Lesson 3

Continuity and the Definition of the Derivative

Sections
2.6, 3.1, 3.2

How Continuity is Defined

So far in your study of mathematics, continuity has been defined as being able to draw the graph of a function from start to finish without picking up your pencil. In other words, a function is continuous if its graph has no holes or breaks in it. We can now define continuity more precisely with our knowledge of limits. A function can be continuous at a single point, or on an interval.

Continuity at a Point

We will start with a discussion of continuity of f at a single point a.

Definition 3.1: Continuity at a Point

- **Continuity at a Point**: A function $f(x)$ is said to be **continuous at a point** $x = a$ if
$$\lim_{x \to a} f(x) = f(a)$$
In other words we must meet *all* of the following conditions:
  1. $f(a)$ exists;
  2. $\lim_{x \to a} f(x)$ exists;
  3. $\lim_{x \to a} f(x) = f(a)$;

  If even one of these cases fails, then $f(x)$ fails to be continuous at a and $x = a$ is known as a **point of discontinuity**.

This definition of continuity requires that *both* $\lim_{x \to a} f(x)$ and $f(a)$ exist. If one fails to exist then $f(x)$ cannot be continuous there. So to establish if a function $f(x)$ is continuous at a point a you need to verify all three statements. This leads us to the following theorem.

Theorem 3.1

If $f(x)$ is continuous at $x = a$ then
$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a^-} f(x) = f(a), \qquad \lim_{x \to a^+} f(x) = f(a)$$

Note that this theorem is telling us that if we know f is continuous at a then we know the value of the three limits that are listed.

Example 3.1: Determining Continuity from a Graph

Examine the graph in Figure 3.1 and determine if $f(x)$ is continuous at the values

$$x = 1, \quad x = 2, \quad \text{and} \quad x = 4$$

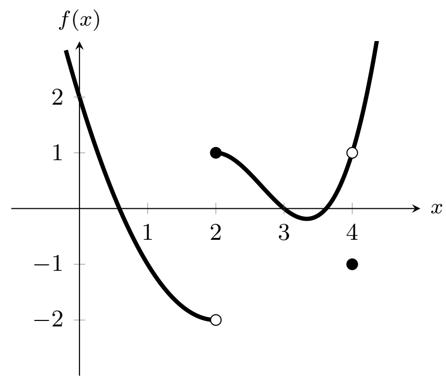


Fig. 3.1: Graph of $f(x)$

a.) $x = 1$

Solution. We see that $f(1) = -1$ and that

$$\lim_{x \to 1^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 1$$

And so we see that the value of the function at $x = 1$ and the limit as $x \to 1$ are the same. In other words, we have

$$\lim_{x \to 1} f(x) = f(1)$$

Therefore, since f meets the 3 conditions of the definition then $f(x)$ is continuous at $x = 1$.

b.) $x = 2$

Solution. We see that $f(2) = 1$ but

$$\lim_{x \to 2^-} f(x) = -1 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 1$$

Since the left and right hand limits are not equal. In other words,

$$\lim_{x \to 2} f(x) \text{ does not exist.}$$

Since f fails the second condition of the definition, it is not continuous at $x = 2$.

This is known as a **jump discontinuity**. We can see from the graph that at $x = 2$ the graph "jumps" from -1 to 1.

c.) $x = 4$

Solution. We see that

$$f(4) = -1 \quad \text{and} \quad \lim_{x \to 4} f(x) = 1$$

We see that the function value is not equal to the value of the limit. So f fails the third part of the definition at $x = 4$. Therefore, $f(x)$ is not continuous at $x = 4$.

This type of discontinuity is known as a **removable discontinuity**. We can see from the graph that at $x = 4$ we have a "hole" in the graph.

In this example we got the limit values from the graph. Without a graph we would calculate the limit using strategies we have learned so far.

Continuity on an Interval

Definition 3.2: Continuity on an Interval

- **On an Open Interval:** A function $f(x)$ is said to be **continuous on an open interval** (a,b) if it is continuous at each point on the interval.
- **On an Closed Interval:** A function is **continuous on a closed interval** $[a,b]$ if it is continuous on the open interval (a,b) and if
  - The function f is **continuous from the right** at a, i.e. if
  $$\lim_{x \to a^+} f(x) = f(a)$$
  and
  - The function f is **continuous from the left** at b, i.e. if
  $$\lim_{x \to b^-} f(x) = f(b)$$

We will not spend any time specifically dealing with continuity on an interval. Since we have seen how to show a function f is continuous at a point a we can simply extend this to continuity on open and closed intervals.

