On Parameters of Quadratization of Scalar Polynomial ODE's

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Abstract

Quadratization is a mathematical process that allows us to reduce defined scalar polynomial ODE's into quadratic form via the introduction of a new variable. In other words, quadratization is a quadratic reformulation of a polynomial $\dot{x}$ with the introduction of new variables. The process of quadratization involves introducing new variables $z_1, z_2, z_3, ..., z_k$ with the hope of using each of these $z_i's$ to write $\dot{x}, z_1', z_2', z_3', ..., z_k'$ as quadratic. This process allows to solve optimization problems involving a large degree polynomial more conveniently as there are already well-established methods of solving quadratic polynomials. Quadratization has been recently used for model dimension reduction in [x]. The idea is that although a quadratized system is larger than the original one, there are more powerful reduction techniques available for quadratic systems. In this paper, we attempt to find parameters on defined scalar polynomial ODE's such that they can be quadratized using exactly one new variable. The most significant finding in this research paper is the following theorem: for scalar polynomial ODE $\dot{x}$ of at least degree 5, they can be quadratized using exactly one new variable if and only if some linear transformation of $\dot{x}$ is of the form $x^n + ax^2 + bx$.

1 Introduction

Definition 1. *Consider a scalar polynomial ODE $\dot{x} = p(x)$. Consider new variables $z_1 = z_1(x), z_2 = z_2(x), ..., z_m = z_m(x)$ where each $z_i(x)$ is a polynomial in x. We say that a scalar polynomial ODE $\dot{x} = p(x)$* is quadratized *by polynomials $z_1(x), z_2(x), ..., z_m(x)$ if the following conditions are satisfied:*

1. *$\dot{x} = p(x)$ can be written as a quadratic polynomial in $z_1, z_2, ..., z_m, x$.*
2. *For every i, $\dot{z_i} = z_i'(x)p(x)$ can be written as a quadratic polynomial in $z_1, z_2, ..., z_m, x$.*

Example 1. Consider the scalar polynomial ODE $\dot{x} = x^n$ with $n > 2$. Let $z(x) := x^{n-1}$. We will use z to quadratize $\dot{x}$. We can write:

$$\dot{x} = zx$$

$$\dot{z} = z'(x)\dot{x} = (n-1)x^{n-2}\dot{x} = (n-1)x^{n-2}x^n = (n-1)x^{2n-2} = (n-1)z^2$$

Thus, we have quadratized $\dot{x}$ as both $\dot{x}$ and $\dot{z}$ are can be written as quadratic in x and z.

Example 2. Consider the scalar polynomial ODE $\dot{x} = x^5 + x^4 + x^3 + x^2 + x + 1$. We let $z_1(x) := x^4$ and $z_2(x) := x^3$. It follows that:

$$\dot{x} = z_1x + z_1 + z_2 + x^2 + x + 1$$

$$\dot{z_1} = z_1'(x)\dot{x} = 4x^3\dot{x} = 4(z_1^2 + z_1z_2 + z_2^2 + z_1x + z_1 + z_2)$$

$$\dot{z_2} = z_2'(x)\dot{x} = 3x^2\dot{x} = 3(z_1z_2 + z_2^2 + z_1x + z_1 + z_2 + x^2)$$

Thus, we have quadratized $\dot{x}$ as $\dot{x}$, $\dot{z_1}$, and $\dot{z_2}$ are quadratic in z_1, z_2, x.

2 Scalar Polynomial ODE's of Degree 3 and 4

We will show that all $\dot{x}$ of degree 3 and 4 can be quadratized using exactly 1 new variable.

Proposition 1. *All $\dot{x}$ of degree 3 can be quadratized using exactly 1 new variable.*

Proof. Let $\dot{x} = x^3 + ax^2 + bx + c$ and $z(x) := x^2$. It follows that:

$$\dot{x} = zx + ax^2 + bx + c$$

$$\dot{z} = 2x\dot{x} = 2x(x^3 + ax^2 + bx + c) = 2z^2 + 2azx + 2bx^2 + 2cx$$

Thus, since both $\dot{x}$ and $\dot{z}$ have been quadratized with our new variable z, the proposition is proved. □

Now we will consider scalar polynomial ODE $\dot{x}$ of degree 4.

