University of California, Santa Barbara
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SENIOR THESIS

Analyzing the Hamiltonian of the Golden Chain

College of Creative Studies

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Abstract

The field of topological quantum computing is a rich area for mathematicians and physicists alike. The potential for building a quantum computer whose computations are based on the topological properties of particle interactions would allow for a system that is more stable. The focus of this report will primarily be a certain quantum topological framework known as the Fibonacci anyonic model. In particular, we analyze the generalized Heisenberg Hamiltonian presented in [7] for the Golden Chain, providing explicit computations of the Hamiltonian in order to demonstrate its lack of the frustration-free property. Additionally, we discuss the potential for an alternative approach to computing the Heisenberg Hamiltonian.

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Chapter 1

Introduction

1.1 Brief Word on Anyons

A well-known property in the physics of elementary particles is the classification of particles into what are known as *fermions* and *bosons*. The fundamental difference between these two types of particles is the difference in their exchange statistics: that is, what happens to the wavefunctions when two identical particles are permuted with one another. In our three-dimensional space, the only two possibilities that can occur are an introduction of an overall phase of either 1 or -1, corresponding to bosons and fermions respectively.

In topological quantum computing, however, we consider the interaction of quasiparticles, called anyons, that can be thought of as traditional quantum particles whose movement in space is restricted to two dimensions. Such particles can be classified into two categories: Abelian anyons and non-Abelian anyons. The Abelian anyons correspond to acquiring a phase $e^{i\theta}$ for any $\theta \in [0, \pi]$ (i.e. "any" phase - not restricted just to the $\theta = 0$ or π case corresponding to bosons and fermions respectively). In the non-Abelian case, the exchange of two identical anyons results in an exotic interaction that can be modeled by a $k \times k$ unitary matrix (where $k > 1$) that acts on a finite dimensional Hilbert space. With multiple anyons, we may exchange two of these quasiparticles at a time to generate a product of matrices. In general it is not true that two of these matrices commute with one another, motivating the name *non-Abelian* anyon. Since the anyons are restricted to two dimensional space, we can rephrase the problem by considering the world lines of each particle when we allow one dimension to represent time (i.e. we look at the particles' trajectory twisting around one other through space-time). Graphically, this is denoted graphically as in the following figure:

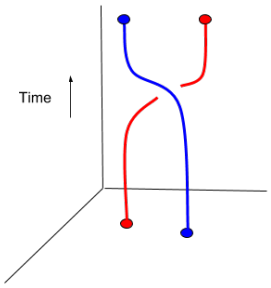


Figure 1.1: Exchanging two 2-dimensional particles through time

With an n number of 2-dimensional quasiparticles, each permutation in this way looks like an element of the braid group on n-strands, B_n. Thus we may appeal to braid theory and develop representations of the braid group to study these "swappings" of the particles.

1.2 Motivation for Quantum Computation

After carefully choosing a meaningful representation for these various elements of the braid group, computations in topological quantum computers amount to exchanging these non-Abelian anyons in a strategic manner and measuring the result, which have been proven to be capable of universal quantum computation.[1] This is an extremely useful paradigm for quantum computing since we now have entirely topological descriptions of physical characteristics: for example, the increased stability of topological computations against small perturbations stems from the fact that these braids are equivalent up to ambient isotopy.

[1] See [3]

Chapter 2

Background

2.1 Formalism

In general, the stage on which topological phases of matter are studied is a mathematical structure called a modular tensor category. In its entirety, this is a very rich and intricate structure that is too much to explore in detail here.[1] Fortunately, our particular interest in this thesis is an extremely simplified case of the general model, so we only highlight the characteristics of a modular tensor category that pertain to us.

2.1.1 Fusion Rules

Definition 1. *Define a label set* $(L, \hat{\ })$ *to be a set* L *that includes an element denoted* $\mathbf{1}$*, together with an operation* $\hat{\ } : L \to L$ *such that* $\hat{\mathbf{1}} = \mathbf{1}$ *and for all* $a \in L$*, composing* $\hat{\ }$ *with itself returns* a *(i.e.* $\hat{\ }$ *is an involution). The elements of* L *are called labels,* $\mathbf{1}$ *being the trivial label, and* $\hat{a}$ *denotes the dual of* a*.*

From here onwards, label sets will be denoted simply by their corresponding set L with the duality $\hat{\ }$ implied.

Definition 2. *Let* L *be a label set, and denote* $\{0,1\}^L$ *as the set of maps* $L \to \{0,1\}$*. For all* $a, b, c \in L$*, let* $N_{ab}^c = N(a,b,c)$ *denote the image of a map from the triple of labels* $(a,b,c) \to \{0,1\}$*. Then a (multiplicity-free) fusion rule is a binary operation* $\otimes : L \times L \to \{0,1\}^L$ *where*

$$a \otimes b = \bigoplus_{c \in L} N_{ab}^c c$$

is a formal sum of labels subject to the following conditions:

[1] We defer the interested reader to Etingof et al. [1] for a more complete discussion of Modular Tensor Categories.

i) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

ii) $N_{a\mathbf{1}}^c = N_{\mathbf{1}a}^c = \delta_{ca}$

iii) $N_{ab}^{\mathbf{1}} = N_{ba}^{\mathbf{1}} = \delta_{b\hat{a}}$

Note. More generally, over the label set L, the tensor product $a \otimes b$ gives rise to an L-graded vector space where each N_{ab}^c is more accurately the dimension of the vector space of the associated Hom space from $a \otimes b$ to c. For the purposes of this paper, the dimension of this vector space will always be either 1 or 0, so the associated Hom space will either be a single vector or the empty space respectively. The only reason that this added structure is mentioned here is due to the interpretation of F moves as change of bases, which will be introduced in section 2.1.2.

For ease of discussion, labels a, b, c where $N_{ab}^c = 0$ are called *inadmissable*, and labels with $N_{ab}^c \neq 0$ are *admissible*. Additionally, the labeling of a trivalent diagram is called admissible if for all trivalent vertices, the associated edges a, b, and $c \in L$ necessarily have $N_{ab}^c \neq 0$.

Example 1. Let L be a label set given by the elements of a group $(G, \cdot)$ union an element $x \notin G$. Define the duality of L by identifying the identity of G, 1_G, with the trivial label $\mathbf{1}$, $\hat{g} = g^{-1}$ for all $g \in G$, and $\hat{x} = x$. Then for all $g, h \in G$, let

$$g \otimes h = g \cdot h \qquad x \otimes g = g \otimes x = x \qquad x \otimes x = \bigoplus_{g \in G} g$$

One can verify that, subject to these conditions, $\otimes$ does indeed define a fusion rule on L.

Most of the strength in defining our label set and fusion rule in such a way is due to the reformulation of these rules in an entirely diagrammatic sense. Pictorially, the elements of our label set are depicted by taking straight lines (directed edges) and labeling the line with an element of the label set L. Duality is represented by reversing the directed edge as in Figure 2.1.

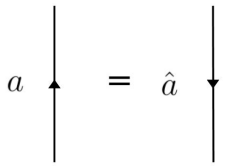
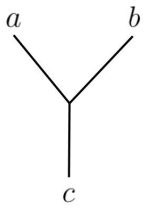


Figure 2.1: **Duality:** Labeling a directed edge with a on the left is equivalent to a "time reversal" (switching the arrow), and relabeling with $\hat{a}$ as on the right. (See the footnote on page 7 for a more precise interpretation of these diagrams.)

In fact, for self-dual label sets (that is, label sets L such that $\hat{a} = a$ for all $a \in L$), the arrows may be dropped. As this is a property for the fusion category of interest in this thesis, this simplification will be made now, leaving out the arrows from here onward.

Fusion rules are diagrammatically represented as trivalent vertices with the edges of the vertex labeled using admissible labels in this framework:

$$a \otimes b = \sum_{c \in L^*}$$



as a formal sum of diagrams, where L^* denotes the subset of L such that $N_{ab}^c \neq 0$[2].

Physically, $a \otimes b$ corresponds to bringing two anyonic particles of type a and b together and fusing them into the resulting particle c. Since, in general, there may be more than one result from the fusion of a and b, the sum over admissible labelings corresponds to considering all of the possible admissible fusion results in our model.

Note. The particular theory of interest to this paper is *unimodal*, meaning the label **1** may be introduced as an extra edge anywhere in the diagram (as long as it connects to an edge of the diagram; connecting to a vertex would violate the trivalent property of the diagram).

2.1.2 F Matrices

We now introduce the concept of an anyonic Hilbert space:

Definition 3. *Given a label set L, a fusion rule on L, and a (possibly already partially labeled) trivalent diagram G, a vector space V over $\mathbb{C}$ can be constructed given by*

$$V := \text{span}\{\text{admissible labeling of } G\}$$

In words, the elements of V are formal linear combinations of admissible labelings over $\mathbb{C}$; thus the labeled admissible diagrams serve as the basis elements for this anyonic vector space.

[2]Emphasizing the more precise interpretation of the Fusion Rules provided in the Note after Definition 2, this formal sum of trivalent diagrams is simply the L-graded vector space of Hom spaces. In this light, in Figure , it would be more accurate to label the top and bottom of each edge. The left hand side represents the vector space End of the object a. The right hand side would then be this vector space's dual space: the vector space End of the object $\hat{a}$. Since there is little ambiguity in this slight abuse in notation, however, it is common to see labels written in this manner in order to shift towards the mentality of simply labeling diagrams.

Consider a line of particles of types 1, 2, 3, ... , n corresponding to the elements $1, 2, 3, ..., n \in L$. Proceed by fusing the particles 1 and 2 together to obtain an admissible element in L. Take this result and then fuse it with the particle 3. Continuing on in this fashion, finally fuse this result with the last particle n. Diagrammatically, the procedure described here can be represented by the following "fusion tree":

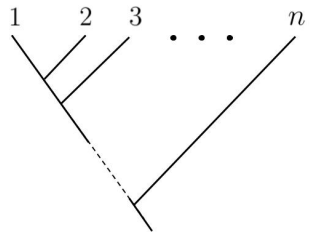


Figure 2.2: A fusion tree depicting a line of anyons being fused together one by one

The dimension of the anyonic Hilbert space associated to this fusion tree for a label set L and a fixed fusion rule on L is then

$$\sum_{e_1, e_2, ..., e_{n-1} \in L} N_{12}^{e_1} N_{e_1 3}^{e_2} ... N_{e_{n-2} n}^{e_{n-1}}$$

Considering the fusion of three labels $a, b, c \in L$ into a certain fixed label $d \in L$ (i.e. in the above diagram when $n = 3$), it should be noted that there are two entirely separate ways of performing this process:

(1) fuse a and b first, then fuse the result with c to attain d

(2) fuse b and c first, then fuse the result with a to attain d

The possibilities that arise from the fusion of a and b may be different from that of b and c, thus the admissible labelings of the diagram as a whole may be different. However, the definition of the fusion rules necessitates that $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ (see Definition 2 condition i). Thus these two diagrams must span the same vector space:

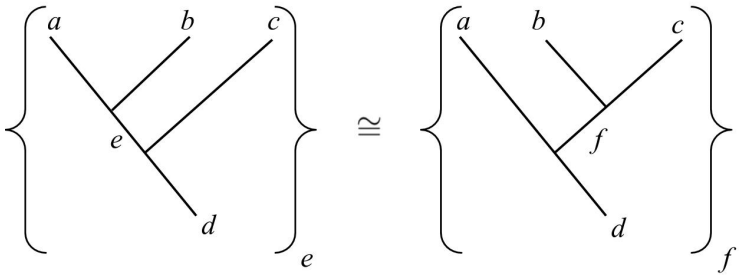


Figure 2.3: **The F move:** The basis elements of admissible labelings on these two diagrams (labels a, b, c, and d are fixed) must span the same vector space

Therefore, these two different fusion trees may be viewed simply as different bases of the same anyonic vector space space, meaning *a priori* a unitary change of basis, called an F matrix, from one fusion tree to the other may be constructed. Viewing these two diagrams as basis elements for the same vector space, the F move fundamentally gives rise to a change of basis matrix. Since this matrix depends on the labels of a, b, c, and d, the matrix is commonly denoted F_d^{abc} with the elements of the matrix being $F_{d;fe}^{abc}$. Furthermore, any number of these arbitrary choices of a fusion tree of n particles must be characterized in a consistent manner. As it so happens, the following Pentagon Relation provides necessary and sufficient conditions for this consistency:

Definition 4 (Wang [8])**.** *For $(a, b, c, d, n, m) \in L^6$, we say this 6-tuple is F-admissible if $(a, b, m), (m, c, d), (b, c, n)$, and (a, n, d) are all admissible. Define a map $F : L^6 \to \mathbb{C}$ satisfying the following conditions:*

1. *Admissibility Criterion:*
   (a) *If (a, b, c, d, n, m) is not admissible, then $F(a, b, c, d, n, m) = 0$.*
   (b) *The matrix F_d^{abc} is invertible.*
2. *Pentagon equation: for all $a, b, c, d, e, f, p, q, m \in L$*

$$\sum_n F_{q;pn}^{bcd} F_{f;qe}^{and} F_{e;nm}^{abc} = F_{f;qm}^{abp} F_{f;pe}^{mcd} \tag{2.1}$$

using the notation $F_{d;nm}^{abc} := F(a, b, c, d, n, m)$, and F_d^{abc} is the matrix whose $(n, m)^{th}$ entry is $F_{d;nm}^{abc}$.

Diagrammatically, the Pentagon relation is equivalent to the statement that the upper and lower paths in Figure 2.4 are equivalent.

In general the pentagon relation amounts to a system of equations that are fairly difficult to solve, even with computer-assisted methods. However, in the

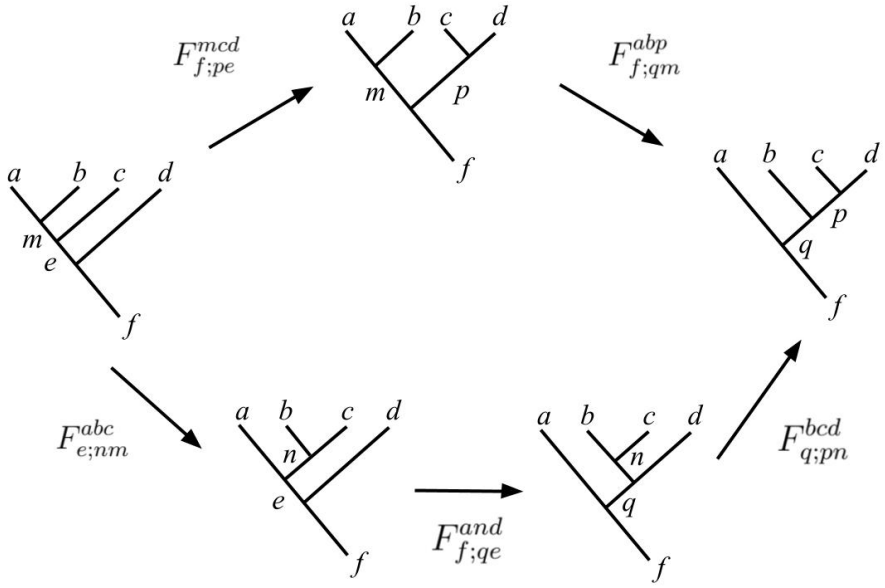


Figure 2.4: **The Pentagon Relation:** Taking the upper "path" of arrows should be equivalent to taking the lower "path" (see Equation 2.1)

particular anyonic model studied in this report, the F matrices have all been computed.

2.2 Fibonacci Anyons

In our studies, we consider a particular anyonic model called the Fibonacci anyon model. Although simple, the model is not unimportant: the model is capable of universal quantum computation[3]. Within this theory, there is the trivial particle type denoted by $\mathbf{1}$ and only one nontrivial type with label τ. As mentioned before, this label set $L = \{\mathbf{1}, \tau\}$ is self-dual, meaning $\hat{\tau} = \tau$, so without loss of generality arrows need not be drawn in the diagrams in this model. The fusion rules for the model are as follows:

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$$
$$\tau \otimes \mathbf{1} = \tau$$
$$\tau \otimes \tau = \mathbf{1} \oplus \tau$$

[3]This means the Fibonacci anyonic model is capable of approximating an arbitrary quantum computer within an arbitrary amount of precision. For more details, see [4]

Diagrammatically, the fact that $\tau \otimes \tau = \mathbf{1} \oplus \tau$ means the following two labelings are admissible:

Figure 2.5: Fusing two τ particles to get either the label $\mathbf{1}$ or τ

Because the fusion rules are so restrictive, most of the F matrices are trivial:

$$F_{\tau}^{\mathbf{1}\tau\tau} = F_{\tau}^{\tau\mathbf{1}\tau} = F_{\tau}^{\tau\tau\mathbf{1}} = F_{\mathbf{1}}^{\tau\tau\tau} = F_{\mathbf{1}}^{\mathbf{111}} = F_{\tau}^{\mathbf{11}\tau} = F_{\tau}^{\tau\mathbf{11}} = F_{\mathbf{1}}^{\mathbf{1}\tau\tau} = F_{\mathbf{1}}^{\tau\tau\mathbf{1}} = 1^4$$

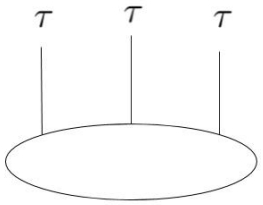
The only other nonzero F matrix is the matrix for all τ anyons, $F_{\tau}^{\tau\tau\tau}$. Solving condition 2.1 of Definition 4 in this case provides the matrix[5]

$$F_{\tau}^{\tau\tau\tau} = \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix}.$$

where ϕ denotes the Golden Ratio.

2.2.1 The Golden Chain

The Golden Chain is a specific diagram that is used to analyze the interactions of Fibonacci anyons. Diagrammatically it is represented as



with L total "spokes"[6] labeled τ projecting from the chain (pictured here is when $L = 3$). In fact, the Golden Chain can be imagined as a fusion tree as given in

[4]It would be a good exercise for the reader to determine which two labels create the sole admissible labeling for each of these matrices. In the case $F_{\mathbf{1}}^{\mathbf{1}\tau\tau}$, for example, the only nonzero element will be $F_{\mathbf{1};\mathbf{1}\tau}^{\mathbf{1}\tau\tau}$

[5]See [7] for a more explicit derivation.

[6]Here L does *not* denote the label set.

2.1.2 where the labels in positions 2 through n are all τ and the particle in the position 1 is equal to the resulting particle (at the root of the tree). In other words, it is akin to a fusion tree of τ anyons with an additional periodicity requirement.

Consider the anyonic Hilbert space spanned by the admissible labelings of the Golden Chain. As an explicit example, the basis for the case $L = 3$ is given below:

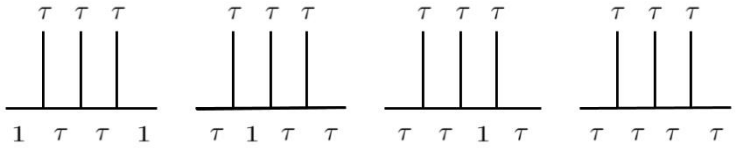


Figure 2.6: Golden Chain Level 3

Since it is understood that each of the spokes has label τ, the basis elements can be denoted by reading off the elements on the circular part of the chain in order. For example, in the basis given in Figure 2.6 above, the basis will be written as $\{(\mathbf{1}, \tau, \tau, \mathbf{1}), (\tau, \mathbf{1}, \tau, \tau), (\tau, \tau, \mathbf{1}, \tau), (\tau, \tau, \tau, \tau)\}$.

Note that when $L = 1$, there is only one admissible labeling for the Golden Chain: $\{(\tau, \tau)\}$[7]. In other words, the anyonic Hilbert space for the Golden Chain at $L = 1$ is 1 dimensional. When $L = 2$, the Hilbert space goes up to 3 dimensions: the basis elements are $\{(\tau, \tau, \tau), (\tau, \mathbf{1}, \tau), (\mathbf{1}, \tau, \mathbf{1})\}$. As shown above, for $L = 3$, the Hilbert space is 4 dimensional. In fact, let the dimension for the Golden Chain anyonic Hilbert space at level n be denoted by GC_n. It is not difficult to see that the construction of the Golden Chain leads to the relation $GC_n = GC_{n-1} + GC_{n-2}$. For completeness, an explicit proof is included in the Appendix. This proposition demonstrates that the dimensionality of the Golden Chain's associated vector space satisfies the well-known Fibonacci recurrence relation, hence the name "Fibonacci anyon model."

2.2.2 Hamiltonians

A Hamiltonian is a self-adjoint linear operator on a Hilbert space whose eigenvalues correspond physically to the energies of the system. For the purposes involved in this paper, the only Hamiltonians considered will act on anyonic Hilbert spaces of finite dimension, thus the self-adjoint operators are simply Hermitian matrices. In real-world scenarios, computing the full Hamiltonian is a very elusive task, so a localized model is necessary in order to create a tractable problem.

[7]The labeling $\{(\mathbf{1}, \mathbf{1})\}$ does not work since this would create a trivalent vertex $(\mathbf{1}, \mathbf{1}, \tau)$: a quick glance at the fusion rules affirms this is inadmissible.

The particular local Hamiltonian $\mathcal{H}_i$ for the Golden Chain is defined as the i^{th} nearest-neighbor spokes' fusion projected onto the vacuum state. Diagrammatically, the definition of $\mathcal{H}_i$ is given in Figure 2.7. The total Hamiltonian is defined as

$$\mathcal{H}^{\text{Fibonacci}}_{\text{Heisenberg}} := -J \sum_{\langle a,b \rangle} \Pi^{ab}_{\mathbf{1}} = -J \sum_{i} \mathcal{H}_i$$

where J is some nonzero constant, and $\Pi^{ab}_{\mathbf{1}}$ is a diagonal projection matrix whose diagonal entries are 1 if $(a, b, \mathbf{1})$ is admissible and 0 otherwise.

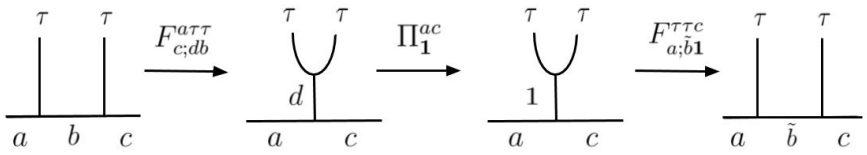


Figure 2.7: The Diagrammatic Procedure for computing the local $\mathcal{H}_i$

The factor of $-J$ is intended to have the Hamiltonian energetically favor fusing to the trivial label. To see what this means, we define the ground state of the Hamiltonian on our anyonic Hilbert space.

