1. $\lim_{x\to 0} \frac{3\sin(4x)}{5x}$

Solution:

Since $\frac{3\sin(0)}{5(0)} = \frac{0}{0}$ $\Rightarrow$ Indeterminate Form $\frac{0}{0}$

So we can use L'Hopital's Rule directly

Evaluating the limit

$$\lim_{x\to 0} \frac{3\sin(4x)}{5x} \overset{H}{=} \lim_{x\to 0} \frac{12\cos(4x)}{5}$$

$$= \frac{12\cos(0)}{5}$$

$$= \frac{12(1)}{5}$$

$$= \frac{12}{5}$$

2. $\lim_{x\to\infty} \frac{e^{3x}}{3e^{3x}+5}$

Solution:

Since $\frac{e^{\infty}}{3e^{\infty}+5} = \frac{\infty}{\infty}$ $\Rightarrow$ Indeterminate Form $\frac{\infty}{\infty}$

So we can use L'Hopital's Rule directly

Evaluating the limit

$$\lim_{x\to\infty} \frac{e^{3x}}{3e^{x}+5} = \lim_{x\to\infty} \frac{3\cancel{e^{3x}}}{9\cancel{e^{3x}}}$$

$$= \lim_{x\to\infty} \frac{3}{9}$$

$$= \frac{3}{9}$$

$$= \frac{1}{3}$$

3. $\displaystyle\lim_{x\to 0^+} \sin(x)\sqrt{\frac{1-x}{x}}$

Solution:

Since $\sin(0)\cdot\sqrt{\frac{1-0}{0}} = 0\cdot\infty \Rightarrow$ Indeterminate Form $0\cdot\infty$

This is Not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

We must rewrite so we get the proper Form to use L'Hopital's Rule

Rewriting: $\displaystyle\lim_{x\to 0^+} \sin(x)\sqrt{\frac{1-x}{x}} = \lim_{x\to 0^+} \sin(x)\sqrt{\frac{x(1-x)}{x^2}}$

$$= \lim_{x\to 0^+} \sin(x)\frac{\sqrt{x(1-x)}}{\sqrt{x^2}}$$

$$= \lim_{x\to 0^+} \frac{\sin(x)}{x}\sqrt{x(1-x)}$$

$$= \lim_{x\to 0^+} \frac{\sin(x)}{x}\cdot\lim_{x\to 0^+}\sqrt{x(1-x)}$$

For the first limit

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = \lim_{x\to 0^+} \frac{\cos(x)}{1} = \cos(0) = 1$$

For the second limit

$$\lim_{x\to 0^+} \sqrt{x(1-x)} = \sqrt{0(1-0)} = 0$$

Combining these two Facts we then have

$$\lim_{x\to 0^+} \sin(x)\sqrt{\frac{1-x}{x}} = \lim_{x\to 0^+} \frac{\sin(x)}{x}\cdot\lim_{x\to 0^+}\sqrt{x(1-x)}$$

$$= 1\cdot 0 = 0$$

4. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Solution:

Since $\left(1+\frac{1}{\infty}\right)^\infty = 1^\infty$ $\Longrightarrow$ Indeterminate Form : 1^∞

This is Not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

This is a fcn of form $y = f(x)^{g(x)}$ Limits of this type are handled in a way similiar to how we took derivatives.

Apply Properties of Logs.

Let $y = \left(1+\frac{1}{x}\right)^x$

take ln of each side

$$\ln(y) = \ln\left(\left(1+\frac{1}{x}\right)^x\right)$$

$$\Longrightarrow \ln(y) = x\ln\left(1+\frac{1}{x}\right)$$

Alternatively you can also let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\Longrightarrow \ln(y) = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)$$

$$= \lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right)$$

Recall we can move this limit "outside" since $\ln(x)$ is a continuous fcn! Then proceed similarly

So our limit becomes

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \Longrightarrow \text{Indet. Form } 0\cdot\infty$$

