An Elementary Analytic Theory of the Degree of Mapping in n-Dimensional Space

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Introduction. Since the publication of BROUWER's fundamental papers[1] much effort has been made in order to establish the principal properties of the degree of a mapping and, in particular, the Brouwer fixed point theorem by analytic methods which do not involve the concepts of combinatorial topology.[2] The first satisfactory approach to this problem was made by HADAMARD [5] by means of the Kronecker integral. Proofs of the Brouwer fixed point theorem based on the theory of analytic functions were also given later by G. D. BIRKHOFF & KELLOGG [3] and quite recently by SEKI [17]. In [11] NAGUMO deduced the main theorems on the Brouwer degree by using results on implicit functions and a lemma of SARD [14] on the critical points of a mapping. Furthermore, he gives a proof of a theorem, due to LERAY ([8] and [9]), on the degree of the composition of two continuous mappings. From quite a different point of view the theory of the Brouwer degree and also the more general homology theory were developed by DE RHAM [13]. His results are derived from the theory of currents on differentiable manifolds, which is also connected with the work of L. SCHWARTZ on distributions (see [15] and [16]).

In the present paper we shall establish the main properties of the Brouwer degree by a method which is closely related to the work of DE RHAM, but seems to be of a more elementary character. In view of the applications to existence problems of analysis[3] we restrict ourselves to the case where the given mapping $y = y(x)$ is defined in the closure of a bounded open set of the n-dimensional space. In §1 we first define the Brouwer degree for continuously differentiable mappings by means of a volume integral (Definition 1) and then extend this concept to the continuous case (Definition 2). The justification of these definitions depends on three simple lemmas (Lemma 1, 2, and 3), the first of which essentially involves an identity which also plays an important rôle in HADA-

[1] See, in particular, [4].

[2] For the topological methods see ALEXANDROFF-HOPF [1].

[3] See LERAY-SCHAUDER [7]. A complete bibliography is found in LERAY [9].

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MARD's classical paper.[4] From these lemmas the main theorems on the degree of a mapping and the Brouwer fixed point theorem are readily deduced. In §2 we give the well known representation of the degree in terms of the zeros of the equation $y(x) = z$ (z fixed) by means of the concept of the index. Finally, in §3, we shall establish a general transformation formula for multiple integrals (Lemma 7) and thence deduce LERAY's product theorem (Theorem 7).

1. Definition and fundamental properties of the Brouwer degree. Let E^n be the n-dimensional space of real vectors $x = (x^1, \cdots, x^n)$ with the norm $|x| = [(x^1)^2 + \cdots + (x^n)^2]^{\frac{1}{2}}$. The boundary and the closure of an open set Ω in E^n will be denoted by $\dot{\Omega}$ and $\bar{\Omega} = \Omega + \dot{\Omega}$, respectively. Furthermore, dx stands for the volume-element in E^n. A mapping $y = y(x)$ of Ω into a subset $y(\Omega) \subset E^n$ is said to be of class C^k ($k \geqq 0$) in Ω if it can be represented in the form

$$y^i = y^i(x^1, \cdots, x^n) \qquad (i = 1, \cdots, n), \tag{1.1}$$

where all functions y^i belong to C^k in Ω. If $y(x)$ is of class C^1 then its Jacobian will be denoted by $J[y(x)]$. Furthermore, $A_{ij}(x)$ is defined to be the cofactor of $a_{ij}(x)$ in the determinant $J[y(x)] = \det(a_{ij}(x))$, where $a_{ij}(x) = \partial y^i/\partial x^j$ ($i, j = 1, \cdots, n$). If $k \geqq 2$ the functions $A_{ij}(x)$ belong to C^1 and satisfy the well known relations.

$$\sum_{i=1}^{n} \frac{\partial A_{ij}(x)}{\partial x^i} = 0 \qquad (j = 1, \cdots, n). \tag{1.2}$$

Our whole theory is based on the following lemma:

Lemma 1. *Hypotheses*: (i) *The mapping $y = y(x)$ is of class C^1 in a bounded open set $\Omega \subset E^n$. Furthermore, it is continuous in $\bar{\Omega}$, and we have*

$$|y(x)| > \epsilon > 0 \tag{1.3}$$

for $x \in \dot{\Omega}$.

(ii) *The real-valued function $\varphi(r)$ is continuous in the interval $0 \leqq r < \infty$. Furthermore, it vanishes for $r \geqq \epsilon$ and in a vicinity of $r = 0$ and satisfies the equation*

$$\int_0^\infty r^{n-1}\varphi(r)\, dr = 0. \tag{1.4}$$

Conclusion: *We have the equation*

$$\int_\Omega \varphi(|y(x)|) J[y(x)]\, dx = 0. \tag{1.5}$$

Proof. On account of the Weierstrass approximation theorem it suffices to prove (1.5) under the additional assumption that $y(x)$ is of class C^2 in Ω. Let us first consider the function

[4] See [5], in particular, p. 455.

$$(1.6) \qquad \psi(r) = \begin{cases} r^{-n} \int_0^r \rho^{n-1}\varphi(\rho)\, d\rho & (0 < r < \infty) \\ 0 & (r = 0). \end{cases}$$

On account of our hypotheses it belongs to C^1 for $0 \leqq r < \infty$ and vanishes identically in a neighborhood of $r = 0$ and in the interval $\epsilon \leqq r < \infty$. Furthermore, it satisfies for $0 \leqq r < \infty$ the differential equation

$$(1.7) \qquad r\psi'(r) + n\psi(r) = \varphi(r).$$

Consequently the functions

$$(1.8) \qquad f^j(y) = \psi(|y|)y^j \qquad (j = 1, \cdots, n)$$

belong to C^1 for $y \varepsilon E^n$, and we have $f^j(y) \equiv 0$ for $|y| \geqq \epsilon$. From this it follows that $f^j(y(x))$ is of class C^1 for $x \varepsilon \Omega$ and vanishes identically in a neighborhood of $\dot{\Omega}$. Using the identities (1.2) we obtain for $x \varepsilon \Omega$ the equation

$$(1.9) \qquad \sum_{i=1}^n \frac{\partial}{\partial x^i} \sum_{j=1}^n A_{ij}(x) f^j(y(x)) = J[y(x)]\left(\sum_{j=1}^n \frac{\partial f^j}{\partial y^j}\right)_{y=y(x)}$$
$$= J[y(x)](r\psi'(r) + n\psi(r))_{r=|y(x)|} = \varphi(|y(x)|)J[y(x)],$$

from which (1.5) follows by integration.

