A description of Quantum Hall states can be built up from stacking multiple microscopic 1D wires. This paper outlines how to generate a hierarchy of states by letting different inter-wire hopping processes dominate the electron dynamics. Common techniques of bosonisation and the renormalisation group allows the behaviour of the system to be analysed for a wide range of perturbations.

The Quantum Hall Effect encapsulates so much of the progress that theoretical condensed matter has made in the past 40 years. From the evolution of topological field theory ideas, to still unanswered questions [1] on the true nature of some plateaus, the field of strongly correlated electron systems seems to have every possibility expressed in this experiment.

It also shows the complementary nature of physics, where different descriptions of the same phenomena allow for better understanding. Each viewpoint has its strengths and weaknesses and can be deployed in further situations. One way is to construct the states from a microscopic basis, not just from a variational (guessed if you're feeling uncharitable) wavefunction or from a generic low-energy theory. Although Kane, Mukhopadhyay and Lubensky were not the first to notice the correspondence between the 1D systems and quantum Hall states, that was Sondhi and Yang [2], they generalised the procedure and considered fractional states.

The underlying model from which our QH states will be derived is known as the sliding Luttinger liquid or smectic metal. This trio had previously looked at sliding Luttinger liquids `https://link.aps.org/doi/10.1103/PhysRevB.64.045120`, named such because of the similarity to the sliding phase in a 2D XY model. Smetic metal comes from terminology of liquid crystals, that have properties of both solids and liquids, namely a long range positional order in at least one direction. Normally this is associated with alternating layers that have order within the layer (think like a quantum Hall system where different layers have different fillings). The paper is about showing that this model is unstable towards the formation of new states – one of which turns out to be quantum Hall states!

1 Integer Fractional Quantum Hall

One very appealing thing about the description of quantum Hall as arising from an array of coupled 1D wires is that the basic behaviour of the integer states (plateaus and current carried around edges of system) can be obtained without much effort. So the first thing to do is recap the construction of the integer states in this paper.

So they start off with the Hamiltonian for spinless electrons in a wire, with the value of the minimally coupled potential depending on where the wire is placed (encapsulated by subscript j),

$$H = \frac{1}{2m}(p + eA_j)^2, \tag{1}$$

If the wires are along the x-axis and stacked up along the y-axis with a magnetic field in the $\hat{z}$ direction, then a gauge can be chosen so that it only varies in the y-axis (ie between wires hence the A_j). The Landau gauge is chosen $A = -By\hat{x}$, and if the separation of the wires is a, then $y = a * j$ where j is an integer that labels the wires. The Hamiltonian for the jth wire becomes

$$H = \frac{1}{2m}(\hbar k - \frac{eBa}{c}j)^2. \tag{2}$$

[1] I haven't kept up with recent papers so I do not know how settled the 5/2 plateau is at the moment - I should really check!

[2] `https://arxiv.org/abs/cond-mat/0007349`

In the case of no tunnelling, there can be only one momentum (that in the x-direction) so $p = \hbar k$ describes the momentum along the wire. The dispersions of each wire is therefore given by

$$E_j(k) = \frac{\hbar^2}{2m}(k - bj)^2. \tag{3}$$

The magnetic field enters the problem through $b = eaB/\hbar c$. This dispersion is described by multiple parabolas where each minimum has been shifted by bj – therefore there are many crossings or equivalently multiple points where there is a twofold degeneracy.

When tunnelling processes are included into the Hamiltonian, the crossings become avoided crossings and a gap opens up in the spectrum at all crossing points. Another way to describe this process is that the tunnelling processes break the symmetry that gives rise to the degeneracies. A final way to express this fact is though a simple eigenvalue calculation; if the states are degenerate then the two states will have the same eigenvalue, for a degeneracy with two points this corresponds to solving the following eigenvalue problem,

$$\begin{pmatrix} \omega - \lambda & 0 \\ 0 & \omega - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4}$$

Taking the determinant gives the solution of $(\omega - \lambda)^2$. This corresponds to the two wires existing independently, where any change to one must be made to the other (its a symmetry!). Breaking this symmetry, or including tunnelling corresponds to adding off-diagonal elements,

