Lab 11: MATLAB Integration Routines & Gauss Quadrature

Math 3341: Introduction to Scientific Computing Lab — Spring 2018

Built-in Integration Functions

A few notes on the differences between functions listed in Table 1.

- Use `trapz` and `cumtrapz` to perform numerical integrations on discrete data sets.
- `trapz` reduces the size of the dimension it's operating on to 1 and returns only the final integration value. In other words, it takes a vector of some length and returns a single value.
- `cumtrapz` returns the intermediate integration values, preserving the size of the dimension it operates on. In other words, it takes a vector of some length and returns a vector of the same length.
- Use `integral, integral2, integral3` if a functional expression for the data is available.

Table 1

Command	Description
`polyint(p,k)`	returns the integral of the polynomial represented by the coefficients in `p` using a constant of integration `k`. Using `polyint(p)` assumes `k=0`.
`trapz(X,Y)`	integrates `Y` with spacing increment `X`
`cumtrapz(X,Y)`	computes the cumulative integral of `Y` with respect to `X` using trapezoidal integration.
`integral(f,xmin,xmax)`	numerically integrates function f from $x_{\min} \leq x \leq x_{\max}$ using global adaptive quadrature and default error tolerances
`integral2(f," ",ymin,ymax)`	approximates the integral of the function $z = f(x, y)$ over the planar region $x_{\min} \leq x \leq x_{\max}$ and $y_{\min} \leq y \leq y_{\max}$
`integral3(f," ", zmin,zmax)`	approximates the integral of the function $f(x, y, z)$ over the planar region $x_{\min} \leq x \leq x_{\max}$, $y_{\min} \leq y \leq y_{\max}$, and $z_{\min} \leq z \leq z_{\max}$

If searching for help online you may encounter references to `quad, dblquad, triplequad`, etc. These refer to old ML functions and will be removed in future versions. They still work for the most recent versions of ML, however it is advised to `integral` and its variations. Always search the ML documentation first before using functions you encounter when using references from online sources or from older textbooks and materials.

Gauss Quadrature

Integration of $f(x)$ on the interval $[-1,1]$ using Gauss Quadrature is given by

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i) \tag{1}$$

where w_i and x_i are chosen so the integration rule is exact for the largest class of polynomials. In this case, we will choose the x_i and w_i to be the roots and coefficients of the Legendre polynomial. These values are given in the table to the right.

It is worth noting that if $P(x)$ is a polynomial of degree less than or equal to $2n$ then Gauss Quadrature will produce an exact result. In other words,

$$\int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

Table 2

n	**Roots** x_i	**Coefficients** w_i
1	0.00000000	2.00000000
2	0.57735027	1.00000000
	-0.57735027	1.00000000
3	0.77459667	0.55555556
	0.00000000	0.88888889
	-0.77459667	0.55555556
4	0.33998104	0.65214515
	0.86113631	0.34785485
	-0.33998104	0.65214515
	-0.86113631	0.34785485
5	0.90617985	0.23692689
	0.53846931	0.47862867
	0.00000000	0.56888889
	-0.53846931	0.47862867
	-0.90617985	0.23692689
6	0.93246951	0.17132449
	0.66120939	0.36076157
	0.23861919	0.46791393
	-0.23861919	0.46791393
	-0.66120939	0.36076157
	-0.93246951	0.17132449

To approximate the integral on the general interval $[a,b]$ we need to use the following change of variables:

$$t = \frac{b-a}{2}x + \frac{b+a}{2}, \quad -1 \le x \le 1 \qquad \Longrightarrow \qquad dt = \frac{b-a}{2}\,dx$$

and so the Gauss Quadrature on a general interval $[a,b]$ is given by

$$\int_a^b f(t)\,dt = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}\,dx \approx \sum_{i=1}^{n} w_i \cdot f\left(\frac{b-a}{2}x_i + \frac{b+a}{2}\right)\cdot\frac{b-a}{2} \tag{2}$$

Before beginning the lab

Download the files in `lab11files.m`. This contains the script file for this lab, the file to write your function for Gauss quadrature, and a text file containing the values given in Table 2 for easy copy/pasting. You will store all of your function calls for this lab in the provided scripts. Record your output using the `diary` function for each script file.

Lab Exercises

I. Built-in Integration Functions

1. Use both `polyint` and `integral` to evaluate $\int_{-1}^{3} (x^2 - 2x + 1)\, dx$
2. Evaluate the previous integral again, now using `trapz` and `cumtrapz`.
3. Use `integral2` to evaluate $\int_{-\pi}^{-\frac{3\pi}{2}} \int_{0}^{2\pi} (y\sin(x) + x\cos(y))\, dy\, dx$.
4. Use `integral3` to evaluate $\int_0^1 \int_{x^2}^{x} \int_{x-y}^{x+y} y\, dz\, dy\, dx$.

II. Gauss Quadrature

1. Write a function named `gaussQuad.m` which computes Gauss quadrature for $n = 1, 2, 3, 4, 5$ as defined in (1). A template of this file is included in **`lab11files.zip`**. Keep in mind that there are several different ways in which you could write this function in depending on how you decide to store and call the necessary x_i and w_i. Values given in the Table 2 are found in the file **`legendrePolyData.txt`** for easy copy & pasting.
2. Use this function to evaluate the integral

$$\int_1^{1.6} \frac{2x}{x^2 - 4}\, dx$$

   for $n = 2, 3, 4, 5$. For efficiency, it is recommended to do this evaluation inside of a for loop.

   Note: This integral is on an interval that is NOT the interval $[-1, 1]$ so we cannot use our `quassQuad.m` function directly. Instead, we must use the change of variables given in (2) to re-define our integral so that it can be evaluated on the interval $[-1, 1]$. There are two main approaches you could take to handle this

   - Recall that Matlab allows us to perform function compositions with anonymous functions. In other words, in our script file before using our function, we can define
     ```
     gCOV = @(t)((b−a)/2).* g( ( (b−a).*t + b + a )/2 )
     ```
     This will calculate the integral properly.
   - Alternatively, you can also choose to re-define `gaussQuad.m` to calculate the change of variables directly as it is defined in on the right hand side of (2). If you take this approach your function will also require the inputs `a` and `b` in addition to `f` and `N`.

Include all of your script files, function file and output files in the provided LaTeX template.