Solving for Equilibrium Arms Levels and Welfare in the Dynamic Model with Endogenous Arming and Exogenous War Costs

Our extension to the (Fearon 2018) model requires us to model war equilibria in addition to the peace equilibria analyzed by Fearon. In addition, we allow states to have a discount rate bounded away from zero, which implies that states will not necessarily fully arm in war equilibria.

Below, we find analytical solutions for equilibrium arms levels and states' expected utilities in war equilibria using the Karush-Kuhn-Tucker (KKT) conditions for constrained optimization. This allows us to run comparative comparative statics on the equilibria, which are useful for proving our propositions related to our Welfare is U-Shaped Under Offensive Advantage (WUO) hypothesis.

As discussed in the main text, depending on the offense defense balance, there are some pure strategy war equilibria when one side attacks and one side defends (when $m < 1$) and there are some where both sides simultaneously attack (when $m > 1$).

We thus take the following steps, which are divided into subsections below:

- We solve for the best response arms levels and expected utility for a state that attacks another defending state as a function of the other state's arms levels
- We solve for the best response arms levels and expected utility for a state that is defending against another state's attack as a function of the other state's arms levels
- We solve for the reduced form Nash Equilibrium arms levels in the attack-defend equilibrium (where one state is attacking, and one is defending)
- We run comparative statics in the attack-defend war equilibrium to determine how welfare evolves under changes in the offense-defense balance
- We solve for the best response arms levels and expected utility for a state that attacks another state while that state is also simultaneously attacking it as a function of the other state's arms levels
- We solve for the reduced form Nash Equilibrium arms levels in the simultaneous attack war equilibrium
- We run comparative statics in the simultaneous attack war equilibrium to determine how welfare evolves under changes in the offense-defense balance

Arms Levels and Expected Utility Under Attack

We can solve analytically for the RHS of equation 3 (the dynamic war constraint) using the Karush-Kuhn-Tucker (KKT) conditions for constrained optimization. We can see that this maximization problem meets the KKT conditions:

1. By assumption, $p^i(a, \hat{a}; m)$ is increasing and concave in a (this follows from CSF assumptions 4 and 5 stipulated above). Therefore the objective function is weakly concave. The objective function is also continuous and differentiable.
2. The choice variable $a \in [0, 1]$ is drawn from a compact set.

Now we can draw up our Lagrangian and first order conditions:

$$L = 1 - a + p^i(a, \hat{a}; m))\phi + \lambda a + \theta(1 - a)$$

Where $\phi = \frac{[\gamma - c + \delta(1+\mu)]}{(1-\delta)}$, i.e. the perpetuity benefit from winning the war.

Not that the suggested functional form for the attack contest success function is:

$$p_o^i(a_i, a_j; m) = \frac{ma_i}{ma_i + a_j} \tag{A6}$$

And note that:

$$\frac{\partial p_o^i(a_i, a_j; m)}{\partial a_i} = \frac{ma_j}{(ma_i + a_j)^2} \tag{A7}$$

Case 1: Interior Solution (Partial Arming)

Start by assuming an interior solution (both complementary slackness conditions bind) and finding the first order condition of the Lagrangian, it follows that:

$$\frac{\partial p^i(a, \hat{a}; m)}{\partial a}\phi = 1 \tag{A8}$$

Where the LHS represents the marginal benefit of arming: the marginal increase in the probability of winning the war multiplied by the perpetuity benefit of war. The RHS is the marginal cost of arming.

We can then use this functional form to solve for arms levels in equilibrium when there is an interior solution:

$$a = \sqrt{\frac{\hat{a}\phi}{m}} - \frac{\hat{a}}{m} \tag{A9}$$

Recall that the expected utility of war is:

$$EU_i(war) = 1 - a_i + p_i(a_i, a_j; m)\phi \tag{A10}$$

We can then plug in equations A6 and A9 to solve for the expected utility of attacking. Simplifying yields the following expression:

$$EU_i(attack) = 1 + \frac{\hat{a}}{m} + \phi - 2\sqrt{\frac{\hat{a}\phi}{m}} \tag{A11}$$

