Rates of Change

1. A particle moves on a vertical line so that its coordinate at time t is $s(t) = t^3 - 12t + 3$, $t \geq 0$.
   (a) Find the velocity and acceleration functions.
   Solution:

$$v(t) = s'(t) = 3t^2 - 12$$

$$a(t) = v'(t) = s''(t) = 6t$$

   (b) When is the particle moving upward and downward?
   Solution:

Direction based on sign of $v(t)$.

Note: $v(t) = 0$ (i.e. when sign possibly changes.

$$\Rightarrow 3t^2 - 12 = 0$$
$$3(t^2-4) = 0$$
$$3(t-2)(t+2) = 0 \quad \Longrightarrow \quad t = -2, +2$$

Since $t \geq 0$ we only use $t = 2$:

For $0 \leq t \leq 2$

Check $t = 1$: $3(1^2 - 4) = -12 < 0$

so $v(t) < 0$ for $0 \leq t \leq 2$

$\Rightarrow$ Particle moving downward on $0 \leq t \leq 2$

For $t \geq 2$

Check $t = 3$

$\Longrightarrow v(3) = 3(3^2-4) = 3(5) = 15 > 0$

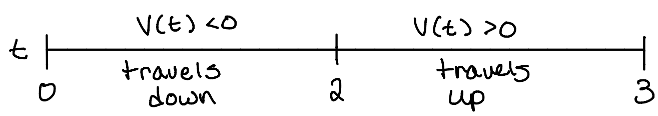
so $v(t) > 0$ for $t > 2$

$\Longrightarrow$ Particle moving upward on $t > 2$

(c) Find the distance the particle travels in the time interval $0 \le t \le 3$.

Solution:

on $0 \le t \le 3$



downward distance : $|s(2) - s(0)| = |3 - (-13)| = 16$

upward distance : $|s(3) - s(2)| = |-6 - (-13)| = 7$

Total distance traveled : $16 + 7 = 23$

(d) Determine when the particle is speeding up and slowing down on the interval $0 \le t \le 3$. (It may help to graph it.)

Solution: Using our results from (b) we know from our $v(t)$ that the particle changes sign at $t = 2$ (Since this is the point when $v(t) = 0$). Algebraically, we can determine when we are speeding up or slowing down by testing values of $v(t)$ and $a(t)$ on the intervals $0 \le t \le 2$ and $2 \le t \le 3$. For each interval we find that:

- $0 \le t \le 2$: $v(t)$ and $a(t)$ have different signs $\Rightarrow$ slowing down
- $2 \le t \le 3$: $v(t)$ and $a(t)$ have same signs $\Rightarrow$ speeding up

In Figure 1 we have graphs of the position, velocity and acceleration functions. We can also use this to determine the sign of $a(t)$ with respect to $v(t)$.

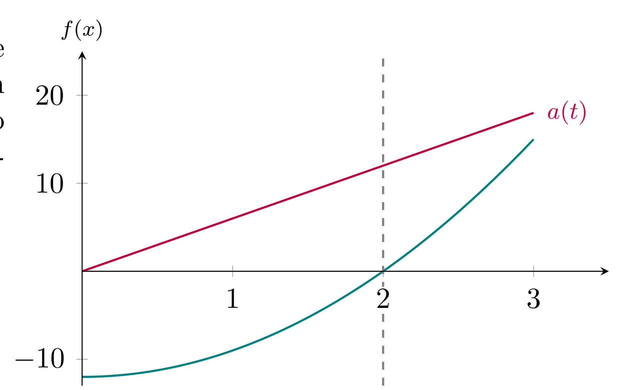


Figure 1: Graphs of $v(t)$, $a(t)$

2. The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t^2)$

   (a) Find the function that governs the *rate of change* of the amount of water in the tank with respect to time.

   Solution:

   The rate of change of a function given by the derivative

   $\implies Q'(t) = 200(-2t) = -400t$

   (b) What is the rate at which water is leaving the tank at the end of 10 minutes?

