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HW - 2

1 $X \sim b(n, p)$

$P(x) = \binom{n}{x} (1-p)^n e^{x \log \frac{p}{1-p}}$

$h(x) = \binom{n}{x}$ $\quad w_i(\theta) = \log \frac{p}{1-p}$

$c(\theta) = (1-p)^n$ $\quad t_i(x) = x$

First theorem $E(x) = np$

from that

$E\left[\frac{d\, w_i(\theta)}{d\theta_i} \cdot t_i(x)\right] = -\frac{d \log (1-p)^n}{dp} = -n \frac{d \log(1-p)}{dp} = \frac{-n}{1-p} = \frac{d \log c(\theta)}{dp}$

$\hookrightarrow$ from here, $\frac{d^2 \log c(\theta)}{d^2 p} = -\frac{n}{(1-p)^2}$

$\frac{d\, w_i(\theta)}{dp} = \frac{d}{dp} \ln\left(\frac{p}{1-p}\right) = \frac{d}{dp}\left(\ln(p) - \ln(1-p)\right) = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$

$\frac{d^2 w_i(\theta)}{dp} = \frac{2p-1}{(p-p^2)^2} = \frac{2p-1}{(1-p)^2 p^2}$ $\quad u = 1$ $\quad v = p - p^2$ $\quad u' = 0$ $\quad v' = 1 - 2p$

$\text{var}\left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j} t_i(x)\right) = -\frac{d^2}{d\theta_j^2} \log c(\theta) - E\left(\sum_{i=1}^{k} \frac{d^2 w_i(\theta)}{d\theta_j^2} t_i(x)\right)$

$\text{Var}(t_i(x)) \left(\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial \theta_j}\right)^2 = -\frac{d^2}{d\theta_j^2} \log c(\theta) - E(t_i(x)) \cdot \sum_{i=1}^{k} \frac{d^2 w_i(\theta)}{d\theta_j^2}$

$\text{Var}(t_i(x)) \left(\frac{1}{p(1-p)}\right)^2 = \frac{n}{(1-p)^2} - \frac{np(2p-1)}{(1-p)^2 p^2}$

$\text{var}(t_i(x)) = p^2(1-p)^2 \cdot \frac{p^2 n - 2p^2 n + np}{p^2(1-p)^2}$

$= np - p^2 n = np(1-p)$

2 $\frac{d}{d\theta}\left(\int_x h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i \theta\, t_i(x)\right)\right) = \frac{d}{d\theta} 1$

derive in report to θ. both sides

$$\int_x h(x) \frac{d}{d\theta} c(\theta) \cdot \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right) + \int_x h(x)\, c(\theta) \cdot \frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i \theta\, t_i(x)\right) \cdot \exp\left(\sum_{i=1}^{k} (w_i(\theta)\, t_i(x)\right) = 0$$

$$-\int_x h(x) \frac{d}{d\theta} c(\theta) \cdot \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right) = \boxed{\int_x h(x) c(\theta) \cdot \boxed{\frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right)} \exp\left(\sum_{i=1}^{k} (w_i \theta) t_i(x)\right)}$$

* If $\int_x h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right) = 1$ then this $= 1$.

$$-\int_x h(x) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right) \frac{d}{d\theta} c(\theta) = \frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)$$

We know $E[g(x)] = \int_x g(x) f(x) dx$ ~ $f(x)$ is pdf.

$g(x) = \frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right)$; $f(x) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right)$ then,

$$E\left[\frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right)\right] = -\int_x h(x) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right) \frac{d}{d\theta} c(\theta) \cdot \frac{c(\theta)}{c(\theta)}$$

$$= -\underbrace{\int_x h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)}_{=1} \frac{d}{d\theta} \log c(\theta)$$

$$E\left[\frac{d}{d\theta}\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)\right] = -\frac{d}{d\theta} \log c(\theta)$$