Lesson 2

Limits and Infinity

Sections
2.4, 2.5

Using Infinity

Before we can talk about limits involving infinity you must understand some key things about infinity. First, it is important to remember that infinity is not a number itself. It represents a concept. If you go on in mathematics you will learn that there are even different types of infinities!

Because of this, we are not always permitted to carry out the usual operations between infinities. How much is $\infty - \infty$? We don't actually know! The expression $\infty - \infty$ is what is known as an **indeterminate form**. We will learn how to deal with limits involving these in a later lesson.

Definition 2.1: Indeterminate Forms

$$1^{\infty} \qquad 0^0 \qquad \infty^0 \qquad 0 \cdot \infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty$$

What are the "legal" operations that can be carried out in regards to infinity? The following are common cases you may see as you evaluate limits. Note that these are not actual operations, merely guidelines for results you may see. These are presented without proof because some involve higher level mathematics knowlege than we see in this course. These should merely be used as a reference as to what is acceptable and what is not when dealing with infinity.

Definition 2.2: Properties of Infinity

Let k be a real number.

$$\infty \pm k = \infty \qquad \infty \cdot \infty = \infty \qquad 0^{\infty} = 0$$

$$\infty + \infty = \infty \qquad (\pm k)\infty = \pm\infty, \text{ for } k \neq 0 \qquad \infty^{\infty} = \infty$$

$$0^k = \begin{cases} 0, & \text{if } k > 0 \\ \infty, & \text{if } k < 0 \end{cases} \qquad k^{\infty} = \begin{cases} \infty, & \text{if } k > 1 \\ 0, & \text{if } 0 < k < 1 \end{cases}$$

The following is an important concept when dealing with real numbers. For positive real numbers, we see that

$$\frac{1}{\text{Small Number}} = \text{Large Number} \qquad \text{and} \qquad \frac{1}{\text{Large Number}} = \text{Small Number}$$

Mathematically, these cases can be represented as follows. (Again, note that these are not actual operations, merely a way of representing the concept of what is happening in each case.)

Definition 2.3: Properties of Division Involving Infinity

Let k be a real number.

$$\frac{0}{k} = 0, \qquad \frac{0}{\infty} = 0, \qquad \frac{k}{\infty} = 0, \qquad \frac{k}{0} = \pm\infty, \qquad \frac{\infty}{k} = \infty, \qquad \frac{\infty}{0} = \infty$$

Remark: When actually writing out your work as you evaluate a limit it is never acceptable to write

$$\lim_{x \to a} f(x) = \frac{1}{\infty}$$

since this is not a correct mathematical statement. However, often you will find yourself writing this very term to keep track of your work. The acceptable way to write this would be

$$\lim_{x \to a} f(x) = \frac{1}{\infty} = 0$$

This way you are properly communicating that you understand this value to be 0 rather than a nonsensical mathematical term. It may seem tedious but notation is important in mathematics so get used to it!

Infinite Limits

Consider the function we saw in Example 1.3 in Lesson 1

$$f(x) = \frac{1}{x^2}$$

We see that as x gets closer and closer to zero, $f(x)$ gets larger and larger, in other words, positive infinity.

The limit as x approaches 0 in this case is known as an **infinite limit**.

Technically speaking, infinite limits do not exist but they occur frequently in the study of calculus. Because of this, it is important to learn how to find them and how to interpret what they mean for the behavior of a function.

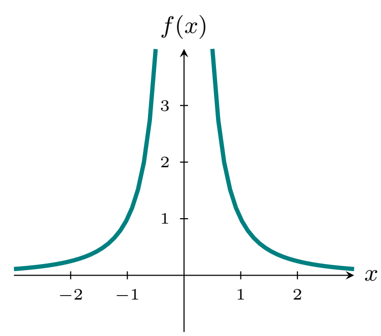


Fig. 2.1: Graph of $f(x) = \dfrac{1}{x^2}$

Remark: Recall that the term **magnitude** refers to distance from zero, rather than the actual numerical value. This is the same concept as the absolute value function. Thus, we make the distinction *in magnitude* when defining limits that approach negative infinity. This means that $-\infty$ is very large in terms of how far it is from zero, even though it's a "smaller" number in terms of its numerical value. Thinking of this limit as "very small" can lead to confusion between limits that are getting smaller *in magnitude* since these types of limits are actually approaching zero!

Definition 2.4: Two-Sided Infinite Limits

Suppose f is defined for all x near a.

- If $f(x)$ grows arbitrarily large for all x sufficiently close, but not equal to a, then

$$\lim_{x\to a} f(x) = \infty$$

is a **(positive) infinite limit** and we say that the limit of $f(x)$ as x approaches a is infinity.

