), we call  $K'$  *essentially golden*/ $\tau$ -*super-golden*. If (2) doesn't hold, we call  $K'$  *almost golden*/ $\tau$ -*super-golden*.

Beware that the first equation is difficult to satisfy—in particular, it requires  $G$  to have class number one. Remark 2.3.4 therefore makes golden adelic subgroups quite rare.

Given  $K' \leq G(\widehat{\mathcal{O}})$  and prime  $\mathfrak{p}$ , define

$$\Gamma := \Gamma^{K'} := G(F) \cap K' \quad \Lambda_\mathfrak{p} := \Lambda_\mathfrak{p}^{K'} := G(F) \cap (K')^\mathfrak{p}. \quad (4.1)$$

**Lemma 4.1.2.** *If  $K' \leq G(\widehat{\mathcal{O}})$  is an almost golden adelic group, then:*

- (1) *We have an isomorphism of  $G_\infty$ -sets*

$$\Gamma \backslash G_\infty \rightarrow G(F) \backslash G(\mathbb{A}) / K', \quad g \mapsto G(F)(g, 1, 1, \dots) K'$$

*inducing the following isomorphism of  $G_\infty$ -representations*

$$L^2(\Gamma \backslash G_\infty) \cong L^2(G(F) \backslash G(\mathbb{A}))^{K'}.$$

- (2) *For primes  $\mathfrak{p}$ , we have an isomorphism of  $G_\infty \times G_\mathfrak{p}$ -sets:*

$$\Lambda_\mathfrak{p} \backslash G_\infty \times G_\mathfrak{p} / K'_\mathfrak{p} \rightarrow G(F) \backslash G(\mathbb{A}) / K', \quad (g_\infty, g_\mathfrak{p}) \mapsto G(F)(g_\infty, g_\mathfrak{p}, 1, \dots) K'$$

*inducing the following isomorphism of  $G_\infty \times G_\mathfrak{p}$ -representations*

$$L^2(\Lambda_\mathfrak{p} \backslash G_\infty \times G_\mathfrak{p})^{K'_\mathfrak{p}} \cong L^2(G(F) \backslash G(\mathbb{A}))^{K'}.$$

*Proof.* For both claims, surjectivity follows since  $G(\mathbb{A}) = G(F)G_\infty K'$  and injectivity since  $G(F) \cap K' = \Gamma$  and  $G(F) \cap (K')^\mathfrak{p} = \Lambda_\mathfrak{p}$ .  $\square$

**Lemma 4.1.3.** *Let  $K' \leq G(\widehat{\mathcal{O}})$  be a almost golden adelic subgroup and let  $\mathfrak{p}$  be a prime. If  $K'$  is golden at  $\mathfrak{p}$  (resp.  $\tau$ -super-golden), then:*

- (1)  $\Lambda_\mathfrak{p}$  acts transitively on  $G_\mathfrak{p} / K'_\mathfrak{p}$  with stabilizer  $\Gamma$
- (2)  $\Lambda_\mathfrak{p} / (Z_{G_\mathfrak{p}}^{\text{spl}} \cap \Lambda_\mathfrak{p})$  acts transitively on  $G_\mathfrak{p} v_0$  (resp.  $G_\mathfrak{p} \tau$ ) with stabilizer  $\Gamma / (Z_{G_\mathfrak{p}}^{\text{spl}} \cap \Gamma)$ .
- (3)  $\Lambda_\mathfrak{p} \cap Z_{G_\mathfrak{p}}^{\text{spl}}$  acts simply transitively on the fibers of  $\Lambda_\mathfrak{p} \rightarrow \Lambda_\mathfrak{p} / (Z_{G_\mathfrak{p}}^{\text{spl}} \cap \Lambda_\mathfrak{p})$ .

*In particular, when  $K'$  is golden, the above actions are all simple transitive.*

*Proof.* Since  $K'_\mathfrak{p}$  is a stabilizer of  $v_0$  or  $\tau$ , respectively, (1) follows from the set equalities

$$G_\mathfrak{p} = \Lambda_\mathfrak{p} \cdot K'_\mathfrak{p} \quad \text{and} \quad \Lambda_\mathfrak{p} \cap K'_\mathfrak{p} = \Gamma,$$

which in turn follow from the assumption on the group  $K'$  (similarly to lemma 2.2.4). (2) follows since  $\tau$  has stabilizer exactly  $K'_\mathfrak{p} Z_{G_\mathfrak{p}}^{\text{spl}}$  and (3) is automatic.  $\square$

## 4.2 Golden Gate Sets

We now discuss how to construct gate sets from golden and super-golden adelic subgroups.

### 4.2.1 Golden Case

**Definition 4.2.1.** Let  $K'$  be an almost golden arithmetic subgroup of  $U_n^{E,H}$  that is golden at  $\mathfrak{p}$ . Then define the gate set and lifted gate set

$$S_{\mathfrak{p}} := S_{\mathfrak{p}}^{K'} := S_{\Lambda_{\mathfrak{p}}^{K'}}, \quad \tilde{S}_{\mathfrak{p}} := \tilde{S}_{\mathfrak{p}}^{K'} := \tilde{S}_{\Lambda_{\mathfrak{p}}^{K'}}$$

and analogous sets coset representatives  ${}^0S_{\mathfrak{p}}, {}^0\tilde{S}_{\mathfrak{p}}$  as in Definition 3.4.1. Note that when  $K$  is golden, we have  ${}^0S_{\Lambda} = S_{\Lambda}$  and  ${}^0\tilde{S}_{\Lambda} = \tilde{S}_{\Lambda}$ .

Proposition 3.4.2 immediately gives that

- ${}^0S_{\mathfrak{p}} \cup \Gamma$  generates  $\bar{\Lambda}_{\mathfrak{p}}$ ,
- $S_{\mathfrak{p}}^{[\ell]} = (S_{\mathfrak{p}} \cup \Gamma)^{(\ell)} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}} : \|a_{\gamma}\|'_{G_{\mathfrak{p}}} = \ell\}$ .

**Example 4.2.2.** If  $\mathfrak{p}_{\text{spl}}$  is split, then by Example 3.3.4,  $\|\cdot\|_{G_{\mathfrak{p}_{\text{spl}}}}$  is the graph distance on the 1-skeleton of  $\mathcal{B}$  so

$$S_{\mathfrak{p}_{\text{spl}}} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}_{\text{spl}}} : \text{dist}(\gamma.v_0, v_0) = 1\}.$$

If  $\mathfrak{p}_{\text{ns}}$  is non-split unramified and  $H_{\mathfrak{p}_{\text{ns}}}$  is also unramified, then by Example 3.3.5,  $\|\cdot\|_{G_{\mathfrak{p}_{\text{ns}}}}$  is half the graph distance on the 1-skeleton of  $\mathcal{B}$ . In addition,  $Z_{G_{\mathfrak{p}_{\text{ns}}}}^{\text{spl}} = 1$  so we can ignore center technicalities. Therefore,

$$S_{\mathfrak{p}_{\text{ns}}} = \tilde{S}_{\mathfrak{p}_{\text{ns}}} = \{\gamma \in \Lambda_{\mathfrak{p}_{\text{ns}}} : \text{dist}(\gamma.v_0, v_0) = 2\}.$$

### 4.2.2 Super-golden Case

In the  $\tau$ -super-golden case, we need to first restrict our choice of  $\tau$ :

**Definition 4.2.3.** Let  $\tau \subseteq \mathcal{B}$  contain a special vertex  $x_0$ . For each  $x \in Gx_0$ , define  $(G\tau)_x \subseteq G\tau$  to be the cosets containing  $g$  such that  $gx_0 = x$ .

Let  $\Sigma_{\tau} \subseteq \Sigma$  be  $\lambda$  such that for each (equiv. any)  $x \in Gx_0$  such that  $\|x - x_0\| = \lambda$ , there is  $\tau_{0,x} \in (G\tau)_{x_0}$ ,  $\tau_{1,x} \in (G\tau)_x$ , and  $r_x \in \mathbb{Z}_{>0}$  such that any automorphism  $\varphi$  of  $\mathcal{B}$  such that  $\varphi(\tau_{0,x}) = \tau_{1,x}$  satisfies  $\varphi^{r_x}(\tau_{0,x}) = \tau_{0,x}$ .

We say that  $\tau$  is *traversable* if  $\Sigma_{\tau}$  is Weyl-complete (as in 3.3.10) and navigable (as in 3.3.12). If  $\Sigma_{\tau} = \Sigma$ , we say that  $\tau$  is *standard traversable*. Otherwise, it is *non-standard traversable*.

Finally if  $\Sigma^{\text{sc}} \neq \Sigma$ , we define analogous notions of *decimated standard/non-standard traversable*.

In this case, there is a particularly nice structure we can give  $C_{\mathfrak{p}}$  and  ${}^0S_{\mathfrak{p}}$ .

**Definition 4.2.4.** Let  $K'$  be a golden arithmetic subgroup of  $U_n^{E,H}$  that is  $\tau$ -super-golden at  $\mathfrak{p}$  for  $\tau$  traversable. Then, following Definitions 3.4.1 and remark 3.4.4, set

$$C_{\mathfrak{p}} := C_{\mathfrak{p}}^{K'} := C_{\Lambda_{\mathfrak{p}}^{K'}}, \quad \tilde{C}_{\mathfrak{p}} := \tilde{C}_{\mathfrak{p}}^{K'} := \tilde{C}_{\Lambda_{\mathfrak{p}}^{K'}}.$$

and

$$S_{\mathfrak{p}} := S_{\mathfrak{p}}^{K'} := S_{\Lambda_{\mathfrak{p}}^{K'}, \Sigma_0}, \quad \tilde{S}_{\mathfrak{p}} := \tilde{S}_{\mathfrak{p}}^{K'} := \tilde{S}_{\Lambda_{\mathfrak{p}}^{K'}, \Sigma_0}.$$

Also set

$$\Gamma = C_{\mathfrak{p}} \cap K_{\mathfrak{p}} / (Z_{G_{\mathfrak{p}}} \cap K_{\mathfrak{p}}), \quad \tilde{\Gamma} = \tilde{C}_{\mathfrak{p}} \cap K_{\mathfrak{p}}.$$

In the decimated traversable case, make analogous definitions following 3.4.5.

Fix a set of representatives  $v_i$  of the orbits  $C_{\mathfrak{p}} \backslash \{v \in Gv_0 : \|v - v_0\|_{G_{\mathfrak{p}}} = 1\}$  and for each  $i$ , fix choices of  $\tau_{0,v_i}$  and  $\tau_{1,v_i}$  as in Definition 4.2.3. Since  $C_{\mathfrak{p}}$  acts transitively on  $(G\tau)_{v_0}$ , we may without loss of generality choose the  $v_i$  so that we may choose all  $\tau_{0,v_i} = \tau$ . Then define:

$$\begin{aligned} T_{\mathfrak{p}} &:= T_{\mathfrak{p}}^{K'} := \{\gamma \in S_{\mathfrak{p}} : \gamma\tau = \tau_{1,v_i} \text{ for some } i\} \\ \tilde{T}_{\mathfrak{p}} &:= \tilde{T}_{\mathfrak{p}}^{K'} := \{\gamma \in \tilde{S}_{\mathfrak{p}} : \gamma\tau = \tau_{1,v_i} \text{ for some } i\} \end{aligned}$$

Choose representatives  ${}^0T_{\mathfrak{p}}$  for the cosets  $T_{\mathfrak{p}}/\Gamma$  and  ${}^0\tilde{T}_{\mathfrak{p}}$  for  $\tilde{T}_{\mathfrak{p}}/\tilde{\Gamma}$  such that each  $\tilde{t}_i \in {}^0\tilde{T}_{\mathfrak{p}}$  lifts a  $t_i \in {}^0T_{\mathfrak{p}}$ .

In the golden case,  ${}^0T_{\mathfrak{p}} = T_{\mathfrak{p}}$  and  ${}^0\tilde{T}_{\mathfrak{p}} = \tilde{T}_{\mathfrak{p}}$ . As some quick properties:

**Lemma 4.2.5.** *The following hold*

- (1)  $C_{\mathfrak{p}}$  is a finite group.
- (2) For each  $v_i$ , there is exactly one  $t_i \in {}^0T_{\mathfrak{p}}$  with  $t_i\tau = \tau_{1,v_i}$  by lemma 4.1.3.
- (3) For each  $t_i \in T_{\mathfrak{p}}$ ,  $t_i^{r_{v_i}} \in \Gamma$ . In particular,  $t_i$  has finite order that divides  $r_{v_i}$  when  $K'$  is golden.
- (4)  $S_{\mathfrak{p}} = \{c_2tc_2^{-1}c_1 : c_1, c_2 \in C_{\mathfrak{p}}, t \in {}^0T_{\mathfrak{p}}\}$  (note here that  $c_2tc_2^{-1}c_1 = c'_1c_2tc_2^{-1}$  for  $c'_1$  in the stabilizer of  $c_2tc_2^{-1}v_0$ ). In particular,  $S_{\mathfrak{p}} \cup C_{\mathfrak{p}}$  is generated by finite-order elements.
- (5)  $\tilde{S}_{\mathfrak{p}} = \{c_2\tilde{t}c_2^{-1}c_1 : c_1, c_2 \in \tilde{C}_{\mathfrak{p}}, \tilde{t} \in {}^0\tilde{T}_{\mathfrak{p}}\}$ .

We can therefore make choices of coset representatives:

$${}^0S_{\mathfrak{p}} \subseteq \{ctc^{-1} : c \in C_{\mathfrak{p}}, t \in {}^0T_{\mathfrak{p}}\}, \quad {}^0\tilde{S}_{\mathfrak{p}} \subseteq \{c\tilde{t}c^{-1} : c \in C_{\mathfrak{p}}, \tilde{t} \in {}^0\tilde{T}_{\mathfrak{p}}\}.$$

Note that these are both exact equalities in the golden case. Either way, Proposition 3.4.2 and remark 3.4.4 then give that

- ${}^0S_{\mathfrak{p}} \cup C_{\mathfrak{p}}$  generates  $\bar{\Lambda}_{\mathfrak{p}}$ ,
- $S_{\mathfrak{p}}^{[\ell]} := {}^0S_{\mathfrak{p}}^{(\ell)} C_{\mathfrak{p}} = (S_{\mathfrak{p}} \cup \Gamma)^{(\ell)} = \{\gamma \in \bar{\Lambda}_{\mathfrak{p}} : \|a_{\gamma}\|'_{\Sigma_0} = \ell\}$ .

**Example 4.2.6.** Here are some traversable  $\tau$  and the corresponding  $C_{\mathfrak{p}}$  and  $T_{\mathfrak{p}}$ :

- (1)  $G = \mathrm{GL}_n$  and  $\tau \ni x_0$  a chamber in  $\mathcal{B}_G$ :  $\Sigma_{\tau} = \Sigma$ , the  $\tau_{0,x}$  range over  $\Omega_{x_0}$ -translates of  $\tau$ , and the  $\tau_{1,x}$  are rotations of  $\tau_{0,x}$ .

Then  $C_{\mathfrak{p}}$  acts transitively on the chambers containing  $x_0$ . The  $t_i$  are all of the form  $r^i$ , where  $r$  has order  $n$  and acts on  $\tau$  as an  $n$ -cycle on its vertices.

Note that the  $\mathrm{GL}_2$ -case is what was used in [PS18] where  $\mathcal{B}$  is a tree and there is a single  $t_i$  of order two that flips an edge.

- (2)  $G$  a quasisplit, unramified unitary group and  $\tau \ni x_0$  a chamber in  $\mathcal{B}_G$ :  $\Sigma_{\tau} = \Sigma$ , the  $\tau_{0,x}$  range over Weyl translates of  $\tau$ , and the  $\tau_{1,x}$  range over  $\Omega_y$ -translates of  $\tau_{0,x}$  where  $y$  is another vertex of  $\tau_{0,x}$ .

Then  $C_{\mathfrak{p}}$  acts transitively on the chambers containing  $x_0$ . There are many different  $t_i$  which are complicated to describe.

- (3)  $G$  a quasisplit, unramified  $U_4$  and  $\tau \ni x_0$  a two-edge path connecting  $x_0$  to a hyperspecial vertex through a non-special vertex:  $\Sigma_\tau$  is the  $\Sigma_0$  from Example 3.3.14, the  $\tau_{0,x}$  range over  $\Omega_{x_0}$ -translates of  $\tau$  and the  $\tau_{1,x}$  are reflections of  $\tau_{0,x}$  across the middle non-special vertex.

Then  $C_p$  acts transitively on the hyperspecial vertices at graph-distance two from  $x_0$  through a non-special vertex. There is one  $t_i$  whose square is in  $C_p$ .

- (4)  $G$  a quasisplit, ramified  $U_4$  and  $\tau \ni x_0$  an edge connecting  $x_0$  to a special vertex:  $\Sigma_\tau$  is the  $\Sigma_0$  from Example 3.3.15, the  $\tau_{0,x}$  range over  $\Omega_{x_0}$ -translates of  $\tau$  and the  $\tau_{1,x}$  are reflections of  $\tau_{0,x}$  across the middle.

Then  $C_p$  acts transitively on special vertices at graph-distance one from  $x_0$ . There is one  $t_i$  whose square is in  $C_p$ .

- (5)  $G$  a quasisplit, ramified  $U_4$  and  $\tau \ni x_0$  a two-edge path connecting  $x_0$  to a special vertex through a non-special vertex:  $\Sigma_\tau^{\text{sc}}$  is  $\{(1,0)\}$  as in Example 3.4.7, and the corresponding super-golden gate set behaves similarly to (3) except for being decimated.

- (6)  $G$  a non-quasisplit  $U_4$  and  $\tau$  the union of two edges connected at a vertex not in  $Gx_0$ :  $\Sigma_\tau = \Sigma$ , the  $\tau_{0,x}$  range over  $(G\tau)_{x_0}$ , and the  $\tau_{1,x}$  are reflections of the  $\tau_{0,x}$  across their middles.

Then  $C_p$  acts transitively on the distance-2 vertices from  $x_0$  and there is only one  $t_i$  whose square is in  $C_p$ .

### 4.3 Some Useful Computations

#### 4.3.1 Finding Gates

Fix  $K'$  an almost golden arithmetic subgroup of  $U_n^{E,H}$ . We explain some cases of how to find the gate sets  $S_p$  at each place  $p$  where  $K'$  is golden or  $\tau$ -super-golden for  $\tau$  traversable. The various sub-pieces  ${}^0S_p$ ,  $C_p$ ,  $T_p$ , etc. can be easily constructed from the full set  $S_p$ .

**Example 4.3.1.** Assume  $p|q$  is unramified and  $H$  is unramified. The  $G_p = U^{E_q, H_p}(F_p)$  and the gates  $S_p$  are exactly the  $g \in K' \cap U^{E,H}(F)$  such that  $\|a_g\|_p = 1$ . The  $a_g$  can be computed by the  $U^{E,H}(F_p)$ - or equivalently  $\text{GL}_n(E_q)$ -Cartan decomposition and  $\|a_g\|_p = 1$  is equivalent to largest entry of  $a_g$  being 1 as in Example 3.3.4.

By the standard algorithm for computing Cartan decompositions for  $\text{GL}_n$ ,  $\|a_g\|_p = 1$  is therefore equivalent to  $g \in \pi_q^{-1}M_n(\mathcal{O}_p) \setminus M_n(\mathcal{O}_p)$ , where  $\pi_q$  is a uniformizer for  $E_q$  (equiv.  $F_p$ ). Assume further that the scalar matrix  $\pi_q \in (K')^v$  (e.g, without loss of generality if  $E$  has class number one and  $K'$  is a principal congruence subgroup). Then scaling by  $\pi_q$  gives:

$$S_p = \{\pi_q^{-1}g : g^*Hg = \pi_q^2H, g \in M_4(\mathcal{O}_p) \times (K')^p, g \notin \pi_q K'\},$$

where without loss of generality  $\pi_p = \pi_q$  is a uniformizer for  $F_p$ .

**Example 4.3.2.** Assume  $p = p_1 p_2$  is split and  $K' = U^{E,H}(\mathcal{O}_p)$  is golden at  $v$ . Then  $G_p = \text{GL}_n(E_{p_1})$ . Fix lifts  $\tilde{\Sigma}$  to have smallest coordinate 0. Then the gates  $S_p$ —exactly the  $g \in K' \cap U^{E,H}(F)$  such that  $\|a_g\|_p = 1$ —are also exactly the  $g$  with the largest coordinate of  $a_g$  being 1 by Example 3.3.5.

As in Example 4.3.1,  $\|a_g\|_p = 1$  is therefore equivalent to  $g \in \pi_{p_1}^{-1}M_n(\mathcal{O}_p) \setminus M_n(\mathcal{O}_p)$ , where  $\pi_{p_1}$  is a uniformizer for  $E_{p_1}$ . If we similarly assume further that  $\pi_{p_1} \in (K')^p$ , then we get

$$S_p = \{\pi_{p_1}^{-1}g : g^*Hg = \pi_p H, g \in M_4(\mathcal{O}_p) \times (K')^p, g \notin \pi_{p_1} K'\},$$

where  $\pi_p = \pi_{p_1} \pi_{p_1}^*$  is a uniformizer for  $F_p$ .



**Example 4.3.3.** The other cases when  $\mathfrak{p}|\mathfrak{q}$  is non-split work similarly to Example 4.3.1 except inputting the computations of modified Cartan norms from Examples 3.3.7, 3.3.8, and 3.3.9. We get

$$S_{\mathfrak{p}} = \{\pi_{\mathfrak{q}}^{-1}g : g^*Hg = \pi_{\mathfrak{p}}^e H, g \in M_4(\mathcal{O}_{\mathfrak{p}}) \times (K')^{\mathfrak{p}}, g \notin \pi_{\mathfrak{q}}K'\},$$

where

$$e = \begin{cases} 2 & E_{\mathfrak{q}}/F_{\mathfrak{p}} \text{ unramified} \\ 1 & E_{\mathfrak{q}}/F_{\mathfrak{p}} \text{ ramified} \end{cases}.$$

**Example 4.3.4.** The non-standard gates of Examples 3.3.14 and 3.3.15 can be selected from the corresponding standard gates of Example 4.3.3 as those where the second coordinate of their Cartan invariant is 0.

**Example 4.3.5.** When  $n$  is even and  $\mathfrak{p}|\mathfrak{q}$  is ramified, the gates in the standard decimated case of Example 3.4.7 are the (proper) subset:

$$S_{\mathfrak{p}} \subsetneq \{\pi_{\mathfrak{q}}^{-2}g : g^*Hg = \pi_{\mathfrak{p}}^2 H, g \in M_4(\mathcal{O}_{\mathfrak{p}}) \times (K')^{\mathfrak{p}}, g \notin \pi_{\mathfrak{q}}K'\}$$

of the  $s$  with Cartan invariants having first two coordinates  $(1/2, 1/2, \dots)$  or  $(1, 0, \dots)$ .

If  $n = 4$ , in the non-standard decimated case we instead get (proper) subset

$$S_{\mathfrak{p}} \subsetneq \{\pi_{\mathfrak{q}}^{-1}g : g^*Hg = \pi_{\mathfrak{p}}^2 H, g \in M_4(\mathcal{O}_{\mathfrak{p}}) \times (K')^{\mathfrak{p}}, g \notin \pi_{\mathfrak{q}}K'\}$$

of the  $s$  with Cartan invariant  $(1/2, 1/2)$ .

### 4.3.2 Sizes and Growth Rates

To understand the sizes of gate sets  ${}^0S_{\mathfrak{p}}$  and  $S_{\mathfrak{p}}^{[\ell]}$ , we recall the following well-known fact about buildings:

**Proposition 4.3.6.** *The degree  $\deg(v_0)$  of a hyperspecial vertex in the 1-skeleton of the building  $\mathcal{B}$  for group  $H/\mathbb{F}$  is the number of maximal proper parabolics in  $H(k_{\mathbb{F}})$  for the corresponding integral model.*

**Proposition 4.3.7** ([Cas95, Prop. 1.5.2]). *Let  $v_0$  be a special vertex of  $\mathcal{B}$  and  $\lambda \in X_{1+}(\bar{A}_G)$ . Let  $B_{\lambda}$  be the number of points  $v \in \mathcal{B}$  such that  $v - v_0 = \lambda \in X_+(\bar{A}_G)$ . Then*

$$C_1 q_{\mathbb{F}}^{\langle \lambda, 2\rho_G \rangle} \leq B_{\lambda} \leq C_2 q_{\mathbb{F}}^{\langle \lambda, 2\rho_G \rangle}$$

for some constants  $C_1, C_2$  depending on  $H$  and where  $\rho_G$  is the half-sum of positive roots.

