



### Tensor tensión de Cauchy

$$\mathbf{T}^{(n)} = \mathbf{n} \cdot \boldsymbol{\sigma}, \quad T_j^{(n)} = \sigma_{ij} n_i$$

$$(\boldsymbol{\sigma} - \lambda_i \mathbf{I}) \mathbf{n}_i = \mathbf{0}, \quad |\boldsymbol{\sigma} - \lambda \mathbf{I}| = 0$$

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$

$$I_1 = \sigma_{kk}$$

$$I_2 = \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji})$$

$$I_3 = \det(\sigma_{ij})$$

$$\boldsymbol{\sigma}' = \mathbf{A} \boldsymbol{\sigma} \mathbf{A}^T, \quad \sigma'_{ij} = a_{im} a_{jn} \sigma_{mn}$$

### Círculo de Mohr en tensión plana

$$\begin{aligned} \sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta, \\ \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta, \\ \tau_{x'y'} &= -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta. \end{aligned}$$

$$r = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

### Relaciones tensión-deformación-temperatura

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0, \quad \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

$$\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} - \left( \lambda + \frac{2}{3} \mu \right) \alpha \Delta T \delta_{ij}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha \Delta T \delta_{ij}$$

	$E$ : módulo de Young $\nu$ : coeficiente de Poisson	$K$ : módulo de compresibilidad $G$ : módulo de rigidez	$\lambda$ : 1.º coeficiente de Lamé $\mu$ : 2.º coeficiente de Lamé
$(E, \nu)$	---	$K = \frac{E}{3(1-2\nu)}$ $G = \frac{E}{2(1+\nu)}$	$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ $\mu = \frac{E}{2(1+\nu)}$
$(K, G)$	$E = \frac{9KG}{3K+2G}$ $\nu = \frac{3K-2G}{2(3K+G)}$	---	$\lambda = K - \frac{2G}{3}$ $\mu = G$
$(\lambda, \mu)$	$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ $\nu = \frac{\lambda}{2(\lambda+\mu)}$	$K = \lambda + \frac{2\mu}{3}$ $G = \mu$	---

### Tensión de Von Mises $\sigma_{VM}$

$$\sigma_{VM} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}},$$

donde  $\sigma_1, \sigma_2, \sigma_3$ , son las tensiones principales.

### Energía de deformación en cuerpos elásticos

$$U = \frac{1}{2} \int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} dV.$$

### Sistema lineal-elástico

$$\delta_i = \frac{\partial U}{\partial P_i}, \quad y \quad \phi_i = \frac{\partial U}{\partial M_i},$$

### Estados planos en coordenadas cartesianas

$$\begin{aligned} \varepsilon_x &= \frac{1}{E} (\sigma_x - \tilde{\nu} \sigma_y), & \sigma_x &= \frac{\tilde{E}}{1 - \tilde{\nu}^2} (\varepsilon_x - \tilde{\nu} \varepsilon_y), \\ \varepsilon_y &= \frac{1}{\tilde{E}} (\sigma_y - \tilde{\nu} \sigma_x), & \sigma_y &= \frac{\tilde{E}}{1 - \tilde{\nu}^2} (\varepsilon_y - \tilde{\nu} \varepsilon_x), \end{aligned}$$

	Tensión plana	Deformación plana
$\tilde{\nu}$	$\nu$	$\frac{\nu}{1-\nu}$
$\tilde{E}$	$E$	$\frac{E}{1-\nu^2}$
$\sigma_z$	0	$\frac{\nu E}{1-\nu^2} (\varepsilon_x - \varepsilon_y)$
$\varepsilon_z$	$-\frac{\nu}{E} (\sigma_x + \sigma_y)$	0

### Función de tensiones de Airy

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial y \partial x}.$$

$$\nabla^4 \phi = \phi_{,1111} + 2\phi_{,1122} + \phi_{,2222} = 0$$

### Estados planos en coordenadas polares

$$\begin{aligned} \sigma_{r,r} + \frac{1}{r} \tau_{r\theta,\theta} + \frac{\sigma_r - \sigma_\theta}{r} &= 0, \\ \frac{1}{r} \sigma_{\theta,\theta} + \tau_{r\theta,r} + \frac{2\tau_{r\theta}}{r} &= 0. \end{aligned}$$

$$\varepsilon_r = u_{,r} \quad \varepsilon_\theta = \frac{u}{r} + \frac{1}{r} v_{,\theta} \quad \gamma_{r\theta} = \frac{1}{r} u_{,\theta} + v_{,r} - \frac{v}{r}$$

