Optimal Design of Chirp Signals with Applications to Frequency Response Measurement

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Abstract

This paper explores the possibility of testing the frequency response using Chirp signals. First, the spectral properties of the Chirp signal are studied. Second, based on the properties and specifications, a constrained optimization problem is formulated and numerically solved to find the parameters of the desired signal. Finally, the frequency response test procedure is provided using Chirp excitation. Simulation results show that high measurement precision could be achieved by the proposed approach.

1 Introduction

The input signal design problem has recently received much research attention [1-2], especially used as drive signals in the modelling problem. Take the wind turbine system for instance, the mode established would decrease the risk and cost for the system design and be required in some nation grid codes for characteristics estimation of wind turbine system.

In practice, the continuous signals are usually sampled into discrete signals and the Fourier transforms are implemented by the efficient fast Fourier transforms (FFTs) [3]. This may result in undesired loss of accuracy, if the input signal is not chosen properly; hence the design of excitation signals is of crucial importance in the FFT-based identification methods. Several aspects need to be considered, including aliasing, leakage and the crest factor. To avoid aliasing, the signal should be sufficiently band-limited [3], while periodic signals or time-limited signals are required to avoid leakage [4-5]. The crest factor is defined as the ratio between the peak value and the root mean square (RMS) value of the signal. Excitation signals with low crest factor are desirable in transfer function measurements, since this allows the maximization of the signal-to-noise ratios (SNRs) for given allowable amplitude ranges of the signals [6]. Besides, lower crest factor may also mean smaller influence of process nonlinear distortions on identification accuracy [5].

Various signals have been employed as excitations in the literature. The multi-sines are proved to be universal, very flexible signals, which can be used to solve a lot of measurement problems with minimal time consumption and low crest factor [5-6]. However, these periodical signals have infinite duration and the frequency response cannot be calculated directly through the data obtained by respectively extracting a single period from the periodical input and output signals, because the length of a two-signal linear convolution is much longer than that of each single signal [7]. Hence aperiodic signals are advised to be used. Step and Ramp signals have also been employed [8-9]; however, these signals are not sufficiently band-limited and it is also impossible to create customized amplitude spectrum with these signals.

In this paper, the excitation is chosen as a modified aperiodic chirp. No leakage exists as this signal has only finite duration. A weighting-function design procedure is employed to ensure the proposed signal has good band-limiting property and low crest factor. Another advantage is the possibility of creating customized amplitude spectrum with this signal, as will be shown in this paper. A constrained optimization problem is formulated and numerically solved, so that the parameters of the desired Chirp signal could be calculated. Using the designed signal, the frequency response test procedure is proposed. Discussions on the choice of parameters are also provided, so that the precision could be guaranteed.

The rest of the paper is organized as follows: Section II provides the spectral properties of the Chirp signal. In Section III, the design considerations and specifications are discussed and summarized. The signal design problem is formulated and solved in Section IV. Section V presents the frequency response measurement procedure, followed by some concluding remarks in Section VI.

2 Spectral properties of chirp signal

The linear frequency modulation (FM) chirp is one of the most ubiquitous signals used in radar signal processing [10]. It provides a frequency sweep of a prior known frequency band centered at the carrier frequency [11] and has been used as an excitation signal in various fields [12, 13].

The chirp is defined as

$$c(t) = \mathrm{rect}\left(\frac{t-0.5T}{T}\right)\sin\left(2\pi f_0 t + \frac{B}{T}\pi t^2\right) \tag{1}$$

$$\mathrm{rect}(z) = \begin{cases} 1 & |z| < 0.5 \\ 0 & |z| \geq 0.5 \end{cases} \tag{2}$$

where T is the duration, B is the bandwidth, and f_0 is the carrier frequency. Since the frequency response at low frequencies is used, f_0 is set to 0. Besides, $D=BT$ is defined as the dispersion factor, $\lambda=f/B$ is defined as the normalized frequency variable [11] and $\tau=t/T$ is defined as the normalized time variable. The chirp is sufficiently band-limited and also has a low crest factor when the dispersion factor is sufficiently large (e.g., D=500). However, large dispersion factor is not desired, because larger dispersion factor will lead to either larger bandwidth or longer duration. As will be shown, larger bandwidth will do harm to the identification accuracy while longer duration means greater time-consumption. This problem can be solved by adding a proper weighing window; therefore the modified chirp in Equation 3 is introduced.