*Proof.* Let  $M$  be the centralizer of  $A_G$  and  $m \in M$  such that  $a_m = \lambda$ . Let  $\tilde{K}$  be the stabilizer of  $v_0$ . Then by the Cartan decomposition,  $B_{\lambda} = |\tilde{K}m\tilde{K}/\tilde{K}|$ .

Let  $P$  be a parabolic for  $M$  and  $N$  the corresponding unipotent with Lie algebra  $\mathfrak{b}$ . We now apply [Cas95, Prop. 1.5.2]:  $|\det \text{Ad}_{\mathfrak{n}} m| = \langle \lambda, 2\rho_G \rangle$  so the result follows.  $\square$

**Corollary 4.3.8.** *Let  $K' < G^{\infty}$  be almost golden at  $\mathfrak{p}$  (resp. almost  $\tau$ -super-golden for  $\tau$  standard traversable) and  $S_{\mathfrak{p}}$  the corresponding gate set. Let  $M = \max_{\lambda \in \Sigma} \langle \lambda, 2\rho_{G_{\mathfrak{p}}} \rangle$  and  $k$  the number of  $\lambda \in \Sigma$  realizing this maximum. Then  $|S_{\mathfrak{p}}^{[\ell]}| \asymp \ell^{k-1} q_{\mathfrak{p}}^{M\ell}$ .*

*Proof.* By lemma 4.1.3 and the versions of Proposition 3.4.2 as used above,

$$|S_{\mathfrak{p}}^{[\ell]}| = |C_{\mathfrak{p}}| |\{v \in G_{\mathfrak{p}} v_0 \subseteq \mathcal{B} : \|v - v_0\| = \ell\}|$$

By Proposition 4.3.7, if we let  $\lambda_0$  realize the maximum value of  $\langle \lambda, 2\rho_G \rangle$ , this gives

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp \sum_{\substack{\lambda \in X_+(\bar{A}_{G_{\mathfrak{p}}}) \\ \|\lambda\| = \ell}} q_{\mathfrak{p}}^{\langle \lambda, 2\rho_G \rangle} \asymp \binom{\ell + k - 1}{k - 1} q_{\mathfrak{p}}^{\langle \ell \lambda_0, 2\rho_G \rangle},$$

where the last line is by counting how many terms in the sum are equal to  $q_{\mathfrak{p}}^{\langle \ell \lambda_0, 2\rho_G \rangle}$  and noting that the number of terms equal to  $q_{\mathfrak{p}}^{\langle \ell \lambda_0, 2\rho_G \rangle - c}$  is larger by a factor that is most polynomial in  $c$  and that  $\sum_{c=0}^{\infty} c^n q_{\mathfrak{p}}^{-c}$  converges for all  $n$ .  $\square$

**Corollary 4.3.9.** *Let  $K' < G^{\infty}$  be almost  $\tau$ -super-golden at  $\mathfrak{p}$  for  $\tau$  non-standard traversable. Let  $M = \max_{\lambda \in \Sigma} w_{\Sigma_{\tau}}(\lambda)^{-1} \langle \lambda, 2\rho_{G_{\mathfrak{p}}} \rangle$  recalling the definition of  $w_{\Sigma_{\tau}}$  from 3.3.11. Let  $k$  be the number of  $\lambda \in \Sigma$  realizing this maximum. Then  $|S_{\mathfrak{p}}^{[\ell]}| \asymp \ell^{k-1} q_{\mathfrak{p}}^{M\ell}$ .*

*Proof.* This is the same argument as lemma 4.3.8.  $\square$

**Example 4.3.10.** If  $\mathfrak{p}$  is split then  $G_{\mathfrak{p}} = GL_n/F_{\mathfrak{p}}$ . If  $K'$  is almost golden at  $\mathfrak{p}$ , then  $|S_{\mathfrak{p}}| = \deg(v_0)$ . This is the sum of the sizes of the Grassmanians  $G(n, k)$  for  $1 \leq k \leq n-1$  over the residue field of  $F_{\mathfrak{p}}$  giving

$$|S_{\mathfrak{p}}| = \sum_{i=1}^{n-1} \binom{n}{i}_{q_{\mathfrak{p}}} := \sum_{i=1}^{n-1} \frac{(1 - q_{\mathfrak{p}}^n) \cdots (1 - q_{\mathfrak{p}}^{n-k+1})}{(1 - q_{\mathfrak{p}}) \cdots (1 - q_{\mathfrak{p}}^k)} = (1 + \mathbf{1}_{n \text{ odd}}) q_{\mathfrak{p}}^{\lfloor n^2/4 \rfloor} + O(q_{\mathfrak{p}}^{\lfloor n^2/4 \rfloor - 1}).$$

In comparison,  $2\rho_G = (n-1, n-3, \dots, -n+1)$ , which has maximized pairing with the middle fundamental weight in  $\Sigma$  (there are 2 if  $n$  is odd). This gives

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp \ell^{\mathbf{1}_{n \text{ odd}}} q_{\mathfrak{p}}^{\lfloor n^2/4 \rfloor \ell}$$

in both the golden and super-golden cases.

**Example 4.3.11.** If  $\mathfrak{p}$  is non-split, then  $G_{\mathfrak{p}}$  is a unitary group. Coordinatize

$$X_*(A_G) = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : m_i = -m_{n+1-i}\}$$

with  $m_{n/2} = m_{n/2+1} = 0$  if  $G_{\mathfrak{p}}$  is non-quasisplit. We still have  $2\rho_G = (n-1, n-3, \dots, -n+1)$ . Then, inspecting Examples 3.3.5-3.3.9, the maximized pairing is with the  $(m_i) \in \Sigma$  with  $m_i = (1 + \mathbf{1}_{\mathfrak{p} \text{ ramified}})^{-1}$  for  $1 \leq i \leq \lfloor n/2 \rfloor - \mathbf{1}_{G_{\mathfrak{p}} \text{ n.q.s.}}$ . This gives

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp q_{\mathfrak{p}}^{A\ell}, \quad A = \begin{cases} 2\lfloor n^2/4 \rfloor & \mathfrak{p} \text{ unramified, } G_{\mathfrak{p}} \text{ quasisplit} \\ \lfloor n^2/4 \rfloor & \mathfrak{p} \text{ ramified, } G_{\mathfrak{p}} \text{ quasisplit} \\ n^2/2 - 2 & \mathfrak{p} \text{ unramified, } G_{\mathfrak{p}} \text{ non-quasisplit} \\ n^2/4 - 1 & \mathfrak{p} \text{ ramified, } G_{\mathfrak{p}} \text{ non-quasisplit} \end{cases}.$$

**Example 4.3.12.** In the non-standard  $\tau$  for  $U_4$  case of Example 3.3.14/4.2.6(3),  $2\rho_G = (3, 1, -1, 3)$  and the pairing in 4.3.9 is maximized for  $\lambda = (1, 0, 0, -1)$ . This gives

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp q_{\mathfrak{p}}^{6\ell}.$$

In the similar non-standard case of Example 3.3.15/4.2.6(4), we similarly get

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp q_{\mathfrak{p}}^{3\ell}.$$

Finally, in the decimated non-standard case of Example 3.4.7/4.2.6(5), the maximizing  $\lambda$  is  $(1/2, 1/2, -1/2, -1/2)$ , giving

$$|S_{\mathfrak{p}}^{[\ell]}| \asymp q_{\mathfrak{p}}^{4\ell}.$$

#### 4.4 Growth, Navigation, and Approximation

Next we summarize how the gate sets that correspond to golden adelic groups satisfy the last three properties in the Definition 1.2.1 of golden gate sets; namely, growth, navigation and approximation.

**Theorem 4.4.1.** *Let  $G = U_n^{E,H}$ , let  $K' \leq G(\widehat{\mathcal{O}})$  be an almost golden adelic subgroup, let  $\mathfrak{p}$  be a prime at which  $K'$  is golden (resp.  $\tau$ -super-golden for  $\tau$  traversable), and let  $S_{\mathfrak{p}}$  (resp.  ${}^0S_{\mathfrak{p}} \cup C_{\mathfrak{p}}$  which is generated by elements of finite order by 4.2.5(4)) be the gate set corresponding to  $K'$  as in definition 4.2.1 (resp. 4.2.4). Then  $S_{\mathfrak{p}}$  satisfies the following properties:*

- (1) Growth:  $|S_{\mathfrak{p}}^{[\ell]}|$  grows exponentially in  $\ell$ .
- (2) Navigation: The group generated by  $S_{\mathfrak{p}}$  is  $\bar{\Lambda}_{\mathfrak{p}} := \bar{\Lambda}_{\mathfrak{p}}^{K'}$  and it has the following efficient solution for its word problem: Given  $1 \neq g \in \bar{\Lambda}_{\mathfrak{p}}$ , find an element  $s \in {}^0S_{\mathfrak{p}}$  such that  $a_{sg} \prec a_g$ , and proceed by induction on  $a_g$ . The algorithm will terminate in  $O(|{}^0S_{\mathfrak{p}}| \cdot \|a_g\|)$  time which (for a fixed  $\mathfrak{p}$ ) is polynomial in the input  $g$ .
- (3) Approximation: If  $\mathcal{O}_F$  is Euclidean, there exists a heuristic polynomial algorithm such that given  $g \in PU(n)$ ,  $\varepsilon > 0$ , and  $\ell \in \mathbb{N}$ , if  $B(g, \varepsilon) \cap S^{[\ell]} \neq \emptyset$ , then the algorithm outputs an element in  $B(g, \varepsilon) \cap S^{[\ell, N]}$ , where  $N = \dim PU(n)$ .

*Proof.* In order:

Growth:

This follows from Corollary 4.3.8.

Navigation:

Both termination and run time follow by the proof of 3.4.2(1). By uniqueness of the Cartan decomposition 3.1, we can compute  $a_g$  at each step by the integer normal form algorithm on  $G_{\mathfrak{p}} = GL_n(F_{\mathfrak{p}})$  when  $v$  is split or on the bigger group  $GL_n(E_{\mathfrak{q}}) \supset G_{\mathfrak{p}}$  when  $w$  lies over  $v$  and  $v$  is non-split.

Approximation:

This follows as a consequence (Corollary 4.4.3) of the algorithm of Ross and Selinger [RS15, PS18] for approximating elements in  $U(2)$  by matrices with integer coefficients. We sketch this in the subsequent part of this section.

To apply those results, Proposition 3.4.2(2) and the work of §4.3.1 find  $i$  so that  $S_{\mathfrak{p}}^{[k]} \subseteq U_n^{E/H}(F) \cap \varpi_{\mathfrak{p}}^{-i} M_n(\mathcal{O}_E \otimes \mathcal{O}_{F_{\mathfrak{p}}})$  if and only if  $k \leq \ell$ . Furthermore, by Gram-Schmidt, we can find a diagonal  $H'$  such that  $U_n^{E,H}/F \cong U_n^{E,H'}/F$  so we can find  $r \in \mathbb{N}$  with only ramified factors such that  $U_n^{E/H}(\mathcal{O}_F) \subseteq r^{-1} M_n(\mathcal{O}_E) \cap U^{H'}(n)$ . These two together provide the inputs  $m, H'$  in the lemmas.

Since  $S_{\mathfrak{p}}^{[\ell]}$  is then a finite fraction of  $\varpi_{\mathfrak{p}}^{-i} r^{-1} M_n(\mathcal{O}_E) \cap U^{H'}(n)$ , sampling enough outputs will heuristically generate one in  $S_{\mathfrak{p}}^{[\ell]}$  with high probability.  $\square$

**Theorem 4.4.2** ([RS15], [PS18, Thm 2.6, §2.3]). *Let  $K$  be a totally real number field such that its ring of integers  $\mathcal{O}_K$  is Euclidean and let  $d \in \mathbb{N}$ . Let  $H'$  be a diagonal definite Hermitian form for  $K[\sqrt{-d}]/K$ . Then there is a randomized, heuristic efficient algorithm, such that given  $\varepsilon > 0$ ,  $k, m \in \mathbb{N}$ , and a diagonal unitary matrix*

$$g = \begin{pmatrix} x_1 + ix_2 & \\ & x_1 - ix_2 \end{pmatrix} \in U^{H'}(2),$$

it finds up to  $k$  different

$$h \in M_2(\mathcal{O}_K[\sqrt{-d}]) \quad : \quad \tilde{h} = m^{\frac{-1}{2}} \cdot h \in U^{H'}(2) \quad , \quad 1 - \frac{\text{Trace}(g^* \cdot \tilde{h})}{2} < \varepsilon.$$

if that many exist and all such  $h$  otherwise.

*Proof.* The original statement of the Theorem 2.6 in [PS18] is with  $d = 1$  and  $H = I$ , but their arguments works just as well for any fixed  $d$  by modifying the lattices studied and any fixed diagonal  $H'$  by the discussion after (3.13) therein.  $\square$

The above algorithm can be generalized to arbitrary elements of higher dimensional unitary groups:

**Corollary 4.4.3.**  *$K$  be a totally real number field such that its ring of integers  $\mathcal{O}_K$  is Euclidean and let  $d \in \mathbb{N}$ . Let  $H'$  be a diagonal definite Hermitian form for  $K[\sqrt{-d}]/K$ . Then, there is a heuristic efficient algorithm such that given  $\varepsilon > 0$ ,  $k, m \in \mathbb{N}$ , and a unitary matrix,  $g \in U(n)$ , it finds  $k$  different tuples of matrices*

$$h_1, \dots, h_N \in M_n(\mathbb{Z}[\sqrt{-d}]) \quad : \quad \tilde{h}_i = m^{\frac{-1}{2}} \cdot h_i \in U(n) \quad , \quad 1 - \frac{|\text{Trace}(g^* \cdot (\tilde{h}_1 \cdots \tilde{h}_N))|}{n} < \varepsilon$$

if that many exist and all such tuples otherwise.

*Proof.* The claim follows from the fact that for each,  $g \in U(n)$ , there exists  $N$  unitary matrices,  $g_1, \dots, g_N$  which are 1-parameter elements (namely,  $g_i \in H_i \leq U(n)$ , where  $H_i \cong U(1)$ ), such that  $g = g_1 \cdots g_N$  (see also [NC11] section 4.5.1, for a similar though longer decomposition and the discussion after (3.13) in [PS18]). Using the fact that

$$d : U(n) \times U(n) \rightarrow \mathbb{R}_{\geq 0} \quad , \quad d(g, h) = 1 - \frac{|\text{Trace}(g^* \cdot h)|}{n},$$

is a bi-invariant metric on  $U(n)$ , we get that it is enough to approximate each  $g_i$  individually, which is done by Theorem 4.4.2.  $\square$

## 4.5 Explicit Constructions of Golden Adelic Groups

Let us present several examples of golden and super-golden adelic groups.

### 4.5.1 Previous Constructions

**Example 4.5.1.** In dimension  $n = 2$ , several constructions were given in [PS18, Sar15b]. Let us present just one example of a golden adelic group which is super-golden at  $p = 3$ :

$$G = U_2^{\mathbb{Q}[\sqrt{-1}], I} / \mathbb{Q}, \quad K' = \left\{ g \in G \left( \widehat{\mathbb{Z}} \right) \mid g \equiv I \pmod{2} \right\}.$$

**Example 4.5.2.** In dimension  $n = 3$ , several constructions were given in [EP22, BEMP23]. Here are two such examples of golden adelic groups:

$$G = U_3^{\mathbb{Q}[\sqrt{-1}], I} / \mathbb{Q}, \quad K' = \left\{ g \in G \left( \widehat{\mathbb{Z}} \right) \mid \forall i, g_{i,i} \equiv 1 \pmod{2 + 2i} \right\},$$

$$G = U_3^{\mathbb{Q}[\sqrt{-3}], I} / \mathbb{Q}, \quad K' = \left\{ g \in G \left( \widehat{\mathbb{Z}} \right) \mid \forall i, g_{i,i} \equiv 1 \pmod{3} \right\}.$$

Furthermore, as a consequence of the work of Mumford [Mum79] (see [BEMP23] for the details), one get that for the  $3 \times 3$  Hermitian positive definite matrix

$$H = \begin{pmatrix} 3 & \lambda & \lambda \\ \bar{\lambda} & 3 & \lambda \\ \bar{\lambda} & \bar{\lambda} & 3 \end{pmatrix} \quad \text{where} \quad \lambda = \frac{1 + \sqrt{-7}}{2},$$

the following is a golden adelic group which is super-golden at  $p = 2$ :

$$G = U_3^{\mathbb{Q}[\sqrt{-7}], H} / \mathbb{Q}, \quad K' = \left\{ g \in G(\widehat{\mathbb{Z}}) \mid \forall i > j, g_{i,j} \equiv 0 \pmod{2} \right\}.$$

#### 4.5.2 New Constructions: $\mathbb{Q}(\sqrt{-3})$

Below we present our constructions of golden and super-golden groups. We start with the case of  $E/F = \mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ .

**Proposition 4.5.3.** *For the  $4 \times 4$  Hermitian positive definite matrix,*

$$H_3 = 2 \cdot \begin{pmatrix} I & A \\ -A & I \end{pmatrix} \quad \text{where} \quad A = \frac{\sqrt{-3}}{3} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

define

$$G := U_4^{\mathbb{Q}[\sqrt{-3}], H_3} / \mathbb{Q}.$$

Then the following are essentially golden adelic groups:

$$K_1^{(2)} := \{g \in G(\widehat{\mathbb{Z}}) \mid g \equiv I \pmod{2}\}, \quad K_1^{(\sqrt{3})} := \{g \in G(\widehat{\mathbb{Z}}) \mid g \equiv I \pmod{\sqrt{3}}\}$$

*Proof.* We note that  $H_3$  was chosen so that  $U^{E, H_3}$  is quasisplit at all places and  $U^{E, H_3}(\mathbb{Z}_p)$  is special at  $p = 3$  and hyperspecial at all other  $p$ , e.g, as in Remark 3.2.3. In particular  $\text{Ram}(H) = \emptyset$  so all the  $\lambda_\ell$  in 2.3.2 are 1. Plugging in the  $L$ -values:

$$R(G) = 155520^{-1}.$$

We may also compute that  $|G(\mathbb{Z})| = 155520$ . One method that runs in a few minutes on a personal laptop is to first minimally scale  $H$  to have integral entries, note that the diagonal entries are then all 3, find all<sup>(†)</sup>  $v \in \mathcal{O}_E^4$  of norm 3, and finally build up all possible  $\mathcal{O}_E$ -matrices preserving  $H$  through choosing the rows one-by-one from this set of  $v$ . Therefore, by lemma 2.3.3,  $G$  has class number one.

Since  $K_1^{(2)}$  is the kernel of the reduction map  $r_2 : G(\widehat{\mathbb{Z}}) \rightarrow G(\mathbb{Z}/2)$ , it therefore suffices to show that  $r_2|_{G(\mathbb{Z})}$  is surjective with kernel contained in  $Z_G(\widehat{\mathbb{Z}})$ . The prime 2 is unramified in  $G$ , hence  $G(\mathbb{Z}/2\mathbb{Z}) \cong U_4(\mathbb{F}_2)$  and by a standard formula,  $|U_4(\mathbb{F}_2)| = 77760 = |G(\mathbb{Z})|/2$ . Another computer check finally shows that  $G(\mathbb{Z}) \cap \ker r_2 = \{\pm 1\}$ .

For  $K_1^{(\sqrt{3})}$ , since  $G(\widehat{\mathbb{Z}})$  is special at 3,  $G(\mathbb{Z}/3\mathbb{Z}) \cong \text{Sp}_4(\mathbb{F}_3[x]/x^2)$  with reductive quotient  $\text{Sp}_4(\mathbb{F}_3)$ . Furthermore, this composite map  $G(\widehat{\mathbb{Z}}) \rightarrow \text{Sp}_4(\mathbb{F}_3)$  can be described exactly by reducing matrix entries mod  $\sqrt{3}$ . Therefore, by a standard formula,  $[K : K_1^{(\sqrt{3})}] = |\text{Sp}_4(\mathbb{F}_3)| = 51840 = |G(\widehat{\mathbb{Z}})|/3$ . A computer check then gives that  $G(\mathbb{Z}) \cap K_1^{(\sqrt{3})} = \{1, \zeta_3, \zeta_3^2\}$ , which has 3 elements.  $\square$

<sup>(†)</sup>there are 240 of these corresponding to the roots of an interpretation of  $\mathcal{O}_E^4$  as the  $E_8$  lattice

This example gives gate sets  $S_p$  for all  $p$  with growth rates:

$$|S_p^{[\ell]}| \asymp \begin{cases} p^{4\ell} & p \equiv 1 \pmod{3} \\ p^{8\ell} & p \equiv 2 \pmod{3}, \\ 81^\ell & p = 3 \end{cases}.$$

It also can produce super-golden examples:

**Proposition 4.5.4.** *Consider  $G = U_4^{\mathbb{Q}[\sqrt{-3}], H_3} / \mathbb{Q}$  as in Proposition 4.5.3. Choose  $X_2 \in \mathrm{GL}_4(\mathbb{Z}_2[\sqrt{-3}])$  such that*

$$X_2^* H_3 X_2 \equiv \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \pmod{4},$$

for example,

$$X_2 = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2} + \frac{i}{2\sqrt{3}} & \frac{1}{2} - \frac{i}{2\sqrt{3}} & \frac{2i}{\sqrt{3}} \\ -\frac{1}{2} - \frac{i}{2\sqrt{3}} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{2i}{\sqrt{3}} & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \\ 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \\ \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 & 0 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{pmatrix}.$$

Define

$$K_s^{(2)} := \left\{ g \in G(\widehat{\mathbb{Z}}) \left| X_2^{-1} g X_2 \in \begin{pmatrix} \mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} \\ 2\mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} \\ 2\mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} & \mathcal{O}_{E_2} \\ 4\mathcal{O}_{E_2} & 2\mathcal{O}_{E_2} & 2\mathcal{O}_{E_2} & \mathcal{O}_{E_2} \end{pmatrix} \right. \right\}.$$

Then  $K_s^{(2)}$  is almost  $\tau$ -super golden for (non-standard traversable)  $\tau$  as in Example 4.2.6(3).

*Proof.* Note that  $X_2^{-1} G(\mathbb{Q}_2) X_2$  then has a chamber corresponding to lattices  $\mathcal{O}_{E_2}^{4-i} \oplus 2\mathcal{O}_{E_2}^i$  for  $0 \leq i \leq 3$ . The non-special vertex corresponds to  $i = 1, 3$  which are dual to each other under  $X_2^* H_3 X_2$ . Therefore lattices  $\mathcal{O}_{E_2}^4$  and  $\mathcal{O}_{E_2} \oplus 2\mathcal{O}_E^2 \oplus 4\mathcal{O}_2$  are hyperspecial connected through a two-step path through a non-special vertex and  $K^{(2),s}$  exactly stabilizes both.

