Reinaldo Daniswara
604840665

Stat 100B HW 7

a $f(\underline{y}) = \frac{1}{(2\pi)^{n/2}} |\sigma^2 \underline{V}|^{-1/2} e^{-\frac{1}{2}(\underline{y}-\mu\underline{1})'(\sigma^2 V)^{-1}(\underline{y}-\mu\underline{1})}$

$= \frac{1}{(2\pi)^{n/2}} (\sigma^2)^{-n/2} |V|^{-1/2} e^{-\frac{1}{2\sigma^2}(\underline{y}-\mu\underline{1})' V^{-1}(y-\mu\underline{1})}$

$(2\pi\sigma^2)^{-n/2}$

$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}(\underline{y}-\mu\underline{1})' V^{-1}(\underline{y}-\mu\underline{1}) - \frac{1}{2}\ln|V|$

$= -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left[y V^{-1} y - \mu y V^{-1} 1 - \mu 1' V^{-1} y + \mu^2 \underline{1}' \underline{V}^{-1}\underline{1}\right] - \frac{1}{2}\ln|V|$

$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left[y' V^{-1} y - 2\mu 1' V^{-1} y + \mu^2 1' V^{-1} 1\right]$

$\frac{\partial \ln L}{\partial \mu} = \frac{1}{2\sigma^2}\left(-2\,1' V^{-1} y + 2\mu\, 1' V^{-1} 1\right) = 0$

$\boxed{\mu = \frac{1' V^{-1} y}{1 V^{-1} 1}}$ if $\underline{V} = 1$

$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(y-\hat{\mu}1)' V^{-1}(y-\hat{\mu}1) = 0$

$\boxed{\hat{\sigma}^2 = \frac{(y-\hat{\mu}1)' V^{-1}(y-\hat{\mu}1)}{n}}$

b $E(\hat{\mu}) = \frac{1' V^{-1} E y}{1' V^{-1} 1} = \mu$

$Var(\hat{\mu}) = Var\left(\frac{1' V^{-1} y}{1 V^{-1} 1}\right)$ Var $Ax = A\, Var(x)\, A'$

$= \frac{1}{(1' V^{-1} 1)^2} Var(1' V^{-1} y) = \frac{(1' V^{-1})\, Var(y)\, (1' V^{-1})'}{(1' V^{-1} 1)^2}$

$= \frac{(1' V^{-1})\, \sigma^2 V\, (1' V^{-1})'}{(1' V^{-1} 1)^2}$

$E(\hat{\sigma}^2) = E\left(\frac{(y-\mu 1)' V^{-1}(y-\mu 1)}{n}\right) = E\, \frac{tr(y-\mu 1) V^{-1}(y-\mu 1)}{n}$

$= tr\, E\, \frac{V^{-1}(y-\mu 1)(y-\mu 1)'}{n} = tr\, \frac{V^{-1} E(y-\mu 1)(y-\mu 1)'}{n}$

$= tr\, \frac{V^{-1}(\sigma^2 V + 00')}{n} = \sigma^2 \frac{tr\, I_{n\times n}}{n} = \sigma^2$

c

c $I(\theta) = -E\begin{pmatrix} \frac{\partial^2 \ln L}{\partial^2 \mu} & \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \\ \frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} & \frac{\partial^2 \ln L}{\partial^2 \sigma^2} \end{pmatrix}$

$$\frac{\partial^2 \ln L}{\partial^2 \mu} = \frac{2 1' V' 1}{2\sigma^2} = \frac{1' V' 1}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} = \frac{2(1' V^{-1} Y) - 2\mu(1' V^{-1} 1)}{2\sigma^4}$$

$$\frac{\partial^2 \ln L}{\partial^2 \sigma^2} = \frac{n}{2\sigma^4} - \frac{[(y - \mu 1)' (\sigma^2 V)^{-1} (Y - \mu 1)]}{\sigma^4}$$

$$E\left(\frac{1' V' 1}{\sigma^2}\right) = \frac{1' V' 1}{\sigma^2}$$

$$E\left(\frac{(1' V^{-1} y) - \mu(1' V^{-1} 1)}{\sigma^4}\right) = \frac{E(1' V^{-1} y)}{\sigma^4} - \frac{\mu(1' V^{-1} 1)}{\sigma^4}$$

$$= \frac{1' V^{-1} E(y) - \mu(1' V^{-1} 1)}{\sigma^4} = \frac{1' V^{-1} \mu 1 - 1' V^{-1} \mu 1}{\sigma^4}$$

$$E\left(\frac{\partial^2 \ln L}{\partial^2 \sigma^2}\right) = \frac{n}{2\sigma^4} - \frac{E[(Y - \mu 1)'(\sigma^2 V)^{-1}(Y - \mu 1)]}{\sigma^4} = \frac{n}{2\sigma^4} - \frac{n}{\sigma^4} = -\frac{n}{\sigma^4}$$