Case 2: Full Arming

When there is full arming ($a = 1$), the FOC becomes:

$$\frac{\partial p^i(a, \hat{a}; m)}{\partial a}\phi - \theta = 1$$

For the Lagrange multiplier $\theta > 0$, which represents the shadow price of increasing arms beyond the level where $a = 1$. This implies that the marginal benefit of arming at $a = 1$ is greater than the marginal cost of arming:

$$\frac{\partial p^i(a,\hat{a};m)}{\partial a}\phi\Big|\,|_{a=1} > 1$$

Plugging in the preferred functional form for the contest success function, we find that this occurs when:

$$\phi > \frac{(m+\hat{a})^2}{(m+\hat{a})m - m^2}$$

Which simplifies to:

$$\phi > \frac{(m+\hat{a})^2}{m\hat{a}} \tag{A12}$$

In this case the expected value of attacking is:

$$EU_i(war) = \frac{\phi}{1+\frac{\hat{a}}{m}} \tag{A13}$$

Case 3: No Arming

When there is no arming ($a = 0$), the FOC becomes:

$$\frac{\partial p^i(a,\hat{a};m)}{\partial a}\phi + \lambda = 1$$

For the Lagrange multiplier $\lambda > 0$, which represents the shadow price of relaxing the constraint where the players can spend no less on arms than $a = 0$. This implies that the marginal benefit of arming at $a = 0$ is less than the marginal cost of arming:

$$\frac{\partial p^i(a,\hat{a};m)}{\partial a}\phi\Big|\,|_{a=0} < 1$$

Plugging in the preferred functional form of the contest success function, we find that this occurs when:

$$\phi < \frac{\hat{a}}{m} \tag{A14}$$

By the assumptions we made for the war CSF, an unarmed state loses with 100% certainty, so the expected value of war is:

$$EU_i(war) = 1 - 0 + 0 * \phi$$

Which simplifies to::

$$EU_i(war) = 1 \tag{A15}$$

Summary

We can summarize the results in figure A2:

FIGURE A2.

Optimal Arms Levels Under Attack

As a function of the other sides' arms levels $\hat{a}$

	Occurs When	Arms Levels	$EU_i(attack)$
No Arming $a = 0$	$\phi \leq \frac{\hat{a}}{m}$	$a = 0$	1
Partial Arming $a \in (0,1)$	$\frac{\hat{a}}{m} < \phi < \frac{(m+\hat{a})^2}{m\hat{a}}$	$a = \sqrt{\frac{\hat{a}\phi}{m}} - \frac{\hat{a}}{m}$	$1 + \frac{\hat{a}}{m} + \phi - 2\sqrt{\frac{\hat{a}\phi}{m}}$
Full Arming $a = 1$	$\phi \geq \frac{m}{\hat{a}} + 2 + \frac{\hat{a}}{m} = \frac{(m+\hat{a})^2}{m\hat{a}}$	$a = 1$	$\frac{1}{1+\frac{\hat{a}}{m}}\phi$

Arms Levels and Welfare Under Defense

When the war constraint is not satisfied and defense is advantaged (scenario 2: Rope-a-Dope) there are two anti-coordination equilibria and one mixed strategy equilibrium. Consistent with the rest of the analysis, we will focus on the pure strategy equilibria.

To find arms levels and welfare in this scenario, we will consider the situation in which one side maximizes their arms levels for defense and the other side maximizes their arms levels for attack.

We already found optimal arms levels under attack in section , so now we will need to solve analytically for optimal arms levels under defense as a function of the other sides' arms using the Karush-Kuhn-Tucker (KKT) conditions for constrained optimization as we did in section .

The results of this optimization are shown in figure A3:

FIGURE A3

Optimal Arms Levels Under Defense

As a function of the other sides' arms levels

	Occurs When	Arms Levels	$EU_i(defend)$
No Arming $a_i = 0$	$\phi \leq ma_j$	$a_i = 0$	1
Partial Arming $a_i \in (0,1)$	$ma_j < \phi < \frac{(ma_j+1)^2}{ma_j}$	$a_i = \sqrt{ma_j\phi} - ma_j$	$1 + \phi + ma_j - 2\sqrt{ma_j\phi}$
Full Arming $a_i = 1$	$\phi \geq \frac{(ma_j+1)^2}{ma_j}$	$a_i = 1$	$\frac{\phi}{ma_j+1}$

Below is the derivation for figure A3. Assume that player i is defending against player j's attack. The contest success function for player i is:

$$p^i_d(a_i, a_j; m) = \frac{a_i}{a_i + ma_j} \tag{A16}$$

Note that the partial derivative with respect to a_i is:

$$\frac{\partial p^i_d}{\partial a_i} = \frac{ma_j}{(a_i + ma_j)^2} \tag{A17}$$

We use the same Lagrangian as in section , but with the contest success function described above. We will skip the Lagrangian steps for the general contest success function (see for those steps) and skip right to the steps where we plug in the defender's contest success function to solve for their optimal arms levels and expected utility.