We can also compute that  $[K : K_s^{(2)}] = [K_2 : K_s^{(2)}] = 90 = G(\mathbb{Z})/1728$  (this is counting the choices of  $\tau$  starting at the hyperspecial vertex  $K$ : there are 45 choices for the first edge and 2 per each of these for the second). Therefore, since  $K$  is almost golden,  $K_s^{(2)}$  is almost golden if and only if  $|G(\mathbb{Z}) \cap K_s^{(2)}| = 1728$ , which holds.  $\square$

**Proposition 4.5.5.** *Consider  $G = U_4^{\mathbb{Q}[\sqrt{-3}], H_3} / \mathbb{Q}$  as in Proposition 4.5.3. Choose  $X_3 \in \mathrm{GL}_4(\mathbb{Z}_3[\sqrt{-3}])$  such that for some  $C \in (\mathbb{Z}/3)[\sqrt{-3}]$ ,*

$$X_3^* H_3 X_3 \equiv C \begin{pmatrix} & & & \sqrt{-3} \\ & & \sqrt{-3} & \\ & -\sqrt{-3} & & \\ -\sqrt{-3} & & & \end{pmatrix} \pmod{3},$$

for example,

$$X_3 = \begin{pmatrix} 4 & -3 & i\sqrt{3} & -2i\sqrt{3} \\ 3 & 4 & -2i\sqrt{3} & -i\sqrt{3} \\ -4i\sqrt{3} & -i\sqrt{3} & -1 & -4 \\ -i\sqrt{3} & 4i\sqrt{3} & 4 & -1 \end{pmatrix}.$$

Define

$$K_s^{(3)} := \left\{ g \in G(\widehat{\mathbb{Z}}) \left| X_3^{-1} g X_3 \in \begin{pmatrix} \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} \\ \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} \\ 3\mathcal{O}_{E_{\sqrt{-3}}} & 3\mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} \\ 3\mathcal{O}_{E_{\sqrt{-3}}} & 3\mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} & \mathcal{O}_{E_{\sqrt{-3}}} \end{pmatrix} \right. \right\}.$$

Then  $K_s^{(3)}$  is almost  $\tau$ -super golden for (decimated, non-standard traversable)  $\tau$  as in Example 4.2.6(5).

*Proof.* Note that  $X_3^{-1}G(\mathbb{Q}_3)X_3$  then has a chamber corresponding to lattices  $\mathcal{O}_{E_{\sqrt{-3}}}^{4-i} \oplus \sqrt{3}\mathcal{O}_{E_{\sqrt{-3}}}^i$  for  $0 \leq i \leq 3$ . The non-special vertex corresponds to  $i = 2$ . Therefore lattices  $\mathcal{O}_{E_{\sqrt{-3}}}^4$  and  $\mathcal{O}_{E_{\sqrt{-3}}}^2 \oplus 3\mathcal{O}_{E_{\sqrt{-3}}}^2$  are special connected through a two-step path through a non-special vertex and  $K_s^{(3)}$  exactly stabilizes both.

We then compute as in Proposition 4.5.4 that  $[K : K^{(3),s}] = [K_3 : K_3^{(3),s}] = 120 = |G(\mathbb{Z})|/1296$  (there are 40 choices for the first edge and 3 per each of these for the second). It therefore suffices to check by computer that  $|G(\mathbb{Z}) \cap K^{(3),s}| = 1296$  which holds.  $\square$

$K_s^{(3)}$  and  $K_s^{(2)}$  therefore both produce super-golden gate sets for  $PU(4)$  with finite group

$$C_p = G(\mathbb{Z})/\langle \zeta_6 \rangle \cong {}^2A_2(2) \cong C_2(3).$$

and a single extra element that we call  $T_{E,2}$  and  $T_{E,3}$  respectively. They have growth rates

$$|S_p^{[\ell]}| \asymp \begin{cases} 64^\ell & K^{(2),s} \\ 81^\ell & K_s^{(3)} \end{cases}.$$

Let  $K_s^{(\sqrt{-3})}$  be the same as  $K_s^{(3)}$  except the congruence condition is only mod  $\sqrt{-3}$ . If we let

$$X_M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \end{pmatrix},$$

then  $X_M^{-1}K_s^{(\sqrt{-3})}X_M$  is  $\{\pm 1\}$  times the group of monomial matrices with entries that are 3rd roots of unity.

Using the isomorphism  $(K \cap G(\mathbb{Q}))/\langle \zeta_3 \rangle \cong \text{Sp}_4(\mathbb{F}_3)$  from the proof of Proposition 4.5.3 and noting that it maps  $(K_s^{(\sqrt{-3})} \cap G(\mathbb{Q}))/\langle \zeta_3 \rangle$  to a parabolic subgroup,  $K \cap G(\mathbb{Q})$  is generated by  $K_s^{(\sqrt{-3})} \cap G(\mathbb{Q})$  and the preimage  $C_W$  of a long Weyl element. We may choose  $C_W$  so that

$$X_M^{-1}C_W X_M = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \end{pmatrix}.$$

Using the techniques of 4.3.3, we may choose (possibly without loss of generality conjugating  $K_s^{(3)}$  within  $K_s^{(\sqrt{-3})}$ ):

$$X_M^{-1}T_{E,3}X_M = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$



This gives Theorem 1.2.4.

Finally we note that while the hermitian positive definite matrix  $H_3$  gives golden and super golden gate sets with large finite group  $C_p \cong {}^2A_2(2)$ , one can instead take the identity matrix  $I$  and also get a golden gate sets with a smaller finite group,  $C_p \cong \langle \zeta_6 \rangle^{4-1} \rtimes S_4$ . At this time it is not clear whether super golden gate sets can be obtained from this form.

**Proposition 4.5.6.** *Define*

$$G := U_4^{\mathbb{Q}[\sqrt{-3}], I} / \mathbb{Q}.$$

*Then  $G(\widehat{\mathbb{Z}})$  is an almost golden adelic group.*

*Proof.* We note that  $G$  is quasisplit at all places and  $G(\mathbb{Z}_p)$  is hyperspecial at all  $p \neq 3$ . At  $p = 3$ , we have that  $G(\mathbb{Z}_3)$  is a maximal parahoric subgroup (see Lemma 5.8 in [BEMP23], it is stated for  $n = 3$  but the proof works verbatim for all  $n \geq 2$ ). By [GHY01] we get that  $\lambda_3 = \frac{3^{N(Sp_4/\mathbb{F}_3)} \cdot Sp_4(\mathbb{F}_3)}{3^{N(G/\mathbb{F}_3)} \cdot G(\mathbb{F}_3)} = \frac{1}{5}$ , and for any  $p \neq 3$  we have  $\lambda_p = 1$ , hence  $\lambda(G) = \frac{1}{5}$ . Since  $G$  is an inner form of  $U_4^{\mathbb{Q}[\sqrt{-3}], H_3}$ , and since  $L(G)$  is invariant under taking inner forms, we get that  $L(G) = 155520^{-1}$ , hence  $R(G) = 31104^{-1}$ . On the other hand,  $G(\mathbb{Z})$  is comprised of exactly the monomial matrices with non-zero coefficients in the unit group of the ring of integers, hence  $G(\mathbb{Z}) \cong \langle \zeta_6 \rangle^4 \rtimes S_4$ , and therefore  $|G(\mathbb{Z})| = 6^4 \cdot 4! = 31104$ . Therefore, by lemma 2.3.3,  $G$  has class number one.  $\square$

#### 4.5.3 New Constructions: Rank-8

Next, we present the rank-8 golden group of [MSG12].

**Proposition 4.5.7.** *For the  $8 \times 8$  Hermitian positive definite matrix*

$${}^8H_3 = \begin{pmatrix} H_3 & 0 \\ 0 & H_3 \end{pmatrix}$$

*the following is an almost golden adelic group*

$$G = U_8^{\mathbb{Q}[\sqrt{-3}], {}^8H_3} / \mathbb{Q}, K = G(\widehat{\mathbb{Z}})$$

*Proof.* Then  $U_8^{\mathbb{Q}[\sqrt{-3}], {}^8H_3}$  is quasisplit at all places and  $U_8^{\mathbb{Q}[\sqrt{-3}], {}^8H_3}(\mathbb{Z}_p)$  is hyperspecial at all  $p \neq 3$  and special at  $p = 3$ . The result then follows from the main result of [MSG12]. We can also compute that  $R(G) = 48372940800^{-1}$  and that  $U_8^{\mathbb{Q}[\sqrt{-3}], {}^8H_3}(\widehat{\mathbb{Z}})$  is  $U_4^{\mathbb{Q}[\sqrt{-3}], H_3}(\widehat{\mathbb{Z}})$  from 4.5.3 squared semidirect  $\mathbb{Z}/2$  which has size  $2 \cdot 155520^2 = 48372940800$ .  $\square$

This gives golden gate sets for 3 qubits.

#### 4.5.4 New Constructions: $\mathbb{Q}(i)$

Moving on to  $E/F = \mathbb{Q}(i)/\mathbb{Q}$ :

**Proposition 4.5.8.** *For the  $4 \times 4$  Hermitian positive definite matrix*

$$H_4 = \begin{pmatrix} 4 & 0 & 2+i & 1+i \\ 0 & 4 & -1+i & 2-i \\ 2-i & -1-i & 2 & 0 \\ 1-i & 2+i & 0 & 2 \end{pmatrix},$$

*define*

$$G = U_4^{\mathbb{Q}(i), H_4} / \mathbb{Q}.$$

*Then  $G(\widehat{\mathbb{Z}})$  is an almost golden adelic subgroup of  $G$ .*

*Proof.*  $H_4$  is selected so that  $G(\mathbb{Z}_p)$  is hyperspecial at all  $p \neq 2$  and special at  $p = 2$  following remark 3.2.3. Note that the different ideal of  $\mathbb{Q}_2(i)/\mathbb{Q}_2$  is  $(2)$  so we can choose  $H_4$  to be equivalent to the antidiagonal matrix of all ones over  $\mathbb{Z}_2[i]$ .

Then the argument follows applying lemma 2.3.3 as in Proposition 4.5.3: here, both  $R(G)^{-1}$  and  $|G(\mathbb{Z})|$  are 46080.  $\square$

$H_4$  also produces some super-golden gate sets:

**Proposition 4.5.9.** *Consider  $G = U_4^{\mathbb{Q}[i], H_4}/\mathbb{Q}$  as in Proposition 4.5.8. Choose  $X_2 \in \text{GL}_4(\mathbb{Z}_2[i])$  such that*

$$X_2^* H_4 X_2 \equiv \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \pmod{4},$$

for example,

$$X_2 = \begin{pmatrix} -1 & -2 & -2 & -1 \\ 1 & 1 & -2 & 1 \\ 2 & 2 & 1-2i & 2-i \\ -1-2i & -i & 3 & -i \end{pmatrix}.$$

Define

$$K' := \left\{ g \in G(\widehat{\mathbb{Z}}) \left| X_2^{-1} g X_2 \in \begin{pmatrix} \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ 2\mathcal{O}_{E_{1+i}} & 2\mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ 2\mathcal{O}_{E_{1+i}} & 2\mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \end{pmatrix} \right. \right\}$$

Then  $K'$  is almost  $\tau$ -super golden for (decimated non-standard traversable)  $\tau$  as in Example 4.2.6(5).

*Proof.* As in Proposition 4.5.5,  $X_2^{-1} G(\mathbb{Q}_2) X_2$  exactly stabilizes a particular two-step path through a non-special vertex. In the wildly ramified setting, we additionally need to ensure that this two-step path is actually in  $\mathcal{B}(G_{\mathbb{Q}_2})$  instead of intersecting barbs—this is guaranteed by the congruence condition on  $X_2$  being mod 4 instead of mod 2.

Then, arguing as in Proposition 4.5.4,  $[K : K'] = [K_2 : K'_2] = 30 = |G(\mathbb{Z})|/1536$  (there are 15 choices for the first edge and 2 per each of these for the second by standard formulas) and by computer check,  $|G(\mathbb{Z}) \cap K'| = 1536$ .  $\square$

The example in Proposition 4.5.9 is particularly interesting since here,  $C = G(\mathbb{Z})/\langle i \rangle$  is the 2-qubit Clifford group<sup>(†)</sup> (this is guaranteed by its size and a classification of maximal finite subgroups of  $PU(4)$ , e.g., [KT24, Appx D]). Therefore, this provides a single element that we call  $T_G$  that together with the 2-qubit Clifford group gives a super-golden gate set on  $PU(4)$  with growth rate

$$|S_2^{[\ell]}| \asymp 16^\ell.$$

**Proposition 4.5.10.** *In the notation of proposition 4.5.9, Let*

$$K'' = \left\{ g \in G(\widehat{\mathbb{Z}}) \left| X_2^{-1} g X_2 \in \begin{pmatrix} \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ (1+i)\mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ (1+i)\mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \\ (1+i)\mathcal{O}_{E_{1+i}} & (1+i)\mathcal{O}_{E_{1+i}} & (1+i)\mathcal{O}_{E_{1+i}} & \mathcal{O}_{E_{1+i}} \end{pmatrix} \right. \right\}$$

<sup>(†)</sup>Beware that  $G(\mathbb{Z})$  on the other hand is a different lift of the projective Clifford group to  $U(4)$  than is standard in the literature.

Then  $K' \subseteq G$  is almost  $\tau$ -super golden for (non-standard traversable)  $\tau$  as in Example 4.2.6(4).

*Proof.* Let  $K_1 \subseteq K$  be the subgroup of elements that are trivial mod  $(1+i)$ . Then,  $K/K_1$  is the reductive quotient in the special fiber of the parahoric model corresponding to  $K'$  and isomorphic to  $\mathrm{Sp}_4(\mathbb{F}_2)$ . Inside this,  $K''/K_1$  is a parabolic subgroup, so  $K''$  is the stabilizer of an edge emanating from  $x_0$ . Since  $K''$  doesn't contain  $K'$ , this edge must connect to a special vertex.

Then, arguing as in Proposition 4.5.4, we may compute  $[K : K''] = [K_2 : K_2''] = 15 = |G(\mathbb{Z})|/3072$  and that  $|G(\mathbb{Z}) \cap K''| = 3072$ .  $\square$

The super-golden gate set  $S_{2'}$  from 4.5.10 is again the 2-qubit Clifford group with an added element we now call  $T'_G$ . It has growth rate

$$|S_{2'}^{[\ell]}| \asymp 8^\ell.$$

While this growth rate is much worse than the other examples, the extra gate  $T'_G$  has much better properties for fault-tolerant implementation:

**Lemma 4.5.11.** *In the notation above,  $T'_G$  is in the 3rd level of the Clifford hierarchy.*

*Proof.* As in the proof of Proposition 4.5.10, let  $K_1$  be the elements of  $K$  that are trivial mod  $(1+i)$  so that  $K/K_1 \cong \mathrm{Sp}_4(\mathbb{F}_2)$ . We can also compute:

$$[G(\mathbb{Z}) : G(\mathbb{Z}) \cap K_1] = [K : K_1] \implies G(\mathbb{Z})/G(\mathbb{Z}) \cap K_1 = K/K_1 \cong \mathrm{Sp}_4(\mathbb{F}_2).$$

Identifying  $G(\mathbb{Z})/i$  with the 2-qubit Clifford group, this characterizes  $(G(\mathbb{Z}) \cap K_1)/i$  as the 2-qubit Pauli group.

Next, since  $T'_G$  acts as an involution of the edge containing  $x_0$  stabilized by  $K''$ ,  $T'_G K'' (T'_G)^{-1}$  stabilizes  $x_0$  and is therefore contained in  $K$ . In total

$$T'_G(K'' \cap G(\mathbb{Q}))(T'_G)^{-1} \subseteq G(\mathbb{Z}).$$

Since  $K_1 \subseteq K''$ , this shows that  $T'_G$  conjugates the Pauli group  $(G(\mathbb{Z}) \cap K_1)/i$  into the Clifford group  $G(\mathbb{Z})/i$ .  $\square$

The 2-qubit Pauli group has a standard presentation generated by tensors of two matrices chosen from

$$I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad X = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad Y = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and the scalars  $\pm 1, \pm i$ . If we set:

$$X_P := \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ i & 2-i & -1 & 1 \\ -1 & i & i & 1-2i \end{pmatrix},$$

then  $X_P^{-1}(K_1 \cap G(\mathbb{Q}))X_P$  is the Pauli group in this standard presentation.

As in Example 4.3.4, we can make a choice (possibly without loss of generality conjugating  $K''$  inside  $K$ ; note that conjugation doesn't change the normal  $K_1$ ):

$$T'_G := \begin{pmatrix} \frac{3}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} & \frac{3}{2} + \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{3i}{2} & -\frac{1}{2} - \frac{3i}{2} & \frac{1}{2} + \frac{3i}{2} & -\frac{1}{2} + \frac{i}{2} \end{pmatrix}.$$

such that  $X_P^{-1}T'_G X_P$  is the CS gate<sup>(†)</sup>, giving Theorem 1.2.5. We may in fact take the 2-qubit Clifford group together with any additional matrix  $T$  such that  $T'_G = X_P T X_P^{-1}$  is in  $G(\mathbb{Z}[1/2])$  and has Cartan invariant  $(1/2, 0)$ —for example, many other controlled Clifford gates or monomial matrices work.

#### 4.5.5 New Constructions: $\mathbb{Q}(\sqrt{-7})$

Finally, for  $E/F = \mathbb{Q}(\sqrt{-7})/\mathbb{Q}$ , Proposition 4.5.12 produces the example in Theorem 1.2.4.

**Proposition 4.5.12.** *For the  $4 \times 4$  Hermitian positive definite matrix*

$$H_7 = \begin{pmatrix} 7 & 0 & -3i\sqrt{7} & -2i\sqrt{7} \\ 0 & 7 & 2i\sqrt{7} & -3i\sqrt{7} \\ 3i\sqrt{7} & -2i\sqrt{7} & 14 & 0 \\ 2i\sqrt{7} & 3i\sqrt{7} & 0 & 14 \end{pmatrix},$$

*the following is an almost golden adelic group:*

$$G = U_4^{\mathbb{Q}(\sqrt{-7}), H_7} / \mathbb{Q}, \quad K' = \{g \in G(\widehat{\mathbb{Z}}) \mid g \text{ upper triangular} \pmod{(1 + \sqrt{-7})/2}\}$$

*It is also almost  $\tau$ -super-golden at  $p = 2$  for  $\tau$  a complete chamber.*

*Proof.* Here,  $H_7$  was chosen so that  $U^{E, H_7}(\mathbb{Z}_p)$  is special at  $p = 7$  and hyperspecial at all other  $p$  as in remark 3.2.3. Plugging in  $L$ -values

$$R(G) = 5040^{-1}.$$

As in Proposition 4.5.3, we can also compute that  $|G(\mathbb{Z})| = 5040$ , though the computation takes significantly longer to run<sup>(‡)</sup>. Therefore by lemma 2.3.3,  $G$  has class number one.

Next, in the embedding  $G_2 \subseteq \mathrm{GL}_4(E_{(1+\sqrt{-7})/2}) \times \mathrm{GL}_4(E_{(1-\sqrt{-7})/2})$ , projection onto the first coordinate is an isomorphism,  $K'_2$  is an Iwahori subgroup. This is the stabilizer of a chamber.

Finally, we need to show that  $[G(\mathbb{Z}) : G(\mathbb{Z}) \cap K'] = [G(\widehat{\mathbb{Z}}) : K']$ . The Iwahori in  $\mathrm{GL}_2(\mathbb{F}_2)$  has index 315 by standard formulas for general linear groups over finite fields. Therefore, it suffices to check that  $|G(\mathbb{Z}) \cap K'| = 5040/315 = 16$ , which holds.  $\square$

The chamber  $\tau$  is standard traversable as in Example 4.2.6(1). This gives a super-golden gate set with growth rate

$$|S_2^{[\ell]}| \asymp 16^\ell.$$

The group  $C_2$  is the alternating group  $A_7$  (resp.  $\tilde{C}_2$  is the double cover usually denoted  $2.A_7$ ). Inside this  $\Gamma$  (resp.  $\tilde{\Gamma}$ ) is the Sylow-2 subgroup. Finally, we can write down explicit

<sup>(†)</sup>In particular, the gate set of 4.5.10 is equivalent to the gate set studied in [GRT21]. In fact, the exact synthesis algorithm of [GRT21] can be interpreted as using the exceptional Lie group isomorphism therein to realize the Clifford group as the points of an integral model of  $\mathrm{SO}_6$  and then applying the general Bruhat-Tits theoretic algorithm of Theorem 4.4.1(2). Here, we instead get the realization as integral points through conjugation by  $X_P$ , which also gives us far stronger covering bounds by automorphic techniques.

<sup>(‡)</sup>the norm-7 vectors again correspond to the 240 roots of the  $E_8$  lattice while the norm-14 vectors are the 2160 of second-smallest length

generators

$$T_2 = \left\{ T_K^i := \begin{pmatrix} -o & 2o+2 & -o+2 & o-2 \\ -2 & o-2 & 0 & -o \\ 1 & 4o-4 & o+2 & -o-2 \\ -o+2 & o-4 & o & -o \end{pmatrix}^i, 1 \leq i < 4 \right\}$$

$$C_2 = \left\langle \begin{pmatrix} 0 & -2o+1 & 1 & -2 \\ -o+1 & -o & 0 & -1 \\ o+1 & -2 & -o+1 & o \\ -1 & -4 & -o+1 & 2o-1 \end{pmatrix}, \begin{pmatrix} 2 & -o+1 & -o+1 & -o \\ 1 & 2o & -1 & -o+2 \\ -o+1 & -3 & -o-1 & o-1 \\ -2o+2 & -o+2 & -1 & -o-2 \end{pmatrix} \right\rangle$$

where  $o = (1 + \sqrt{-7})/2$ . The element  $T_K$  acts as an order-4 rotation cyclically permuting the vertices of a chamber.

#### 4.5.6 New Constructions: Non-Quasisplit

Finally, we present examples where  $G$  is non-quasisplit at some places:

**Proposition 4.5.13.** *For the  $4 \times 4$  Hermitian positive definite matrix*

$$H_{3R} := \text{diag}(1, 1, 1, 2),$$

Define

$$G := U_4^{\mathbb{Q}(\sqrt{-3}), H_{3R}} / \mathbb{Q}.$$

Then  $G$  is non-quasisplit at  $p = 2, 3$  and  $G(\widehat{\mathbb{Z}})$  is almost golden.

*Proof.* Non-quasisplitness can be tested by computing  $\text{disc}(H_{3R})$ . In addition  $G(\widehat{\mathbb{Z}})$  was chosen as in remark 3.2.3 so that it is extraspecial at 3, special at 2, and hyperspecial at all other places.

Next,  $G(\mathbb{Z})$  is easily computable as  $\mu_6^4 \times S^3$  where  $\mu_6$  is the group of 6th roots of unity and the  $S_3$  acts on the first three coordinates. This has size  $6^5 = 7776$  and we may also compute  $R(G) = 7776^{-1}$  so the result follows by lemma 2.3.3.  $\square$

The gate sets  $S_2$  and  $S_3$  from 4.5.13 are particularly interesting:  $\langle S_2 \rangle$  acts transitively on every other vertex of an infinite 9-regular tree while  $\langle S_3 \rangle$  acts transitively on the degree-10 vertices of an infinite  $(10, 4)$ -biregular tree.

**Proposition 4.5.14.** *For the  $4 \times 4$  Hermitian positive definite matrix*

$$H_{4R} := \text{diag}(1, 1, 1, 3),$$

define

$$G := U_4^{\mathbb{Q}(i), H_{4R}} / \mathbb{Q}.$$

Then  $G$  is non-quasisplit at  $p = 2, 3$  and  $G(\widehat{\mathbb{Z}})$  is almost golden.

*Proof.* This is a similar argument to 4.5.13 with  $R(G)^{-1} = |G(\mathbb{Z})| = 1536$ .  $\square$

In the case of 4.5.14,  $\langle S_2 \rangle$  acts transitively on the degree-5 vertices of an infinite  $(5, 3)$ -biregular tree while  $\langle S_3 \rangle$  acts transitively on every other vertex of an infinite 28-regular tree.

**Remark 4.5.15.** The example in 4.5.13 can be modified to give a  $K'$  that is  $\tau$ -super-golden at 2 for  $\tau$  a *single* edge in  $\mathcal{B}(G_2)$ . However, this  $\tau$  is far less convenient than Example 4.2.6(6) since we do not have an element in  $G_2$  that flips an edge—it requires 8 additional elements  $t_i$  corresponding to the other 8 edges neighboring at a vertex.

Moving to the setting of  $GU_4^{\mathbb{Q}(\sqrt{-3}), H_{3R}}$  may provide the desired edge-flip. However, we cannot prove our automorphic bound 8.2.3 for  $GU$  since we do not then have access to the endoscopic classification. Therefore we postpone analyzing this until the requisite automorphic results are known to be true.