Definition 5. *The ground state is defined as the span of the eigenvectors associated to the lowest eigenvalue of the Hamiltonian.*

In quantum physics, the eigenvalues of the Hamiltonian correspond to the measurable quantities of energy in the system. The lowest eigenvalues are simply the lowest possible measurable energies for the system in question, and the ground state is in some ways the "most stable" or physically favorable state of the system.

Definition 6. *A frustration-free Hamiltonian $\mathcal{H}$ is one such that a vector $|\psi\rangle$ is a ground state for $\mathcal{H}$ if and only if $|\psi\rangle$ is a ground state for each local $\mathcal{H}_i$.*

Intuitively, a frustration-free Hamiltonian has the benefit that the analysis of its ground state vectors can be reduced to an analysis of the ground states of its localized constituents. In this report, the aim is to prove that the Hamiltonian for the Golden Chain is frustration-free. Additionally, some discussion will be presented on the goal of developing an alternative construction of the Total Hamiltonian in hopes that perhaps that a purely diagrammatic computation exists for deriving the ground state vector.

Chapter 3

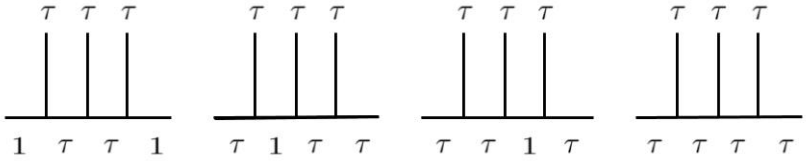
Computations

In this chapter, we consider the Golden Chain composed of 3, 4, and 5 sites and compute explicitly the local and total Hamiltonians and their ground states in order to show that the system is not in general frustration free.

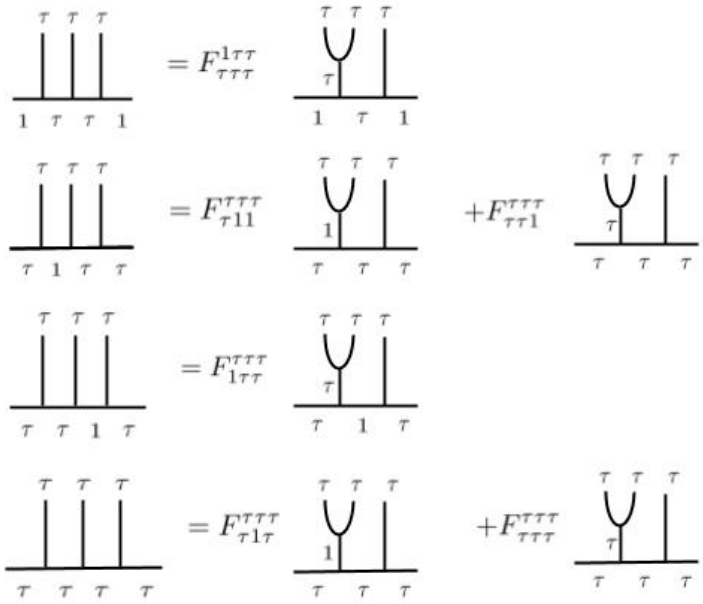
3.1 Golden Chain Level 3

3.1.1 Explicit Calculation of the First Local Hamiltonian

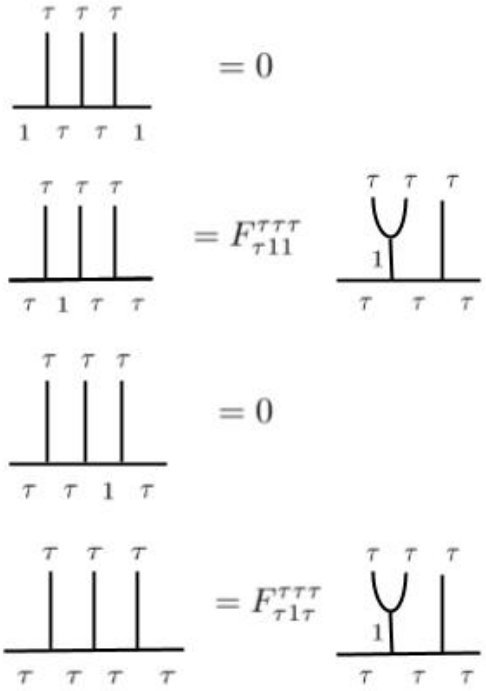
With three sites, we consider the ordered basis



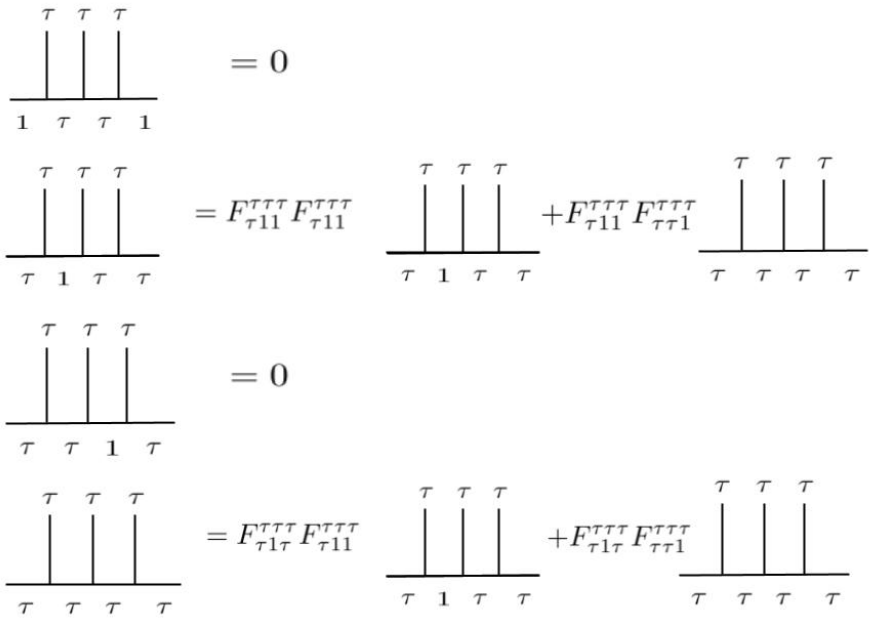
which from now on will be denoted as $\{1, \tau, \tau, 1\}, \{\tau, 1, \tau, \tau\}, \{\tau, \tau, 1, \tau\}$ and $\{\tau, \tau, \tau, \tau\}$. We now explicitly give the steps of computing the first local Hamiltonian, operating on the first two sites of each basis vector. First we perform an F move on the first two sites:



Then we project onto the vacuum state, meaning only the F moves that result in the trivial label are considered (all others are multiplied by 0):



Finally we take the inverse F move to switch back to our original basis to get



Resulting in the first local Hamiltonian matrix (H_1)

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^{\tau\tau\tau}_{\tau 11} F^{\tau\tau\tau}_{\tau 11} & 0 & F^{\tau\tau\tau}_{\tau 11} F^{\tau\tau\tau}_{\tau\tau 1} \\ 0 & 0 & 0 & 0 \\ 0 & F^{\tau\tau\tau}_{\tau 1\tau} F^{\tau\tau\tau}_{\tau 11} & 0 & F^{\tau\tau\tau}_{\tau 1\tau} F^{\tau\tau\tau}_{\tau\tau 1} \end{pmatrix}$$

Using the fact that the only nontrivial F matrix is $F^{\tau\tau\tau}_{\tau} = \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix}$ (where ϕ is the golden ratio) our matrix becomes

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/\phi^2 & 0 & 1/\phi^{3/2} \\ 0 & 0 & 0 & 0 \\ 0 & 1/\phi^{3/2} & 0 & 1/\phi \end{pmatrix}$$

3.1.2 Results of Computation for Remaining Level 3 Matrices

Proceeding in a similar manner for the other two matrices, we get

$$H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\phi^2 & 1/\phi^{3/2} \\ 0 & 0 & 1/\phi^{3/2} & 1/\phi \end{pmatrix}$$

$$H_3 = \begin{pmatrix} 1/\phi^2 & 0 & 0 & 1/\phi^{3/2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/\phi^{3/2} & 0 & 0 & 1/\phi \end{pmatrix}$$

$$H_{Total} = H_1 + H_2 + H_3 = \begin{pmatrix} 1/\phi^2 & 0 & 0 & 1/\phi^{3/2} \\ 0 & 1/\phi^2 & 0 & 1/\phi^{3/2} \\ 0 & 0 & 1/\phi^2 & 1/\phi^{3/2} \\ 1/\phi^{3/2} & 1/\phi^{3/2} & 1/\phi^{3/2} & 3/\phi \end{pmatrix}$$

We have that the characteristic polynomial for H_2 is

$$\begin{aligned} (H_2 - \lambda I) &= \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 1/\phi^2 - \lambda & 1/\phi^{3/2} \\ 0 & 0 & 1/\phi^{3/2} & 1/\phi - \lambda \end{vmatrix} \\ &= \lambda^2[(1/\phi^2 - \lambda)(1/\phi - \lambda) - 1/\phi^3] \\ &= \lambda^3 \left(\lambda - \frac{1+\phi}{\phi^2}\right) \\ &= \lambda^3(\lambda - 1) \end{aligned}$$

whose roots are 0 (with algebraic multiplicity 3) and 1. Interchanging columns 2 and 3 and rows 2 and 3 of H_2 results in H_1, and interchanging columns 1 and 3 and rows 1 and 3 of H_2 results in H_3, so the characteristic polynomials of these matrices (and thus the eigenvalues) will be identical. The characteristic polynomial for H_{Total} is

$$\begin{aligned} (H_{Total} - \lambda I) &= \begin{vmatrix} 1/\phi^2 - \lambda & 0 & 0 & 1/\phi^{3/2} \\ 0 & 1/\phi^2 - \lambda & 0 & 1/\phi^{3/2} \\ 0 & 0 & 1/\phi^2 - \lambda & 1/\phi^{3/2} \\ 1/\phi^{3/2} & 1/\phi^{3/2} & 1/\phi^{3/2} & 3/\phi - \lambda \end{vmatrix} \\ &= \frac{(\lambda(\lambda\phi^2 - 1)^2(\phi(\lambda\phi - 3) - 1))}{\phi^6} \\ &= 1/2\lambda(15 - 7\sqrt{5} - 3\lambda + 3\sqrt{5}\lambda - 6\lambda^2 + 2\lambda^3) \end{aligned}$$

whose roots are 0, $1/\phi^2$ (with algebraic multiplicty of 2), and $\sqrt{5}$.