Case 1: Interior Solution (Partial Arming)

From , we find that in the partial arming ($0 < a_i < 1$) scenario:

$$\frac{\partial p_i}{\partial a_i}\phi = 1$$

Plugging in the defender's CSF:

$$\frac{ma_j}{(a_i + ma_j)^2}\phi = 1$$

Solving for arms levels:

$$a_i = \sqrt{ma_j\phi} - ma_j \tag{A18}$$

And plugging in arms levels to the equation for expected utility under war (equation A10), expected utility is:

$$EU_i(defend) = 1 - \sqrt{ma_j\phi} + ma_j + \frac{\sqrt{ma_j\phi} - ma_j}{ma_j + \sqrt{ma_j\phi} - ma_j}\phi$$

Simplifying, we get:

$$EU_i(defend) = 1 + \phi + ma_j - 2\sqrt{ma_j\phi} \tag{A19}$$

Case 2: Full Arming

From , we find that in the full arming ($a_i = 1$) scenario:

$$\frac{\partial p_i}{\partial a_i}\bigg|\,|_{a_i=1} \geq 1$$

Plugging in equation A17:

$$\frac{ma_j}{(1 + ma_j)^2}\phi \geq 1$$

This leads to the condition:

$$\phi \geq \frac{(ma_j + 1)^2}{ma_j} \tag{A20}$$

Plugging in $a_i = 1$ into equation A10, the expected value is:

$$EU_i(defend) = \frac{\phi}{ma_j + 1} \tag{A21}$$

Case 3: No Arming

From , we find that in the no arming ($a_i = 0$) scenario:

$$\frac{\partial p_i}{\partial a_i}\bigg|\,|_{a_i=0} \leq 1$$

Plugging in equation A17:

$$\frac{\phi}{ma_j}\phi \leq 1$$

This leads to the condition:

$$\phi \leq ma_j \tag{A22}$$

And the expected value is:

$$EU_i = 1 \tag{A23}$$

Reduced Form Nash Equilibrium in Attack-Defend Dyad

Now that we've solved for optimal arms levels under both attack and defend as a function of the other side's arms levels, we can plug in these solutions to find the Nash Equilibrium reduced form arms levels and expected utility for the attack-defend dyad in figure A4, and the derivation is shown below.:

FIGURE A4

Nash Equilibrium Reduced Form Arms Levels and Expected Utility under Attack-Defend Dyad

	Arms Levels	Individual Expected Utility	Total Expected Utility ($EU_o + EU_d$)
No arming $\phi \leq 0$	$a_{o,d} = 0$	$EU_o = 1 + \frac{\gamma}{2}; EU_d = 1 + \frac{\gamma}{2}$	$2 + \gamma$
Full Arming $\phi \geq \frac{(m+1)^2}{m}$	$a_{o,d} = 1$	$EU_o = \frac{\phi}{1 + \frac{1}{m}}; EU_d = \frac{\phi}{m+1}$	ϕ
Partial Arming $0 < \phi < \frac{(m+1)^2}{m}$	$a_{o,d} = \frac{m\phi}{(1+m)^2}$	$EU_o = 1 + \frac{m^2\phi}{(1+m)^2}$ $EU_d = 1 + \frac{\phi}{(1+m)^2}$	$2 + \frac{\phi(m^2+1)}{(1+m)^2}$

Start by assuming that player o (attacker) attacks player d (defender). We already solved for the attacker's arms levels as a function of the defender's arms levels and the defender's arms levels as a function of the attacker's arms levels. Now we can combine these results to find the Nash equilibrium such that both players are playing best responses to the strategy of the other side.

Case 1: Full Arming

Is there an equilibrium where one side partially arms while the other is fully arming? Let's begin by considering the case where the attacker is fully arming, and see if there is an equilibrium in which the defender is partially arming.

Assume that the defender chooses arms such that:

$$\phi \geq \frac{(m + a_d)^2}{m a_d} \tag{A24}$$

Recall that in this case, the attacker's best response is to choose arms levels $a_o = 1$.

Now consider the defender's best response. Recall that full arming is the defender's best response if:

$$\phi \geq \frac{(m a_o + 1)^2}{m a_o}$$

Since $a_o = 1$ by assumption, then:

$$\phi \geq \frac{(m + 1)^2}{m}$$

Therefore, for the defender to partially arm, it must be the case that:

$$\frac{(m + 1)^2}{m} > \phi \tag{A25}$$

So now we have two conditions (A27) and (A25) that must hold for the attacker to fully arm and for the defender to partially arm.