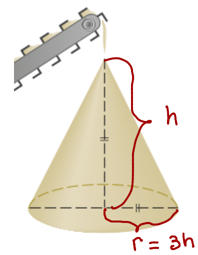
   Solution:

   At the end of 10 minutes rate is

   $Q'(10) = -400(10) = -4000$ gal/min

Related Rates

3. Sand falls from an overhead bin and accumulates in a conical pile with a radius that is always three times its height. Suppose the height of the pile increases at a rate of $2\,\mathrm{cm/s}$ when the pile is $12\,\mathrm{cm}$ high. At what rate is the sand leaving the bin at that instant.



Solution:

Given: Volume of cone $\Longrightarrow$ $V = \frac{1}{3}\pi r^2 h$

Radius 3 times height $\Longrightarrow$ $r = 3h$

rate height increases $\Longrightarrow$ $\frac{dh}{dt} = 2\ \mathrm{cm/sec}$

Asked

Rate sand leaves bin
i.e. added to pile
i.e. volume increases $\Longrightarrow$ $\frac{dV}{dt} = ?$ when $h = 12\mathrm{cm}$

Compute

Since $r = 3h$

$$\Longrightarrow V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi (3h)^2 \cdot h = \frac{1}{3}\pi(9h^3) = 3\pi h^3$$

Differentiate:

$$\frac{dV}{dt} = 3\pi(3h^2)\cdot\frac{dh}{dt}$$
$$= 9\pi h^2 \frac{dh}{dt}$$

When $h = 12$ & $\frac{dh}{dt} = 2$

$$\Longrightarrow \frac{dV}{dt} = 9\pi(12)^2(2) = 2592\pi\ \frac{\mathrm{cm}}{\mathrm{sec}}$$

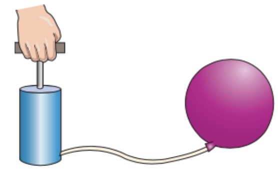
Answer

Sand is leaving the bin @ rate of $2592\pi\ \frac{\mathrm{cm}^3}{\mathrm{sec}}$ when $h = 12\mathrm{cm}$.

Possible?

$$2592\pi = 9\pi h^2(2)$$
$$\Longrightarrow \frac{2592\pi}{18\pi} = h^2$$
$$\Longrightarrow h^2 = 144$$
$$\Longrightarrow h = \pm 12 \quad \checkmark$$

4. A spherical ballon is inflated and its volume increases at a rate of $15\,\mathrm{in}^3/\mathrm{min}$. How fast is the radius increasing when the radius is $10\,\mathrm{in}$.



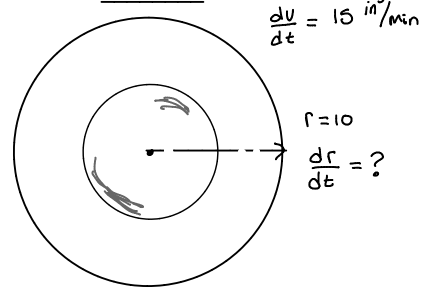
Solution:

Given: Rate Volume increasing $15\,m^3/\text{min}$

$\Longrightarrow \frac{dV}{dt} = 15$

Volume of a sphere: $V = \frac{4}{3}\pi r^3$

Sketch



Asked: Rate radius increasing i.e. $\frac{dr}{dt} = ?$
when $r = 10$ in

Compute

We know: $V = \frac{4}{3}\pi r^3$

Differentiate w.r.t. t:

$$\frac{d}{dt}[V] = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$
$$= \frac{4}{3}\frac{d}{dt}\left[r^3\right]$$
$$= \frac{4}{3}\left(3r^2 r'\right)$$

Note:

r is a fcn of time i.e. $r = r(t)$

$$\Longrightarrow \frac{d}{dr}\left[(r(t))^3\right] = (r(t))^2 r'(t)$$
$$= r^2 \frac{dr}{dt}$$

by the Chain Rule

So $\frac{dV}{dt} = 4r^2\frac{dr}{dt}$ Since $\frac{dV}{dt} = 15$ when $r = 10$

$$\Longrightarrow 15 = 4(10)^2\frac{dr}{dt}$$
$$15 = 400\frac{dr}{dt} \Longrightarrow \frac{dr}{dt} = \frac{15}{400} = \frac{3}{80} \approx 0.0375$$

