Lesson 6

Rates of Change and Related Rates

Sections
3.6, 3.11

Rates of Change

Up until now, we have seen the derivative interpreted as the slope of the tangent line. Another common use of the derivative is as a rate of change. That is, the rate of change of one variable with respect to another variable, usually time. Rates of change are seen everywhere in the real world. In almost every field we have "stuff" that happens over time. Whether that is a chemical reaction, distance an object travels, growth of a population, etc. We can track changes happening over time much more easily than we can record any other kind of data. This means that in real world applications we often actually have information on the derivative and not the function itself! More about this idea comes toward the end of the course and is studied much more in depth in an ordinary differential equations course.

Average vs Instantaneous Rate of Change

Since we only know how to take derivatives at this point the examples in this lesson will start with the function itself and then determine a rate of change. A key piece of notation you will see throughout this section is Δ which means "change in". For example,

$$\Delta t = t_2 - t_1$$

would be the change in time. This notation is seen in the definition below where we begin with a discussion of the average rate of change of a function.

Definition 6.1: Average Rate of Change

The **average rate of change** in f between $t = a$ and $t = a + \Delta t$ is the slope of the corresponding secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$ given by

$$m_{av} = \frac{f(a + \Delta t) - f(a)}{a + \Delta t - a} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

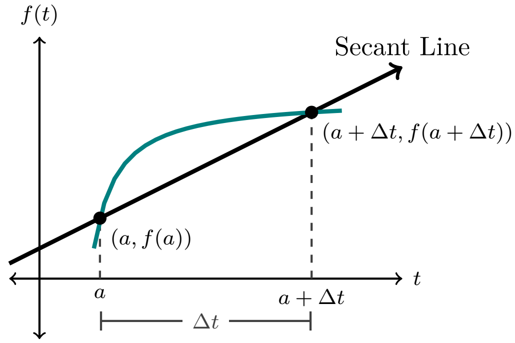


Fig. 6.1: Slope of the secant line

The average rate of change can graphically be described as the slope of the secant line. The rate of change at a single point is known as the **instantaneous** rate of change.

How do we find the rate of change at exactly one point? To find the instantaneous rate of change we want the distance Δt to become as small as possible. In other words, we want to obtain the tangent line.

You should recognize that this is exactly how we began to define the definition of the derivative. This is the same definition we used before only instead of h we call this distance Δt.

Definition 6.2: Instantaneous Rate of Change

The **instantaneous rate of change** of the function $f(t)$ at a is the slope of the tangent line given by

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t}$$

Objects in Motion

A common example we see regarding rates of change has to do with the model of an object in motion. We will begin by providing some basic definitions and then demonstrate their use through some examples. We start with a position function which lets us know where a particle may be at a given time t. In many textbooks the position function is denoted by $s(t)$ but any notation is acceptable.

Definition 6.3: Position and Displacement of an Object

- A function $s(t)$ that gives the position of an object as function of time t is known as the **position function**.
- The **displacement** of an object between a time t and a time Δt is given by

$$\Delta s = s(t + \Delta t) - s(t)$$

In the real world, we often do not have the position function. Think about when you drive in a car. Our speedometer gives us information on our positive velocity which is how fast we are traveling in the forward direction. In the United States, we record the speed at which a car is traveling as miles per hour (mi/hr). In other words, the distance traveled over the time we have been traveled. This is just a rate of change or in other words, the derivative.

In the following examples we will be given a position function and use that to determine velocity function. Extending this idea again, we can find the rate of change of the velocity, or in other words, the acceleration function. These are summarized in the following definition.

Definition 6.4: Velocity and Acceleration

- The instantaneous rate of change of the position function is known as the **velocity function**, $v(t)$, and so

$$v(t) = \frac{d}{dt}[s(t)] = s'(t)$$

- The instantaneous rate of change of the velocity function is given by the **acceleration function**, $a(t)$, and so

$$a(t) = v'(t) = \frac{d}{dt}[s'(t)] = s''(t)$$

Example 6.1: Finding Velocity an Object in Motion

The position of a particle with respect to time t in seconds (s) is given in meters (m) by the function

$$s(t) = t^3 - 12t^2 + 36t$$

Determine the following.

a.) Find the velocity function.

