1. Find the linear approximation to $f(x) = 3xe^{2x-10}$ at $x = 5$.

Solution:

Linear approx is given by

$$L(x) = f(a) + f'(a)(x-a)$$

We have

$$a = 5$$
$$f(a) = f(5) = 3(5)e^{2(5)-10} = 15$$

$$f'(x) = \frac{d}{dx}[p \cdot q] = p'q + pq'$$

$$p = 3x \qquad q = e^{2x-10}$$
$$p' = 3 \qquad q' = \frac{d}{dx}[e^u] = u'e^u = 2e^{2x-10}$$
$$u = 2x - 10$$
$$u' = 2$$

$$\Rightarrow f'(x) = 3e^{2x-10} + 6xe^{2x-10}$$

$$f'(a) = f'(5) = 3e^{2(5)-10} + 6(5)e^{2(5)-10} = 3 + 30 = 33$$

So our linear approximation is

$$L(x) = f(5) + f'(5)(x-5)$$
$$= 15 + 33(x-5)$$

2. Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$.

Solution:

The actual change in y is Δy:

$$\Delta y = y(2.03) - y(2)$$
$$= \cos((2.03)^2+1) - (2.03) - [\cos((2)^2+1) - (2)]$$
$$= 0.083581127$$

Find dy:

$$dy = y'(x)\,dx$$
$$= (-2x\sin(x^2+1) - 1)\,dx$$

Since $\Delta x = 2.03 - 2 = 0.03$,

we assume that $dx \approx \Delta x = 0.03$

$$\Rightarrow dy = (-2(2)\sin((2)^2+1) - 1)(0.03)$$
$$= 0.085070913$$

Then we see that if Δx is small, we will have $dy \approx \Delta y$

3. The sides of a cube are found to be 6 feet in length with a possible error of no more than 1.5 inches. What is the maximum possible error in the volume of the cube if we use this value of the length of the side to compute the volume?

Solution:

Given: Let x be side of cube

Volume of cub: $V(x) = x^3$

Side of cube: 6 ft

Side length error:

$$dx \approx \Delta x = 1.5 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} = 0.125 \text{ ft}$$

Asked: max possible error

i.e. max of $\Delta V \approx dV$

Compute: $\Delta V \approx dV = V'(x)\, dx$

$$= 3x^2\, dx$$

Since

$$dx \approx \Delta x = 0.125 \text{ ft}$$

$$\Longrightarrow \Delta V \approx dV = 3(6)^2(0.125) = 13.5 \text{ ft}^3$$

Answer

If there is a maximum possible error of 1.5 in in the length of a side, then there will be a max possible error of 13.5 ft^3 in the volume.

4. Find two numbers whose difference is 100 and whose product is a minimum.

Solution:

Given Difference of two #'s:

$$x - y = 100 \quad \Rightarrow \text{ constraint equation}$$

Asked

Minimum product: $P = x \cdot y$

Compute

Use constraint eqn to write P as fcn only of

$$x - y = 100 \quad \Rightarrow \quad x = 100 + y$$

Plug this into P:

$$P = (100 + y)y = 100y + y^2$$

Find Derivative

$$P' = 100 + 2y$$

Critical pts: $P' = 0$

$$100 + 2y = 0 \quad \Rightarrow \quad y = \frac{-100}{2} = -50$$

Verify its a minimum:

By 2nd D. test: $P'' = 100 > 0$

so $y = -50$ is a minimum

Find x: $x = 100 + (-50) = 50$

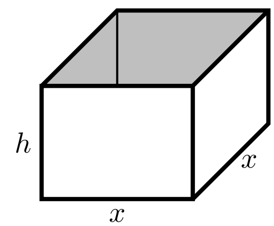
Answer

The difference of $x = 50$ & $y = -50$ is 100

& their product is a minimum: $P = (50)(-50) = -2500$

5. A square-based, box-shaped shipping crate is designed to have a volume of $16\,\text{ft}^3$. The material used to make the base costs twice as much per square foot as the material in the sides, and the material used to make the top costs half as much per square foot as the material in the sides. What are the dimensions of the crate that minimize the cost of materials?