#### 4.5.7 Clifford Gates

We end this section by discussing the well-studied and standard Clifford+T gates and their generalizations from the point of view of arithmetic unitary groups and golden adelic groups (see [Sar15b] and the references there).

The classical Clifford+T gates are a finite set of unitary  $2 \times 2$  matrices with coefficients in the ring  $\mathbb{Z}[\sqrt{-1}, \sqrt{2}^{\pm 1}]$ . By [KMM13], the set of elements in  $U(2)$  that are synthesisable (i.e. generated by matrix multiplication and tensoring) by the Clifford+T gates is precisely the full 2-arithmetic group  $\Lambda$  of unitary  $2 \times 2$  matrices with coefficients in the ring  $\mathbb{Z}[\sqrt{-1}, \sqrt{2}^{\pm 1}]$ ; i.e.  $\Lambda = G(\mathbb{Z}[\sqrt{2}^{\pm 1}])$  where

$$G = U_2^{\mathbb{Q}[\sqrt{2}, \sqrt{-1}], I} / \mathbb{Q}[\sqrt{2}].$$

The group  $G$  has class number one and moreover gives rise to super golden gate sets (see Section 4.1.3 in [PS18]).

The multiqubit Clifford+T gates are a finite set of unitary  $2^n \times 2^n$  matrices with coefficients in the ring  $\mathbb{Z}[\sqrt{-1}, \sqrt{2}^{\pm 1}]$ . By [GS13], the group of elements in  $U(2^n)$  that are synthesisable by the multiqubit Clifford+T gates is  $\Lambda = G(\mathbb{Z}[\sqrt{2}^{\pm 1}])$ , where

$$G = U_{2^n}^{\mathbb{Q}[\sqrt{2}, \sqrt{-1}], I} / \mathbb{Q}[\sqrt{2}].$$

However, for  $n \geq 2$ , the group  $G$  is not of class number one—in particular  $\Lambda$  does not act transitively on the special vertices of the corresponding Bruhat-Tits building of  $G(\mathbb{Q}[\sqrt{2}]/\sqrt{2})$ .

The Clifford+cyclotomic gates are a finite set of unitary  $2 \times 2$  matrices with coefficients in the ring  $R_m = \mathbb{Z}[\zeta_m][\frac{1}{2}]$ , where  $\zeta_m = e^{\frac{2\pi i}{m}}$ ,  $m \in \mathbb{N}$ . These matrices sit inside the full 2-arithmetic subgroup of

$$G = U_2^{\mathbb{Q}[\zeta_m], I} / \mathbb{Q}[\zeta_m + \zeta_m^{-1}].$$

By [FGKM15, IJK<sup>+</sup>21], the group of elements in  $U(2)$  that are synthesizable by the Clifford+cyclotomic gates is a 2-arithmetic subgroup of  $G$  if and only if  $m = 4, 8, 12, 16$  or  $24$ . Note that being a 2-arithmetic subgroup only implies that the class number is finite, not necessarily that the class number is one (it is one for  $m = 4, 8$ ).

#### 4.6 Super-Golden Gate Set Comparisons

The case of super-golden gates on  $PU(4)$  is particularly interesting for applications. We compare the covering rate produced by our 2-qubit super-golden gates versus using 1-qubit super-golden gates together with specific other 2-qubit gates.

Assume first we have a set of super-golden gates  $S_p$  for  $PU(4)$  such that  $|S_p^{[\ell]}| \asymp R^\ell$ . We want to find the minimum  $\ell$  such that there is an element of  $S_p^{[\ell]}$  within distance  $\epsilon$  of

some  $A \in PU(4)$ ; in other words, such that  $B(A, \epsilon^{15}) \cap S_{\mathfrak{p}}^{[\ell]} \neq \emptyset$ . Up to decreasing  $C$  by an arbitrarily small factor, the covering property gives

$$\ell \leq \log_R(\epsilon^{-15}) + A = 15(\log_2 R)^{-1} \log_2(1/\epsilon) + A$$

for some constant  $A$ .

On the other hand, assume we have a set of super-golden gates  $S_{\mathfrak{p}}$  for  $PU(2)$  such that  $|S_{\mathfrak{p}}^{[\ell]}| \asymp C^\ell$ . There is an embedding  $L = U(2) \times U(2)/U(1) \hookrightarrow PU(4)$  that restricts to two embeddings  $L_1, L_2 : U(2) \hookrightarrow PU(4)$ . The paper [ZVSW04] gives a way to write elements of  $PU(4)$  as a product of a minimal number of elements in  $L_1 \cup L_2$  together with copies of a fixed additional matrix (called  $B$ ). This requires 6 elements of  $L_1 \cup L_2$ .

To approximate  $A \in PU(4)$  within distance  $\epsilon$ , each of the factors  $A_i$  in the  $L_i$  needs to be approximated within distance  $\approx \epsilon/6$ ; in other words we need to find an element of  $S_{\mathfrak{p}}^{[\ell]}$  in  $B(A_i, 1/\epsilon^3)$  (together with a phase shift). This implies that we need  $\ell$  gates in total with

$$\ell \leq 6 \log_R(\epsilon^{-3}) + A \approx 18(\log_2 R)^{-1} \log_2(1/\epsilon) + A$$

for some constant  $A$ .

The CNOT gate is sometimes preferred instead of  $B$  since it is contained in the 2-qubit Clifford group. Three copies of CNOT together with 8 elements of  $L_1 \cup L_2$  suffice to give any element of  $PU(4)$  (see [VD04]). This approximation strategy therefore instead requires  $\ell$  gates with:

$$\ell \leq 24(\log_2 R)^{-1} \log_2(1/\epsilon) + A.$$

In Table 4.1, we summarize details of selected gate sets constructed from either 1-qubit super-golden sets from [PS18] or the 2-qubit sets here. Each can be thought of as a finite group together with some extra finite-order elements—which ones are relevant for how well the gates can be (or could be with future work) realized in a fault-tolerant way with respect to some quantum error correction scheme.

More specifically, for the added gates to be implementable with the teleportation procedure of [GC99], we need  $K' \cap G(\mathbb{Q})$  to linearly span all  $4 \times 4$  complex matrices. For compatibility with the current stabilizer-code paradigm for fault-tolerant implementation, we need the much stronger property that the finite group is Clifford and that the added gates are in the 3rd level of the Clifford hierarchy. We mark these cases.

We also present, as in the calculations above, the growth rate for the gate set  $L$  and the constant determining how efficiently elements of  $PU(4)$  can be approximated. We emphasize that this “covering efficiency” is the theoretical optimal. The best known practically computable approximations are a factor of  $7/3$  larger in the 1-qubit cases as in [BS23]. The 2-qubit cases here are worse by a much larger factor since the algorithm in Theorem 4.4.1(3) is very far from optimized.

**Remark 4.6.1.** Expanding the notion of super-golden gates may be worthwhile. As one example, we can consider the case of  $K_2^\infty \subseteq K_1^\infty \subseteq G^\infty$  and place  $\mathfrak{p}$  such that

- $K_1^\infty$  has class number 1,
- $K_2^\infty$  has class number  $n$ ,
- $K_{1,\mathfrak{p}}$  is special,
- $K_{2,\mathfrak{p}}$  is a stabilizer of  $\tau$  as in Example 4.2.6(1,3-6).



Then, we get a super-golden gate set requiring  $n$  different added gates  $T_i$ . The weakening of the class-number-1 condition on  $K_2^\infty$  allows for examples with dramatically better growth rates, though it is unclear whether this is worth the cost of extra  $T_i$ . The best growth rates should come from Example 4.2.6(3) giving  $R = q_p^6$  as in Example 4.3.11.

The best general framework for constructing super-golden gate sets is currently unclear.

	Gate Set	Fin. Group	Added Gates	$R$	Covering Efficiency
1-qb	Cliff. + $T$ + CNOT**	2-qb Cliff.	$T$	2	24
	Cliff. + $T$ + $B$	(1-qb. Cliff.) <sup>2</sup>	$T, B$	2	18
	Oct. + CNOT	2-qb Cliff.	$T_{24}$	23	$24(\log_2 23)^{-1} \approx 5.31$
	Icos. + CNOT	(Icos.) <sup>2</sup>	$T_{60}, \text{CNOT}$	59	$24(\log_2 59)^{-1} \approx 4.08$
	Icos. + $B$	(Icos.) <sup>2</sup>	$T_{60}, B$	59	$18(\log_2 59)^{-1} \approx 3.06$
2-qb	4.5.10 (Cliff.+CS)**	2-qb Cliff.	$T'_G \sim \text{CS}$	8	$15(\log_2 8)^{-1} = 5$
	4.5.9*	2-qb Cliff.	$T_G$	16	$15(\log_2 16)^{-1} = 3.75$
	4.5.12	$\text{Alt}_7$	$T_K, T_K^2, T_K^3$	16	$15(\log_2 16)^{-1} = 3.75$
	4.5.4	$C_2(3)$	$T_{E,2}$	64	$15(\log_2 64)^{-1} = 2.5$
	4.5.5*	$C_2(3)$	$T_{E,3}$	81	$15(\log_2 81)^{-1} \approx 2.37$

**Note:** Starred entries are where  $K' \cap G(\mathbb{Q})$  spans  $\text{Mat}_{4 \times 4} \mathbb{C}$  so teleportation can implement the added gates assuming an implementation of the finite group. Double starred entries are where, in addition, the finite group is Clifford and the added gates are at the 3rd level of the Clifford hierarchy.

Table 4.1: Gate Set Comparisons

## 5 Automorphic Representations Background

We now enter the second half of the paper: proving the covering property of our golden gate sets using the theory of automorphic representations. In this section we recall some basic facts and notations concerning automorphic representations.

### 5.1 Automorphic representations

Throughout this section,  $F$  is a number field with ring of integers  $\mathcal{O} = \mathcal{O}_F$  and adele ring  $\mathbb{A} = \mathbb{A}_F$ . Let  $v$  denote a place of  $F$ , let  $F_v$  be the  $v$ -completion of  $F$ , and, when  $v$  is finite, let  $\mathcal{O}_v$  be the ring of integers of  $F_v$  with uniformizer  $\varpi_v$  and order of residue field  $q_v := |\mathcal{O}_v / \mathfrak{p}_v \mathcal{O}_v|$ .

Let  $G$  be a connected reductive group over  $F$ . For simplicity, assume that the maximal split torus in the center of the real group  $G_\infty$  is trivial so that  $G(F) \backslash G(\mathbb{A})$  has finite volume. Fix a  $k$ -embedding  $G \hookrightarrow GL_n(k)$ . For any place  $v$ , denote  $G_v := G(k_v)$ . When  $v$  is finite, denote  $K_v = G(k_v) \cap GL_n(\mathcal{O}_v)$  and, for any  $m \in \mathbb{N}$ , denote  $K_n(\mathfrak{p}_v^m) := \ker(K_v \rightarrow GL_n(\mathcal{O}_v / \mathfrak{p}_v^m \mathcal{O}_v))$ .

Consider the right regular  $G(\mathbb{A})$ -representation on  $L^2(G(F) \backslash G(\mathbb{A}))$ . An  $F$ -automorphic representation of  $G$  is an irreducible  $G(\mathbb{A})$ -representation  $\pi$  which is weakly contained in  $L^2(G(F) \backslash G(\mathbb{A}))$  and whose central character is unitary. Denote by  $\mathcal{AR}(G)$  the set of  $F$ -automorphic representations of  $G$ . Consider the decomposition of  $\mathcal{AR}(G)$  into its cuspidal



$\mathcal{AR}_{\text{cusp}}(G)$ , residual  $\mathcal{AR}_{\text{res}}(G)$ , discrete  $\mathcal{AR}_{\text{disc}}(G)$ , and continuous  $\mathcal{AR}_{\text{cont}}(G)$  parts:

$$\begin{aligned}\mathcal{AR}(G) &= \mathcal{AR}_{\text{cusp}}(G) \oplus \mathcal{AR}_{\text{res}}(G) \oplus \mathcal{AR}_{\text{cont}}(G), \\ \mathcal{AR}_{\text{disc}}(G) &= \mathcal{AR}_{\text{cusp}}(G) \oplus \mathcal{AR}_{\text{res}}(G).\end{aligned}$$

Any  $\pi \in \mathcal{AR}(G)$  decomposes as a restricted tensor product  $\pi = \otimes'_v \pi_v$ , where  $\pi_v$ , called the *local-factor* of  $\pi$  at  $v$ , is an irreducible admissible  $G_v$ -representation (cf. [Fla79]). Let  $\sigma(\pi_v)$  be the infimum over  $\sigma \geq 2$ , such that each  $K_v$ -finite matrix coefficient of  $\pi_v$  is in  $L^{\sigma+\epsilon}(G_v)$  for any  $\epsilon > 0$ . Say that  $\pi$  is tempered at  $v$  if  $\sigma(\pi_v) = 2$ . The Generalized Ramanujan Conjecture for  $G = GL_n$  states the following (see [Sar05]):

**Conjecture 5.1.1.** (*GRC*) *Let  $F$  be a number field,  $N \in \mathbb{N}$ , and let  $\pi \in \mathcal{AR}_{\text{cusp}}(GL_N)$ . Then the local component  $\pi_v$  is tempered at every place  $v$  of  $F$ .*

The conjecture is open for any  $N \geq 2$  and any number field  $F$ . However, there are special cases of cuspidal automorphic representations for which it is a theorem:

**Definition 5.1.2.** Let  $\pi \in \mathcal{AR}(n)$ . Say that  $\pi$  is cohomological if, for an Archimedian place  $v$  of  $F$ , the  $v$ -factor of  $\pi$  has an infinitesimal character of a finite dimensional representation. When  $F$  is totally real (resp. CM), say that  $\pi$  is  $F$  self dual (resp.  $F$ -conjugate self-dual) if it is isomorphic to its (resp.  $F$ -conjugate of its) contragredient representation  $\tilde{\pi}(g) := \pi((g^t)^{-1})$ .

**Theorem 5.1.3.** [HTT01, Shi11, Clo13, Car12] *Let  $F$  be a CM field,  $n \in \mathbb{N}$ , and let  $\pi \in \mathcal{AR}_{\text{cusp}}(GL_n)$  be both cohomological and  $F$ -conjugate self-dual (see Definition 5.1.2). Then  $\pi_v$  is tempered at every place  $v$  of  $F$ .*

**Remark 5.1.4.** For  $F = \mathbb{Q}$  and  $n = 2$ , this Theorem was first proved by Eichler [Eic54], for weight  $k = 2$ , and by Deligne [Del74], for general weights. [Car12] gives the result in all cases we need, extending the results of [HTT01], [Clo13], and [Shi11] needing progressively weaker technical assumptions.

According to the Langland functoriality conjecture, for any  $G$  with dual  $\widehat{G} \leq GL_n$ , the set  $\mathcal{AR}(G)$  should be encoded in  $\mathcal{AR}(GL_n/F)$ . We therefore begin by describing the classification of automorphic representations of  $G = GL_n$  over a number field  $F$ . Fix a global field  $F$  and, for any  $N \in \mathbb{N}$ , denote  $\mathcal{AR}(n) := \mathcal{AR}(GL_n/F)$  and  $\mathcal{AR}_\star(n) := \mathcal{AR}_\star(GL_n/F)$  for  $\star = \text{cusp}, \text{res}$  or  $\text{cont}$ .

**Definition 5.1.5.** Define an (unrefined) shape of  $n$  to be a sequence of pairs of positive integers,  $\square = ((T_1, d_1), \dots, (T_k, d_k))$ , such that  $\sum_i T_i \cdot d_i = n$ . Let  $M_\square := \prod_{i=1}^k GL_{T_i}^{d_i} \leq P_\square \leq GL_n$  be the corresponding Levi (block diagonal) and parabolic (block upper triangular) subgroups of shape  $\square$ .

**Theorem 5.1.6.** [Lan06, MW89] *For any shape of  $n$ ,  $\square = ((T_i, d_i))_{i=1}^k$ , there is a map,  $I_\square : \prod_{i=1}^k \mathcal{AR}_{\text{cusp}}(T_i) \rightarrow \mathcal{AR}(n)$ , called the automorphic parabolic induction of shape  $\square$  and satisfying:*

- (1) *For any  $\pi \in \mathcal{AR}(n)$ , there exist a unique shape  $\square = ((T_i, d_i))_{i=1}^k$  and a unique (up to order) sequence of cuspidal representations  $(\pi_i)_{i=1}^k \in \prod_{i=1}^k \mathcal{AR}_{\text{cusp}}(T_i)$  such that  $\pi = I_\square((\pi_i)_{i=1}^k)$ , in which case  $\pi$  is said to be of shape  $\square$ . Moreover,  $\pi$  lies in the discrete (resp. cuspidal) part of  $\mathcal{AR}(n)$  if and only if  $k = 1$  (resp.  $k = 1$  and  $d_1 = 1$ ).*

- (2) Let  $\pi = I_{\square}((\pi_i)_{i=1}^k)$ , where  $\square = ((T_i, d_i))_{i=1}^k$  and  $(\pi_i)_{i=1}^k \in \prod_{i=1}^k \mathcal{AR}_{\text{cusp}}(T_i)$ . Then, for any place  $v$  of  $k$ , the local component  $\pi_v$  is a subquotient of the (unitary) parabolic induction

$$\text{Ind}_{P_{\square}(\mathbb{Q}_v)}^{GL_n(\mathbb{Q}_v)} \left( \bigotimes_{i=1}^k \left( |\cdot|_v^{\frac{d_i-1}{2}} \pi_{i,v} \otimes |\cdot|_v^{\frac{d_i-3}{2}} \pi_{i,v} \otimes \dots \otimes |\cdot|_v^{\frac{1-d_i}{2}} \pi_{i,v} \right) \right).$$

**Definition 5.1.7.** Let  $\square = ((T_i, d_i))_{i=1}^k$ . Then we shorthand  $I_{\square}(\tau_1, \dots, \tau_k)$  by the formal expression

$$I_{\square}(\tau_1, \dots, \tau_k) =: \bigoplus_{i=1}^k \tau_i[d_i].$$

## 5.2 Endoscopic Classification and Shapes

The unitary endoscopic classification of [Mok15] extended to non-quasisplit unitary groups in [KMSW14] lets us decompose  $\mathcal{AR}_{\text{disc}}(G)$  for our unitary groups  $U_n^{E/F, H}$  into pieces corresponding to shapes  $\square$  for  $GL_n$ . Fix CM quadratic extension  $E/F$ .

**Definition 5.2.1.** An automorphic representation  $\bigoplus_i \tau_i[d_i]$  of  $\text{Res}_F^E GL_n$  (this is the same as one of  $GL_n/E$ ) is said to be *elliptic* if each  $\tau_i$  is conjugate self-dual and the individual  $\tau_i[d_i]$  are all distinct.

Let  $\tilde{\Psi}_{\text{ell}}(n)$  be the set of elliptic automorphic representations  $\psi$  of  $\text{Res}_F^E GL_n$ . These are usually referred to as elliptic (global) Arthur parameters.

Attached to  $\text{Res}_F^E GL_n$  are a set of elliptic twisted endoscopic groups  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}^E(n)$  described in [Mok15, §2.1]. These each come with  $L$ -embeddings

$${}^L G^* \hookrightarrow {}^L \text{Res}_F^E GL_n.$$

We will only care about a specific “simple” element:  $U_n^+ \in \tilde{\mathcal{E}}_{\text{ell}}^E(n)$ . The main result of [Mok15] is a decomposition

$$\tilde{\Psi}_{\text{ell}}(n) = \bigsqcup_{G^* \in \tilde{\mathcal{E}}_{\text{ell}}^E(n)} \Psi_{\text{ell}}(G^*)$$

and a description of discrete automorphic representations of each  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}^E(n)$  in terms of  $\Psi_{\text{ell}}(G^*)$ .

The paper [KMSW14] generalizes Mok’s classification to “extended pure inner” forms  $G$  of each  $G^*$ . These are enumerated in [KMSW14, §0.3.3]. In particular, the extended pure inner forms of  $U_n^+$  include all the definite unitary groups  $U_n^{E, H}$  we consider here. We recall that if  $G$  is an inner form of  $G^*$ , then  ${}^L G = {}^L G^*$ .

We recall all parts of [KMSW14]’s classification that are needed to explain our results and point readers to [DGG23, §2] for a full summary geared towards trace formula applications.

**Theorem 5.2.2** ([KMSW14] partial summary of main result). *Let  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}^E(n)$  and  $G$  an extended pure inner form of  $G^*$ . Then*

- (1) *To each  $\psi \in \Psi_{\text{ell}}(G^*)$  there is subset  $\Pi_{\psi}^G$  called the (global) Arthur packet such that*

$$\mathcal{AR}_{\text{disc}}(G) = \bigsqcup_{\psi \in \Psi_{\text{ell}}(G^*)} \Pi_{\psi}^G.$$

*This  $\Pi_{\psi}^G$  is empty unless  $\psi$  satisfies a condition of being relevant as in [KMSW14, §0.4.1.2].*

- (2) Let  $\psi = \bigoplus_{i=1}^k \tau_i[d_i]$  and fix a place  $V$ . Through the local Langlands correspondence, each  $\tau_i$  is associated to a (local)  $L$ -parameter

$$\tau_i : WD_{F_v} \rightarrow {}^L(\text{Res}_F^E GL_{T_i})_v$$

from the Weil-Deligne group of  $F_v$ . Define the local  $A$ -parameter

$$\psi_v = \bigoplus_i \tau_i \boxtimes [d_i] : WD_{F_v} \times \text{SL}_2 \rightarrow {}^L(\text{Res}_F^E GL_n)_v$$

where  $[d_i]$  is the  $d_i$ -dimensional representation of  $\text{SL}_2$ . Then  $\psi_v$  factors through  ${}^L G$ .

- (3) There is a finite set of representations  $\Pi_{\psi_v}^G$ , depending only on  $\psi_v$ , called the local  $A$ -packet such that for all  $\pi \in \Pi_{\psi_v}^G$ , we have  $\pi_v \in \Pi_{\psi_v}^G$ .
- (4) If  $\psi_v$  is generic (i.e. all  $d_i = 1$ ), then the assignment  $\psi_v \rightarrow \Pi_{\psi_v}$  satisfies all the desired properties of a local Langlands correspondence. In particular:
- (a) If  $v|\infty$ , then the infinitesimal character of  $\pi_v \in \Pi_{\psi_v}$  is the same as that of  $\psi_v$  through the embedding  $\widehat{G} \hookrightarrow GL_n \mathbb{C} \times GL_n \mathbb{C}$ .
  - (b)  $\pi_v \in \Pi_{\psi_v}$  is tempered if and only if  $\psi_v$  is (for  $v \nmid \infty$ , this means that the  $\tau_i$  are bounded/correspond to unitary supercuspidals).

We can now make our key definition, following [DGG24, §5]:

**Definition 5.2.3.** Let  $\square$  be a shape for  $GL_n/E$ . If  $\pi \in \mathcal{AR}_{\text{disc}}(G)$  with parameter  $\psi \in \Psi_{\text{ell}}(n)$  such that  $\psi \in \square$ , then we say  $\pi$  has shape  $\square$  or  $\pi \in \square$ .

**Remark 5.2.4.** The definition of shape in [DGG24, §5] is actually a list of triples  $\square = ((T_i, d_i, \eta_i))_{i=1}^k$  for some signs  $\eta_i$ . In general, the  $\eta_i$  are needed to determine the  $G$  such that  $\psi \in \Psi_{\text{ell}}(G^*)$  for all  $\psi \in \square$ . However, when  $G^*$  is a simple twisted endoscopic group (e.g.  $U_n^+$ ), there is always a unique choice of  $\eta_i$  that determines  $\psi \in \Psi_{\text{ell}}(G^*)$ .

In our case, we require a priori that  $\psi \in \Psi_{\text{ell}}(U_n^+)$  and can therefore ignore the data of the  $\eta_i$ . Nevertheless, the induction in the black-boxed proof of Theorem 7.1.1 requires keeping track of them.

