Lesson 1

Introduction to Limits and their Properties

Sections
2.1, 2.2, 2.3

Introduction to Limits

Consider the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

pictured in Figure 1.1. We see that $f(x)$ in has a "hole" at $x = 2$. In other words, $f(x)$ is defined for every value of x except $x = 2$.

Even though $f(x)$ is not defined at $x = 2$, we still want to know its behavior as we get closer and closer to this x value. We do this by looking at the behavior on the left hand side and on the right hand side of $x = 2$.

In Table 1.1 we see some values of $f(x)$ for x values close to $x = 2$.

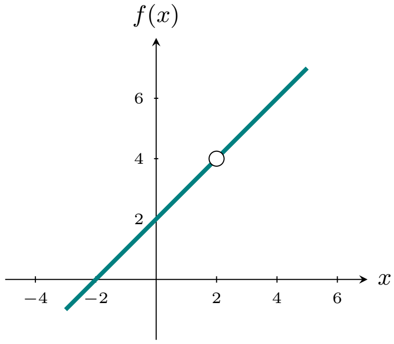

Fig. 1.1: Graph of $f(x) = \frac{x^2-4}{x-2}$

x approaches 2 from the left → ← x approaches 2 from the right

x	1.9	1.99	1.999	2	2.0001	2.001	2.01
$f(x)$	3.9	3.99	3.999	?	4.0001	4.001	4.01

$f(x)$ approaches 4 → ← $f(x)$ approaches 4

Table 1.1: Some values of $f(x) = \frac{x^2-4}{x-2}$

We observe that $f(x)$ approaches 4 as x approaches 2 from *both* the left and the right hand sides. The value that $f(x)$ approaches is know as the **limit** of $f(x)$ as x approaches 2.

Definition 1.1: Limit of a Function

Let $f(x)$ be defined on an open interval about a (except possibly at a itself). If $f(x)$ gets arbitrarily close to L for all x sufficiently close to a (but not equal to a), we write

$$\lim_{x \to a} f(x) = L$$

and say that the limit of $f(x)$ as x approaches a is equal to L.

Example 1.1: Evaluating a Limit Numerically

Evaluate the function $f(x) = \frac{x}{\sqrt{x+1}-1}$ at several points near $x = 0$ and use the result to estimate

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$$

Solution. We list our values in a table to more easily examine the behavior.

x approaches 0 from the left → ← x approaches 0 from the right

x	-0.01	-0.001	0	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.999950	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2 → ← $f(x)$ approaches 2

From the results in the table we estimate that

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1} = 2$$

We can also see this behavior by examining the graph of $f(x)$

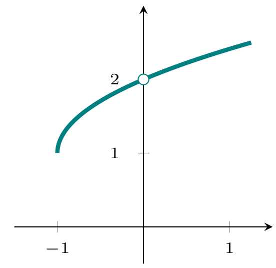

Fig. 1.2: Graph of $f(x) = \frac{x}{\sqrt{x+1}-1}$

Limits that Do Not Exist

There are several situations where limits do not exist. We will demonstrate these through some examples. Before we begin, we need an important definition. We have mentioned $\lim_{x\to a} f(x) = L$. This is what is known as a **two sided limit** since we examine behavior as we approach a from the left *and* from the right according to its graph. It is also possible to talk about these different "sides" of the limit as limits themselves.

Definition 1.2: One Sided Limits

Suppose that f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to a value L as x approaches a from the left then the **left side limit** is written as

$$\lim_{x\to a^-} f(x) = L$$

Suppose that f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to a value L as x approaches a from the right then the **right side limit** is written as

$$\lim_{x\to a^+} f(x) = L$$

Example 1.2: $f(x)$ has Differing Left and Right Side Limits

Determine whether $\lim_{x\to 0} \frac{|x|}{x}$ exists.

Solution. Let $f(x) = \frac{|x|}{x}$. Since $f(x)$ contains the absolute value we must examine two cases; when $x > 0$ and when $x < 0$.