This lemma gives rise to the following definition:

Definition 1. *Let the mapping $y = y(x)$ be of class C^1 in a bounded open set $\Omega \subset E^n$ and continuous in $\bar{\Omega}$. Furthermore, let $y(x) \neq z$ for $x \varepsilon \dot{\Omega}$, where z is fixed in E^n, and let a real-valued function $\Phi(r)$ be chosen such that the following conditions are satisfied:*

(i) *$\Phi(r)$ is continuous in the interval $0 \leqq r < \infty$. Furthermore, it vanishes in a neighborhood of $r = 0$ and for $\epsilon \leqq r < \infty$, where $0 < \epsilon < \min_{x \varepsilon \dot{\Omega}} |y(x) - z|$.*

(ii) *We have*

$$(1.10) \qquad \int_{E^n} \Phi(|x|)\, dx = 1.$$

Then the Brouwer degree $d[y(x); \Omega, z]$ is uniquely defined by the equation

$$(1.11) \qquad d[y(x); \Omega, z] = \int_\Omega \Phi(|y(x) - z|)J[y(x)]\, dx.$$

In order to justify this definition we denote by D the linear space of functions $\Phi(r)$ which satisfy (i), and set

$$(1.12) \qquad L\Phi = \int_0^\infty r^{n-1}\Phi(r)\, dr,$$

$$(1.13) \qquad M\Phi = \int_{E^n} \Phi(|x|)\, dx,$$

and

$$N\Phi = \int_\Omega \Phi(|y(x) - z|)J[y(x)]\,dx. \tag{1.14}$$

Then L, M, and N are linear functionals on D. Applying the preceding lemma to the mappings $y = x$ ($|x| < 2\epsilon$) and $y = y(x) - z$ ($x \in \Omega$), it follows that the equation $L\Phi = 0$ ($\Phi \in D$) implies $M\Phi = N\Phi = 0$. Now let Φ_1 and Φ_2 be two functions of D with $M\Phi_1 = M\Phi_2 = 1$. Then, since

$$L(L\Phi_2 \cdot \Phi_1 - L\Phi_1 \cdot \Phi_2) = 0 \tag{1.15}$$

holds, we have

$$L\Phi_2 \cdot M\Phi_1 - L\Phi_1 \cdot M\Phi_2 = 0, \tag{1.16}$$

hence

$$L(\Phi_1 - \Phi_2) = 0. \tag{1.17}$$

The last equation implies

$$N(\Phi_1 - \Phi_2) = 0 \tag{1.18}$$

or

$$N\Phi_1 = N\Phi_2 . \tag{1.19}$$

This shows the uniqueness of our definition.

In order to define the degree for continuous mappings use is made of the following two lemmas:

Lemma 2. *Let Ω be a bounded open set in E^n and $y = y_i(x)$ ($i = 1, 2$) be two mappings, which are of class C^1 in Ω and continuous in $\bar{\Omega}$. Furthermore, let the inequalities*

$$|y_i(x) - z| > 7\epsilon \qquad (x \in \dot{\Omega};\, i = 1, 2) \tag{1.20}$$

and

$$|y_1(x) - y_2(x)| < \epsilon \qquad (x \in \bar{\Omega}) \tag{1.21}$$

be satisfisd, where z is fixed in E^n and ϵ is a positive number. Then we have the equation

$$d[y_1(x);\, \Omega, z] = d[y_2(x);\, \Omega, z]. \tag{1.22}$$

Proof. Since

$$d[y_i(x);\, \Omega, z] = d[y_i(x) - z;\, \Omega, 0] \qquad (i = 1, 2) \tag{1.23}$$

holds by Definition 1, we may suppose, without loss of generality, that $z = 0$. Let $f(r)$ be a real-valued function of class C^1 for $0 \leqq r < \infty$ with

$$f(r) = 1 \qquad (0 \leqq r \leqq 2\epsilon), \tag{1.24}$$

$$f(r) = 0 \qquad (3\epsilon \leqq r < \infty), \tag{1.25}$$

and

$$0 \leqq f(r) \leqq 1 \qquad (0 \leqq r < \infty), \tag{1.26}$$

and let us consider the auxiliary mapping

$$y_3(x) = [1 - f(|y_1(x)|)]y_1(x) + f(|y_1(x)|)y_2(x). \tag{1.27}$$

Obviously it is continuous in $\bar{\Omega}$ and belongs to C^1 for $x \varepsilon \Omega$. Furthermore, on account of (1.20), (1.21), and (1.26) we have the estimates

$$|y_i(x) - y_k(x)| < \epsilon \qquad (x \varepsilon \bar{\Omega}; i, k = 1, 2, 3) \tag{1.28}$$

and

$$|y_i(x)| > 6\epsilon \qquad (x \varepsilon \dot{\Omega}; i = 1, 2, 3), \tag{1.29}$$

and from (1.24) and (1.25) it follows that

$$y_3(x) = y_1(x) \quad \text{if} \quad |y_1(x)| > 3\epsilon \tag{1.30}$$

and

$$y_3(x) = y_2(x) \quad \text{if} \quad |y_1(x)| < 2\epsilon. \tag{1.31}$$

Now let $\Phi_i(r)$ $(i = 1, 2)$ be two real-valued continuous functions in $0 \leqq r < \infty$, which vanish in a vicinity of $r = 0$ and satisfy the equations

$$\Phi_1(r) = 0 \qquad (0 \leqq r \leqq 4\epsilon; 5\epsilon \leqq r < \infty) \tag{1.32}$$