Properties of Continuity

Now for a discussion of some of the properties of continuity. The results presented here are really just a summary of knowledge we already have from algebra

Theorem 3.2: Continuity Rules

If f and g are continuous at a then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

$$f+g \qquad f-g \qquad cf$$

$$fg \qquad (f(x))^n \qquad \frac{f}{g}, \text{ provided } g(a) \neq 0$$

The next theorem is a consequence of these rules and summarizes what we already know about polynomial and rational functions in terms of their continuity.

Theorem 3.3: Continuity of Polynomial & Rational Functions

- A polynomial function is continuous for all x.
- A rational function of the form $\frac{p}{q}$ where p and q are polynomials, is continuous for all x for which $q(x) \neq 0$.

This next property is essential for evaluating limits of composite functions which we will see in the next section.

Theorem 3.4: Continuity of Composite Functions

If g is continuous at a and f is continuous at $g(a)$, then the composite function given by
$$(f \circ g)(x) = f(g(x))$$
is also continuous at a.

Example 3.2: Applying Continuity Rules

For what values of x is
$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$
continuous?

Solution. This is a rational function. The values not in the domain of this function are those for which the denominator is equal to zero. Setting the denominator equal to zero we see that
$$5 - 3x = 0 \quad \Longrightarrow \quad x = \frac{5}{3}$$
and so we must have that $x \neq \frac{5}{3}$.
Thus, by Theorem 3.3 we know that $f(x)$ is continuous everywhere except the point $x = \frac{5}{3}$.

Additionally, we can discuss the continuity of transcendental functions on their domains. The following theorem states the continuity of these functions.

Theorem 3.5: Continuity of Transcendental Functions

The following functions are continuous on their domains:

- **Trigonometric Functions:** $\sin x$, $\cos x$, $\tan x$, $\csc x$ $\sec x$, $\cot x$.
- **Inverse Trig Functions:** $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.
- **Exponential Functions:** b^x, e^x
- **Logarithmic Functions:** $\log_b(x)$, $\ln(x)$

Limit of a Function Composition

Now that we have established a more precise definition for a function to be continuous and discussed some properties of continuous functions we can discuss the limit of a function composition.

Theorem 3.6: Limit of Function Composition

If $f(x)$ is continuous at $x = b$ and $\lim_{x\to a} g(x) = b$ then,

$$\lim_{x\to a} f\left(g(x)\right) = f\left(\lim_{x\to a} g(x)\right)$$

Example 3.3: Limit of a Function Composition

Find $\lim_{x\to 0} e^{(\sin x)}$.

Solution. We know that $e^{(\sin x)}$ is the composition of the functions e^x and $\sin x$. By the previous theorem we have

$$\lim_{x\to 0} e^{(\sin x)} = e^{\left(\lim_{x\to 0} \sin x\right)}$$

We know that $\lim_{x\to 0} \sin x = \sin(0) = 0$. Thus the previous theorem we have

$$\lim_{x\to 0} e^{(\sin x)} = e^{\left(\lim_{x\to 0} \sin x\right)} = e^0 = 1$$

The Intermediate Value Theorem

A common problem addressed in mathematics involves finding the roots of a function. Recall that the *roots* of a function f are also referred to as *zeros* of a function f or as *solutions* to $f(x) = 0$. These are all equivalent statements.

Suppose we have a continuous function f for which we want to find the roots.

If we set $f(x) = 0$ then if $f(a)$ and $f(b)$ are of opposite signs (regardless of their value), we know that the function had to cross the x axis at some point on the interval.

This implies that at some point the function was equal to zero at least once on the interval, thus locating a solution to the equation. An example of this is pictured in Figure 3.2.