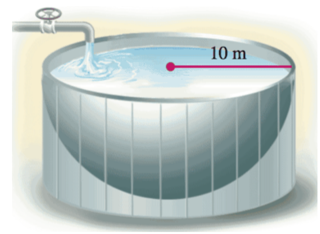
Answer

When the radius is 10 in the radius is increasing at a rate of 0.0375 in/min

Possible?

$$\frac{dV}{dt} = 4(10)^2\left(\frac{3}{80}\right) = 15\ \checkmark$$

5. A hemispherical tank with a radius of 10 m is filled from an inflow pipe at a rate of 3 m^3/min.
   (a) How fast is the water level rising when the water level is 5 m from the bottom of the tank? (Hint: the volume of a cap of thickness h sliced from a sphere of radius r is $\frac{\pi}{3}h^2(3r-h)$)
   Solution:



Given: radius: 10 m $\Rightarrow$ $r = 10\text{m}$

rate filled: 3 m^3/min $\Rightarrow$ $\frac{dV}{dt} = 3 \ \text{m}^3/\text{min}$

Volume of cap w/ thickness h: $V = \frac{\pi}{3}h^2(3r-h)$

Asked How fast is the H_2O level rising?

i.e. $\frac{dh}{dt} = ?$ when $h = 5\text{m}$

Compute

Volume of slice w/ radius $r = 10\text{m}$

$$V = \frac{\pi}{3}h^2(3(10)-h) = \frac{\pi}{3}h^2(30-h)$$
$$= 10\pi h^2 - \frac{\pi}{3}h^3$$

Differentiating:

$$\frac{dV}{dt} = 2(10\pi)h\frac{dh}{dt} - 3\frac{\pi}{3}h^2 \cdot \frac{dh}{dt} = \frac{dh}{dt}(20\pi h - \pi h^2)$$

When $h = 5$ & $\frac{dV}{dt} = 3$

$$\Rightarrow \quad 3 = \frac{dh}{dt}(20\pi(5) - \pi(5)^2)$$
$$3 = \frac{dh}{dt}(100\pi - 25)$$
$$3 = 75\pi \cdot \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{75\pi} = \frac{1}{25\pi} \ \text{m}/\text{min}$$

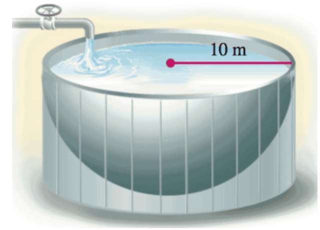
Answer

The height of the H_2O is changing at a rate of $\frac{1}{25\pi}$ m/min

Possible?

$$\frac{dV}{dt} = \left(\frac{1}{25\pi}\right)(20\pi(5) - \pi(5)^2) = \frac{1}{25\pi}(75\pi) = 3 \checkmark$$

(b) What is the rate of change of the area of the exposed surface of the water when the water is 5 m deep?



Solution:

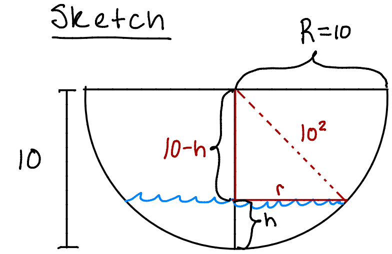
Given: Fixed Radius of tank $\Rightarrow r = 10$

$h =$ height of H_2O

$r =$ Changing radius of H_2O

rate tank Filled $\Rightarrow \frac{dV}{dt} = 3 \text{ m}^3/\text{min}$

Sketch



Asked: rate surface area of H_2O changes i.e. rate area of circle changes

$\frac{dA}{dt} = ?$ when $h = 5\text{m}$

Compute

Area of circle: $A = \pi r^2$

Differentiate:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \Rightarrow \text{Need to find } \frac{dr}{dt} \text{ \& } r \text{ in terms of } h$$

