PHYS 5794: Computational Physics HW 5

All program and output files can be found in: /home/k-class/rdcunha2/hw5

Method:
A grid was chosen such that h=0.05, with 21 grid points in the x direction and 20001 grid points in the y direction, such that y varied {0,1000} and x varied {0,1}.
The grid was iterated over, using the finite difference formula to update the φs, using the φs calculated from before.
Formula:

$$\varphi^{new}(i, j) = (1-\omega)\varphi^{old}(i, j) - (\omega/4)*(\varphi(i+1, j) + \varphi(i, j+1) + \varphi(i-1, j) +\varphi(i, j-1))$$

The energy functional used to compute the final formula for phi was also calculated for each iteration.
Formula:

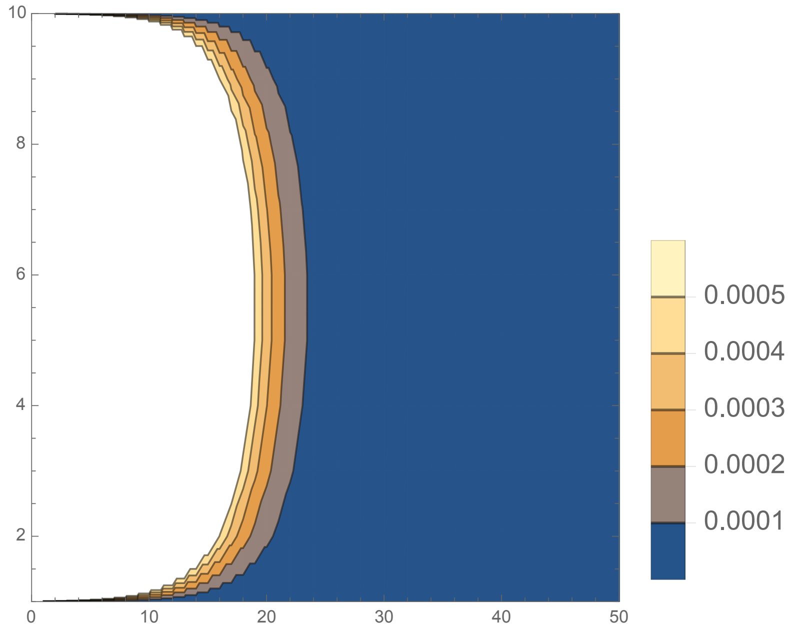
$$E = (h^2/2)*[(\{\varphi(i+1, j) - \varphi(i, j) \}/h)^2+(\{\varphi(i+1, j) - \varphi(i, j) \}/h)^2]$$

Convergence criterion was $(\varphi^{new}- \varphi^{old})/n_i*n_j < 10^{-9}$

Verification of program:
The matrix φ drops off exponentially from the boundary conditions and approaches zero. It has larger values towards x=0.5 and drops down to 0 at both ends.

Data:

Figure 1. Contour plot of phi



Analysis:
For h=0.05

ω	E from program	E from analytical solution
0.5	0.0567881	0.03333
1.0	0.0567879	0.03333
1.5	0.0567879	0.03333

For ω=0.5

Grid spacing h	E from program	E analytical solution
0.01	0.068188*	0.03333
0.05	0.0567881	0.03333
0.1	0.0611994	0.03333

*This value took too long to converge, so the criterion was relaxed to 10^{-6}, which may have resulted in a higher value than with a smaller convergence criterion.

The analytical solution was calculated(attached).

Interpretation:
For the same grid spacing, changing the value of ω from 0.5 to 1.0 and 1.5 does not change the value of the converged energy functional E(Table 2)
The size of the grid spacing relative to the size of ω changes the value of the energy.(Table 3) A smaller grid spacing means a more converged energy, however, the time taken by the program increases, and the convergence criteria must be changed

Log:
8 hours

$$\frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} = 0$$

$\phi(x,y) = X(x)\,Y(y)$

Separating the variables,

$$Y(y)\frac{\delta^2 X(x)}{\delta x^2} + X(x)\frac{\delta^2 Y(y)}{\delta y^2} = 0$$

$$\Rightarrow \frac{1}{X}\frac{\delta^2 X}{\delta x^2} = -\frac{1}{Y}\frac{\delta^2 Y}{\delta y^2}$$

These are dependent on different variables, so we can equate both sides to a constant, λ^2.

$$\frac{1}{X}\frac{\delta^2 X}{\delta x^2} = -\lambda^2 \Rightarrow \frac{\delta^2 X}{\delta x^2} = -\lambda^2 X$$

$$\frac{1}{Y}\frac{\delta^2 Y}{\delta y^2} = \lambda^2 \Rightarrow \frac{\delta^2 Y}{\delta y^2} = \lambda^2 Y$$

With boundary conditions $\phi(0,y) = 0$

$\phi(1,y) = 0$

$X(x) = \sin n\pi x$

With boundary condition $\phi(x,\infty) = 0$

$Y(y) = e^{-n\pi y}$

But for the boundary condition $\phi(x,0) = x(1-x)$ we must have a linear combination of sin functions making up $x(1-x)$

So $\sum_n^\infty A_n \sin n\pi x = x(1-x)$

$$\Rightarrow A_n = \int_0^1 x(1-x)\sin n\pi x \, dx$$

$$= \frac{2(1-\cos n\pi)}{n^3\pi^3}$$

$$\therefore \phi(x, y) = \sum_n^{\infty} A_n X(x)\, Y(x)$$

$$= \sum_n^{\infty} \frac{2(1-\cos n\pi)}{n^3 \pi^3} \cdot \sin n\pi x \; e^{-n\pi y}$$

$$E = \int_0^1 dx \int_0^{\infty} dy \, \frac{1}{2}\left[(\nabla\phi)^2 - \cancel{S_{ij}\phi}^{\,0}\right]$$

$$= \int_0^1 dx \int_0^{\infty} dy \, \frac{1}{2}\left[(\nabla\phi)^2\right]$$

$$= \sum_n^{\infty} \frac{\cancel{4}^2(1-\cos n\pi)^2}{n^{\cancel{4}}\pi^{\cancel{4}}\,\cancel{2}} \int_0^{\infty} dy \left(\int_0^1 \cos n\pi x \, dx - \int_0^1 \sin n\pi x \, dx \right)^2 e^{-2n\pi y}$$

$$= \sum_n^{\infty} \frac{2(1-\cos n\pi)}{n^4 \pi^4} \int_0^{\infty} dy \left(\frac{2}{n\pi}\right)^2 e^{-2n\pi y}$$

$$= \sum_n^{\infty} \frac{8(1-\cos n\pi)}{n^6 \pi^6} \cdot \frac{1}{n\pi}$$

$$= \sum_n^{\infty} \frac{8(1-\cos n\pi)}{n^7 \pi^7}$$

$$\approx 0.03333$$