Egger and Grabert

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January 27, 2021

- Context for the paper

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- Closer Look at the Screening Behaviour

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1 Context of the Paper

The problem of how to include the effect of lead/reservoirs into a description of one dimensional systems was one that was dealt with in the latter half of the 90's. The 95 papers of Maslov/Stone and Ponomarenko dealt with the issue through having the Luttinger parameter vary in space with the system - reaching the values of $K = 1$ and v_F corresponding to a Fermi gas outside the wire. Safi/Schulz dealt with this in using a Landauer approach rather than the Kubo formalism, but both rely on having a sharp distinction between the parameters in the wire and then taking the d.c. limit of the conductance. Taking this limit essentially blurs out the deviation in the parameter, resulting in no effect of the wire and its properties depending entirely on the reservoirs. There is another way of formulating the problem which resulted in Egger and Grabert radiative boundary conditions.

The boundary conditions were first mentioned in their 1996 paper [1] with the average density of left and right movers being determined by the chemical potential of the reservoirs. The effect of an external voltage field is then found through the using a generating functional and the absence of renormalisation of the conductance by interactions is shown. This derivation is not very physically clear but is quite mathematically concrete. It is in their 98 paper [2] that a more physically obvious description is given.

The 98 paper however relies on insights developed in a 97 paper [3] that describes the screening of electron charge in 1D (with an erratum correction to the 96 [4]). This approach is close to Kuwbata's reasoning for why there is no renormalisation of the conductance where the voltage is redefined by the screening because of the Luttinger Liquid. It is this explanation that I will present first in the notes as I think it shows the core of the derivation.

2 The 98 Paper - Voltage as a Boundary Condition

This paper aims to generalise the Landauer approach to situations where there is strong interactions between the electrons, considering a single channel wire with a single impurity that is adiabatically connected to external voltage sources. They look into the voltage drop across the impurity (an assumption Kane and Fisher requires) and if this is always the case. They claim that this Landauer approach is more general than Kubo based investigations and doesnt require the assumption that $K = 1$ models ideal reservoirs.

2.1 Setting Up Our Hamiltonian

Considering a system of a quantum wire, $-L/2 < x < L/2$, with an impurity at the origin. The first question to ask is what is causing the interactions to be short range and in effect justify the use of the Luttinger Liquid model. It is in their 97 paper that they look into this question (so more detail about this assumption will be looked into

in that section) and they conclude that the 'screening gate' is responsible for this screening of interactions. This screening gate is separate from the wire and I assume corresponds to the 'side gate' in experiments - the thing that causes the constriction.

If we are using a Luttinger liquid model to describe our system, we by design have contact interactions that are a delta function in real space. This has been reduced from the usual long range Coulomb interaction by screening by other electrons which introduces a length scale over which the interaction decays. The interaction will only have this contact form if the length scales we are considering are larger than the screening length imposed by the gate. So introducing the displacement field $\theta(x)$ and its dual field $\phi(x)$ and bosonising our kinetic Hamiltonian that describes free electrons in 1D, we end up with the usual Luttinger Liquid Hamiltonian form,

$$H_0 = \frac{\hbar v_F}{2} \int_{-L/2}^{L/2} dx [(\partial_x \phi)^2 + (\partial_x \theta)^2] \tag{1}$$

see Luttinger Liquid notes for details. We can include interactions with $H_{int} = u_0/2 \int dx \rho^2(x)$ (so already assuming contact interactions) and $\rho(x)$ is the difference in density from its usual equilibrium value k_F/π. The backscattering processes (ie the $2k_F$ part in the interaction) is found to the RG irrelevant and the density in the interacting part of the Hamiltonian can be taken to be the lowest order $\rho = \rho_L + \rho_R$ and means that the spacial derivative of our displacement field is the density $\sqrt{\pi}\rho = \partial_x \theta$. We therefore get our full interacting Hamiltonian,

$$H_{LL} = \frac{\hbar \nu}{2} \int_{-L/2}^{L/2} dx [K(\partial_x \phi)^2 + (\partial_x \theta)^2/K] \tag{2}$$

where we assume translational symmetry in our system so that our velocity of excitations equals $v = v_F/K$. An impurity is included in its usual form of electron backscattering, which when bosonised gives $H_{imp} = \lambda \cos(\sqrt{4\pi}\theta(x=0))$.

2.2 Looking Into Reservoirs

This wire is connected to reservoirs held at chemical potentials $\mu_{L,R}$, where good screening of higher dimensional ($d > 1$) Fermi liquids means that the difference between the two chemical potentials divided by charge is the two terminal voltage (Not quite sure why the screening is needed?).

Our aim here is to re-express the interaction of the Luttinger Liquid as an interaction with a spatially varying electrochemical field, which we will call $\varphi(x)$. This reformulation allows us to visualise the system and the gate that provides the constriction as a capacitor. The figure below shows this[5],
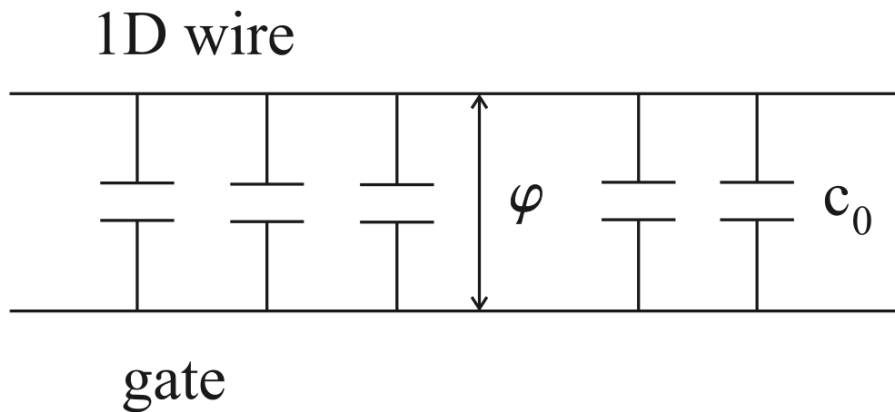
This allows to to relate how the capacitor system would behave with extra charge to the underlying question of how Luttinger Liquids behave.

The Luttinger Liquid parameter has its usual form $K = (1 + u_0/\pi\hbar v_F)^{-1/2}$, where u_0 is the strength of the contact interaction. They then relate the interaction term of the Hamiltonian to the charging energy of a capacitor,

$$H_{int} = \frac{u_0}{2}\int dx \rho^2(x), \qquad E = \frac{1}{2}CV^2 = \frac{Q^2}{2C} = \frac{e^2}{2c}\int dx \rho^2(x) \tag{3}$$

which if we want these to be equal enforces that the capacitance per unit length is $c = e^2/u_0$. This then motivates them to identify the electrostatic potential $\varphi(x)$ in the wire by comparing the way electrostatic potentials normally enter Hamiltonians (linearly coupled to the density) to our interacting Hamiltonian

$$H_V = \frac{e}{2}\int dx\ \rho(x)\varphi(x), \qquad e\varphi(x) = u_0\rho(x) \tag{4}$$

essentially obtaining a 'short ranged' Poisson equation that relates the electrostatic potential to the density. This potential is important as this directly gives the local potential drop between the wire and the object that provides the screening for the interactions (screening backgate). If there is no interaction ($u_0 = 0$ or $K = 1$) then $\varphi(x) = 0$ so there is no voltage drop between the screening backgate and the wire - they claim this means that the backgate is within the wire. If there is no gate and the interactions are not screened then putting $K \to 0$ (so very strong interactions) will work in the long wavelength limit.

So now we consider right and left moving densities being injected into the system $\rho^0_{L,R}$, with ρ^0_R being injected from the left reservoir. This will charge the gate-wire capacitor and cause a voltage drop in accordance with our new short range Poisson equation, shifting the energy of the band upwards by $e\varphi(x)$ due to the increased local energy from the electrostatic field. This is using our knowledge of the capacitor system, which we will now try to relate back to the Luttinger Liquid outlook.