and

$$\Phi_2(r) = 0 \qquad (\epsilon \leqq r < \infty). \tag{1.33}$$

Furthermore, let

$$\int_{E^n} \Phi_i(|x|)\, dx = 1 \qquad (i = 1, 2). \tag{1.34}$$

Then from (1.28) and (1.30)–(1.33) we conclude that for $x \varepsilon \Omega$ the equations

$$\Phi_1(|y_3(x)|)J[y_3(x)] = \Phi_1(|y_1(x)|)J[y_1(x)] \tag{1.35}$$

and

$$\Phi_2(|y_3(x)|)J[y_3(x)] = \Phi_2(|y_2(x)|)J[y_2(x)] \tag{1.36}$$

hold. On integrating these with respect to x and taking account of (1.29), (1.32), (1.33), and (1.34) we obtain by Definition 1 the relations

$$d[y_3(x); \Omega, 0] = d[y_1(x); \Omega, 0] \tag{1.37}$$

and

$$d[y_3(x); \Omega, 0] = d[y_2(x); \Omega, 0], \tag{1.38}$$

hence

$$d[y_1(x);\ \Omega,\ 0] = d[y_2(x);\ \Omega,\ 0], \tag{1.39}$$

which is the desired result.

Lemma 3.[5] *Let $g(x)$ be a continuous function on a bounded closed set $F \subset E^n$. Then there exists a continuous function $h(x)$ in E^n which coincides with $g(x)$ on F.*

Proof. For $x \not\in F$ and $a \in E^n$ the function

$$\rho(x, a) = \min\left\{2 - \frac{|x - a|}{\min\limits_{y \in F} |x - y|},\ 0\right\} \tag{1.40}$$

is continuous, and we have $0 \leqq \rho(x, a) \leqq 2$. Hence, if $\{a_k\}$ is a dense sequence of points in F, then the function

$$h(x) = \begin{cases} g(x) & (x \in F) \\ \left\{\sum\limits_{k=1}^{\infty} 2^{-k}\rho(x, a_k)\right\}^{-1} \sum\limits_{k=1}^{\infty} 2^{-k}\rho(x, a_k)g(a_k) & (x \not\in F) \end{cases} \tag{1.41}$$

is a required extension of $g(x)$.

We are now in a position to define the degree for arbitrary continuous mappings:

Definition 2. *Let the mapping $y = y(x)$ be continuous for $x \in \bar{\Omega}$ and $y(x) \neq z$ for $x \in \dot{\Omega}$, where Ω is a bounded open set in E^n and z is fixed. Furthermore, let $\{y_k(x)\}$ $(k = 1, 2, \cdots)$ be a sequence of mappings which are of class C^1 for $x \in E^n$ and satisfy the relations*

$$y_k(x) \neq z \qquad (x \in \dot{\Omega}) \tag{1.42}$$

and

$$\lim_{k \to \infty} y_k(x) = y(x) \qquad (x \in \bar{\Omega}), \tag{1.43}$$

where the convergence is uniform on $\bar{\Omega}$.[6] *Then the Brouwer degree $d[y(x);\ \Omega,\ z]$ is uniquely defined by the equation*

$$d[y(x);\ \Omega,\ z] = \lim_{k \to \infty} d[y_k(x);\ \Omega,\ z]. \tag{1.44}$$

We shall now establish the principal properties of the degree $d[y(x);\ \Omega,\ z]$:

Theorem 1. *Let Ω_1 and Ω_2 be two disjoint bounded open sets in E^n. Furthermore, let the mapping $y = y(x)$ be continuous in $\bar{\Omega}_1 + \bar{\Omega}_2$ and $y(x) \neq z$ for $x \in \dot{\Omega}_1 + \dot{\Omega}_2$,*

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[5] This is a special case of the well known Tietze extension theorem (see TIETZE [18], also KELLEY [6] for further references). The proof given below is due to NAGUMO (see [12], p. 509).

[6] The existence of such a sequence follows from the preceding lemma and the Weierstrass approximation theorem.

where z is fixed in E^n. Then we have the equation

$$d[y(x); \Omega_1 + \Omega_2, z] = d[y(x); \Omega_1, z] + d[y(x); \Omega_2, z]. \tag{1.45}$$

Proof. This follows immediately from Definitions 1 and 2.

Theorem 2. *Let the mapping $y = y(x)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$ and $y(x) \neq z$ for $x \in \dot{\Omega}$, where z is fixed. Furthermore, let $d[y(x); \Omega, z] \neq 0$. Then there exists at least one point $\xi \in \Omega$ such that $y(\xi) = z$.*

Proof. Assume that $y(x) \neq z$ for $x \in \Omega$. Then from our hypotheses it follows that an inequality of the form

$$|y(x) - z| > \epsilon \tag{1.46}$$

holds for $x \in \bar{\Omega}$, where ϵ is a positive number. Consequently, if $\{y_k(x)\}$ is an approximating sequence of mappings satisfying the conditions of Definition 2, we have

$$|y_k(x) - z| > \epsilon \tag{1.47}$$

for $x \in \bar{\Omega}$ and $k \geqq k_0$. Choosing now the function $\Phi(r)$ such that the conditions (i) and (ii) of Definition 1 are satisfied, we obtain from (1.47) the equation

$$d[y_k(x); \Omega, z] = \int_\Omega \Phi(|y_k(x) - z|)J[y_k(x)]\, dx = 0 \tag{1.48}$$

for $k \geqq k_0$. Hence we conclude

$$d[y(x); \Omega, z] = \lim_{k\to\infty} d[y_k(x); \Omega, z] = 0, \tag{1.49}$$

which contradicts our hypotheses. This completes the proof of our theorem.