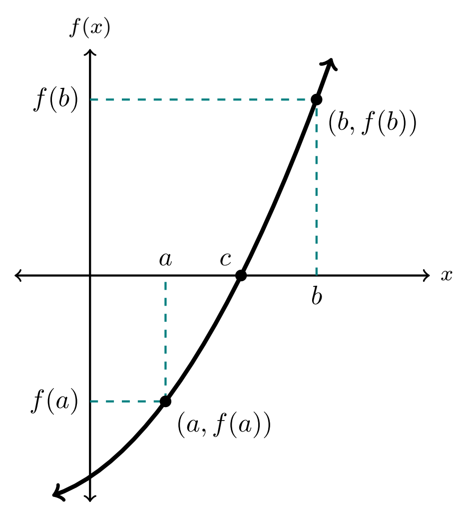


Fig. 3.2: Finding $f(c) = 0$ on (a, b)

While this may seem like a straightforward problem at this level, in the real world and in higher level mathematics, we don't know information about roots (or even how to find them) for many functions. So before going through a bunch of work to find the zeros of a function f it would be beneficial to know if they even exist!

In many situations it does not even matter what the actual value of the solution is, it is enough to know that at least one exists on an interval. Determining the location of a root (or solution to an equation) is made possible by the Intermediate Value Theorem.

Theorem 3.7: Intermediate Value Theorem

Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and that $f(a) \neq f(b)$. Let L be any number between $f(a)$ and $f(b)$. Then there exists at least one number c such that

$$a < c < b \qquad \text{and} \qquad f(c) = L$$

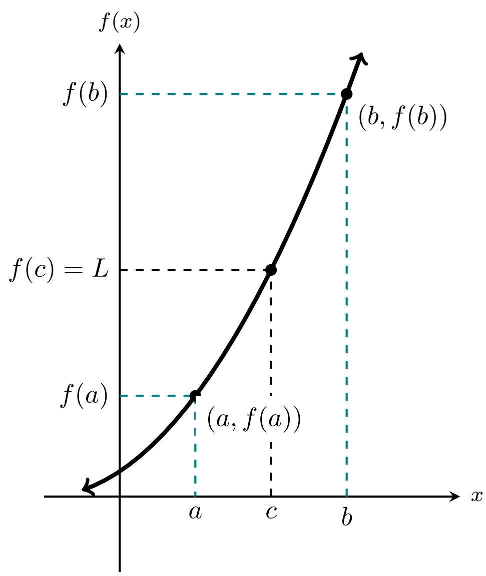


Fig. 3.3: Visual of the Intermediate Value Theorem

This theorem tells us that a *continuous* function on an interval $[a, b]$ will take on all values between $f(a)$ and $f(b)$.

Note that we talk about values between $f(a)$ and $f(b)$. This means we can either have the case that $f(a) < f(b)$ or we can have the case that $f(b) < f(a)$.

A visual representation of what the Intermediate Value Theorem states can be seen in Figure 3.3. Here we have that $f(a) < f(b)$ on the interval (a, b)

Hopefully, from this figure it is clear that if f is discontinuous at any point in the interval $[a, b]$ (i.e. there is a "break" in our graph), then we can not determine that such a value L exists. In other words, continuity of f is necessary for us to use the Intermediate Value Theorem.

You should also note that the Intermediate Value Theorem tells us that there is *at least* one value L. This means that it is possible for there to be *more* than one place where f has the value L.

We will see a formal example of the application of the Intermediate Value Theorem in the following example.

Example 3.4: Applying the Intermediate Value Theorem

Show
$$p(x) = 2x^3 - 5x^2 - 10x + 5$$
has a root in $[-1, 2]$.

Solution. First, we know that p is continuous since it is a polynomial and polynomials are continuous everywhere. Next we want to know if $p(x) = 0$ between $x = -1$ and $x = 2$. In other words, we want a number c for which

$$-1 < c < 2 \qquad \text{and} \qquad f(c) = 0$$

so we want $p(-1) < 0 < p(2)$ or equivalently $p(2) < 0 < p(-1)$. We see that $p(-1) = 8$ and $p(2) = -19$. So we have

$$-19 = p(2) < 0 < p(-1) = 8$$

Thus, $p(x) = 0$ somewhere between $p(-1)$ and $p(2)$. So by the Intermediate Value Theorem, there must be a number c in this interval for which $p(c) = 0$. Therefore, a root exists in the interval $[-1, 2]$.

Remark: With our knowledge of algebra, we know that we could attempt to factor this polynomial to find its roots. However, not all polynomials are able to be factored. Furthermore, the equation for finding solutions to cubics is far more complicated than our friend the quadratic formula. Plus, there doesn't even exist formulas for polynomials with degree higher than 4! This is where the Intermediate Value Theorem comes in handy.