To compute the ground state of the localized matrices, we only consider the eigenvalue 1[1]. In each of the three matrices, there are four nonzero entries, so when H_i is multiplied by a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, we will have two nonzero entries to the vector in the product $H\mathbf{x}$. To find when 1 is the eigenvalue, we then solve the system for $H\mathbf{x} = \mathbf{x}$ which forces all of the entries in $\mathbf{x}$ to be 0 except for the two corresponding nonzero rows of H. Finding the remaining two values amounts to solving the system

$$1/\phi^2 x_i + 1/\phi^{3/2} = x_i$$
$$1/\phi^{3/2} x_i + 1/\phi x_j = x_j$$

where i and j are the index of the nonzero rows of H (j in this case is 4). Solving this (and using the fact that $\phi^{1/2}(\phi - 1) = 1/\sqrt{\phi}$), we get a solution when $x_i = 1/\sqrt{\phi}$ and $x_j = 1$. This technique will be helpful in looking at larger cases.
To compute the ground state of the total matrix, we defer to Mathematica since the computations aren't particularly enlightening.
For completeness, we list the rest of the eigensystems for these matrices:

Matrix	Eigenvalue	Eigenvector
H_1	1	$\begin{pmatrix} 0 \\ 1/\sqrt{\phi} \\ 0 \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ -\sqrt{\phi} \\ 0 \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
	0	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Matrix	Eigenvalue	Eigenvector
H_3	1	$\begin{pmatrix} 1/\sqrt{\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} -\sqrt{\phi} \\ 0 \\ 0 \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

[1]The eigenvectors with the *maximum* associated eigenvalue: we take the maximum to account for the fact that $\mathcal{H} = -J\sum_{\langle i,j \rangle} \Pi^1_{ij}$ has a nonzero $-J$ term in order to favor the trivial channel.

Matrix	Eigenvalue	Eigenvector
H_2	1	$\begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{\phi} \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ 0 \\ -\sqrt{\phi} \\ 1 \end{pmatrix}$
	0	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
	0	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Matrix	Eigenvalue	Eigenvector
H_{Total}	$\frac{1}{\phi^2}$	$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
	$\frac{1}{\phi^2}$	$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
	0	$\begin{pmatrix} -\sqrt{\phi} \\ -\sqrt{\phi} \\ -\sqrt{\phi} \\ 1 \end{pmatrix}$
	$\sqrt{5}$	$\begin{pmatrix} \frac{1}{3\sqrt{\phi}} \\ \frac{1}{3\sqrt{\phi}} \\ \frac{1}{3\sqrt{\phi}} \\ 1 \end{pmatrix}$

Thus the ground state vectors are the following:

$$H_1 : \left(0, 1/\sqrt{\phi}, 0, 1\right)^T \qquad H_3 : (1/\sqrt{\phi}, 0, 0, 1)^T$$

$$H_2 : \left(0, 0, 1/\sqrt{\phi}, 1\right)^T \qquad H_{Total} : \left(1/(3\sqrt{\phi}), 1/(3\sqrt{\phi}), 1/(3\sqrt{\phi}), 1\right)^T$$

3.2 Golden Chain Level 4

We now consider the ordered basis

$$\{1,\tau,1,\tau,1\},\{1,\tau,\tau,\tau,1\},\{\tau,1,\tau,1,\tau\},\{\tau,1,\tau,\tau,\tau\},\{\tau,\tau,1,\tau,\tau\},$$
$$\{\tau,\tau,\tau,1,\tau\},\{\tau,\tau,\tau,\tau,\tau\}$$

using the notation introduced in part 1.1

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi} & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi} \end{pmatrix} \quad H_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi} \end{pmatrix} \quad H_4 = \begin{pmatrix} \frac{1}{\phi^2} & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & \frac{1}{\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$H_{Total} = \begin{pmatrix} 2+\frac{2}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ \frac{1}{\phi^{3/2}} & \frac{1}{\phi}+\frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 2+\frac{2}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & 0 \\ 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi}+\frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & \frac{1}{\phi}+\frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi}+\frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} \\ 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{4}{\phi} \end{pmatrix}$$

As before it is not difficult to see that we may permute an even total number of rows and columns of any of the local matrices in order to arrive at the block diagonal matrix H_3, which takes the form of H_2 in the Level 3 case. Then we know that all of these matrices have the same characteristic polynomial and thus the same eigenvalues. In fact, because the blocks are the same as the H_2 of level 3, we collect one 1 from the row with a single nonzero value, 2 zeros from the rows with

no nonzero values, and one 1 and one 0 for each block. The eigenvalues are thus 0 (alg multiplicity 4) and 1 (alg multiplicity 3) again. For the total Hamiltonian H_{Total}, the characteristic polynomial is

$$-(((-3+\lambda)(-1+\lambda)^2(208+336\phi+\lambda(-4(249+403\phi)+\lambda((-7+\lambda)\lambda(89+144\phi)+4(343+555\phi)))))/\phi^{12})$$

whose roots are 1 (alg mult 2), $4-2\phi, 3, \frac{1}{2}(3-\sqrt{57-32\phi})$ (alg mult 2), and 2ϕ. The ground state eigenvector will then be the one associated with the eigenvalue 2ϕ since this is the greatest numerical value.

3.2.1 Ground States

$$\begin{aligned}
H_1:&\ (1,0,0,0,0,0,0)^T,\\
&\ (0,0,0,1/\sqrt{\phi},0,0,1)^T,\\
&\ (0,0,1/\sqrt{\phi},0,0,1,0)^T\\
H_2:&\ (0,0,1,0,0,0,0)^T,\\
&\ (0,0,0,0,1/\sqrt{\phi},0,1)^T,\\
&\ (1/\sqrt{\phi},1,0,0,0,0,0)^T
\end{aligned}
\qquad
\begin{aligned}
H_3:&\ (1,0,0,0,0,0,0)^T,\\
&\ (0,0,0,0,0,1/\sqrt{\phi},1)^T,\\
&\ (0,0,1/\sqrt{\phi},1,0,0,0)^T\\
H_4:&\ (0,0,1,0,0,0,0)^T,\\
&\ (0,1/\sqrt{\phi},0,0,0,0,1)^T,\\
&\ (1/\sqrt{\phi},0,0,0,1,0,0)^T
\end{aligned}$$

$$H_{Total} = \left(\frac{\phi}{2}, \frac{1}{2\sqrt{\phi}}, \frac{\phi}{2}, \frac{\sqrt{-1+\phi}}{2}, \frac{1}{2\sqrt{\phi}}, \frac{\sqrt{-1+\phi}}{2}, 1\right)^T$$

3.3 Golden Chain Level 5

Our ordered basis with 5 spokes is:

$$\{1,\tau,1,\tau,\tau,1\},\{1,\tau,\tau,1,\tau,1\},\{1,\tau,\tau,\tau,\tau,1\},\{\tau,1,\tau,1,\tau,\tau\},\{\tau,1,\tau,\tau,1,\tau\},$$
$$\{\tau,1,\tau,\tau,\tau,\tau\},\{\tau,\tau,1,\tau,1,\tau\},\{\tau,\tau,1,\tau,\tau,\tau\},\{\tau,\tau,\tau,1,\tau,\tau\},\{\tau,\tau,\tau,\tau,1,\tau\},$$
$$\{\tau,\tau,\tau,\tau,\tau,\tau\}$$

again using the notation introduced in part 3.1

3.3.1 Matrices

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{1}{\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{1}{\phi} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$H_2 = \begin{pmatrix} \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\phi^2} & 0 & \frac{1}{\phi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} & \end{pmatrix}$$

$$H_5 = \begin{pmatrix} \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi} & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$H_{Total} = \begin{pmatrix} 1+\frac{2}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 \\ 0 & 1+\frac{2}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{2}{\phi}+\frac{1}{\phi^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 1+\frac{2}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{2}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{2}{\phi}+\frac{1}{\phi^2} & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1+\frac{2}{\phi^2} & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & 0 \\ \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{2}{\phi}+\frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & 0 & \frac{2}{\phi}+\frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & 0 & \frac{2}{\phi}+\frac{1}{\phi^2}+\frac{1}{\phi^{3/2}} \\ 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{5}{\phi} \end{pmatrix}$$

The characteristic polynomial of each of the local Hamiltonians are $-(-1+\lambda)^4\lambda^7$, so the eigenvalues are 0 with algebraic multiplicity 7 and 1 with algebraic multiplicity 4. In fact the Fibonacci recurrence in the algebraic multiplicities of these eigenvalues can actually be shown to hold due to the Fibonacci recurrence in the basis states themselves. The characteristic polynomial of the total Hamiltonian is

$$-(-2+\lambda)^2(250(-5+3\phi)+\lambda(125(-48+31\phi)+\lambda(50(-117+62\phi)+\lambda(5550-2550\phi$$

$$+\lambda(390-1160\phi+\lambda(-800+1105\phi+\lambda(2-260\phi+\lambda(77+(-16+\lambda)\lambda+20\phi)))))))$$

whose roots (the eigenvalues of H_{Total}) are $2, 2\phi - 1, 3 - \phi, \frac{1}{2}(-\phi - \sqrt{30 - 17\phi} + 5), \frac{1}{2}(-\phi + \sqrt{30 - 17\phi} + 5), \phi - \frac{1}{2}\sqrt{37 - 16\phi} + \frac{1}{2}$, and $\frac{1}{2}(2\phi + \sqrt{37 - 16\phi} + 1)$. The highest eigenvalue of these (the ground state energy) is $\frac{1}{2}(2\phi + \sqrt{37 - 16\phi} + 1)$. The corresponding eigenvector (the ground state) is too long to explicitly write out the exact form here, but it is of the form $(a, a, b, a, a, b, a, b, b, b, 1)^T$ where $a = 0.13748188392343136380336081614 6$ and $b = 0.28590325073987432183025639997 7$ has been verified correct to 30 decimal points.

3.4 Conclusion: Non-Frustration Free

In the above three examples we have that the total Hamiltonian's ground state does not appear in any of the local Hamiltonians' ground state eigenvectors, thus the system is not frustration free.

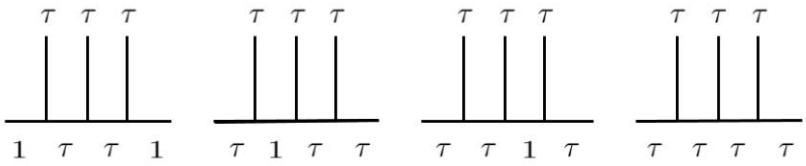
Chapter 4

Diagrammatic Discussion and Future Work

The goal in this chapter is to introduce and analyze different diagrammatic methods of computing a multi-tiered "global" operation that will act on the Golden Chain with the hope that one method will converge (as the number of tiers n goes to infinity) in a meaningful way to the standard Heisenberg Hamiltonian given by $-J\sum_{\langle ij\rangle}\Pi^1_{ij}$.