- When $x > 0$ we have

$$\frac{|x|}{x} = \frac{x}{x} = 1$$

using limit notation we write

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

- When $x < 0$ we have

$$\frac{|-x|}{-x} = \frac{x}{-x} = -1$$

using limit notation we write

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

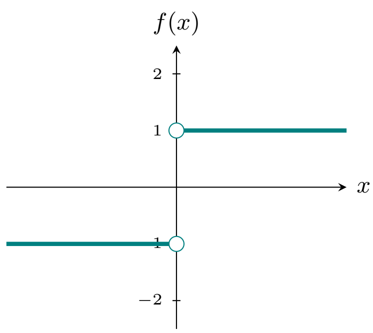

Fig. 1.3: Graph of $f(x) = \frac{|x|}{x}$

By examining the graph of $f(x)$ we can see that as we approach the value $x = 0$ our $f(x)$ approaches two different values. Which value do we use for $\lim_{x\to 0} f(x)$? The answer is neither.

Since we have conflicting information for which value $f(x)$ is approaching we say that *the limit does not exist*. This is also most commonly abbreviated by DNE.

Our findings in this example are precisely stated in the following theorem.

Theorem 1.1: Existence of Two Sided Limits

$\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

In other words, a limit exists iff its left side limit is equal to its right side limit.

Example 1.3: $f(x)$ Increases or Decreases without Bound

Determine whether $\lim_{x \to 0} \frac{1}{x^2}$ exists.

Solution. Lets establish a table of function values as x approaches 0.

x approaches 0 from the left $\Rightarrow$ $\Leftarrow$ x approaches 0 from the right

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	100	10000	1000000	?	1000000	10000	100

$f(x)$ approaches ∞ $\Rightarrow$ $\Leftarrow$ $f(x)$ approaches ∞

We see that as x gets closer and closer to zero, $f(x)$ gets larger and larger.

In this case, we see say that $f(x)$ **increases without bound**. This means that the limit L approaches infinity, which is not a real number!

Thus, according to our definition this limit *does not exist*. This is also true for functions that **decreases without bound**.

We can also see this in the graph of $f(x) = \frac{1}{x^2}$ in Figure 1.4.

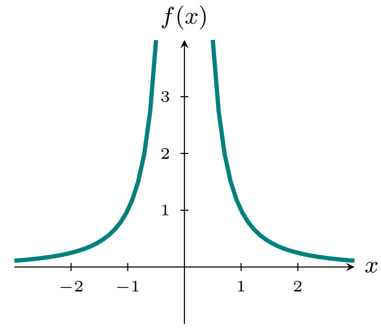

Fig. 1.4: Graph of $f(x) = \dfrac{1}{x^2}$

We will examine these types of limits in more detail in Lesson 2. Next, we discuss a typical example that often arises when dealing with periodic functions like the trigonometric functions.

Example 1.4: $f(x)$ has Oscillating Behavior

Determine whether $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ exists.

Solution. The graph of $f(x) = \sin\left(\frac{1}{x}\right)$ is seen in Figure 1.5.

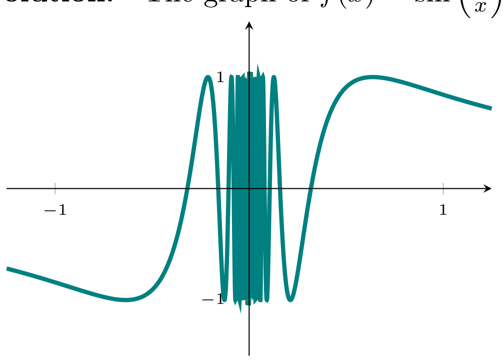

Fig. 1.5: Graph of $f(x) = \sin\left(\frac{1}{x}\right)$

We observe that as x gets closer and closer to zero, $f(x)$ oscillates between -1 and 1 infinitely as often as x approaches 0. So $f(x)$ doesn't approach a fixed value as x approaches 0.

In this case we also have a limit that *does not exist*.

To summarize the results from these examples, a limit does not exist when

- $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$
- $f(x)$ increases or decreases without bound as $x \to a$
- $f(x)$ oscillates between two fixed values as $x \to a$

Remark: It is not always possible (or wise) to find limits numerically using a table. In most cases, we must find limits analytically. The word "analytic" in mathematics essentially means exact. In other words, this involves using algebraic methods to find a solution rather than numerical ones.

