Lab 12: Romberg Integration

Math 3341: Introduction to Scientific Computing Lab — Spring 2018

In this lab you will write a function which performs the Romberg Integration procedure.

Romberg Integration

Composite Trapezoidal rule for approximating the integral of a function $f(x)$ on an interval $[a, b]$ using m subintervals is

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{m-1} f(x_j)\right] - \frac{(b-a)}{12}h^2 f''(\xi)$$

where $a < \xi < b$ and $h = (b-a)/m$ and $x_j = a + jh$ for each $j = 0, 1, \ldots, m$.

Finding approximations for $m_1 = 1$, $m_2 = 2\, m_3 = 4$, $\ldots$, $m_n = 2^{n-1}$ for $n \in \mathbb{N}$. The corresponding step size h_k for each m_k is then given by $h_k = \frac{b-a}{m_k} = \frac{b-a}{2^{k-1}}$. The trapezoidal rule then becomes

$$\int_a^b f(x)\,dx = \frac{h_k}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{2^{k-1}-1} f(a + ih_k)\right] - \frac{(b-a)}{12}h^2 f''(\xi_k)$$

where $\xi_k \in (a, b)$. Here we'll use the notation $R_{k,1}$ to denote the portion used for the trapezoidal approximation. In other words,

$$R_{1,1} = \frac{h_1}{2}[f(a) + f(b)] = \frac{b-a}{2}[f(a) + f(b)]$$
$$R_{2,1} = \frac{h_2}{2}[f(a) + f(b) + 2f(a + h_2)] = \frac{b-a}{2}\left[f(a) + f(b) + 2f\left(a + \frac{b-a}{2}\right)\right] = \frac{1}{2}[R_{1,1} + h_1 f(a + h_2)]$$
$$R_{3,1} = \frac{1}{2}[R_{2,1} + h_1[f(a + h_3) + f(a + 3h_3)]]$$

This leads to the Trapezoidal rule in the general form

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k_1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right] \quad \text{for } k = 2, 3, \ldots, n$$

This method converges very slowly on its own. A technique called Richardson's Extrapolation is applied to speed convergence. Essentially, this performs a method of averaging previously calculated entries to obtain the next entry in the table. This is given in general form

$$R_{k,j} = R_{k,j} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$$

This method will give us the following entries of R in a tabular format. The number of rows is determined by the value n that we desire.

$R_{1,1}$					
$R_{2,1}$	$R_{2,2}$				
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\ldots$	$R_{n,n}$

Algorithm 1: Romberg Integration

Approximates the integral $I = \int_a^b f(x)\,dx$, select an integer $n > 0$.

INPUT: $f(x)$, integer n, endpoints a, b.
OUTPUT: Array R *(Compute R by rows; only last 2 rows are stored)*

Set $h = b - a$

$R_{1,1} = \frac{h}{2}\left(f(a) + f(b)\right)$

OUTPUT $(R_{1,1})$.

for $i = 2, \ldots, n$ **do**

  Set $R_{2,1} = \dfrac{1}{2}\left[R_{1,1} + h\displaystyle\sum_{k=1}^{2^{i-2}} f(a + (k - 0.5)h)\right]$ ▷ Trapezoidal Rule

  for $j = 2, \ldots, i$ **do**

    Set $R_{2,j} = R_{2,j-1} + \dfrac{R_{2,,j-1} - R_{1,j-1}}{4^{j-1} - 1}$ ▷ Richardson Extrapolation

    OUTPUT ($R_{2,j}$ for $j = 1, 2, \ldots, i$)

  end

  Set $h = \frac{h}{2}$.

  for $j = 1, 2, \ldots, i$ **do**

    Set $R_{1,j} = R_{2,j}$ ▷ Update Row 1 of R

  end

end

Lab Exercises

1. Download the files in `lab12files.m`. This contains the script file for this lab and the function file to write your function for Romberg integration.
2. The pseudocode for Romberg integration is above. Use this to write a version in Matlab in the provided function file `romberg.m`.
3. Run the script file `lab12.m` to verify that your function is working. Create a diary to store this output.
4. Submit your function, script file, and output file in the provided LaTeX template