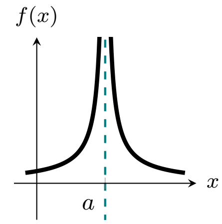


Fig. 2.2: $\lim_{x\to a} f(x) = \infty$

- If $f(x)$ is *negative* and grows arbitrarily large *in magnitude* for all x sufficiently close, but not equal to a, then

$$\lim_{x\to a} f(x) = -\infty$$

is a **(negative) infinite limit** and we say that the limit of $f(x)$ as x approaches a is *negative* infinity.

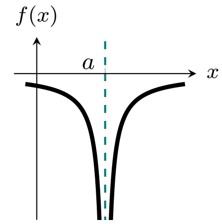


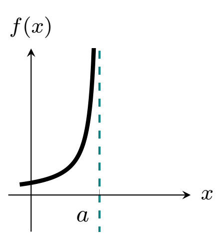
Fig. 2.3: $\lim_{x\to a} f(x) = -\infty$

In other words, as we get closer and closer to $x = a$ from both the left and the right, our value for $f(x)$ gets larger and larger (or smaller and smaller in the case of $-\infty$). We also have conditions for defining one-sided infinite limits.

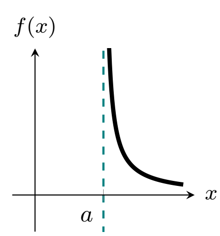
Definition 2.5: One-Sided (Positive) Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close to a (but not equal to a) then

- For $x < a$ we write, $\lim_{x\to a^-} f(x) = \infty$
- For $x > a$ we write, $\lim_{x\to a^+} f(x) = \infty$



$\lim_{x\to a^-} f(x) = \infty$

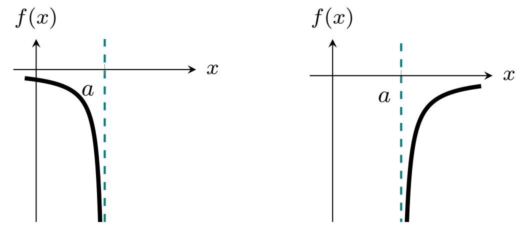


$\lim_{x\to a^+} f(x) = \infty$

Definition 2.6: One-Sided (Negative) Infinite Limits

If $f(x)$ is *negative* and grows arbitrarily large in *magnitude* for all x sufficiently close to a (but not equal to a) then

- For $x < a$ we write, $\lim_{x \to a^-} f(x) = -\infty$
- For $x > a$ we write, $\lim_{x \to a^+} f(x) = -\infty$



$\lim_{x \to a^-} f(x) = -\infty$ $\lim_{x \to a^+} f(x) = -\infty$

Vertical Asymptotes

From our study of algebra we known that the vertical line at the point $x = a$ is a **vertical asymptote**. Now we can define these asymptotes by using our understanding of limits.

Definition 2.7: Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of a function $f(x)$ if at least one of the following is true.

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 2.1: Determining Infinite Limits Graphically

Find all limits of

$$f(x) = \frac{1}{x^2 - 1}$$

Solution. First we re-write $f(x)$ with a factored denominator

$$f(x) = \frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)}$$

The zeros of the denominator are $x = -1$, $x = 1$. These are the vertical asymptotes of $f(x)$. We see the graph of $f(x)$ in Figure 2.4.

We observe the following facts:

- As $x \to -1$ from the left $f(x) \to \infty$
- As $x \to -1$ from the right $f(x) \to -\infty$
- As $x \to 1$ from the left $f(x) \to -\infty$
- As $x \to 1$ from the right $f(x) \to \infty$

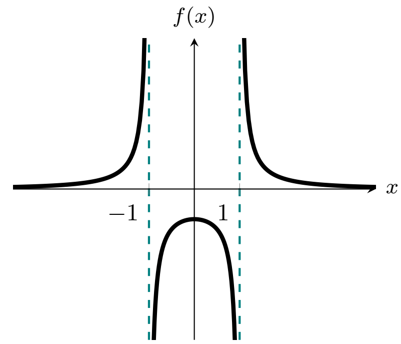


Fig. 2.4: Graph of $f(x) = \frac{1}{x^2-1}$

Thus, our limits are

$$\lim_{x\to -1^-} f(x) = \infty, \qquad \lim_{x\to -1^+} f(x) = -\infty, \qquad \lim_{x\to 1^-} f(x) = -\infty, \qquad \lim_{x\to 1^+} f(x) = \infty$$

Observe that both $\lim\limits_{x\to -1} f(x)$ and $\lim\limits_{x\to 1} f(x)$ do not exist since the limits on the left and the right are approaching different values.