Properties of Limits

We have seen that $\lim\limits_{x\to a} f(x)$ does not necessarily depend on the value of f at $x = a$ since often it is not even defined there. However, for functions that are continuous at x we can have

$$\lim_{x\to a} f(x) = f(a)$$

This is known as evaluating a limit by **direct substitution**. We simply plug in a to f and evaluate to get our result. For many limits, we must carry out some algebraic manipulation before we can actually use direct substitution.

Before we can start manipulating limits, we need to know what we can and cannot do with limits. First we have some basic properties.

Theorem 1.2: Basic Properties of Limits

1. $\lim\limits_{x\to a} b = b$ **2.** $\lim\limits_{x\to a} x = a$ **3.** $\lim\limits_{x\to a} x^n = a^n$

Example 1.5: Evaluating Basic Limits

Evaluate the following limits.

a.) $\lim\limits_{x\to 2} 3$

Solution. Since $f(x)$ is always 3 it doesn't matter what x-value we approach. In other words, we have $f(2) = 3$ thus, $\lim\limits_{x\to 2} 3 = 3$

b.) $\lim\limits_{x\to -4} x$

Solution. We know that $f(x) = x$ is a continuous function and that $f(-4) = -4$. Thus, $\lim\limits_{x\to -4} x = -4$

c.) $\lim\limits_{x\to 2} x^2$

Solution. Since $f(x) = x^2$ is a continuous function and we know that $f(2) = (2)^2$. Thus,

$$\lim_{x\to 2} x^2 = 2^2 = 4$$

Now what about when we have multiple terms with several different operations involved in a limit? The next properties listed will allow us to handle these cases. They are commonly referred to as the **limit laws**.

Theorem 1.3: The Limit Laws

Let c be a real number, let $m > 0$ and $n > 0$ be integers. Assuming that

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist, then the following hold.

1. Sum $\quad \lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. Difference $\quad \lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. Constant Multiple $\quad \lim_{x \to a} c(f(x) = c \lim_{x \to a} f(x)$

4. Product $\quad \lim_{x \to a}(f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$

5. Quotient $\quad \lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad$ where $\lim_{x \to a} g(x) \neq 0$

6. Power $\quad \lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n$

7. Fractional Power $\quad \lim_{x \to a}(f(x))^{n/m} = \left(\lim_{x \to a} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a if n/m is reduced to lowest terms and m is even.

We now examine some examples applying the limit laws. It may seem tedious to write out every step in these examples but it is important to recognize why direct substitution works in certain cases.

Example 1.6: Applying Limit Laws

Evaluate the following limits.

a.) $\lim_{x\to 5}(2x^4 - 3x + 4)$

Solution.

$$\begin{aligned}
\lim_{x\to 5}(2x^4 - 3x + 4) &= \lim_{x\to 5} 2x^4 - \lim_{x\to 5} 3x + \lim_{x\to 5} 4 && \text{Apply laws 1 \& 2} \\
&= 2\lim_{x\to 5} x^4 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Apply law 3} \\
&= 2\left(\lim_{x\to 5} x\right)^4 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Apply law 6} \\
&= 2(5)^4 - 3(5) + 4 = 39 && \text{Use direct substitution}
\end{aligned}$$

b.) $\lim_{x\to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution.

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)} && \text{Law 5} \\
&= \frac{\lim_{x\to -2} x^3 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - \lim_{x\to -2} 3x} && \text{Laws 1 \& 2} \\
&= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x} && \text{Law 3} \\
&= \frac{\left(\lim_{x\to -2} x\right)^3 + 2\left(\lim_{x\to -2} x\right)^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x} && \text{Law 6} \\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11} && \text{Direct substitution}
\end{aligned}$$

By applying the limit laws in the previous examples we have demonstrated another valuable theorem for finding limits of polynomial and rational functions. This theorem essentially tells us when we can actually use direct substitution in a limit evaluation.