$$\begin{aligned} x(t) &= c(t)\cdot w(\frac{t}{T}) \\ &= rect(\frac{t-0.5T}{T})\sin(2\pi f_0 t+\frac{B}{T}\pi t^2)\cdot w(\frac{t}{T}) \end{aligned} \tag{3}$$

where $w(t/T)$ is a weighting window and $w(t/T)$ is

$$w(\frac{t}{T}) = \begin{cases} 1 & 0<t<kT \\ \{0.5+0.5\cos[\frac{\pi}{(1-k)}\frac{t}{T}]\}^2 & kT<t<T \\ 0 & \text{others} \end{cases} \tag{4}$$

where k is a regulatory factor and $0<k<1$.

Let $\boldsymbol{A}(j\omega)$ denote the Fourier transform of an arbitrary signal $a(t)$. The Fourier transform of the signal in Equation 3 becomes

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^T w(\frac{t}{T})\cdot\sin(\frac{B}{T}\pi t^2)e^{-j\omega t}dt \tag{5}$$

Because the calculation of Equation 5 includes the evaluation of the Fresnel integral, the Fourier transform in Equation 5 cannot be solved analytically. Fortunately, several useful properties of the spectrum can still be summarized without the analytical solution.

According to the principle of stationary phase, an approximate relationship exists between time domain envelope $w(t/T)$ and the magnitude spectrum of the modified chirp [14]; that is,

$$\left|X(j\omega)\right| \doteq \frac{1}{2\sqrt{B/T}}w(\frac{t_\omega}{T}),\ t_\omega = \frac{T}{2\pi B}\omega \tag{6}$$

This property reveals that arbitrary amplitude spectrum can be created through appropriate choice of a weighting window. This property motivates the weighting window in the form of Equation 4 to make compromise between the band-limiting property and low crest factor. Another important property is that the shape of the spectrum is fully determined by the dispersion factor and regulatory factor, as is described in Theorem 1. Note that this property forms the basis for the design and use of the excitation signal.

Theorem 1. Suppose the relative amplitude spectrum of the modified chirp is defined as $X_1(j\omega)=\sqrt{2B/T}\,|X(j\omega)|$, $X_1(j\omega)$ is uniquely determined by the dispersion factor D and the regulatory factor k, on the scale of the normalized frequency.

The proof of this theorem is presented in Appendix A. Note that this property forms the basis for the design and use of the excitation signal.

3 Design indexes and design parameters

Because modified chirp has finite duration, no leakage exists for this signal and the frequency response can be accurately calculated through the ratio between the FFTs of the output and input signals, with proper zero-padding on the input signal [7]. To avoid aliasing and guarantee low crest factor, design indexes are chosen and specifications are made.

3.1 Aliasing

If a signal is not strictly band-limited, aliasing problem always exists. For a sampled signal $x_p(t)$ and the related continuous signal $x(t)$, Equation 7 holds. [3]

$$X_p(j\omega) = \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s)) \tag{7}$$

where ω_s is the sampling frequency, $X_p(j\omega)$ and $X(j\omega)$ are the Fourier transforms of $x_p(t)$ and $x(t)$, respectively. When $x(t)$ is sufficiently band-limited and only the positive frequency part of the spectrum is considered, aliasing is mainly caused by the shifted replica of $X(j\omega)$ centered at $\omega=\omega_s$, as is illustrated in Figure 1. It is possible to evaluate the aliasing problem in the form of the aliasing rate $\gamma=(\omega)$, which is defined as

$$\gamma(\omega) = X(j(\omega-\omega_s))/X(j\omega),\ 0\le\omega<\omega_s/2 \tag{8}$$