Recall that when the defender is partially arming, their best response arms levels are determined by:

$$a_d = \sqrt{m a_o \phi} - m a_o$$

Plugging this into equation (A27):

$$\phi \geq \frac{(m + \sqrt{m a_o \phi} - m a_o)^2}{\sqrt{m a_o \phi} - m a_o}$$

Because $a_o = 1$ by assumption:

$$\phi \geq \frac{(m + \sqrt{m \phi} - m)^2}{\sqrt{m \phi} - m a_o}$$

Simplifying, we get the condition:

$$\phi \geq \frac{(m+1)^2}{m} \tag{A26}$$

However, this condition is the complement of (A25) and it is impossible to satisfy both conditions. Therefore, there is no equilibrium in which the attacker fully arms, but the defender partially arms.

Now we consider the case where the defender is fully arming, and check to so if there is an equilibrium in which the attacker partially arms.

Assume that the attacker chooses arms such that:

$$\phi \geq \frac{(ma_o+1)^2}{ma_o} \tag{A27}$$

Recall that the defender's best response under this condition is to choose arms such that $a_d = 1$.

Now consider the attacker's best response. Recall that full arming is the attacker's best response if:

$$\phi \geq \frac{(m+a_d)^2}{ma_d}$$

Because $a_d = 1$ by assumption, then:

$$\phi \geq \frac{(m+1)^2}{m}$$

Therefore, for the attacker to partially arm, it must be the case that:

$$\frac{(m+1)^2}{m} > \phi \tag{A28}$$

Now, we have two conditions (A27) and (A28) that must hold for the defender to fully arm and for the attacker to partially arm.

Recall that when the attacker's best response under partial arming is given by:

$$a_o = \sqrt{\frac{a_d\phi}{m}} - \frac{a_d}{m}$$

Because $a_d = 1$ by assumption, then:

$$a_o = \sqrt{\frac{\phi}{m}} - \frac{1}{m}$$

Plugging this into equation (A27):

$$\phi \geq \frac{[m(\sqrt{\frac{\phi}{m}} - \frac{1}{m}) + 1]^2}{m(\sqrt{\frac{\phi}{m}} - \frac{1}{m})}$$

Simplifying, we get:

$$\phi \geq \frac{(m+1)^2}{m} \tag{A29}$$

However, this condition is the complement of (A28) and it is impossible to satisfy both conditions. Therefore, there is not an equilibrium in which the defender fully arms, but the attacker partially arms.

We have now proved that there is no condition where one side fully arms while the either side partially arms. The only equilibrium with full arming involves both sides fully arming.

When both states fully arm:

$$EU_o = \frac{\phi}{1 - 1/m} \tag{A30}$$

$$EU_d = \frac{\phi}{m+1} \tag{A31}$$

$$EU_{total} = \phi \tag{A32}$$

Case 2: No arming, no war

Consider the case where:

$$\phi \leq 0$$

In this case there is never any war because the expected value from winning a war when spending nothing on arms in negative. In this case, states will spend nothing on arms.

$$EU_o = 1 + \frac{\gamma}{2} \tag{A33}$$

$$EU_d = 1 + \frac{\gamma}{2} \tag{A34}$$

$$EU_{total} = 2 + \gamma \tag{A35}$$

Case 3: Partial Arming

Recall that under the partial arming condition, the attacker's best response arms levels are:

$$a_o = \sqrt{\frac{a_d \phi}{m}} - \frac{a_d}{m} \tag{A36}$$

Recall that under partial arming, the defender's best response arms levels are:

$$a_d = \sqrt{m a_o \phi} - m a_o \tag{A37}$$

We can then plug equation A37 into equation A36 to solve for a_o as a function of exogenous variables.

After a good deal of algebra, we find:

$$a_o = \frac{m\phi}{(1+m)^2} \tag{A38}$$

We can then plug equation A38 into equation A37. After a good deal of algebra, we find that defenders will invest the same amount in their arms under partial arming:

$$a_d = \frac{m\phi}{(1+m)^2} \tag{A39}$$

Now that we solved for a_o and a_d as a function of exogenous variables, we can plug these into equation A10 to find expected utility for the attacker and defender as a function of exogenous variables. After a good deal of algebra, we find that expected utility for the attacker is:

$$EU_o = 1 + \frac{m^2\phi}{(1+m)^2} \tag{A40}$$

After a good deal of algebra shown, we find that expected utility for the defender is:

$$EU_d = 1 + \frac{\phi}{(1+m)^2} \tag{A41}$$

The total utility under partial arming in the attack-defend dyad is:

$$EU_{total} = 2 + \frac{\phi(m^2+1)}{(1+m)^2} \tag{A42}$$

Comparative Statics

Importantly, because we now have a reduced-form solution for total welfare (i.e. total expected utility) in figure A4, we can do comparative statics to see how welfare in the attack-defend dyad changes with the offense-defense balance.