Assuming this shift is small compared to the Fermi energy so that the density of states remains constant ($1/\pi\hbar v_F$) then the density will be shifted by $-e\varphi(x)/\pi\hbar v_F$. The actual density is then self consistently determined,

$$\rho(x) = \rho_R(x) + \rho_L(x) = \rho^0_R(x) + \rho^0_L(x) - e\varphi(x)/\pi\hbar v_F \tag{5}$$

which can be reexpressed in terms of the LL interaction parameter

$$\rho(x) + e\varphi(x)/\pi\hbar v_F = \rho^0_R + \rho^0_L$$
$$\rho(x) + \frac{u_0\rho(x)}{\pi\hbar v_F} = \rho^0_R + \rho^0_L$$

which gives the following relation

$$\rho(x) = K^2[\rho^0_L + \rho^0_R] = \rho_L(x) + \rho_R(x) \tag{6}$$

as our full density can always be split up into left and right movers. This is the crucial expression that is examined in more detail in the other papers, and in my notes on those papers! Note that the electrostatic potential only enters the equations as the sum of the densities of right and left movers, which implies that the difference of these densities (current) remains invariant,

$$\rho_R - \rho_L = \rho^0_R - \rho^0_L. \tag{7}$$

Multiplying this by the charge and Fermi velocity gives the current which can be computed at any point because

of the continuity equation.

Having got these relations between the bare and full densities, we can rewrite our equations in terms of the bare left/right densities.

$$2\rho_R^0(x) = (K^{-2} + 1)\rho_R(x) + (K^{-2} - 1)\rho_L(x) \tag{8}$$

$$2\rho_L^0(x) = (K^{-2} - 1)\rho_R(x) + (K^{-2} + 1)\rho_L(x) \tag{9}$$

We can then formulate the boundary condition of the bare injected densities in terms of the chemical potential of the reservoir, which at zero temperature and when applied voltage is much less than the band width (so that we can assume that the density of states is constant) will be:

$$\rho_L^0(-L/2) = eU_1/2\pi\hbar v_F, \qquad \rho_R^0(L/2) = eU_2/2\pi\hbar v_F \tag{10}$$

The factor of 1/2 comes from the fact that we only consider left and right movers, halving the usual number of states at a particular energy. These boundary conditions will not depend on any properties of the quantum wire. Substituting this in gives the our description where the effect of the reservoirs are included as boundary conditions for the full right and left densities.

2.2.1 Length Scale Interlude

Having got this result relating the injected density to the full density, we shall take a step back to see the assumptions that underline what we have done. The first most obvious one is that the density of states is constant which was spelled out explicitly in the derivation - implying low temperature and small 2 terminal voltage (ie integrating over a small energy range). The other assumptions are more hidden.

Next, the equating of the injected density which has no dependence on the wire (Eq 10) to the full, self consistently determined density implicitly assumes something. This is done at two points (ie $x = \pm L/2$) so we are essentially assuming that the interaction in sharply switched on as we would expect some decay into reservoirs and there to be some dependence on the properties of the wire for Eq 10 if they were smoothly switched on. This assumption can be made if the spacial extent over which the interactions are turned on is much smaller than the density scale (which will soon be related to the displacement field).

The other assumption that they outline is the adiabaticity of the turning on of the interactions which refers to requiring that the interactions are turned on slowly in comparison to the Fermi wavelength so that there is no single-electron backscattering as they enter the wire. Both of these conditions can simultaneously be satisfied if the Fermi wavelength is **much much** smaller than the density fluctuations which is usually the case.

This however is not all of it, as there is the screening length of the external gate that provides the constriction to consider as well. The length scales we consider must also be larger than screening length in order for our Luttinger Liquid assumptions to be appropriate! So this method although it seems quite physically appealing hides a lot of the assumptions under its hood. It is worth noting that in the 96 formulation of the boundary conditions, they set the field to asymptotically approach the boundary value as $x \to \pm\infty$ rather than equally at a point.

2.2.2 Displacement Field Formulation

Returning to the actual boundary conditions, they can be rewritten in a more familiar way by using the correspondence between the sum and difference of the full right/left densities and the density and current. They chose the normalise it in the following way,

$$\rho_R + \rho_L = \partial_x \theta / \sqrt{\pi}, \qquad \rho_R - \rho_L = \partial_t \theta / v_F \sqrt{\pi} \tag{11}$$

These densities (the sum) and current (the difference) can be expressed as the time/spatial derivative of the bosonic displacement field θ. We therefore arrive at the radiative boundary conditions:

$$\left(\frac{1}{K^2} \partial_x + \frac{1}{v_F} \partial_t \right) \langle \theta(x = -L/2, t) \rangle = \frac{eU_1}{\sqrt{\pi} \hbar v_F} \tag{12}$$

$$\left(\frac{1}{K^2} \partial_x - \frac{1}{v_F} \partial_t \right) \langle \theta(x = L/2, t) \rangle = \frac{eU_2}{\sqrt{\pi} \hbar v_F} \tag{13}$$

These conditions have to be fulfilled for all times in the stationary non-equilibrium state. "They hold provided that <u>ideal reservoirs</u> are <u>adiabatically connected</u> to the QW and we are in the <u>low-energy regime</u> where both the applied voltage and temperature are very small in comparison to Fermi energy."

2.3 Applying These Conditions To Different Situations

2.3.1 Clean Case

We will first look into the clean case where there is no impurity. Recovering the result that the conductance is not renormalised, we first need to note that in a clean system with no impurities, we have full translational invariance so both injected and actual densities are independent of coordinate I guess as there is no dependence on x in the lagrangian then it's like a cyclic variable and therefore a symmetry and all quantities derived from the Lagrangian should respect this. Therefore in equation 9, the x dependence can be dropped. Rearranging these equations to so that we can subtract the two full densities gives:

$$2\rho_R = (\rho_R^0 - \rho_L^0) + K^2(\rho_R^0 + \rho_L^0) \tag{14}$$

$$2\rho_L = (\rho_L^0 - \rho_R^0) + K^2(\rho_R^0 + \rho_L^0) \tag{15}$$

which outlines the effect of charging on the full densities - that there will always be left movers in the wire, even if there is none being injected from the left into the wire because of this shift of the bottom of the band. Subtracting these two expressions gives the current (although as the current is not changed by the charging effects this can be done from the start) as $I = e^2 U / h$ which gives the quantisation of conductance in terms of the conductance quantum.

The excess density that charges the capacitor system can be found and through our short range Poisson equation, the overall electrostatic field that acts on the system can be seen

$$\rho = \frac{K^2 e(U_1 + U_2)}{2\pi \hbar v_F}, \qquad \varphi = (1 - K^2) \frac{U_1 + U_2}{2}. \tag{16}$$

The electrostatic field does not fully compensate for the average potential shift $U_1 + U_2/2$ which leaves the fraction K^2 of the bare density present in the wire. For any value of K there is no associated electric field as the electrostatic potential, being defined by the density, is independent of coordinate.

2.3.2 Perfect Backscattering

The same analysis can be done for a case with perfect backscattering, but the condition is now that no current may flow hence $\rho_L = \rho_R$. Therefore if the two chiral densities are equal either side of the barrier, the full density can

be caluclated either side as,

$$\rho(x < 0) = \frac{K^2 e U_1}{\pi \hbar v_F}, \qquad \rho(x > 0) = \frac{K^2 e U_2}{\pi \hbar v_F} \tag{17}$$

the density drop across the barrier is therefore the difference between these two quantities $\Delta \rho = K^2 e (U_1 - U_2)/\pi \hbar v_F$. From our short-range Poisson equation we can find the voltage drop over the impurity

$$\Delta \varphi = (1 - K^2)(U_1 - U_2) \tag{18}$$

which is the two terminal voltage drop multiplied by the underscreening factor. If we considered it in a slightly different way, starting from the potential drop between the screening backgate and the wire, then to the left of the impurity we would have $(1 - K^2)U_1$ and correspondingly $(1 - K^2)U_2$ to the right. Then because the screening backgate will all be at the same voltage, we can change this situation into the voltage drop across the impurity by subtracting these two. Rewriting the density drop in another way,

$$\Delta \rho = \frac{e\Delta U - e\Delta \varphi}{\pi \hbar v_F} \tag{19}$$

we can show how much of the density change comes from the change in chemical potential and how much comes from the electrostatic field.