Theorem 3. *Let Ω be a bounded open set in E^n and I be a closed interval $a \leqq \tau \leqq b$. Furthermore, let the mapping $y = y(x, \tau)$ be continuous for $(x, \tau) \in \bar{\Omega} \times I$, and $y(x, \tau) \neq z$ for $(x, \tau) \in \dot{\Omega} \times I$, where z is fixed in E^n. Then $d[y(x, \tau); \Omega, z]$ is a constant for $\tau \in I$.*

Proof. From our hypotheses it follows that an inequality of the form

$$|y(x, \tau) - z| > 7\epsilon > 0 \tag{1.50}$$

holds for $x \in \dot{\Omega}$ and $\tau \in I$. Furthermore, we can determine a positive number $\delta = \delta(\epsilon)$ such that for $\tau_i \in I$ $(i = 1, 2)$, $|\tau_1 - \tau_2| \leqq \delta(\epsilon)$, and $x \in \bar{\Omega}$ the inequality

$$|y(x, \tau_1) - y(x, \tau_2)| < \epsilon \tag{1.51}$$

is satisfied. The numbers τ_1 and τ_2 will be kept fixed in the sequel. Let us now approximate $y(x, \tau_1)$ and $y(x, \tau_2)$ by sequences of mappings $\{y_{1k}(x)\}$ and $\{y_{2k}(x)\}$ in the manner described in Definition 2. Then from (1.50) and (1.51) it follows that we can choose an integer k_0 such that for $k \geqq k_0$ the estimates

$$|y_{ik}(x) - z| > 7\epsilon > 0 \qquad (i = 1, 2; x \in \dot{\Omega}) \tag{1.52}$$

and

$$|y_{1k}(x) - y_{2k}(x)| < \epsilon \qquad (x \in \bar{\Omega}) \tag{1.53}$$

hold. Applying now Lemma 2 we infer

$$d[y_{1k}(x); \Omega, z] = d[y_{2k}(x); \Omega, z] \qquad (k \geqq k_0), \tag{1.54}$$

hence by Definition 2

$$d[y(x, \tau_1); \Omega, z] = d[y(x, \tau_2); \Omega, z]. \tag{1.55}$$

Since this holds for any pair of numbers τ_1 , $\tau_2 \in I$ with $|\tau_1 - \tau_2| \leqq \delta(\epsilon)$, we conclude that $d[y(x, \tau); \Omega, z]$ is a constant for $\tau \in I$, which proves our theorem.

As an immediate consequence of the last theorem we get a sharpened form of Lemma 2, namely

Theorem 3′. *Let the mappings $y = y_i(x)$ $(i = 1, 2)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$. Furthermore, let the inequality*

$$|y_1(x) - y_2(x)| < |y_1(x) - z| \tag{1.56}$$

be satisfied for $x \in \dot{\Omega}$, where z is fixed in E^n. Then we have the equation

$$d[y_1(x); \Omega, z] = d[y_2(x); \Omega, z]. \tag{1.57}$$

Proof. This follows at once from Theorem 3 by putting

$$y(x, \tau) = y_1(x) + \tau(y_2(x) - y_1(x)) \qquad (0 \leqq \tau \leqq 1). \tag{1.58}$$

Theorem 3′, together with Lemma 3, permits us to define the degree for any continuous mapping defined on $\dot{\Omega}$. We shall, however, not make use of this fact. We conclude this section with a proof of BROUWER's fixed point theorem:

Theorem 3″. *Let $y = y(x)$ be a continuous mapping of the closed ball $|x| \leqq 1$ into itself. Then there exists a point ξ with $|\xi| \leqq 1$ and $y(\xi) = \xi$.*

Proof. We shall prove the assertion by a *reductio ad absurdum*. Let Ω be the open ball $|x| < 1$, and let us assume that $y(x) \neq x$ holds for $x \in \bar{\Omega}$. If we put

$$y(x, \tau) = x - \tau y(x) \qquad (x \in \bar{\Omega}; 0 \leqq \tau \leqq 1), \tag{1.59}$$

then on account of our hypotheses we have

$$|y(x, \tau)| \geqq |x| - \tau\,|y(x)| \geqq 1 - \tau > 0 \tag{1.60}$$

for $x \in \dot{\Omega}$ and $0 \leqq \tau < 1$, hence $y(x, \tau) \neq 0$ for $x \in \dot{\Omega}$ and $0 \leqq \tau \leqq 1$. Furthermore, by Theorem 3 and Definition 1 we have the equation

$$d[y(x, 1); \Omega, 0] = d[y(x, 0); \Omega, 0] = +1. \tag{1.61}$$

Applying now Theorem 2 we conclude from (1.61) that there exists a point $\xi \in \Omega$ with $\xi - y(\xi) = 0$, which is a contradiction. This completes the proof of the theorem.

2. The index of a mapping. In this section we shall evaluate the degree $d[y(x); \Omega, z]$ in terms of the zeros of the equation $y(x) = z$ (Theorem 4 and 5) and then prove that it is an integer (Theorem 6). This is done by using the well known concept of the index of a mapping. The possibility of defining the index depends on the following lemma:

Lemma 4. *Let the mapping $y = y(x)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$. Furthermore, let F be the closed set of all $x \varepsilon \bar{\Omega}$ such that $y(x) = z$, where z is fixed in E^n, and let Ω_0 be an open subset of Ω containing F. Then we have*

$$d[y(x); \Omega, z] = d[y(x); \Omega_0, z]. \tag{2.1}$$

Proof. By hypothesis we have $y(x) \neq z$ for $x \varepsilon \bar{\Omega} - \Omega_0$. Since $\bar{\Omega} - \Omega_0$ is a bounded closed set in E^n, there exists a positive number ϵ such that the estimate

$$|y(x) - z| > \epsilon \tag{2.2}$$

holds for $x \varepsilon \bar{\Omega} - \Omega_0$. Consequently, if we approximate $y(x)$ by a sequence of mappings $\{y_k(x)\}$ in the manner described in Definition 2, we have for $k \geqq k_0(\epsilon)$ and $x \varepsilon \bar{\Omega} - \Omega_0$ the inequality

$$|y_k(x) - z| > \epsilon. \tag{2.3}$$

Choosing now the function $\Phi(r)$ such that the conditions (i) and (ii) of Definition 1 are satisfied, we have

$$\begin{aligned} d[y_k(x); \Omega, z] &= \int_\Omega \Phi(|y_k(x) - z|)J[y_k(x)]\,dx \\ &= \int_{\Omega_0} \Phi(|y_k(x) - z|)J[y_k(x)]\,dx = d[y_k(x); \Omega_0, z] \end{aligned} \tag{2.4}$$

for $k \geqq k_0(\epsilon)$. If we pass to the limit $(k \to \infty)$ we obtain (2.1), which proves the lemma.