$$I(\theta) = \begin{bmatrix} -\left(\frac{1' V' 1}{\sigma^2}\right) & 0 \\ 0 & \frac{n}{\sigma^4} \end{bmatrix} \qquad I^{-1}(\theta) = \begin{bmatrix} \frac{\sigma^2}{1' V' 1} & 0 \\ 0 & \frac{\sigma^4}{n} \end{bmatrix}$$

$$\mathrm{Var}(\hat{\mu}) = \mathrm{Var}\left[\frac{1' V^{-1} y}{1 V^{-1} 1}\right] = \frac{(1' V^{-1})\,\mathrm{Var}(y)\,(1' V^{-1})'}{(1 V^{-1} 1)^2}$$

$$= \frac{(1' V^{-1})\,\sigma^2 V\,(1' V^{-1})'}{(1' V^{-1} 1)^2} = \frac{\sigma^2 V}{1' V^{-1} 1}$$

f. find the

d $Y_i \sim N(i\theta, i\sigma)$

$$f(y) = \frac{1}{i\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_j - i\theta}{i\sigma}\right)^2}$$

$$= (2\pi i^2\sigma^2)^{-1/2} e^{-\frac{1}{2i^2\sigma^2}(x_j - i\theta)^2}$$

$$L = (2\pi i^2\sigma^2)^{-n/2} e^{-\frac{1}{2i^2\sigma^2}\Sigma(x_j - i\theta)^2}$$

$$\ln L = -\frac{n}{2}\ln(2\pi i^2\sigma^2) - \frac{1}{2i^2\sigma^2}\Sigma(x_j - i\theta)^2$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{2i(\Sigma(x_j - i\theta))}{2i^2\sigma^2} = \frac{\Sigma x_j - in\theta}{i\sigma^2} = 0$$

$$\Sigma x_j = in\theta$$

$$\frac{\Sigma x_j}{ni} = \hat{\theta}$$

$$E(\hat{\theta}) = \frac{E(\Sigma x_j)}{ni} = \frac{ni\theta}{ni} = \theta$$

$$\frac{\partial^2 \ln L}{\partial^2\theta} = \frac{-2i^2 n}{2i^2\sigma^2} = -\frac{n}{\sigma^2}$$

$$-E\left(\frac{\partial^2 \ln L}{\partial^2\theta}\right) = \frac{n}{\sigma^2} \qquad \text{MVEU} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$$

→ Because of $E(\theta) = \theta$ and $E(\hat{\theta}) = \theta$,

Yes! It is an efficient estimator of θ

e. $\epsilon \sim N(0, \sigma)$ $\quad E(\epsilon) = 0 \quad Var(\epsilon) = \sigma^2$

$$E(\epsilon^2) = Var(\epsilon) + (E(\epsilon))^2 = \sigma^2$$

Actual area of circle $= \pi r^2$

$$E(\pi r^2) = \pi r^2$$

Area of circle with an error $= \pi(r+\epsilon)^2$

$$E(\pi(r+\epsilon)^2) = E(\pi r^2 + \pi 2r\epsilon + \pi\epsilon^2)$$

$$= E(\pi r^2) + 2\pi r E(\epsilon) + \pi E(\epsilon^2)$$

$$= \pi r^2 + \pi\sigma^2$$

flip

Since we want to eliminate $\pi\sigma^2$, then I assume unbiase estimator is $\pi(r+\epsilon)^2 - \pi\sigma^2$.

Proof $\Rightarrow$ $E(\pi(r+\epsilon)^2 - \pi\sigma^2) = E(\pi(r+\epsilon)^2) - E(\pi\sigma^2)$

$= \pi r^2 + \cancel{\pi\sigma^2} - \cancel{\pi\sigma^2}$

$= \pi r^2$

f $\quad F(y|\theta) = \left(\frac{2y}{\theta}\right) \exp\left(-\frac{y^2}{\theta}\right)$

$F(y^2 \le y) = F(y \le \sqrt{y}) = F_y(\sqrt{y})$

$F_y(y) = \frac{d}{dy} F_y(\sqrt{y}) = \frac{1}{2}\sqrt{y}\; F_y(\sqrt{y})$

$= \frac{1}{2}\sqrt{y}\; \frac{2\sqrt{y}}{\theta} \exp\left(-\frac{y}{\theta}\right)$

$= \frac{1}{\theta} \exp\left(-\frac{y}{\theta}\right)$

$\hookrightarrow$ follow the distribution of exponential $\frac{1}{\theta}$

Thus, $E(y^2) = \theta$

$Var(y^2) = \theta^2$

$E(\hat{\theta}) = \frac{E\left(\sum_{i=1}^{n} Y_i^2\right)}{n} = \frac{E(y_1^2) + \dots + E(y_n^2)}{n} = \frac{n\theta}{n} = \theta$