Finding Infinite Limits Analytically

Our use of the property explained in the introduction is key to helping us deduce whether we have an infinite limit or not.

Example 2.2: Determining Infinite Limits Analytically

Let

$$f(x) = \frac{-4}{x+2}$$

and evaluate the following limits analytically.

a.) $\lim\limits_{x\to -2^-} f(x)$

Solution. For this limit we are approaching $x = -2$ from the left. We investigate what happens to this function for values $x < -2$. Now

$$x < -2 \implies x + 2 < 0$$

This means we have a negative denominator that is getting closer and closer to zero for values of x to the left of -2. Thus, we have a negative number divided by a smaller and smaller negative number. This tells us that our value of $f(x)$ is approaching a larger and larger positive number. Therefore,

$$\lim_{x\to -2^-} \frac{-4}{x+2} = \infty$$

b.) $\lim_{x \to -2^+} f(x)$

Solution. Now we are approaching $x = -2$ from the right. We investigate what happens to this function for values $x > -2$. Now

$$x > -2 \implies x + 2 > 0$$

This means we have a positive denominator that is getting closer and closer to zero for values of x to the right of -2. Thus, we have a negative number divided by a smaller and smaller positive number. This tells us that our value of $f(x)$ is approaching a larger and larger (in magnitude) negative number. Therefore,

$$\lim_{x \to -2^+} \frac{-4}{x+2} = -\infty$$

c.) $\lim_{x \to -2} f(x)$

Solution. Since

$$\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$$

the limit as $x \to -2$ does not exist.

Do not be quick to decide that all zeros of the denominator yield vertical asymptotes. The following example demonstrates when this may not be the case.

Example 2.3: Determining Infinite Limits Analytically

Let

$$f(x) = \frac{x^2 - x}{x^2 - 4x + 3}$$

and evaluate the following limits analytically and find all vertical asymptotes.

a.) $\lim_{x \to 1} f(x)$

Solution. First re-write $f(x)$ in factored form

$$f(x) = \frac{x^2 - x}{x^2 - 4x + 3} = \frac{x(x-1)}{(x-1)(x-3)}$$

We see that the term $(x - 1)$ cancels from both the numerator and denominator giving us

$$f(x) = \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = \frac{x}{x-3}$$

From algebra, we know that the point $x = 1$ is a "hole" in the graph of $f(x)$. Examining the limit here we see that

$$\lim_{x \to 1} \frac{x}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$$

Since this value is not $\pm\infty$ (i.e. is a finite value) the point $x = 1$ is '*not* a vertical asymptote of $f(x)$.

b.) $\lim\limits_{x\to 3^-} f(x)$

Solution. We are approaching $x = 3$ from the left. For $x < 3$ we have

$$x < 3 \implies x - 3 < 0$$

This means we have a negative denominator that is getting closer and closer to zero. Note that close to 3 our numerator will be positive. Thus, we have a positive number divided by a smaller and smaller negative number. This tells us that our value of $f(x)$ is approaching a larger and larger (in magnitude) negative number. Therefore,

$$\lim_{x\to 3^-} f(x) = -\infty$$

c.) $\lim\limits_{x\to 3^+} f(x)$

Solution. Now we are approaching $x = 3$ from the right. Now for $x > 3$ we have

$$x > 3 \implies x - 3 > 0$$

This means we have a positive denominator that is getting closer and closer to zero. Again, close to 3 our numerator will be positive. Thus, we have a positive number divided by a smaller and smaller positive number. This tells us that our value of $f(x)$ is approaching a larger and larger positive number. Therefore,

$$\lim_{x\to 3^+} f(x) = \infty$$

Limits at Infinity

Again consider the function $f(x) = \dfrac{1}{x^2}$

Now suppose that we let x get larger and larger. In other words let x approach infinity. In Table 2.1 we see some values for $f(x)$ as $x \to \infty$.

x goes to positive infinity

x	1	10	100	1000	100000	...	∞
$f(x)$	1	0.1	0.001	0.0001	0.00001	...	0

$f(x)$ approaches 0

Table 2.1: Values of $f(x) = \frac{1}{x^2}$ as $x \to \infty$

In this case we see that our function value will get smaller and smaller, so $f(x)$ gets closer and closer to zero. Letting x approach positive or negative infinity is what is known as a **limit at infinity**.

Definition 2.8: Limits at Infinity

If $f(x)$ becomes arbitrarily close to a finite number L

- For all x sufficiently large and positive, then

$$\lim_{x\to\infty} f(x) = L$$

and we say that the limit of $f(x)$ as x goes to infinity is L.

