1. Find the maximum and minimum values of

$$h(\theta) = 3\sin(\theta), \qquad 0 \le \theta \le 2\pi$$

Solution:

First, we need the derivative:

$$h'(\theta) = 3\cos(\theta)$$

Critical points are

$$h'(\theta) = 0 \implies 3\cos(\theta) = 0$$
$$\implies \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Since θ is on a closed interval $[0, 2\pi]$ to find max & min values we test compare values of the function @ critical pts & @ interval endpoints

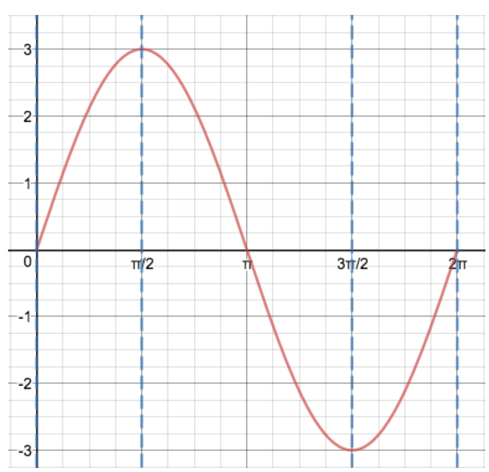
$$h(0) = 3\sin(0) = 0$$
$$h\left(\tfrac{\pi}{2}\right) = 3\sin\left(\tfrac{\pi}{2}\right) = 3(1) = 3 \implies \text{Absolute max}$$
$$h\left(\tfrac{3\pi}{2}\right) = 3\sin\left(\tfrac{3\pi}{2}\right) = 3(-1) = -3 \implies \text{Absolute min}$$
$$h(2\pi) = 3\sin(2\pi) = 3(0) = 0$$

so

Absolute Max of function is $h\left(\frac{\pi}{2}\right) = 3$

Absolute Min of function is $h\left(\frac{3\pi}{2}\right) = -3$

These results can be confirmed by looking at the graph



2. The population p of a species at time t is given by the equation

$$p(t) = At^3(B - t), \qquad 0 \le t < 2B$$

where A and B are positive constants. At what time will the population reach its maximum?

Solution: First, we need the derivative:

$$p'(t) = \frac{d}{dx}[f \cdot g] = f'g + fg'$$

$$f = At^3 \qquad g = B - t$$

$$f' = 3At^2 \qquad g' = -1$$

$$\Rightarrow p'(t) = 3At^2(B-t) + At^3(-1)$$

$$= 3At^2(B-t) - At^3$$

$$= 3ABt^2 - 3At^3 - At^3$$

$$= 3ABt^2 - 4At^3$$

$$= 3At^2\left(B - \frac{4}{3}t\right)$$

Critical points:

$$3At^2(B - 4At) = 0$$

$$\Rightarrow 3At^2 = 0 \quad \Rightarrow \quad t = 0$$

$$B - \frac{4}{3}t = 0 \quad \Rightarrow \quad t = \frac{3}{4}B$$

We are not on a closed interval so we must use a derivative test to confirm max/min values.

By 1st Derivative test:

Interval	$(0, \frac{3}{4}B)$	$(\frac{3}{4}B, 2B)$
test value	$\frac{1}{2}B$	B
Value of $f'(x)$	$\frac{AB^3}{4}$	$-AB^3$
Sign of $f'(x)$	$+$	$-$
Behavior	Increasing	Decreasing

Since $p(t)$ goes from increasing to Decreasing @ $t = \frac{3}{4}B$ then this is when the population will meet its maximum value.

By 2nd Derivative test:

Note that $p''(t) = 6At(B - 2t)$

Since $p''\left(\frac{3}{4}B\right) = 6A\left(\frac{3}{4}B\right)\left(B - 2\left(\frac{3}{4}B\right)\right) = -\frac{9}{4}AB^2 < 0$

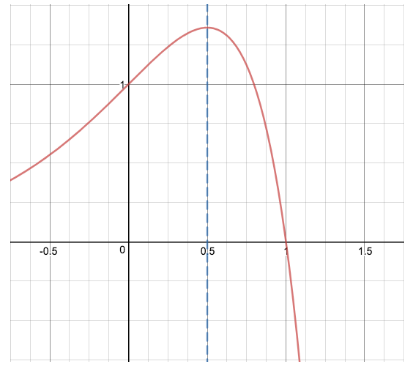
there is a maximum value @ $t = \frac{3}{4}$

For problems 3 to 4: answer the following for each of the given functions.

3. $f(x) = (1 - x)e^{2x}$

(a) State the first and second derivative of the function. (You do not need to show your work.)

Solution:



$$f'(x) = 2e^{2x}(1-x) - e^{2x}$$

$$f''(x) = 4e^{2x}(1-x) - 4e^{2x}$$

(b) Find the critical points.

Solution:

$f'(x)$ exists every where

$\Rightarrow$ Only C.P's are when $f'(x) = 0$

$$\Rightarrow 2e^{2x}(1-x) - e^{2x} = 0$$

$$\Rightarrow e^{2x}(2(1-x) - 1) = 0$$

$$\Rightarrow e^{2x}(1 - 2x) = 0 \quad \Rightarrow x = \frac{1}{2}$$

Critical point is $x = \frac{1}{2}$

(c) Find the interval(s) of increase and/or decrease.