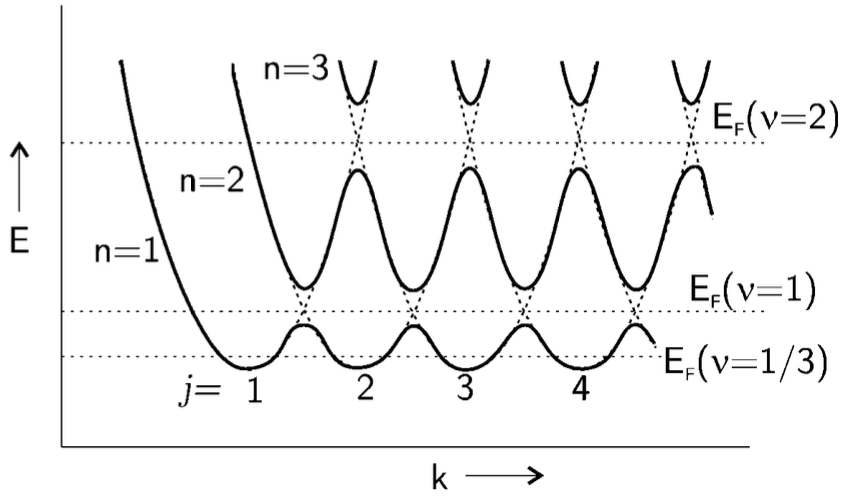
$$\begin{pmatrix} \omega - \lambda & \Gamma \\ \Gamma & \omega - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

This now has energies of $\omega = \lambda \pm \Gamma$ and has two distinct solutions. For more details of how this occurs, there are multiple good beginner solid state textbooks [3]. This broad strokes approach saves us a lot of time on performing actual calculations, whilst providing a lot of understanding about how the system will behave due to the multiple gaps opening up in the spectrum.

Next we consider how the interacting system behaves at different integer fillings. The first gap will occur at values of k where $E_j(k) = E_{j+1}(k)$, with the nth gap occurring at $E_j(k) = E_{j+n}(k)$. Solving this equation gives that the nth crossings will occur at $2k = 2bj + bn$. If each wire is filled up to $k_F = \pi n_e$, then the ratio of electron filling to magnetic field $\nu = 2k_F/b = n$ will be an integer at the crossings. This is as the first crossing point with respect to each wire is the $j = 0$ one. If tunnelling occurs between the wires that are n apart, a gap will open at this crossing point and having the filling ν be an integer means the Fermi energy will be in a gap. This figure in the original paper explains this point pretty well

There are however some possible states available to excite when the Fermi energy is in this 'gap' (inverted commas as there are actually states!). These are the two edge wires which do not have another parabola either side to intersect with, therefore there are gapless excitations that exist on the edge of the system. This explains the basic behaviour of the integer quantum hall state, where there are plateaus in the conductance, corresponding to the Fermi energy traversing the gapped crossings. In these plateaus, the current is carried around the edges of the system, with the number of channels possible for transport increasing at higher gaps - draw a line across the second gap and see that there are now two edge modes either side.

[3] Kittel, just look up avoided crossings!



2 Fractional Quantum Hall

Although this model easily extracts the behaviour of integer states, which corresponds to different fillings of the wires, to understand what happens at non-integer fillings will require a lot more work. First what happens when the non-interacting system is at non-integer fillings? This can be again seen by appealing to the dispersion figure, with each wire having gapless excitations. Expressing this rather simple fact in complicated RG language, this is a trivial sliding Luttinger liquid fixed point[4].

Introducing interactions at non-integer fillings will often mess up this nice descriptions and result in the system flowing to a new fixed point. If you are unfamiliar with the RG terminology, essentially it means that the description is unstable towards interactions and if they are strong enough the underlying low energy description of the model won't be a slight change of the non-interacting case, but tend to completely different behaviour. See my notes of RG to get a background as RG analysis forms the backbone of the analysis, although it is possible to just take those parts of trust.

2.1 Bosonising the System

To progress, we use a technique that defines 1D physics - bosonisation. There is plenty of subtlety about this procedure, but it amounts to describing chiral fermionic fields ψ_η, where $\eta = \{R, L\}$ in terms of bosonic chiral ones θ_η. This happens for each wire i and so the same amount of fields are introduced.

$$\psi_{\eta,i}(x) \sim e^{i(bj \pm k_F)x} e^{i\theta_{\eta,i}(x)}. \tag{6}$$

See my various writings on Luttinger liquids for much more detail about this and how to bosonise the underlying quadratic Hamiltonian, which is actually the difficult part; it's actually simple when applying it to backscattering [5] [6]. After you know about linearisation and bosonisation the next started result should be familiar.