- For all x sufficiently large *in magnitude*, then

$$\lim_{x\to-\infty} f(x) = L$$

and we say that the limit of $f(x)$ as x goes to negative infinity is L.

Horizontal Asymptotes

Similar to infinite limits, limits at infinity allow us to find **horizontal asymptotes** of functions.

Definition 2.9: Horizontal Asymptotes

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to\infty} f(x) = L$$

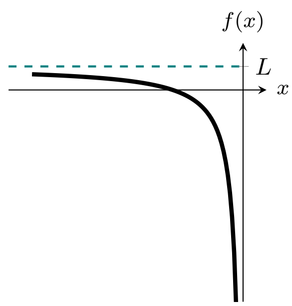


Fig. 2.5: $\lim_{x\to-\infty} f(x) = L$

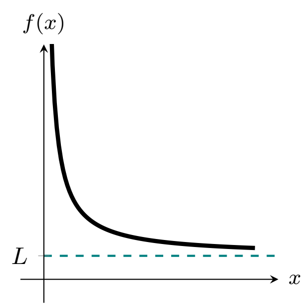


Fig. 2.6: $\lim_{x\to\infty} f(x) = L$

Slant Asymptotes

In addition to horizontal and vertical asymptotes, it is possible for a function to have a **slant asymptote** (also referred to as oblique asymptotes). To find a slant asymptote you must carry out polynomial long division for $f(x)$. In other words, divide the numerator by the denominator. If you need to review polynomial long division consult page 142 of the *Just in Time* text.

Evalutating Limits at Infinity

Similar strategies for finding finite limits are used here. All of the same limit laws apply. The behavior of the function $f(x) = \frac{1}{x^2}$ leads us to an important theorem in regards to limits at infinity.

Theorem 2.1

If $n > 0$ is a rational number then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

If $n > 0$ is a rational number such that x^n is defined for all x then

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Example 2.4: Limits at Infinity and Limit Laws

Evaluate $\displaystyle\lim_{x\to\infty}\left(5+\frac{1}{x}\right)$

Solution.

$$\begin{aligned} \lim_{x\to\infty}\left(5+\frac{1}{x}\right) &= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} && \text{Sum of limits} \\ &= 5+0=5 && \text{Limit of a constant and Theorem 2.1} \end{aligned}$$

The following example demonstrates.

Example 2.5: Limits of Polynomials at Infinity

Evaluate $\displaystyle\lim_{x\to\infty}\left(2x^4-x^2-8x\right)$

Solution. We are not able to use direct substitution here. If we were to just plug in ∞ we would obtain

$$\lim_{x\to\infty}\left(2x^4-x^2-8x\right) = 2(\infty)^4-(\infty)^2-8(\infty)$$

We have seen previously that this yields an indeterminate form and so is not a mathematically correct statement. Instead we first factor out the highest degree term

$$\begin{aligned} \lim_{x\to\infty}\left(2x^4-x^2-8x\right) &= \lim_{x\to\infty}\left(x^4\left(2-\frac{1}{x^2}-\frac{8}{x^3}\right)\right) && \text{Factor out } x^4 \\ &= \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty}\left(2-\frac{1}{x^2}-8\frac{1}{x^3}\right)\right) && \text{Apply limit laws} \\ &= \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x^2} - 8\lim_{x\to\infty}\frac{1}{x^3}\right) \\ &= (\infty)^4(2-(0)-8(0)) = \infty && \text{By Theorem 2.1} \end{aligned}$$

By using the properties given at the beginning of the lesson we see that this limit is positive infinity.

This example leads to another important theorem

Theorem 2.2: Limits of Polynomials at Infinity

Let n be a positive integer and let $p(x)$ be an nth degree polynomial with leading term $a_n x^n$.

a.) $\lim\limits_{x\to\infty} x^n = \infty$

b.) $\lim\limits_{x\to-\infty} x^n = \infty$, when n is even

c.) $\lim\limits_{x\to-\infty} x^n = -\infty$, when n is odd

d.) $\lim\limits_{x\to\pm\infty} p(x) = \lim\limits_{x\to\pm\infty} a_n x^n = \pm\infty$. Where the sign depends on the degree of the polynomial and the sign of the leading coefficient.

When dealing with rational functions as $x \to \pm\infty$, the behavior of the numerator with respect to the denominator is not always clear initially. In order to analyze these limits, we ideally want them to have the form of functions for which we know the limit as $x \to \pm\infty$, namely $\frac{1}{x^n}$. We can achieve this form by dividing both the numerator and denominator by the highest power term in the denominator.