Given a threshold $\varepsilon<<1$, the aliasing can be considered negligible within frequency region $[0, \omega_{\max}]$ that satisfies $\gamma(\omega)<\varepsilon$. For the modified chirp signal, an aliasing factor $F_a=\omega_{\max}/2\pi B$ is defined. When ω_s is chosen $2\pi B$, $F_a\in[0,1]$. This quantity is chosen as a design index to evaluate the band-limiting property and the excitation efficiency of the signal. As a corollary of Theorem 1, the aliasing factor of the modified chirp is determined by the dispersion factor D and the regulatory factor k.

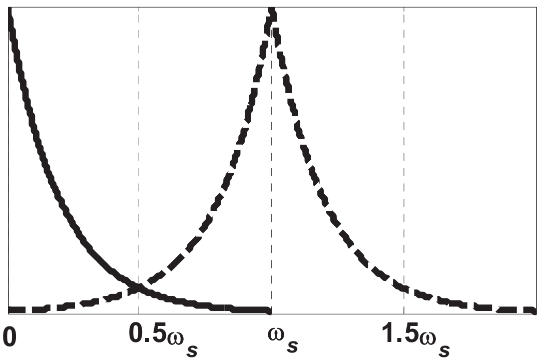

Figure1: Effect of Frequency Domain of Aliasing: (---) $X(j(\omega-\omega_s))$, (—) $X(j\omega)$

3.2 Crest factor

The crest factor is chosen as a design index to ensure the modified chirp has good immunity from noise and non-linear distortions. It is greater than or equal to 1 and low crest factor usually has a typical value of 1.4 to 1.7 [5]. The crest factor is calculated as:

$$F_c = \frac{1}{\sqrt{\frac{1}{T}\int_0^T x^2(t)dt}} \tag{9}$$

where $x(t)$ is the modified chirp and is the duration. The numerator is chosen one as it provides the upper bound of the peak value, according to Equations 3 and 4. For the modified chirp, the crest factor is independent of T for a fixed dispersion factor D, as is described in Theorem 2:

Theorem 2. For the modified chirp described in Equations 3 and 4, the crest factor is fully determined by the dispersion factor D and the regulatory factor k.

The proof of this theorem is presented in Appendix B.

3.3 Design parameters

According to theorem 1 and theorem 2, the key factors that determine the time and frequency characteristics of the modified chirp are the dispersion factor D and the shape of the weighting window, equivalently, the regulatory factor k. Therefore these remain to be the design parameters. These parameters will be tuned iteratively to meet specifications on the aliasing factor and the crest factor.

4 Design procedure and results

The design of the modified chirp can be formulated into an optimization problem to find design parameters that minimize an objective function

$$J = \frac{1}{2}\sum_{i=1}^{n} \lambda_i (P_i(\rho) - P_i^*)^2 \qquad (10)$$

where ρ is the vector of the design parameters of dimension n_ρ, P is the vector of the design indexes of dimension n, $P_i(\rho)$ indicates the ith design index related to the design parameters, and P_i^* denotes the ith design specifications. λ_i is a weighting factor, which is chosen as $(1/P_i^*)^2$, to normalize the terms in the objective function. For this problem, $n_\rho=n=2$, $\rho=[\rho_1, \rho_2]=[k, D]$ and $P=[P_1, P_2]=[F_a, F_c]$.

4.1 Iterative solution

The objective function is minimized iteratively using the Gauss-Newton method [15]. The design parameter vector is gradually evolved according to Equation 11.

$$\rho_{m+1} = \rho_m - \gamma_m H(\rho_m)^{-1} J'(\rho_m) \qquad (11)$$

where m is the iteration number, γ_m is the step size, $H(\rho)$ is the Hessian of the objective function, and $J'(\rho)$ is the gradient of the objective function with respect to ρ.

