Lesson 5

Implicit and Logarithmic Differentiation

Sections
3.8, 3.9, 3.10

Implicit vs. Explicit Functions

So far we have been taking derivatives of explicit functions. In other words, functions like

$$f(x) = \sqrt{x+2}, \quad g(x) = 3x^2 + 7x + 9, \quad h(x) = 3, \text{ etc.}$$

A function can also be implied by an equation. These types of functions are known as *implicit functions*. These are functions like

$$y = \sqrt{x+y}, \quad xy = \sin(y^2) + 2x + 6y, \quad x^2 + y^2 = 1, \text{ etc.}$$

In other words, we have an equation where we see y inside another function or on both sides of our equation. An implicit function can sometimes be written explicitly like

$$x^2 + y^2 = 1 \implies y = \pm\sqrt{1 - x^2}$$

The issue with doing this is that now we need to be careful about our domain. Often implicit functions *cannot* be written as an explicit function. An example of this is the function

$$y = \sqrt{x+y}$$

There is no way to get it to be a function that is y on one side and terms only involving the variable x on the other. For the equation just given we will always have a y term on both sides of the equation. A special type of differentiation allows us to find the derivatives of these implicit functions. The process is referred to as *implicit differentiation*. It allows us to completely avoid writing a function explicitly.

The Implicit Differentiation Process

Here is one of those times where it may be helpful to use the $\frac{dy}{dx}$ notation. More specifically, recall that this notation is saying "differentiate the function y with respect to x". In this course we are only dealing with *single variable* calculus. This means that y will be treated as a function and not as a variable. In other words, $y = y(x)$. This is different from treating y as if it were a variable. These types of functions are handled in multivariable calculus where you see how to handle functions with multiple variables. So for this course remember, ***y* is function of *x***.

Example 5.1: Differentiate with Respect to x

Differentiate the following.

a.) $\dfrac{d}{dx}[x^4]$

Solution. Note that here the variable of the term we are differentiating with respect to "matches" the variable in $\frac{d}{dx}$. In this case, we simply differentiate as we have before using the power rule we learned in the last lesson. So,

$$\frac{d}{dx}[x^4] = 4x^3$$

b.) $\dfrac{d}{dx}[y^4]$

Solution. Here our variables do not "match". This means we are differentiating an implicit function y, with respect to x. In other words, we are taking the derivative of a function that contains other functions and so we need to use the chain rule. Your textbook uses $\frac{dy}{dx}$ to represent $y'(x)$. Both notations are presented below.

Using $\frac{dy}{dx}$ notation:

$$\frac{d}{dx}[y^4] = 4y^3\frac{dy}{dx}$$

Using prime notation:

$$\frac{d}{dx}[y^4] = 4y^3y'$$

If you have trouble remembering to treat y as a function when using prime notation you can always write is as $y = y(x)$, i.e.

$$\frac{d}{dx}[y^4] = \frac{d}{dx}[(y(x))^4] = 4(y(x))^3 \cdot y'(x)$$

Remark: For the remainder of this section I will exclusively use prime notation. Keep in mind that your text will use $\dfrac{dy}{dx}$ notation.

Implicit differentiation often involves using several differentiation rules. I recommend splitting up your equation into a left hand side (LHS) and right hand side (RHS). This way you can find the derivative of each side and then put the pieces back together to solve for y'. This process will be demonstrated in the next example.

Example 5.2: Implicit Differentiation

Differentiate $x^3 + y^3 = 9xy$.

Solution. Solving this function explicitly for y can be done but it is too complicated to do by hand. It is easier to use implicit differentiation. We begin by taking the derivative of each side.

