Developing Algorithmic Machinery to Explore the Cosmological Horizon Problem by Numerically Solving Maxwell's Equations in the Kasner Metric

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A. Rationale

1.1 The Early Universe and the Cosmic Microwave Background

Beginning as an initial singularity of apparently infinite density and temperature, the early universe can be described by the Big Bang Theory, but there are still uncertainties about its earliest phases. (Guth, 2000)

Evidencing the occurrence of the Big Bang, the Cosmic Microwave Background (CMB) is remnant electromagnetic radiation from the early universe produced by the decoupling of photons from hot plasma, when photons were able to travel long distances for the first time in the universe's history. This occurred because photons were emitted when electrons combined with protons to form hydrogen atoms, during a period of the universe's expansion and cooling. These photons are still traveling today, and are observed as microwaves in the CMB. (Spergel and Zaldarriaga, 1997)

1.2 The Horizon Problem

The CMB shows that the universe is isotropic (uniform with respect to angle) and homogenous (uniform with respect to position), as the average CMB is a steady 2.726 ± 0.001 K throughout the entire sky. Consider an observer at any point in the universe examining opposite ends of the horizon (Figure 1). The view from the two ends of the observable universe demonstrates that light from opposite sides of the CMB has traveled 13 billion light-years to reach the observer. However, Einstein's Theory of Special Relativity establishes that no information can travel faster than the speed of light in a vacuum. Thus, it can be said that these opposites ends were outside of each other's horizons when the light was emitted from either area, meaning that the universe has not existed long enough for information traveling at the speed of light to connect these two regions, so there is a difficulty in explaining the apparent homogeneity of the CMB across causally disconnected areas in space. (Guth, 1981)

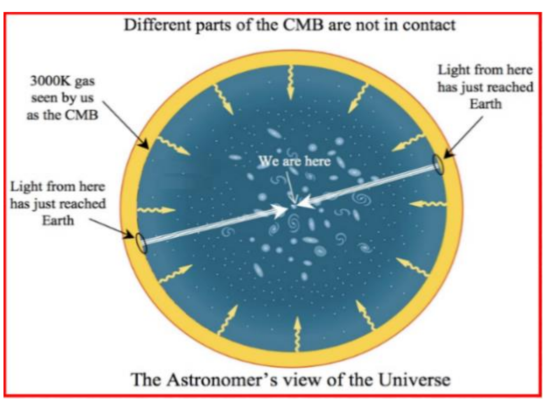

Figure 1. Areas from opposite ends of the observable universe are too far apart to have communicated, so they are "causally disconnected". (Whittle, 2008)

The evenness of the CMB poses the cosmological problem: areas in the universe too far apart to have communicated without violating causality appear to be in equilibrium. Since the universe was not initially homogenous, how could the CMB be at such a uniform temperature? (Brandenberger, 2000)

The inflationary model is a common attempt to solve the Horizon Problem by proposing that there was a 10^{-36} second epoch after the Big Bang Singularity during which exponential expansion allowed for the universe to equilibrate in temperature. However, this paradigm is ambiguous, as the physical mechanics behind inflation are unknown and unproven. For example, no inflatons (hypothetical inflation particles theorized to drive inflation), or signs in the CMB of primordial gravitational waves ("B-Modes") have been observed. (Akrami et al., 2019)

1.3 The Kasner Metric and Mixmaster Universe

Positing that there was a chaotic epoch where the early universe "churned" to evenly distribute matter and radiation, the Mixmaster Universe is Misner's (1969) solution to Einstein's field equations of General Relativity. Modelling oscillating intervals of the universe's expansion, the Mixmaster Universe describes growing and shrinking across three dimensions of space. This may have allowed for distant regions of space to come in contact with each other, so the early universe was able to achieve the homogeneity observed in the CMB. (Pettini, 2018)

Each of these chaotic, oscillating epochs can be described as exhibiting Kasner behavior, as the Mixmaster model consists of an infinite number of Kasner epochs, alternating between an expanding and contracting universe at different rates in different directions. To solve Einstein's field equations for gravitational behavior in this model, the Kasner metric is a function that defines spacetime interval *ds* in a vacuum as a function of time *t* as follows: (Brevik and Pettersen, 1997):

$$ds^2 = -c^2\,dt^2 + t^{2p_x}\,dx^2 + t^{2p_y}\,dy^2 + t^{2p_z}\,dz^2 \qquad \text{(Equation 1).}$$

The Kasner exponents must satisfy the following conditions:

$$p_x + p_y + p_z = 1 \ \text{ and } \ p_x^2 + p_y^2 + p_z^2 = 1 \qquad \text{(Equations 2, 3).}$$

The two interesting sets of integer solutions to Equations 2 and 3 are:

$$(0, 0, 1) \text{ and } \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right) \qquad \text{(Equation 4).}$$

These solutions are significant because they exhibit cylindrical symmetry in spacetime, because two indices are the same, resulting in symmetry across two axes of space.