As in [DGG24], we also associate to shape  $\square = ((T_i, d_i))_{i=1}^k$  the group

$$G_F(\square) := \prod_{i=1}^k U_{T_i}^+. \quad (5.1)$$

This is not in general an element of  $\widetilde{\mathcal{E}}_{\text{ell}}(n)$  and can be thought of as the smallest group through which  $\psi \in \square$  functorially factor through. It will appear in bounds on sizes of automorphic families intersected with  $\square$ .

Finally,

**Definition 5.2.5.** Let

$$L_{\square}^2 := \bigoplus_{\pi \in \square} m_{\pi} \pi,$$

where  $m_{\pi}$  is the multiplicity of  $\pi$  in  $L^2(G(F) \backslash G(\mathbb{A}))$ , and

$$\mathbb{P}_{\square} : L^2(G(F) \backslash G(\mathbb{A})) \rightarrow L_{\square}^2$$

be the orthogonal projection operator.

If  $K$  is an open compact subgroup of  $G(\mathbb{A})$ , we will also use  $\mathbb{P}_{\square}$  to denote the restriction of this projection operator to the subspace  $L^2(G(F) \backslash G(\mathbb{A}))^K$ .

### 5.3 Infinitesimal Characters

#### 5.3.1 Definitions and Relation to Shapes

Formulas later on will involve infinitesimal characters. Consider again  $G$  that is an extended pure inner form of  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}(n)$ .

Any finite dimensional representation  $\pi_\infty$  of  $G_\infty$  has an associated infinitesimal character  $\lambda$  that is a semisimple conjugacy class in  $\hat{\mathfrak{g}}_\infty$ . Since  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}(n)$ , there is a map  $\hat{G} \hookrightarrow GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$  restricted from  ${}^L G \hookrightarrow {}^L \text{Res}_F^E GL_n$ . It is in particular determined by its first coordinate so the infinitesimal character can also be represented by a semisimple conjugacy class in  $M_N(F_\infty \otimes_{\mathbb{R}} \mathbb{C})$ . This can then be represented as an unordered sequence of eigenvalues,

$$\lambda = (\lambda_1, \dots, \lambda_N),$$

with each  $\lambda_i = (\lambda_{i,v})_{v|\infty} \in F_\infty \otimes_{\mathbb{R}} \mathbb{C}$ .

It is also sometimes useful to package the tuple  $\lambda_v = (\lambda_{1,v}, \dots, \lambda_{n,v})$  as the generating function  $\sum_j X^{\lambda_{j,v}}$ , which, by abuse of notation, we will also denote by  $\lambda_v$ . In this way, if  $\bigoplus_i \tau_{i,v}[d_i]$  is a local Arthur parameter such that each  $\tau_{i,v}$  has infinitesimal character  $\lambda_v^{(i)} = \sum_{j=1}^{n_i} X^{\lambda_{j,v}^{(i)}}$ , then we have infinitesimal character assignment

$$\left( \bigoplus_i \tau_{i,v}[d_i] \right)_\infty \mapsto \square((\lambda_v^{(i)})_i) := \sum_i \lambda_v^{(i)} \sum_{l=1}^{d_i} X^{\frac{d_i+1}{2}-l}. \quad (5.2)$$

It can be seen from this that the character of  $\tau[d]$  determines that of  $\tau$ .

**Definition 5.3.1.** If  $\lambda$  matches the infinitesimal character of a finite-dimensional representation, we say that it is *regular integral*.

Regular integral is equivalent to two conditions:

- (Regular) For each  $v$ , the  $\lambda_{i,v}$  are distinct.
- (Integral) If  $N$  is even, the  $\lambda_{i,v} \in \mathbb{Z}$  and if  $N$  is odd, the  $\lambda_{i,v} \in \mathbb{Z} + 1/2$ .

We without loss of generality order regular integral  $\lambda$ :

$$\lambda_{1,v} > \dots > \lambda_{N,v}.$$

We also make some convenient definitions:

**Definition 5.3.2.** Let  $\lambda_\square$  be the set of possible regular, integral infinitesimal characters of  $\psi_\infty$  with  $\psi \in \square$  (i.e. the image of (5.2)). We say  $\lambda \in \square$  as shorthand for  $\lambda \in \lambda_\square$ .

Let  $\square^{-1}(\lambda)$  be the set of possible assignments of infinitesimal characters  $(\lambda^{(i)})_{i=1}^k$  to each  $(\tau_{i,\infty})_{i=1}^k$  so that  $\bigoplus_{i=1}^k \tau_{i,\infty}[d_i]$  has infinitesimal character  $\lambda$ .

#### 5.3.2 Norms of Infinitesimal Characters

Choose distinguished infinite place  $v_0$  and consider infinitesimal character  $\lambda$  of  $G_{v_0}$  for  $G$  an extended pure inner form of  $G^* \in \tilde{\mathcal{E}}_{\text{ell}}(n)$ .

Let  $\Phi_+(G)$  be the standard set of positive coroots of  $G_{v_0}$ . We will need to compare/recall three different norms of  $\lambda_{v_0}$ :

- The dimension of the finite dimensional representation corresponding to  $\lambda$ :

$$\dim \lambda := C_{G,1} \prod_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle,$$

- $\lambda$  paired with itself with by the Killing form:

$$\|\lambda\| := C_{G,2} \left( \sum_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle^2 \right)^{1/2},$$

- A minimum

$$m(\lambda) := \min_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle = \min_{v|\infty} \min_{1 \leq i \leq N-1} (\lambda_{i,v} - \lambda_{i+1,v}), \quad (5.3)$$

where the constants  $C_{G,i}$  only depend on  $G$ .

Given unrefined shape  $\square$ , we can also define

$$\dim_{\square} \lambda := \max_{(\lambda_i)_{i \in \square^{-1}(\lambda)}} \prod_i \dim \lambda_i.$$

Note that this can be 0 if  $\square^{-1}(\lambda)$  is empty. All such definitions can be made analogously for  $G_{\infty}$  as a whole.

Given a group  $G$ , there are some key dimensions to keep track of  $N_G = \dim G$ ,  $r_G = \text{rank } G$ , and  $N_G^{\text{der}} = \dim G^{\text{der}}$ . From these we can compute the number of positive roots:

$$P_G := \frac{1}{2}(N_G - r_G).$$

As some bounds (recalling the definition (5.1) of  $G_F(\square)$ ):

**Lemma 5.3.3.**

$$\dim_{\square} \lambda \leq (\dim \lambda) m(\lambda)^{P_{G_F(\square)} - P_G}.$$

*Proof.* The factors  $\langle \alpha, \lambda \rangle$  in the Weyl dimension formula for  $\dim_{\square} \lambda$  are always a subset of those in  $\dim \lambda$ . However,  $\dim_{\square} \lambda$  has  $P_G - P_{G_F(\square)}$  fewer factors.  $\square$

**Lemma 5.3.4.**

$$\dim_{\square} \lambda \leq C \|\lambda\|^{P_{G_F(\square)}}$$

for some constant  $C$  depending only on  $G$  and  $\square$ .

*Proof.*  $\dim_{\square} \lambda$  is a product of some subset of size  $P_{G_F(\square)}$  of the  $\langle \alpha, \lambda \rangle$  for  $\alpha \in \Phi_+(G)$ . Therefore, by the RMS-AM-GM math-contest inequality,  $(\dim_{\square} \lambda)^{1/P_{G_F(\square)}}$  is bounded above by the root-mean-square of this subset. This is further bounded above by a constant times  $\|\lambda\|$  where the constant depends only on  $P_{G_F(\square)}$  and  $P_G$ .  $\square$

For  $\square = \Sigma_{\eta}$ , the bound 5.3.4 has the optimal exponent on  $\|\lambda\|$  when there is  $C$  such that  $C \min_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle \geq \max_{\alpha \in \Phi_+(G)} \langle \alpha, \lambda \rangle$ . This is an asymptotically positive proportion of all  $\lambda$  in a  $\|\cdot\|$ -ball as the ball's radius goes to infinity.

However, the  $\lambda \in \square$  do not satisfy this property if  $\square$  has non-trivial  $\text{SL}_2$  as some of the  $\langle \alpha, \lambda \rangle$  are then bounded. Therefore, we will also need a slight variant of the bound:

**Lemma 5.3.5.** Choose constant  $m$  and subset  $S \subseteq \Phi_+(G)$ . Then for all  $\lambda$  such that  $\langle \alpha, \lambda \rangle \leq m$  for all  $\alpha \in S$ ,

$$\dim \lambda \leq C m^{|S|} \|\lambda\|^{P_G - |S|}$$

for some constant  $C$  depending only on  $G$  and  $|S|$ .

*Proof.* This is a slight variant of the argument of lemma 5.3.4 where we apply RMS-AM-GM to the set of  $\langle \alpha, \lambda \rangle$  for  $\alpha \in \Phi_+(G) - S$ .  $\square$

Applying this to  $\lambda_\square$ , define for  $\square = ((T_i, d_i))_i$  a correction

$$e(\square) := \sum_i \frac{1}{2} T_i d_i (d_i - 1). \quad (5.4)$$

Then we get our tightening:

**Corollary 5.3.6.** *Choose unrefined shape  $\square$ . Then for all  $\lambda \in \square$  and all  $\delta > 0$ ,*

$$\dim \lambda \leq C \|\lambda\|^{P_G - e(\square)}$$

for some constant  $C$  depending only on  $G$  and  $\square$ .

*Proof.* By inspecting formula (5.2), we see that for each  $(T_i, d_i)$  pair making up  $\square$ , any  $\lambda \in \square$  has  $1/2 T_i d_i (d_i - 1)$  different  $\alpha \in \Phi_+$  with  $\langle \alpha, \lambda \rangle \leq d_i - 1$ . The result then follows from lemma 5.3.5 after noting that the  $m$  and  $|S|$  just depend on  $\square$ .  $\square$

## 5.4 Automorphic Families and the Density Hypothesis

In this subsection, we introduce the notion of an automorphic family (following [SST16]) and the statements of the Ramanujan conjecture and density hypothesis for such automorphic families (following [SX91, Sar90]).

Recall our notations  $f \lesssim g$  if for any  $\epsilon > 0$ , there exists  $c_\epsilon > 0$  such that  $f(x) \leq c_\epsilon \cdot \max\{g(x)^{1+\epsilon}, g(x)^{1-\epsilon}\}$  for any  $x \geq 0$  and  $f \sim g$  if both  $f \lesssim g$  and  $g \lesssim f$ .

**Definition 5.4.1.** Let  $G/F$  be a reductive group over a number field. A (discrete) *automorphic family*  $\mathcal{F}$  for  $G$  is a weighted subset of  $\mathcal{AR}_{\text{disc}}(G)$ : i.e. a function

$$\mathcal{F} : \mathcal{AR}_{\text{disc}}(G) \rightarrow \mathbb{R}_{\geq 0}.$$

**Example 5.4.2.** Let  $K' < G^\infty$  be compact open. Then the family of automorphic forms at level  $K'$  is

$$\mathcal{F}_{K'}(\pi) = m_\pi \dim \left( (\pi^\infty)^{K'} \right).$$

This models the vector space of automorphic forms on  $G$  of level  $K'$

**Definition 5.4.3.** We say automorphic family  $\mathcal{F}$  satisfies the *Ramanujan conjecture* if all  $\pi \in \mathcal{AR}_{\text{disc}}(G)$  with  $\mathcal{F}(\pi) \neq 0$  are tempered.

If  $\mathcal{F}$  is the family of all cuspidal automorphic representations, this is called the “naïve Ramanujan conjecture” and it was shown to be false even just on  $\text{Sp}_4$  in [HPS79]. The expected correction is that Ramanujan holds for the families of *generic* automorphic representations—this is the generalized Ramanujan conjecture. In the case of  $GL_n$ , cuspidal implies generic so this difference is irrelevant.

**Definition 5.4.4.** Let  $G/F$  be a reductive group over a number field. An asymptotic family  $\mathcal{F}$  for  $G$  is an indexed sequence of automorphic families  $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$  together with a “conductor” function  $m : \Lambda \rightarrow \mathbb{R}$  such that:

- Each  $\mathcal{F}_\lambda$  has finite total weight.
- The size of the support of  $\mathcal{F}_\lambda$  goes to infinity as  $m(\lambda) \rightarrow \infty$ .

**Example 5.4.5.** Let  $K'$  a compact open subgroup of  $G^\infty$ . Then the *weight-aspect family of level- $K'$  automorphic forms* on  $G$  is

$$\mathcal{F}_{K',\lambda}(\pi) = m_\pi \mathbf{1}_{\text{infchar}(\pi_\infty)=\lambda} \dim \left( (\pi^\infty)^{K'} \right)$$

indexed over the set of regular, integral infinitesimal characters  $\lambda$  of  $G_\infty$ . Its conductor function is the  $m$  from (5.3).

This models the space of automorphic forms on  $G$  of level  $K'$  as the infinitesimal character at infinity gets larger and larger.

**Example 5.4.6.** Let  $I \subseteq \infty$  contain all infinite places at which  $G$  is non-compact. Let  $K = K^\infty G_{\infty \setminus I}$  for open compact  $K^\infty \subseteq G^\infty$ . Then the connected component of the identity in  $G(F) \backslash G(\mathbb{A})/K$  is  $\Gamma \backslash G_I$  for some discrete, cofinite volume  $\Gamma$ .

Pick an invariant metric on  $G_I$  and let  $B(\delta)$  be the ball of volume  $\delta$  around the identity in  $\Gamma \backslash G_I$ . Then define the  $\delta$ -ball family in  $G_I$  by:

$$\mathcal{F}_\delta(\pi) := m_\pi \frac{\| \mathbb{P}_{\pi_I}(\mathbf{1}_{B(\delta)}) \|_2^2}{\| \mathbf{1}_{B(\delta)} \|_2^2} \mathbf{1}_{\pi_{\infty \setminus I} \text{ triv.}} \dim \left( (\pi^\infty)^{K^\infty} \right) = m_\pi \frac{1}{\| \mathbf{1}_{B(\delta)} \|_2^2} \text{tr}_\pi((f_\epsilon \star f_\epsilon^*) \bar{\mathbf{1}}_{G_{\infty \setminus I}} \bar{\mathbf{1}}_K),$$

with  $m(\delta) = 1/\delta$ .

This models the decompositions of the indicators of smaller and smaller balls in the automorphic spectrum.

**Definition 5.4.7.** We say an asymptotic family  $\mathcal{F}_\lambda$  eventually satisfies the Ramanujan conjecture if there  $L$  such that  $\mathcal{F}_\lambda$  satisfies the Ramanujan conjecture whenever  $m(\lambda) \geq L$ .

Fix place  $v$  so that  $(G_{\text{sc}})_v$  has no anisotropic factors. We say an asymptotic family  $\mathcal{F}_\lambda$  satisfies the density hypothesis at  $v$  if for all  $\sigma \geq 2$  and  $\epsilon > 0$ ,

$$\sum_{\substack{\pi \in \mathcal{AR}_{\text{disc}}(G) \\ \sigma(\pi, v) \geq \sigma}} \mathcal{F}_\lambda(\pi) \lesssim \left( \sum_{\pi \in \mathcal{AC}(G)} \mathcal{F}_\lambda(\pi) \right)^{1 - \frac{2}{\sigma}} \left( \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_\lambda(\pi) \right)^{\frac{2}{\sigma}},$$

where  $\mathcal{AC}(G)$  is the set of automorphic characters (1-dimensional automorphic representations of  $G$ ) and  $\lesssim$  is interpreted asymptotically in  $m(\lambda)$ .

Note that  $\sigma(\pi_v) = \infty$  is equivalent to  $\pi_v$  being a character which, under our conditions on  $G_v$ , is further equivalent to  $\pi$  being a character (see e.g. [KST16, Lem 6.2]). Therefore, this can be thought of as an interpolation between the case  $\sigma = 2$  and  $\sigma = \infty$ .

The automorphic density hypothesis was raised as a conjecture in [SX91, Sar90] as a possible substitute for the failure of the naïve Ramanujan conjecture. In recent years this conjecture was proven in several special instances [Blo23, Mar14, MS19, GK22, Sar15a]. Here, we will specifically be applying methods from [DGG23, DGG24].

**Example 5.4.8.** Let  $G$  be one of the  $U_n^{E,H}$ . Then weight-aspect families on  $G$  can be seen to eventually satisfy the Ramanujan conjecture through the endoscopic classification [KMSW14]: if  $m(\lambda) \gg 1$ , then formula (5.2) shows that all  $\pi$  with infinitesimal character  $\lambda$  at infinity are necessarily of shape  $\square = ((T_i, d_i))_{i=1}^k$  with all  $d_i = 1$  (i.e. they have generic parameters). Then, Theorem 5.1.3 can be used to show that they are all tempered.

**Example 5.4.9.** The density hypothesis for a slight variant of the  $\delta$ -ball family will be the key input towards proving the optimal covering property for our set of gates.



## 6 Matrix Coefficient Decay

We next need to understand how the shape of an  $A$ -parameter of an automorphic representation controls the decay of the matrix coefficients of its local components at finite places. This will require the very serious black-box inputs of Theorem 5.1.3 and results from explicit constructions of  $A$ -packets.

Fix some finite place  $v$  of quadratic extension  $E/F$  and unitary group  $G = U_n^{E/H}$ . Assume first that  $v$  is non-split. A much simpler version of this argument works for  $v$  split (see remark 6.4.3)—we only present the full details of the more complicated non-split case for brevity.

### 6.1 Exponents

Fix a minimal parabolic  $P_0$  of  $G_v$  containing minimal Levi  $M_0$ . We will consider the set of standard Levi's  $M \supseteq M_0$  and standard parabolics  $P_M \subseteq P_0$ , where  $P_M$  has Levi factor  $M$ . Given a representation  $\pi$  of such  $M$ , define  $\mathcal{I}_M^G(\pi) := \mathcal{I}_{P_M}^G(\pi)$  to be the normalized parabolic induction of [BZ77, §1.8(a)] and  $\mathcal{R}_M^G(\pi)$  to be the normalized Jacquet functor of [BZ77, §1.8(b)]; see [KT20, §3.2] for a modern summary.

Let  $\pi$  be an irreducible representation of  $G_v$ . Then by the Langlands classification ([KT20, §3.3] for a modern exposition), there is a standard Levi  $M$  of  $G_v$ , tempered irreducible representation  $\sigma$  of  $M$ , and unramified character  $\lambda$  of  $M$  such that  $\pi$  is a subrepresentation of  $\mathcal{I}_{P_M}^G(\sigma \otimes \lambda)$ .

Since  $G_v$  is unitary splitting over  $E_w$ ,  $M$  is of the form:

$$M = \text{Res}_{F_v}^{E_w} GL_{n_1} \times \cdots \times \text{Res}_{F_v}^{E_w} GL_{n_k} \times G'_v,$$

where  $G'_v$  is a smaller unitary group splitting over  $E_w$  (see e.g. [Mín11, §3.2.3]). Therefore, by the Bernstein-Zelevinsky classification ([Zel80], [LM16, §2] for a modern exposition), we can actually choose  $(M, \sigma)$  so that  $\sigma \otimes \lambda$  is of the form

$$\sigma \otimes \lambda = \text{St}(\rho_1, a_1) |\det|^{-x_1} \boxtimes \cdots \boxtimes \text{St}(\rho_k, a_k) |\det|^{-x_k} \boxtimes \pi_{\text{temp}}, \quad (6.1)$$

where  $\text{St}(\rho_i, a_i)$  are Steinberg representations built out of supercuspidals  $\rho_i$  of  $GL_{T_i} E_w$  with  $T_i a_i = n_i$ ,  $\pi_{\text{temp}}$  is a tempered representation<sup>(†)</sup> of  $G'_v$ , and  $x_1 \geq \cdots \geq x_k > 0$ . Such  $\sigma \otimes \lambda$  is unique up to permuting factors  $i$  with equal  $x_i$ 's.

Furthermore, by the endoscopic classification 5.2.2,  $\pi_{\text{temp}}$  has a tempered  $L$ -parameter:

$$\varphi_{\pi_{\text{temp}}} = \bigoplus_j \tau_j \boxtimes [b_j] \quad (6.2)$$

for  $\tau_j$  unitary supercuspidals of some  $GL_{R_j}(E_w)$ .

Following [Mœg09], we can now define some invariants of  $\pi$ :

**Definition 6.1.1.** For  $\pi$  an irreducible representation of  $G_v$  as above, define:

- (1) The Langlands data for  $\pi$  is the data  $((\rho_i, a_i, x_i)_i, \pi_{\text{temp}})$  from (6.1),
- (2) The extended supercuspidal support for  $\pi$  is the multiset produced by taking a union of

$$(\rho_i |\det|^{l-x_i}, \bar{\rho}_i^\vee |\det|^{l+x_i} : l \in \{(a_i - 1)/2, (a_i - 3)/2, \dots, (1 - a_i)/2\})$$

---

<sup>(†)</sup>we say  $\pi_{\text{temp}} = 0$  in the case when the last factor of  $M$  doesn't appear



over the  $i$  from (6.1) together with

$$(\tau_j | \det |^l : l \in \{(b_j - 1)/2, (b_j - 3)/2, \dots, (1 - b_j)/2\})$$

over the  $j$  from (6.2).

To understand matrix coefficient decay, we also need a slightly different invariant:

**Definition 6.1.2.** For  $\pi$  an irreducible representation of  $G_v$  as above, define:

- (1) the coarse exponents  $\bar{L}_\pi$  to be the list of  $x_i$  from (6.1) in non-increasing order,
- (2) the exponents  $L_\pi$  to be the list of each  $x_i$  from (6.1) repeated  $n_i$  times in non-increasing order.

Lists of exponents can be compared:

**Definition 6.1.3.** If  $L = (l_i)_{i=1}^k$  is a non-increasing list of numbers:

- (1) define

$$\sigma_i(L) := \sum_{j=1}^i x_j,$$

where for indexing purposes,  $x_j = 0$  when  $j$  is out-of-bounds.

- (2) We say that  $L_1 \succeq L_2$  if for all  $i$ ,  $\sigma_i(L_1) \geq \sigma_i(L_2)$ .

## 6.2 Exponents and Matrix Coefficient Decay

The exponents of a representation control its matrix coefficients' decay. Pick representation  $\pi$  of  $G_v$  and define the corresponding  $\lambda, \sigma, M$  as in (6.1); i.e, in terms of the Langlands data:

$$\begin{aligned} \lambda : M = \text{Res}_{F_v}^{E_w} GL_{n_1} \times \cdots \times \text{Res}_{F_v}^{E_w} GL_{n_k} \times G'_v &\rightarrow \mathbb{C} : \\ g_1 \times \cdots \times g_k \times g &\mapsto |\det g_1|^{-x_1} \cdots |\det g_k|^{-x_k}. \end{aligned}$$

For any standard Levis  $M_1 \supseteq M_2$ , let  $\delta_{M_2}^{M_1}$  be modulus character of  $P_{M_2} \cap M_1$ —i.e the choice of  $P_0$  determines a set of (absolute) positive roots  $\Phi_M^+$  for each  $M$  and we take a product:

$$\delta_{M_2}^{M_1} := \prod_{\alpha \in \Phi^+(M_1) \setminus \Phi^+(M_2)} |\alpha|.$$

This extends to a character of  $M_2$ . Note that for  $z$  in the center  $Z_{M_2}$  of  $M_2$ :

$$\delta_{M_2}^{M_1}(z) = \delta^{M_1}(z) := \prod_{\alpha \in \Phi^+(M_1)} |\alpha(z)|$$

and for  $z \in Z_{M_1}$ ,  $\delta_{M_2}^{M_1}(z) = \delta^{M_1}(z) = 1$ .

Let  $T$  be a maximal torus of the minimal standard Levi  $M_0$  and  $T^\mathcal{O}$  be the subset on which all algebraic characters take values with norm 1. For any  $M$ , define  $Z_M^\mathcal{O}$  similarly in the center of  $M$  and let  $Z_M^-$  be the set of all  $z \in Z_M$  such that  $|\alpha(z)| \leq 1$  for all  $\alpha \in \Phi^+(G) \setminus \Phi^+(M)$ .