Continuity is an important property of functions. Many operations or processes cannot be carried out on a function unless it is continuous. It is a key component in establishing the existence of a *derivative*, the main topic studied in the first semester of calculus.

Tangent Lines

There are two classic problems in calculus. The tangent line problem and the area problem. Now that we know some things about limits we are prepared to talk about the tangent line problem.

Definition 3.3: Tangent Line

Graphically, a **tangent line** can be described as the line that touches, but does not cross a graph at a certain point. This is demonstrated in Figure 3.4.

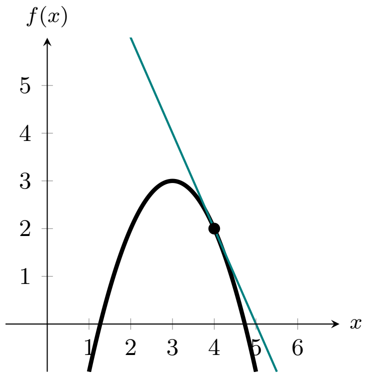


Fig. 3.4: A tangent line

Recall that to find the equation of a line we need the slope of the line and a point on the line. Given a slope m and a point on the line (x_1, y_1) we can construct the **point-slope** form of a line. Recall that the point-slope form of a line is given by

$$y - y_1 = m(x - x_1)$$

So in order to find a line tangent to a graph at a certain point we need to find the slope of the tangent line at that point.

In algebra, the slope and equation of a line is usually discussed using the points (x_1, y_1) and (x_2, y_2). Here we will instead use the points $(x, f(x))$ and $(x+h, f(x+h))$. In other words, the slope of a line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

We can approximate the tangent line using the something called the **secant line**. The definition of a secant line is given below.

Definition 3.4: Secant Line

A line drawn between any two points $(x, f(x))$ and $(x+h, f(x+h))$ on a graph is known as a **secant line**. The secant line is pictured in Figure 3.5.

The slope of the secant line is then given by the same slope formula,

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h}$$

When the slope formula is seen in this form it is sometimes referred to as the **difference quotient**. It is a crucial piece when discussing the definition of the derivative.

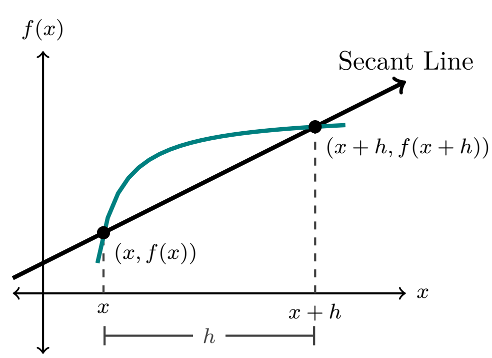


Fig. 3.5: Slope of the secant line

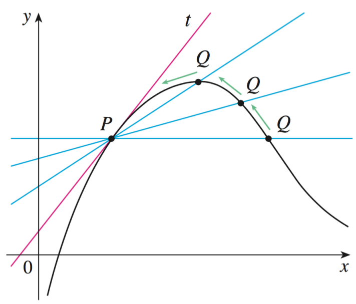


Fig. 3.6: Secant and Tangent lines

The secant line can help us approximate the tangent line. How can we do this? Take the graph of the function shown in Figure 3.6. A single secant line defined by the points P and Q does not do a good job of estimating the tangent line given at point P.

What if we move Q closer to the point P? The next secant line will have a better approximation to the tangent line. If we keep doing this with points that are closer and closer together, eventually the distance between them will be zero. This will give us the slope of the tangent line.

What we really want is for the distance h in Fig. 3.5 to be as small as possible. The smallest h could be is of course zero. We know that we can't divide by zero, so what do we do? This is where we can use our newly found knowledge of limits! By evaluating the limit as h goes to zero we will be able to find the slope of our tangent line.

Definition 3.5: Slope of the Tangent Line

For any point x in the domain of a function f, the **slope of the tangent line**, denoted $m_{\tan}$, is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

What if we want to know the value of the slope of the tangent line for a function given any value x? It would be nice to find an equation for which we could just plug in a value and obtain our slope. The equation that will give us this information is known as the **derivative**.

Definition of the Derivative

Instead of finding the slope of the line tangent to f at a single point every time we would like to have a *function* that will allow us to find this information more easily. Luckily, we can find the function governing the slope of the tangent line using the same idea given above to calculate the slope of the tangent line.

