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HW 4

a Sample Mean $\bar{x} = \frac{\sum x_i}{n} = \frac{I}{n}$

$M_{\bar{x}} = M_{\frac{x_1 + x_2 + \dots + x_n}{n}}(t)$

$= M_{x_1 + x_2 + \dots + x_n}\left(\frac{t}{n}\right)$

Bcs it's independent,

$= M_{x_1} \cdot M_{x_2} \cdot \dots \cdot M_{x_n}\left(\frac{t}{n}\right)$

$= \left(1 - B\frac{t}{n}\right)^{-\alpha} \cdot \left(1 - B\frac{t}{n}\right)^{-\alpha} \cdot \dots \left(1 - B\frac{t}{n}\right)^{-\alpha}$

$= \left(1 - B\frac{t}{n}\right)^{-n\alpha}$

let $\bar{x} = \frac{2tn}{\beta}$

$$M_{\bar{x}}\left(\frac{2tn}{B}\right) = \left(1 - B \cdot \frac{2tn}{Bn}\right)^{-n\alpha} = (1 - 2t)^{-n\alpha}$$

This follows pdf of $\Gamma(n\alpha, 2)$

then, the degree of freedom of this transformation is $2n\alpha$.

b x has uniform distribution (0,1)

$$\begin{bmatrix} x \\ x^2 \end{bmatrix} = \begin{pmatrix} \mu_x \\ \mu_{x^2} \end{pmatrix}$$

$\mu_x = \frac{a+b}{2} = \frac{0+1}{2} = \frac{1}{2}$ $\quad E(x) = \frac{1}{2}$ $\quad Var(x) = \frac{1}{12}$

$Var(x) = E(x^2) - [E(x)]^2$

$\frac{1}{12} = E(x^2) - \frac{1}{4}$

$E(x^2) = \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = \frac{1}{3}$

$$\begin{bmatrix} x \\ x^2 \end{bmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}$$

$Cov(x, x_2) = E[x x^2] - E(x)E(x^2) = E[x^3] - E(x)E(x^2)$

$E[x^3] = \int g(x) f(x)$ ← pdf

$= \int_0^1 x^3 \cdot 1 = \frac{x^4}{4}\Big]_0^1 = \frac{1}{4}$

$Cov(x, x^2) = \frac{1}{4} - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{3-2}{12} = \frac{1}{12}$

$Var(x^2) = E(x^4) - (E(x^2))^2$

$E(x^4) = \int_0^1 x^4 = \frac{1}{5}x^5 = \frac{1}{5}$

$Var(x^2) = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{9-5}{45} = \frac{4}{45}$

$$\begin{pmatrix} x \\ x^2 \end{pmatrix} \rightarrow \begin{pmatrix} 1/12 & 1/12 \\ 1/12 & 4/45 \end{pmatrix}.$$

C $Y = 2.\ X_1^{1/2}\ X_2^{1/2}$

$E(Y) = 2E(X_1^{1/2})\ E(X_2^{1/2})$

$E(X_1^{1/2}) = \frac{\Gamma(\alpha+\frac{1}{2})B^{1/2}}{\Gamma(\alpha)}$

$\hookrightarrow \Gamma(\alpha,1) \Rightarrow E(X_1^{1/2}) = \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}$

$\hookrightarrow \Gamma(\alpha+\frac{1}{2},1) \Rightarrow E(X_2^{1/2}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})}$

$E(Y) = 2 \cdot \frac{\Gamma(\cancel{\alpha+\frac{1}{2}})}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\cancel{\alpha+\frac{1}{2}})} = 2\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = 2 \cdot \frac{\alpha!}{(\alpha-1)!}$

$= 2 \cdot \frac{\alpha\,\cancel{(\alpha-1)!}}{\cancel{(\alpha-1)!}} = 2\alpha.$

$Var(y) = E(y^2) - E(y)^2.$

$Y^2 = 4X_1X_2$

$E(Y^2) = 4E(X_1)\ E(X_2)$

$\hookrightarrow E(X_1) = \alpha \quad E(X_2) = \alpha + \frac{1}{2}$

$E(Y^2) = 4 \cdot (\alpha^2 + \frac{1}{2}\alpha) = 4\alpha^2 + 2\alpha = 2\alpha(2\alpha+1)$

$Var(y) = 4\alpha^2 + 2\alpha - 4\alpha^2 = 2\alpha.$

d. Joint distribution :

$$M_{\bar{x}\bar{y}}(s,t) = Ee^{\bar{x}s+\bar{y}t} = Ee^{(x_1+x_2+\dots+x_n)\frac{s}{n} + (y_1+y_2+\dots+y_n)\frac{t}{n}}$$

$$= Ee^{\frac{x_1 s}{n}+\frac{y_1 t}{n}} \cdot Ee^{\frac{x_2 s}{n}+\frac{y_2 t}{n}} \cdot \dots \cdot Ee^{\frac{x_n s}{n}+\frac{y_n t}{n}}$$

$$= Ee^{(x_1 y_1)\binom{s/n}{t/n}} \cdots Ee^{(x_n y_n)\binom{s/n}{t/n}}$$

$$= \left[Ee^{(x_i\, Y_i)\binom{s/n}{t/n}}\right]^n$$

From $M_y(t) = Ee^{ty} = Ee^{(\Sigma t_i y_i)} = e^{t'\mu + \frac{1}{2} t \Sigma t'}$

$$= \left(e^{(s/n\ \ t/n)\binom{\mu_1}{\mu_2} + \frac{1}{2}(s/n\ \ t/n)\Sigma\binom{s/n}{t/n}}\right)^n$$

$$= e^{(s\ t)\binom{\mu_1}{\mu_2} + \frac{1}{2}(s/n\ \ t/n)\Sigma\binom{s}{t}}$$

$$= e^{(s\ t)\binom{\mu_1}{\mu_2} + \frac{1}{2}(s\ t)(\Sigma/n)\binom{s}{t}}$$

Thus, $\bar{x}, \bar{y} \sim N\left[\binom{\mu_1}{\mu_2}, (\Sigma/n)\right]$

$(x-\mu)' \Sigma^{-1} (x-\mu) \sim \chi_n^2$

$n(\bar{x}-\mu_1, \bar{y}-\mu_2)\Sigma^{-1}\binom{\bar{x}-\mu_1}{\bar{y}-\mu_2}$

$= (\bar{x}-\mu_1, \bar{y}-\mu_2)\left(\frac{\Sigma}{n}\right)^{-1}\binom{\bar{x}_1-\mu_1}{\bar{y}-\mu_2} \rightarrow$ degree of freedom 2