The next two lemmas apply to arbitrary  $p$ -adic reductive groups  $G$ . Lemma 6.2.1 is the key technical idea that lets us input the fact of  $\sigma$  being tempered to tighten bounds as much as possible.

**Lemma 6.2.1.** *In the notation above, let  $L$  be another standard Levi of  $G$ . Let  $\chi$  be the central character of an irreducible subquotient of  $\mathcal{R}_L^G \mathcal{I}_M^G(\sigma \otimes \lambda)$ . Then there is Weyl element  $w$  such that  $\chi = \chi_1 \otimes (\lambda \circ w)|_{Z_L}$ , where  $|\chi_1(z)| \leq 1$  for all  $z \in Z_L^- \setminus Z_G T^\mathcal{O}$ .*

*Proof.* This follows from Bernstein's geometric lemma [BZ77, pg 448]:

Any  $\chi$  is, in the notation therein, a central character of a subquotient of some

$$F_w(\sigma \otimes \lambda) := \mathcal{I}_{L'}^L \circ w \circ \mathcal{R}_{M'}^M(\sigma \otimes \lambda),$$

with  $w$  some Weyl element and  $w(M') = L'$ . Computing,  $\mathcal{R}_{M'}^M \mathcal{I}_M^G(\sigma \otimes \lambda)$  has central characters of the form  $\chi'_1 \otimes \lambda|_{Z_{M'}}$  where  $\chi'_1$  is a central character of  $\mathcal{R}_{M'}^M \sigma$ . Further applying  $w$  produces those of the form  $\chi_1 \otimes (w \circ \lambda)|_{Z_{L'}}$  where  $\chi_1$  is a character in  $\mathcal{R}_{L'}^{wM}(w\sigma)$ . Applying the induction then gives characters of the form  $\chi_1 \otimes (w \circ \lambda) \otimes (\delta^L)^{1/2}|_{Z_L} = \chi_1 \otimes (\lambda \circ w)|_{Z_L}$ .

Finally, [Cas95, Cor 4.4.6] gives the desired property of  $\chi_1$  since  $w\sigma$  is tempered. Note that we cancel out the  $(\delta_{L'}^{wM})^{-1/2}$  in the reference by using normalized Jacquet modules.  $\square$

**Proposition 6.2.2.** *In the notation above, the matrix coefficients of  $\mathcal{I}_M^{G_v}(\sigma \otimes \lambda)$  are  $L^{p+\epsilon}$  mod center for all  $\epsilon > 0$  if for all negative dominant  $\nu \in X_*(Z_{M_0}) \setminus X_*(Z_G)$ ,*

$$|\lambda(\nu(\varpi))\delta^G(\nu(\varpi))^{1/2-1/p}| \leq 1.$$

*Proof.* The parameter  $p$  in the application of Corollary 4.4.5 in the proof of Theorem 4.4.6 in [Cas95] can be any value instead of just 2 producing a test for  $L^p$  matrix coefficients.

Recalling that our Jacquet modules are normalized, it therefore suffices to check that for any Levi  $L$ , the central characters  $\chi$  of  $\mathcal{R}_L^G \mathcal{I}_M^G(\sigma \otimes \lambda)$  satisfy that

$$|\chi(a)\delta^G(a)^{1/2-1/p}| \leq 1 \tag{6.3}$$

for all  $a \in Z_L^- \setminus Z_G T^\mathcal{O}$ . However, by lemma 6.2.1, there is a Weyl element  $w$  such that

$$|\chi(a)\delta^G(a)^{1/2-1/p}| \leq |\lambda(wa)\delta^G(a)^{1/2-1/p}| \leq |\lambda(a)\delta^G(a)^{1/2-1/p}|$$

by the ordering of the  $x_i$ . Then, since  $\lambda$  is trivial on  $Z_L^\mathcal{O}$ , we only need to check the condition (6.3) for the listed coset representatives  $\nu(\varpi)$  of  $Z_L^-/Z_L^\mathcal{O}$ , ignoring those intersecting  $Z_G$ .

The result follows from noting that  $Z_L \subseteq Z_{M_0}$ .  $\square$

We rephrase this slightly in our specific situation:

**Corollary 6.2.3.** *In the notation above,*

$$\frac{2}{\sigma(\pi)} \geq 1 - \max_{1 \leq i \leq \lfloor n/2 \rfloor} \frac{2\sigma_i(L_\pi)}{i(n-i)}.$$

*Proof.* Since  $\pi \subseteq \mathcal{I}_M^{G_v}(\sigma \otimes \lambda)$ , it suffices to check which  $p$  satisfy condition of Proposition 6.2.2. We also without loss of generality consider all negative dominant  $\nu \in X_*(A) \setminus X_*(G)$  for a maximally split torus  $A$ .

Parameterize  $A$  as diagonal matrices  $(t_1, \dots, t_n)$  for  $t_i \in E_w$  with  $t_i^{-1} = \bar{t}_{N-i}$ . When  $G_v$  isn't quasisplit,  $n$  is even and we further require  $t_{n/2} = t_{n/2+1} = 1$ . Then  $X_+(A) \setminus X_*(Z_G)$  is generated as a semigroup by the fundamental weights for  $1 \leq i \leq \lfloor n/2 \rfloor$  (resp.  $n/2 - 1$  when  $G_v$  isn't quasisplit):

$$\xi_i : F_v^\times \rightarrow T : t \mapsto (t, \dots, t, 1, \dots, 1, t^{-1}, \dots, t^{-1})$$

where the breakpoints are at indices  $i$  and  $n - i + 1$ .

By the inequality,

$$b, d > 0 \text{ and } \frac{a}{b} \leq \frac{c}{d} \implies \frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d},$$

it suffices to only check  $\nu = -\xi_i$  in 6.2.2. Then

$$\log_{q_v} |\lambda(-\xi_i(\varpi))| = 2\sigma_i(L_\pi), \quad \log_{q_v} |\delta^G(-\xi_i(\varpi))| = -2i(n-i),$$

so  $\pi$  has matrix coefficients in  $L^p$  if

$$\sigma_i(L_\pi) \leq \left(\frac{1}{2} - \frac{1}{p}\right) i(n-i) \iff \frac{2}{p} \leq 1 - \frac{2\sigma_i(L_\pi)}{i(n-i)}$$

for all  $1 \leq i \leq \lfloor n/2 \rfloor$  (without loss of generality adding in an irrelevant term when  $G_v$  isn't quasisplit). The result follows.  $\square$

### 6.3 Exponents and Parameters

Parameters also have a notion of exponents. Let tempered local parameter  $\psi_v$  decompose as a representation of  $W_{F_v} \times \mathrm{SL}_2 \times \mathrm{SL}_2$  as

$$\bigoplus_i \tau_i \boxtimes [a_i] \boxtimes [d_i],$$

where the  $\tau_i$  are unitary supercuspidals of  $GL_{T_i} E_w$ . Let  $\tau_i$  have dimension  $T_i$ .

**Definition 6.3.1.** In the notation above, the coarse exponents  $\bar{L}_{\psi_v}$  for  $\psi_v$  is the concatenation of the lists

$$\bigsqcup_i ((d_i - 1)/2, (d_i - 3)/3, \dots, (d_i - 2\lfloor d_i/2 \rfloor)/2),$$

where the result is reordered to be non-increasing.

The exponents  $L_{\psi_v}$  for  $\psi_v$  are the same except we repeat the  $i$ th list  $T_i a_i$  times:

$$\bigsqcup_i ((d_i - 1)/2, (d_i - 3)/3, \dots, (d_i - 2\lfloor d_i/2 \rfloor)/2)^{(T_i a_i)},$$

with the result ordered to be non-increasing.

**Definition 6.3.2.** In the notation above, the extended supercuspidal support of  $\psi_v$  is the multiset produced by taking a union of

$$(\tau_i | \det |^l : l \in \langle a_i \rangle \boxplus \langle b_i \rangle)$$

over all  $i$ . We use shorthand

$$\langle r \rangle := ((r-1)/2, (r-3)/2, \dots, (1-r)/2)$$

and define  $A \boxplus B$  to be the multiset  $(a+b : a \in A, b \in B)$ .

We can now state a key input of Mœglin:

**Theorem 6.3.3** ([Mœg09, Thm 7.2, Prop 4.1]). *Let  $\psi_v \in \Psi_{G_v}$  and  $\pi_v \in \Pi_{\psi_v}$ . Then:*

- (1)  $\bar{L}_{\pi_v} \preceq \bar{L}_{\psi_v}$ .
- (2) The extended supercuspidal support of  $\pi_v$  is the same as that of  $\psi_v$ .

*Proof.* For ease of the reader, the groups to which [Mœg09] applies are specified in the first paragraph of the introduction: all inner forms of unitary, symplectic, and orthogonal groups.  $\square$

This is not good enough to bound matrix coefficient decay; what we actually desire is:

**Conjecture 6.3.4.** *Let  $\psi_v \in \Psi_{G_v}$  and  $\pi_v \in \Pi_{\psi_v}$ . Then  $L_{\pi_v} \preceq L_{\psi_v}$ .*

**Remark 6.3.5.** Conjecture 6.3.4 would follow from the closure-order conjecture [Xu2405, Conj 2.1] as shown in [HLLZ25, Thm 4.11(2)].

The closure-order conjecture is known in case of symplectic and orthogonal groups by [HLLZ25]. The argument depends on algorithms computing the set of  $A$ -packets containing a representation from [Ato23] in the symplectic/orthogonal case—these are expected to analogize to the unitary case, though the details have not been completed as-of-this-writing.

The closure-order conjecture also holds for the ABV packets of [CFM<sup>+</sup>22] which are conjectured (see Conjecture 8.1 therein) to be the same as the  $A$ -packets we use from [Mok15].

However, for our applications, we only need special cases:

**Corollary 6.3.6.** *Conjecture 6.3.4 holds in the following cases:*

- (1)  $\pi_v$  is unramified,
- (2)  $n = 4$ .

*Proof.* Fix a  $\pi_v$ . If the  $N_i = T_i a_i$  in (6.1) are all 1, then  $L_{\pi_v} = \bar{L}_{\pi_v} \preceq \bar{L}_{\psi_v} \preceq L_{\psi_v}$  by 6.3.3(1) so 6.3.4 always holds. This covers case (1) and all  $\pi_v$  for case (2) except those with Langlands data of the form  $((\chi, 2, x), 0)$  for some character  $\chi$  of  $E_w^\times$ .

Then, such  $\pi_v$  has extended supercuspidal support

$$(\bar{\chi}^\vee | \det |^{1/2+x}, \bar{\chi}^\vee | \det |^{-1/2+x}, \chi | \det |^{1/2-x}, \chi | \det |^{-1/2-x})$$

By 6.3.3(2), this needs to match that of  $\psi_v$  which can only happen if  $x = 0, 1/2, 1$ .

If  $x = 0$ ,  $L_{\pi_v}$  is all 0's so we are done. If  $x = 1/2$ , then  $\chi = \bar{\chi}^\vee$  and  $\psi_v = \chi[1][3] + \chi[1][1]$  (using the natural shorthand). Therefore,  $L_{\psi_v} = (1, 0, 0)$  and  $L_{\pi_v} = (1/2, 1/2)$  which satisfies the bound. Finally,  $x = 1$  would force  $\chi = \bar{\chi}^\vee$  and  $\psi_v = \chi[1][4]$  which has  $A$ -packet containing just characters. This contradicts.  $\square$

The case  $n = 8$  will be resolved later in Corollary 7.3.4.

## 6.4 Bounding Decay by Shape

Note that for global parameter  $\psi$ , each  $L_{\psi_v}$  only depends on the restriction to the Arthur-SL<sub>2</sub>. Therefore, for any shape  $\square$  and place  $v$ ,  $L_{\psi_v}$  is constant over  $\psi \in \square$ .

**Definition 6.4.1.** For  $\square$  a shape for  $G$ , let  $L_\square$  be the common value of  $L_{\psi_v}$  for non-split, unramified  $v$  and  $\psi \in \square$ .

As a consequence of all the above work and the deep input of Theorem 5.1.3, we get our final result:

**Theorem 6.4.2.** *Let  $\square$  be a shape for  $G$  and  $\psi \in \square$  such that  $\psi_\infty$  has regular, integral infinitesimal character.*

*Then for all non-split places  $v$  and  $\pi \in \Pi_{\psi_v}$  such that conjecture 6.3.4 holds:*

$$\frac{2}{\sigma(\pi)} \geq 1 - \max_{1 \leq i \leq \lfloor n/2 \rfloor} \frac{2\sigma_i(L_\square)}{i(n-i)}.$$

*Proof.* Let  $\psi = \bigoplus_i \tau_i \boxtimes [d_i]$  with  $\tau_i$  cuspidal. Then, since  $\psi$  has infinitesimal character at infinity matching that a finite-dimensional representation, all the  $\tau_i$  also do and are therefore cohomological (see also, [NP21, Cor 4]). Therefore, by Theorem 5.1.3, the  $\tau_{i,v}$  are all tempered. Then,  $\psi_v$  decomposes as  $\bigoplus_j \sigma_j \boxtimes [a_j] \boxtimes [d_j]$  for  $\sigma_j$  unitary supercuspidal, so it is in  $\Psi_v$  (instead of the larger  $\Psi_v^+$  of [Mok15]).

The result then follows from the equality  $L_{\psi_v} = L_{\square}$ , Conjecture 6.3.4 and Corollary 6.2.3.  $\square$

**Remark 6.4.3.** When  $v$  is split, analogous notions of exponents for representations  $\pi_v$  of  $G_v$  can be defined using the Bernstein-Zelevinsky classification. The analogue of Corollary 6.2.3 then still holds. If  $\psi_v$  is a parameter of  $G_v$  then it corresponds to a irreducible representation  $\psi_0 \boxtimes \psi_0^\vee$  on  $GL_n(F_v \otimes_F E) \cong GL_n(E_w)^2$  where  $w$  lies over  $v$ . Then  $\Pi_{\psi_v}$  is a singleton containing only the representation  $\pi_{\psi}$  corresponding to  $\psi_0$  (see e.g. [DGG23, lem 6.1.1]). In particular,  $L_{\psi} = L_{\pi_{\psi}}$  so Conjecture 6.3.4 always holds. Therefore, Theorem 6.4.2 always holds as well.

Motivated by the above:

**Definition 6.4.4.** Let  $\square$  be a shape for  $G$ . Then define  $\sigma_{\square}$  by

$$\frac{2}{\sigma_{\square}} := 1 - \max_{1 \leq i \leq \lfloor n/2 \rfloor} \frac{2\sigma_i(L_{\square})}{i(n-i)}.$$

In particular, Theorem 6.4.2 gives that if  $\psi \in \square$  and  $\pi \in \Pi_{\psi_v}$ , then  $\sigma(\pi) \leq \sigma_{\square}$ .

## 7 Density Hypothesis Proof

Let  $G = U_N^{E/F,H}$  be a definite arithmetic unitary group and  $v_0$  a distinguished infinite place. Choose open compact  $K' < G^{\infty}$ . Let  $\Gamma = G(F) \cap K$  as a subgroup of  $G_{v_0}$  so that there is a map

$$\rho^{K'} : G_{v_0} \twoheadrightarrow \Gamma \backslash G_{v_0} \hookrightarrow G(F) \backslash G(\mathbb{A}) / KG_{\infty \backslash v_0}.$$

In our eventual application when  $K'$  is golden,  $\Gamma = 1$  and the second map will be an bijection.

In this section, we prove the density hypothesis for a variant of the  $\delta$ -ball family on  $G_{v_0}$ : we define functions  $f_{v_0}^{\epsilon, Z}$  on  $G_{v_0}$  that are approximately indicator functions of balls of radius  $\epsilon$  and consider families

$$\begin{aligned} \mathcal{F}_{\epsilon, Z}^{K'}(\pi) &:= m_{\pi} \frac{\|\mathbb{P}_{\pi_{v_0}}(f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2} \mathbf{1}_{\pi_{\infty \backslash v_0} \text{ triv.}} \dim \left( (\pi^{\infty})^{K'} \right) \\ &= m_{\pi} \frac{1}{\|f_{\epsilon}\|_2^2} \text{tr}_{\pi}((f_{\epsilon} \star f_{\epsilon}^*) \bar{\mathbf{1}}_{G_{\infty \backslash v_0}} \bar{\mathbf{1}}_{K'}) \end{aligned} \quad (7.1)$$

in conductor  $m(\epsilon) = 1/\epsilon$  and for various choices of  $Z$ . We can also consider  $f_{v_0}^{\epsilon, Z}$  as an element of  $L^2(G(F) \backslash G(\mathbb{A}))^{K'}$  through summing over fibers of  $\rho^{K'} : G_{v_0} \twoheadrightarrow \Gamma \backslash G_{v_0}$  to get an alternate interpretation:

$$\mathcal{F}_{\epsilon, Z}^{K'}(\pi) = m_{\pi} \frac{\|\mathbb{P}_{\pi}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2}.$$

The proof is from comparing two bounds: first in Theorem 7.2.5, we bound

$$\sum_{\pi \in \square} \mathcal{F}_{\epsilon, Z}^{K'}(\pi) = \frac{\|\mathbb{P}_{\square}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2}, \quad (7.2)$$

(recalling notation from Definition 5.2.5). This requires the very serious black-boxed input of the endoscopic classification as used in Theorem 7.1.1. We then compare this to the matrix-coefficient decay bound Theorem 6.4.2.

## 7.1 Input Bound

We now state our black-box input bound.

Fix infinitesimal character  $\lambda$  for  $G_\infty$  and let  $V_\lambda$  be the corresponding finite dimensional representation. If  $\square$  is a shape and  $K' < G^\infty$  is an open compact, define asymptotic automorphic family

$$\mathcal{F}_{\lambda, \square}^{G, K'}(\pi) := m_\pi \mathbf{1}_{\pi \in \square} \mathbf{1}_{\pi_\infty = V_\lambda} \dim \left( (\pi_\infty)^{K'} \right).$$

The paper [DGG23, §7-9] used the endoscopic classification of [KMSW14] through an inductive analysis of [Taï17] to upper bound asymptotics of the total mass of  $\mathcal{F}_{\lambda, \square}^{G, K'}$  for certain sequences of  $K' \rightarrow 1$ .

The paper [DGG24, §5-6] used the same techniques in the much simpler case of  $m(\lambda) \rightarrow \infty$ . We recall the result:

**Theorem 7.1.1** (Special case of [DGG24, Thm 6.5.1]). *In the notation above,*

$$\sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_{\lambda, \square}^{G, K'}(\pi) \leq (\dim_\square \lambda) (\Lambda(G, \square, K') + O_{G, \square, K'}(m(\lambda)^{-1}))$$

for some constant  $\Lambda(G, \square, K')$  depending only on the three arguments.

*Proof.* Recall the definitions of  $I_\square^G(\text{EP}_\lambda \bar{\mathbf{1}}_{K'})$  and  $S_\square^{G*}(\text{EP}_\lambda \bar{\mathbf{1}}_{K'})$  from [DGG24, §6.1].

Since  $V_\lambda$  is the only representation of compact  $G_\infty$  with infinitesimal character  $\lambda$ ,

$$\text{tr}_\pi(\text{EP}_\lambda \bar{\mathbf{1}}_{K'}) = \mathbf{1}_{\pi_\infty = V_\lambda} \dim \left( (\pi_\infty)^{K'} \right).$$

Therefore,

$$\sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_{\lambda, \square}^{G, K'}(\pi) = I_\square^G(\text{EP}_\lambda \bar{\mathbf{1}}_{K'}).$$

The result then follows from the second bound of [DGG24, Thm 6.5.1].

For the reader's convenience, we very roughly sketch the argument of [DGG24, Thm 6.5.1]. By an implementation in [DGG23] of an inductive strategy from [Taï17] inputting the endoscopic classification [KMSW14, Mok15], our count of representations can be upper bounded by a linear combination of terms on the geometric side of Arthur's discrete-at- $\infty$  trace formula from [Art89].

All these [Art89] terms are sums of smaller terms of the form

$$C_\gamma \Phi_{\lambda'}^H(\gamma),$$

where  $H$  ranges over groups that are  $G_F(\square)$  or smaller, the  $\Phi_{\lambda'}^H$  are certain character sums on maximal tori related to traces against the finite dimensional representation on  $H$  with infinitesimal character  $\lambda'$  derived from  $\lambda$ ,  $\gamma$  ranges over some fixed finite set of rational conjugacy classes of  $H$  depending only on  $H, \square$  and  $K'$ , and  $C_\gamma$  are some inexplicit constants that nevertheless depend only on  $H, \square, K'$ , and  $\gamma$ .

Finally, we apply the analysis of [ST16] to these terms—in particular [ST16, Lem 6.10(ii)] bounds the  $\Phi_{\lambda'}^H$ , thereby showing that a term for  $\gamma = 1$  on  $G_F(\square)$  itself dominates. For this term specifically,  $\Phi_\lambda^H(\gamma) = \dim_\square \lambda$  and  $C_\gamma$  can be made more precise.  $\square$

**Remark 7.1.2.** We will in fact only need that

$$(\dim_{\square} \lambda)^{-1} \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_{\lambda, \square}^{G, K'}(\pi) = (\dim_{\square} \lambda)^{-1} I_{\square}^G(\text{EP}_{\lambda} \bar{\mathbf{1}}_{K'})$$

is bounded by a constant independent of  $\lambda$ .

## 7.2 Indicators of Balls

We now define the functions  $f_{v_0}^{\epsilon, Z}$  defining our variant of the  $f$ -ball family and bound (7.2). Our main technical tool here is Kirilov's orbit-method character formula; see [Ros78, Ver79] for the full proof and [Kir04, Ch 5] for a textbook summary. Our  $f_{v_0}^{\epsilon, Z}$  are close to but not exactly indicator functions, instead chosen specifically to simplify the orbit-method computations.

### 7.2.1 Modified Indicator Functions

First, consider the case of  $H$  a compact, semisimple, and simply connected Lie group. Let  $\mathfrak{h}$  be the real Lie algebra for  $H$ ,  $\dim \mathfrak{h} = N$ , and  $\text{rank } H = r$ . Define on  $\mathfrak{h}$ :

$$j(X) := \det \left( \frac{\sinh(\text{ad } X/2)}{\text{ad } X/2} \right),$$

Consider test functions

$$f^{\epsilon} \circ \exp := \mathbf{1}_{B_{\epsilon}(0)} j^{1/2}$$

on  $H$  and where balls are defined using the Killing form. Note that  $f^{\epsilon}$  is supported on the ball  $\exp(B_{\epsilon}(0))$ , is analytic, and takes values close to 1 for small enough  $\epsilon$ .

We use the Kirilov character formula to compute traces of  $f^{\epsilon}$  against the finite dimensional representation  $V_{\lambda}$ . If  $\mathfrak{t}$  is a Cartan for  $\mathfrak{h}$ , the Killing form gives an embedding  $\mathfrak{t}^* \hookrightarrow \mathfrak{h}^*$  so  $i\lambda$  can be interpreted as a point in  $i\mathfrak{h}^*$ . To define a Fourier transform, pick a measure on  $\mathfrak{h}$  that is Plancherel self-dual through the Killing form isomorphisms  $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$  and associate  $x \in \mathfrak{h}^*$  to the multiplicative character  $e^{2\pi i x(\cdot)}$  on  $\mathfrak{h}$ . Then for small enough  $\epsilon$ :

$$\text{tr}_{\pi_{\lambda}}(f^{\epsilon}) = \int_{\mathcal{O}_{\lambda/(2\pi i)}} \widehat{\mathbf{1}_{B_{\epsilon}(0)}} d\omega, \quad (7.3)$$

where the coadjoint orbit  $\mathcal{O}_{\lambda/(2\pi i)} \subseteq i\mathfrak{h}^*$  is given its canonical measure as an integral symplectic manifold with total volume  $\dim \lambda$ .