[4] Sliding Luttinger liquids are essentially a bunch of stacked Luttinger liquids, where all perturbations that tend to different kinds of order (ie superconductivity, charge density wave, Fermi liquid etc) are irrelevant. It is exactly this idea of each wire having gapless excitations that exist in their own wire.

[5] This is if you are ignoring difficulties with Klein factors which give the bosinised fields actual Fermionic commutation relations, they don't matter here though!

[6] The short distance cut-off is also excluded here for clarity.

The most general quadratic Hamiltonian that results from this process can take into account forward scattering $\mathcal{H}_{fs}$ (density-density interactions) and the kinetic part $\mathcal{H}_0$

$$\mathcal{H}_0 + \mathcal{H}_{fs} = \pi v_F \sum_i [n_{R,i}^2(x) + n_{L,i}^2(x)] + \sum_{ij} (n_{R,i}, n_{L,i}) \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix} \begin{pmatrix} n_{R,j} \\ n_{L,j} \end{pmatrix} \tag{7}$$

where the density is of the chiral bosonic field $n_{\eta,i} = \pm\partial_x \theta_{\eta,i}(x)/2\pi$. The system is described by the way in which the densities in each chiral channel couple to each other in each wire (the matrix structure that is explicitly written out) and in-between each wire (the structure in the i,j indices). The leading order terms that arise from the non-interacting Hamiltonian are the interaction between densities of the same chirality on the same wire. This result is just the usual one for a multichannel Luttinger liquid. Getting to this model from the original description of the wires is not easy but if you've seen it before, it is a familiar form. There are lots of resources online on finding this form, and is the starting point for most treatments of 1D physics, so go and learn it if you haven't!

2.2 Backscattering

The previous description the system is not exhaustive as there are certain interaction processes that could occur between the wires that cannot be written as a quadratic density interaction. The most general type of single-particle interaction looks like

$$\mathcal{O}_{s_p^L, s_p^R} \sim \sum_j \prod_p (\psi_{R,j+p})^{s_p^R} (\psi_{L,j+p})^{s_p^L}. \tag{8}$$

The generic interaction is between wires that are p apart with $2(p+1)$ possible different chiral fermionic fields that the interaction could be made from. Ie for wires two apart a possible interaction could be $\psi_{L,j}^\dagger \psi_{R,j+1} \psi_{L,j+1} \psi_{R,j+2}^\dagger$ – with the most general one containing multiple of the same fields. Therefore, there are two $(p+1)$ strings of integers that describe the most general possible interaction, one for the right fields and one for the left ones. In the example considered above $s_p^R = \{s_0^R, s_1^R, s_2^R\} = \{0, 1, 1\}$. This is then applied across all sets of wires through the sum over j. It is also taken that $\psi^s = (\psi^\dagger)^{|s|}$ for negative s (roughly the inverse of the creation is the annihilation, though it isn't strictly true in general but using the bosonisation formula shows why this is useful.)

There are two important restrictions on the allowed indices. The first must always be true, and that is the electron that is being scattered around cannot be destroyed or created (unless we start considering open systems!). Charge conservation is given by $\sum_p s_p^R + s_p^L = 0$. The second requirement is that of momentum conservation. This corresponds to the first exponential in the bosonisation formula, which for a non-zero exponent, will oscillate rapidly as longer length scales are considered (we're increasing x). Bear in mind that we are interested in low-energy physics, which corresponds to long length scales. Any rapidly oscillating exponential will average to zero when the integral over space is performed; this will happen when taking the average over these operators. Therefore, we only need to consider those operators what have that exponent equal to zero. The term freely commutes as it is not an operator and is just,

$$k_F \sum_p (s_p^R - s_p^L) + b \sum_p (s_p^R + s_p^L) = 0. \tag{9}$$

The types of operators can then be further differentiated into degenerate and non-degenerate ones. Non-degenerate

ones are the operators that upon 'rotating' by 180 degrees, they are equivalent[7]. This is simply a way to guarantee that the interaction commutes with itself on the different wires. That is that the operator can be written as,

$$\mathcal{O}_{s_p^L,s_p^R} = \lambda \sum_j \cos(\Xi_j), \qquad \text{where,} \quad \Xi_j = \sum_p s_p^R \theta_{R,j+p} + s_p^L \theta_{L,j+p}. \tag{10}$$