2.4 Generic Effect of Voltage Sources - Integrating Out

Now the paper moves to a much more mathematical section, where the quadratic modes around the impurity is integrated out in order to investigate precisely how the voltage drop occurs and how it should be added into the Hamiltonian. The procedure for doing this is outlined in a couple of different places (both of the 96 papers [6, 1]) and wont be focused on in this section and only the results will be mentioned. Go check out the notes on the 96 paper to see the gory details. This paper does however use the Keldysh contour, although this does not really affect things as we are simply finding an effective action. Another difference that instead over being left in Matsubara form as in the paper in 96, they explicitly change back to real frequencies - which is why there is a $\tanh$ looking term in the latter paper.

Integrating out the theta field at all points apart from the impurity, is a standard technique. A delta function is included so that our new variable is always equal to the field at the impurity. Solving this with respect to the boundary conditions means we have a part of the field (the particular solution and the homogeneous part) and it turns out that the average density is given by the derivative of the particular solution (again see the notes on 96 for details). This has the form,

$$\langle \rho(x) \rangle = \frac{K^2 e (U_1 + U_2)}{2\pi \hbar v_F} - \frac{K^2 e V}{2\pi \hbar v_F} \text{sgn}(x) \tag{20}$$

which when we think a little shows that the coefficient in front of the sgn(x) term must be half the drop in density over the impurity. V here is related to the zero mode of the Lagrange multiplier field by zero mode here we just mean the value of the constant part of the field (ie the Fourier transform is a delta so it must be at zero frequency). In essence the V is just a redefinition of our constant Lagrangian multiplier field (constant in time but can vary dependent on the systems conditions), setting $\Lambda_p = -ieV/2\pi$. The reason for this redefinition is described below.

Substituting this density drop into the short range Poisson ($e\varphi = u_0\rho$) allows us to express the voltage drop over the impurity as $\Delta \varphi = (1 - K^2)V$. Rephrasing the density drop in terms of chemical potential change (as total density can be related to the density of states, that is assumed to be constant, over an energy difference) we get $\Delta \mu = K^2 eV$. If we here to measure the four-terminal voltage (using Buttiker ideas), we would see that it has the

form $\Delta\mu/e + \Delta\varphi$. This gives V as the four terminal voltage and allows a more physical interpretation of what is happening.

Doing the integrating out procedure in full gives an effective action that contains effects from the boundaries (as this is how we are including the voltage effects) and these cause two effects. The first is a term in the action that looks like

$$\frac{eV}{2\pi} \int_C dz q(z) \tag{21}$$

where C is the Keldysh contour and $q(z)$ is our introduced variable. This indicates a **local** potential drop over the impurity and therefore this can be incorporated by adding into the Hamiltonian $eV\theta(0)/\sqrt{\pi}$ where the four terminal voltage is used rather than the two terminal one. The second term looks like,

$$\lambda \int_C dz \cos(q(z) + e(U_1 - U_2 - V)z/\hbar) \tag{22}$$

and looks like a Josephson-esque term. We have a term that is linear in time Vz, that cannot be incorporated into a model by simply adding terms into the Hamiltonian. Now to consider what happens to the four terminal voltage in the two extremal cases.

In the clean case, the field q describes a massless particle - there will be no mass term in the action (remember that mass in high energy QFT is equivalent to cosine terms in condensed matter). This arises becuase of the infrared diveregence in the kinetic term (as the Green's function is proportional to $1/\omega$ so and the exitations are the poles so as $\omega \to 0$.)

In the perfect backscattering case, when $\lambda \to \infty$, the action term above will start to have an infinite action cost to it. So it will want to be minimised, especially in time. This enforces that $U_1 - U_2 = V$ and the two terminal voltage is equal to the four terminal one. There should be a crossover between these behaviours as we vary the impurity strength λ. Therefore to investigate further, we can solve the problem exactly for a specific value of K which will give us an idea of how the cross over between these behaviour occurs and the rest can be argued by continuity.

2.5 Refermionisation

The 98 paper then looks at the exact solution to the problem that can be found when refermionising the problem when $K = 1/2$. They start this by giving the boundary conditions on the field in terms of the chiral bosonic fields (rather than the usual θ and ϕ) which are defined as

$$\phi_R(x) = \sqrt{\pi}\Big(\theta(x)/\sqrt{K} + \sqrt{K}\phi(x)\Big), \qquad \phi_L(x) = \sqrt{\pi}\Big(-\theta(x)/\sqrt{K} + \sqrt{K}\phi(x)\Big), \tag{23}$$

this results in the Hamiltonian becoming,

$$H = \frac{\hbar\nu}{8\pi} \int dx (\partial_x\phi_L)^2 + (\partial_x\phi_R)^2 + \lambda\delta(x)\cos\Big(\sqrt{K}(\phi_R - \phi_L)\Big) \tag{24}$$

and the boundary conditions as defined in Eq 9 - essentially returning to our original description. We then define slightly weird non-local fields in the following way,

$$\tilde{\phi}_\pm(x) = \frac{1}{\sqrt{2}}(\phi_R(x) \mp \phi_L(-x)), \qquad \tilde{\rho}_\pm = \frac{1}{2\pi}\partial_x\tilde{\phi}_\pm = \frac{1}{\sqrt{2K}}(\rho_R(x) \mp \rho_L(-x)) \tag{25}$$

which then results in the new Hamiltonian where we have successfully decomposed it into separate fields, now choosing $K = 1/2$,

$$H = \frac{\hbar\nu}{8\pi} \int dx (\partial_x \tilde{\phi}_+)^2 + (\partial_x \tilde{\phi}_-)^2 + \lambda\delta(x)\cos\left(\tilde{\phi}_+\right) \tag{26}$$

with the following boundary conditions obtained from adding and subtracting eq 9.

$$5\tilde{\rho}_-(-L/2) + 3\tilde{\rho}_-(L/2) = e(U_1 + U_2)/\pi\hbar v_F, \qquad 5\tilde{\rho}_+(-L/2) + 3\tilde{\rho}_+(L/2) = e(U_1 - U_2)/\pi\hbar v_F, \tag{27}$$

so it is these equations which we are now trying to solve. Here it is clear to see that the impurity term only couples to the $\tilde{\phi}_+$ sector and the applied voltage difference only affects this sector too. The corresponding $\tilde{\phi}_-$ sector is associated with the shift in density that arises from $U_1 \neq -U_2$ and we will from now assume that the voltage drop is symmetric across the wire, essentially setting $U_1 = -U_2$ allowing us to ignore this term. There is also no impurity in the - sector so the density must be constant along the wire. Having reduced the complexity of the problem, we now can start to refermionise $\tilde{\phi}_+$.

They do this by introducing,

$$\psi = (c + c^\dagger)\sqrt{\frac{1}{2\pi a}} \exp\left(i\tilde{\phi}_+\right) \tag{28}$$

where to ensure the commutation relations of fermions we have included $c + c^\dagger$ which is a Majorana mode (as it is its own Hermitian conjugate). This then results in the refermionised Hamiltonian of

$$H = -i\hbar\nu \int dx \psi^\dagger \partial_x \psi + \sqrt{\hbar\nu\lambda_B/2}(c + c^\dagger)(\psi(0) + \psi^\dagger(0)) \tag{29}$$

with the effective impurity strength of $\lambda_B = \pi a\lambda^2/\hbar\nu$. The kinetic term has that particular form as that is a sum over the modes with a linear dispersion (see Luttinger Liquid notes for context). Owing to the choosing of $K = 1/2$, the 'refermions' are now free rather than interacting and we have a delta function interacting barrier at $x = 0$. Therefore this is now just a scattering problem, which can be solve in a variety of ways. In this paper they choose use the equation of motion method and essentially find the equations of motion for the system, where the majorana mode has been renamed f

$$-i\partial_t\phi = [H, \phi] = i\partial_x\phi + \sqrt{\hbar\nu\lambda_B/2}f\delta(x), \qquad -i\partial_t\phi^\dagger = [H, \phi] = i\partial_x\phi^\dagger + \sqrt{\hbar\nu\lambda_B/2}f\delta(x), \tag{30}$$

and the time dependence of the Majorana mode must also be found and solved for,

$$-i\partial_t f = [H, f] = \sqrt{\hbar\nu\lambda_B/2}(\psi(0) + \psi^\dagger(0)) \tag{31}$$