Solution. The velocity function is found by taking the derivative of the position function. In other words, $v(t) = s'(t)$ so we have

$$v(t) = s'(t) = 3t^2 - 24t + 36$$

b.) Find the particle's velocity after 3 seconds.

Solution. Since $v(t) = 3t^2 - 24t + 36$ then

$$v(3) = 3(3)^2 - 24(3) + 36 = -9\,\mathrm{m/s}$$

Example 6.2: Finding Acceleration of a an Object in Motion

Consider the same particle in Example 6.1. Find the following.

a.) The acceleration function.

Solution. The acceleration function is found by taking the derivative of the velocity function. In other words, $a(t) = v'(t) = s''(t)$. Since $v(t) = 3t^2 - 24t + 36$ we have

$$a(t) = v'(t) = 6t - 24$$

b.) The particle's acceleration after 3 seconds.

Solution. Since $a(t) = 6t - 24$ then

$$a(3) = 6(3) - 24 = -6\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}^2}$$

The direction in which the particle is moving at any time t is found by using the velocity function. The three possible cases are covered in the following definition.

Definition 6.5: Direction in Which an Object Travels

Suppose particle is moving with velocity given by $v(t)$. Then when

$v(t) > 0$	the particle is moving in the **postive direction**.
$v(t) = 0$	the particle is **at rest**.
$v(t) < 0$	the particle is moving in the **negative direction**

It may seem trivial to calculate the distance traveled by an object. However, we often will not be talking about objects traveling in a straight line and we will deal with objects that are not always traveling forward. To calculate total distance then we need to consider intervals where an object is moving forward *and* moving backward.

Definition 6.6: Distance Traveled by an Object

The **total distance** traveled by an object on an interval from $t = t_1$ to $t = t_2$ is given by

$$|s(t_2) - s(t_1)|$$

Example 6.3: Determining Distance Traveled by an Object in Motion

Consider the same particle in Example 6.1. Find the following.

a.) When the particle is at rest.

Solution. The particle is at rest when $v(t) = 0$. This occurs when

$$3t^2 - 24t + 36 = 3(t - 2)(t - 6) = 0$$

and so the particle is at rest when $t = 0$ and $t = 6$. A few things to note here. There is no such thing as negative time and so negative t values should be excluded.

Also, since $v(t)$ is zero at these points, these are also the points where the velocity function may change sign.

b.) When the particle is moving in the positive and negative direction.

Solution. Since the particle is at rest at $t = 2$ and $t = 6$ we will test the intervals

$$0 < t < 2, \quad 2 < t < 6, \quad t > 6$$

to determine the sign of the velocity function $v(t)$. We evaluate $v(t)$ at $t = 1, t = 3, t = 6$.

$$\begin{aligned}
v(1) &= 3\Big((1) - 2\Big)\Big((1) - 6\Big) = 15 && v(t) > 0 \\
v(3) &= 3\Big((3) - 2\Big)\Big((3) - 6\Big) = -9 && v(t) < 0 \\
v(7) &= 3\Big((7) - 2\Big)\Big((7) - 6\Big) = 15 && v(t) > 0
\end{aligned}$$

We see that $v(t)$ is moving in the positive direction on the intervals $0 \leq t \leq 2$ and $t \geq 6$. The particle is moving in the negative direction on $2 \leq t \leq 6$.

c.) The total distance traveled by the particle from $t = 0$ seconds to $t = 8$ seconds.

Solution. We find the following:

$$\begin{aligned}
0 \leq t \leq 2 :& \quad |s(2) - s(0)| = |32 - 0| = 32 \\
2 \leq t \leq 6 :& \quad |s(6) - s(2)| = |0 - 32| = 32 \\
6 \leq t \leq 8 :& \quad |s(8) - s(6)| = |32 - 0| = 32
\end{aligned}$$

And so the total distance traveled is $32 + 32 + 32 = 96$ meters.

Definition 6.7: Speed of an Object

The **speed** of an object is given by the absolute value of the velocity

$$\text{Speed} = |v(t)|$$

An object is **speeding up** when the velocity and acceleration have the same sign. In other words, when

$$v(t) > 0 \text{ and } a(t) > 0 \quad \text{or} \quad v(t) < 0 \text{ and } a(t) < 0$$

An object is **slowing down** when the velocity and acceleration have different signs. In other words, when

$$v(t) > 0 \text{ and } a(t) < 0 \quad \text{or} \quad v(t) < 0 \text{ and } a(t) > 0$$

Example 6.4: Determining Speed of an Object in Motion

In Figure 6.2 we have graphs of the position, velocity and acceleration functions of the particle in Example 6.1. Find the following.

a.) When the particle is speeding up.

b.) When the particle is slowing down.