This means we can find an equation that will give us the value of the slope along the entire graph of the function f. This is known as the **derivative of a function** f.

Definition 3.6: Derivative of a Function

The **derivative** of f is the function given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists and that x is in the domain of f.

The symbol f' above is read as "f prime". This is only one way to denote the derivative. There are actually several different ways to denote the derivative of a function which we will discuss later. First, we will look at a simple example of how to apply the definition of the derivative.

Example 3.5: Compute the Derivative using the Definition

Find the derivative of the function $g(x) = 2x + 3$.

Solution. We need to use the definition of the derivative of a function given in Definition 3.6.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{[2x + 2h + 3] - [2x + 3]}{h} && \text{Plug in } g(x+h) \text{ \& } g(x) \\
&= \lim_{h\to 0} \frac{2x + 2h + 3 - 2x - 3}{h} && \text{Distribute} -1 \text{ to terms of } g(x) \\
&= \lim_{h\to 0} \frac{\cancel{2x} + 2h + \cancel{3} - \cancel{2x} - \cancel{3}}{h} && \text{Combine like terms} \\
&= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}} && \text{Cancel common terms} \\
&= \lim_{h\to 0} 2 = 2
\end{aligned}$$

Therefore, $g'(x) = 2$.

Now lets look at some slightly more complicated examples of computing the derivative.

Example 3.6: Computing Derivatives using the Definition

Compute the derivative of the following functions using the definition.

a.) $f(x) = x^2 - x$

Solution. We need to use the definition of the derivative of a function given in Definition 3.6. So we want to find

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For complicated functions it can sometimes be helpful to calculate $f(x+h)$ before proceeding with your limit evaluation. So we note that

$$f(x+h) = (x+h)^2 - (x+h) = (x+h)^2 - (x+h) = x^2 + 2hx + h^2 - x - h$$

Now we find $f'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{[x^2+2hx+h^2-x-h]-[x^2-x]}{h} && \text{Plug in } f(x+h) \text{ \& } f(x) \\
&= \lim_{h\to 0} \frac{x^2+2hx+h^2-x-h-x^2+x}{h} && \text{Distribute } -1 \text{ to terms of } f(x) \\
&= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2-\cancel{x}-h-\cancel{x^2}+\cancel{x}}{h} && \text{Combine \& cancel common terms} \\
&= \lim_{h\to 0} \frac{2hx+h^2-h}{h} && \text{Factor out } h \\
&= \lim_{h\to 0} \frac{\cancel{h}(h+2x-1)}{\cancel{h}} && \text{Cancel common terms} \\
&= \lim_{h\to 0} (h+2x-1) \\
&= (0)+2x-1 && \text{Direct substitution} \\
&= 2x-1
\end{aligned}$$

b.) $f(x) = \sqrt{9-x}$

Solution. We again use the definition of the derivative given in Definition 3.6. It's important not to forget strategies we have learned for evaluating limits. Here we will need to multiply by the conjugate.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h}\right)\left(\frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}}\right) \\
&= \lim_{h\to 0} \frac{(9-(x+h))-(9-x)}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{\cancel{9}-\cancel{x}-h-\cancel{9}+\cancel{x}}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \frac{-1}{\sqrt{9-(x+(0))}+\sqrt{9-x}} \\
&= \frac{-1}{\sqrt{9-x}+\sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}
\end{aligned}$$

Therefore, $f'(x) = \dfrac{-1}{2\sqrt{9-x}}$

Derivative Notation

Indicating the derivative of a function f by $f'(x)$ is known as "prime" notation. The textbook mostly uses $\frac{d}{dx}$ notation to indicate the derivative, this is known as Leibniz notation. Several different notations exist for the derivative because Calculus was independently developed by several different people at the same time in history.

Each type of notation can be more useful in some situations over others. Often, it is just the person's preference as to which is used. I tend to stick to prime notation in most cases. I find it more simple to use and it leaves less letters to keep track of! Ultimately, it's up to you as to which you use, but you should be able to recognize all of them.

Here is a summary of common derivative notations used.

- $f'(x)$ is read as "f prime of x"
- $\frac{dy}{dx}$ is read as "derivative of y with respect to x"
- y' is read as "y prime"
- $\frac{d}{dx}[f(x)]$ is read as "derivative of $f(x)$ with respect to x"

Finding Tangent Lines

Since we can obtain $f'(x)$, where x represents some real number, we can of course find $f'(a)$ for a specific value a. This leads us to an alternative form of the definition of the derivative.