On account of this lemma the following definition is justified:

Definition 3. *Let the mapping $y = y(x)$ be continuous in the closure of the open ball $B(x_0, \rho) = \{|x - x_0| < \rho\}$ and assume that $y(x) \neq y(x_0)$ holds for $x \neq x_0$ and $x \varepsilon \bar{B}(x_0, \rho)$. Then the index $i[y(x); x_0]$ of the mapping $y = y(x)$ at the point x_0 is uniquely defined by the equation*

$$i[y(x); x_0] = d[y(x); B(x_0, \rho), y(x_0)]. \tag{2.5}$$

Now we have

Theorem 4. *Let the mapping $y = y(x)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$. Furthermore, let the equation $y(x) = z$ ($x \varepsilon \bar{\Omega}$, z fixed) have p distinct solutions $x_1, \cdots, x_p$ which belong to Ω. Then we have the representation*

$$d[y(x); \Omega, z] = \sum_{\nu=1}^{p} i[y(x); x_\nu]. \tag{2.6}$$

Proof. This is an immediate consequence of Lemma 4 and Theorem 1.

We shall now evaluate the index of a mapping in an important special case. For the sake of convenience we write $j(x)$ in place of $J[y(x)]$.

Lemma 5. *Let the mapping $y = y(x)$ be of class C^1 in a neighborhood V of the point x_0. Furthermore, let $j(x_0) \neq 0$. Then we have*

$$i[y(x); x_0] = \frac{j(x_0)}{|j(x_0)|}. \tag{2.7}$$

Proof. First of all we can choose a positive number ρ_0 such that $\bar{B}(x_0, \rho_0)$ is contained in V and for $x \in \bar{B}(x_0, \rho)$ $(0 < \rho \leqq \rho_0)$ the equation

$$y(x) - y(x_0) = A(x - x_0) + R(x) \tag{2.8}$$

holds. Here A is a linear mapping of E^n onto E^n, represented by a square matrix of order n with det $A = j(x_0) \neq 0$, and $R(x)$ is a mapping of class C^1 in V such that for $x \in \bar{B}(x_0, \rho)$ the estimate

$$|R(x)| \leqq \eta(\rho)\, |x - x_0| \tag{2.9}$$

holds with

$$\lim_{\rho \to 0} \eta(\rho) = 0. \tag{2.10}$$

By a well known theorem of linear algebra we have a representation of the form

$$A = SP, \tag{2.11}$$

where S and P are real square matrices of order n. Furthermore, S is orthogonal and P is symmetric and positive-definite. Now consider the auxiliary mapping

$$y(x, \tau) = y(x_0) + S((1 - \tau)P(x - x_0) + \tau(x - x_0)) + (1 - \tau)R(x), \tag{2.12}$$

where $x \in \bar{B}(x_0, \rho)$ and $0 \leqq \tau \leqq 1$. Obviously we have an estimate of the form

$$|(1 - \tau)Pz + \tau z| \geqq c\,|z| \tag{2.13}$$

for $z \in E^n$ and $0 \leqq \tau \leqq 1$, where c is a fixed positive constant. Hence we obtain for $x \in \bar{B}(x_0, \rho)$ and $0 \leqq \tau \leqq 1$ the inequality

$$\begin{aligned} |y(x, \tau) - y(x_0)| &\geqq |S((1 - \tau)P(x - x_0) + \tau(x - x_0))| - (1 - \tau)\,|Rx| \\ &= |(1 - \tau)P(x - x_0) + \tau(x - x_0)| - (1 - \tau)\,|Rx| \geqq (c - \eta(\rho))\,|x - x_0|. \end{aligned} \tag{2.14}$$

If ρ is chosen such that $0 < \rho \leqq \rho_0$ and $\eta(\rho) < c$, then it follows that

$$y(x, \tau) \neq y(x_0) \tag{2.15}$$

holds for $x \neq x_0$, $x \in \bar{B}(x_0, \rho)$, and $0 \leqq \tau \leqq 1$. Applying Theorem 3 to the family of mappings $y = y(x, \tau)$ $(0 \leqq \tau \leqq 1)$ we obtain

$$\begin{aligned} d[y(x); B(x_0, \rho), y(x_0)] &= d[y(x, 0); B(x_0, \rho), y(x_0)] \\ &= d[y(x, 1); B(x_0, \rho), y(x_0)] = d[S(x - x_0); B(x_0, \rho), 0]. \end{aligned} \tag{2.16}$$

Let us now choose a real-valued function $\Phi(r)$ such that the conditions (i) and (ii) of Definition 1 are satisfied with $\epsilon < \rho$. Then we have

$$
\begin{aligned}
d[S(x - x_0); B(x_0, \rho), 0] &= \int_{B(x_0,\rho)} \Phi(|S(x - x_0)|) J[S(x - x_0)]\, dx \\
&= \det S \int_{|x-x_0|<\rho} \Phi(|x - x_0|)\, dx = \det S \int_{E^n} \Phi(|x|)\, dx \qquad (2.17)\\
&= \det S = \frac{\det A}{|\det A|} = \frac{j(x_0)}{|j(x_0)|}.
\end{aligned}
$$

Combining (2.16) and (2.17) and taking account of Definition 3 we arrive at the desired equation (2.7). The lemma is thus proved.

As an immediate consequence of Theorem 4 and Lemma 4 we have

Theorem 5. *Hypotheses*: (i) *The mapping $y = y(x)$ is continuous in the closure of a bounded open set $\Omega \subset E^n$, and the equation $y(x) = z$ ($x \in \bar{\Omega}$, z fixed) has a finite number of distinct solutions $x_1, \cdots, x_N$ which belong to Ω.*

(ii) *The mapping $y = y(x)$ is of class C^1 in a vicinity of each point x_ν ($\nu = 1, \cdots, N$), and the Jacobian $j(x)$ does not vanish for $x = x_\nu$ ($\nu = 1, \cdots, N$). Furthermore, N^+ or N^- is the number of points of the set $x_1, \cdots, x_N$, where $j(x)$ is positive or negative, respectively.*

Conclusion: *We have the equation*

$$d[y(x); \Omega, z] = N^+ - N^-. \qquad (2.18)$$

If $y = y(x)$ is a suitable simplicial approximation of a given continuous mapping, we obtain Brouwer's original definition of the degree (see [4]).

