Lab 06: LU Decomposition

Math 3341: Introduction to Scientific Computing Lab   Spring 2018

The LU Decomposition

Consider the system of equations

$$
\begin{array}{rcrcrcrcr}
10x_1 & - & x_2 & + & 2x_2 & & & = & 6 \\
-x_1 & + & 11x_2 & - & x_3 & + & 3x_4 & = & 25 \\
2x_1 & - & x_2 & + & 10x_3 & - & x_4 & = & -11 \\
& & 3x_2 & - & x_3 & + & 8x_4 & = & 15
\end{array}
$$

In matrix form we have the equation $A\mathbf{x} = \mathbf{b}$

$$
\underbrace{\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}}_{\mathbf{b}} \tag{1}
$$

The LU decomposition allows us to factor the matrix A into two matrices, a lower triangular matrix L and an upper triangular matrix U.

$$
A = LU = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{41} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}.
$$

Upper and lower triangular systems are easy to solve using forward or back substitution algorithms. To solve the matrix equation $A\mathbf{x} = \mathbf{b}$ we perform the following:

1. Solve the system
$$
L\mathbf{z} = \mathbf{b} \tag{2}
$$

   for the vector $\mathbf{z}$. In Matlab, use `z = L\b`. Then

2. Solve the system
$$
U\mathbf{x} = \mathbf{z} \tag{3}
$$

   for the vector $\mathbf{x}$. In Matlab, use `x = U\z`

this vector $\mathbf{x}$ will be the solution to our system.

Commands for Solving Systems

The following is a list of commands used in this lab along with a brief description of what they do.

Command	Description
`lu(A)`	finds the LU decomposition of a matrix A
`x = A\b`	solves the equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$
`x(i)`	calls the ith element of a vector $\mathbf{x}$
`A(i,j)`	calls the ijth element of a matrix A

Lab Exercises

I. Solving a System with LU Decomposition

1. Open the script file `lab06script01.m`. This file defines the matrix A and the vector b given in (1).

2. Calculate the LU decomposition of the matrix `A`.

3. Solve the system given in (2) using the backslash operator.

4. Then solve the system given in (3).

5. Check your solution by calculating `A*x`.

II. Varying the Vector b

Suppose we want to solve solve the system for each integer value of m in between $m = 0$ and $m = 20$. This time use the LU-decomposition of the system matrix; perform the decomposition only once and use the lower and upper triangular factors repeatedly to find each successive solution. Then generate a plot of the second component of the solution, y, versus the integer m.

$$\begin{cases} 3x + y + z = m \\ x - 5y + 2z = 5 \\ 2x + y + 5z = 10 \end{cases} \tag{4}$$

To do this you'll follow the steps below.

1. Open the script file `lab06script02.m`. This file defines the matrix A, the vector b given in (4) as well as some additional items.

2. Calculate the LU decomposition of the matrix `A` where indicated.

3. Type following code for a for loop which will solve the system on each loop and store the output of the second element of your resulting solution vector `x` in a vector `y`.

```
for k = 1:length(m)
    b = [m(k); 5; 10];
    z = L\b;
    x = U\z;
    fprintf('%2d | % 5.4f | % 5.4f | % 5.4f |\n',m(k),x(1),x(2),x(3))
    y(k) = x(2);
end
```

4. The basic plot is included in your code. Remember to display the plot title, axes and legend.

III. Writing Better Solutions with LaTeX

For this section, you must go to your LaTeX document for Homework 1. Copy and paste your solution to Problem 1 in the appropriate section of the Lab Report. Edit and/or rewrite portions of your solution so that it is not only formatted nicely in LaTeX, but also well written. Ensure that it follows the requirements outlined in the `Homework Requirements.pdf`. Review your feedback from your graded Homework 1 if you need to see what needs to be improved.

Ask questions regarding elements you are not sure about. Be aware that you may not be able to finish this task completely during the lab session. Work on it outside of class. The due date will be extended to accommodate the fact that you have a current homework assignment. Note the date as given by WyoCourses.