Away from $x = 0$ the field can be found and solved for as it must satisfy the equation $(i\partial_t + i\partial_x)\psi = 0$, with the form of ψ being able to be found by taking the Fourier transform. This must can be different either side of $x = 0$ so in general our solution will be,

$$\psi(x, t) = \frac{1}{L}\sum_k e^{ik(\nu t - x)} \begin{cases} a_k, & (x < 0) \\ b_k, & (x > 0) \end{cases} = \frac{1}{L}\sum_k e^{ik(\nu t - x)}(\Theta(-x)a_k + \Theta(x)b_k) \tag{32}$$

To find the boundary condition to join these two solutions together we must solve for f which involves defining $\psi(0)$. To ensure commutation relations are satisfied (as ψ is bosonic and f makes the combination $f\psi$ fermionic) we define $\psi(0) = (\psi(0^+) + \psi(0^-))/2$. Therefore solving for f with this definition we can find that,

$$f_\omega = \frac{\sqrt{\hbar\nu\lambda_B/2}}{\omega}(\psi_\omega(0) + \psi_\omega^\dagger(0)) = \frac{\sqrt{\hbar\nu\lambda_B/2}}{2\omega}(a_k + b_k + a_k^\dagger + b_k^\dagger) \tag{33}$$

Now we plug back into the equations of motion, splitting it up into $\psi_L(x, t) = \frac{1}{L}\sum_k e^{ik(\nu t - x)}a_k$ and ψ_R respec-

tively, joined together by a theta function. This gives,

$$(i\partial_t + i\partial_x)(\psi_L\Theta(-x) + \psi_R\Theta(x)) = 2\sqrt{\hbar\nu\lambda_B/2}f\delta(x) \tag{34}$$

$$\Theta(-x)(i\partial_t + i\partial_x)\psi_L + \Theta(x)(i\partial_t + i\partial_x)\psi_R + i\psi_L\partial_x\Theta(-x) + i\psi_R\partial_x\Theta(x) = 2\sqrt{\hbar\nu\lambda_B/2}f\delta(x) \tag{35}$$

$$i(\psi_R - \psi_L)\delta(x) = 2\sqrt{\hbar\nu\lambda_B/2}f(x)\delta(x) \tag{36}$$

therefore we can equate the two coefficients of the delta function and take the Fourier transform to obtain,

$$i(b_k - a_k) = -\frac{\hbar\nu\lambda_B}{2\omega}(a_k + b_k + a_k^\dagger + b_k^\dagger) \tag{37}$$

and the procedure can be repeated on the equation of motion for the conjugate $\psi^\dagger$. This gives two equations from which we can eliminate $b_k^\dagger$ to obtain a relations between the scattering components,

$$2b_k = (1 + \frac{i\hbar\nu k - \lambda_B}{i\hbar\nu k + \lambda_B})a_k + (1 - \frac{i\hbar\nu k - \lambda_B}{i\hbar\nu k + \lambda_B})a_k^\dagger \tag{38}$$

As a quick note, this method is a lot more clear in reference [7] which is quite a well written paper, but however does not focus on boundary conditions so will be currently not focused on. From here they then incorporate the boundary conditions as their new 'non local density' $\tilde{\rho}_+$ can be represented as $\psi^\dagger\psi$ from the definition of our (un)bosonised field. This therefore translates into the boundary condition,

$$\frac{1}{L}\sum_k 5\left\langle a_k^\dagger a_k \right\rangle - 3\left\langle b^\dagger b_k \right\rangle = e(U_1 - U_2)/\pi\hbar v_F \tag{39}$$

where we can used the boundary conditions of $\tilde{\rho(x)}$. The averages are non-equilibrium ones and as we have chosen the Luther-Emery point to refermionise, our a_k operators correspond to non-interacting fermions and therefore their average is the Fermi distribution for some self consistently determined k^* (ie the k corresponding to the chemical potential). Our matching of the operators at $x = 0$ gives a way to express what the average of b is in terms of a,

$$2\langle b_k^\dagger b_k\rangle = (1 + \cos(\alpha_k))n_k(k^*) \tag{40}$$

$$+ (1 - \cos(\alpha_k))n_k(-k^*) \tag{41}$$

To get this result we can show that $\langle a_{-k}a_{-k}^\dagger$ by commutation relations. It is useful to calculate $\cos(\alpha)$ which is evaluated to give,

$$\cos(\alpha_k) = \frac{\hbar^2\nu^2 k^2 - \lambda_B^2}{\lambda_B^2 + \hbar^2\nu^2 k^2}$$

and so substituting it all in gives an expression,

$$\frac{1}{L}\sum_k 2n_k(k^*) + \frac{3}{2}(n_k(k^*) - n_k(-k^*))(1 - \frac{\hbar^2\nu^2 k^2 - \lambda_B^2}{\lambda_B^2 + \hbar^2\nu^2 k^2}) = \frac{e(U_1 - U_2)}{\hbar\pi v_F} \tag{42}$$

$$\tag{43}$$

$$k^* + \frac{3\pi}{L}\sum_k \frac{n_k(k^*) - n_k(-k^*)}{1 + (\hbar\nu k/\lambda_B)^2} = \frac{e(U_1 - U_2)}{\hbar v_F} \tag{44}$$

Where we have taken the first sum (assuming it the fermi function is close enough to a step function) as the sum is over integer multiples of $2\pi/L$ up to k^* - ie $1 \times k^*L/2\pi = n^*$. In the second sum, I am sticking my neck out to say I think there is a typo in the paper where they have written the Fermi function at zero 'chemical potential'

Finally, we make the assumption of a long quantum wire so that the other sum can be converted into an integral and this integral can be evaluated using contour integration.

This contour integration is however pretty difficult, and its not helped by converting the answer into a digamma function, $\psi(z)$, which helps when taking asymptotics. The relation we need to understand where this comes from is

$$\frac{1}{\pi}\left(\frac{1}{1 + e^{-2\pi b}} - \frac{1}{1 + e^{2\pi b}}\right) = \frac{1}{\pi}\tanh(\pi b) = \text{Im}\left(\psi(\frac{1}{2} + ib)\right) \tag{45}$$

From this we can see the similarity between our Fermi function and the digamma function. So taking the contour integral on each Fermi function separately with semi-circle contours going in opposite directions allows us to cancel out the two sets of countably infinite number of poles that come from each Fermi function. That leaves us with

$$k^* + \frac{3\lambda_B}{\hbar\nu}\text{Im}\left(\psi(\frac{1}{2} + \frac{\lambda_B + i\hbar\nu k^*}{2\pi k_B T})\right) = \frac{e(U_1 - U_2)}{\hbar v_F} \tag{46}$$

So we have managed to obtain an expression for

$$k^*$$

in terms of the strength of the scatterer. So we can find the limits of what our 'chemical momentum potential' is in the strong and weak scattering limits. As $\lambda_B \to 0$ this gives $k^* = eU/\hbar v_F$.

In the infinite scattering limit we need to find the asymptotics of our digamma function which is the logarithm of the argument (look it up if you want). The imaginary part of the logarithm is then the phase of the complex argument. This is then arctan of the ratio of the imaginary and real parts.

The ratio of the imaginary and real parts is

$$\frac{\hbar\nu k^*/2\pi k_B T}{\frac{1}{2} + \lambda_B/2\pi k_B T} \sim \frac{\hbar\nu k^*}{\lambda_B}$$

as we are taking the $\lambda_B \to \infty$ limit we can perform this simplification. Finally, we use that $\arctan(x) \sim x$ to get our result of $k^* = eU/4\hbar v_F$.