Theorem 1.4: Limits of Polynomial and Rational Functions

Assume that p and q are polynomials and a is constant.

a. Polynomial Functions $\lim_{x\to a} p(x) = p(a)$

b. Rational Functions $\lim_{x\to a} \dfrac{p(x)}{q(x)} = \dfrac{p(a)}{q(a)}$ provided $q(a) \neq 0$

The limit laws given are also true for one sided limits, with the exception of fractional powers. This rule can be re-stated as follows

Theorem 1.5: One Sided Limits with Fractional Powers

Assume that $m > 0$, $n > 0$ are integers and that n/m is reduced to lowest terms. Then the following hold.

a. $\displaystyle\lim_{x\to a^+} (f(x))^{n/m} = \left(\lim_{x\to a^+} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a with $x > a$ if m is even.

b. $\displaystyle\lim_{x\to a^-} (f(x))^{n/m} = \left(\lim_{x\to a^-} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a with $x < a$ if m is even.

Example 1.7: Finding One-Sided Limits Analytically

Consider the function

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ (x-2)^2, & x \geq 1 \end{cases}$$

Find the following.

a.) $\displaystyle\lim_{x\to 1^-} f(x)$

Solution. For $x \to 1^-$ we are approaching 1 from the left hand side, where $x \leq 1$ so we examine $f(x) = x^2 + 1$. We have

$$\lim_{x\to 1^-} x^2 + 1 = (-1)^2 + 1 = 2$$

b.) $\displaystyle\lim_{x\to 1^+} f(x)$

Solution. For $x \to 1^-$ we are approaching 1 from the right hand side, where $x \geq 1$ so we examine $f(x) = (x-2)^2$. We have

$$\lim_{x\to 1^+} (x-2)^2 = (1-2)^2 = (-1)^2 = 1$$

c.) $\displaystyle\lim_{x\to 1} f(x)$

Solution. Since

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

then $\displaystyle\lim_{x\to 1} f(x)$ *does not exist* (DNE)

Techniques for Finding Limits

There are two main methods used to algebraically manipulate a limit in order to reach an analytic solution.

Factor and Cancel

This method is most often used when given a rational function. Simply factor the numerator or denominator and then cancel common terms.

Example 1.8: Finding a Limit by Factoring and Canceling

Find $\lim\limits_{x\to -3} \dfrac{x^2+x-6}{x+3}$.

Solution. We cannot use direct substitution here since it would result in a zero for the denominator.

$$\begin{aligned}
\lim_{x\to -3} \frac{x^2+x-6}{x+3} &= \lim_{x\to -3} \frac{(x+3)(x-2)}{x+3} && \text{Factor the numerator} \\
&= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} && \text{Cancel common terms} \\
&= \lim_{x\to -3} x-2 \\
&= -3-2 = -5 && \text{Use direct substitution}
\end{aligned}$$

Multiplying by the Conjugate

Recall that when we rationalize a denominator (or a numerator) of an algebraic expression we multiply by it's **conjugate**. For example, the expression $a+b$ has the conjugate $a-b$. Multiplying conjugates together gives us

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

You should recognize this as the **difference of squares**. We most commonly use this technique when dealing with radicals. The difference of squares property since it allows us to eliminate the radical. This frees up terms so that they can (potentially) be eliminated.

Remark: In previous math courses you may have been required to "rationalize the denominator" when you have a result like $\frac{2}{\sqrt{2}}$. There is *no* mathematical necessity in writing results in this form! Usually it is actually more convenient to leave the radical in the denominator!

Example 1.9: Finding Limits Using Conjugates

Find $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$.

Solution. We cannot use direct substitution here since it would result in the denominator being equal to zero. In order to find this limit we will use the conjugate of $\sqrt{x+1}-1$. This will allow us to free the terms under the radical.

The conjugate of this algebraic expression is $\sqrt{x+1}+1$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) && \text{Multiply by the conjugate} \\
&= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}
\end{aligned}$$

Remark: You may be tempted to distribute terms at this step. Don't! Remember that our goal is ultimately cancel the x term in the denominator!

$$\begin{aligned}
&= \lim_{x\to 0} \frac{(\sqrt{x+1})^2-(1)^2}{x(\sqrt{x+1}+1)} && \text{Apply difference of squares} \\
&= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} \\
&= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} && \text{Cancel common terms} \\
&= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \\
&= \frac{1}{\sqrt{(0)+1}+1} = \frac{1}{2} && \text{Use direct substitution}
\end{aligned}$$

Limits of Trigonometric Functions

The following are some helpful limits to recognize for common transcendental functions.