As expected, expected utility for the attacker is strictly increasing in offensive advantage:

$$\frac{\partial EU_o}{\partial m} = \frac{2m\phi}{(1+m)^3} \tag{A43}$$

As expected, expected utility for the defender is strictly decreasing in offensive advantage:

$$\frac{\partial EU_d}{\partial m} = \frac{-2\phi}{(1+m)^3} \tag{A44}$$

Total expected utility is decreasing in m when $m < 1$, but increasing in m when $m > 1$:

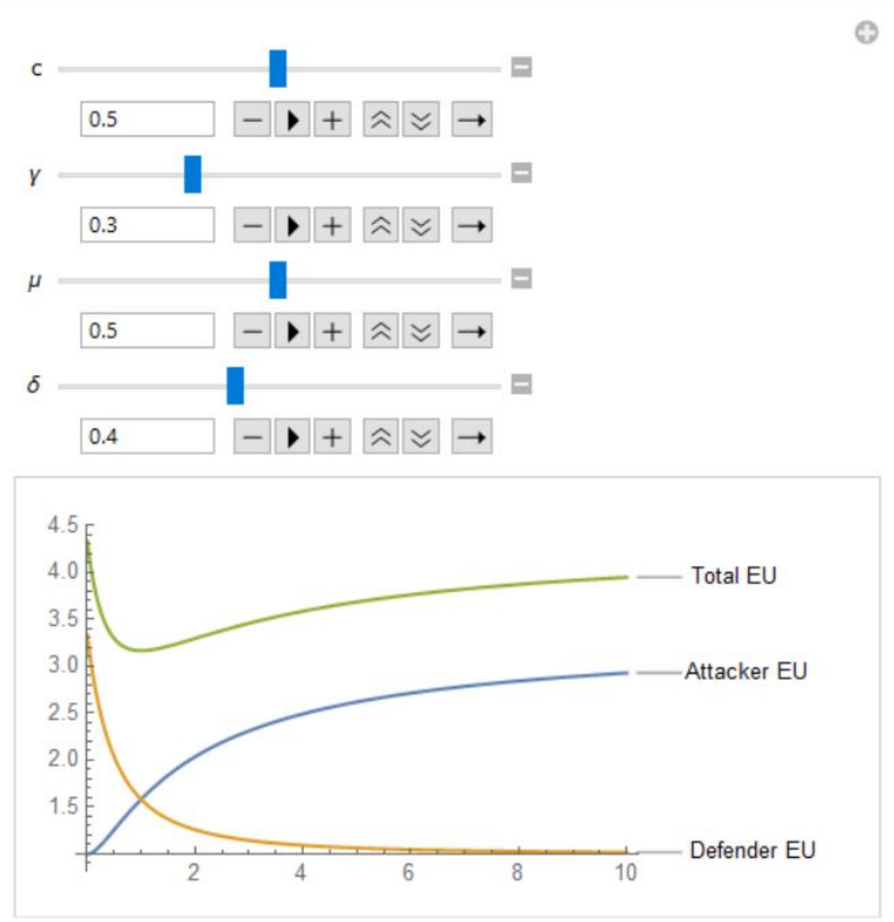
$$\frac{\partial EU_{total}}{\partial m} = \frac{2\phi(m-1)}{(1+m)^3} \tag{A45}$$

The driver of the U-shape of total expected utility under offensive advantage is the equilibrium arms levels. Under partial arming, arms levels for both the attacker and defender are increasing when $m < 1$, which decreases total welfare. They are increasing when $m > 1$, which increases total welfare:

$$\frac{\partial(a_o = a_d)}{\partial m} = \frac{\phi(1-m)}{(1+m)^3} \tag{A46}$$

The full chart that shows comparative statics of welfare under expected utility in the attack-defend Dyad is shown below.

FIGURE A5. *Welfare Under Offensive Advantage in the Attack-Defend Dyad*



Arms Levels and Welfare Under Simultaneous Attack

Under simultaneous attack, we assume there is a 50% chance that a given side ends up on the attack and a 50% chance they end up on the defense. Formally:

$$p_s^i(a_i, a_j; m) = (\frac{1}{2})(\frac{ma_i}{a_i + ma_j}) + (\frac{1}{2})(\frac{a_i}{ma_i + a_j}) \tag{A47}$$

And note that:

$$\frac{\partial p_s^i(a_i, a_j; m)}{\partial a_i} = (\frac{1}{2})(\frac{ma_j}{(ma_i + a_j)^2}) + (\frac{1}{2})(\frac{ma_j}{(a_i + ma_j)^2}) \tag{A48}$$

Partial Arming

Recall the general Karush Kuhn Tucker conditions for the choice of arms levels in the war scenarios.