2.5.1 Four terminal Voltage

With our new formula relating the operators either side of the impurity, we can find the density drop over the impurity which was shown to be related to the four terminal voltage in the previous section. This density drop can obviously be related to our expectation values of operators a_k and b_k,

$$\tilde{\rho}_+(x < 0) - \tilde{\rho}_+(x > 0) = \int \frac{dk}{2\pi}\langle a_k^\dagger a_k - b_k^\dagger b_k\rangle \tag{47}$$

$$= \int \frac{dk}{2\pi}(1 - \cos(\alpha))(n_k(\mu) - n_k(0)) \tag{48}$$

which is the same as the the integral calculated earlier (ie just the digamma term). So this allows us to relate the four terminal voltage to our refermionised results,

$$eV = 2\lambda_B\text{Im}\left(\psi(\frac{1}{2} + \frac{\lambda_B + i\hbar\nu k^*}{2\pi k_B T})\right) \tag{49}$$

To do a quick sanity check we can find that in the clean limit ($\lambda_B \to 0$) that $V = 0$. This makes sense as if there is no impurity there should be no voltage drop! For the strong scattering limit, we again use the expansion of the digamma function in the exact same way as we did earlier with our strong scattering result for k^* to get $V = U$ which also agrees with our previous analysis.

So having checked this answer seems reasonable, we can combine it with our boundary effect to find an expression for k^*

$$k^* = \frac{e}{\hbar v_F}(U - \frac{3V}{4}) \tag{50}$$

Therefore we can substitute this into our four terminal voltage to find a self constant equation which can be solved (numerically) to give the behaviour!

Finally to get the quantity we actually want to compare, the current, we again consider the expectation values again. The current will be given by (see all of the above as to where this comes from),

$$I = ev_F \langle \tilde{\rho}_+ \rangle = \frac{ev_F}{4} \int \frac{dk}{2\pi} \langle (a_k^\dagger b_k^\dagger)(a_k + b_k) \rangle \tag{51}$$

which gives the result $I = \frac{e^2}{h}(U - V(U, T, \lambda_B))$ for the self consistently determined four terminal voltage. Therefore this is the solution to our non-linear current-voltage characteristics and is what they calculate in their paper to show the cross over between different strengths of impurity/temperature for a specific value of the electron-electron interaction. So now we can look at the current voltage dependence
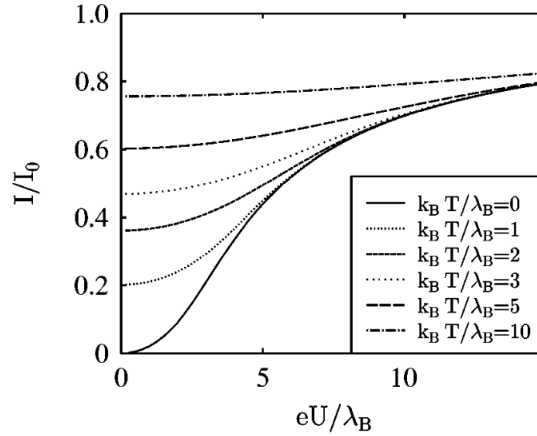
The current is normalised to $I_0 = Ue^2/h$. Important features to note are that as the scattering strength decreases we will have more current flowing for the same two terminal voltage. We recover the Kane-Fisher result of no current for zero temperature with a scatterer present as expected and see the two terminal voltage overcome this. All results that we wanted!

The notes on this paper are intended to show how to apply these radiative boundary conditions to two different techniques. The first is a more general technique of integrating out quadratic modes subject to the conditions on the field caused by these boundary conditions. The second is when refermionising the fields at a specific interaction parameter value and expressing all quantities of interest as expectation values of operators. So having looked at the 98 paper, it is time to look a bit closer at the nature of screening and why the density is a fraction of its normal value and the lead up to this formulation of the boundary conditions.

3   The 97 Paper - Screening in 1D

In this paper, they look at how a Luttinger Liquid reacts to the presence of external charges and find that the response to the charge does not give electro-neutrality but instead only a fraction $(1 - K^2)$ is compensated. Their reasoning for this is that the external gates that cause the finite nature of the interactions in the Luttinger Liquid also react to the external charge.

Screening is a property that underlies a lot of the assumptions that we make in physics. The fact that charge will rearrange itself to eliminate the field at long distances is what allows us to assume that interactions are short distanced which is often assumed for many condensed matter system - it is crucial for obtaining the Luttinger Liquid model. Thomas-Fermi (putting an exponential decay on the Coulomb potential) screening is a common model that can be derived from a full treatment of the polarisation operator. However, we will approach the problem from finding the dielectric function which relates the displacement field to the applied field. This is just a redefinition of some variables and will be exactly stated later when needed. Assuming that linear screening is accurate (ie the effects of external charges on the system is given by linear response theory) means that the dielectric function $\epsilon(k, \omega)$ describes all the behaviour. It is this function that we will attempt to find for a general 1D system that can be described by Luttinger Liquids.

Here we will treat the interaction of the electrons with each other in a more careful way than the normal Luttinger approach where the interaction is set to be a contact interaction. So consider the Hamiltonian,

$$H_0 = \frac{v_F}{2} \int dx \left(\Pi^2(x) + (\partial_x \theta)^2 \right) + \frac{1}{2\pi} \int dx dx' \partial_x \theta(x) U(x - x') \partial_{x'} \theta(x') \tag{52}$$

where $\Pi(x)$ is the conjugate momentum to θ. Here in the unscreened limit $U(x - x')$ can either be an unscreened Coulomb interaction (behaving as $1/|x|$) or a finite range potential that is screened by external charges. We will assume that the $q = 2k_F$ component of the Fourier transform of the interaction is small - this is will simplify the calculation but doing it with the backscattering will yield the same behaviour.

In the case of Luttinger Liquids, where the interaction $U(x)$ is a contact interaction, this means that its strength is equal to the Fourier transform of the interaction at zero momenta $U(q = 0) \equiv U_0$ and zero elsewhere. The Luttinger liquid parameter is therefore its usual form $K = (1 + U_0/\pi v_F)^{-1/2}$ which is obtained from absorbing all the $\partial_x \theta$ terms into one parameter.

We now will consider some external time-dependent charge distribution $Q(x, t)$ brought into the system, with the interaction with the metal being given by

$$H_Q(t) = \int dx dx' Q(x, t) U(x - x') \rho(x'), \quad H_Q(\omega) = \frac{1}{2\pi\beta} \sum_{q,\omega} Q(q, \omega) U(q) \rho(-q, -\omega) \tag{53}$$

which we will add to our total Hamiltonian and calculate the average density $\langle \rho(q, \omega) \rangle$. Ignoring the $2k_F$ component that should arise in the interaction with the external charge (as we care about the long range response of the electrons - this is why we assumed it was small earlier) we can rewrite the derivative of the field as $\sqrt{\pi}$ multiplied by the density. The first thing to notice is that if we want the average of the density, then taking the functional derivative of the partition function with respect to $U(q)Q(q)$ will give the average of the density (and dividing it by just the partition function). Now we want to complete the square on our action (also calculating this in terms of actions as it is what I understand better but this same process can be followed in an operator formalism) so we start with our full action

$$S = \frac{1}{(2\pi)\beta} \sum_{\omega, q} \left(\frac{v_F}{2} \omega^2 + \frac{1}{2v_F} \omega^2(q) \right) |\theta(q, \omega)|^2 + Q(q, \omega) U(q) q \theta(q, \omega) \tag{54}$$

which is done by just taking the Fourier transform and collecting up relevant terms. We have defined the plasmon frequency

$$\omega(q) = v_F |q| \sqrt{1 + U(q)/\pi v_F} \tag{55}$$

to clean up notation. Then we complete the square,

$$S = \frac{1}{2\pi\beta} \sum_{\omega,q} \left(\frac{v_F}{2}\omega^2 + \frac{1}{2v_F}\omega^2(q) \right) \left(\theta(q,\omega) + \frac{Q(q,\omega)U(q)q}{v_F\omega^2 + \frac{1}{v_F}\omega^2(q)} \right)^2 - \frac{1}{2} \frac{Q^2(q,\omega)U^2(q)q^2}{v_F\omega^2 + \frac{1}{v_F}\omega^2(q)} \tag{56}$$