So we rewrite again

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \Longrightarrow \text{Now has indet. form } \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} 1+\frac{1}{x}$$

$$= 1+\frac{1}{\infty} = 1+0 = 1$$

So we have $\lim_{x\to\infty}\ln(y) = 1$

Exponentiating each side : $\lim_{x\to\infty} e^{\ln(y)} = e^1 \Longrightarrow \lim_{x\to\infty} y = e$

So our final result is: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

5. $\lim_{x\to 0^+} x^{1/\ln(x)}$

Solution:

Since $(0)^{\frac{1}{0}} = 0^{\infty}$ $\Longrightarrow$ Indeterminate Form: 0^{∞}

This is <u>Not</u> $\frac{0}{0}$ or $\frac{\infty}{\infty}$

This is a fcn of form $y = f(x)^{g(x)}$ Limits of this type are handled in a way similiar to how we took derivatives.

Let $y = \lim_{x\to 0^+} x^{\frac{1}{\ln(x)}}$ taking natural log of both sides:

$$\begin{aligned}\ln(y) &= \ln\left(\lim_{x\to 0^+} x^{1/\ln(x)}\right)\\ &= \lim_{x\to 0^+} \ln(x^{1/\ln(x)})\\ &= \lim_{x\to 0^+}\left(\frac{1}{\cancel{\ln(x)}}\cancel{\ln(x)}\right)\\ &= \lim_{x\to 0^+} 1 = 1\end{aligned}$$

Note that this process is equivalent to the one used in #4. We're just starting a little differently. Use whichever way you prefer/makes more sense to your brain

So we have

$$\ln(y) = 1$$

Exponentiating both sides

$$e^{\ln(y)} = e^1 \quad \Longrightarrow \quad y = e$$

$$\Longrightarrow \quad y = \lim_{x\to 0^+} x^{\frac{1}{\ln(x)}} = e$$

6. $\displaystyle\lim_{x\to 1^+}\left(\frac{1}{\ln(x)}-\frac{1}{x-1}\right)$

Solution:

Since $\frac{1}{\ln(1)} - \frac{1}{1-1} = \frac{1}{0} - \frac{1}{0}$
$= \infty - \infty$

$\Rightarrow$ Indeterminate Form: $\infty - \infty$

This is <u>Not</u> $\frac{0}{0}$ or $\frac{\infty}{\infty}$

We cannot use L'Hopital's Rule
We must rewrite so we get the proper Form

Write as a single fraction by giving a common denom.

Note: $\frac{1}{\ln(x)}\left(\frac{x-1}{x-1}\right) - \frac{1}{(x-1)}\left(\frac{\ln(x)}{\ln(x)}\right) = \frac{x-1-\ln(x)}{(x-1)\ln(x)}$

So limit becomes

$\displaystyle\lim_{x\to 1^+}\frac{x-1-\ln(x)}{(x-1)\ln(x)}$ since $\frac{1-1-\ln(1)}{(1-1)\ln(1)} = \frac{0}{0}$

$\Rightarrow$ Indeterminate Form: $\frac{0}{0}$

Now we can apply L'Hopital's Rule

Evaluating the limit:

$$\lim_{x\to 1^+}\frac{x-1-\ln(x)}{(x-1)\ln(x)} \overset{H}{=} \lim_{x\to 1^+}\left(\frac{1-\frac{1}{x}}{\ln(x)+\frac{1}{x}(x-1)}\right)$$

$$= \lim_{x\to 1^+}\frac{1-\frac{1}{x}}{\ln(x)+1-\frac{1}{x}}$$

$$= \lim_{x\to 1^+}\frac{\cancel{\frac{1}{x}}(x-1)}{\cancel{\frac{1}{x}}(x\ln(x)+x-1)}$$

$$= \lim_{x\to 1^+}\frac{x-1}{x\ln(x)+x-1}$$

We still have an indet. form $\frac{0}{0}$
$\Rightarrow$ use Hopital's again

$$= \lim_{x\to 1^+}\frac{1}{\ln(x)+x\left(\frac{1}{x}\right)+1}$$

$$= \lim_{x\to 1^+}\frac{1}{\ln(x)+1+1}$$

$$= \frac{1}{\ln(1)+1+1} = \frac{1}{2}$$