$Var(\hat{\theta}) = \frac{Var\left(\sum_{i=1}^{n} Y_i^2\right)}{n} = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}$

$L = \left(\frac{2y}{\theta}\right)^n \exp\left(-\frac{\Sigma y^2}{\theta}\right)$

$\ln L = n \ln\left(\frac{2y}{\theta}\right) - \frac{\Sigma y^2}{\theta}$

$\frac{\partial \ln L}{\partial \theta} = n\left[\frac{-2y\theta^{-2}}{2y\theta^{-1}}\right] + (\Sigma y^2)\theta^{-2} = -\frac{n}{\theta} + \frac{\Sigma y^2}{\theta^2}$

$\frac{\partial^2 \ln L}{\partial^2 \theta} = \frac{n}{\theta^2} - \frac{2\Sigma y^2}{\theta^3}$

$-E\left(\frac{n}{\theta^2} - \frac{2\Sigma y^2}{\theta^3}\right) = -\frac{n}{\theta^2} + \frac{2E(\Sigma y^2)}{\theta^3} = -\frac{n}{\theta^2} + \frac{2n\cancel{\theta}}{\theta^{3\,2}} = \frac{n}{\theta^2}$

$MVUE = \frac{1}{-E\left(\frac{\partial^2 \ln L}{\partial^2 \theta}\right)} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$

Since $E(\hat{\theta}) = \theta$ & $MVUE = Var(\hat{\theta}) = \frac{n}{\theta^2}$

It's an efficient estimator

9. $X_1, \ldots, X_n$ iid $N(\theta, \theta)$, $\theta > 0$.

$E(T-\theta)^2 = Var(T) + Bias^2$ $\qquad \frac{\theta^2}{n}$

$= Var(\alpha_1 \bar{X} + \alpha_2(cS)) + Bias^2$

$= \alpha_1^2 \, Var(\bar{x}) + c^2\alpha_2^2 \, Var(S) + Bias^2$

$E(\bar{x}) = \theta \qquad Var(\bar{x}) = \frac{\theta^2}{n}$.

$Bias = (\alpha_1 + \alpha_2)\theta - \theta$

$E(cS) = \theta = c\,E(S) \rightarrow E(S) = \frac{\theta}{c}$.

$E(S^2) = E\left(\frac{(X^2_{n-1})\sigma^2}{(n-1)}\right) = \frac{\sigma^2}{n-1} E(X^2_{n-1}) = \sigma^2$

$Var(S) = E(S^2) - (E(S))^2 = \sigma^2 - \frac{\theta^2}{c^2}$

$Var(T) = \alpha_1^2 \cdot \frac{\theta^2}{n} + \alpha_2^2 c^2 \cdot \left(\sigma^2 - \frac{\theta^2}{c^2}\right)$

$= \alpha_1^2 \frac{\theta^2}{n} + \alpha_2^2 c^2 \sigma^2 - \alpha_2^2 \theta^2$

$Bias^2 = (\alpha_1\theta + \alpha_2\theta - \theta)(\alpha_1\theta + \alpha_2\theta - \theta)$

$= (\alpha_1^2\theta^2 + \alpha_1\alpha_2\theta^2 - \alpha_1\theta^2 + \alpha_1\alpha_2\theta^2 + \alpha_2^2\theta^2 - \alpha_2\theta^2 -$
$\alpha_1\theta^2 - \alpha_2\theta^2 + \theta^2)$

$= \alpha_1^2\theta^2 + 2\alpha_1\alpha_2\theta^2 - 2\alpha_1\theta^2 - 2\alpha_2\theta^2 + \alpha_2^2\theta^2 + \theta^2$

$\frac{\partial}{\partial \alpha_1} = 2\alpha_1 \frac{\theta^2}{n} + 2\alpha_1\theta^2 + 2\alpha_2\theta^2 - 2\theta^2 = 0$

$= \alpha_1\left(\frac{2\theta^2}{n} + 2\theta^2\right) = -2\alpha_2\theta^2 + 2\theta^2$

$\alpha_1 = \dfrac{-2\alpha_2\theta^2 + 2\theta^2}{\frac{2\theta^2}{n} + 2\theta^2} = \dfrac{n(\theta^2 - \alpha_2\theta^2)}{\theta^2 + n\theta^2}$

$\frac{\partial}{\partial \alpha_2} = 2\alpha_2 c^2\sigma^2 - 2\alpha_2\theta^2 + 2\alpha_1\theta^2 - 2\theta^2 + 2\alpha_2\theta^2$

$= \alpha_2(2c^2\sigma^2) + 2\alpha_1\theta^2 - 2\theta^2 = 0$

$\alpha_2 = \dfrac{2\theta^2 - 2\alpha_1\theta^2}{2c^2\sigma^2} = \dfrac{\theta^2 - \alpha_1\theta^2}{c^2\sigma^2}$