from here we can perform the functional integral over the theta field to get a determinant that goes as $\det\left(\frac{v_F}{2}\omega^2 + \frac{1}{2v_F}\omega^2(q) \right)$ which will be cancelled out in the averaging. So the average of the density is the functional derivative as stated earlier and taking the functional derivative with respect to $U(q)Q(q)$ and diving by the partition function we get,

$$\langle \rho(q,\omega) \rangle = \frac{v_F}{\pi} \frac{q^2 U(q)}{\omega^2 - \omega^2(q)} Q(q,\omega) \tag{57}$$

where we have analytically continued to real frequencies. This calculation can also be done by evaluating the field that minimises the functional and then taking the spatial derivative to get the density. This can be done because the action is quadratic Therefore we have linear screening where the average induced density is proportional to the charge density multiplied by the Fourier transform of the interaction with the relation between them being the polarisation $\chi(q,\omega)$, where $\langle \rho(q,\omega) \rangle = U(q)\chi(q,\omega)Q(q,\omega)$. Therefore our polarisation is,

$$\chi(q,\omega) = \frac{v_F}{\pi} \frac{q^2}{\omega^2 - \omega^2(q)} \tag{58}$$

3.1   Screening - Dielectric function

As we are interested in the screening properties, we relate this to the dielectric function which we define to be related to the polarisation,

$$\epsilon(q,\omega)^{-1} = 1 + U(q)\chi(q,\omega), \quad \implies \quad \epsilon^{-1}(q,\omega) = 1 + \frac{v_F}{m} \frac{q^2 U(q)}{\omega^2 - \omega^2(q)} \tag{59}$$

where the static $\omega = 0$ case gives $\epsilon(q) = 1 + U(q)/\pi v_F$. The important thing to note is that throughout this derivation we require the external charge to not be too large as the bosonisation approach will not work in this case. We can define the screened interaction potential by dividing the Fourier transform of the interaction by the dielectric function $U_{eff}(q) = U(q)/\epsilon(q)$. Most importantly the static limit of the dielectric function gives the result that $U_{eff} = K^2 U(q)$ as $q \to 0$. To see this,

$$U_{eff} = U(q)(1 + U(q)/\pi v_F)^{-1}, \qquad K = (1 + U(0)/\pi v_F)^{-1/2} \tag{60}$$

relating this to our average induced density which is proportional to the Fourier transform of the interaction. This leads to the conclusion that the induced density is K^2 different to the unscreened interaction. This is another point towards the crucial relation that we used in the last paper relating the two densities inside and outside the wire. To be a bit more direct, consider a point charge sitting at $x = 0$ so $Q(x,t) = Q\delta(x)$. The total screening charge is therefore,

$$eQ_S = e \int dx \, \langle \rho(x) \rangle, \quad \implies \quad Q_S = -(1 - K^2)Q \tag{61}$$

which is obviously reduced in comparison to the bare charge (which affects the density linearly). This therefore looks like electroneutrality can be violated (ie the impurity charge at $x = 0$ plus the screening charge) or as another way of phrasing this the Friedel sum rule is violated. The reason for this apparent problem is that we have

not considered the screening gates, but before we show that considering the entirety of system solves the issue we first look at a phase derivation of the same effect.

3.2 Screening - Phase Shift Argument

Instead of relying on the dielectric function, we can formulate the argument in another way. Considering how the external charge comes into the system when our external charge is localised $Q\partial(x)$, we see that performing the unitary transformation

$$U = \exp\left(-i\sqrt{\pi}\int dx\alpha(x)\phi(x)\right), \qquad \alpha(q) = -\frac{U(q)/\pi v_F}{1 + U(q)/\pi v_F}Q \tag{62}$$

for the dual field ϕ to our original field. Therefore comparing this to the density result we got from the quadratic Hamiltonian in the static limit $\omega = 0$, we see that the average density is the phase of the transformation $\langle\rho(x)\rangle = \alpha(x)$. Performing the transformation leads to an overall phase shift in the $2k_F$ part of the density,

$$\eta(x) = \pi\int dx'\text{sgn}(x - x')\alpha(x'), \qquad \eta(x \to \infty)/\pi = \int dx\alpha(x) = Q_s \tag{63}$$

where the last line is because the average density is equal to α. This shows that Friedel's phase shift sum rule still holds (Fill in what exactly this is), which implies that the phase shift is related to a screened impurity charge, not the full charge of the system. By susbstituting in for $\alpha(q)$, we can determine the screening of the charge result again.

3.3 Screening Gates

To see more clearly why electroneutrality is 'violated' it is worth giving a more concrete example. So consider the system shown in the diagram below - of a 1D wire with 2D screening gates located a distance D away. The bare impurity charge is Q, the screened response is Q_s and the induced charge on the gate is Q_g
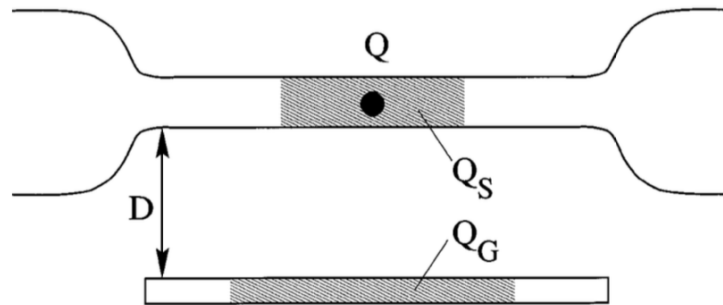
So to make our first step, we can express the Luttinger Liquid interaction parameter as a function of the width of the wire and separation of the wire and screening gate. This is normally done by introducing a cut off to the $1/|x|$ interaction at a particular distance. This normally means the interaction has the real space form of $1/\sqrt{x^2 - c^2}$. When Fourier transformed, this has the form of a modified Bessel function $K_0(qc)$ which to leading order is given by a log $U(q) = 2e^2|\ln(qc)|$. Therefore our interaction parameter becomes,

$$K = \left(1 + \frac{2e^2}{\pi v_F}\ln(2D/d)\right)^{-1/2} \tag{64}$$

for distance separation of the screening gate and the wire D and width of the wire d. Finish off these calculations Therefore our new condition of electroneutrality is

$$Q_s + Q + Q_g = 0 \tag{65}$$

and as mentioned before the induced charge on the screening gate must be included. The final part of the paper looks into the $2k_F$ response by looking at the Lindhard formula. However the breakdown of linear screening due to the Peierls instability means that the approach of using the dielectric function breaks down. These fast oscillations are argued to ignored with the separation of fast and slow mode that bosonisation allows (ie $q \approx 0, 2k_F$) being the reason we can ignore this.

However, the most direct proof of the idea is shown in their 96 paper where they explicitly solve the problem using generating functionals. There was a correction to the 96 paper which includes the screening that they demonstrated in this paper which means that some of their results need to be multiplied by K^2 - the process is the same however!

4 The 96 paper - Integrating out

In the final paper of these notes (and the earliest one released), the boundary conditions were originally introduced. However, there are some differences in how they are derived - and they also turn out to be wrong! The set up of the system is similar to the other papers with a Luttinger liquid action that contains an localised impurity at $x = 0$. This 1D quantum wire is connected to the higher dimensional reservoirs by adiabatically widening the constriction (ie the Glazmann paper), which is the requirements for Landauer's ideal reservoirs - alongside the need for them to be reflectionless.

The reservoir are at the spacial infinities and a voltage is applied, resulting in an imbalance in the density of left/right movers. Defining the two terminal voltage $eU = \mu_{-\infty} - \mu_{\infty}$, the conditions from the injected densities from the reservoir are,

$$\langle \rho_\pm \rangle = \pm \frac{eU}{4\pi v_F}, \quad x \to \mp\infty \tag{66}$$

The mean chemical potential corresponds to the background charge k_F/π, and we have also assumed that the density drop is symmetric so $\mu_{\pm\infty} = \mp eU/2$. In a clean system the densities of the movers that are injected by the reservoirs spread homogeneously along the system, but if an impurity is present then we do not know the densities without figuring them out - which we will now proceed to do.