Under an interior solution:

$$\frac{\partial p^i(a_i, a_j; m)}{\partial a_i}\phi = 1$$

Where the LHS represents the marginal benefit of arming: the marginal increase in the probability of winning the war multiplied by the perpetuity benefit of war. The RHS is the marginal cost of arming.

In the attack-defend dyad, the attacker and the defender had different CSFs. However, in the simultaneous attack dyad, the CSFs of both sides have the same functional form, and we can use this to our advantage. Because ϕ is identical for both players by assumption, we end up with the condition:

$$\frac{\partial p^i(a_i, a_j; m)}{\partial a_i} = \frac{\partial p^j(a_j, a_i; m)}{\partial a_j} = \frac{1}{\phi} \tag{A49}$$

Plugging in the functional form for the CSF:

$$(\frac{1}{2})[\frac{ma_j}{(ma_i + a_j)^2} + \frac{ma_j}{(a_i + ma_j)^2}] = (\frac{1}{2})[\frac{ma_i}{(ma_j + a_i)^2} + \frac{ma_i}{(a_j + ma_i)^2}] = \frac{1}{\phi} \tag{A50}$$

We can see that this equation is solved when both sides arm equally, i.e. $a_i = a_j$. We will call this arms level a. When we plug in a single arms levels a, we get the following condition:

$$\frac{ma}{(a + am)^2} + \frac{ma}{(a + am)^2} = \frac{2}{\phi}$$

Simplifying and solving for a:

$$a = \frac{m\phi}{(m+1)^2} \tag{A51}$$

Plugging this reduced-form solution for arms into the expression for expected utility in war, $EU_i(war) = 1 - a_i + p_i(a_i, a_j; m)\phi$ yields:

$$EU_i(simultaneous) = \frac{2+\phi}{2} - \frac{m\phi}{(m+1)^2} \tag{A52}$$

Full Arming

In a corner solution with full arming, recall the general Karush Kuhn Tucker condition:

$$\frac{\partial p^i(a_i, a_j); m)}{\partial a_i}\phi - \theta = 1$$

For the Lagrange multiplier $\theta > 0$, which represents the shadow price of increasing arms beyond the level where $a = 1$. This implies that the marginal benefit of arming at $a = 1$ is greater than the marginal cost of arming:

$$\frac{\partial p^i(a_i, a_j; m)}{\partial a_i}\phi \Big| |_{a_i=1} \geq 1$$

Because both players have identical CSF functional forms and ϕ values, the threshold for where full arming occurs is equal for both players:

$$\frac{\partial p^i(a_i, a_j; m)}{\partial a_i}\Big| |_{a_i=1} = \frac{\partial p^j(a_j, a_i; m)}{\partial a_j}\Big| |_{a_j=1} = \frac{1}{\phi} \tag{A53}$$

Plugging in functional forms:

$$(\frac{1}{2})[\frac{ma_j}{(m+a_j)^2} + \frac{ma_j}{(1+ma_j)^2}] = (\frac{1}{2})[\frac{ma_i}{(m+a_i)^2} + \frac{ma_i}{(1+ma_i)^2}] = \frac{1}{\phi} \tag{A54}$$

We can see that this equation is solved when $a_j = a_i$. Since we assumed that $a_i = 1$ and $a_j = 1$, we can plug in this into the full arming condition with the functional form:

$$(\frac{1}{2})[\frac{m}{(m+1)^2} + \frac{m}{(1+m)^2}] \geq \frac{1}{\phi}$$

Simplifying, we get the condition that full arming occurs when:

$$\phi \geq 2(m+1) \tag{A55}$$

And the expected utility under this condition is:

$$EU_i(simultaneous) = \frac{\phi}{2} \tag{A56}$$

No Arming

And recall that when the perpetuity benefit of winning the war is negative, $\phi \leq 0$, then neither side will arm and there will be peace.