Definition 3.7: Definition of the Derivative at a Point

The **derivative** of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Why have another form? This form gives us a convenient way to find the derivative using the definition at a single point. It can be much easier to use in certain situations like showing differentiability of a function discussed in the next lecture.

At this point in the course, the main take away is the following statement.

Theorem 3.8: Derivative is the Slope of the Tangent Line

The slope of the tangent line to the graph of the function $y = f(x)$ at the point $(a, f(a)$ is $f'(a)$.

We know that the slope of a tangent line at a point $(a, f(a))$ is given by $f'(a)$. The general equation of the equation of a tangent line for any function $f(x)$ at $x = a$ can be defined as follows.

Definition 3.8: Equation of a Tangent Line at a Point

The **equation of the tangent line** at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

We now examine an example of how to calculate the tangent line for a given function at a specified point.

Example 3.7: Finding the Equation of a Tangent Line

Find an equation of the line tangent to the function $f(x) = 9 - 2x^2$ at the point $P(2, 1)$.

Solution. We want to find the slope of the tangent line when $x = 2$, i.e. using Definition 3.7 we have $a = 2$. Now we evaluate this limit.

$$\begin{aligned} \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x\to 2} \frac{(9 - 2x^2) - (1)}{x - 2} = \lim_{x\to 2} \frac{8 - 2x^2}{x - 2} \\ &= \lim_{x\to 2} \frac{-2\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} = \lim_{x\to 2}[-2(x + 2)] = -2 \cdot 4 = -8 \end{aligned}$$

So we have found that $f'(2) = -8$, this is the slope of our tangent line at the point $(2, 1)$. To find the equation of this line we simply plug it into the equation given in Definition 3.8 and obtain

$$y = 1 - 8(x - 2) \implies y = -8x + 17$$

this is the equation of the line tangent to $f(x)$ at the point $(2, 1)$.

Differentiability

When a function has a derivative on an interval or at a point it is said to be *differentiable* there.

Definition 3.9: Differentiability

A function f is

- **differentiable at a point** x if it's derivative exists at x.
- **differentiable on an interval** (a,b) if it is differentiable at every point in the interval.

Now that we know how to find a derivative using the definition we can talk about when we are able to actually do this. Since the definition involves a limit, we will only be able to compute derivatives when this limit exists. Previously, we have seen some cases when the limit does not exist. These same cases will affect our ability to find a derivative at a point or along an interval.

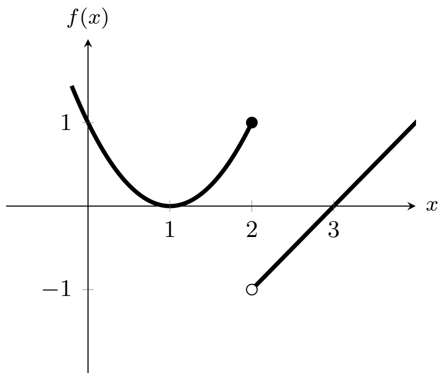
When is a function not differentiable?

Just as before, in order for this limit to exist we need both the left hand limit and the right hand limit to be equal. We will use the alternative form of the derivative in the following examples. In other words, for f to be differentiable at a point $x = c$ we will need

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

We will refer to these expressions as the **derivative from the left** and the **derivative from the right** respectively.

Example 3.8: Discontinuous Functions



Consider the function

$$f(x) = \begin{cases} (x-1)^2, & \text{if } x \leq 2 \\ x - 3, & \text{if } x > 2 \end{cases}$$

Note that

$$f(2) = (2-1)^2 = 1$$

Solution. To determine differentiability we will examine the derivative from the left and the derivative from the right at $a = 2$.

Derivative from the left:

$$\begin{aligned}
\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2^-} \frac{(x-1)^2 - 1}{x - 2} \\
&= \lim_{x \to 2^-} \frac{x^2 - 2x}{x - 2} && \text{Expand and combine like terms} \\
&= \lim_{x \to 2^-} \frac{x(x-2)}{x - 2} && \text{Factor out common terms} \\
&= \lim_{x \to 2^-} x && \text{Cancel common terms} \\
&= 2 && \text{Direct Subsititution}
\end{aligned}$$

Derivative from the right:

$$\begin{aligned}
\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2^-} \frac{(x-3) - 1}{x - 2} \\
&= \lim_{x \to 2^+} \frac{x - 4}{x - 2} && \text{Expand and combine like terms} \\
&= \frac{-2}{+\text{small number}} \\
&= -\infty
\end{aligned}$$

Since

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = 2 \quad \text{and} \quad \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = -\infty$$

then $\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ does not exist. Therefore, $f(x)$ is *not* differentiable at $x=2$.