Proposition 2. *All scalar polynomial ODE's $\dot{x}$ of degree 4 can be quadratized using exactly 1 new variable.*

Proof. Let $\dot{x} = ax^4 + bx^3 + cx^2 + dx + e$ and $z(x) := x^3$. It follows that:

$$\dot{x} = azx + bz + cx^2 + dx + e$$

$$\dot{z(x)} = (3x^2)(\dot{x}) = 3x^2(ax^4 + bx^3 + cx^2 + dx + e) = 3az^2 + 3bx^5 + 3czx + 3dz + 3ex^2$$

Notice that the $3bx^5$ in $\dot{z(x)}$ term is the only problematic term when we let $z(x) := x^3$ as we cannot quadratize that term using only x and z. Thus, we attempt to eliminate the 3rd degree term or depress our scalar polynomial ODE using a change of variables. We will let $x = y - \lambda$.
It follows that:

$$\begin{aligned}\dot{x} &= a(y-\lambda)^4 + b(y-\lambda)^3 + c(y-\lambda)^2 + d(y-\lambda) + e = a(y^4 - 4\lambda y^3 + 6\lambda^2 y^2 - 4\lambda^3 y + \lambda^4)\\ &+ b(y^3 - 3\lambda y^2 + 3\lambda^2 y - \lambda^3) + c(y^2 - 2\lambda y + \lambda^2) + d(y-\lambda) + e = ay^4 + (-4a\lambda + b)y^3\\ &+ (6a\lambda^2 - 3b\lambda + c)y^2 + (-4a\lambda^3 + 3b\lambda^2 - 2c\lambda + d)y + (a\lambda^4 - b\lambda^3 + c\lambda^2 - d\lambda + e)\end{aligned}$$

Thus, to get rid of the third-degree term in the equation above, we must have:

$$-4a\lambda + b = 0$$

$$\lambda = \frac{b}{4a}$$

Hence, we are left with:

$$\dot{x} = ay^4 + (-\frac{3b^2}{8a} + c)y^2 + (\frac{b^3}{8a^2} - \frac{bc}{2a} + d)y + (-\frac{3b^4}{256a^3} + \frac{cb^2}{16a^2} - \frac{bd}{4a} + e)$$

Now, introducing $z(y) := y^3$, we can write:

$$\dot{x} = azy + (-\frac{3b^2}{8a} + c)y^2 + (\frac{b^3}{8a^2} - \frac{bc}{2a} + d)y + (-\frac{3b^4}{256a^3} + \frac{cb^2}{16a^2} - \frac{bd}{4a} + e)$$

$$\dot{z} = 3y^2(\dot{x}) = 3az^2 + (-\frac{9b^2}{8a} + 3c)zy + (\frac{3b^3}{8a^2} - \frac{3bc}{2a} + 3d)z + (-\frac{9b^4}{256a^3} + \frac{3cb^2}{16a^2} - \frac{3bd}{4a} + 3e)y^2$$

Since all 4^{th} degree polynomials can be shifted such that the third degree term is removed, the proposition has been proved. □

3 Experimentation: Gröbner Bases

For scalar polynomial ODE's $\dot{x}$ of degree 5, our experimentation involved python-programmed Gröbner Bases via SageMath. Gröbner Bases is a method that encompasses Gaussian Elimination, Euclidean Algorithm for polynomials, and the Simplex Algorithm of linear programming in order to answer the following query: given two polynomials p and q with coefficients $p_i's$ and $q_j's$, respectively, and a set of conditions on these two polynomials expressed through the determinants of a matrix, what are the constraints on the coefficients of p_i. The conditions we may set are wide ranging.
For more info on Gröbner Bases, you may view the following article
`https://math.berkeley.edu/~bernd/what-is.pdf`. The main way we apply a Gröbner Basis is for elimination and simplification of terms. There are many ways for one to simplify a system and find conditions on that system. Gröbner Bases is just one method. Before we delve into experimentation using Gröbner Bases, we illustrate a generic process of elimination to find the intersection of two circles

Example 3. We define the following system:

$$x^2 + y^2 = 4$$

$$(x+2)^2 + (y-1)^2 = 9$$

In order to solve such a system, we subtract the top equation from the bottom equation and are left with:

$$4x + 4 - 2y + 1 = 5$$

$$2x = y$$

Given this relationship between x and y, we can substitute back into one of our original equations and solve. Doing so, we get:

$$x^2 + (2x)^2 = 4$$

$$x = \sqrt{\frac{4}{5}} = 2\sqrt{\frac{1}{5}}$$

$$y = 2x = 4\sqrt{\frac{1}{5}}$$

Thus, we have concluded our example.