The gradient of the objective function is given by

$$J'(\rho) = \frac{dJ}{d\rho} = \sum_{i=1}^{n} \lambda_i (P_i(\rho) - P_i^*) \frac{dP_i(\rho)}{d\rho} \qquad (12)$$

and the Hessian matrix is defined as

$$H(\rho) = \sum_{i=1}^{n} \lambda_i \frac{dP_i(\rho)}{d\rho} \frac{dP_i(\rho)}{d\rho}^T \qquad (13)$$

The success of the Gauss-Newton method will depend on the importance of the neglected the second-order derivatives of $P_i(\rho)$ with respect to ρ in the calculation of the Hessian matrix [15]. Fortunately, these terms are usually small, especially in the neighborhood of the optimum.

4.2 Calculation of derivatives

According to Equation 11-13, the calculation of Equation 11 requires the estimation of the derivatives of $P_i(\rho)$ with respect to ρ. Analytical solutions are difficult to calculate; therefore numerical differentiation is employed to evaluate these derivatives. The approximation of the derivatives of $P_i(\rho)$ with respect to ρ are expressed as:

$$\frac{dP_i(\rho)}{d\rho_j} = \frac{P_i(\rho + \Delta\rho) - P_i(\rho)}{\alpha\rho_j} \quad i, j \in N \text{ and } i, j \le n_\rho \qquad (14)$$

where $|\alpha| \ll 1$, $\Delta\rho = [\Delta\rho_1, \Delta\rho_2, \ldots, \Delta\rho_{n_\rho}]$ and

$$\Delta\rho_k = \begin{cases} 0 & k \ne j \\ \alpha\rho_j & k = j \end{cases} \quad k \in N^+ \text{ and } k \le n_\rho \qquad (15)$$

$P_i(\rho+\Delta\rho)$and $P_i(\rho)$ can be calculated numerically according to the definition of the aliasing factor and the crest factor.

4.3 Results

In this paper, the specified values are chosen. $P^*=[F_a^*, F_c^*]$ $=[0.85, 1.55]$. The threshold to calculate the aliasing factor is chosen $\varepsilon=0.005$ and the step size is 1. The initial values of the design parameters are $\rho_0=[k_0, D_0]=[0.4, 24]$, the corresponding indexes are $P_0=[F_{a0}, F_{c0}]=[0.828, 1.97]$,. The value of the objective function finally decreases below and the design results are $\rho=[k, D]=[0.869, 62.4]$, the corresponding indexes are $P=[F_a, F_c]=[0.886, 1.51]$. This signal will be used as the excitation in the identification of process frequency response. Since the designed signal is sufficiently bandlimited, the sampling rate f_s is chosen twice of the bandwidth. Choice of the remaining parameters, including the bandwidth, will be discussed in Section 3. Note that the choice of the bandwidth does not alter band-limiting property and the crest factor of the designed signal, according to theorem 1 and theorem 2.

5 Identification method

In this section, the identification procedure is introduced, followed by some instructions on the choice of parameters.

5.1 Identification procedure

A simple experiment is proposed to identify the process frequency response. Figure 2 illustrates the identification setup. As feedback is involved in the setup, this test enjoys the benefits of closed-loop identification [16]. Suppose the transfer functions of the controller and the process are $K(s)$ and $G(s)$, respectively. The sensitivity function is denoted as $S(s)$ and $S(s)=1/[1+G(s)GK(s)]$.To perform this test, initial values of controller parameters are required, and therefore $K(s)$ is assumed to be known.