Theorem 1.6: Limits of Trancendental Functions

Let a be a real number in the domain of the given function.

1. $\lim_{x\to a} \sin x = \sin a$

2. $\lim_{x\to a} \cos x = \cos a$

3. $\lim_{x\to a} \tan x = \tan a$

4. $\lim_{x\to a} b^x = b^a$

5. $\lim_{x\to a} \csc x = \csc a$

6. $\lim_{x\to a} \sec x = \sec a$

7. $\lim_{x\to a} \cot x = \cot a$

8. $\lim_{x\to a} \ln x = \ln a$

The following are some special limits that can be helpful. These results are found by applying the **squeeze theorem** which is covered in the next section.

Theorem 1.7: Special Limits

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ **3.** $\lim_{x\to 0} (1+x)^{1/x} = e$

The Squeeze Theorem

We have now developed some techniques for finding limits analytically. Unfortunately, much of the time there are limits that we cannot find using these direct methods. The **squeeze theorem** (also sometimes referred to as the sandwich or pinching theorem) is an important tool in finding some of these limits.

Suppose that we have a function $f(x)$ for which we cannot directly find $\lim_{x\to a} f(x)$. In order to use the squeeze theorem we must know two functions that will "squeeze" $f(x)$ between them near the point $x = a$.

Theorem 1.8: The Squeeze Theorem

Suppose we have three functions f, g, and h where

$$g(x) \leq f(x) \leq h(x)$$

If $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$ then,

$$\lim_{x\to a} f(x) = L$$

The best way to see how the squeeze theorem works is to look at an example of how it may be applied.

Example 1.10: Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} x^2 \cos(20\pi x) = 0$$

Solution. Let's break this down a bit. With our knowledge of graph transformations we know that $\cos(20\pi x)$ is just $\cos(x)$ with a horizontal stretch. The range remains unchanged. We also know that $\cos(x)$ has range $[-1, 1]$. In other words, $-1 \leq \cos(x) \leq 1$ and so we have

$$-1 \leq \cos(20\pi x) \leq 1$$

If we multiply each part of this inequality by x^2 we get

$$-x^2 \leq x^2 \cos(20\pi x) \leq x^2$$

In the middle of this inequality we have our function $f(x)$ for which we want to find the limit.

Our two outside functions which we are "squeezing" $f(x)$ at $x = 0$ are $g(x) = -x^2$ and $h(x) = x^2$. This is seen in Figure 1.6.

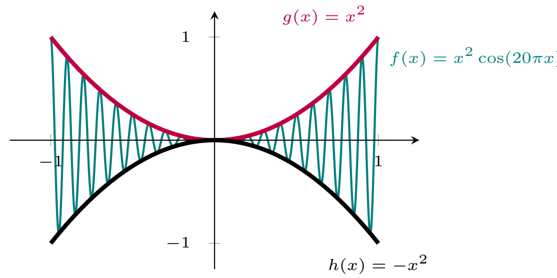

Fig. 1.6: Graph of g, h and f

Now we have the 3 functions required by the definition.

We know by direct substitution that

$$\lim_{x \to 0} -x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} x^2 = 0$$

So if we take the limit of each part of this inequality as $x \to 0$ we have

$$\lim_{x \to 0} -x^2 \leq \lim_{x \to 0} x^2 \cos(20\pi x) \leq \lim_{x \to 0} x^2$$

and this gives us that

$$0 \leq \lim_{x \to 0} x^2 \cos(20\pi x) \leq 0$$

Now the only value that can exist between 0 and 0? It's zero! So *by the squeeze theorem*, we must have

$$\lim_{x \to 0} x^2 \cos(20\pi x) = 0$$

Practice Problems

<u>Section 2.1</u>
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<u>Section 2.2</u>
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<u>Section 2.3</u>
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