Reduced Form Nash Equilibrium in Simultaneous Attack Dyad

FIGURE A6

Nash Equilibrium Reduced Form Arms Levels and Expected Utility under Simultaneous Attack Dyad

	Arms Levels	Individual Expected Utility	Total Expected Utility $(EU_i + EU_j)$
No arming $\phi \leq 0$	$a_{i,j} = 0$	$EU_i = 1 + \frac{\gamma}{2}; EU_j = 1 + \frac{\gamma}{2}$	$2 + \gamma$
Full Arming $\phi \geq 2(m+1)$	$a_{i,j} = 1$	$EU_i = \frac{\phi}{2}; EU_j = \frac{\phi}{2}$	ϕ
Partial Arming $0 < \phi < 2(m+1)$	$a_{i,j} = \frac{m\phi}{(m+1)^2}$	$EU_{i,j} = \frac{2+\phi}{2} - \frac{m\phi}{(m+1)^2}$	$2 + \phi - \frac{2m\phi}{(m+1)^2}$

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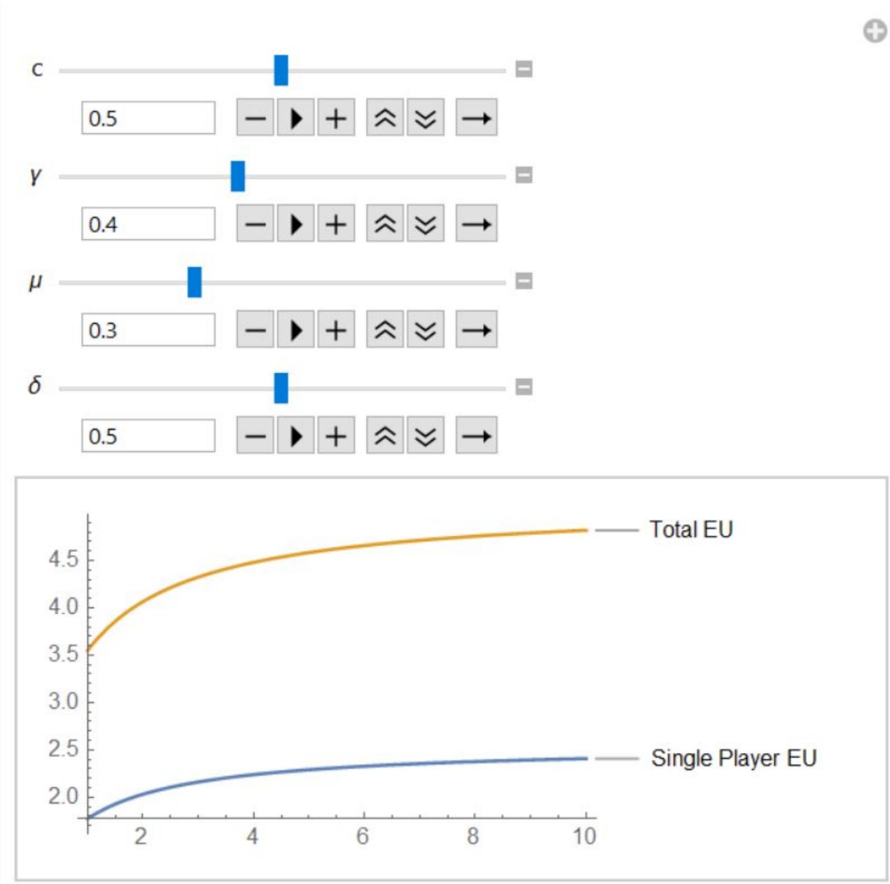
FIGURE A7. *Welfare Under Offensive Advantage in the Simultaneous Attack Dyad*



Comparative statics

Importantly, because we now have a reduced-form solution for total welfare (i.e. total expected utility) in figure A6, we can do comparative statics to see how welfare in the attack-defend dyad changes with the offense-defense balance:

$$\frac{\partial EU_{total}}{\partial m} = \frac{\phi}{(1+m)^2} \tag{A57}$$

This figure is strictly positive as $m > 0$ and $\phi > 0$ by assumption, so therefore total EU is strictly increasing in offensive advantage for the simultaneous attack war equilibrium.

Under partial arming, equilibrium arms levels are decreasing when $m > 1$. This drives higher welfare under offensive advantage, since players choose to spend less on arms as offensive advantage increases.

$$\frac{\partial a}{\partial m} = \phi(1-m^2) \tag{A58}$$

Solving the Dynamic Model with Endogenous Arming and Endogenous War Costs

Let's start with the decision to arm discussed in section .

Recall that the Lagrangian for that decision problem was:

$$L = 1 - a + p^i(a, \hat{a}; m))\phi + \lambda a + \theta(1-a)$$

Where $\phi = \frac{[\gamma - c + \delta(1+\mu)]}{(1-\delta)}$, i.e. the perpetuity benefit from winning the war. The only difference in our setup now is that ϕ is now longer completely exogenous, as c is now a function of π_s, which is a function of arms levels.