Now, we move on to experimentation with Gröbner Bases. In order to carry out our experimentation of quadratization, we invoke linear algebra. We want to find whether our 5^{th} degree polynomial p can be quadratized using another polynomial q. In order to do this, we must first understand the idea of linear dependence. Linear dependence means that for vectors $v_1, v_2, ..., v_n$, there exists constants $c_1, c_2, ..., c_n$, not all zero, such that:

$$c_1v_1 + c_2v_2 + ... + c_nv_n = 0$$

Linear dependence happens if and only if the matrix of these vectors has determinant 0.

So, we strive to find conditions on when our degree 5 polynomial p can be quadratized with just one new variable q.

We let:

$$p = p_5x^5 + p_4x^4 + p_3x^3 + p_2x^2 + p_1x + 1$$

$$q = x^4 + q_3x^3 + q_2x^2 + q_1x + q_0$$

$$q' = (p)(4x^3 + 3q_3x^2 + 2q_2x + q_1)$$

where p is the degree 5 polynomial we wish to quadratize and q is the new variable we introduce in order to quadratize. To find whether p can be written as a linear combination of $1, x, q, x^2, xq$, we have basis vectors

$\{1, x, x^2, x^3, x^4, x^5\}$ along the rows and we have basis vectors $\{1, x, x^2, q, xq, p\}$ along the columns. We will call this matrix our p matrix.

$$\text{p matrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3 & q_2 & p_3 \\ 0 & 0 & 0 & 1 & q_3 & p_4 \\ 0 & 0 & 0 & 0 & 1 & p_5 \end{pmatrix}$$

To find whether q' can be written as a linear combination of $1, x, x^2, q, xq, pq', q^2$, we have basis vectors $\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\}$ along the rows and we have basis vectors $\{1, x, x^2, q, xq, pq', q^2\}$ along the columns. We will call this matrix our q' matrix.

$$\text{q' matrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_3 & q_2 & (p_3q_1 + 2p_2q_2 + 3p_1q_3 + 4p_0) & (2q_3q_0 + 2q_2q_1) \\ 0 & 0 & 0 & 1 & q_3 & (p_4q_1 + 2p_3q_2 + 3p_2q_3 + 4p_1) & (q_2^2 + 2q_1q_3 + 2q_0) \\ 0 & 0 & 0 & 0 & 1 & (p_5q_1 + 2p_4q_2 + 3p_3q_3 + 4p_2) & (2q_1 + 2q_2q_3) \\ 0 & 0 & 0 & 0 & 0 & (4p_3 + 3p_4q_2p_5q_2) & (q_3^2 + 2q_2) \\ 0 & 0 & 0 & 0 & 0 & (3p_5q_3 + 4p_4) & (2q_3) \\ 0 & 0 & 0 & 0 & 0 & 4p_5 & 1 \end{pmatrix}$$

We use Gröbner Bases to find conditions on when the determinant of the two matrices are 0. We find that for the determinants of the matrices above to be zero, we must have the following conditions:

$$p_0 - \frac{1}{5}p_1p_4 + \frac{1}{25}p_2p_4^2 - \frac{6}{3125}p_4^5 = 0$$

$$p_3 - \frac{2}{5}p_4^2 = 0$$

Notice that from the two equations above: if $p_4 = 0$, then it follows that $p_0 = 0, p_3 = 0$. Since we can always shift a 5^{th} degree polynomial such that the x^4 term is gone, the above statement holds (see Fact 1 in Section 4). So, we have that for a degree 5 polynomial p, it can only be quadratized with a single new variable q if a linear transformation of p is of the form $p = x^5 + p_2x^2 + p_1x$.

The conditions given the Gröbner Bases for higher degrees have not been given in this paper as there are so many conditions. We have included the rest of this experimentation and the code used to conduct this experimentation in the Appendix. This experimentation gives us reason to believe that a scalar polynomial ODE $\dot{x} = p(x)$ of degree n can only be quadratized using exactly one new variable $z = z(x)$ if and only if some linear transformation of $\dot{x}$ is of the form $\dot{x} = x^n + ax^2 + bx$. We will prove so in the following section.