However, previous work involving the Kasner metric typically assumed a restrictive simplification by approximating light as rays, rather than as waves. For a ray-tracing approximation (geometrical optics approximation), *ds² = 0*, and solving for $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ provides the light-ray propagation speeds. Instead, this study models light as waves rather than rays, allowing it to retain physical properties such as diffraction. (Montani et al., 2008)

1.4 EM Propagation and the Kasner Metric

Serving as the foundations for classical electromagnetism, Maxwell's equations describe the unifying relationships between electricity and magnetism. (Maxwell, 2013)

The electromagnetic wave equations in vacuum space can be derived from Maxwell's equations. However, Maxwell's equations are usually written for flat spacetime (in the absence of gravity), so they must be applied to the curved-space Kasner metric. Assuming that there is no medium to conduct charges, zero is substituted for both charge and current in Maxwell's equations. After making this substitution and using vector field operations and identities, the result is a second-order partial differential equation, which could be further simplified because symmetry results from two of the Kasner exponents being equal (Equation 4).This gives the time-dependent differential equation of the electric or magnetic field $F_i(t)$ in the direction normal to the plane of symmetry, where k_x, k_y, k_z are defined as the electromagnetic wave vector and overdots are time derivatives (Bochner 2015):

$$\left[(t^{-2p_x})k_x^2 + (t^{-2p_y})k_y^2 + (t^{-2p_z})k_z^2\right]F_i(t) + \left(\frac{p_i}{t}\right)\dot{F}_i(t) + \ddot{F}_i(t) = 0 \qquad \text{(Equation 5).}$$

Since $p_x = p_y$ in both Kasner cases (Equation 4), the x-y plane is the plane of symmetry, so $F_i(t)$ is in the z-direction (e.g., the z-component of the vector field *F*). Applying the Kasner metric (Equation 1) with the $(p_x, p_y, p_z) = (0, 0, 1)$ exponent case to the 4-dimensional wave equation (Equation 5) in spacetime results in the following modification to Equation 5:

$$\frac{1}{c^2}\ddot{F}_z(t) + \frac{1}{t}\dot{F}_z(t) + \left(k_x^2 + k_y^2 + \frac{k_z^2}{t^2}\right)F_z(t) = 0 \qquad \text{(Equation 6)}$$

However, the *(0, 0, 1)* Kasner case is physically trivial and does not give insight into the Horizon Problem because the *x* and *y* components are constant if the p_x and p_y exponents are 0, and z is directly proportional to time *t* because the p_z exponent is 1. As a result, it can be shown that this case does not represent expanding spacetime.

On the other hand, applying Kasner metric (Equation 1) with the $(p_x, p_y, p_z) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ exponent case results in:

$$\frac{1}{c^2}\ddot{F}_z(t) - \frac{1}{3t}\dot{F}_z(t) + \left(\frac{k_x^2+k_y^2}{t^{\frac{4}{3}}} + t^{\frac{2}{3}}k_z^2\right)F_z(t) = 0 \qquad \text{(Equation 7)}$$

More complex and interesting physics can be derived from Equation 7; its solutions resembling Bessel functions (oscillating functions associatied with cylindrical waveguides), but unfortunately, Equation 7 has no known exact analytical solution. Thus, the study will focus numerically finding solutions to Maxwell's equations in this physically interesting Kasner metric case.

B.

1. Research Question

In order to model the propagation speed of light waves in the early universe, can the solutions to the equation from the physically significant *(2/3, 2/3, -1/3)* Kasner case be numerically determined?

2. Hypothesis

It is hypothesized that, to test this study's simulation program, the light-ray expectation of the velocity of the models will match the result produced by a light wave model. This demonstrates the successful development of a calculation tool, which will be extended to another model with radiation-filled cosmology in future work, prospected to reveal unforeseen properties of light as waves that deviate from properties attributed to light as rays.

3. Expected Outcomes

Numerically determined wave speeds are expected to match the predicted ray speeds because the Kasner metric describes a vacuum universe. Matching wave and ray speeds will demonstrate the successful development of a calculation tool that can be used in a non-vacuum universe.

C.