### Estado plano de tensiones

$$\begin{aligned} \varepsilon_r &= \frac{1}{E} (\sigma_r - \nu \sigma_\theta), \\ \varepsilon_\theta &= \frac{1}{E} (\sigma_\theta - \nu \sigma_r), \\ \gamma_{r\theta} &= \frac{1}{G} \tau_{r\theta}. \end{aligned}$$

### Distribuciones de tensión axisimétricas

$$\begin{aligned}\phi &= C_1 + C_2 \log r + C_3 r^2 + C_4 r^2 \log r, \\ \sigma_r &= \frac{1}{r} \phi_{,r} = \frac{C_2}{r^2} + 2C_3 + C_4(2 \log r + 1), \\ \sigma_\theta &= \phi_{,rr} = -\frac{C_2}{r^2} + 2C_3 + C_4(2 \log r + 3).\end{aligned}$$

Cilindro de pared gruesa sometido a presión uniforme  
 $C_4 = 0$

$$\sigma_r(r=a) = -p_i, \quad \sigma_r(r=b) = -p_e.$$

$$\begin{aligned}u &= \frac{2(1-\nu)}{E} C_3 r - \frac{(1+\nu)}{E} \frac{C_2}{r}, \\ \sigma_r &= \frac{p_i a^2 - p_e b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_e - p_i)}{r^2 (b^2 - a^2)}, \\ \sigma_\theta &= \frac{p_i a^2 - p_e b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_e - p_i)}{r^2 (b^2 - a^2)}.\end{aligned}$$

Pequeños agujeros circulares en placas tensionadas

$$\sigma_r = \sigma_\infty \left[ 1 - \left( \frac{R}{r} \right)^2 \right], \quad \sigma_\theta = \sigma_\infty \left[ 1 + \left( \frac{R}{r} \right)^2 \right].$$

**Discos rotantes**, velocidad angular  $\omega$

$$\begin{aligned}\sigma_r &= \frac{3+\nu}{8} \rho \omega^2 \left( a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2 \right), \\ \sigma_\theta &= \frac{3+\nu}{8} \rho \omega^2 \left( a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right).\end{aligned}$$

**Cilindros rotantes**, velocidad angular  $\omega$

$$\begin{aligned}\sigma_r &= \frac{3+\nu}{8} \rho \omega^2 \left( a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2 \right), \\ \sigma_\theta &= \frac{3+\nu}{8} \rho \omega^2 \left( a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right), \\ u &= r \varepsilon_\theta = \frac{r}{E} \left[ \sigma_\theta - \nu(\sigma_r + \sigma_z) \right].\end{aligned}$$

Deformación plana  $\varepsilon_z = 0$

$$\sigma_z = \frac{3-2\nu}{4(1-\nu)} \nu \rho \omega^2 \left( a^2 + b^2 - \frac{2r^2}{3-2\nu} \right).$$

Deformación plana generalizada  $\varepsilon_z = cte$

$$\int_0^{2\pi} \int_a^b \sigma_z r dr d\theta = 0,$$

$$\begin{aligned}\varepsilon_z &= -\frac{\nu \rho \omega^2}{2E} (a^2 + b^2), \\ \sigma_z &= \frac{\nu \rho \omega^2}{4(1-\nu)} (a^2 + b^2 - 2r^2).\end{aligned}$$

**Discos rotantes de espesor variable**,  $h = cr^{-\beta}$ ,  $c$  y  $\beta$  constantes.

$$\begin{aligned}\phi &= C_1 r^{q_1} + C_2 r^{q_2} - \frac{3+\nu}{8-(3+\nu)\beta} c \rho \omega^2 r^{3-\beta}, \\ \sigma_r &= \frac{C_1}{c} r^{q_1+\beta-1} + \frac{C_2}{c} r^{q_2+\beta-1} - \frac{(3+\nu)\rho \omega^2 r^2}{8-(3+\nu)\beta}, \\ \sigma_\theta &= \frac{C_1}{c} q_1 r^{q_1+\beta-1} + \frac{C_2}{c} q_2 r^{q_2+\beta-1} - \frac{(1+3\nu)\rho \omega^2 r^2}{8-(3+\nu)\beta}.\end{aligned}$$