4 Parameters on $\dot{x}$ for Quadratization Using Only One New Variable

Lemma 1. *Assume that a scalar polynomial ODE $\dot{x} = x^n + q(x)$ with $n \geq 5$ and $\deg q(x) \leqslant n-1$ can be quadratized by a single new variable. Then the degree of the new variable $z = z(x)$ must be $n-1$.*

Proof. The first term of $\dot{x}$, x^n, must be quadratized as a quadratic in x and z. For $n \geqslant 5$, only terms z, xz, and z^2 may involve x^n, and this might happen only if $\deg z \geqslant 3$. Thus, $\deg z < \deg xz < \deg z^2$, so x^n must be the leading monomial of one of them. Thus, $\deg z \in \{n, n-1, n/2\}$.

If $\deg z = n$, then $\deg \dot{z} = 2n-1$. Since $\deg z^2 > 2n-1$ and the degree of any other quadratic monomial in x and z is less than $2n-1$, $\dot{z}$ cannot be quadratized.

If n is odd, the only remaining option is $\deg z = n-1$, so we are done. Consider the case of even n and $\deg z = \frac{n}{2}$. Let $z(x) := \alpha x^{\frac{n}{2}} + r(x)$ and $\dot{x} = x^n + q(x)$ where $\deg r(x) < n/2$, $\deg q(x) < n$, and $\alpha \neq 0$.

Since $\dot{z}$ must be quadratized, we have that:

$$\dot{z} = z'(x)\dot{x} = \left(\alpha\frac{n}{2}x^{\frac{n-2}{2}} + \mathrm{o}(x^{\frac{n-2}{2}})\right)(x^n + \mathrm{o}(x^n)) = \alpha\frac{n}{2}x^{\frac{3n-2}{2}} + \mathrm{o}(x^{\frac{3n-2}{2}}). \tag{1}$$

Since $\deg z(x) \geqslant 3$, the degree of any quadratic polynomial in x and $z(x)$ is at most $2\deg z(x) \leqslant n$. Therefore, a quadratic polynomial in x and $z(x)$ cannot have the degree $\frac{3n-2}{2}$ for $n \geqslant 6$. □

Fact 1. *All scalar polynomial ODE's $\dot{x} = x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$ can be shifted uniquely such that the $n-1$ term disappears.*

Proof. Let $x = y - \lambda$. Substituting for x in $\dot{x}$, we get:

$$\dot{x} = (y-\lambda)^n + a_{n-1}(y-\lambda)^{n-1} + ... + a_2(y-\lambda)^2 + a_1(y-\lambda) + a_0 \tag{2}$$

After expanding, we find that the coefficient behind the $n-1$ term is $n\lambda + a_{n-1}$. Thus, it follows that for the $n-1$ term to tend to zero, we must have $\lambda = \frac{-a_{n-1}}{n}$. Thus, since $n > 0$, this shift is always possible and is unique as we only get one value for λ. □

Theorem 1. *For $\dot{x}$ of at least degree 5, it can be quadratized using exactly one new variable if and only if some linear transformation of $\dot{x}$ is of the form $x^n + ax^2 + bx$.*

Proof. We will first prove the easier statement: if $\dot{x}$ is of the form $\dot{x} = x^n + ax^2 + bx$, then it can be quadratized using one new variable. Let $z := x^{n-1}$. It follows that:

$$\dot{x} = zx + ax^2 + bx \tag{3}$$

$$\dot{z} = (n-1)x^{n-2}(\dot{x}) = (n-1)(x^{2n-2} + ax^n + bx^{n-1}) = (n-1)(z^2 + azx + bz) \tag{4}$$

Now, we will prove the converse: if $\dot{x}$ can be quadratized using one new variable, then some linear transformation of $\dot{x}$ must be of the form $x^n + ax^2 + bx$. We let:

$$\dot{x} = x^n + q(x) \tag{5}$$

By Fact 1, we can uniquely shift $\dot{x}$ such that the $n-1$ term goes away, so we can further assume that $\deg q(x) \leqslant n-2$. Using Lemma 1, we know that our new polynomial $z(x)$ must begin with the term x^{n-1}. So we let:

$$z := x^{n-1} + r(x) \tag{6}$$

Since $\dot{x}$ must be quadratized by z, we obtain the following.