Solution:

Interval	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, \infty)$		
Test value	0	1		
Value of $f'(x)$	1	$-e^{-2}$		
Sign of $f'(x)$	$+$	$-$		
Behavior	increasing	Decreasing		

$f(x)$ is increasing on $(-\infty, \frac{1}{2})$

$f(x)$ is dereasing on $(\frac{1}{2}, \infty)$

3. (Continued)

(d) Find the interval(s) where the function is concave up and/or concave down.

Solution: $f''(x)$ exists everywhere

$f''(x)=0$: $e^{-2x}(4(1-x)-4)=0$ since $e^{-2x}\neq 0$:

$\Rightarrow e^{-2x}(-4x)=0 \Longrightarrow -4x=0 \Rightarrow x=0$

Interval	$(-\infty,0)$	$(0,\infty)$		
Test value	-1	1		
Value of $f''(x)$	$4/e^2$	$-4e^2$		
Sign of $f''(x)$	$+$	$-$		
Behavior	Concave up $\cup$	Concave down $\cap$		

$f(x)$ is concave up on $(-\infty,0)$

$f(x)$ is concave down on $(0,\infty)$

(e) Determine the point(s) of inflection (if any).

Solution:

Since $f(x)$ changes concavity @ $x=0$ then there is an inflection pt @ $x=0$.

(f) Use an appropriate *derivative test* to determine any maximum or minimum values. This means you must clearly indicate which test you are applying by name.

Solution:

<ins>Using 1st Derivative Test:</ins>

Since $f(x)$ goes from increasing to Decreasing $\Longrightarrow$ @ $x=\frac{1}{2}$ is a relative max.

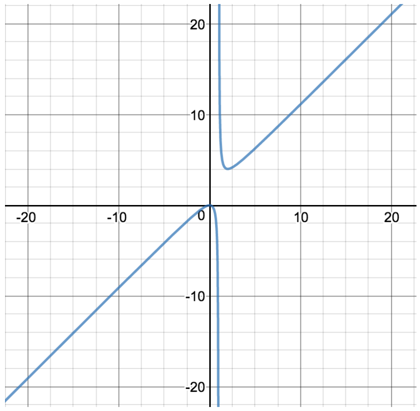
<ins>Using 2nd Derivative Test:</ins>

$$f''\left(\tfrac{1}{2}\right)=4e^{2(\frac{1}{2})}\left(1-\tfrac{1}{2}\right)-4e^{2(\frac{1}{2})}=-2e$$

Since $f''\left(\frac{1}{2}\right)<0$ then there is a relative max @ $x=\frac{1}{2}$

4. $g(x) = \dfrac{x^2}{x-1}$

(a) State the first and second derivative of the function. (You do not need to show your work.)



Solution:

$$g'(x) = \frac{x(x-2)}{(x-1)^2}$$

$$g''(x) = \frac{2}{(x-1)^3}$$

(b) Find the critical points.

Solution:

$g'(x) = 0$ when $x(x-2) = 0$

$\Rightarrow x = 0,\ x = 2$

$g'(x)$ DNE when $x - 1 = 0$

$\Rightarrow x = 1$

Critical points are $x = 0, 1, 2$

(c) Find the interval(s) of increase and/or decrease.

Solution:

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
Test value	-1	$\frac{1}{2}$	$\frac{3}{2}$	3
Value of $g'(x)$				
Sign of $g'(x)$	$+$	$-$	$-$	$+$
Behavior	Increasing	Decreasing	Decreasing	Increasing

$g(x)$ is increasing on $(-\infty, 0)$ & $(2, \infty)$

$g(x)$ is decreasing on $(0, 1)$ & $(1, 2)$

4. (Continued)

(d) Find the interval(s) where the function is concave up and/or concave down.

Solution:

$g''(x) = 0$ nowhere $\quad g''(x)$ DNE @ $x = 1$

Interval	$(-\infty, 1)$	$(1, \infty)$		
Test value	0	2		
Value of $g''(x)$				
Sign of $g''(x)$	$-$	$+$		
Behavior	Concave Down $\cap$	Concave up $\cup$		

$g(x)$ is concave up on $(1, \infty)$

$g(x)$ is concave down on $(-\infty, 1)$

(e) Determine the point(s) of inflection (if any).

Solution:

$g(x)$ changes concavity @ $x = 1$ but $g(x)$ is not defined there

$\Rightarrow$ no inflection points

(f) Use an appropriate *derivative test* to determine any maximum or minimum values. This means you must clearly indicate which test you are applying by name.

Solution:

Using 1^{st} Derivative Test:

@ $x = 0$ $g(x)$ goes from increasing to Decreasing $\Rightarrow$ Relative max

@ $x = 2$ $g(x)$ goes from decreasing to Increasing $\Rightarrow$ Relative min

Using 2^{nd} Derivative Test:

$g''(0) = \frac{2}{(0-1)^3} = -2 < 0 \Longrightarrow$ Relative max.

$g''(2) = \frac{2}{(2-1)^3} = 2 > 0 \Longrightarrow$ Relative min.