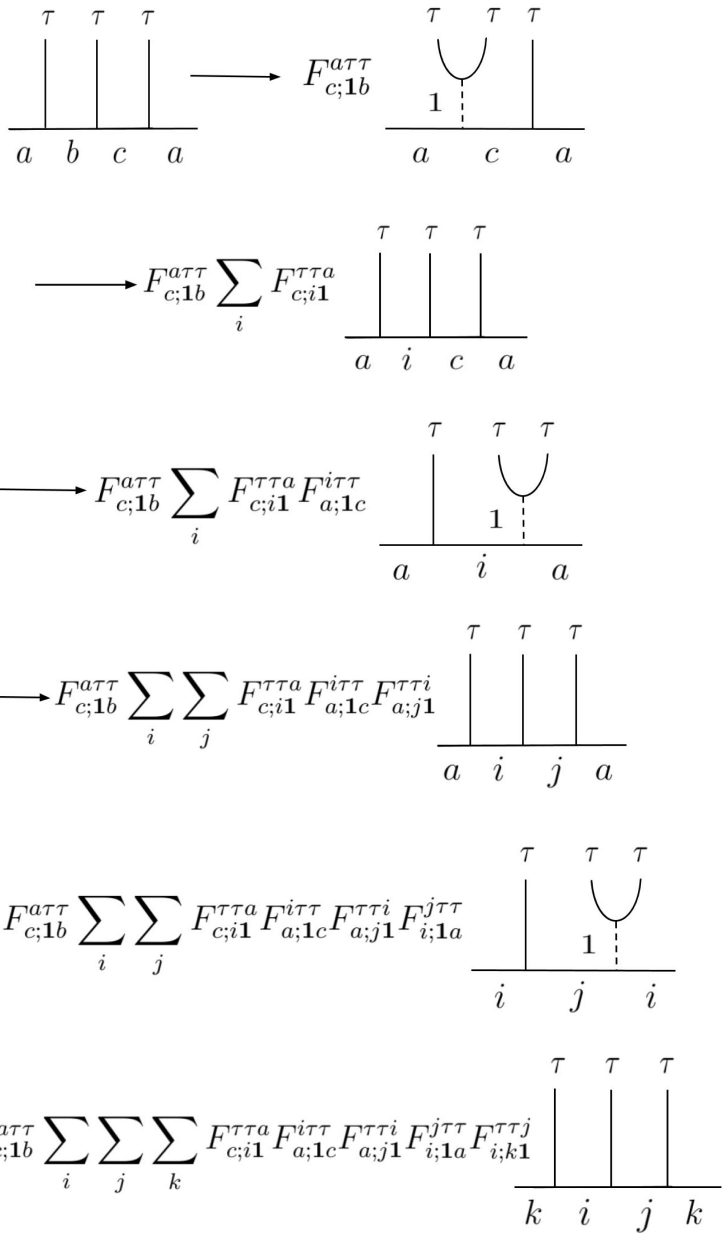
4.1 Introduction

To consider several possible interpretations, the following sections will explicitly compute a few various methods on the level 3 chain for the first tier and attempt to discover a pattern that might occur for higher tiers to see if it looks to be a promising method for computing the Hamiltonian. As usual, the ordered basis considered is the following:



A preliminary introduction to several diagrammatic approaches will be followed by a section of an analysis on each.

4.1.1 Up/Down Method:

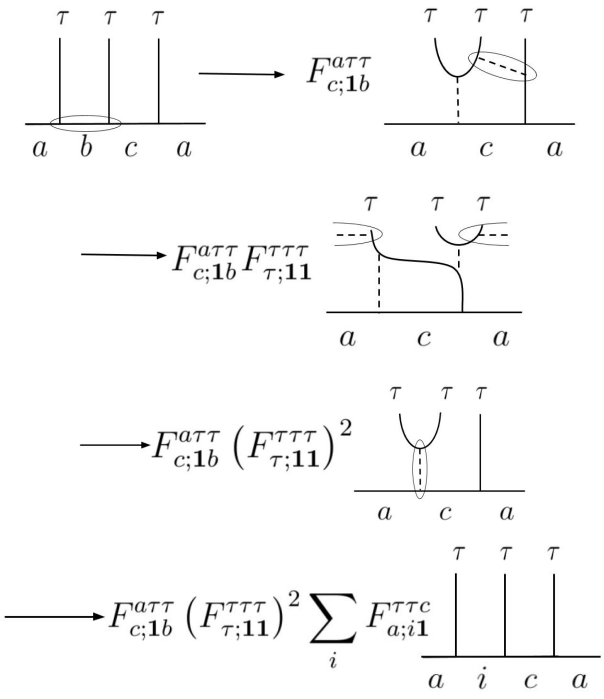


In words, begin by fusing the first two τ strands, proceed by projecting onto the trivial state, and then finally perform an F move to go back to a basis element, just as is done in the standard localized Hamiltonian for the Golden Chain. What differs from the usual method is that the exact same procedure is performed on the next two τ strands and continued on until the entire chain has been completed. In this computation, the trivial label $\mathbf{1}$ was accompanied with a dashed corresponding

edge. In the future, an edge with a label **1**, an edge being a dashed line, or an edge having both characteristics will be interchangeable.

4.1.2 Up, Up, and Away Method:

Recall that, as noted in Chapter 2, our model is unimodal, so an edge with label **1** may be introduced as desired in the diagram without having any affect. A circled edge will be used to designate that an F move will take place along that corresponding edge in the proceeding step.



Here, the idea is to fuse each τ strand to the other and project onto the trivial label until all strands are fused. Then we take the final edge and F move along the vacuum state to return to our basis.

4.2 Analysis of the Methods

4.2.1 Up/Down Summary:

Performing the computation for $n = 1$ (i.e. each pair of local sites gets acted on only once), we get the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \left(F^{\tau\tau\tau}_{\tau;\mathbf{11}}\right)^2 & 0 & 0 & F^{\tau\tau\tau}_{\tau;\mathbf{11}}\left(F^{\tau\tau\tau}_{\tau;\tau\mathbf{1}}\right)^3\left(F^{\tau\tau\tau}_{\tau;\mathbf{1}\tau}\right)^2 \\ 0 & 0 & 0 & 0 \\ \left(F^{\tau\tau\tau}_{\tau;\mathbf{1}\tau}\right)^4 F^{\tau\tau\tau}_{\tau;\tau\mathbf{1}} & 0 & 0 & \left(F^{\tau\tau\tau}_{\tau;\mathbf{1}\tau}\right)^3\left(F^{\tau\tau\tau}_{\tau;\tau\mathbf{1}}\right)^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\phi} & 0 & 0 & \frac{1}{\phi^{7/2}} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\phi^{5/2}} & 0 & 0 & \frac{1}{\phi^3} \end{pmatrix}$$

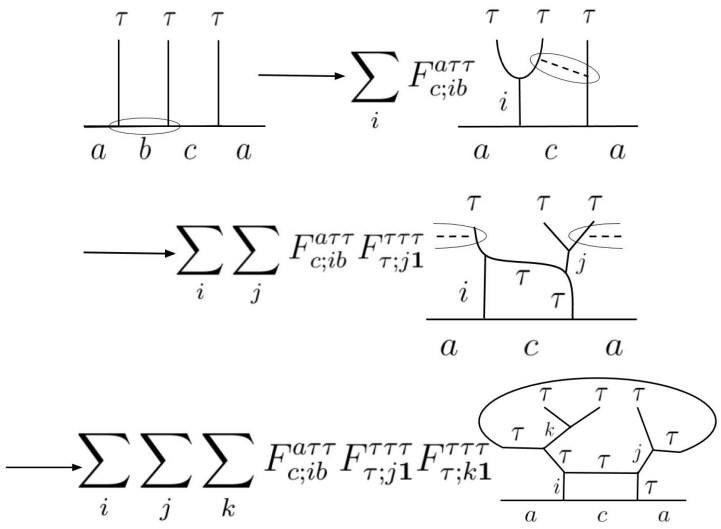
In performing the next tier, the matrix elements will simply be the exact same form but exponentiated. This doesn't appear promising since the Heisenberg Hamiltonian of the Golden Chain contains more nonzero values, yet as n goes to infinity, the zero values will not change and the nonzero values will simply tend to 0 as they're exponentiated. In addition, this matrix isn't symmetric, meaning it isn't even a proper Hamiltonian. It is possible to consider perhaps "starting at" a different section in the chain, i.e. fusing the second two τ strands first and working around the chain from there, but the matrix will be the same up to a permutation of the elements.

4.2.2 Up, Up, and Away Summary

In the $n = 1$ case, we get the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(F^{\tau\tau\tau}_{\tau;\mathbf{11}}\right)^4 & 0 & \left(F^{\tau\tau\tau}_{\tau;\mathbf{11}}\right)^3 F^{\tau\tau\tau}_{\tau;\tau\mathbf{1}} \\ 0 & 0 & 0 & 0 \\ 0 & F^{\tau\tau\tau}_{\tau;\mathbf{1}\tau}\left(F^{\tau\tau\tau}_{\tau;\mathbf{11}}\right)^3 & 0 & F^{\tau\tau\tau}_{\tau;\mathbf{1}\tau}\left(F^{\tau\tau\tau}_{\tau;\mathbf{11}}\right)^2 F^{\tau\tau\tau}_{\tau;\tau\mathbf{1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^4} & 0 & \frac{1}{\phi^{7/2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^{7/2}} & 0 & \frac{1}{\phi^3} \end{pmatrix}$$

This matrix is actually symmetric, so it may be a good potential candidate to explore. However once again the surviving entries will simply be an exponentiated $\left(F^{\tau\tau\tau\tau}_{\tau;\tau\mathbf{1}}\right)^{3n-1}$ for the n^{th} tier (since we fuse the τ strands over the trivial state a total of $3n-1$ times), but otherwise will have the same F moves aside from this. Thus there are very few nonzero entries, due mostly to the fact that we project onto the trivial state in the first move. Since the first and third basis elements have a $\mathbf{1}$ in the a and c position respectively, the state is annihilated in the first step. This motivated considering a more general case. What if the projection took place at some other step after performing the fusion of all of the τ strands? Then the diagram looks as the final step in the computation below:



Looking at the *actual* Heisenberg Hamiltonian for the level 3 Golden Chain,

$$\begin{pmatrix} 1/\phi^2 & 0 & 0 & 1/\phi^{3/2} \\ 0 & 1/\phi^2 & 0 & 1/\phi^{3/2} \\ 0 & 0 & 1/\phi^2 & 1/\phi^{3/2} \\ 1/\phi^{3/2} & 1/\phi^{3/2} & 1/\phi^{3/2} & 3/\phi \end{pmatrix}$$

perhaps there exists some way of carefully selecting the projections so that an entire row is not annihilated immediately from the first F move. This might be a way to approach the Up/Down method as well, and future work may determine a clever way of manipulating the label choices resulting in a matrix with nonzero entries in the correct locations. At the end of the Mathematica code located in the Appendix, there are some lines of code that may help with the visualization of the nonzero entries in the Total Hamiltonian for working backwards to this effect.

Appendix

Proofs of Propositions

Proposition 1. *Let GC_n denote the dimension of the anyonic Hilbert space of the Golden Chain at Level n. Then $GC_n = GC_{n-1} + GC_{n-2}$.*

Proof. The proof of this claim can be proven using a slightly stronger claim. First, as notation, denote the basis elements in the form discussed above with an added subscript n to designate in this proof the number of labels in the basis element under consideration. The strengthened claim is as follows: the number of basis elements whose last label is τ (i.e. basis elements of the form $\{\tau, ..., \tau\}_n$) and the number of those whose last label is $\mathbf{1}$ (i.e. basis elements of the form $\{\mathbf{1}, ..., \mathbf{1}_n\}$) each follow the recurrence relation given by the statement of the proposition. Once this claim has been shown, the proposition follows since if the number of admissible elements whose last labels are τ and $\mathbf{1}$ follow the relation, their sum does as well. Beginning with the elements of the form $\{\tau, ..., \tau\}_n$, observe that the claim holds for the case $n = 3$, as computed explicitly in the discussion above. Inductively, assume the claim holds up to the case $n-1$. Then for level n, the elements of the form $\{\tau, ..., \tau\}_n$ occur in two scenarios:

1. the penultimate label is a τ, meaning the element in question appears as $\{\tau, ..., \tau, \tau\}_n$. In this case, the elements can be formed by simply adding a τ label as the second-to-last label to any of the elements of the form $\{\tau, ..., \tau\}_{n-1}$ in the $n-1$ case.