$$\dot{x} = x^n + q(x) = xz + ax^2 + bx + c = x^n + xr(x) + ax^2 + bx + c \tag{7}$$

Thus, from equation (7), it follows that:

$$q(x) = xr(x) + ax^2 + bx + c \tag{8}$$

From equation (8), we see that $d := \deg r(x) \leqslant n-3$. Thus we can write $r(x) = c_dx^d + c_{d-1}x^{d-1} + ... + c_0$. Then $r'(x) = (d)c_dx^{d-1} + (d-1)c_{d-1}x^{d-2} + ... + c_1x$. Now, we look to write $\dot{z}$ in two different ways.

$$\begin{aligned}\dot{z} &= z'(x)\dot{x} = ((n-1)x^{n-2} + r'(x))(x^n + xr(x) + ax^2 + bx + c) = (n-1)x^{2n-2} + (n-1)x^{n-1}r(x) \\ &+ x^nr'(x) + \mathrm{o}(x^{n-1+d}) = (n-1)x^{2n-2} + (n-1)c_dx^{n-1+d} + (d)c_dx^{n-1+d} + \mathrm{o}(x^{n-1+d})\end{aligned} \tag{9}$$

Now, since $\dot{z}$ must be quadratized, it must have the form:

$$\begin{aligned}\dot{z} &= (n-1)z^2 + a_1zx + a_2z + a_3x^2 + a_4x + a_5 = (n-1)x^{2n-2} + 2(n-1)x^{n-1}r(x) + \mathrm{o}(x^{n-1+d}) \\ &= (n-1)x^{2n-2} + 2(n-1)c_dx^{n-1+d} + \mathrm{o}(x^{n-1+d})\end{aligned} \tag{10}$$

for constants a_1, a_2, a_3, a_4, a_5. Setting equations (9) and (10) equal to each other and simplifying, we obtain the following first order differential equation on r(x):

$$(n-1)c_d x^{n-1+d} + (d)c_d x^{n-1+d} + \mathrm{o}(x^{n-1+d}) = 2(n-1)c_d x^{n-1+d} + \mathrm{o}(x^{n-1+d}) \tag{11}$$

If one would like to view the long-hand expanded first order differentiual equation on r(x), please see Appendix.

Rearranging and simplifying (11) in order to analyze the highest degree terms on both sides of the equation, we have:

$$(n-1)c_d = (d)c_d \tag{12}$$

Given equation (12), we show that the only solutions to equation (11) is $r(x) = 0, r'(x) = 0$.

Claim 1. $r(x) = 0, r'(x) = 0$

We will use proof by contradiction. We assume that $r(x)$ is nonzero. Thus, we say it has degree d for d in $\{0, 1, 2, 3..., n-3\}$. By definition of the degree of a polynomial, $c_d \neq 0$. Thus, we have:

$$(n-1)c_d = (d)c_d \tag{13}$$

Since $n-1$ is not a possible value for d, it follows that c_d must equal zero for this equation to hold true. However, this is a contradiction as we had that $c_d \neq 0$. Thus, it follows that our assumption that $r(x)$ is nonzero is false. So, $r(x) = 0$. So, we have proved the claim.

Now, we can let $r(x) = 0$. We attempt to quadratize. We let:

$$z(x) := x^{n-1}, \quad \dot{x} = x^n + q(x), \quad q(x) = ax^2 + bx + c$$

It follows that:

$$\dot{x} = zx + ax^2 + bx + c \tag{14}$$

We can also write:

$$\dot{z} = z'(x)\dot{x} = ((n-1)x^{n-2})(x^n + ax^2 + bx + c) = (n-1)x^{2n-2} + (na-a)x^n + (nb-b)x^{n-1} + (nc-c)x^{n-2} \tag{15}$$

Since $\dot{z}$ must be quadratized, it must be of the form:

$$\dot{z} = (n-1)z^2 + a_1 zx + a_2 z + a_3 x^2 + a_4 x + a_5 \tag{16}$$

Notice in equation (15), we have the term $(nc-c)x^{n-2}$. This term cannot be quadratized for n at least 5 because based on equation (16), there is no x^{n-2} term. Thus, we find that $nc - c = 0$. Since $n \geq 5$, the only way $nc - c$ is zero is if c is 0. All other terms in equation (15) can be quadratized using some quadratic combination of z and x. Now, we have our desired form of $x' = x^n + ax^2 + bx$ for n at least 5. Thus, we have proved the conjecture. □