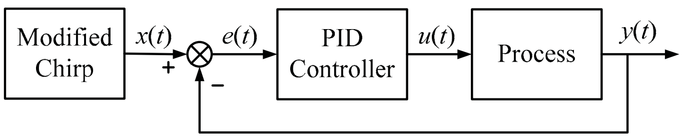

Figure 2: Closed-Loop Experiment For Frequency Response Identification

The proposed identification procedure is established as follows:

a) The designed chirp is generated and inserted into the closed loop system.

b) The signals $e(t)$ and $x(t)$ are sampled and recorded by a digital signal processing system. If this system also acts as the chirp generator, only signal $e(t)$ needs to be sampled and recorded. Since $e(t)=y(t)-x(t)$ and $y(t)$ is the linear convolution of $x(t)$ and the impulse response of the closed loop system, the duration of signal $e(t)$ is longer than that of $x(t)$. Hence the data acquisition process does not stop until $e(t)$ diminishes to zero.

c) The discrete Fourier transforms (DFTs) of the sampled $e(t)$ and $x(t)$ are calculated by FFT algorithm. The corresponding samples of the sensitivity function $S(s)$ is then evaluated by

$$S(s)\big|_{s=j\frac{2\pi k}{NT_s}} = E[k]/X[k],\ k=0,2,\cdots,N-1 \quad (16)$$

where $E[k]$ and $X[k]$are the DFTs of $e(t)$ and $x(t)$, respectively. T_s is the sampling period and N is the length of FFT. To prove this, notice that

$$E[k]=1/T_s\ E(j\omega)\big|_{\omega=\frac{2\pi k}{NT_s}},\ X[k]=1/T_s\ X(j\omega)\big|_{\omega=\frac{2\pi k}{NT_s}}$$

and $S(j\omega)= E(j\omega)/ X(j\omega)$

where $E(j\omega)$and $X(j\omega)$are the Fourier transforms of $e(t)$ and $x(t)$, respectively.

d) The process frequency response is derived from the sensitivity function, that is,

$$G(j\omega)=(\frac{1}{S(j\omega)}-1)/K(j\omega) \quad (17)$$

Since the structure and parameters of $K(s)$ is known, $G(j\omega)$ can be calculated directly.

5.2 Choice of parameters

a) Choice of bandwidth and FFT length.

The identification accuracy can be estimated by the interval between two sampled frequencies. This interval is denoted as $\Delta\omega$ and stands for the frequency resolution. According to the discrete Fourier theory [7],

$$\Delta\omega=\frac{2\pi f_s}{N} \quad (18)$$

As the sampling rate f_s is chosen twice of the bandwidth, Equation 18 becomes

$$\Delta\omega=\frac{4\pi B}{N} \quad (19)$$

b) Choice of the bandwidth

The bandwidth should be chosen large enough to provide sufficient high frequency excitations. This value is also limited by the identification accuracy according to Equation 19. Simulations show that the phase crossover frequency can serve as a good reference on the choice of B. The phase crossover frequency ω_c of the control loop can be easily acquired using the relay feedback test and B is chosen 3 to 5 times of ω_c/F_a. The aliasing factor F_a is also considered in the choice of B because the aliasing is assured negligible within $[0,2\pi BF_a]$.

c) Choice of the FFT length N

According to Equation 19, when B is fixed, N should be chosen in accordance with the specified identification precision and the bandwidth. Meanwhile, N should also make compromise with the calculation and storage capability of the digital signal processing system. Besides, it's best to choose N as an integer power of 2, since efficient FFT algorithm exists for these values [7].

5.3 Examples

Consider a three-order process

$$G(s)=\frac{1}{(s+1)(s+2)(s+3)} \quad (20)$$

The process frequency response is identified, which is illustrated in Figure 3. The parameters are chosen as N=4096 and B=4Hz. For comparison, the exact frequency response at low frequencies is also plotted. It is shown that frequency response can be estimated accurately by the proposed method.