By triangle in Red: $r^2 + (10-h)^2 = 10^2 \Rightarrow r^2 = 10^2 - (10-h)^2$

$= 20h - h^2$

Differentiate to Find $\frac{dr}{dt}$: $\frac{d}{dt}[r^2] = \frac{d}{dt}[20h - h^2]$

$$\Rightarrow 2r\frac{dr}{dt} = 20\frac{dh}{dt} - 2h \cdot \frac{dh}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{2r} \cdot \frac{dh}{dt}(20 - 2h)$$

When $h = 5$

$r^2 = 20(5) - (5)^2 = 75 \Rightarrow r = \sqrt{75}$

From part (a): $\frac{dh}{dt} = \frac{1}{25\pi} \Rightarrow \frac{dr}{dt} = \frac{1}{2\sqrt{75}}\left(20 - 2\left(\frac{1}{25\pi}\right)\right) = \frac{1}{5\pi\sqrt{75}}$

Now we can find $\frac{dA}{dt}$: $\frac{dA}{dt} = 2\cancel{\pi}(\cancel{\sqrt{75}})\left(\frac{1}{5\cancel{\pi}\cancel{\sqrt{75}}}\right) = \frac{2}{5} \text{ m}^2/\text{min}$

Often its easier (& always more accurate) to use exact values in intermediate steps!

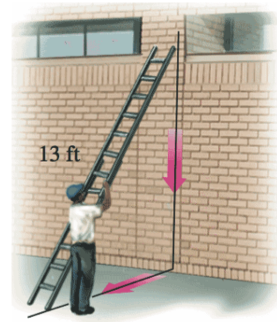
Answer

Rate @ which surface area of H_2O changes is $\frac{2}{5}$ m²/sec when the height of H_2O is 5 m.

Possible?

$$\frac{2}{5} = 2\pi(\sqrt{75})\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{\cancel{2}}{5}\left(\frac{1}{\cancel{2}\pi\sqrt{75}}\right) = \frac{1}{5\pi\sqrt{75}} \checkmark$$

6. A 13 foot ladder is leaning against a vertical wall when Jack begins pulling the foot of the ladder away form the wall at a rate of 0.5 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 5 feet from the wall?



Solution:

Given:

rate ladder moves away from wall

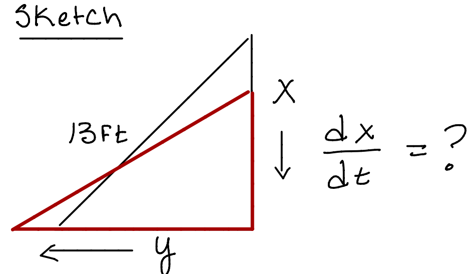
$\Longrightarrow \frac{dy}{dt} = 0.5$ in

Pythagorean Theorem: $c^2 = x^2 + y^2$

where c is length of the ladder

distance from wall: $y = 5$ ft

Sketch



$\frac{dy}{dt} = 0.5$ ft

$y = 5$ft

Asked

rate @ which height of ladder decreasing

$\Longrightarrow \frac{dx}{dt} = ?$ when $y = 5$ft

Compute

Differentiating: $13^2 = x^2 + y^2$

$$0 = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$\Longrightarrow \frac{dx}{dt} = -\frac{2y}{2x}\frac{dy}{dx} = -\frac{y}{x}\frac{dy}{dx}$$

Note when $y = 5$ft

We need to find x when $y = 5$

$$13^2 = x^2 + (5)^2 \Longrightarrow x = \sqrt{13^2 - 25} = 12$$

Using $y = 5$, $\frac{dy}{dt} = 0.5$, $x = 12$

$$\Longrightarrow \frac{dx}{dt} = -\frac{5}{12}(0.5) = -\frac{5}{12}\left(\frac{1}{2}\right) = -\frac{5}{24}$$

Answer

Height of the ladder is decreasing @ rate of $-\frac{5}{24}$ ft/sec when ladder is 5 ft from the wall.

Possible?

$$2(12)\left(-\frac{5}{24}\right) + 2y\left(\frac{1}{2}\right) = 0 \Longrightarrow -5 = -y \Longrightarrow y = 5 \checkmark$$