We shall now show that the degree is always an integer (Theorem 6). For this purpose we need the following lemma:

Lemma 6.[7] *Let the mapping $y = y(x)$ be of class C^1 in Ω and continuous in $\bar{\Omega}$, where Ω is a bounded open set in E^n. Furthermore, let $y(x) \neq z_0$ for $x \in \dot{\Omega}$, where z_0 is fixed. Then to each positive number $\epsilon < \min_{x \in \dot{\Omega}} |y(x) - z_0|$ there exists a point $z \in E^n$ with $|z - z_0| \leqq \epsilon$ such that the following conditions are satisfied:*

(i) *The equation $y(x) = z$ ($x \in \bar{\Omega}$) has at most a finite number of solutions $x_1, \cdots, x_p$, which belong to Ω.*[8]

(ii) *We have $j(x_\nu) \neq 0$ for $\nu = 1, \cdots, p$.*

Proof. Let us denote by F the closed set of points $x \in \bar{\Omega}$, where $|y(x) - z_0| \leqq \epsilon$. Furthermore, let F_0 be the closed subset of points of F, where the Jacobian $j(x)$ vanishes. Since by hypothesis F is contained in Ω, the image set F_0^* of F_0 under the transformation $x \to y(x)$ has Jordan measure zero.[9] Consequently

[7] See LOEWNER [10], pp. 318–319, and DE RHAM [13], p. 96.

[8] If there is no solution we set $p = 0$.

[9] See SARD [14], also DE RHAM [13], pp. 10–11. These authors operate with Lebesgue measure. However, it can easily be shown that the set under consideration has Jordan measure zero.

there exists a point z in the closed ball $|z - z_0| \leqq \epsilon$ which does not belong to F_0^*. Hence $y(\xi) = z$ and $\xi \in \bar{\Omega}$ imply $\xi \in F$ and $j(\xi) \neq 0$. We shall now show that the equation $y(x) = z$ $(x \in \bar{\Omega})$ has at most a finite number of solutions. Otherwise we would have a sequence $\{x_\nu\}$ of distinct points in $\bar{\Omega}$ such that the relations

$$\lim_{\nu \to \infty} x_\nu = \xi \tag{2.19}$$

and

$$y(x_\nu) = z \qquad (\nu = 1, 2, \cdots) \tag{2.20}$$

are satisfied. From this we conclude $y(\xi) = z$ and $\xi \in \bar{\Omega}$, hence $\xi \in F$ and $j(\xi) \neq 0$, which clearly contradicts (2.19) and (2.20). The lemma is thus established.

Now we have

Theorem 6. *Let the mapping $y = y(x)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$, and $y(x) \neq z_0$ for $x \in \dot{\Omega}$, where z_0 is fixed. Then the degree $d[y(x);\, \Omega,\, z_0]$ is an integer.*

Proof. According to Definition 2 it suffices to prove the assertion under the additional assumption that $y(x)$ is of class C^1 in E^n. From Lemma 6, Theorem 2, and Theorem 5 it follows that there exists a sequence of points $z_k \in E^n$ $(k = 1, 2, \cdots)$ tending to z_0 such that $y(x) \neq z_k$ for $x \in \dot{\Omega}$ and $d[y(x);\Omega, x_k]$ is an integer. Since $d[y(x);\, \Omega,\, z]$ is a continuous function of z in a vicinity of $z = z_0$, the degree $d[y(x);\, \Omega,\, z_0]$ must also be an integer, which proves the theorem.

3. The product theorm. We shall now apply the results of §1 in order to prove a theorem, due to LERAY,[10] on the degree of the composition of two continuous mappings (Theorem 7).[11] As a preliminary step we shall first extend our definition of the degree and then prove a generalized transformation formula for multiple integrals (Lemma 7).

Definition 4. *Let the mapping $y = y(x)$ be continuous in the closure of a bounded open set $\Omega \subset E^n$. Furthermore, let $y(\dot{\Omega})$ be the image set of $\dot{\Omega}$ under the transformation $x \rightarrow y(x)$, and let D be a component*[12] *of the open set $E^n - y(\dot{\Omega})$. Then the degree $d[y(x);\, \Omega,\, D]$ is defined by the equation*

$$d[y(x);\, \Omega,\, D] = d[y(x);\, \Omega, z], \tag{3.1}$$

where z is an arbitrary point in D.

This definition is justified by Theorem 3, since we have $y(x) - z \neq 0$ for $x \in \dot{\Omega}$ and $z \in D$, and $d[y(x);\, \Omega,\, z] = d[y(x) - z;\, \Omega,\, 0]$.

Lemma 7. *Hypotheses*: (i) *The mapping $y = y(x)$ is continuous in the closure*

[10] See [8] and [9], also NAGUMO [11], [12] and BERS [2].

[11] The possibility of deducing this result from Definition 1 of this paper was suggested to me by H. L. ROYDEN.

[12] *I. e.* a maximal connected open subset.

of a bounded open set $\Omega \subset E^n$, and $\{y_k(x)\}$ is a sequence of mappings which belong to C^1 for $x \in E^n$ and converge uniformly to $y(x)$ on $\bar{\Omega}$. Furthermore, D_i $(i = 1, 2, \cdots)$ are the components of the open set $E^n - y(\dot{\Omega})$, where $y(\dot{\Omega})$ is the image of $\dot{\Omega}$ with respect to the mapping $y = y(x)$.