Although earlier p was taken to be $\{0,1,2\}$, for this section it is easier to change that to $\{-1,0,1\}$. It is just an indexing set anyway so doesn't need specific values. I think this slight subtlety is why they never give it a form in the paper; the rotations for an interaction that spans an even number of wires (eg. 4) wires requires a weird shift of j to maintain a 'symmetric' p indexing (ie the centre of rotation is inbetween wires and therefore there is no zero term in the indexing set $\{-2,-1,1,2\}$). Anyway this is because the rotation is performed around the jth wire sending $\theta_{R,j+p} \to \theta_{L,j-p}$ (now hopefully you see why the indexing set is chosen to have equal positive and negative values). This places another restriction on the indices that $s_p^R = \pm s_{-p}^L$. Note that one sign must be chosen for all of them, in constast to there being a different plus or minus for each value of p, due to the cosine not caring about overall sign so,

$$\cos\left(\sum_p s_p^R \theta_{R,j+p} + s_p^L \theta_{L,j+p}\right) \to \cos\left(\sum_p s_p^R \theta_{L,j-p} + s_p^L \theta_{R,j-p}\right) = \cos\left(\pm \sum_p s_{-p}^R \theta_{L,j+p} + s_{-p}^L \theta_{R,j+p}\right). \tag{11}$$

This restriction means that $[\Xi_i, \Xi_j] = 0$. This is required if all modes want to be frozen independently of each other which is the main point of the analysis. The 'degenerate' case, means that these operators do not commute and therefore cannot be simultaneously minimised and is not considered for the rest of this paper [8]. Future papers by Kane do consider these types of operators.

2.3 Minimising operators

If the strength of the cosine backscattering term, λ, dominates the quadratic part of the Hamiltonian, then the backscattering would suppress fluctuations of the field and try to pin the fields to a definite value, such that the cosine is minimised. Any fluctuations around the cosine minimum will have a huge energetic cost to perform. If there are multiple different cosines then the fields can only be simultaneously minimised if they commute (it's the uncertainty principle, specifying the position of the field!). If fluctuations about the minimum field configuration, set by the cosine, are heavily penalised then the cosine can be expanded to quadratic order in the fields. This allows us to understand the effect of the cosine when it dominates the system - a gap opens up.

For example, let's consider a cosine acting on a single-channel Luttinger liquid, described by dual fields $\theta(x), \phi(x)$ and Luttinger parameters ν, K,

[7] I really dislike that they call these non-degenerate because normally if there is a symmetry then there is a degeneracy, but they are inverting the usual meaning here so there is a lot of inverted commas in the paper when they talk about 'degenerate'

[8] In the paper, they write that they must come in pairs, which I don't like - as the Hermitian conjugate of the operator almost always exists which will have the opposite sign to every term and form a cosine. Also, why should the operator rotated by 180 degrees exist and be relevant? Symmetry requirements shold also apply for Hermitian conjugates, not just rotated operators. This is just my personal gripes with the set-up, everything they say is true!

$$H = \frac{\nu}{2\pi K}\int dx(\partial_x\theta(x))^2 + K^2(\partial_x\phi(x))^2 + \lambda\cos(\theta(x)) \tag{12}$$

$$\approx \frac{\nu}{2\pi K}\int dx(\partial_x\theta(x))^2 + K^2(\partial_x\phi(x))^2 + \frac{\lambda}{2}\theta(x)^2. \tag{13}$$

Then going to the action (using $p\dot{q}$ term of $\partial_x\phi\partial_t\theta$), by integrating out ϕ, and then swapping to the Fourier transformed field $\theta_{k,\omega}$ we get,

$$S = \frac{\pi}{2\nu K}\sum_{k,\omega}|\theta_{k,\omega}|^2(\omega^2 - \frac{\nu^2}{\pi^2}k^2 - \frac{\lambda\nu K}{\pi}) \tag{14}$$

The dispersion is given when the bracket is zero, so $\omega = \pm\frac{\nu}{\pi}\sqrt{k^2 + \lambda\nu\pi K}$. This has a gap at $k = 0$ and the size of that gap is related to λ. Therefore, strong backscattering means that suddenly there are no more states to occupy and the same state will be occupied until the gap is exceeded. Generalising this to a multichannel case is simple due to the requirement that all the operators commute. All fields are promoted to a $N-$dimensional vector, where N is a number of wires, and in the 'direction' of the backscattering vector (defined by array of s_p^R) there are no states to occupy and the dynamics are frozen until the gap is exceeded.