5 Conclusion

We have proved that all scalar polynomial ODE's $\dot{x}$ of degree 3 and 4 can be quadratized using exactly one new variable. The major theorem that is proved in this paper is: for scalar polynomial ODE's $\dot{x}$ of at least degree 5, they can be quadratized using exactly one new variable if and only if some linear transformation of $\dot{x}$ is of the form $x^n + ax^2 + bx$. In order to arrive at such a conclusion, we experimented not only by hand, but with the assistance of python-programmed Groebner Bases. In the proof of this theorem, we mainly used algebraic and combinatorial methods to analyze differential equations. Further work will be taken on the investigation of quadratization using more than one new variable. We know that all scalar polynomial ODE's $\dot{x}$ with degree at most 6 can be quadratized using exactly two new variables. This is because we can form all terms x^i for $i \in \{0, 1, 2, ..., 10\}$ using quadratic combinations of $z_1 := x^5$, $z_2 := x^4$, and x. Furthermore, we know that all scalar polynomial ODE's $\dot{x}$ with degree at most 8 can be quadratized using exactly three new variables. We strive to investigate the parameters on $\dot{x}$ such that it can be quadratized using exactly k variables.

6 Appendix

The longer version of equation (11) is the following:

$$\begin{aligned}&(n-1)x^{2n-2}+(n-1)x^{n}+(n-1)x^{n-1}(r(x)+1)+c(n-1)x^{n-2}+r'(x)(x^{n}+xr(x)+ax^{2}+bx+c)\\&=(n-1)x^{2n-2}+(n-1)r^{2}(x)+2x^{n-1}r(x)(n-1)+a_{1}(x^{n}+xr(x))+a_{2}(x^{n-1}+r(x))+a_{3}x^{2}+a_{4}x+a_{5}\end{aligned}$$

As an extension upon our Gröbner Bases section, we provide the conditions for quadratization obtained for degree 6, 7, and 8 polynomial. We also provide the general code for degree n polynomial Gröbner Bases. The conditions for quadratization for degree 6 polynomial are the following:

$$p_0-\frac{1}{6}p_1p_5=0$$
$$p_3=0$$
$$p_4=0$$
$$p_5^2=0$$

Notice that from the 4 equations above, it follows that for a degree 6 polynomial p to be quadratized using exactly one new variable q, all of p_0, p_3, p_4, p_5 must be zero. Thus, a polynomial p of degree 6 can only be quadratized if it is of the form $p = x^6 + p_2x^2 + p_1x$.

For degree 7 polynomial, the conditions of quadratizations given by our Gröbner Basis are:

$$p_0^3-\frac{1}{343}p_1^3p_6^3=0$$
$$p_0^2p_1-\frac{1}{49}p_1^3p_6^2+\frac{2}{343}p_1^2p_2p_6^3+\frac{124}{386561}p_2^3p_6^5=0$$
$$p_0^2p_6-\frac{1}{49}p_1^2p_6^3+\frac{464}{55223}p_2^2p_6^5=0$$
$$p_0p_1^2p_6-\frac{1}{7}p_1^3p_6^2+\frac{1}{49}p_1^2p_2p_6^3-\frac{240}{4889}p_2^3p_6^5=0$$
$$p_0p_1p_6^2-\frac{1}{7}p_1^2p_6^3+\frac{156}{7889}p_2^2p_6^5=0$$
$$p_0p_3=0$$
$$p_0p_4-\frac{5}{1127}p_2p_6^5=0$$
$$p_0p_5-\frac{41}{49}p_0p_6^2+\frac{1}{7}p_1p_5p_6-\frac{1}{343}p_1p_6^3-\frac{3}{49}p_2p_5p_6^2+\frac{43}{2401}p_2p_6^4=0$$
$$p_0p_6^3-\frac{9}{161}p_2p_6^5=0$$
$$p_1^2p_5p_6-\frac{3}{7}p_1^2p_6^3-\frac{960}{7889}p_2^2p_6^5=0$$
$$p_1p_5p_6^2-\frac{7}{23}p_2p_6^5=0$$
$$p_1p_6^4-\frac{121}{161}p_2p_6^5=0$$
$$p_3^3=0$$
$$p_3p_4=0$$
$$p_3p_5=0$$
$$p_3p_6-\frac{5}{161}p_6^5=0$$

$$p_4^2 = 0$$
$$p_4 p_5 - \frac{40}{1127} p_6^5 = 0$$
$$p_4 p_6 + \frac{23}{56} p_5 p_6^2 - \frac{1}{8} p_6^4 = 0$$
$$p_5^2 + \frac{22}{49} p_5 p_6^2 - \frac{1}{7} p_6^4 = 0$$
$$p_5 p_6^3 - \frac{7}{23} p_6^5 = 0$$
$$p_6^6 = 0$$