- ## Methodologies

Algorithms in Wolfram Mathematica Version 11.3 will be written to conduct the numerical programs. All procedures will first be used on the following differential equation:

$$\frac{d^2y}{dt^2} = -y \qquad \text{(Equation 8)}$$

Equation 8 will be used as a simple test differential equation because it has well known solutions of sines and cosines. Afterwards, the same procedures will be conducted on the more complicated Equation 7, where the constant speed of light in a vacuum will be set as *c = 1* to simplify calculations.

3.1 Orthonormalizing Functions

Two solutions to the second-order differential equation of form Equation 8 will be found using the `NDSolve` function, and initial conditions will e varied so that the solution functions will not necessarily be orthogonal. This is to show that orthogonality can be found numerically, which will later be used on the more complex solutions to Equation 7. One of the solutions, *f(t),* will be normalized so that:

$$\int_{t_n}^{t_{n+2}} f(t)^2 = 1 \qquad \text{(Equation 9)}$$

across one period. The period will be found by finding a zero *n,* and finding the second zero *n+2* that occurred after *N.*

Then, a second solution to the differential equation, $g_0(t)$ will be modified with the following integral, to produce orthogonalized function $g(t)$:

$$g(t) = g_0(t) - \int_{t_n}^{t_{n+2}} f(t) g_0(t) \quad \text{(Equation 10)}$$

This will result in orthogonality such that:

$$\int_{t_n}^{t_{n+2}} f(t) g(t) = 0 \quad \text{(Equation 11)}$$

Then, $g(t)$ will be normalized using the same process applied to *f(t)*. All integration will be performed numerically using the `NIntegrate` function.

3.2 Defining the Wave Function and the Wavefront

The two orthonormalized functions, *f(t)* and $g(t)$, will be combined to form the wave function:

$$w(t) = f(t)\cos(k_i x + \phi_R) + g(t)\sin(k_i x + \phi_R) \quad \text{(Equation 12)}$$

As introduced in Equation 5, k_i is the component of wave vector (k_x, k_y, k_z) in the direction that the wave is propagating. For simplicity, this study will define the magnitude of k_i to be 1. Then, to define a wavefront at *w(0) = 0,* the phase shift ϕ_R can be solved to be:

$$\phi_R = \tan^{-1}\left(-\frac{f(0)}{g(0)}\right) \quad \text{(Equation 13)}$$

3.3 Propagating the Wavefront

To propagate the wavefront forwards, the wavefront *x(t)* will be defined as a function of time, based on the wave equation (Equation 12):

$$x(t) = \frac{1}{k_i}\left(\tan^{-1}\left(-\frac{f(t)}{g(t)}\right) - \phi_R\right) \quad \text{(Equation 14)}$$

Since $tan^{-1}(t)$ is a discontinuous function that “resets” every period π, a periodic step function $Q\pi$ will be added to Equation 14 to ensure that it would propagate forwards, where Q would increment by one, starting at zero, every time *x(t)* encounters a discontinuity. As a result, the propagating wavefront will be produced:

$$x(t) = \frac{1}{k_i}\left(\tan^{-1}\left(-\frac{f(t)}{g(t)}\right) - \phi_R\right) + Q\pi \quad \text{(Equation 15)}$$

The wavefront function *x(t)* will be manipulated by defining *x(t)* as a list of values, with a step size of $\Delta t = 0.0001$.

3.4 Finding Wave Velocity

To find wave velocity, the derivative of Equation 15 will be numerically approximated by iteratively taking the following:

$$x(t) = \frac{x(t+\Delta t) - x(t)}{\Delta t} \qquad \text{(Equation 16)}$$

Due to machine limitations, to ensure that Equation 16 will be calculated within a reasonable time frame, this study will use $\Delta t = 0.0001$. Then, the result will be plotted along with the predicted velocity produced from ray tracing (ie., solving Equation 1 for $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$ when $ds^2 = 0$).

Afterwards, residuals will be plotted to precisely show deviations from the ray prediction, using the equation:

$$Residual = speed_{wave} - speed_{ray} \qquad \text{(Equation 17).}$$

- **Risk and Safety**
  1. *Human Participants Research – N/A*
  2. *Vertebrate Animal Research – N/A*
  3. *Potentially Hazardous Biological Agents – N/A*
  4. *Hazardous Chemicals, Activities, and Devices – N/A*

- **Data Analysis**

  Data will be analyzed in Mathematica software by calculating the residual between experimentally determined wave speed and predicted wave speed, using Equation 17. Then, the residuals will be plotted.

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Project Summary:

NO ADDENDUMS EXIST