Solution:



Given $V = 16\text{ ft}^3$

Volume of box : $V = x^2 h$

Constraint : $x^2 h = 16 \implies h = \frac{16}{x^2}$

Price (p) of material:

$\implies p$ = cost of side material

$\implies 2p$ = cost of base material

$\implies \frac{1}{2}p$ = cost of top material

Asked

minimize cost (C) of box :

surface area of box : base + 4 sides + top

$x^2 \quad 4xy \quad + \quad x^2$

$$\implies C = 2px^2 + 4pxy + \frac{1}{2}px^2$$

$$= \frac{5}{2}px^2 + 4pxy \Longleftarrow \text{want minimum}$$

Compute

Since $h = \frac{16}{x^2}$

$$\implies C = \frac{5}{2}px^2 + 4px\left(\frac{16}{x^2}\right)$$

$$= \frac{5}{2}px^2 + \frac{64p}{x} = \frac{5}{2}px^2 + 64px^{-1} = p\left(\frac{5}{2}x^2 + 64x^{-1}\right)$$

Find C.P's : $C' = p(5x - 64x^{-2})$

For $C' = 0 \implies p(5x - 64x^{-2}) = 0 \implies 5x - 64x^{-2} = 0$

$\implies 5x = 64x^{-2}$

$\implies x^3 = \frac{64}{5}$

$\implies x = \sqrt[3]{\frac{64}{5}} = \frac{4}{\sqrt[3]{5}}$

By the 2nd Derivative Test:

$$C'' = p(5 + 128x^{-3})$$

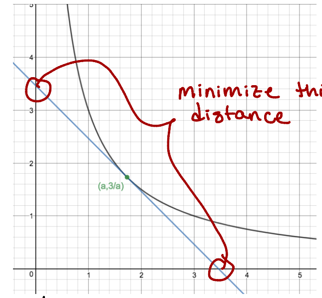
$$C''\left(\frac{4}{\sqrt[3]{5}}\right) = p\left(5 + 128\left(\frac{4}{\sqrt[3]{5}}\right)^{-3}\right) = 15p > 0 \quad \text{since we know } p > 0$$

So $x = \frac{4}{\sqrt[3]{5}}$ is a minimum value.

Answer Dimensions that will minimize cost are

$$x = \frac{4}{\sqrt[3]{5}} \quad \& \quad h = \frac{16}{(\sqrt[3]{5})^2} = 4\cdot\sqrt[3]{5}$$

6. What is the shortest possible length of the line segment that is cut off by the first quadrant and is tangent to the curve $y = \dfrac{3}{x}$ at some point?



Solution:

Given $\quad y = \frac{3}{a}$

Asked $\quad$ Shortest length of line tangent to $y = \frac{3}{x}$ & cut by 1st quadrant

i.e. shortest distance btwn intercepts of tangent line.

Compute $\quad$ General pt on $y = \frac{3}{x}$ is $\left(a, \frac{3}{a}\right)$

Find tangent line @ pt $\left(a, \frac{3}{a}\right)$: $\quad L(x) = y(a) + y'(a)(x-a)$

where $y(a) = \frac{3}{a}$

$y'(x) = -3x^{-2} \Rightarrow y'(a) = -3a^{-2}$

$$\Rightarrow L(x) = \frac{3}{a} - 3a^{-2}(x-a)$$
$$= -3a^{-2}x + \frac{6}{a}$$

Intercepts of Tangent line: $\quad$ y intercept: $\frac{6}{a}$

x intercept: $-3a^{-2}x + \frac{6}{a} = 0 \Rightarrow \frac{6}{a} = \frac{3}{a^2}x \Rightarrow \frac{6}{a}\left(\frac{a^2}{3}\right) = x \Rightarrow x = 2a$

So we need distance btwn pts $\left(0, \frac{6}{a}\right)$ & $(2a, 0)$ to be a minimum

Distance btwn two pts:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \left((2a - 0)^2 + \left(\frac{6}{a} - 0\right)^2\right)^{1/2}$$
$$= \left(4a^2 + \frac{36}{a^2}\right)^{1/2}$$

Let $d^2 = h \Rightarrow h = 4a^2 + \frac{36}{a^2}$ $\quad$ if we minimize h we minimize d

b/c derivative of d is more complicated!

Find critical pts of h: $\quad h' = 8a - 72a^{-3}$

$h'(a) = 0 \Rightarrow 8a - 72a^{-3} = 0$

$8a = 72a^{-3}$

$a^4 = 9 \Rightarrow a = \sqrt{3}$

By 2nd Derivative test: $\quad h'' = 8 + \frac{216}{a^4}$

$h''(\sqrt{3}) = 8 + \frac{216}{(\sqrt{3})^4} > 0 \Rightarrow a = \sqrt{3}$ is a minimum

Answer $\quad$ For $a = \sqrt{3}$ distance of tangent line btwn its intercepts is given by $d = \sqrt{h(a)} = \sqrt{24}$