This analysis was only technically in the limit that the backscattering effect is strong, but this is where the magic of RG allows our results to be extended for a larger slice of parameter space. For some certain strength of inter/intra-channel interaction, considering a lower energy version of the model means the backscattering effect gets stronger. This is known as a relevant operator and is possible when the scaling dimension of the operator is larger than 2 - to actually find the specific algebraic value of the interaction when the different low energy theory become appropriate requires knowledge of RG. The main point is that, as we are interested in the low-energy behaviour, for a certain strength of interactions in the quadratic part of the Hamiltonian the backscattering will get stronger and stronger as we consider lower and lower energies. Therefore, the resulting gap will happen even if the backscattering is not originally strong.

3 New Phases

The first case considered, they call crystalline phases, defined by $\sum_p s_p^R = \sum_p s_p^L = 0$ and $\sum_p p(s_p^R + s_p^L) = 0$. The means that both the right and left moving charges are independently conserved. In this case, the requirement for momentum conservation is satisfied by any magnetic field. They give multiple suggestions of interaction, but this really isn't the focus on the paper so let's skip to the main course

The other phase corresponds to the quantum Hall states, where $\sum_p s_p^L \neq \sum_p s_p^R$. This means that momentum is only conserved for certain magnetic fields, which is defined by the filling factor,

$$\nu = 2\frac{\sum_p p(s_p^R + s_p^L)}{\sum_p s_p^R - s_p^L}. \tag{15}$$

Replacing the cosine of the operator with a quadratic term results in the bulk being gapped, as seen previously, with a gapless edge mode. Using $\sum_j \cos(\phi_j - \phi_{j+1})$ recreates the integer quantum Hall result; now on a solid mathematical footing rather than just inferences from the dispersion.

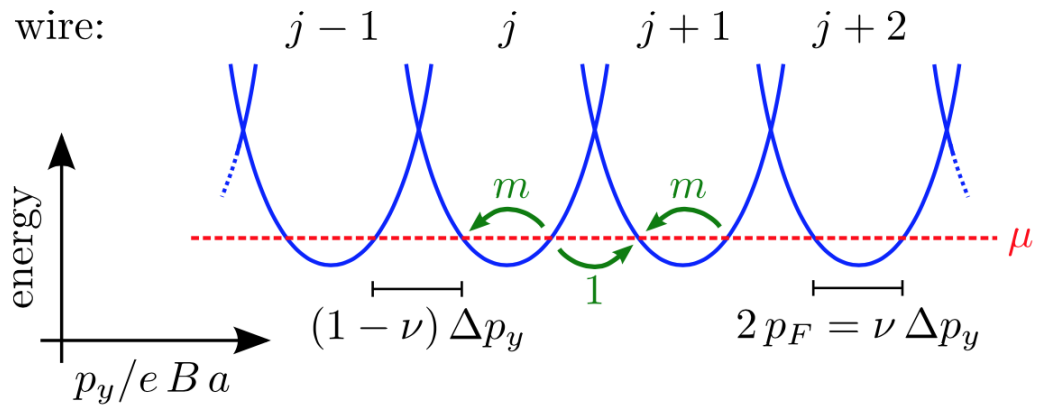


Figure 1: Taken from Meng (2020), `https://arxiv.org/abs/1906.09771`

Fractional quantum Hall states are possible for correlated hoppings. If an electron hops to an adjacent wire and backscatters $(m-1)/2$ electrons on each wire in the process, then the resulting low-energy theory of the sliding Luttinger liquid and the backscattering operator corresponds to the $1/m$ Fractional Quantum Hall states - where m is an odd integer. It has to be an odd integer, otherwise the hopping process would be backscattering half an electron on each wire! But how do we know that this produces this FQH state? We must determine the topological order, which is done by looking at the quasiparticles of the low-energy system and the structure of the gapless modes.