(ii) *$f(y)$ is a real-valued continuous function in E^n, which vanishes identically in a vicinity of $y(\dot{\Omega})$ and for $|y| \geqq R$, where R is a positive number.*

Conclusion: There exists a positive integer k^* such that for $k \geqq k^*$ the functions $f(y_k(x))$ vanish in a neighborhood of $\dot{\Omega}$, and we have the equation

$$\int_\Omega f(y_k(x))J[y_k(x)]\,dx = \sum_i d[y(x);\,\Omega,\,D_i]\int_{D_i} f(z)\,dz. \tag{3.2}$$ [13]

Proof. Let C_f be the closure of the set of all points in E^n, where $f(y) \neq 0$. Then, since $y(\dot{\Omega})$ and C_f are bounded, closed, and disjoint, we have an inequality of the form

$$|y(x) - z| > \epsilon_0 > 0 \quad \text{for} \quad x \in \dot{\Omega} \quad \text{and} \quad z \in C_f\,, \tag{3.3}$$

hence

$$|y_k(x) - z| > \epsilon_0 > 0 \quad \text{for} \quad x \in \dot{\Omega}, \quad z \in C_f\,, \quad \text{and} \quad k \geqq k_0\,. \tag{3.4}$$

Now let a non-negative function $\Phi_\epsilon(r)$ be chosen such that the conditions (i) and (ii) of Definition 1 are satisfied (with $0 < \epsilon < \epsilon_0$). Then from (3.4) it follows that we have for $z \in C_f$ and $k \geqq k_0$ the equation

$$d[y_k(x);\,\Omega,\,z] = \int_\Omega \Phi_\epsilon(|y_k(x) - z|)J[y_k(x)]\,dx. \tag{3.5}$$

If we now apply Theorem 3′ we infer from (3.4) and (3.5) that there exists an integer $k^* \geqq k_0$ such that for $z \in C_f$, $k \geqq k^*$, and $0 < \epsilon < \epsilon_0$ the equation

$$d[y(x);\,\Omega,\,z] = \int_\Omega \Phi_\epsilon(|y_k(x) - z|)J[y_k(x)]\,dx \tag{3.6}$$

holds. Now let $\theta(x)$ be the characteristic function of Ω; *i.e.* $\theta(x) = 1$ for $x \in \Omega$ and $\theta(x) = 0$ for $x \notin \Omega$, and let us define $d[y(x);\,\Omega,\,z]$ for $z \in y(\dot{\Omega})$ by the equation

$$d[y(x);\,\Omega,\,z] = 0. \tag{3.7}$$

Then from (3.6) we obtain

$$d[y(x);\,\Omega,\,z]f(z) = \int_{E^n} f(z)\Phi_\epsilon(|y_k(x) - z|)\,\theta(x)J[y_k(x)]\,dx \tag{3.8}$$

for $z \in E^n$ and $k \geqq k^*$. Since the integrand on the right-hand side is continuous for $(x, z) \in E^n \times E^n$, we infer

$$\int_{E^n} d[y(x);\,\Omega,\,z]f(z)\,dz = \int_{E^n} f_\epsilon(y_k(x))\,\theta(x)J[y_k(x)]\,dx \tag{3.9}$$

[13] In case there are denumerably many components D_i the sum on the right-hand side of this equation consists only of a finite number of non-vanishing terms, since we evidently have $f(z) \equiv 0$ for $z \in D_k$, if k exceeds a certain number.

for $k \geqq k^*$ and $0 < \epsilon < \epsilon_0$, where $f_\epsilon(y)$ is defined by the equation

$$f_\epsilon(y) = \int_{E^n} \Phi_\epsilon(|y - z|)f(z)\,dz \qquad (y \in E^n). \tag{3.10}$$

On account of $\Phi_\epsilon(r) \geqq 0$ $(0 \leqq r < \infty)$ and conditions (i) and (ii) of Definition 1 we have

$$\begin{aligned} |f_\epsilon(y) - f(y)| &= \left|\int_{E^n} \Phi_\epsilon(|y - z|)(f(z) - f(y))\,dz\right| \\ &\leqq \int_{E^n} \Phi_\epsilon(|y - z|)\,|f(z) - f(y)|\,dz \\ &\leqq \operatorname*{l.u.b.}_{|z-y|\leqq\epsilon} |f(z) - f(y)| \cdot \int_{E^n} \Phi_\epsilon(|z - y|)\,dz \\ &= \operatorname*{l.u.b.}_{|z-y|\leqq\epsilon} |f(z) - f(y)| \cdot \int_{E^n} \Phi_\epsilon(|z|)\,dz \leqq \eta(\epsilon) \end{aligned} \tag{3.11}$$

uniformly for $y \in E^n$, where $\lim_{\epsilon\to 0} \eta(\epsilon) = 0$. Since by (3.4) the functions $f(y_k(x))$ $(k \geqq k^*)$ vanish in a vicinity of $\dot\Omega$, we obtain

$$\lim_{\epsilon\to 0} \int_{E^n} f_\epsilon(y_k(x))\theta(x)J[y_k(x)]\,dx = \int_\Omega f(y_k(x))J[y_k(x)]\,dx \tag{3.12}$$

for $k \geqq k^*$. Combining now (3.9) with (3.12) we conclude

$$\begin{aligned} \int_\Omega f(y_k(x))J[y_k(x)]\,dx &= \int_{E^n} d[y(x);\,\Omega, z]f(z)\,dz \\ &= \sum_i \int_{D_i} d[y(x);\,\Omega, z]f(z)\,dz = \sum_i d[y(x);\,\Omega, D_i] \int_{D_i} f(z)\,dz, \end{aligned} \tag{3.13}$$

which proves our lemma.