Now the decision problem becomes:

$$\max_a 1 - a + p^i(a, \hat{a}; m)\frac{[\gamma + \delta(1+\mu)]}{(1-\delta)} - p^i(a, \hat{a}; m)\frac{k}{(1-\pi_s(m))(1-\delta)}$$

Let's start by just considering the internal solution where $a \in (0, 1)$. Recall that the attacker will choose arms levels such that the marginal benefit of arms equals the marginal cost of arms. When ϕ was completely exogenous, then this condition was:

$$\frac{\partial p^i(a, \hat{a}; m)}{\partial a}\phi = 1$$

Now that ϕ is partially endogenous, we have to split it up into its component parts, and the condition becomes:

$$\frac{\partial p^i(a,\hat{a};m)}{\partial a}\frac{[\gamma+\delta(1+\mu)]}{(1-\delta)}-\frac{\partial p^i(a,\hat{a};m)}{\partial a}\frac{k}{(1-\pi_s(m))(1-\delta)}=1 \tag{A59}$$

This new condition still equates marginal benefit to marginal cost, but the attacker also needs to consider the effect that their arming will have on the cost of war.

To simplify the notation, we will specify θ as the perpetuity benefit of winning the war that excludes the war cost:

$$\theta=\frac{[\gamma+\delta(1+\mu)]}{(1-\delta)}$$

From here, we can use the same steps as the previous section to solve the model. We use the above condition to solve for optimal arms levels for the attacker as a function of the defender's arms levels. Then we do the same analysis from the persepective of the defender. Then we can solve for the Nash Equilibrium arms levels as a function of exogenous variables shown in the figures below.

FIGURE A8

Nash Equilibrium Reduced Form Arms Levels and Expected Utility under Simultaneous Attack Dyad

	Arms Levels	Individual Expected Utility	Total Expected Utility $(\boldsymbol{EU_i+EU_j})$
No arming $\theta-k-\frac{\mathrm{k}}{m}\leq 0$	$a_{i,j}=0$	$EU_i=1+\frac{\gamma}{2}; EU_j=1+\frac{\gamma}{2}$	$2+\gamma$
Full Arming $\theta-k-\frac{\mathrm{k}}{m}\geq 2(m+1)$	$a_{i,j}=1$	$EU_i=\frac{(\theta-k-\frac{\mathrm{k}}{m})}{2}; EU_j=\frac{(\theta-k-\frac{\mathrm{k}}{m})}{2}$	$\theta-k-\frac{\mathrm{k}}{m}$
Partial Arming $0<\theta-k-\frac{\mathrm{k}}{m}<2(m+1)$	$a_{i,j}=\frac{m(\theta-k-\frac{\mathrm{k}}{m})}{(m+1)^2}$	$EU_{i,j}=\frac{2+(\theta-k-\frac{\mathrm{k}}{m})}{2}-\frac{m(\theta-k-\frac{\mathrm{k}}{m})}{(m+1)^2}$	$2+\theta-k-\frac{\mathrm{k}}{m}-\frac{2m(\theta-k-\frac{\mathrm{k}}{m})}{(m+1)^2}$

FIGURE A9

Nash Equilibrium Reduced Form Arms Levels and Expected Utility under Attack-Defend Dyad

	Arms Levels	Individual Expected Utility	Total Expected Utility ($EU_o + EU_d$)
No arming $\theta - \frac{k(m-1)}{m(1-\delta)} \le 0$	$a_{o,d} = 0$	$EU_o = 1 + \frac{\gamma}{2}; EU_d = 1 + \frac{\gamma}{2}$	$2 + \gamma$
Full Arming $\theta - \frac{k(m-1)}{m(1-\delta)} \ge \frac{(m+1)^2}{m}$	$a_{o,d} = 1$	$EU_o = \frac{\theta - \frac{k(m-1)}{m(1-\delta)}}{1 + \frac{1}{m}}; EU_d = \frac{\theta - \frac{k(m-1)}{m(1-\delta)}}{m+1}$	$\theta - \frac{k(m-1)}{m(1-\delta)}$
Partial Arming $0 < \theta - \frac{k(m-1)}{m(1-\delta)} < \frac{(m+1)^2}{m}$	$a_{o,d} = \frac{m(\theta - \frac{k(m-1)}{m(1-\delta)})}{(1+m)^2}$	$EU_o = 1 + \frac{m^2(\theta - \frac{k(m-1)}{m(1-\delta)})}{(1+m)^2}$ $EU_d = 1 + \frac{\theta - \frac{k(m-1)}{m(1-\delta)}}{(1+m)^2}$	$2 + \frac{(\theta - \frac{k(m-1)}{m(1-\delta)})(m^2+1)}{(1+m)^2}$

We see a similar WUO curve to the one shown in section , except a much more pronounced increase of welfare under offensive advantage, including in situations where $\delta \to 1$ because the cost of war is now decreasing in offensive advantage.