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[9xy]$$

Since implicit differentiation can be complicated we're going to split this equation into the left hand side (LHS) and the right hand side (RHS). Let

$$\text{LHS} = \frac{d}{dx}[x^3 + y^3] \qquad \text{and} \qquad \text{RHS} = \frac{d}{dx}[9xy]$$

For LHS:

Using our basic differentiation rules we're going to break the LHS up so that we can see exactly what is going on with each term.

$$\text{LHS} = \frac{d}{dx}[x^3 + y^3] = \underbrace{\frac{d}{dx}[x^3]}_{(1)} + \underbrace{\frac{d}{dx}[y^3]}_{(2)}$$

(1) This derivative is only in terms of x so we just apply the power rule.

$$\frac{d}{dx}[x^3] = 3x^2$$

(2) Remember that $y = y(x)$ so to take the derivative of (2) we need to use the chain rule. We have

$$\frac{d}{dx}[y^3] = \frac{d}{dx}[(y(x))^3] = 3(y(x))^2 \cdot y'(x) = 3y^2y'$$

So for LHS we have

$$\begin{aligned} \text{LHS} &= \frac{d}{dx}[x^3 + y^3] \\ &= \frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] \\ &= 3x^2 + 3y^2y' \end{aligned}$$

For RHS:

Now let RHS $= \frac{d}{dx}[9xy]$. Note that this is a product of two functions! To take the derivative we need to use the product rule.

$$\text{RHS} = \frac{d}{dx}[9xy] = \frac{d}{dx}[fg] = f'g + fg'$$

$$\begin{array}{lll} \text{Let} & f = 9x \quad , & g = y \\ & f' = 9 \quad , & g' = y' \end{array}$$

So RHS $= \frac{d}{dx}[9xy] = 9y + 9xy'$.

Now we bring our LHS and RHS back together.

$$\begin{aligned}\frac{d}{dx}[x^3+y^3] &= \frac{d}{dx}[9xy]\\ \implies 3x^2+3y^2y' &= 9y+9xy'\end{aligned}$$

We now need to solve for y'. To do this we collect all terms with y' together on one side and move remaining terms to the other side.

$$\begin{aligned}3x^2+3y^2y' &= 9y+9xy' && \\ 3y^2y'-9xy' &= 9y-3x^2 && \text{Isolate } y' \text{ terms}\\ y'(3y^2-9x) &= 9y-3x^2 && \text{Factor out a } y'\\ y' &= \frac{9y^2-3x^2}{3y-9x} && \text{Solve for } y'\\ &= \frac{3y^2-x^2}{y-3x} && \text{Simplifying}\end{aligned}$$

Note that in this example our derivative is also an implicit function. This may or may not be the case when carrying out implicit differentiation.

Remember that it is important to write out all of your steps. This may seem like a lot of work just to take a derivative. Be warned that the most common mistake students make is to forget to use the chain rule when taking the derivative of a function involving y. The more you write out your work the easier it will be to identify errors you have made and quickly correct them. When first practicing these problems do not skip any steps and take up a minimum of a half page if not more!

Example 5.3: Implicit Differentiation with Trigonometric Functions

Differentiate $x+y=\cos(y)$.

Solution. This is an example of a function that cannot be solved explicitly for y. So again we will need to use implicit differentiation. Taking the derivative of both sides we have

$$\frac{d}{dx}[x+y]=\frac{d}{dx}[\cos(y)]$$

The LHS is simple in this case. Let

$$\text{LHS}=\frac{d}{dx}[x+y]=\frac{d}{dx}[x]+\frac{d}{dx}[y]=1+y'$$

Now for the RHS we will need to apply the chain rule. So

$$\text{RHS}=\frac{d}{dx}[\cos(y)]=-\sin(y)y'$$

Now we bring our LHS and RHS back together

$$\frac{d}{dx}[x+y]=\frac{d}{dx}[\cos(y)] \implies 1+y'=-\sin(y)y'$$

Now solve for y'

$$\begin{aligned} y' + \sin(y)y' &= -1 && \text{Isolate } y' \text{ terms} \\ y'(1+\sin(y)) &= -1 && \text{Factor our } y' \\ y' &= \frac{-1}{1+\sin(y)} && \text{Solve for } y' \end{aligned}$$

Tangent Lines of Implicit Functions

We can also find lines tangent to graphs of implicit functions.