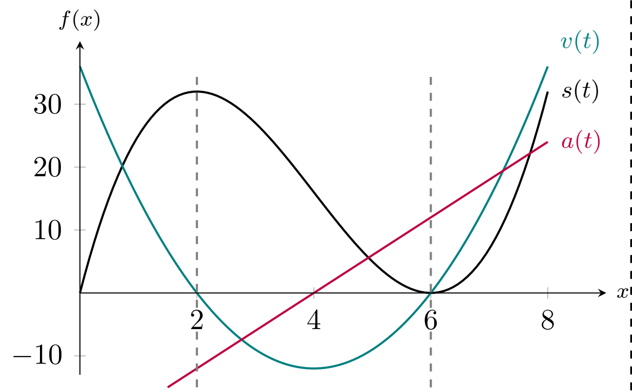


Fig. 6.2: Graphs of $s(t)$, $v(t)$, $a(t)$

Solution. From the graph we can see that $a(t) < 0$ on the interval $0 < t < 4$ and $a(t) > 0$ on the interval $4 < t < 8$. To compare the signs of $v(t)$ and $a(t)$ we can look at the graph but it may be easier to organize our information into a "sign chart" to better compare $v(t)$ and $a(t)$. You can use the information you found in Example 6.3 for $v(t)$ and carry out a similar procedure for $a(t)$ and then compare the intervals.

Interval (t)	$0 \le t \le 2$	$2 \le t \le 4$	$4 \le t \le 6$	$6 \le t \le 8:$
Sign of $v(t)$	$+$	$-$	$-$	$+$
Sign of $a(t)$	$-$	$-$	$+$	$+$

a.) To determine when the particle is speeding up we need to find when $v(t)$ and $a(t)$ have the same sign. We can see that this occurs on the intervals $0 \le t \le 4$ and $6 \le t \le 8$.

Thus, the particle is speeding up on the intervals $2 < t < 4$ and $6 < t < 8$.

b.) From the table we also observe that $a(t)$ and $v(t)$ have differing signs on the intervals $0 < t < 2$ and $4 < t < 6$.

Thus, the particle is slowing down on the intervals $0 < t < 2$ and $4 < t < 6$.

When using graphs always use technology to ensure that your graphs are accurate.

Rates of Growth

Many of the applications involving rates of change have to do with rates of *growth*. The same interpretations of average rate of change an instantaneous rate of change apply here. We demonstrate this through a simple example involving population growth.

Example 6.5: Determining Growth Rates of a Population

The population in Georgia (in thousands) from 1995 to 2005 is modeled by the function

$$p(t) = -0.27t^2 + 101t + 7055$$

Determine the following

a.) Determine the average growth rate of the population from 1995 to 2005.

Solution. The average rate of change of this function is given by the slope of the line between t_1 and t_2.

$$\frac{\Delta p}{\Delta t} = \frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

We are asked to find the average growth rate from 1995 to 2005. Note that this model only gives the population between these two years. So we have that for 1995 $t = 0$ and for 2005 $t = 10$. So the average growth rate is

$$\frac{\Delta p}{\Delta t} = \frac{p(t_2) - p(t_1)}{t_2 - t_1} = \frac{p(10) - p(0)}{10 - 0} = \frac{8038 - 7055}{10} = 98.3 \text{ thousand people/year}$$

b.) Determine the growth rate of the population in 1997.

Solution. The growth rate in any single year will be given by the instantaneous growth rate. We know that this is given by the derivative $p'(t)$ which we find to be

$$p'(t) = -0.54t + 101$$

In 1997 we have $t = 2$ and so the growth rate in 1997 is

$$p'(2) = -0.54(2) + 101 = 99.2 \text{ thousand people/year}$$

c.) Determine the growth rate in 2005.