3.1 Transformations and more transformations

So far, we've managed to sidestep most of the maths and quote what the low-energy modes correspond to, but we must now face the task head on. They first introduce the usual density and phase fields in Luttinger liquids,

$$\varphi_j(x) = (\phi_{R,j}(x) + \phi_{L,j}(x))/2, \quad \theta_j(x) = (\phi_{R,j}(x) - \phi_{L,j}(x))/2$$

The density is given by $n(x) = \partial_x \theta_j(x)/\pi$ and $\varphi(x)$ is then the conjugate phase. The interaction term for the fractional quantum states (as described earlier) in these density and phase fields becomes,

$$\mathcal{O}_{\text{Laughlin}} \sim \sum_j \cos(\varphi_j - \varphi_{j+1} + m(\theta_j + \theta_{j+1})). \tag{16}$$

Now having done one transformation, we should keep on doing them! This time it's to find new chiral fields. These are defined as fields that have a non-zero commutator with themselves but commute with the other chirality - each chirality have a commutator with themselves and have the opposite sign to each other. The most general form of $A_R(x), A_L(x)$ for constant z,

$$[A_R(x), A_R(x')] = +z\text{sgn}(x-x'), \quad [A_L(x), A_L(x')] = -z\text{sgn}(x-x'), \quad [A_R(x), A_L(x')] = 0 \tag{17}$$

This is motivated by seeing that the operator must overall commute with itself (that was one of the original restrictions), yet considering the terms at j, they have a non-zero commutator with themselves. So there must be

some way in which the commutator of the j and $j+1$ terms cancel each other out - they have a different sign, therefore chiral fields! Why are they called chiral? Well when you compute $\partial_t \tilde{\rho}_{R,L} = i[H, \tilde{\rho}_{R,L}] = \pm v \partial_x \tilde{\rho}_{R,L}$, for some density operator of the chiral fields $\tilde{\rho}_{R,L}$, you can find that the densities (and hence the fields) only depend on $x \pm vt$.

Dropping notation for spatial dependence, the chiral fields,

$$\tilde{\Phi}_{R/L,j} = \pm\theta_j + \varphi_j/m, \qquad [\tilde{\Phi}_{R/L,j}(x), \tilde{\Phi}_{R/L,j'}(x')] = \pm i\pi \text{sgn}(x - x')/m \tag{18}$$

What's the interaction in this basis?

$$\mathcal{O}_{\text{Laughlin}} \sim \sum_j \cos\Big(\tilde{\Phi}_j - \tilde{\Phi}_{j+1}\Big) \tag{19}$$

This now looks the same as the integer quantum hall backscattering interaction (quoted earlier) except for the underlying fields being weird combinations of the original fields. As the argument for the cosine tries to be minimised, then the value of $\tilde{\phi}_j = \tilde{\phi}_{j+1}$ locking neighbouring fields to the same value at each x. To get more clarity, it's time for another transformation! This time it is to variables that are defined between wires (called links in later papers) and having the plus $1/2$ in the position index of the field indicates that these are not localised to a wire.

$$\tilde{\varphi}_{j+1/2} = (\tilde{\Phi}_{R,j} + \tilde{\Phi}_{L,j+1})/2, \qquad \tilde{\theta}_{j+1/2} = (\tilde{\Phi}_{R,j} - \tilde{\Phi}_{L,j+1})/2 \tag{20}$$

They then write the full Hamiltonian in this new basis using $\tilde{\varphi}, \tilde{\theta}, \tilde{\Phi}$. Its given by the sum of the backscattering, free and interaction terms $H = H_{bs} + H_{fs} + H_e$,

$$H_{bs} = \sum_{j=1}^{\infty} \nu[(\partial_x \tilde{\phi}_{j+1/2})^2 + (\partial_x \tilde{\theta}_{j+1/2})^2] - u\cos\Big(2m\tilde{\theta}_{j+1/2}\Big), \qquad H_e = \nu_e(\partial_x \tilde{\Phi}_{L,1})^2 \tag{21}$$

The forward scattering is then just quadratic in the three fields so can be ignored - at least until we want to find the precise boundaries of the RG flow. Having got the problem into this form, we now can analyse the problem.

3.2 Bulky Bulk

As discussed previously in many cases the backscattering terms will flow to strong coupling and the argument of the cosine must be minimised. Therefore $2m\tilde{\theta}_{j+1/2}$ must be a multiple of 2π. Although variations in the field are energetically penalised, there is a type of solution, called a soliton that is in the different minima of the Hamiltonian at $x = \pm\infty$. See my notes (or anyone elses) for why this is a solution and a common feature when there are multiple degenerate minima of a Hamiltonian[9].