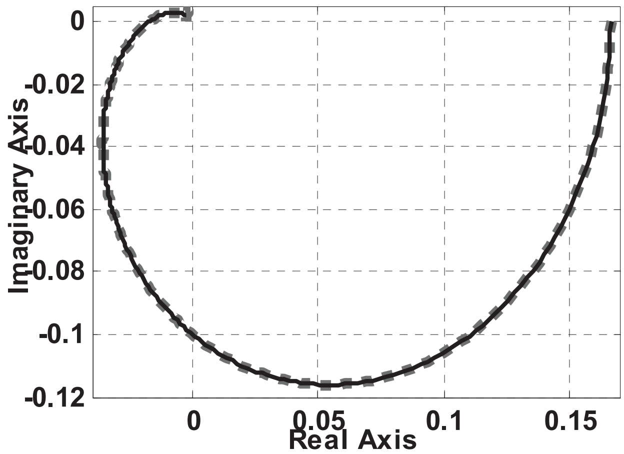

Figure 3: Frequency Response of the Tested Process: (---)Exact Values ,(—) Identified Values

6 Conclusions

In this paper, a constrained optimization problem is formulated to design a Chirp signal with pre-specified performance. With the aid of the designed signal, a frequency response test procedure is proposed. It is shown that high measurement precision could be achieved, given the parameters are chosen properly. However, the proposed optimization problem is nonconvex, and thus convex relaxation to the problem is a topic for future work.

Appendix A

Proof of theorem 1: according to Equation 5, Normalizing the time variable t and frequency variable f according to $\tau=t/T$ and $\lambda=f/B$, Equation 5 can be rearranged as

$$\begin{aligned} X(j\omega)&=X(j2\pi B\lambda)=T\int_0^1 w(\tau)\bullet\sin(D\pi\tau^2)e^{-j2\pi D\lambda\tau}d\tau \\ &=T\int_0^k \sin(D\pi\tau^2)e^{-j2\pi D\lambda\tau}d\tau \\ &+T\int_k^1 \{\frac{1}{2}+\frac{1}{2}\cos[\frac{\pi}{(1-k)}\tau]\}^2\sin(D\pi\tau^2)e^{-j2\pi D\lambda\tau}d\tau \end{aligned} \quad (A.1)$$

Let

$$\tilde{X}(\lambda,D,k)=\sqrt{2D}\,|\int_0^k \sin(D\pi\tau^2)e^{-j2\pi D\lambda\tau}d\tau + \int_k^1 \{\frac{1}{2}+\frac{1}{2}\cos[\frac{\pi}{(1-k)}\tau]\}^2 \sin(D\pi\tau^2)e^{-j2\pi D\lambda\tau}d\tau| \tag{A.2}$$

The relative amplitude spectrum can be denoted as

$$X_1(j\omega)=\sqrt{2B/T}\,|X(j\omega)|=\tilde{X}(\lambda,D,k) \tag{A.3}$$

This shows that the relative amplitude spectrum is a function of the dispersion factor D, the normalized frequency parameter λ and the regulatory factor k, independent of any particular values of the duration and the bandwidth.

Appendix B

Proof of theorem 2: according to the Parseval's Theorem,

$$\sqrt{\frac{1}{T}\int_0^T x^2(t)dt}=\sqrt{\frac{1}{2\pi T}\int_{-\infty}^{+\infty}|X(j\omega)|^2\,d\omega} \tag{B.1}$$

From Equation A.3, it follows that

$$\begin{aligned}\sqrt{\frac{1}{T}\int_0^T x^2(t)dt} &= \sqrt{\frac{1}{2\pi T}\int_{-\infty}^{+\infty}\frac{T}{2B}X_1(j\omega)^2\,d\omega} \\ &= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi\tilde{X}(\lambda,D,k)^2\,d\lambda}\end{aligned} \tag{B.2}$$

Hence the RMS value of the modified chirp is a function of the dispersion factor D and regulatory factor k; therefore the crest factor defined by Equation 9 can be fully determined by the dispersion factor D and regulatory factor k.

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