By a classical result:

$$\widehat{\mathbf{1}_{B_{\epsilon}(0)}}(\xi/(2\pi)) = \epsilon^N \|\epsilon \xi\|^{-N/2} J_{N/2}(\|\epsilon \xi\|) = \epsilon^{N/2} \|\xi\|^{-N/2} J_{N/2}(\epsilon \|\xi\|),$$

where  $J_{N/2}$  is the classical Bessel function of the first kind. Since the adjoint action preserves the Killing form, the integral in (7.3) is constant so:

**Lemma 7.2.1.** *For  $H$  a compact, semisimple, and simply connected Lie group and in the above notation:*

$$\text{tr}_{V_{\lambda}}(f^{\epsilon}) = (\dim \lambda) \epsilon^{N/2} \|\lambda\|^{-N/2} J_{N/2}(\epsilon \|\lambda\|).$$

for small enough  $\epsilon$ .

**Remark 7.2.2.** We can understand the factors in lemma 7.2.1 through lemma 5.3.4,

$$\dim \lambda \leq \|\lambda\|^{P_H},$$

and the Bessel function asymptotics:

$$|J_{N/2}(x)| \leq \begin{cases} Cx^{N/2} & x \ll \sqrt{N/2 + 1} \\ Cx^{-1/2} & x \gg \sqrt{N/2 + 1} \end{cases}.$$

In particular, this term should be thought of as order

$$\mathrm{tr}_{V_\lambda}(f_\infty^\epsilon) = \begin{cases} O(\epsilon^N \|\lambda\|^{1/2(N-r)}) & \|\lambda\| \ll \epsilon^{-1} \\ O(\epsilon^{(N-1)/2} \|\lambda\|^{-1/2(r+1)}) & \|\lambda\| \gg \epsilon^{-1} \end{cases}.$$

We can generalize this to our  $G_{v_0}$  that is compact and (topologically) connected. Then  $G_{v_0} = G_{v_0}^{\mathrm{der}} \times Z_{G_{v_0}}/Z_{G_{v_0}^{\mathrm{der}}}$  on points and we have a corresponding canonical factorization on Lie algebras  $\mathfrak{g} = \mathfrak{g}^{\mathrm{der}} \times \mathfrak{z}$ .

Then for any small enough  $\epsilon$  and subset  $Z \subseteq \mathfrak{z}$  on which  $\exp$  to  $G_{v_0}$  is injective, define

$$f_{v_0}^{\epsilon, Z} \circ \exp := j^{1/2} \mathbf{1}_{B_\epsilon(0) \times Z}. \quad (7.4)$$

The Kirillov character formula a priori computes the trace character of  $V_\lambda$  pulled back to  $(G_{v_0}^{\mathrm{der}})_{\mathrm{sc}}$ . However, for small enough  $\epsilon$ , this is the same as its trace against  $G_{v_0}^{\mathrm{der}}$ .

We can in addition integrate over  $Z_{G_{v_0}}$  to compute traces of the pullback to  $G_{v_0}^{\mathrm{der}} \times Z_{G_{v_0}}$ , noting that  $V_\lambda$  has central character  $\lambda|_{\mathfrak{z}}$  on  $\mathfrak{z}$ . Since the diagonal embedding of  $Z_{G_{v_0}^{\mathrm{der}}}$  intersects  $\exp(B_\epsilon(0) \times Z) \subseteq G_{v_0}^{\mathrm{der}} \times Z_{G_{v_0}}$  trivially for small enough  $\epsilon$ , this is the same as computing traces on  $G_{v_0}$ .

In total:

**Lemma 7.2.3.** *In our case where  $G_{v_0}$  is compact and (topologically) connected:*

$$\mathrm{tr}_{V_\lambda}(f_{v_0}^{\epsilon, Z}) = (\dim \lambda) \epsilon^{N^{\mathrm{der}}/2} \|\lambda\|^{-N^{\mathrm{der}}/2} J_{N^{\mathrm{der}}/2}(\epsilon \|\lambda\|) \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}})$$

for small enough  $\epsilon$  and  $Z \subseteq \mathfrak{z}$  on which  $\exp$  to  $G_{v_0}$  is injective. Here,  $\mathfrak{z}$  is given the measure that corresponds to unit Haar measure on  $Z_{G_{v_0}}$  and recall that  $N^{\mathrm{der}} = \dim \mathfrak{g}^{\mathrm{der}}$ .

We also need to understand traces against  $(f_{v_0}^{\epsilon, Z})^* \star f_{v_0}^{\epsilon, Z}$ . By Theorem 10 on page 174 of [Kir04],

$$(f_{v_0}^{\epsilon, Z})^* \star_{G_{v_0}} f_{v_0}^{\epsilon, Z} = (\mathbf{1}_{B_\epsilon(0) \times (-Z)} \star_{\mathfrak{g}^{\mathrm{der}} \times \mathfrak{z}} \mathbf{1}_{B_\epsilon(0) \times Z}) j^{1/2},$$

so using that abelian Fourier transform takes convolution to product, a similar computation gives:

**Lemma 7.2.4.** *In our case where  $G_{v_0}$  is compact and (topologically) connected:*

$$\mathrm{tr}_{V_\lambda}((f_{v_0}^{\epsilon, Z})^* \star f_{v_0}^{\epsilon, Z}) = (\dim \lambda) \epsilon^{N^{\mathrm{der}}} \|\lambda\|^{-N^{\mathrm{der}}} J_{N^{\mathrm{der}}/2}(\epsilon \|\lambda\|)^2 \left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2$$

for small enough  $\epsilon$  and  $Z \subseteq \mathfrak{z}$  on which  $\exp$  to  $G_{v_0}$ . Here,  $\mathfrak{z}$  is given the measure that corresponds to unit Haar measure on  $Z_{G_{v_0}}$  and recall that  $N = \dim \mathfrak{g}^{\mathrm{der}}$ .

Finally, note that pulling back to the Lie algebra gives

$$\|f_{v_0}^{\epsilon, Z}\|_{G_{v_0}}^2 = \|\mathbf{1}_{B_\epsilon(0) \times Z}\|_{\mathfrak{g}}^2 = \mathrm{vol}(Z) \frac{\pi^{N^{\mathrm{der}}/2}}{\Gamma(N^{\mathrm{der}}/2 + 1)} \epsilon^{N^{\mathrm{der}}}. \quad (7.5)$$



### 7.2.2 Projection Bounds

With the above Kirilov formula computation, we can now input Theorem 7.1.1 and bound  $\|\mathbb{P}_{L_\square} f_{v_0}^{\epsilon, Z}\|_2^2$ . We will consider two possible  $Z$ : either  $Z_\epsilon := (-\epsilon/2, \epsilon/2)$  or  $Z_1 := (-1/2, 1/2)$ . Here,  $\mathfrak{z}$  is parameterized so that Lebesgue measure matches unit Haar measure: i.e. intervals of length 1 exactly cover  $Z_{G_{v_0}} = U_1$ .

First, since  $G_{v_0}$  is compact, we can choose the function  $\text{EP}_\lambda$  to be the matrix coefficient of the finite dimensional representation  $V_\lambda$  with infinitesimal character  $\lambda$ . In particular, by Peter-Weyl, any  $f_{v_0}$  always has the same orbital integrals as a function of the form

$$\sum_{\lambda} a_{\lambda} \text{EP}_{\lambda}.$$

Since by definition  $\text{tr}_{V_{\mu}} \text{EP}_{\lambda} = \mathbf{1}_{\mu=\lambda}$  for any two finite-dimensional reps  $V_{\mu}$  and  $V_{\lambda}$ , comparing traces solves for the coefficients and gives:

$$I(f_{v_0}) = I\left(\sum_{\lambda} (\text{tr}_{V_{\lambda}} f_{v_0}) \text{EP}_{\lambda}\right)$$

for any invariant distribution  $I$ .

In addition, the Plancherel formula gives

$$\|\mathbb{P}_{\square} f_{v_0}\|_2^2 = \text{tr}_{L_{\square}^2}(f_{v_0}^* \star f_{v_0}) = I_{\square}^G((f_{v_0}^* \star f_{v_0}) \bar{\mathbf{1}}_{G_{\infty \setminus v_0}} \bar{\mathbf{1}}_{K'}).$$

Therefore:

$$\begin{aligned} & \|\mathbb{P}_{\square}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2 \\ &= I_{\square}^G(((f_{v_0}^{\epsilon, Z})^* \star f_{v_0}^{\epsilon, Z}) \bar{\mathbf{1}}_{G_{\infty \setminus v_0}} \bar{\mathbf{1}}_{K'}) \\ &= \sum_{\lambda \in \square} \text{tr}_{V_{\lambda}}((f_{v_0}^{\epsilon, Z})^* \star f_{v_0}^{\epsilon, Z}) I_{\square}^G(\text{EP}_{\lambda} \bar{\mathbf{1}}_{G_{\infty \setminus v_0}} \bar{\mathbf{1}}_{K'}) \\ &\stackrel{7.2.4}{=} \sum_{\lambda \in \square} (\dim \lambda) \epsilon^{N^{\text{der}}} \|\lambda\|^{-N^{\text{der}}} J_{N^{\text{der}}/2}(\epsilon \|\lambda\|)^2 \left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2 I_{\square}^G(\text{EP}_{\lambda} \bar{\mathbf{1}}_{G_{\infty \setminus v_0}} \bar{\mathbf{1}}_{K'}) \\ &\stackrel{7.1.1}{\leq} \epsilon^{N^{\text{der}}} \sum_{\lambda \in \square} (\dim \lambda) \|\lambda\|^{-N^{\text{der}}} J_{N^{\text{der}}/2}(\epsilon \|\lambda\|)^2 \left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2 \dim_{\square}(\lambda) (\Lambda + O(m(\lambda)^{-1})) \\ &\leq C \epsilon^{N^{\text{der}}} \sum_{\lambda \in \square} (\dim \lambda) \|\lambda\|^{-N^{\text{der}}} J_{N^{\text{der}}/2}(\epsilon \|\lambda\|)^2 \left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2 \dim_{\square}(\lambda) \\ &\leq C \epsilon^{N_G^{\text{der}}} \sum_{\lambda \in \square} \|\lambda\|^{-N_G^{\text{der}} + P_G + P_{G_F(\square)} - e(\square)} J_{N_G^{\text{der}}/2}(\epsilon \|\lambda\|)^2 \left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2, \end{aligned} \tag{7.6}$$

for some constant  $C$  depending only on  $G$ ,  $\square$ , and  $K'$  and where the last step uses lemma 5.3.4 and Corollary 5.3.6. Recall the convention  $\lambda \in \square$  to mean that  $\lambda$  is a possible total infinitesimal character for a parameter of shape  $\square$  and also recall formula (5.4) defining  $e(\square)$ .

Consider first the case  $Z = [-\epsilon/2, \epsilon/2]$ . Note that  $\widehat{\mathbf{1}}_Z$  is zero on any character that sends  $\lambda(-1) = -1$ , so the lattice of possible  $\lambda$  is of the form  $L_G \times L_Z$  where  $L_Z$  is a character of  $U_1/\pm 1$  and  $L_G$  are regular, integral infinitesimal characters for  $(G_{v_0})_{\text{ad}}$ . Summing over  $L_Z$  using Poisson summation on  $\mathfrak{z}$  turns the  $\left| \widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}}) \right|^2$  into  $\mathbf{1}_Z \star (\mathbf{1}_Z)^*(0) = \epsilon$  as long as  $\epsilon$  is small enough. Therefore, our estimate (7.6) becomes:

$$C \epsilon^{N_G} \sum_{\substack{\lambda \in \square \\ \lambda \text{ for } (G_{v_0})_{\text{ad}}}} \|\lambda\|^{-N_G^{\text{der}} + P_G + P_{G_F(\square)} - e(\square)} J_{N_G^{\text{der}}/2}(\epsilon \|\lambda\|)^2. \tag{7.7}$$

In the other case  $Z = [-1/2, 1/2]$ , note that  $\widehat{\mathbf{1}}_Z(\lambda|_{\mathfrak{z}})$  is an indicator function if  $\lambda|_{\mathfrak{z}} = 1$ , so this simply changes the  $\epsilon^{N_G}$  coefficient on the sum back into  $\epsilon^{N_G^{\text{der}}}$ .

Now, we input the asymptotics for  $J_{N/2}$  to evaluate the sum. For  $\|\lambda\| \ll 1/\epsilon$ , the terms in the sum are

$$\ll \epsilon^{N_G^{\text{der}}} \|\lambda\|^{P_G + P_{G_F(\square)} - e(\square)}.$$

The  $\lambda \in \square$  that are integral on  $(G_{v_0})_{\text{ad}}$  form an  $(r_{G_F(\square)} - 1)$ -dimensional sublattice of all  $\lambda$  (shifted by a small fixed vector depending only on  $\square$  that becomes negligible as  $1/\epsilon \rightarrow \infty$ ). Assume  $r_{G_F(\square)} \geq 2$ . Then, summing over the  $1/\epsilon$ -ball in this subspace gives something of order

$$\begin{aligned} &\ll \epsilon^{N_G^{\text{der}} - P_G - P_{G_F(\square)} + e(\square) - r_{G_F(\square)} + 1} \\ &= \epsilon^{N_G - P_G - P_{G_F(\square)} - r_{G_F(\square)} + e(\square)} = \epsilon^{(N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}, \end{aligned}$$

after approximating the sum by an integral, which we can do since  $r_{G_F(\square)} \geq 2$  means the number of terms in the sum with  $\|\lambda\| \ll 1/\epsilon$  goes to infinity as  $\epsilon \rightarrow 0$ .

On the other side,  $\|\lambda\| \ll 1/\epsilon$ , the terms in the sum are

$$\ll \epsilon^{-1} \|\lambda\|^{-N_G^{\text{der}} + P_G + P_{G_F(\square)} - e(\square) - 1},$$

which after summing gives something of the same order

$$\ll \epsilon^{N_G^{\text{der}} - P_G - P_{G_F(\square)} - r_{G_F(\square)} + e(\square) + 1} = \epsilon^{(N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

We substitute this into the sum from (7.7), first in the case of  $Z = (-\epsilon/2, \epsilon/2)$ . Then, when  $r_{G_F(\square)} \geq 2$ :

$$\|\mathbb{P}_{\square}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2 \ll \epsilon^{N_G + (N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

When  $r_{G_F(\square)} = 1$ , there is a single  $\lambda \in \square$ . We can therefore treat  $\|\lambda\|$  as a constant and reproduce the same formula (note that for this shape  $e(\square) = P_G$ ).

The case of  $Z_1$  just removes a power of  $\epsilon$ . Unifying the two cases by noting that  $\|f_{v_0}^{\epsilon, Z_{\epsilon}}\|_2^2 \asymp \epsilon^{N_G}$  and  $\|f_{v_0}^{\epsilon, Z_1}\|_2^2 \asymp \epsilon^{N_G - 1}$ , we get:

**Theorem 7.2.5.** *Normalize  $\mathfrak{z}$  so that intervals of length 1 exactly cover  $Z_{G_{v_0}} = U_1$ . Then, if  $Z$  is either  $(-\epsilon/2, \epsilon/2)$  or  $(-1/2, 1/2)$ :*

$$\frac{\|\mathbb{P}_{\square}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2} \ll \epsilon^{(N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square)}.$$

As two special cases, we get  $\epsilon^0$  when  $\square = (n, 1)$  is the trivial shape and  $\epsilon^{N_G - 1}$  when  $\square = (1, n)$  is the shape for 1-d representations.

### 7.3 Density Hypothesis Proof

Now we can put together Theorems 7.2.5 and 6.4.2 to prove the density hypothesis.

First,

$$\sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \mathcal{F}_{\epsilon, Z}(\pi) = \frac{\|f_{v_0}^{\epsilon, Z}\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2} = 1.$$

The automorphic characters restricted to  $\Gamma \backslash G_{v_0}$  span  $L^2(\Gamma \backslash G_{v_0} / G_{v_0}^{\text{der}})$ , so we can calculate projections:

$$\mathbb{P}_{\mathcal{AC}(G)} f(x) = \int_{g \in G_{v_0}^{\text{der}}} f(xg) dg.$$

This allows us to compute

$$\sum_{\pi \in \mathcal{AC}(G)} \mathcal{F}_{\epsilon, Z}(\pi) = \frac{\|\mathbb{P}_{\mathcal{AC}(G)}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z}\|_2^2} \asymp \text{vol}(Z) \epsilon^{N_G - 1}.$$

Since there are finitely many possible  $\square$  for each  $N$ , Theorem 6.4.2 gives that the density hypothesis for  $\mathcal{F}_{\epsilon, Z_1}$  at some finite, non-split place  $v$  would be implied by:

**Theorem 7.3.1.**

$$\sum_{\pi \in \square} \mathcal{F}_{\epsilon, Z_1}(\pi) = \frac{\|\mathbb{P}_{\square}(\rho_*^{K'} f_{v_0}^{\epsilon, Z})\|_2^2}{\|f_{v_0}^{\epsilon, Z_1}\|_2^2} \ll \epsilon^{(N_G - 1)(1 - \frac{2}{\sigma_{\square}})}.$$

*Proof.* The inequality we want to show is that

$$R_G(\square) := (N_G - P_G) - (N_{G_F(\square)} - P_{G_F(\square)}) + e(\square) \geq (N_G - 1) \left(1 - \frac{2}{\sigma_{\square}}\right) =: S_G(\square). \quad (7.8)$$

Define the Arthur-SL<sub>2</sub> of a shape  $\square$  to be the partition  $Q$  determined by the restriction of  $\psi \in \square$  to the Arthur-SL<sub>2</sub>. Note that the left-hand side of the inequality only depends on the Arthur-SL<sub>2</sub> so call it  $S_G(Q)$ . Let  $R_G(Q)$  be the minimum of  $R_G(\square)$  over  $\square$  with Arthur-SL<sub>2</sub> given by  $Q$ —this is achieved for the unique such  $\square = ((T_i, d_i))$  with all  $d_i$  distinct.

It therefore suffices to show that  $R_G(Q) \geq S_G(Q)$  for all  $Q$ . Let  $d$  be the maximum size of a part of  $Q$ . Then

$$R_G(Q) \geq R_G(d, 1, \dots, 1) = \frac{n(n+1)}{2} - 1 - \frac{(n-d)(n-d+1)}{2} + \frac{d(d-1)}{2} = nd - 1.$$

Recall the definitions of  $Q_d$  and  $Q'_d$  from [DGG23, lem 12.4.3], which also gives that if we also have that  $Q \neq Q_d$ , then

$$S_G(Q) \leq S_G(Q'_d) = (n^2 - 1) \frac{d-1}{n - \lfloor n/d \rfloor + 1}$$

using [DGG23, 12.4.4] for the equality. By a computer check, this always gives  $R_G(Q) \geq S_G(Q)$ .

It remains to check the case when  $Q = Q_d$ . Let  $r = \lfloor n/d \rfloor$  and  $q = n - rd$ . Then

$$R_G(Q_d) = \frac{n(n+1)}{2} - \frac{r(r+1)}{2} - \mathbf{1}_{q \neq 0} + \frac{rd(d-1)}{2} + \frac{q(q-1)}{2}$$

and by [DGG23, 12.4.4],

$$S_G(Q_d) = (n^2 - 1) \frac{d-1}{n - \lfloor n/d \rfloor}.$$

By computer check again, we always have that  $R_G(Q_d) \geq S_G(Q_d)$ . □

Summarizing the final result:

**Corollary 7.3.2.** *Let  $G = U_n^{E, H}$  be a definite unitary group and  $K' < G^\infty$  be open compact. Pick infinite place  $v_0$  and define  $f_{v_0}^{\epsilon, Z_1}$  as in (7.4). Then the family*

$$\mathcal{F}_{\epsilon, Z_1}^{K'}(\pi) := m_\pi \frac{\|\mathbb{P}_{\pi_{v_0}}(f_{v_0}^{\epsilon, Z_1})\|_2^2}{\|f_{v_0}^{\epsilon, Z_1}\|_2^2} \mathbf{1}_{\pi_{\infty \setminus v_0} \text{ triv. dim}} \left( (\pi^\infty)^{K'} \right)$$

*satisfies the density hypothesis 5.4.7 at finite place  $v$  in the following cases:*

- $n = 4$ ,
- $K'_v$  is hyperspecial,
- $v$  is split,
- Conjecture 6.3.4 holds for Arthur-type representations of  $G_v$  with a  $K'_v$ -fixed vector.

*Proof.* This follows from Theorem 7.3.1 together with Theorem 6.4.2, remark 6.4.3 and Corollary 6.3.6.  $\square$

**Remark 7.3.3.** Note that the  $\epsilon$  safety factor in the exponents in the density hypothesis can be removed in Corollary 7.3.2—i.e, the  $\lesssim$  of Definition 5.4.7 can be improved to an  $\ll$ .

Furthermore, the inequality (7.8) is only tight when  $\square$  corresponds to tempered representations or characters. Therefore, for  $\sigma \neq 2, \infty$ , we can tighten the exponents on the right-hand side of 7.3.1 by some small constant depending on  $N_G$ .

**Corollary 7.3.4.** *Corollary 7.3.2 holds when  $n = 8$*

*Proof.* For every restriction of a parameter  $\psi_v$  to the Arthur and Deligne- $\mathrm{SL}_2$ 's, we by computer list out all the possible exponents of  $\pi_v$  satisfying the conditions of Theorem 6.3.3. Bounding these potential  $\sigma(\pi_v)$  by 6.2.3, the only cases of Langlands data that violate the bound in Theorem 7.3.1 with  $\sigma_\square$  replaced by  $\sigma(\pi_v)$  are:

$$\pi_v \subseteq [3] \cdot ||^{-1} \rtimes \pi_{\text{temp}} \text{ in packet } \psi_v = [5][1] + [1][3], \quad (7.9)$$

$$\pi_v \subseteq [2] \cdot ||^{-1} \rtimes [2] \cdot ||^{-1} \rtimes 0 \text{ in packet } \psi_v = [4][1] + [1][4]. \quad (7.10)$$

in the natural shorthand describing exponents and Arthur/Deligne- $\mathrm{SL}_2$ -pieces. We show that both these cannot occur.

For the case (7.9), the infinitesimal character of  $\pi_v$  always has a factor of the form  $\rho||^0$  or  $\rho||^{1/2}$  coming from the choice of 2-dimensional  $\pi_{\text{temp}}$ . The  $\rho||^{1/2}$  cannot occur since the infinitesimal character of the packet has only integral powers of  $||$  and the  $\rho||^0$  cannot occur because the two zero powers of  $||$  in the infinitesimal character of the packet  $\psi_v$  are already accounted for by the  $[3] \cdot ||^{-1}$ .

The case (7.10) cannot occur since it violates [Mœg09, Thm 6.3]—it corresponds to partition  $(2, 2, 2, 2)$  while the packet restricted to the Deligne- $\mathrm{SL}_2$  corresponds to partition  $(4, 1, 1, 1, 1)$ .  $\square$

## 8 Optimal covering

Here we translate the spectral analysis of previously constructed gate sets in  $PU(n)$  to the settings of automorphic representation theory and show that the density Theorem 7.3.2 implies the optimal covering property. Theorem 1.2.2 from the introduction will then follow from the main result 8.2.3 of this section and Theorem 1.2.3 from the main result combined with Proposition 4.5.3. Note that while Definition 1.2.1 was stated just for Lie groups for simplicity, it also applies to Lie groups mod discrete lattices.

We make a technical assumption that  $\mathcal{O}_F$  is Euclidean for the approximation property from 4.4.1 to hold.

**Theorem 8.0.1.** *Let  $G = U_N^{E/F, H}$  be a definite unitary group and choose distinguished Archimedean place  $v_0$  of  $F$ . Let  $K^{v_0} = K'G_{\infty \setminus v_0}$  for  $K' \leq G(\hat{\mathbb{Z}})$  an almost golden adelic group that is almost golden at  $\mathfrak{p}$  (resp. almost  $\tau$ -super-golden for  $\tau$  traversable).*

Recall the definition of the gate set  $S_{\mathfrak{p}}$  from 4.2.1 (resp.  ${}^0S_{\mathfrak{p}}$  and  $C_{\mathfrak{p}}$  from 4.2.4 and the discussion afterwards). Then  $S_{\mathfrak{p}} \cup \Gamma$  is a golden gate set (resp. the finite subgroup  $C_{\mathfrak{p}}$  and finite-order elements  ${}^0S_{\mathfrak{p}}$  form a super-golden gate set) of  $G(F_{v_0})/U_1 = PU(n)$ .