2. the penultimate label is a $\mathbf{1}$: $\{\tau, ..., \mathbf{1}, \tau\}_n$. In this case, we may not simply add a $\mathbf{1}$ label to the elements $\{\tau, ..., \tau\}_{n-1}$ since the penultimate label here may include a $\mathbf{1}$. Two $\mathbf{1}$ labels are not permitted to follow in succession since this would violate the inadmissibility of $(\mathbf{1}, \tau, \mathbf{1})$. To circumvent this issue, we may simply add the sequence of labels $(\tau, \mathbf{1})$ to any element of the form $\{\tau, ..., \tau\}_{n-2}$ to get $\{\tau, ..., \tau, \mathbf{1}, \tau\}_n$.

Thus the total number of elements $\{\tau, ..., \tau\}_n$ are the number of elements of the form $\{\tau, ..., \tau, \tau\}_{n-1}$ added to the number of elements $\{\tau, ..., \mathbf{1}, \tau\}_{n-2}$. Through the inductive hypothesis, this proves the claim for the τ case. The $\mathbf{1}$ case follows

similarly, except adding instead the labels $(\mathbf{1}, \tau)$ to the $n-2$ case to fulfill scenario (2). □

Mathematica Notebook

For purposes of reproducibility of computed results, what appears in the following pages is an annotated Mathematica notebook (a .nb file) that was used to derive the matrices and values in the preceding chapters. This notebook should be available online to download and manipulate for interested readers at {insert URL}.

Computing the Hamiltonian of the Golden Chain

Code written by Richard Carini
Advised by Wade Bloomquist and Zhenghan Wang
Notebook available for download at _____

Initializing the Golden Chain Basis and Information Relevant to the Fibonacci Anyonic Model

In the following, τ is denoted by 2 and the vacuum vector is denoted by 1. GoldenChainBasis[n] returns an ordered basis that contains all admissible labelings of the ring with n spokes.

```
GoldenChainBasis[1] = {{2, 2}}
GoldenChainBasis[2] = {{1, 2, 1}, {2, 1, 2}, {2, 2, 2}}
GoldenChainBasis[L_] :=
 GoldenChainBasis[L] =
  Module[{PrevChain = GoldenChainBasis[L - 1], SecondPrevChain = GoldenChainBasis[L - 2],
    GoldenTemp = {}},
   Do[If[PrevChain[[i]][[2]] == 2 && PrevChain[[i]][[1]] == 2,
     AppendTo[GoldenTemp, {Insert[PrevChain[[i]], 2, 2],
       Insert[PrevChain[[i]], 1, 2]}],
     AppendTo[GoldenTemp, {Insert[PrevChain[[i]], 2, 2]}],
     Return["Something went wrong"]], {i, 1, Length[PrevChain]}];
   Do[If[SecondPrevChain[[i]][[1]] == 1,
     AppendTo[GoldenTemp, {Flatten[PrependTo[SecondPrevChain[[i]], {1, 2}], 1]}],
     Continue[], Return["Something went wrong"]], {i, 1, Length[SecondPrevChain]}];
   Sort[Flatten[GoldenTemp, 1]]]
```

Examples:

```
GoldenChainBasis[3]
{{1,2,2,1},{2,1,2,2},{2,2,1,2},{2,2,2,2}}
GoldenChainBasis[4]
{{1,2,1,2,1},{1,2,2,2,1},{2,1,2,1,2},{2,1,2,2,2},{2,2,1,2,2},{2,2,2,1,2},{2,2,2,2,2}}
GoldenChainBasis[5]
{{1,2,1,2,2,1},{1,2,2,1,2,1},{1,2,2,2,2,1},{2,1,2,1,2,2},{2,1,2,2,1,2},{2,1,2,2,2,2},{2
```

Initialize a "Fusion Array" such that FusionArrayFib[[a]][[b]][[c]] returns 1 if the triple (a,b,c) is admissible and 0 otherwise:

```
FusionArrayFib = {{{1, 0}, {0, 1}}, {{0, 1}, {1, 1}}}
```

Note that the R moves of the Fibonacci Anyonic model are not explicitly used for the purpose of this thesis, but are included here for completeness.

```
R[2, 2, 1] = Exp[4 i Pi / 5]
R[2, 2, 2] = Exp[-3 i Pi / 5]
```

VerifyFAdmissibleFib[a,b,c,d,e,f] returns 1 if the F move that contains the corresponding 6-tuple is admissible and 0 otherwise.

```
VerifyFAdmissibleFib[a_, b_, c_, d_, e_, f_] :=
 FusionArrayFib[[a]][[b]][[e]] FusionArrayFib[[c]][[d]][[e]] FusionArrayFib[[b]][[c]][[f]]
  FusionArrayFib[[a]][[d]][[f]]
```

FindPositionsFib is a helper function for computing F matrices. Returns an array of all possible tuples {e,f} such that VerifyFAdmissibleFib[a,b,c,d,e,f] is 1

```
FindPositionsFib[a_, b_, c_, d_] :=
 Module[{Positions = {}},
  Do[If[VerifyFAdmissibleFib[a, b, c, d, i, j] == 1, AppendTo[Positions, {i, j}]],
   {i, 1, 2}, {j, 1, 2}];
  Return[Positions]]
```

Example:

```
FindPositionsFib[1,2,2,1] = {{2,1}}
```

F Matrices:

```
FMat[1, 2, 2, 2] = SparseArray[{FindPositionsFib[1, 2, 2, 2][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 1, 2, 2] = SparseArray[{FindPositionsFib[2, 1, 2, 2][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 2, 1, 2] = SparseArray[{FindPositionsFib[2, 2, 1, 2][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 2, 2, 1] = SparseArray[{FindPositionsFib[2, 2, 2, 1][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[1, 1, 2, 2] = SparseArray[{FindPositionsFib[1, 1, 2, 2][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 1, 1, 2] = SparseArray[{FindPositionsFib[2, 1, 1, 2][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[1, 2, 2, 1] = SparseArray[{FindPositionsFib[1, 2, 2, 1][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 2, 1, 1] = SparseArray[{FindPositionsFib[2, 2, 1, 1][[1]]} → {1}, 2]
```

SparseArray[Specified elements: 1 Dimensions: {2, 2}]

```
FMat[2, 2, 2, 2] =
 SparseArray[
  {FindPositionsFib[2, 2, 2, 2][[1]], FindPositionsFib[2, 2, 2, 2][[2]],
    FindPositionsFib[2, 2, 2, 2][[3]], FindPositionsFib[2, 2, 2, 2][[4]]} →
   {ϕ^(-1), ϕ^(-1 / 2), ϕ^(-1 / 2), -ϕ^(-1)}, 2]
```

SparseArray[Specified elements: 4 Dimensions: {2, 2}]

Example:

```
MatrixForm[FMat[2,2,2,2]] =
```

$$\begin{pmatrix} \frac{1}{\phi} & \frac{1}{\sqrt{\phi}} \\ \frac{1}{\sqrt{\phi}} & -\frac{1}{\phi} \end{pmatrix}$$

Code for Hamiltonians

HamiltonianArray[*n*] returns a tuple where the first element is an array of all Local Hamiltonians 1-*n* and the second element is the Total Hamiltonian (the sum of all Local Hamiltonians)

```
HamiltonianArray[L_] :=
 HamiltonianArray[L] =
  Module[{Basis = GoldenChainBasis[L], LocalHamiltonians = {},
    TotalHamiltonian =
     SparseArray[{}, {Length[GoldenChainBasis[L]], Length[GoldenChainBasis[L]]}]},
   Do[AppendTo[LocalHamiltonians,
     Table[If[Drop[Basis[[i]], {k + 1}] == Drop[Basis[[j]], {k + 1}],
       FMat[Basis[[i]][[k]], 2, 2, Basis[[i]][[k + 2]]][[Basis[[i]][[k + 1]]]][[1]]
        FMat[2, 2, Basis[[i]][[k]], Basis[[i]][[k + 2]]][[1]][[Basis[[j]][[k + 1]]]], 0],
      {i, 1, Length[Basis]}, {j, 1, Length[Basis]}]];
    TotalHamiltonian += LocalHamiltonians[[k]], {k, 1, L - 1}];
   AppendTo[LocalHamiltonians,
    Table[If[Take[Basis[[i]], {2, L}] == Take[Basis[[j]], {2, L}],
      FMat[Basis[[i]][[L]], 2, 2, Basis[[i]][[2]]][[Basis[[i]][[1]]]][[1]]
       FMat[2, 2, Basis[[i]][[L]], Basis[[i]][[2]]][[1]][[Basis[[j]][[1]]]], 0],
     {i, 1, Length[Basis]}, {j, 1, Length[Basis]}]];
   TotalHamiltonian += LocalHamiltonians[[L]];
   Return[{LocalHamiltonians, TotalHamiltonian}]]
```

Example:

$$\texttt{HamiltonianArray[3][[1]][[1]]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\phi^{3/2}} & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$\texttt{HamiltonianArray[3][[1]][[2]]} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} \\ 0 & 0 & \frac{1}{\phi^{3/2}} & \frac{1}{\phi} \end{pmatrix}$$

$$\texttt{HamiltonianArray[3][[1]][[3]]} = \begin{pmatrix} \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\phi^{3/2}} & 0 & 0 & \frac{1}{\phi} \end{pmatrix}$$

$$\texttt{HamiltonianArray[3][[2]]} = \begin{pmatrix} \frac{1}{\phi^2} & 0 & 0 & \frac{1}{\phi^{3/2}} \\ 0 & \frac{1}{\phi^2} & 0 & \frac{1}{\phi^{3/2}} \\ 0 & 0 & \frac{1}{\phi^2} & \frac{1}{\phi^{3/2}} \\ \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{1}{\phi^{3/2}} & \frac{3}{\phi} \end{pmatrix}$$