The soliton solutions advance the $2m\tilde{\theta}$ field by 2π. So for integer n,

[9] Search sine-Gordon solitons for more information on the solitons of this particular model.

$$\tilde{\theta}(x) = \begin{cases} \frac{2\pi n}{2m}, & \text{for } x < x_0 \\ \frac{2\pi n}{2m} + \frac{\pi}{m}, & \text{for } x > x_0 \end{cases} \tag{22}$$

So what operator inserts a kink in our field $\tilde{\theta}$ of size π/m at position x_0? One way to understand it is to integrate the density over all space to get the charge,

$$\pi Q = \pi \int_{x'}^{x''} dx\rho = \tilde{\theta}(x'') - \tilde{\theta}(x'), \tag{23}$$

so kinks are equivalent to inserting a particle as $\tilde{\theta}(x'') - \tilde{\theta}(x')$ will be zero unless there is a jump (kink) between them. How to insert a charge then? It is worth recapping the definition of a creation operator A in quantum mechanics and its relation to density p. For generic constant charge c, if the commutator is equal to

$$[\rho, A] = Ac, \qquad \rho_A = \rho_0 + c, \tag{24}$$

then the density of the A state is c more than the previous state. Therefore, if we find an operator A such that $[n_j, A] = A/m$, where $n_j = \partial_x \tilde{\theta}_j/\pi$ (where $\tilde{\theta}_j$ is defined by Eq. 20 with both fields on the same wire), then that will put a kink into the system. This is done by $A(x) = e^{i\tilde{\phi}_j(x)}$ which therefore puts a particle of charge e/m into the system. The position variable of the field has been suppressed for a while but this will be inserted at the position of the field.

To understand how this argument gets generalised to dealing with a kink appearing in the true $\tilde{\theta}_{j+1/2}$ field, let us first rewrite the density operator

$$n_j = \frac{1}{2\pi}\partial_x(\tilde{\phi}_{j+1/2} + \tilde{\theta}_{j+1/2} - \tilde{\phi}_{j-1/2} + \tilde{\theta}_{j-1/2}) \tag{25}$$

Total charge will contain the sum over all j, so if there is a kink in the $\tilde{\theta}_{j'+1/2}$ link, then this will appear twice in the sum resulting in a total charge for the kink of e/m. This is the confirmation that the state we are dealing with corresponds to the FQH state with Laughlin quasiparticles.

Additional operators can be built that hop these bulk excitation around. Though using strings of these hopping operators, the quasi-particles can be braided around each other to find the statistics of the excitations - revealed to be anyonic. This is an area that a lot of the further papers on this coupled-wire construction explore.

3.3 Edge modes

To get a low-energy description of the system, we can integrate out the bulk (where we have used the large λ limit to make the cosine term quadratic). However, the only term that will affect the edge states is the field $\Phi_{\tilde{R},1}$ which when integrated out can affect the value of $\mathcal{H}_{fs}$. This in turn renormalises the speed of the edge mode. The integer that appears in the commutation relation characterises the edge state and should be able to be experimentally accessed through the tunnelling density of states. To create an excitation, we would use the operator

$$\psi_e = e^{im\tilde{\Phi}_{L,1}} \sim \psi_{L,1}(\psi^\dagger_{R,1}\psi_{L,1})^{(m-1)/2} \tag{26}$$

.

This comes directly from using Eq. 20 to express $\tilde{\Phi}_{L,1} = \frac{m+1}{2}\Phi_{L,1} - \frac{m-1}{2}\Phi_{R,1}$. If we try to do the same thing for just $\Phi_{L,1}$, it must contain $\tilde{\Phi}_{R,1}$ and therefore a component of $\tilde{\phi}_{j+3/2}$ - it must be gapped!

We can actually get a full description of how $\tilde{\Phi}_{L,1}$ is distributed on the original chiral fields. To do this we rewrite $\tilde{\Phi}_{L,1}$ in terms of $\tilde{\Phi}_{R,1}$ and $\Phi_{L,1}$. Then we can use the freezing condition to iterate along the wire,

$$m\tilde{\Phi}_{L,j} = \frac{2}{m+1}m\Phi_{L,j} - \frac{m-1}{m+1}m\tilde{\Phi}_{L,j+1} \tag{27}$$

Doing the same for $\tilde{\Phi}_{R,N}$ in terms of $\Phi_{R,j}$ and $\tilde{\Phi}_{L,N}$ allows us to get for $\alpha = m - 1/m + 1$

$$m\tilde{\Phi}_{L,1}(1 + (-1)^N(\alpha)^{2N}) = \frac{2}{m+1}\sum_{j=1}^{N}(-\alpha)^{j-1}\Phi_{L,j} + (-\alpha)^{2N-j}\Phi_{R,j}. \tag{28}$$

Showing that the edge mode is localised to the edge, decreasing exponentially by $\alpha < 1$ as we move away from the edge. Although interestingly, as $m \to \infty$ then $\alpha \to 1$ so it becomes progressively less localised to the edge.