After careful analysis, we found that p_6, p_5, p_4, p_3, p_0 all must equal zero for p with degree 7 to be quadratized using exactly one new variable. So, it follows that $p = x^7 + p_2 x^2 + p_1 x$.

Below is the code for the Gröbner Basis experimentation for degree 6 scalar polynomial ODE.

```
R.<q0, q1, q2, q3, q4, p0, p1, p2, p3, p4, p5, x> = PolynomialRing(QQ, order = "lex")

def coeffs_in_x(poly, deg):
    return [ poly.coefficient({x : i}) for i in range(deg + 1) ]

p = p0 + p1 * x + p2 * x^2 + p3 * x^3 + p4 * x^4 + p5 * x^5 + x^6
q = q0 + q1 * x + q2 * x^2 + q3 * x^3 + q4 * x^4 + x^5

p_matrix = matrix([
    coeffs_in_x(R(1), 5),
    coeffs_in_x(x, 5),
    coeffs_in_x(x^2, 5),
    coeffs_in_x(q, 5),
    coeffs_in_x(x * q, 5),
    coeffs_in_x(p, 5)
])

q_prime_matrix = matrix([
    coeffs_in_x(R(1), 10),
    coeffs_in_x(x, 10),
    coeffs_in_x(x^2, 10),
    coeffs_in_x(q, 10),
    coeffs_in_x(x * q, 10),
    coeffs_in_x(q^2, 10),
    coeffs_in_x(q.derivative(x) * p, 10)
])

polys = p_matrix.minors(5) + q_prime_matrix.minors(7)
print(polys)

I = ideal(polys)
gb = I.groebner_basis()
only_in_p = [ poly for poly in gb if (poly in QQ[p0, p1, p2, p3, p4, p5]) ]
for poly in only_in_p:
    print(poly)
```

Below is the code for the Gröbner Basis experimentation for degree 7 scalar polynomial ODE. One may change the code in order to arrive at conditions for quadratization of higher degree scalar polynomial ODE.

```
R.<q0, q1, q2, q3, q4, q5, p0, p1, p2, p3, p4, p5, p6, x> = PolynomialRing(QQ, order = "lex")

def coeffs_in_x(poly, deg):
    return [ poly.coefficient({x : i}) for i in range(deg + 1) ]

p = p0 + p1 * x + p2 * x^2 + p3 * x^3 + p4 * x^4 + p5 * x^5 + p6 * x^6 + x^7
q = q0 + q1 * x + q2 * x^2 + q3 * x^3 + q4 * x^4 + q5 * x^5 + x^6

p_matrix = matrix([
    coeffs_in_x(R(1), 5),
    coeffs_in_x(x, 5),
    coeffs_in_x(x^2, 5),
    coeffs_in_x(q, 5),
    coeffs_in_x(x * q, 5),
    coeffs_in_x(p, 5)
])

q_prime_matrix = matrix([
    coeffs_in_x(R(1), 12),
    coeffs_in_x(x, 12),
    coeffs_in_x(x^2, 12),
    coeffs_in_x(q, 12),
    coeffs_in_x(x * q, 12),
    coeffs_in_x(q^2, 12),
    coeffs_in_x(q.derivative(x) * p, 12)
])

polys = p_matrix.minors(5) + q_prime_matrix.minors(7)
print(polys)

I = ideal(polys)
gb = I.groebner_basis()
only_in_p = [ poly for poly in gb if (poly in QQ[p0, p1, p2, p3, p4, p5, p6]) ]
for poly in only_in_p:
    print(poly)
```

7 Reference Links

1. https://arc.aiaa.org/doi/10.2514/1.J057791

2. https://math.berkeley.edu/~bernd/what-is.pdf