Analysis of Golden Chain Hamiltonians

Easily compute eigenvalues and characteristic polynomials for Level 3:

```
Roots[FullSimplify[CharacteristicPolynomial[HamiltonianArray[3][[1]][[1]], λ]] == 0,
  λ] // TraditionalForm
Roots[FullSimplify[CharacteristicPolynomial[HamiltonianArray[3][[1]][[2]], λ]] == 0,
  λ] // TraditionalForm
Roots[FullSimplify[CharacteristicPolynomial[HamiltonianArray[3][[1]][[3]], λ]] == 0,
  λ] // TraditionalForm
Roots[FullSimplify[CharacteristicPolynomial[HamiltonianArray[3][[2]], λ]] == 0, λ] //
 TraditionalForm
FullSimplify[CharacteristicPolynomial[HamiltonianArray[3][[2]], λ] /.
   ϕ → GoldenRatio] // TraditionalForm
```

$$\lambda = 0 \vee \lambda = 0 \vee \lambda = 0 \vee \lambda = \frac{\phi + 1}{\phi^2}$$

$$\lambda = 0 \vee \lambda = 0 \vee \lambda = 0 \vee \lambda = \frac{\phi + 1}{\phi^2}$$

$$\lambda = 0 \vee \lambda = 0 \vee \lambda = 0 \vee \lambda = \frac{\phi + 1}{\phi^2}$$

$$\lambda = 0 \vee \lambda = \frac{1}{\phi^2} \vee \lambda = \frac{1}{\phi^2} \vee \lambda = \frac{3\,\phi + 1}{\phi^2}$$

$$\frac{1}{2}\lambda\left(\lambda\left(2\,(\lambda - 3)\,\lambda + 3\left(\sqrt{5}\, - 1\right)\right) - 7\,\sqrt{5}\, + 15\right)$$

Legible/simplified eigensystems for Level 4:

```
Transpose[Eigensystem[HamiltonianArray[4][[1]][[1]]]] // TraditionalForm
```

$$\begin{pmatrix} 0 & \left\{0, 0, 0, -\sqrt{\phi}\,, 0, 0, 1\right\} \\ 0 & \left\{0, 0, -\sqrt{\phi}\,, 0, 0, 1, 0\right\} \\ 0 & \{0, 0, 0, 0, 1, 0, 0\} \\ 0 & \{0, 1, 0, 0, 0, 0, 0\} \\ 1 & \{1, 0, 0, 0, 0, 0, 0\} \\ \frac{\phi+1}{\phi^2} & \left\{0, 0, 0, \frac{1}{\sqrt{\phi}}, 0, 0, 1\right\} \\ \frac{\phi+1}{\phi^2} & \left\{0, 0, \frac{1}{\sqrt{\phi}}, 0, 0, 1, 0\right\} \end{pmatrix}$$

```
Transpose[Eigensystem[HamiltonianArray[4][[1]][[1]]]] // TraditionalForm
```

$$\begin{pmatrix} 0 & \left\{0, 0, 0, -\sqrt{\phi}\,, 0, 0, 1\right\} \\ 0 & \left\{0, 0, -\sqrt{\phi}\,, 0, 0, 1, 0\right\} \\ 0 & \{0, 0, 0, 0, 1, 0, 0\} \\ 0 & \{0, 1, 0, 0, 0, 0, 0\} \\ 1 & \{1, 0, 0, 0, 0, 0, 0\} \\ \frac{\phi+1}{\phi^2} & \left\{0, 0, 0, \frac{1}{\sqrt{\phi}}, 0, 0, 1\right\} \\ \frac{\phi+1}{\phi^2} & \left\{0, 0, \frac{1}{\sqrt{\phi}}, 0, 0, 1, 0\right\} \end{pmatrix}$$

```
Transpose[Eigensystem[HamiltonianArray[4][[1]][[1]]]] // TraditionalForm
```

$$\begin{pmatrix} 0 & \left\{0,0,0,-\sqrt{\phi},0,0,1\right\} \\ 0 & \left\{0,0,-\sqrt{\phi},0,0,1,0\right\} \\ 0 & \{0,0,0,0,1,0,0\} \\ 0 & \{0,1,0,0,0,0,0\} \\ 1 & \{1,0,0,0,0,0,0\} \\ \frac{\phi+1}{\phi^2} & \left\{0,0,0,\frac{1}{\sqrt{\phi}},0,0,1\right\} \\ \frac{\phi+1}{\phi^2} & \left\{0,0,\frac{1}{\sqrt{\phi}},0,0,1,0\right\} \end{pmatrix}$$

```
Transpose[Eigensystem[HamiltonianArray[4][[1]][[1]]]] // TraditionalForm
```

$$\begin{pmatrix} 0 & \left\{0,0,0,-\sqrt{\phi},0,0,1\right\} \\ 0 & \left\{0,0,-\sqrt{\phi},0,0,1,0\right\} \\ 0 & \{0,0,0,0,1,0,0\} \\ 0 & \{0,1,0,0,0,0,0\} \\ 1 & \{1,0,0,0,0,0,0\} \\ \frac{\phi+1}{\phi^2} & \left\{0,0,0,\frac{1}{\sqrt{\phi}},0,0,1\right\} \\ \frac{\phi+1}{\phi^2} & \left\{0,0,\frac{1}{\sqrt{\phi}},0,0,1,0\right\} \end{pmatrix}$$

```
Transpose[
  ToRadicals[FullSimplify[Eigensystem[HamiltonianArray[4][[2]]] /. ϕ → GoldenRatio]] /.
    Sqrt[5] → 2 Inactive[ϕ] - 1 // Simplify // Activate] // TraditionalForm
```

$$\begin{pmatrix} 1 & \{0,0,0,-1,0,1,0\} \\ 1 & \{0,-1,0,0,1,0,0\} \\ 4-2\phi & \left\{\sqrt{2\phi-3},-1,-\sqrt{2\phi-3},1,-1,1,0\right\} \\ 3 & \left\{-2\sqrt{2\phi+1},-1,2\sqrt{2\phi+1},1,-1,1,0\right\} \\ \frac{1}{2}\left(3-\sqrt{57-32\phi}\right) & \left\{\frac{1}{4}\left(4\phi+\sqrt{57-32\phi}-7\right),-\frac{\sqrt{26\phi+\sqrt{885-328\phi}-23}}{4\sqrt{2}},\right. \\ & \quad \frac{1}{4}\left(4\phi+\sqrt{57-32\phi}-7\right),-\frac{\sqrt{26\phi+\sqrt{885-328\phi}-23}}{4\sqrt{2}}, \\ & \quad \left.-\frac{\sqrt{26\phi+\sqrt{885-328\phi}-23}}{4\sqrt{2}},-\frac{\sqrt{26\phi+\sqrt{885-328\phi}-23}}{4\sqrt{2}},1\right\} \\ \frac{1}{2}\left(\sqrt{57-32\phi}+3\right) & \left\{\phi-\frac{1}{4}\sqrt{57-32\phi}-\frac{7}{4},\frac{\sqrt{26\phi-\sqrt{885-328\phi}-23}}{4\sqrt{2}},\right. \\ & \quad \phi-\frac{1}{4}\sqrt{57-32\phi}-\frac{7}{4},\frac{\sqrt{26\phi-\sqrt{885-328\phi}-23}}{4\sqrt{2}}, \\ & \quad \left.\frac{\sqrt{26\phi-\sqrt{885-328\phi}-23}}{4\sqrt{2}},\frac{\sqrt{26\phi-\sqrt{885-328\phi}-23}}{4\sqrt{2}},1\right\} \\ 2\phi & \left\{\frac{\phi}{2},\frac{1}{2\sqrt{\phi}},\frac{\phi}{2},\frac{\sqrt{\phi-1}}{2},\frac{1}{2\sqrt{\phi}},\frac{\sqrt{\phi-1}}{2},1\right\} \end{pmatrix}$$

Level 5 Ground State Eigenvector (not simplified; computation time is lengthy):

```
JordanDecomposition[HamiltonianArray[5][[2]] /. ϕ → GoldenRatio][[1]][[All, 11]] /.
    Sqrt[5] → 2 Inactive[ϕ] - 1 // Simplify // Activate // Simplify //
 TraditionalForm
```

$$\left\{\left(8\left(12\,230\,683\,946\,816\,\phi+1\,803\,859\,384\,818\sqrt{185-80\,\phi}+4\,033\,552\,206\,304\sqrt{37-16\,\phi}-\right.\right.\right.$$
$$\left.\left.\left.978\,297\,980\,501\sqrt{5}\sqrt{\phi\,(21\,\phi-16)}-2\,187\,540\,786\,651\sqrt{\phi\,(21\,\phi-16)}-298\,168\,717\,513\sqrt{5}\right.\right.\right.$$
$$\left.\left.\left.\sqrt{-\phi\left(336\,\phi^2-1033\,\phi+592\right)}-666\,725\,521\,123\sqrt{-\phi\left(336\,\phi^2-1033\,\phi+592\right)}+7\,558\,978\,384\,790\right)\right)\Big/$$
$$\left(5\,\phi^9\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$\left(64\left(-5\,344\,769\,637\,719\,000\,\phi+77\,717\,629\,388\,727\sqrt{5}\sqrt{697-231\,\phi}+173\,781\,902\,363\,329\sqrt{697-231\,\phi}-\right.\right.$$
$$809\,198\,814\,841\,264\sqrt{185-80\,\phi}-1\,809\,423\,557\,297\,332\sqrt{37-16\,\phi}+362\,936\,515\,836\,263\sqrt{5}\sqrt{5\,\phi+21}+$$
$$811\,550\,720\,926\,813\sqrt{5\,\phi+21}+103\,459\,128\,214\,155\sqrt{5}\sqrt{-80\,\phi^2-151\,\phi+777}+$$
$$231\,341\,643\,579\,717\sqrt{-80\,\phi^2-151\,\phi+777}+22\,154\,282\,726\,007\sqrt{5}\sqrt{3696\,\phi^2-19\,699\,\phi+25\,789}+$$
$$\left.\left.49\,538\,482\,168\,101\sqrt{3696\,\phi^2-19\,699\,\phi+25\,789}-3\,303\,249\,298\,148\,804\right)\right)\Big/$$
$$\left(5\,\phi^8\left(6\,\phi+\sqrt{185-80\,\phi}+3\sqrt{37-16\,\phi}-6\right)\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)^2\right.$$
$$\left.\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$\left(16\left(21\,698\,981\,718\,766\,\phi+3\,306\,741\,749\,129\sqrt{185-80\,\phi}+7\,394\,099\,335\,089\sqrt{37-16\,\phi}+13\,410\,708\,223\,460\right)\right)\Big/$$
$$\left(5\,\phi^{29/2}\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$\left(512\left(90\,378\,238\,466\,202\,\phi+13\,623\,810\,789\,703\sqrt{185-80\,\phi}+30\,463\,767\,038\,371\sqrt{37-16\,\phi}+55\,856\,823\,215\,456\right)\right)\Big/$$
$$\left(5\,\phi^7\left(6\,\phi+\sqrt{185-80\,\phi}+3\sqrt{37-16\,\phi}-6\right)\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)^2\right.$$
$$\left.\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$-\left(\left(32\left(421\,114\,176\,267\,870\,\phi+62\,881\,559\,865\,951\sqrt{185-80\,\phi}+140\,607\,442\,391\,489\sqrt{37-16\,\phi}-\right.\right.\right.$$
$$45\,189\,119\,233\,101\sqrt{5}\sqrt{\phi\,(21\,\phi-16)}-101\,045\,942\,448\,557\sqrt{\phi\,(21\,\phi-16)}-$$
$$\left.\left.\left.13\,623\,810\,789\,703\sqrt{5}\sqrt{-\phi\left(336\,\phi^2-1033\,\phi+592\right)}-30\,463\,767\,038\,371\sqrt{-\phi\left(336\,\phi^2-1033\,\phi+592\right)}+\right.\right.\right.$$
$$\left.\left.\left.260\,262\,874\,077\,958\right)\right)\Big/\left(5\,\phi^7\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)^2\right.$$
$$\left.\left.\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right)\right),$$