Solution. From part (b) we know that the derivative of $p(t)$ is

$$p'(t) = -0.54t + 101$$

In 2005 we have $t = 10$ and so the growth rate in 2005 is

$$p'(10) = -0.54(10) + 101 = 95.6 \text{ thousand people/year}$$

There of course many other examples of the derivative's use in determining rates of change in a variety of areas. It is worth working through some of the other examples covered in the exercises in your textbook. This will give you a better idea of rates of change in your particular discipline.

Related Rates

In Lesson 5 we saw that the chain rule can be used to find a derivative implicitly. The chain rule can also be used to find rates of change of several related variables which change with respect to time. These types of problems are known as *related rates* problems.

It is very important to review geometry formulas and basic trigonometry before proceeding. Most of the difficulty in these problems lies in setting them up properly, from there the calculus is usually very straight forward.

Chain Rule in Leibniz Notation

Recall from Lesson 4 that the chain rule is defined as

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

We now show the chain rule using *Leibniz Notation*. You may find this notation helpful when working on these types of problems.

Definition 6.8: Chain Rule in Leibniz Notation

Let $f = f(u(x))$ and $u = u(x)$ both be differentiable functions. Then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

For example, the volume of a sphere is given by the formula

$$V(r) = \frac{4}{3}\pi r^3$$

If the radius r changes over time then we can say that the radius is a function of time. In other words, $r(t)$. So really we have

$$V(r(t)) = \frac{4}{3}\pi(r(t))^3$$

Taking the derivative of this function will require that we use the chain rule. We obtain

$$V'(r(t)) = \frac{4}{3}\pi[3(r(t))^2\cdot]r'(t) = 4\pi(r(t))^2 \cdot r'(t)$$

In Leibniz notation we would write $r'(t) = \frac{dr}{dt}$ and so

$$\frac{d}{dt}V = 4\pi r \cdot \frac{dr}{dt}$$

This is one of the few times I find this notation helpful. Think of the d as meaning *difference*. In other words, $\frac{dr}{dt}$ is the *difference in radius* with respect to the *difference in time*. You can also think of d as meaning *change in*.

We now proceed by showing a few examples. Students often find that related rates are the most challenging since they can be difficult to set up and organize your work. For nearly all word problems like this, I use a process known as "GASCAP". This may be a method similar to something you already do or you may have seen something like it in another course. A separate handout will be made available which outlines this method for word problems.

Note that in the figures accompanying each problem the dotted lines with arrows indicate quantities that are changing and the "direction" in which they are changing. For shapes, solid lines indicate where an object "started" and the dashed lines indicate what it is changing to.

Examples of Related Rates

Example 6.6: Area of a Circle

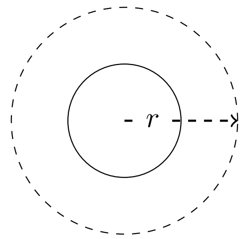
The radius r of a circle is increasing at a rate of $4\,{}^{\text{cm}}\!/\!{}_{\text{min}}$. Find the rates of change of the area when $r = 8\,\text{cm}$ and $r = 32\,\text{cm}$.

Solution.

Given:
We are given the rate at which the radius is changing: $\frac{dr}{dt} = 4\,{}^{\text{cm}}\!/\!{}_{\text{min}}$.

We also know that the area of a circle is given by $A = \pi r^2$.

Sketch:



Asked:
We need to find the rate the *area* is changing when $r = 8$ and $r = 32$. i.e.

$$\frac{dA}{dt} = ? \quad \text{when } r = 8, \text{ and } r = 32$$

Compute: First we differentiate $A = \pi r^2$ implicitly with respect to t. We get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Since $\frac{dr}{dt} = 4$ then

$$\text{When } r = 8: \quad \frac{dA}{dt} = 2\pi(8)(4) = 64\pi\,{}^{\text{cm}^2}\!/\!{}_{\text{min}}$$

$$\text{when } r = 32: \quad \frac{dA}{dt} = 2\pi(32)(4) = 256\pi\,{}^{\text{cm}^2}\!/\!{}_{\text{min}}$$

Answer:
If the radius is increasing at a rate of $4\,{}^{\text{cm}}\!/\!{}_{\text{min}}$ then the area is increasing at a rate of $64\pi\,{}^{\text{cm}^2}\!/\!{}_{\text{min}}$ when $r = 8$ and at a rate of $256\pi\,{}^{\text{cm}^2}\!/\!{}_{\text{min}}$ when $r = 32$.