For convenience, assume  $F = \mathbb{Q}$  on the first read so  $v_0$  is the sole infinite place and  $K' \leq G(\widehat{\mathbb{Z}})$ . Also assume on a first read that  $K'$  is golden instead of just almost golden so that  $\Gamma = 1$ .

## 8.1 A Hecke Operator

We start by interpreting as a Hecke operator the operation of averaging over translates by the set  $S_{\mathfrak{p}}^{[\ell]}$  approximately of words in gates with minimum representation of length  $\ell$  (recall all precise definitions from Proposition 3.4.2).

First, consider  $K'$  an almost golden adelic group and define  $\Gamma, \Lambda_{\mathfrak{p}}$  as in §4.1. Then we get the following identifications by lemma 4.1.2:

$$\begin{aligned} L^2(\Gamma \backslash U(n)) &\cong L^2(\Lambda_{\mathfrak{p}} \backslash G(F_{v_0}) \times G(F_{\mathfrak{p}}))^{K'_{\mathfrak{p}}} \\ &\cong L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}^{\infty}) \times G(F_{v_0}))^{K'G_{\infty \backslash v_0}} \cong L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))^{K'G_{\infty \backslash v_0}}. \end{aligned} \quad (8.1)$$

In particular, we can decompose

$$V = L^2(\Gamma \backslash U(n)) = \bigoplus_{\substack{\pi \in \mathcal{AR}_{\text{disc}}(G) \\ \pi_{\infty \backslash v_0} \text{ trivial}}} \pi_{v_0} \boxtimes (\pi^{\infty})^{K'},$$

where the right-translation action corresponds to the action on the left factor of the  $\boxtimes$  as representations of  $G(F_{v_0}) = U(n)$ . We also get corresponding subspaces  $V_{\square}$  and restricted projections  $\mathbb{P}_{\square} : V \rightarrow V_{\square}$ .

Through the right factor of the  $\boxtimes$ , this decomposition also respects an action of Hecke operators:

**Definition 8.1.1.** Let  $\mathfrak{p}$  be a finite place of  $F$ ,  $\pi_{\mathfrak{p}}$  a  $G_{\mathfrak{p}}$ -representation. Then any finite set  $S \subset G_{\mathfrak{p}}$  defines a Hecke operator  $\mathbb{1}_{K'_{\mathfrak{p}}SK'_{\mathfrak{p}}} \in C_c(K'_{\mathfrak{p}} \backslash G_{\mathfrak{p}}/K'_{\mathfrak{p}})$ , which acts on  $\pi_{\mathfrak{p}}^{K'_{\mathfrak{p}}}$ :

$$\mathbb{1}_{K'_{\mathfrak{p}}SK'_{\mathfrak{p}}}.u := \int_{g \in K'_{\mathfrak{p}}SK'_{\mathfrak{p}}} \pi_v(g).u \, dg = \sum_{s \in K'_{\mathfrak{p}}SK'_{\mathfrak{p}}/K'_{\mathfrak{p}}} \pi_v(s).u,$$

where the integral is normalized by  $\text{vol}(K'_{\mathfrak{p}}) = 1$ .

Recall the definition of  $S_{\mathfrak{p}}^{[\ell]}$  from Proposition 3.4.2: when  $K'$  is golden at  $\mathfrak{p}$ , this is simply the set of words in  $S_{\mathfrak{p}}$  of length precisely  $\ell$  in their shortest representation. As a technicality when  $G_{\mathfrak{p}}$  has non-anisotropic center, recall also from Proposition 3.4.2 their lifts  $\widetilde{S}_{\mathfrak{p}}^{[\ell]}$  and define the following operator  $T_{S_{\mathfrak{p}}^{[\ell]}}$ :

$$\left(T_{S_{\mathfrak{p}}^{[\ell]}} f\right)(g) = \frac{1}{|S_{\mathfrak{p}}^{[\ell]}|} \sum_{s \in \widetilde{S}_{\mathfrak{p}}^{[\ell]}} f(s^{-1}g) \quad (f \in L^2(U(n)), g \in U(n)).$$

This can be interpreted as a Hecke operator on  $V = L^2(\Gamma \backslash U(n))$ :

**Lemma 8.1.2.** For any  $\ell \in \mathbb{N}$ :

(1) the inclusion  $\widetilde{S}_{\mathfrak{p}}^{[\ell]}/\Gamma \hookrightarrow K'_{\mathfrak{p}}\widetilde{S}_{\mathfrak{p}}^{[\ell]}K'_{\mathfrak{p}}/K'_{\mathfrak{p}}$  is a bijection,

(2) The operator  $T_{S_p^{[\ell]}}$  descends to  $L^2(\Gamma \backslash U(n))$  where it is equal to the normalized Hecke operator  $|\tilde{S}_p^{[\ell]}|^{-1} \mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p}$  acting on the  $G_p$  component through the isomorphisms (8.1).

*Proof.* First, (2) follows from (1): for descent, (1) in particular gives that  $\Gamma \tilde{S}_p^{[\ell]} = \tilde{S}_p^{[\ell]}$  so  $(T_{S_p^{[\ell]}} f)(g) = (T_{\tilde{S}_p^{[\ell]}} f)(\gamma^{-1}g)$  for all  $\gamma \in \Gamma$ . For the comparison to a Hecke operator, in the identification  $L^2(U(n)) = L^2(\Lambda_p \backslash U(n) \times G_p)^{K'_p}$ , the  $s$ -action on the  $G_p$  coordinate is equivalent to an  $s^{-1}$  action on the left on the  $U(n)$  coordinate. Note that each argument  $s^{-1}g \in \Gamma \backslash U(n)$  appears  $|\Gamma|$  times in the sum defining  $T$ .

Next, we prove (1). By Proposition 3.4.2 as used in Section 4.2,  $\tilde{S}_p^{[\ell]}.v_0$  is the set of all  $v \in \mathcal{B}$  (or lifts  $v$  of depending on  $v - v_0 \in X_+(\hat{A}_H)$  and the choice of  $\tilde{\Sigma}$ ) such that  $\|v - v_0\| = \ell$ . In the super-golden case, we instead look at elements  $g\tau$  for  $g \in G_p$  such that  $\|gv_0 - v_0\| = \ell$  (again possibly lifted depending on  $gv_0 - v_0 \in X_1+(\hat{A}_H)$ ).

Since  $K_p$  preserves the  $v - v_0 \in X_1+(\bar{A}_H)$ , we get that  $K'_p \tilde{S}_p^{[\ell]}.v_0 = \tilde{S}_p^{[\ell]}.v_0$  (resp.  $K'_p \tilde{S}_p^{[\ell]}. \tau = \tilde{S}_p^{[\ell]}. \tau$ ), hence there is a bijection between  $K'_p \tilde{S}_p^{[\ell]} K'_p / K'_p$  and  $\tilde{S}_p^{[\ell]} / \Gamma$ . This completes the proof.  $\square$

We now prove a key input bounding operator norms of Hecke operators acting on  $\pi_v$ :

**Proposition 8.1.3.** [Kam16] *Let  $K'_p$  be a compact open subgroup of  $G_p$  and fix constant  $M$  and unitary central character  $\chi$ . Then there is constant  $c$  such that for all  $\sigma \geq 2$  and finite  $S \subseteq G_p$  such that  $S \rightarrow S/Z_{G_p}^{\text{spl}}$  has fibers of size  $M$ ,*

$$\|\mathbb{1}_{K'_p SK'_p} \|_{L^\sigma(G_p, \chi)}\|_{op} \ll (\log X)^c X^{\frac{\sigma-1}{\sigma}}, \quad X = |K'_p SK'_p / K'_p|.$$

*Proof.* Without loss of generality, we can assume  $K'_p$  is contained in some Iwahori  $I_p$  by replacing it with a smaller subgroup and noting that this scales  $|K'_p SK'_p / K'_p|$  by a constant factor. For  $f \in L^\sigma(G_p, \chi)$ , denote by  $|f| \in L^\sigma(G_p, \mathbf{1})$  the function  $g \mapsto |f(g)|$ . Then since all operators are convolutions with positive-valued functions:

$$\frac{\|\mathbb{1}_{K'_p SK'_p} \star f\|_p}{\|f\|_p} \leq \frac{\|\mathbb{1}_{K'_p SK'_p} \star |f|\|_p}{\||f|\|_p} \leq \frac{\|\mathbb{1}_{I_p SI_p} \star |f|\|_p}{\||f|\|_p}.$$

Noting similarly to before that  $|K'_p SK'_p / K'_p| \geq C_{K'_p} |I_p SI_p / I_p|$  for some constant  $C_{K'_p}$ , it therefore suffices to prove a similar bound for  $\|\mathbb{1}_{I_p SI_p} \|_{L^\sigma(G_p)}\|_{op}$ ,

We can write  $\mathbb{1}_{K'_p SK'_p}$  as a sum of terms of the form  $h_w$  in the notation of [Kam16]. Since the number of double cosets  $h_w$  with  $l(w) = l_0$  is polynomial in  $l_0$  and each has size  $q^{l_0}$ , we have  $|I_p SI_p / I_p| \asymp l_{\max}^{c'} q^{l_{\max}}$  where  $l_{\max}$  is the maximum of  $l(w)$  for  $w$  appearing in this sum and  $c'$  is some constant. This then follows from [Kam16, Thm 1.9] (which can be generalized to non-semisimple  $G_p$  by noting that the action of  $G_p$  on  $\mathcal{B}$  factors through  $G_p/Z_{G_p}^{\text{spl}}$  and using trivial central character).  $\square$

**Corollary 8.1.4.** *In the notation above, for all  $\sigma \geq 2$ , if unitary irreducible representation  $\pi_p$  of  $G_p$  has matrix coefficients in  $L^{\sigma+\epsilon}$  mod center for all  $\epsilon > 0$ , then*

$$\|\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p} \|_{\pi_p}\|_{op} \lesssim |\tilde{S}_p^{[\ell]}|^{\frac{\sigma-1}{\sigma}}.$$

*Proof.* By lemma 3.4.2(3) and its variants in various non-standard and decimated cases,  $(\tilde{S}_p^{[\ell]})^{-1} = \tilde{S}_p^{[\ell]}$  in  $G_p/Z_{G_p}$ , so the operator  $\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p}$  is normal. Therefore its operator norm on Hilbert space  $\pi_p$  is equal to its spectral radius which is bounded above by

$\|\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p} |_{L^{\sigma+\epsilon}(G_p, \chi)}\|_{op}$  for all  $\epsilon > 0$  and  $\chi$  the central character of  $\pi_p$ . In addition, by definition, the fibers of  $\tilde{S}_p^{[\ell]} \rightarrow \tilde{S}_p^{[\ell]}/Z_{G_p}^{spl}$  are singletons.

The result then follows from Proposition 8.1.3, computing  $|K'_p \tilde{S}_p^{[\ell]} K'_p / K'_p|$  through lemma 8.1.2(1).  $\square$

We need a technical strengthening when  $\sigma = 2$ :

**Corollary 8.1.5.** *In the notation above, if irreducible unitary representation  $\pi_p$  of  $G_p$  is tempered (i.e. has matrix coefficients in  $L^{2+\epsilon}$  mod center for all  $\epsilon > 0$ ), then there is constant  $c$  such that*

$$\|\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p} |_{\pi_p}\|_{op} \ll \left(\log |S_p^{[\ell]}|\right)^c |S_p^{[\ell]}|^{\frac{1}{2}}.$$

*Proof.* Then by [CHH88],  $\pi_p$  is weakly contained in  $L^2(G_p, \chi)$  where  $\chi$  is the central character of  $\pi_p$ . Therefore,  $\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p} |_{\pi_p}$  has spectrum contained in that of  $\mathbb{1}_{K'_p \tilde{S}_p^{[\ell]} K'_p} |_{L^2(G_p, \chi)}$ . The result follows from 8.1.3 similarly to the proof above of 8.1.4 using that normal operators on Hilbert spaces have operator norm equal to their spectral radii.  $\square$

## 8.2 Optimal Covering Proof

We can now put everything together to prove Theorem 8.0.1 using the identifications

$$L_{\square}^2 = V_{\square} := \bigoplus_{\substack{\pi \in \square \\ \pi_{\infty \setminus v_0} \text{ trivial}}} m_{\pi} \pi_{v_0} \boxtimes (\pi^{\infty})^{K^{\infty}}.$$

We start with a corollary of the density hypothesis, which allows us to interpolate an inequality from  $2/\sigma = 1$  and  $2/\sigma = 0$  to all values in between:

**Proposition 8.2.1.** *If  $n = 4, 8$ , for any shape  $\square$ , any  $\ell$ , and  $\epsilon^{N_G-1} \lesssim |S_p^{[\ell]}|^{-1}$ ,*

$$\|T_{S_p^{[\ell]}} | V_{\square}\|_{op}^2 \cdot \|\mathbb{P}_{\square} f_{v_0}^{\epsilon, Z_1}\|_2^2 \lesssim |S_p^{[\ell]}|^{-1} \cdot \|f_{v_0}^{\epsilon, Z_1}\|_2^2.$$

*Proof.* This is a reformulation of Corollaries 7.3.2 and 7.3.4.

We expand all factors in terms of  $\epsilon$  and the  $2/\sigma_{\square}$  from 6.4.4: Theorem 7.3.1 upper-bounds  $\|\mathbb{P}_{\square} f_{v_0}^{\epsilon, Z_1}\|_2^2 / \|f_{v_0}^{\epsilon, Z_1}\|_2^2$ . Theorem 6.4.2 and the extra computations in the proof of 7.3.4 lower bounds  $2/\sigma(\pi_p)$  for any  $\pi \in \square$  by  $2/\sigma_{\square}$ . Therefore, we can input  $\sigma = \sigma_{\square}$  into Corollary 8.1.4 to upper bound  $\|T_{S_p^{[\ell]}} | V_{\square}\|_{op}^2$ .

This gives desired inequality  $\epsilon^{A(\sigma_{\square})} \lesssim \epsilon^{B(\sigma_{\square})}$ , where  $A$  and  $B$  are linear in  $2/\sigma_{\square}$ . We can easily check that  $A \geq B$  when  $2/\sigma_{\square} = 0, 1$ .  $\square$

While conceptually cleaner, Proposition 8.2.1 will only give a  $|S^{(\ell)}|^{\epsilon}$  in the numerator of the covering condition 1.2.1(1). To improve this to a log factor:

**Proposition 8.2.2.** *If  $n = 4, 8$ , for any shape  $\square$ , any  $\ell$ , and  $\epsilon^{N_G-1} \ll |S_p^{[\ell]}|^{-1}$ , there is a  $c > 0$  such that*

$$\|T_{S_p^{[\ell]}} | V_{\square}\|_{op}^2 \cdot \|\mathbb{P}_{\square} f_{v_0}^{\epsilon, Z_1}\|_2^2 \ll \left(\log |S_p^{[\ell]}|\right)^c |S_p^{[\ell]}|^{-1} \cdot \|f_{v_0}^{\epsilon, Z_1}\|_2^2.$$

*Proof.* The only bound in the proof of 8.2.1 that requires the  $\lesssim$  instead of  $\ll$  is that on  $\|T_{S_p^{[\ell]}} | V_{\square}\|_{op}^2$  coming from Corollary 8.1.4. For  $2/\sigma_{\square} \neq 0, 1$ , remark 7.3.3 allows us to tighten the bound 7.3.1 on  $\|\mathbb{P}_{\square} f_{v_0}^{\epsilon, Z_1}\|_2^2$  to nevertheless improve the  $\lesssim$  to an  $\ll$ . For  $2/\sigma_{\square} = 0$ , we can use the trivial bound  $\|T_{S_p^{[\ell]}}\|_{op}^2 \ll 1$ . For  $2/\sigma_{\square} = 1$ , we can use Corollary 8.1.5 instead of Corollary 8.1.4.  $\square$

For convenience, we reindex

$$I_\delta := f_{v_0}^{\epsilon, Z_1}$$

for the  $\epsilon$  such that it has support of volume  $\delta$ . In particular  $\delta \asymp \epsilon^{N_G-1}$  so Proposition 8.2.1 applies to  $\delta \lesssim |S_p^{[\ell]}|^{-1}$ .

Note also that the projection of this support onto  $PU(n)$  has the same volume (normalizing  $\text{vol}(U(n)) = \text{vol}(PU(n)) = 1$ ) and is a ball in an invariant metric. In other words

$$\text{supp}(\bar{f}_{v_0}^{\epsilon, Z_1} : PU(n) \rightarrow \mathbb{C}) = B^{PU(n)}(\delta).$$

In addition, since  $f_{v_0}^{\epsilon, Z_1}$  is analytic, constant on  $U(1)$  orbits, and equal to 1 at the identity,

$$\langle I_\delta, \mathbf{1} \rangle = (\delta + o(\delta)). \quad (8.2)$$

Combining the above propositions, we are now in a position to estimating the covering rate of  $S_p$  in terms of the spectrum of the operators  $T_{S_p^{[\ell]}}$  evaluated on each subspace  $V_\square$  separately.

**Proposition 8.2.3.** *If  $n = 4, 8$ , the analogue of the optimal covering property Definition 1.2.1) for  $PU(n)$  holds for  $S_p \cup \Gamma$  when  $K'$  is almost golden at  $\mathfrak{p}$  (resp. the super version for finite subgroup  $C_p$  and finite-order elements  ${}^0S_p$  when  $K'$  is almost  $\tau$ -super-golden at  $\mathfrak{p}$  for  $\tau$  traversable).*

*In other words, there is a constant  $c$  such that*

$$\mu\left(PU(n) \setminus B\left(S_p^{[\ell]}, \varepsilon_\ell\right)\right) \xrightarrow{\ell \rightarrow \infty} 0, \quad \varepsilon_\ell = \frac{\left(\log |S_p^{[\ell]}|\right)^c}{|S_p^{[\ell]}|},$$

*Proof.* Our parameter is  $\ell$ , and we take  $\delta$  as a function of  $\ell$ :

$$\delta := \frac{(\log |S_p^{[\ell]}|)^c}{|S_p^{[\ell]}|}, \quad h_\delta := I_\delta - \langle I_\delta, \mathbf{1} \rangle \mathbf{1}.$$

Denote by  $\varphi \mapsto \varphi_*$  the normalized (i.e, left-inverse to pullback) pushforward map from  $U(N)$  to  $\Gamma \backslash U(N)$ . Then for any  $\varphi$ , we have  $T_{S_p^{[\ell]}}(\varphi_*) = (T_{S_p^{[\ell]}} \varphi)_*$  since the operator descends to  $L^2(\Gamma \backslash U(n))$ .

Therefore we can compute on the one hand,

$$\|T_{S_p^{[\ell]}}(h_{\delta,*})\|_2^2 = \frac{1}{|\Gamma|} \int_{U(n)} \left[ T_{S_p^{[\ell]}}(I_\delta - \langle I_\delta, \mathbf{1} \rangle \mathbf{1})(x) \right]^2 dx,$$

so since the support of  $T_{S_p^{[\ell]}}(I_\delta)$  is contained in the pullback to  $U(n)$  of  $B(S_p^{[\ell]}, \delta) \subseteq PU(n)$ ,

$$\|T_{S_p^{[\ell]}}(h_{\delta,*})\|_2^2 \geq \frac{1}{|\Gamma|} \int_{PU(n) \setminus B(S_p^{[\ell]}, \delta)} [\langle I_\delta, \mathbf{1} \rangle \mathbf{1}(x)]^2 dx = (\delta + o(\delta))^2 \cdot \mu\left(PU(n) \setminus B(S_p^{[\ell]}, \delta)\right)$$

using (8.2) for the last step.

On the other hand,

$$\begin{aligned} \|T_{S_p^{[\ell]}} h_{\delta,*}\|_2^2 &\leq \|T_{S_p^{[\ell]}} I_\delta\|_2^2 = \left\| \sum_{\square} T_{S_p^{[\ell]}} \mathbb{P}_{\square} I_{\delta,*} \right\|_2^2 = \sum_{\square} \|T_{S_p^{[\ell]}} \mathbb{P}_{\square} I_{\delta,*}\|_2^2 \\ &\leq \sum_{\square} \|T_{S_p^{[\ell]}} \mid V_{\square}\|_{op}^2 \cdot \|\mathbb{P}_{\square} I_{\delta,*}\|_2^2, \end{aligned} \quad (8.3)$$



so noting that the number of possible  $\square$  is a constant depending only on  $n$ , Proposition 8.2.2 gives that for some  $c_0$ :

$$\|T_{S_p^{[\ell]}} h_{\delta,*}\|_2^2 \ll \left(\log |S_p^{[\ell]}|\right)^{c_0} |S_p^{[\ell]}|^{-1} \|I_{\delta,*}\|_2^2 \asymp \left(\log |S_p^{[\ell]}|\right)^{c_0} |S_p^{[\ell]}|^{-1} |\Gamma \cap U(1)|^{-1} \delta.$$

for small  $\delta$ .

Without loss of generality,  $c_0 < c$ . Then, combining the two estimates together, we get

$$\mu\left(PU(n) \setminus B(S_p^{[\ell]}, \delta)\right) \lesssim \frac{1}{(\log |S_p^{[\ell]}|)^{c-c_0}} \xrightarrow{\ell \rightarrow \infty} 0,$$

which proves that  $S_p$  has the optimal covering property.  $\square$

Finally, we note that Theorem 8.0.1 follows from Theorem 4.4.1 and Proposition 8.2.3. We make some further remarks:

**Remark 8.2.4.** If we set

$$\epsilon_\ell = \frac{(\log |S^{[\ell]}|)^c}{|S^{[2\ell]}|} \asymp \frac{(\log |S^{[\ell]}|)^c}{|S^{[\ell]}|^2},$$

then the arguments of [EP24, Prop 4.6]/[PS18, Cor 3.2] show that for large enough  $\ell$ ,

$$\mu\left(PU(n) \setminus B\left(S^{[\ell]}, \epsilon_\ell\right)\right) = 0.$$

In other words, even the small volume of exceptional points that cannot be approximated by words of the “optimal” length  $\ell$  can be approximated by words of length  $2\ell$ .

**Remark 8.2.5.** When  $K'$  does not have class number one, lemma 8.1.2 no longer interprets  $T_{S_p^{[\ell]}}$  as a Hecke operator since we no longer necessarily have that  $KS_p^{[\ell]}K = S_p^{[\ell]}K$ . However, we may instead replace  $T_{S_p^{[\ell]}}$  by the Clozel-Hecke operator of [EP24, Def 4.3] that takes a weighted sum over a set of Clozel-Hecke points also defined therein.

As in the proof of [EP24, Thm 4.4], this Clozel-Hecke operator has operator norm bounded by that of a Hecke operator. This Hecke operator’s norm can further be bounded exactly as in Proposition 8.2.2. The proof of Proposition 8.2.3 then carries through in exactly the same way to prove optimal covering for the weighted set of Clozel-Hecke points.

**Remark 8.2.6.** One might expect that the corresponding  $\tilde{S}_p$  could give golden gate sets for  $U(n)$  by using the argument for 8.2.3 with respect to  $f_{v_0}^{\epsilon, Z_\epsilon}$  instead of  $f_{v_0}^{\epsilon, Z_1}$ .

However, there is an obstruction that as defined here, the determinant is constant on  $\tilde{S}_p$ . In the argument, this shows up in that Theorem 7.3.1 needs to be tightened to a bound instead by  $\epsilon^{N_G(1-2/\sigma_\square)}$ , which turns out to hold for all  $\square$  except the trivial shape  $(n, 1)$  corresponding to automorphic characters. Therefore, in the bound (8.3), the term for  $\square = (n, 1)$  has to be handled separately by noting that  $h_\delta$  has no component along the trivial character and bounding  $\|T_{S_p^{[\ell]}}|\chi|\|_{op}$  for non-trivial characters  $\chi$ .

This operator norm bound can be done by Weyl equidistribution-type estimates, but only if  $\tilde{S}_p$  has elements with determinants differing by a non-root of unity. Therefore, modifying  $\tilde{S}_p$  by multiplying its elements by differing points in  $U(1)$  should suffice to produce a gate set on  $U(n)$ .

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