$$\left(64\left(6\,869\,935\,104\,966\,614\,\phi+1\,035\,588\,849\,661\,021\sqrt{185-80\,\phi}+\right.\right.$$
$$\left.\left.2\,315\,647\,064\,582\,853\sqrt{37-16\,\phi}+4\,245\,853\,395\,375\,444\right)\right)\Big/$$
$$\left(5\,\phi^{33/2}\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)^2\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$\left(4\left(28\,419\,026\,\phi+3\,155\,231\sqrt{185-80\,\phi}+7\,055\,311\sqrt{37-16\,\phi}+17\,563\,924\right)\right)\Big/$$
$$\left(5\,\phi^7\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)\right),$$

$$\left(16\left(285\,463\,503\,554\,\phi+43\,502\,229\,611\sqrt{185-80\,\phi}+97\,273\,942\,583\sqrt{37-16\,\phi}+176\,426\,147\,744\right)\right)\Big/$$
$$\left(5\,\phi^{11/2}\left(142\,\phi+11\sqrt{185-80\,\phi}+25\sqrt{37-16\,\phi}+86\right)\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2\right),$$

$$\frac{8\left(1\,165\,932\,046\,\phi+183\,972\,284\sqrt{185-80\,\phi}+411\,374\,533\sqrt{37-16\,\phi}+720\,585\,633\right)}{5\,\phi^{11/2}\left(11\,142\,\phi+1419\sqrt{185-80\,\phi}+3173\sqrt{37-16\,\phi}+6886\right)^2},$$

$$\Big(32\sqrt{2}\,\Big(12\left(219602+98209\sqrt{5}\right)\phi^5+10\left(294627\sqrt{185-80\phi}+658806\sqrt{37-16\phi}-5476520\sqrt{5}-12245871\right)\phi^4+\left(-38819920\sqrt{185-80\phi}-86803980\sqrt{37-16\phi}+405143294\sqrt{5}+905927946\right)\phi^3+\left(153213415\sqrt{185-80\phi}+342595611\sqrt{37-16\phi}-1229418967\sqrt{5}-2749064383\right)\phi^2+\left(-244321723\sqrt{185-80\phi}-546319981\sqrt{37-16\phi}+1691242689\sqrt{5}+3781733619\right)\phi+143534203\sqrt{185-80\phi}+320952235\sqrt{37-16\phi}-857802033\sqrt{5}-1918103657\Big)\Big)\Big/\Big(\left(1+\sqrt{5}\right)^{9/2}\left(\left(58+26\sqrt{5}\right)\phi^2+2\left(13\sqrt{185-80\phi}+29\sqrt{37-16\phi}-83\sqrt{5}-185\right)\phi-31\sqrt{185-80\phi}-69\sqrt{37-16\phi}+271\sqrt{5}+605\right)\left(\left(7142+3194\sqrt{5}\right)\phi^4+4\left(1597\sqrt{185-80\phi}+3571\sqrt{37-16\phi}-22968\sqrt{5}-51358\right)\phi^3+\left(-48376\sqrt{185-80\phi}-108172\sqrt{37-16\phi}+435850\sqrt{5}+974590\right)\phi^2+2\left(57204\sqrt{185-80\phi}+127912\sqrt{37-16\phi}-387131\sqrt{5}-865651\right)\phi-78431\sqrt{185-80\phi}-175377\sqrt{37-16\phi}+506851\sqrt{5}+1133353\right),1\Big)$$

Computational demonstration of accuracy of numerical approximation of eigenvectors:

```
N[Normalize[Eigensystem[HamiltonianArray[4][[2]] /. ϕ → GoldenRatio, 1][[2]][[1]]],
  50] + Eigensystem[SetPrecision[HamiltonianArray[4][[2]] /. ϕ → GoldenRatio, 50], 1][[
   2]][[1]]
```

```
{0. × 10^-50, 0. × 10^-51, 0. × 10^-50, 0. × 10^-51, 0. × 10^-51, 0. × 10^-51, 0. × 10^-50}
```

```
Eigensystem[SetPrecision[HamiltonianArray[5][[2]] /. ϕ → GoldenRatio, 50], 1][[2]][[
 1]]
```

```
{-0.11213358107934037411519094424637328583604850012088,
 -0.11213358107934037411519094424637328583604850012088,
 -0.23318967148822090037855013823999521386659711047211,
 -0.11213358107934037411519094424637328583604850012088,
 -0.11213358107934037411519094424637328583604850012088,
 -0.23318967148822090037855013823999521386659711047211,
 -0.11213358107934037411519094424637328583604850012088,
 -0.23318967148822090037855013823999521386659711047211,
 -0.23318967148822090037855013823999521386659711047211,
 -0.23318967148822090037855013823999521386659711047211,
 -0.81562441449952506602325816540325784956705984128 99}
```

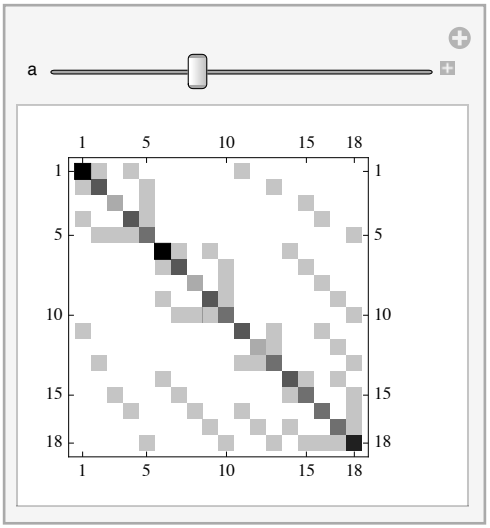
```
Eigensystem[N[HamiltonianArray[6][[2]] /. ϕ → GoldenRatio, 70], 1]
```


```
{{4.69854318470237509718426963826873149293028036194249153614677647080765 5},
 {{-0.435014546216120878422373577407862850672031063145962573863011970392405 2,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.043389983134862319484175079590247745040216875675505473380013943526385 87,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.435014546216120878422373577407862850672031063145962573863011970392405 2,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.043389983134862319484175079590247745040216875675505473380013943526385 87,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.043389983134862319484175079590247745040216875675505473380013943526385 87,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.164934059658294993599548744946523111961082528477420012710456686870335 3,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.175688269737133566139033890257824178704172271719172391622562808664908 1,
   -0.517164330073015641285148357792935473661892597189647869606609190816271 6}}}
```

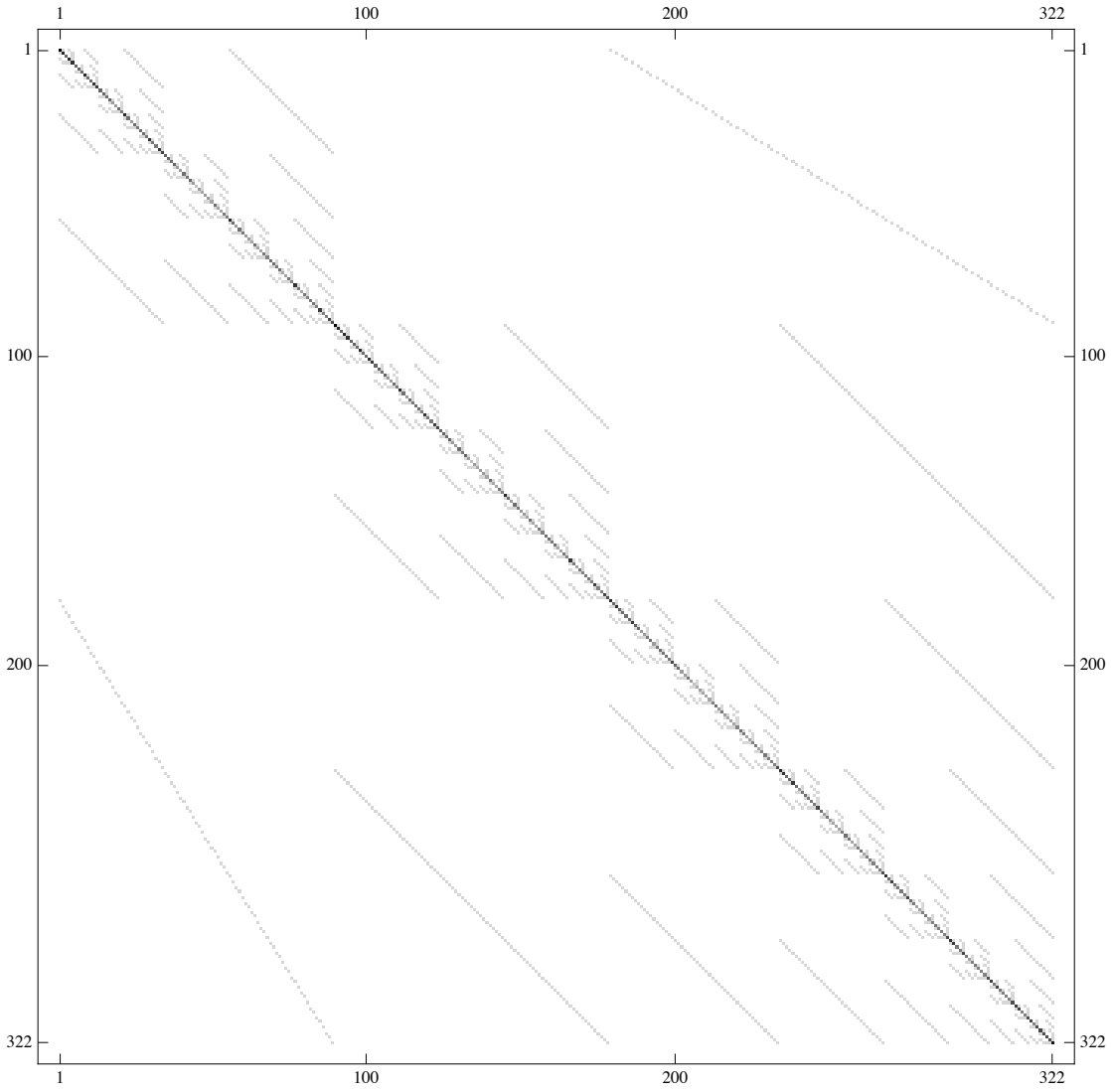

Visualizing Golden Chain Hamiltonian Matrices

In the "Diagrammatic Discussion" of the paper, it was mentioned the potential future work related to analyzing the nonzero entries of the Hamiltonian in hopes of determining an alternative diagrammatic explanation that meaningfully "limits to" the exact form of the Hamiltonian. What follows are lines of Mathematica code whose purpose is to assist in this front.

```
Manipulate[Show[MatrixPlot[1000 * N[HamiltonianArray[a][[2]] /. ϕ → GoldenRatio]],
  ImageSize → Small], {a, 3, 11, 1}]
```



```
Show[MatrixPlot[N[HamiltonianArray[12][[2]] /. ϕ → GoldenRatio]], ImageSize → Large]
```




```
Position[N[HamiltonianArray[8][[2]] /. ϕ → GoldenRatio],
 Max[N[HamiltonianArray[8][[2]] /. ϕ → GoldenRatio]]]
```

```
{{1, 1}, {14, 14}}
```

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