Example 6.7: Melting Snowball

If a snowball melts so that its surface area decreases at a rate of $1\,\mathrm{cm^2/min}$, find the rate at which the diameter decreases when the diameter is $10\,\mathrm{cm}$.

Solution.

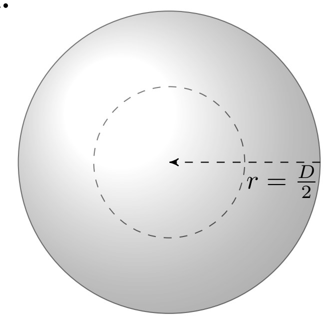
Given:

Surface Area: $S = 4\pi r^2$

Diameter: $D = 2r$

Rate surface area decreases: $\frac{dS}{dt} = -1\,\mathrm{cm^2/min}$

Sketch:



Asked:

We want to find the rate at which the diameter is decreasing when $D = 10\,\mathrm{cm}$. In other words,

$$\frac{dD}{dt} = ? \quad \text{when } D = 8$$

Compute:

Our equation for surface area is given in terms of the radius. Since $D = 2r$ this implies that the radius is given by $r = \frac{D}{2}$. Substituting this into our equation we obtain

$$S = 4\pi\left(\frac{D}{2}\right)^2 = 4\pi\frac{D^2}{4} = \pi D^2$$

Differentiating implicitly with respect to t we have

$$\frac{dS}{dt} = 2\pi D\frac{dD}{dt}$$

Solving for $\frac{dD}{dt}$ we have

$$\frac{dD}{dt} = \frac{1}{2\pi D}\frac{dS}{dt}$$

Since $\frac{dS}{dt} = -1$ when $D = 10$ then

$$\frac{dD}{dt} = \frac{1}{2\pi(10)}(-1) = -\frac{1}{20\pi}\,\mathrm{cm/min}$$

Answer:

When the surface area is decreasing at a rate of $-1\,\mathrm{cm^2/min}$ then the diameter is decreasing at a rate of $-\frac{1}{20\pi}\,\mathrm{cm/min}$.

For these next two problems it is vital to have an understanding of *similar triangles*. In short, when two triangles have two angles that are the same then we can express the lengths of the sides of one triangles as a ratio of the other. This will be demonstrated in the following example.

Example 6.8: Conical Tank

A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

Solution.

Given:

$$\begin{aligned} h &: \text{ height of water} \\ r &: \text{ radius of water} \\ \text{diameter} &= 10 \text{ ft} \\ \text{radius} &= 5 \text{ ft} \\ \text{depth} &= 12 \text{ ft} \\ \text{Rate water flows in: } & \tfrac{dV}{dt} = 10\,\text{ft}^3/\text{min} \end{aligned}$$

Asked:

Need to find the rate of change of the *depth of the water* when the height is $h = 8$ ft. i.e.

$$\frac{dh}{dt} = ? \quad \text{when } h = 8$$

Compute:

We know that volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Since we don't like to handle functions of two variables at this level and we are given a value for h we want to eliminate the r in our equation.

From our sketch of the given information we see that we can construct the triangle pictured in Figure 6.4. This will allow us to use the property of similar triangles to find an expression for r. We obtain

$$\frac{r}{h} = \frac{5}{12} \implies r = \frac{5}{12}h$$

Substituting this into our equation we have

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25}{432}\pi h^3$$

Now we have a function solely of the variable h. Differentiating implicitly with respect to t we obtain

$$\frac{d}{dt}V = \frac{25}{432}\pi\left(3h^2\frac{dh}{dt}\right) = \frac{75}{432}\pi h^2\frac{dh}{dt}$$

Now solving for $\frac{dh}{dt}$ we obtain

$$\frac{dh}{dt} = \frac{432}{75\pi h^2}\frac{dv}{dt}$$

Sketch:

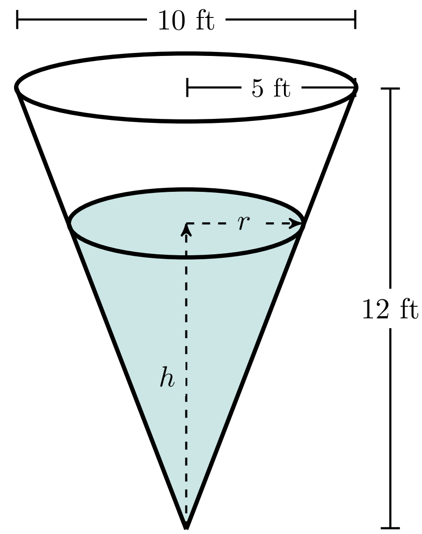


Fig. 6.3: Conical Tank

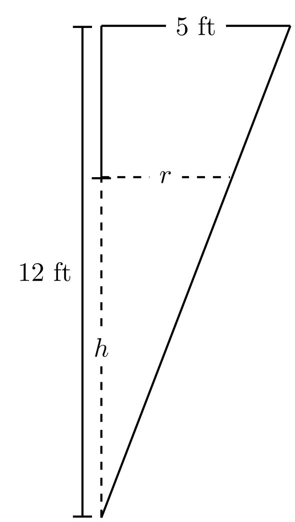


Fig. 6.4: Conical Tank Triangles

We are given that $\frac{dv}{dt} = 10\,\text{ft}^3/\text{min}$ when $h = 8\,\text{ft}$ so

$$\frac{dh}{dt} = \frac{432}{75\pi(8)^2}(10) = \frac{9}{10\pi}\,\text{ft}/\text{min}$$

Answer:
When water is flowing into the tank at $10\,\text{ft}^3/\text{min}$ the height of the water in the tank is increasing at a rate of $\frac{9}{10\pi}\,\text{ft}/\text{min}$.

Example 6.9: Shadow Problem

A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving and at what rate is the length of his shadow changing?

Solution.

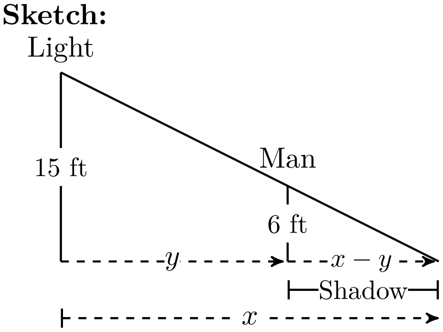
Given:

x:	distance from light to end of shadow
y:	distance from light to man
$x - y$	shadow length
15 ft:	height of light
6 ft:	height of man

Rate distance to man changes:

$$\frac{dy}{dt} = 5\,\text{ft}/\text{sec}$$

Sketch:



Asked:

Rate at which the tip of the shadow is moving: $\dfrac{dx}{dt} = ?$.

Rate at which length of the shadow is changing: $\dfrac{d}{dt}(x - y) = ?$.

Compute:

Again, we must use the property of similar triangles.

$$\frac{15}{6} = \frac{x}{x - y}$$

Simplifying this equation we obtain

$$9x - 15y = 0$$

Differentiating implicitly with respect to t we obtain

$$\frac{d}{dx}(9x - 15y) = 0 \implies 9\frac{dx}{dt} - 15\frac{dy}{dt} = 0$$

Solving for $\frac{dx}{dt}$ we have

$$\frac{dx}{dt} = \frac{15}{9}\frac{dy}{dt}$$

Since $\frac{dy}{dt} = 5\,\text{ft}/\text{sec}$ then

$$\frac{dx}{dt} = \frac{15}{9}(5) = \frac{75}{9} = \frac{25}{3}\,\text{ft}/\text{sec}$$

Now that we have $\frac{dx}{dt}$ we can now find the rate at which the length of the shadow, $x - y$, is changing. We differentiate $x - y$ with respect to t and obtain

$$\frac{d}{dt}(x - y) = \frac{dx}{dt} - \frac{dy}{dt} = \frac{25}{3} - 5 = \frac{10}{3}\,\text{ft}/\text{sec}$$

Answer:
When the distance from the light to the man changes at the rate $5\,\text{ft}/\text{sec}$ the rate at which the tip of the shadow is moving is $\frac{25}{3}\,\text{ft}/\text{sec}$ and the length of the shadow changes at a rate of $\frac{10}{3}\,\text{ft}/\text{sec}$.

Practice Problems

Section 3.6
9-19, 30-36, 41, 43, 44, 48-57

Section 3.11
5-39