3.4 Away from perfect filling

What happens slightly away from these special values of the magnetic field? Lets set the filling to be $\nu = \delta + 1/m$. Recalling the bosonisation formula, there will start to be an oscillation in the field due to the Laughlin operator not fully conserving momentum. But if $\delta k_F \ll 1$ then it will not have much of an effect on the oscillation - this corresponds to looking at length scales δ^{-1}. At shorter lengths, then the operator will renormalise the forward scattering Hamiltonian and not open a gap due to it oscillating too quickly. At larger length scales, then a term can be relevant and cause a change in the low-energy description. The cosine interaction is not possible but crystalline operator (previously ignored) are allowed for any magnetic field/filling. If the interaction is,

$$\cos\Big(2m(\tilde{\theta}_{j+1/2} - \tilde{\theta}_{j-1/2})\Big) \tag{29}$$

This describes a crystal of Laughlin quasi-particles, where all the links fields (that are the Laughlin quasi-particles) are frozen together. This freezing could be the result of weak disorder in the system and will have no dynamics - but there will still be edge modes that corresponds to the edge modes in the FQH state.

3.5 When is this relevant

Now is the time to find out the precise value of the forward scattering when the low-energy description of the model changes from the gapless trivial case to the gapped one that resemble quantum hall states. In this paper, they only consider a specific form of the forward scattering to show that it can be relevant which I'll talk about here. There are papers that examine a much larger slice of parameter space to determine how the Luttinger values become relevant – I'll link some here[10].

Assuming that the original forward scattering Hamiltonian is such that when transformed the new forward scat-

[10]This paper looks the stability of a sliding Luttinger liquid to FQH (called single particle hopping) as well as superconducting and charge density wave instabilities `https://arxiv.org/abs/1711.01639`

tering term becomes,

$$\mathcal{H}_{fs} = w \sum_i (\partial_x \tilde{\theta}_{i+1/2})^2. \tag{30}$$

Then calculating the dimension of the operator (I normally calculate it by integrating out and using Lagrangian, which is all in my thesis somewhere) we get $\Delta = 2m/\sqrt{1 + w/\nu}$ where ν is the strength of the transformed quadratic backscattering term. For the operator to be relevant $\Delta > 2$ (as we are in $1 + 1D$ that we are scaling these operators hence the 2) which can be achieved with a large enough w. This answers the question about the quadratic part of the backscattering, but what about the cosine term? Well it will contain higher powers of $\tilde{\theta}$ or $\tilde{\phi}$ and the dimension of $\exp\left(i\tilde{\phi}\right)$ is $\Delta' = \sqrt{1 + w/\nu}\ /2m$ and will be larger than 2 for large w and therefore irrelevant. So the answer they give here is the backscattering can be relevant and sliding Luttinger Liquid are unstable towards the formation of quantum Hall states.

4 Higher Order States

This process can be extended to deal with other fractions by extending the range of the interaction to many wires. This will be looked at elsewhere in more detail, but there are multiple edge modes, and the value of the commutator between them can be different. They focus on when the interaction spans three wires(resulting in a 2D gapless subspace)and the two edge modes Φ_α, Φ_β will then have a structure encoded by the commutation,

$$[\Phi_\alpha(x), \Phi_\beta(x')] = -i\pi K^{-1}_{\alpha\beta} \text{sgn}(x - x'). \tag{31}$$

The entries of matrix $\mathbf{K}$ are determined by the number of hoppings of the underlying chiral electrons which must be integer. The signs of the entries determine whether both edge states are moving in the same direction. If the move in the opposite direction, this corresponds to second order states in the Haldane-Halperin hierarchy. If they move in the same direction, then this is the hierarchy of Wen-Zee and can be understood as two layers of Laughlin states on top of each other. Further discussions of how the hierarchy works will be saved for another exciting instalment...

5 Resources

Kane has some slides which may be useful! `https://www.physics.upenn.edu/~kane/seminars/sll.pdf`

The start of the Meng paper is also helpful `https://arxiv.org/abs/1906.09771`