Now we have

Theorem 7. *Hypotheses*: (i) *The mapping $y = y(x)$ is continuous in the closure of a bounded open set Ω. Furthermore, $y(\Omega)$, $y(\dot\Omega)$, and $y(\bar\Omega)$ are the images of Ω, $\dot\Omega$, and $\bar\Omega$ under the transformation $x \to y(x)$, respectively, and D_i $(i = 1, 2, \cdots)$ are the components of the open set $E^n - y(\dot\Omega)$.*

(ii) *The mapping $z = z(y)$ is continuous in $y(\bar\Omega)$, and we have $z(y) \neq u_0$ for $y \in y(\dot\Omega)$, where u_0 is fixed in E^n. Furthermore, we set $u(x) = z(y(x))$.*

Conclusion: *We have the equation*

$$d[u(x);\,\Omega, u_0] = \sum_{D_i \subset y(\Omega)} d[y(x);\,\Omega, D_i]\, d[z(y);\,D_i\,, u_0].\text{[14]} \tag{3.14}$$

Proof. Let us approximate $y(x)$ and $z(y)$ by sequences of mappings $\{y_k(x)\}$

[14] If no component D_i belongs to $y(\Omega)$, the right-hand side of this equation is to be interpreted as zero. If there are infinitely many such components, we have $d[z(y);\,D_i\,, u_0] = 0$ for $i \geqq i_0$, where i_0 is a finite positive integer.

and $\{z_k(y)\}$ belonging to C^1 for $x \varepsilon E^n$ such that the relations

$$z_k(y) \to z(y) \qquad (k \to \infty) \tag{3.15}$$

and

$$y_k(x) \to y(x) \qquad (k \to \infty) \tag{3.16}$$

hold uniformly for $y \varepsilon y(\bar{\Omega})$ and $x \varepsilon \bar{\Omega}$, respectively. If we put

$$u_k(x) = z_k(y(x)) \qquad (k = 1, 2, \cdots ; x \varepsilon \bar{\Omega}) \tag{3.17}$$

and

$$u_{kl}(x) = z_k(y_l(x)) \qquad (k, l = 1, 2, \cdots ; x \varepsilon E^n) \tag{3.18}$$

we have

$$\lim_{k\to} u_k(x) = u(x) \tag{3.19}$$

and

$$\lim_{l\to\infty} u_{kl}(x) = u_k(x) \tag{3.20}$$

uniformly in $\bar{\Omega}$. Furthermore, on account of our hypotheses, we have an estimate of the form

$$|z(y) - u_0| > \epsilon > 0 \tag{3.21}$$

for $y \varepsilon y(\dot{\Omega})$. Hence, by (3.15), the inequality

$$|z_k(y) - u_0| > \epsilon > 0 \tag{3.22}$$

holds for $y \varepsilon y(\dot{\Omega})$ and $k \geqq k_0$. (3.21) and (3.22) are equivalent to

$$|u(x) - u_0| > \epsilon > 0 \qquad (x \varepsilon \dot{\Omega}) \tag{3.23}$$

and

$$|u_k(x) - u_0| > \epsilon > 0 \qquad (x \varepsilon \dot{\Omega}; k \geqq k_0). \tag{3.24}$$

Now let $g(u)$ be a real-valued, continuous function in E^n with $g(u) \equiv 0$ for $|u - u_0| \geqq \epsilon$, and $\int_{E^n} g(u)\, du = 1$. Furthermore, let a positive number R be chosen such that the estimates $|y_l(x)| < R$ hold for $x \varepsilon \bar{\Omega}$ and $l = 1, 2, \cdots$, and let $\Lambda(y)$ be a real-valued continuous function in E^n with $\Lambda(y) = 1$ for $|y| \leqq R$ and $\Lambda(y) = 0$ for $|y| \geqq R + 1$. Then from Lemma 7 and Theorem 3′ it follows that to every integer $k \geqq k_0$ there exists a positive number $l_0(k)$ such that the equation

$$d[u_k(x); \Omega, u_0] = \int_\Omega g(u_{kl}(x))J[u_{kl}(x)]\, dx = \int_\Omega h_k(y_l(x))J[y_l(x)]\, dx \tag{3.25}$$

is satisfied for $l \geqq l_0(k)$, where the function $h_k(y)$ is defined by the expression

$$h_k(y) = g(z_k(y))J[z_k(y)]\Lambda(y). \tag{3.26}$$

Since $h_k(y)$ vanishes in a neighborhood of the set $y(\dot{\Omega})$ and for $|y| \geqq R + 1$, we conclude from Lemma 7 that there exists an integer $l_1(k) \geqq l_0(k)$ such that for $l \geqq l_1(k)$ and $k \geqq k_0$ the function $h_k(y_l(x))$ vanishes in a vicinity of $\dot{\Omega}$, and we have the representation

$$\begin{aligned}\int_\Omega h_k(y_l(x))J[y_l(x)]\,dx &= \sum_i d[y(x);\,\Omega,\,D_i]\int_{D_i} h_k(y)\,dy\\ &= \sum_{D_i\subset y(\Omega)} d[y(x);\,\Omega,\,D_i]\int_{D_i} g(z_k(y))J[z_k(y)]\,dy \\ &= \sum_{D_i\subset y(\Omega)} d[y(x);\,\Omega,\,D_i]\,d[z_k(y);\,D_i\,,\,u_0],\end{aligned}\tag{3.27}$$

where the right-hand side of (3.27) is zero if no component D_i belongs to $y(\Omega)$. If we now apply Theorem 3′ to the sequences $\{u_k(x)\}$ and $\{z_k(y)\}$, and remember that $\dot{D}_i$ belongs to $y(\dot{\Omega})$, we conclude that there exists an integer k_1 with $k_1 \geqq k$ such that for $k \geqq k_1$ the equations

$$d[u_k(x);\,\Omega,\,u_0] = d[u(x);\,\Omega,\,u_0]\tag{3.28}$$

and

$$d[z_k(y);\,D_i\,,\,u_0] = d[z(y);\,D_i\,,\,u_0]\tag{3.29}$$

hold, where D_i is any component of $E^n - y(\dot{\Omega})$ which belongs to $y(\Omega)$. Combining (3.25), (3.27), (3.28), and (3.29), we arrive at the desired formula (3.14). The theorem is thus established.

Concluding remarks. As LERAY has shown, the product theorem for the degree can be generalized to infinite dimensional function spaces and yields short and elegant proofs of some fundamental propositions of topology (invariance of domain, Jordan-Brouwer theorem). We shall not consider these applications here, but refer to the papers of LERAY, BERS, and NAGUMO quoted above.

Added in proof. After this paper was submitted, the book *Linear Operators, Part* I: *General Theory* (Interscience Publishers, New York, 1958) by N. DUNFORD & J. T. SCHWARTZ appeared, which contains a simple analytic proof of the Brouwer fixed point theorem on pp. 467–470. This proof has some points in common with that given in the present paper.

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