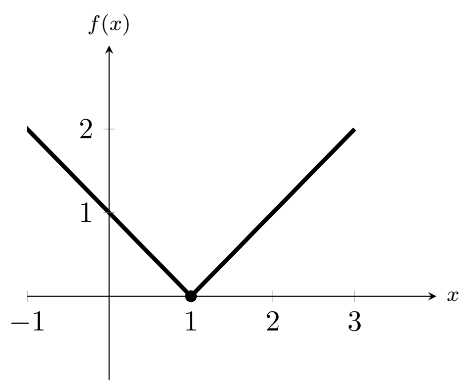
We saw in the previous example that a discontinuous function was not differentiable. This turns out to be the case for any discontinuous function. This gives us the following theorem.

Theorem 3.9: Not Continuous Implies Not Differentiable

If f is not continuous at a, then f is not differentiable at a.

When we have functions with sharp turns or "cusps" if we tried to draw a tangent line at that point there would be an infinite amount of possibilities for our slope. There isn't a single unique line that would be tangent to the curve at the point. The next example demonstrates this case.

Example 3.9: Function with a Sharp Point



Consider the function $g(x)=|x-1|$.

Solution. We can see that the function is continuous at $x=0$ but

$$\begin{aligned}\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} &= \lim_{x\to 1^-} \frac{|x-1|-(0)}{x-1}\\ &= \lim_{x\to 1^-} \frac{-(x-1)}{x-1} = -1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} &= \lim_{x\to 1^+} \frac{|x-1|-(0)}{x-1}\\ &= \lim_{x\to 1^+} \frac{(x-1)}{x-1} = 1\end{aligned}$$

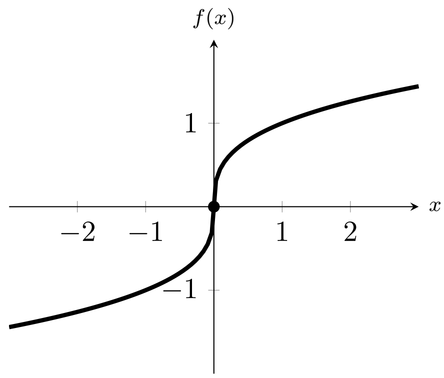
We see that

$$\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} \neq \lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1}$$

so $\lim_{x\to 1} \dfrac{g(x)-g(1)}{x-1}$ does not exist. Therefore, $g(x)$ is *not differentiable* at $x=0$.

Next we consider a graph that has a vertical tangent line at a certain point.

Example 3.10: Function with a Vertical Tangent Line



Consider the function $h(x) = x^{1/3}$.

Solution. This function is continuous at $x = 0$. However,

$$\lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} \frac{x^{1/3} - (0)}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}} = \infty$$

This means that $\displaystyle\lim_{x\to 0} \frac{h(x) - h(0)}{x - 1}$ does not exist. Therefore, $h(x)$ is *not differentiable* at $x = 0$.

What can we say about differentiable functions? It turns out that differentiability is closely linked to whether or not a function is continuous or not, either at a point or on an interval.

Theorem 3.10: Differentiable Implies Continuous

If f is differentiable at a, then f is continuous at a.

In other words, if a function is differentiable at a point or on an interval, then it must be continuous at that point or on that interval.

In Examples 3.9 and 3.10 we saw examples of continuous functions that failed to be differentiable at a point. This means that converse of Theorem 3.10 is *not* true. In other words, a continuous function can fail to be differentiable. This means that just because a function is continuous, it does not necessarily mean it is differentiable there.

We have now explored where the derivative comes from. Using the definition to find derivatives can be a tedious process. In the next lessons we will learn about some important rules that provide us with a more efficient way to find derivatives.

Practice Problems

Section 2.6
13-20, 27-33, 41-56

Section 3.1
15-26, 27-40, 49-52, 57-60

Section 3.2
31-37