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1)

HW 5

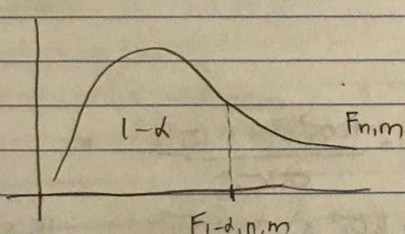
A) $x \sim F_{m,n} \rightarrow \frac{1}{x} \sim F_{n,m}$

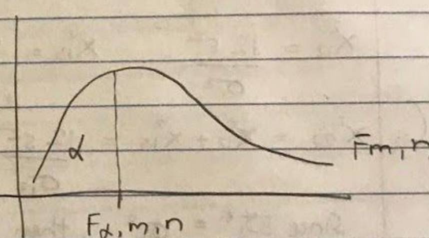
$P(F_{n,m} < F_{1-\alpha,n,m}) = 1-\alpha$

$P\left(F_{m,n} > \frac{1}{F_{1-\alpha;n,m}}\right) = 1-\alpha$

$P\left(F_{m,n} < \frac{1}{F_{1-\alpha;n,m}}\right) = \alpha$

$\left(t_{1-\frac{\alpha}{2};n}\right) = \frac{z}{\sqrt{x_n^2/n}}$

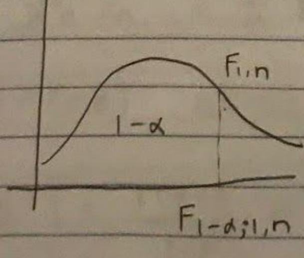


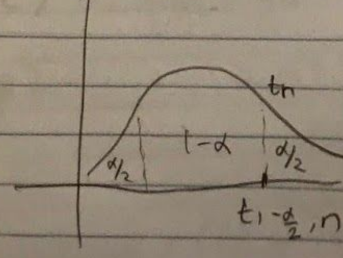


$\left(t_{1-\frac{\alpha}{2};n}\right)^2 = F_{1-\alpha;1,n}$

$t = \frac{z}{\sqrt{x_n^2/n}}$

$t^2 = \frac{z^2}{x_n^2/n} = \frac{(x_1^2)}{(x_n^2/n)} = F_{1,n}$





b $x_1, x_2, \ldots x_{13}$ and $Y_1, Y_2, \ldots Y_{16}$ $\Big\}$ $\sigma_1^2 = \frac{1}{5}\sigma_2^2$

$x \sim N(\mu_1, \sigma_1)$, $Y \sim N(\mu_2, \sigma_2)$

Thus, $\bar{X} - \bar{Y} \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16}\right)$

$$\frac{\sigma_1^2}{13} + \frac{\sigma_2^2}{16} = \frac{16\sigma_1^2}{208} + \frac{13\sigma_2^2}{208} = \frac{16\sigma_1^2 + 65\sigma_1^2}{208} = \frac{81\sigma_1^2}{208}$$

$$X_{12}^2 = \frac{12s^2}{\sigma_1^2} \qquad X_{15}^2 = \frac{15s^2}{\sigma_2^2}$$

$$X_{27}^2 = X_{12}^2 + X_{15}^2 = \frac{12s^2}{\sigma_1^2} + \frac{15s^2}{\sigma_2^2} = \frac{12\sigma_2^2 s^2 + 15\sigma_1^2 s^2}{\sigma_1^2\sigma_2^2} =$$

Since $5\sigma_1^2 = \sigma_2^2$, then $\rightarrow$ $\frac{12 \cdot \cancel{5}\sigma_1^2 s^2 + \cancel{15}^{3}\sigma_1^2 s^2}{\cancel{5}(\sigma_1^2)^2} = \frac{15s^2}{\sigma_1^2}$

$$\frac{\dfrac{(\bar{x}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/13 + \sigma_2^2/16}}}{\sqrt{\dfrac{15s^2/\sigma_1^2}{27}}} = \frac{(\bar{x}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{81\sigma_1^2}{208}}} \times \sqrt{\frac{27}{15s^2/\sigma_1^2}}$$

$$= \sqrt{208}\,\frac{(\bar{x}-\bar{Y}) - (\mu_1 - \mu_2)}{\underset{3}{\cancel{\sqrt{81}}}\,\cancel{\sigma_1}} \times \frac{\sqrt{27}}{\dfrac{\sqrt{15}\,s}{\cancel{\sigma_1}}} = \frac{\sqrt{208}}{3\sqrt{15}}\, t_{27}$$

c $X \sim F_{m,n}$

$$E[X] = E[W/Y] = E[W] \cdot E[Y^{-1}]$$

$$W = \chi^2_m / m$$

$$Y = \chi^2_n / n \to \frac{1}{n} \Gamma(\overset{\alpha}{\frac{n}{2}}, \overset{B}{2}) \to \Gamma(\frac{n}{2}, \frac{2}{n})$$

based on

$$aV \sim \Gamma(\alpha, a\beta)$$

$$E(W) = m$$

$$E(Y^{-1}) = \frac{\Gamma(\alpha+k)\, B^k}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})} \left(\frac{2}{n}\right) = \frac{1}{n-2}$$

Thus $E[X] = E(W) \cdot E(Y^{-1}) = \frac{m}{n-2}$

$$Var(W/Y) = E(W^2)E(Y^{-2}) - [E(W)]^2 [E(Y^{-1})]^2$$

$$E(W^2) = \frac{\Gamma(m/2+2) \cdot 4}{\Gamma(m/2)} = (m/2+1)\, m/2 \cdot 4 = m^2 + 2m$$

$$E(Y^2) = \frac{\Gamma(n/2+2)}{4\Gamma(n/2)} = \frac{1}{(n-2)(n-4)}$$

$$Var(W/Y) = \frac{m^2+2m}{(n-2)(n-4)} - \frac{m^2}{(n-2)^2}$$

$$= \frac{(n-2)(m^2+2m) - m^2(n-4)}{(n-2)^2(n-4)}$$

$$= \frac{2m^2 + 2mn - 4m}{(n-2)^2(n-4)}$$

d $\bar{x} = \frac{1}{n} 1' x$ --- ①

$$\begin{pmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix} = \left(I - \frac{1}{n} 1 1' \right) \text{--- ②}$$

From ① & ②

$$\begin{pmatrix} \frac{1}{n} 1' \\ I - \frac{1}{n} 11' \end{pmatrix} x = Ax \rightarrow Var(Ax) = A \Sigma A'$$

$$\begin{pmatrix} \frac{1}{n} 1' \\ I - \frac{1}{n} 11' \end{pmatrix} \left((1-\rho) I + \rho J \right) \begin{pmatrix} \frac{1}{n} 1' & I - \frac{1}{n} 11' \end{pmatrix}' =$$

Checking the covariance

$= \cdots (1-\rho) \frac{1}{n} 1' + \rho \frac{1}{n} 1' J - (1-\rho) \frac{1}{n^2} 1' 1 1' - \rho \frac{1}{n} 1' J \frac{1}{n} 11'$

$= 0$

Thus, yes they are independent

e $Y \sim N_2 (0, \Sigma) \longrightarrow$

then

$$\left(Y' \Sigma^{-1} Y - \frac{Y_1^2}{\sigma_1^2} \right) \sim \chi_1^2$$

Prove!