**Discos delgados con temperatura no uniforme**

$$\begin{aligned}\sigma_r &= \alpha E \frac{1}{r^2} \left[ \frac{r^2 - a^2}{b^2 - a^2} \int_a^b T r dr - \int_a^r T r dr \right], \\ \sigma_\theta &= \alpha E \frac{1}{r^2} \left[ \frac{r^2 + a^2}{b^2 - a^2} \int_a^b T r dr + \int_a^r T r dr - T r^2 \right], \\ u &= \frac{\alpha}{r} \left[ \frac{r^2(1-\nu) + a^2(1+\nu)}{b^2 - a^2} \int_a^b T r dr + (1+\nu) \int_a^r T r dr \right].\end{aligned}$$

**Cilindros largos con temperatura no uniforme**

$$\begin{aligned}\sigma_r &= \frac{\alpha E}{1-\nu} \frac{1}{r^2} \left[ \frac{r^2 - a^2}{b^2 - a^2} \int_a^b T r dr - \int_a^r T r dr \right], \\ \sigma_\theta &= \frac{\alpha E}{1-\nu} \frac{1}{r^2} \left[ \frac{r^2 + a^2}{b^2 - a^2} \int_a^b T r dr + \int_a^r T r dr - T r^2 \right], \\ u &= r \varepsilon_\theta = \frac{r}{E} \left[ \sigma_\theta - \nu(\sigma_r + \sigma_z) \right] + r \alpha T.\end{aligned}$$

Extremo fijo,  $\varepsilon = 0$

$$\begin{aligned}\sigma_z &= \frac{\alpha E}{1-\nu} \left[ \frac{2\nu}{b^2 - a^2} \int_a^b T r dr - T \right], \\ u &= \frac{1+\nu}{1-\nu} \frac{\alpha}{r} \left[ \frac{(1-2\nu)r^2 + a^2}{b^2 - a^2} \int_a^b T r dr + \int_a^r T r dr \right].\end{aligned}$$

Extremo libre,  $\varepsilon = cte$

$$\begin{aligned}\varepsilon_z &= \frac{2\alpha}{b^2 - a^2} \int_a^b T r dr, \\ \sigma_z &= \frac{\alpha E}{1-\nu} \left[ \frac{2}{b^2 - a^2} \int_a^b T r dr - T \right], \\ u &= \frac{1+\nu}{1-\nu} \frac{\alpha}{r} \left[ \frac{\frac{1-3\nu}{1+\nu} r^2 + a^2}{b^2 - a^2} \int_a^b T r dr + \int_a^r T r dr \right].\end{aligned}$$

### Torsión

Eje sólido circular

$$\sigma_x = \sigma_y = \sigma_z = \tau_{rz} = \tau_{r\theta} = 0, \quad \tau_{\theta z} = G \gamma_{\theta z} = Gr \frac{d\phi}{dz}$$

$$M_t = \int_A r(\tau_{\theta z} dA) = G \frac{d\phi}{dz} \int_A r^2 dA = G \frac{d\phi}{dz} I_z,$$

donde  $I_z = \int_A r^2 dA$ .

$$\frac{d\phi}{dz} = \frac{M_t}{GI_z}, \quad \phi = \int_L \frac{M_t}{GI_z} dz = \frac{M_t L}{GI_z}$$

$$\tau_{\theta z} = \frac{M_t r}{I_z}$$

Eje hueco circular

$$I_z = \frac{\pi r_0^4}{2} \left( 1 - \frac{r_i^4}{r_0^4} \right) = \frac{\pi d_0^4}{32} \left( 1 - \frac{d_i^4}{d_0^4} \right)$$

Energía de deformación torsional

$$\begin{aligned} U &= \frac{1}{2} \int_V \frac{1}{G} \left( \frac{M_t r}{I_z} \right)^2 dV = \frac{1}{2} \int_L \frac{M_t^2}{G I_z^2} dz \int_A r^2 dA \\ &= \frac{1}{2} \int_L \frac{M_t^2}{G I_z} dz \end{aligned}$$