Example 5.4: Finding the Tangent Line of an Implicit Function

Find the tangent line of $x^2 + xy = 3 - y^2$ at point $(1, 1)$.

Solution. First we find the derivative y' using implicit differentiation. Taking the derivative of both sides with respect to x:

$$\frac{d}{dx}[x^2 + xy] = \frac{d}{dx}[3 - y^2]$$

Let LHS $= \frac{d}{dx}[x^2 + xy] = \underbrace{\frac{d}{dx}[x^2]}_{(1)} + \underbrace{\frac{d}{dx}[xy]}_{(2)}.$

(1) $\frac{d}{dx}[x^2] = 2x$

(2) $\frac{d}{dx}[xy] = \frac{d}{dx}[fg] = f'g + fg'$

Let $\begin{array}{lcl} f = x & , & g = y \\ f' = 1 & , & g' = y' \end{array} \implies \frac{d}{dx}[xy] = y + xy'$

So LHS $= \frac{d}{dx}[x^2 + xy] = \frac{d}{dx}[x^2] + \frac{d}{dx}[xy] = 2x + y + xy'$

Now for the RHS $= \frac{d}{dx}[3 - y^2] = \underbrace{\frac{d}{dx}[3]}_{(3)} - \underbrace{\frac{d}{dx}[y^2]}_{(4)}.$

(3) $\frac{d}{dx}[3] = 0$

(4) $\frac{d}{dx}[y^2] = \frac{d}{dx}\left[(y(x))^2\right] = 2y(x) \cdot y'(x) = 2yy'$

So RHS $= \frac{d}{dx}[3 - y^2] = \frac{d}{dx}[3] - \frac{d}{dx}[y^2] = 0 - 2yy' = -2yy'.$

Now we bring our LHS and RHS back together

$$2x + y + xy' = -2yy'$$

and solve for y'

$$\begin{aligned} xy' + 2yy' &= -y - 2x && \text{Isolate } y' \text{ terms} \\ y'(x+2y) &= -y - 2x && \text{Factor our } y' \\ y' &= -\frac{y+2x}{x+2y} && \text{Solve for } y' \end{aligned}$$

The slope of our tangent line at the point $(1,1)$ is

$$y' = -\frac{(1)+2(1)}{(1)+2(1)} = -1$$

Equation of the line is then

$$y - 1 = -(x-1) \implies y = -x + 2$$

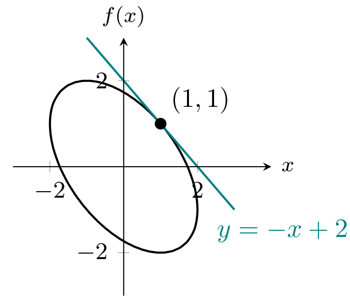

Fig. 5.1: Graph of $x^2 + xy = 3 - y^2$ and $y = -x + 2$

Higher Order Implicit Differentiation

Just as with explicit derivatives, we can also find higher order derivatives of implicit functions.

Example 5.5: Finding the Second Derivative of an Implicit Function

Find the second derivative of $x^2 + y^2 = 9$.

Solution. To find the second derivative we first need to find the first derivative.
First Derivative y':
We differentiate each side with respect to x

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[9]$$

Now on the RHS we just have the derivative of a constant which we know is 0. On the LHS we have

$$\text{LHS} = \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 2x + 2y \cdot y'$$

Putting these pieces together we have

$$2x + 2y \cdot y' = 0$$

Solving for y' we have

$$\begin{aligned} 2yy' &= -2x \\ y' &= \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y} \end{aligned}$$

So our first derivative is $y' = -\dfrac{x}{y}$.