Example 2.6: Limit at Infinity of a Rational Function

a.) Evaluate $\lim\limits_{x\to\infty} \dfrac{11x+2}{2x^3-1}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{11x+2}{2x^3-1} &= \lim_{x\to\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^2}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} && \text{Divide by } x^3 \text{ and simplify} \\
&= \frac{11\lim\limits_{x\to\infty}\frac{1}{x^2} + 2\lim\limits_{x\to\infty}\frac{1}{x^3}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty}\frac{1}{x^3}} && \text{Apply limit laws} \\
&= \frac{11(0)+2(0)}{2-(0)} = \frac{0}{2} = 0 && \text{By Theorem 2.1}
\end{aligned}$$

b.) Evaluate $\lim\limits_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} &= \lim_{x\to\infty} \frac{\frac{5x^2}{x^2}+\frac{8x}{x^2}-\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}} && \text{Divide by } x^2 \\
&= \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} && \text{Simplify}
\end{aligned}$$

$$= \frac{\lim\limits_{x\to\infty} 5 + 8 \lim\limits_{x\to\infty} \frac{1}{x} - 3 \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 3 + 2 \lim\limits_{x\to\infty} \frac{1}{x^2}} \qquad \text{Applying limit laws}$$

$$= \frac{5 + 8(0) + 3(0)}{3 + 2(0)} = \frac{5}{3} \qquad \text{By Theorem 2.1}$$

c.) Evaluate $\lim\limits_{x\to-\infty} \dfrac{2x^2 - 3}{7x + 4}$

Solution.

$$\lim_{x\to\infty} \frac{2x^2 - 3}{7x + 4} = \lim_{x\to\infty} \frac{\frac{2x^2}{x} - \frac{3}{x}}{\frac{7x}{x} + \frac{4}{x}} = \lim_{x\to\infty} \frac{2x - \frac{3}{x}}{7 + \frac{4}{x}} \qquad \text{Divide by } x \text{ and simplify}$$

$$= \frac{2 \lim\limits_{x\to\infty} x - 3 \lim\limits_{x\to\infty} \frac{1}{x}}{\lim\limits_{x\to\infty} 7 + 4 \lim\limits_{x\to\infty} \frac{1}{x}} \qquad \text{Apply limit laws}$$

$$= \frac{2(\infty) - 3(0)}{7 + 4(0)} \qquad \text{By Theorem 2.1}$$

$$= \frac{-\infty}{7} = -\infty \qquad \text{By properties of infinity}$$

Each of the limits in the previous example demonstrates the behavior seen in algebra courses for finding horizontal asymptotes.

Limits at infinity that contain roots can be a little trickier to deal with than the examples we have seen thus far. Note that $\sqrt{x^2}$ is equivalent to $|x|$, the absolute value function. In other words,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

This means that we must pay attention to whether we are approaching positive or negative infinity.

Example 2.7: Limit at Infinity of a Root Function

Evaluate the following limits for $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

a.) $\lim\limits_{x\to\infty} f(x)$

Solution. In this case, dividing by the highest power term in the denominator means that we're actually dividing by x^2 when under the square root in the numerator. It may make more sense to factor and cancel terms rather than divide in problems like this.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2 + \frac{1}{x^2}\right)}}{x\left(3 - \frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2 + \frac{1}{x^2}}}{x\left(3 - \frac{5}{x}\right)}$$

Since $x \to \infty$ we use the fact that $\sqrt{x^2} = x$ for $x > 0$. Thus,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} &= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} && \text{Cancel common terms} \\
&= \frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - 5 \lim\limits_{x\to\infty} \frac{1}{x}} && \text{Apply limit laws} \\
&= \frac{\sqrt{2+0}}{3-5(0)} = \frac{\sqrt{2}}{3} && \text{Theorem 2.1}
\end{aligned}$$

b.) $\lim\limits_{x\to-\infty} f(x)$

Solution. Similar to the last example we get to the point

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

Since we have $x \to -\infty$ we use the fact that $\sqrt{x^2} = -x$ for $x < 0$. Thus,

$$\begin{aligned}
\lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{\cancel{-x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} &= \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} && \text{Cancel common terms} \\
&= \frac{-\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to-\infty} \frac{1}{x^2}}}{\lim\limits_{x\to-\infty} 3 - 5 \lim\limits_{x\to-\infty} \frac{1}{x}} && \text{Apply limit laws} \\
&= -\frac{\sqrt{2+0}}{3-5(0)} = -\frac{\sqrt{2}}{3} && \text{Theorem 2.1}
\end{aligned}$$

Practice Problems

Section 2.4
8-34

Section 2.5
15-34, 41-44, 52-61