4.1 Erratum Correction

Now it is in reformulating the above boundary requirements into boundary conditions for the bosonic displacement field that they made a mistake and it is that there should be an extra factor of $1/K^2$. So originally in the paper, they wrote

$$\rho_+ + \rho_- = \frac{1}{\sqrt{\pi}} \partial_x \theta, \qquad \rho_+ - \rho_- = \frac{i}{v_F \sqrt{\pi}} \partial_t \theta \tag{67}$$

which can be seen by adding and subtracting the equations to lead to

$$\left(\pm \frac{\partial}{\partial x} + \frac{i}{v_F} \frac{\partial}{\partial t} \right) \langle \theta(x \to \mp\infty), t = 0 \rangle = \frac{eU}{2\sqrt{\pi} v_F} \tag{68}$$

where this is true at a specific time t picked to be zero from convention. However, comparing this to the derivation presented from the 98 paper where they write that $\rho(x) = K^2[\rho_L^0 + \rho_R^0] = \rho_L(x) + \rho_R(x)$ we can see that this is where the factor of K^2 comes from on the spacial derivative. The justification to why this change is made is a little hard to see when they argue for the shifting down of the bottom of bands, but considering the underscreening of the charge it becomes a little more clear.

Essentially we are introducing extra charge into the system and this extra charge will cause the 'electrons' to re-arrange to cancel out its long range effects (ie screen the charge). The 1D nature of the system however means it is not completely effective at doing this and can only compensate for $(1 - K^2)$ of the charge with the remaining charge being induced by the electric field on the screening gates which does not affect the transport within the wire (Is that true?). This leaves K^2Q of the charge uncompensated for and as linear screening holds this corresponds to only $K^2\rho$ as the density in system.

To have the densities outside/inside of the wire being the same so we can compare, we must therefore divide the total density by $1/K^2$. This result in the following boundary conditions,

$$\left(\pm \frac{1}{K^2}\frac{\partial}{\partial x} + \frac{i}{v_F}\frac{\partial}{\partial t} \right) \langle \theta(x \to \mp\infty), t = 0 \rangle = \frac{eU}{2\sqrt{\pi}v_F} \tag{69}$$

4.2 Integrating Out

So having formulated the boundary conditions, we now move on to finding the voltage. This is done by considering the generating functional,

$$Z(y, \mu) = \left\langle \exp\left(2\sqrt{\pi}i\mu\theta(y)\right) \right\rangle = \int \mathcal{D}\theta \exp\left(2\sqrt{\pi}i\mu\theta(y)\right)e^{-S_0-S_{int}} \tag{70}$$

as taking the derivative with respect to μ and then setting it to be zero will give the average of θ, from which we can take derivatives as needed to obtain the density/current. This process is outlined in much more detail in another 96 paper [6] but they do not consider the boundary conditions in that one so these notes might jump between the two but will try to be clear throughout. So If we introduce a field $q(t)$ that describes the field at the impurity (the only place where the action is no longer quadratic) and a Lagrange multiplier field $\Lambda(t)$ to enforce the constraint that the displacement field at $x = 0$ is the same as our new introduced (0+1)D field. The total action therefore becomes,

$$S[\theta, \Lambda, q] = \frac{v_F}{2}\int dxdt\left(\frac{(\partial_t\theta)^2}{v_F^2} + \frac{(\partial_x\theta)^2}{K^2}\right) + V\int dt\cos(q) - iq\Lambda$$
$$+ i\int dxdt\, \Lambda 2\sqrt{\pi}\theta\delta(x) - 2\sqrt{\pi}i\mu\theta(y) \tag{71}$$

So now can integrate out the θ field to get a representation in terms of $S_{eff}[\Lambda, q]$ and it will turn out again that this new effective action is quadratic in Λ so we can integrate out this field too. However we also want to make sure that the field that we integrate out also satisfies our boundary conditions (more complicated than our usual must decay at spacial infinity). So they integrate out the fields in a slightly different way, by simply solving the Euler-Lagrange that arises from our action (ie $\delta S/\delta\theta = 0$) and then plugging this solution back into the action.

This obviously turns out to be equivalent to the normal way of integrating out (quadratic so integrating out the fluctuations around the classical minima can be done exactly, leaving only the classical contribution to leading order) but allows us to be more careful with the boundary conditions. So therefore the equations we are trying to solve is

$$\frac{1}{v_F^2}\frac{\partial^2\theta}{\partial t^2} + \frac{1}{K^2}\frac{\partial^2\theta}{\partial x^2} = \frac{2\sqrt{\pi}i}{v_F}\left(\delta(x)\Lambda(t) - \mu\delta(x-y)\delta(t)\right) \tag{72}$$

where θ must satisfy the boundary conditions. We can the decompose the field θ into parts that satisfy the boundary conditions and a part that equals zero (homogeneous/particular solution) with the Λ field being also able to split this up into corresponding parts so that each part of θ equals their respective Λ,

$$\frac{1}{v_F^2}\frac{\partial^2\theta_p}{\partial t^2}+\frac{1}{K^2}\frac{\partial^2\theta_p}{\partial x^2}=\frac{2\sqrt{\pi}i}{v_F}\delta(x)\Lambda_h(t), \qquad \frac{1}{v_F^2}\frac{\partial^2\theta_h}{\partial t^2}+\frac{1}{K^2}\frac{\partial^2\theta_h}{\partial x^2}=\frac{2\sqrt{\pi}i}{v_F}\Big(\delta(x)\Lambda_h(t)-\mu\delta(x-y)\delta(t)\Big) \quad (73)$$

We will now go on to solve both of these PDEs. The most general solution for θ_p that satisfies the boundary conditions requires a time independent Λ (so we have essentially split the original field into $\Lambda=\Lambda_h(t)+\Lambda_p$) and we can find the x form of this by just solving the equation to get

$$\theta_p(x)=\frac{iK^2\sqrt{\pi}}{v_F}\Lambda_p|x|+\theta(t) \tag{74}$$

with a factor of two disappearing due to the definition of a delta function being the derivative of a step function not twice the step function. We can then substitute this into the boundary conditions to get

$$\theta_p(x\to\pm\infty,t)=\frac{iK^2\sqrt{\pi}}{v_F}\Lambda_p|x|+-it(\frac{eU}{2\sqrt{\pi}}-i\sqrt{\pi}\Lambda_p)+const \tag{75}$$

The solution given in the paper is

$$2\sqrt{\pi}\theta_p(x,t)=q_0-\frac{eK^2\varphi}{v_F}|x|-ite(U-\varphi) \tag{76}$$

where the φ is related to the zero mode of the Lagrange constraining field and is $\Lambda_p=ie\varphi/2\pi$. In the paper of 96 there is a factor of $1/K^2$ but i feel that the redefinition of the boundary conditions in the erratum means that is factor does not appear. A tentative guess as to why this is related to the zero mode is that if we set $\partial_t\theta=0$ in our boundary conditions we get something that looks like $\Lambda_p=-ieU/2\pi$ - but i still dont get what a zero mode means of something that we assumed to be independent of time?

