Reinaldo Daniswara
604840665

HW 6

1 $E(\bar{x}) = \dfrac{E(x_1) + E(x_2) + \dots E(x_n)}{n} = \dfrac{n \cdot \lambda}{n} = \lambda$

$$E(s^2) = E \frac{1}{n-1} \Sigma (x_i - \bar{x})^2$$

$$= \frac{1}{n-1} E \Sigma (x_i - \mu - (\bar{x} - \mu))^2$$

$$= \frac{1}{n-1} E\left(\Sigma (x_i - \mu)^2 + n(\bar{x} - \mu)^2 - 2(\bar{x} - \mu)\Sigma(x_i - \mu)\right]$$

$$= \frac{1}{n-1} E\left(\Sigma (x_i - \mu)^2 + n(\bar{x} - \mu)^2 - 2n(\bar{x} - \mu)^2\right]$$

$$= \frac{1}{n-1} E\left(\Sigma (x_i - \mu)^2 - n(\bar{x} - \mu)^2\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 - n\frac{\sigma^2}{n}\right] = \sigma^2$$

$$Var(\bar{x}) = \frac{Var(x_1) + Var(x_2) + \dots Var(x_n)}{n}$$

$$= \frac{n^2 \lambda}{n} = n\lambda$$

$$Var(s^2) = Var\left(\frac{(n-1)}{(n-1)} \cdot \frac{\sigma^2}{\sigma^2} s^2\right)$$

$$= Var\left(\frac{\sigma^2}{n-1}(x^2_{n-1})\right)$$

$$= \left(\frac{\sigma^2}{n-1}\right)^2 \cdot Var(x^2_{n-1}) = \left(\frac{\sigma^2}{n-1}\right)^2 \cdot 2(n-1)$$

$$= \frac{\sigma^4}{(n-1)^{\cancel{2}}} \cdot 2\cancel{(n-1)} = \frac{2\sigma^4}{n-1}$$

Since $n\lambda$ is closer to λ than $\dfrac{2\sigma^4}{n-1}$

then, sample mean $\bar{x}$ is better estimator

b. $\bar{X}$ and cS are unbiased estimator of θ

$E(\bar{x}) = E(cS) = \theta$

$E(\alpha\bar{x} + (1-\alpha)cS) = \alpha E(\bar{x}) + (1-\alpha) E(cS)$

$= \alpha\theta + \theta - \theta\alpha = \theta$

$\hookrightarrow$ unbiased estimator

$Var(\alpha\bar{x} + (1-\alpha)cS) = \alpha^2 Var(\bar{x}) + (1-\alpha)^2 c^2 Var(s)$

$$Var(s) = Var\left[\frac{\sqrt{X^2_{n-1}}\ \sigma}{\sqrt{n-1}}\right] = \frac{\sigma^2}{n-1} Var(\sqrt{X_{n-1}})$$

$\Gamma\left(\frac{n-1}{2}, 2\right) \quad k = 1/2$

$$E(\sqrt{X_{n-1}}) = \frac{\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)\sqrt{2}}{\Gamma\left(\frac{n-1}{2}\right)} = \sqrt{2}\ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$Var(\sqrt{X_{n-1}}) = 2E(\sqrt{X_{n-1}}) = \frac{2\sqrt{2}\ \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$Var[cs] = c^2 Var[s]$$

$$= c^2 Var\left[\frac{\sqrt{X_{n-1}^2}\ \sigma}{\sqrt{n-1}}\right]$$

$$= \frac{c^2\sigma^2}{n-1}\left(\frac{2\sqrt{2}\ \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right)$$

$$= \frac{\sigma^2}{n-1}\left(\frac{\sqrt{n-1}\ \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^2 \frac{2\sqrt{2}\ \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$= \frac{2\sqrt{2}\sigma^2\ \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$Var[cs] = c^2 Var[s] = c^2 Var\left[\frac{\sqrt{X^2_{n-1}}\ \sigma}{\sqrt{n-1}}\right]$$

$$= \frac{c^2}{n-1}\sigma^2 Var(\sqrt{X^2_{n-1}})$$

$$= \frac{\theta^2}{n-1}\left(\frac{\Gamma\left(\frac{n-1}{2}\right)(n-1)}{2\Gamma\left(\frac{n}{2}\right)}\right)^2 \frac{2T_2\ \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{\theta^2(n-1)\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma(n_2)}\sqrt{2}$$

$$= \frac{\theta^2 c\sqrt{2}}{2}$$

HW 6

c $E\left(\frac{1}{\bar{x}}\right) = E\left(\frac{1}{\frac{x_1+x_2+\dots x_n}{n}}\right) = E\left(\frac{n}{x_1+x_2+\dots x_n}\right) = n \times \frac{1}{E(x_1)+E(x_2)+\dots E(x_n)}$

$= \frac{\cancel{n}}{\cancel{n}(\alpha\beta)} = \frac{1}{\alpha\beta}$

$E\left(\frac{\alpha}{\bar{x}}\right) = \alpha\, E\left(\frac{1}{\bar{x}}\right) = \cancel{\alpha}\cdot\frac{1}{\cancel{\alpha}\beta} = \frac{1}{\beta}$

Thus, $\frac{\alpha}{\bar{x}}$ is unbiased estimator of $\frac{1}{\beta}$

d $\sum_{i=1}^{4}(x_i-\bar{x})^2 = \frac{(x_1-x_2)^2}{2} + \frac{\left(x_3-\frac{(x_1+x_2)}{2}\right)^2}{\frac{3}{2}} + \frac{\left(x_4-\frac{(x_1+x_2+x_3)}{3}\right)^2}{4/3}$

- $E(x_1-x_2) = E(x_1)-E(x_2) = 0$ since x_1, x_2 iid
- $Var(x_1-x_2) = Var(x_1)+Var(x_2) = 2.$

$\Big\}\ \frac{(x_1-x_2)^2}{2} \sim \chi_1^2$

- $E\left(x_3-\frac{(x_1+x_2)}{2}\right) = E(x_3) - \frac{1}{2}E(x_1) - \frac{1}{2}E(x_2)$
  $= 0$

  $Var\left(x_3-\frac{(x_1+x_2)}{2}\right) = Var(x_3) + \frac{1}{4}Var(x_1) + \frac{1}{4}Var(x_2)$
  $= \frac{3}{2}$

$\Big\}\ \frac{\left(x_3-\frac{(x_1+x_2)}{2}\right)^2}{3/2} \sim \chi_1^2$

- $E\left(x_4-\frac{(x_1+x_2+x_3)}{3}\right) = 0$

  $Var\left(x_4-\frac{x_1+x_2+x_3}{3}\right) = Var(x_4) + \frac{1}{9}\left(Var(x_1)+Var(x_2)+Var(x_3)\right)$
  $= \frac{4}{3}$

$\frac{\left(x_4-\frac{(x_1+x_2+x_3)}{3}\right)^2}{4/3} \sim \chi_1^2$

$\rightarrow XX' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 \\ -1/3 & -1/3 & -1/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/3 \\ -1 & -1/2 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3/2 & -1/3 \\ 0 & -1/3 & 4/3 \end{bmatrix}$

$\hookrightarrow$ Independence.