The other equation to solve θ_h can be solved by taking the Fourier transform.

$$(\omega^2+\omega_p^2)\theta(p,\omega)=-2\sqrt{\pi}v_F(\Lambda(\omega)-\mu e^{-iky}), \qquad \omega_p=v_F|k|\sqrt{1+U_q/\pi v_F} \tag{77}$$

where the plasmon frequency crops up again. This is due to following the other 96 paper [6] with plasmon frequency being substituted in for the Luttinger Liquid (ie imagining that there is a U_q instead of just the contact interaction $U_{q=0}$ the latter of which corresponds exactly to K). However in the next equation they ditch the more general form to replace it with just K (no idea why) so we let $\omega_k^2=v_F^2k^2/K^2$. Defining the propagators (Green's function) of the LHS in the usual way as the inverse of the diagonal elements, we can then take an inverse Fourier transform,

$$D(x,\omega)=v_F\int dk\frac{e^{ikx}}{\omega^2-\omega_k^2}=\frac{\pi K}{|\omega|}e^{-|K\omega x|/v_F} \tag{78}$$

with the factor of v_F being defined so it cancels out on the RHS. This means that the solution becomes, by letting the Green's function propagate the source term 'forward'

$$\sqrt{\pi}\theta(x,t)=i\int\frac{d\omega}{2\pi}e^{i\omega t}\Big(\Lambda_h(\omega)D(x,\omega)-\mu D(x-y,\omega)\Big) \tag{79}$$

So having solve both of the equations we can now substitute in for $\theta=\theta_p+\theta_h$ which gives us an action quadratic in Λ_h. There is quite a lot to deal with in substituting this result back in, the first is the Luttinger Liquid part which we will write as

$$S_1 = \frac{v_F}{2} \int dx dt - \frac{(\theta_h + \theta_p)}{v_F^2} \partial_t^2 (\theta_h + \theta_p) - \frac{(\theta_h + \theta_p)}{K^2} \partial_x^2 (\theta_h + \theta_p) \tag{80}$$

$$= \int dx dt - (\theta_h + \theta_p) \sqrt{\pi} i \Lambda_p \delta(x) - (\theta_h + \theta_p) \sqrt{\pi} i (\delta(x) \Lambda_h(t) - \mu \delta(x - y) \delta(t) \tag{81}$$

these are the terms from the lagrange multiplier part,

$$S_2 = 2\sqrt{\pi} i \int dt dx (\Lambda_h + \Lambda_p)(\theta_h + \theta_p) \delta(x) - \int dt i (\Lambda_h + \Lambda_p)(q(t) + 2\sqrt{\pi} \theta_p(0, t)) \tag{82}$$

where as $q = 2\sqrt{\pi}\theta(0, t)$ there is an addition of the particular solution part to any time $q(t)$ appears (minus spatial part due to it being evaluated at the origin). So that leaves the chemical potential part which transforms in an obvious way as $2i\sqrt{\pi}(\theta_h(y, 0) + \theta_p(y, 0)$ and the cosine term which will just contain the particular solution added inside the cosine.

That is all the terms and after looking through it a bit more, we see that a lot of the terms in S_1 will subtract from terms in other parts changing the first term in S_2 to being $\sqrt{\pi}$ and the chemical potential part to $\sqrt{\pi}$. From here we can now substitute in the form of the solutions that we have starting with the chemical potential term (also when I'm saying chemical potential I just mean the original term which has the μ and is being averaged over)

$$i\sqrt{\pi}\mu\theta_p(y, 0) - \int d\omega \mu(D(y, \omega)\Lambda(-\omega) - \mu D(0, \omega)) + \cdots \tag{83}$$

Due to the reduction of first term in S_2 to $\sqrt{\pi} i$ the $(\Lambda_h + \Lambda_p)\theta_p$ cancels in S_2. This only leaves $(\Lambda_h + \Lambda_p)\theta_h(0, t)$ and $(\Lambda_h + \Lambda_p)q(t)$. So putting all of these various parts together we now have an action that is quadratic in Λ_h

$$
\begin{aligned}
S = \int \frac{d\omega}{2\pi} \mu^2 D(0, \omega) &- i\sqrt{\pi}\mu\theta_p(y, 0) \\
&+ \int \frac{d\omega}{2\pi} \Lambda_h(\omega) D(0, \omega)\Lambda(-\omega) + \Lambda_h(\omega)(-2\mu D(y, -\omega) + iq(-\omega)) \\
&+ \int dt \cos(q + 2\sqrt{\pi}\theta_p(0, t)) + \Lambda_p q(t)
\end{aligned}
$$

And then we can integrate out to get,

$$
\begin{aligned}
S = -i\sqrt{\pi}\mu\theta_p(y, 0) &+ \int \frac{d\omega}{2\pi} \mu^2 \frac{D^2(0, \omega) - D^2(y, \omega)}{D(0, \omega)} \\
&+ \int \frac{d\omega}{2\pi} \frac{q(\omega)D(y, \omega)}{D(0, \omega)} + \frac{q(\omega)q(-\omega)}{4D(0, \omega)} \\
&+ \int i\Lambda_p q(t) + V \cos(q + 2\sqrt{\pi}\theta_p(0, t))
\end{aligned}
$$

which is the result that they get in the paper. Often the 2nd term in the action is written as $W(x)$ and is an envelope function that does not depend on the properties of the impurity (ie q) and therefore is not of that much interest to us. I feel like i did sneak past the fact that $i\lambda_h\theta_p$ cancels out, but the final answer doesnt contain a term of that form so I imagine that its a missing factor of 2 somewhere.

Doing a final look into this envelope function, when the propagators are that of a Luttinger Liquid we will get,

$$W(x) = \exp\left(K \int_0^\infty d\omega \frac{e^{-2\omega K x / v_F} - 1}{\omega} \right) \tag{84}$$

with the action containing terms that look like $W(x)^{\mu^2}$. This integral has problems in the $\omega \to \infty$ limit because of the $1/\omega$ so we need to introduce a cut off at the Fermi energy, doing this integral isn't completely simple but it will give $(1 + \frac{2KE_F x}{v_F})^{-K}$ and so we can see the length scale associated with the envelope $\alpha = v_F/2KE_F$.

4.3   Now What?

So having got this action we can now read off various features of it. The first feature that can be found is by considering the density, which is related to the derivative of the field. To average over this field we see from the definition of the thing we are averaging over $(\exp(2\sqrt{\pi}i\mu\theta(y,0)))$, that by taking the derivative with respect to μ and then setting it to zero at the end. We also will take the spatial derivative to get the average of the derivative of the field. The general form of the density for 1D systems is $k_F/\pi + \partial_x\theta/\sqrt{\pi}$. So doing all this gives,

$$\rho(x) = k_F/\pi - \frac{K^2 e\varphi}{2\pi v_F}\mathrm{sgn}(x) \tag{85}$$

showing that the average density is discontinuous over the impurity. When the particles are non-interacting and we have an impurity, we get a density that looks exactly like the above equation except with the four-terminal voltage being where φ is so we can associate this to the four-terminal voltage. (Showing this is done in Datta and originally by Buttiker)

Making this associatation, this means that in the case of a clean wire there is always no four-terminal voltage and the average density is a constant - so the densities of right and left movers are also constant. These are determined by the boundary condition and so we get the result that current is not renormalised by interactions $I = e^2 U/h$.

In the paper (which is corrected in the erratum) they claim that if we considered an external voltage coupling into the Hamiltonian as $e\bar{\varphi}\theta(0)/\sqrt{\pi}$ then we are missing a factor of $1/K^2$ but this is actually due to it not appearing in the boundary conditions as they did not consider the screening of density in 1D. However, the addition of the terms that are linear in time within the cosine are correct and cannot be accounted for by this sort of simple substitution into the Hamiltonian and the reservoirs need to be considered as a boundary condition.

So now having had a little explore into how this works, we shall now dive further into this action. The effective action that we have derived depends on φ which is generally a fluctuating quantity with the two limiting cases of zero transmission and zero reflection where the fluctuations vanish. In the case of zero reflection (no scatterer) then we will have the action dominated by the infrared divergence of the kinetic term in the action. Apparently this means that $\varphi = 0$, which is the justification of the earlier part for the current.

The opposite case will have $V \to \infty$ as the scatterer increases in strength, this will cause the cosine term to dominate the action and due to how large it is, it will attempt to get rid of the time dependence due to it costing so much action - this sets $U = \varphi$.

Therefore to extract more results we can expand perturbatively in the limit of weak scattering (and vice versa in strong scattering)

4.3.1   Weak Scattering Regime

To investigate this regime, we can expand the impurity strength and calculate the effect at each order. They decide to investigate the antisymmetric charge distribution $q(x) = -e(\langle\rho(x)\rangle - \langle\rho(-x)\rangle)$

4.3.2 Strong Scattering Regime

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