Second Derivative y':

Recall that the second derivative $y''(x) = \dfrac{d}{dx}[y'(x)]$ so we take the derivative of each side of y'.

$$\frac{d}{dx}[y'] = \frac{d}{dx}\left[-\frac{x}{y}\right]$$

On the LHS we just have y''. On the RHS we have

$$\frac{d}{dx}\left[-\frac{x}{y}\right] = \frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$$

$$\begin{array}{lllll} \text{Let} & f = -x & , & g = y & \\ & f' = -1 & , & g = y' & \end{array} \qquad \Longrightarrow \frac{d}{dx}\left[-\frac{x}{y}\right] = \frac{-y + xy'}{y^2}$$

So putting these pieces back together our second derivative is $y''(x) = \dfrac{-y + xy'}{y^2}$

Derivatives of Logarithmic Functions

We have already seen the derivative of the natural exponential and logarithmic functions. We will now include the rules for these functions of base b. Please refer to your textbook for the derivation of these rules. We simply summarize them below.

Theorem 5.1: Derivatives of Exponential & Logarithmic Functions

- Derivatives of Exponential Functions

$$\frac{d}{dx}[e^u] = e^u u' \qquad \frac{d}{dx}[b^u] = b^u \ln(b) u'$$

- Derivatives of Logarithmic Functions

$$\frac{d}{dx}[\ln(u)] = \frac{u'}{u} \qquad \frac{d}{dx}[\log_b(u)] = \frac{u'}{\ln(b)\cdot u}$$

where $u = u(x)$ is a differentiable function of x.

We will demonstrate use of these derivatives in the following examples.

Example 5.6: Differentiating Logarithmic Functions

Differentiate the following functions.

a.) $f(x) = \ln(5x)$

Solution.

$$f'(x) = \frac{d}{dx}[\ln(5x)] = \frac{\frac{d}{dx}[5x]}{5x} = \frac{\cancel{5}}{\cancel{5}x} = \frac{1}{x}$$

b.) $g(x) = \ln(\cos(x))$

Solution.

$$g'(x) = \frac{d}{dx}[\ln(\cos(x))] = \frac{\frac{d}{dx}[\cos(x)]}{\cos(x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

Now that we have seen how to take derivatives of logarithmic functions we can now use properties of logarithms to simplify the process of taking derivatives of complicated functions. This is known as *logarithmic differentiation*. You can technically use a logarithm with any base in this process but in mathematics we almost always use the natural logarithm. The three main properties you should remember are listed in the following definition.

Definition 5.1: Properties of Natural Logarithms

Product Rule: $\ln(xy) = \ln(x) + \ln(y)$

Quotient Rule: $\ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y)$

Power Rule: $\ln(x^r) = r \cdot \ln(x)$, where r is a real number

Note that these are the *only* rules allowed! Many students make mistakes applying these rules. Remember that the logarithm is just a way of differently representing a function with an exponent. This means that the operations we carry out for exponents are the same ones for logarithms.

Logarithmic Differentiation

Our goal in logarithmic differentiation is to apply properties of logarithms to break up the given function into smaller pieces that are easier to differentiate. This may seem like more work than is needed but it can actually simplify your work in a lot of cases.

Example 5.7: Using Logarithmic Differentiation

Differentiate the function

$$y = x^2\sqrt{x^2+1}$$

Solution. First, we take the natural log (ln) of both sides.

$$\begin{aligned}
\ln(y) &= \ln(x^2\sqrt{x^2+1}) && \\
&= \ln(x^2(x^2+1)^{\frac{1}{2}}) && \text{Rewrite with exponents} \\
&= \ln(x^2) + \ln((x^2+1)^{\frac{1}{2}}) && \text{Apply Product Rule of Logarithms} \\
&= 2\ln(x) + \frac{1}{2}\ln(x^2+1) && \text{Apply Power Rule of Logarithms}
\end{aligned}$$

Now we have the implicit function

$$\ln(y) = 2\ln(x) + \frac{1}{2}\ln(x^2+1)$$

So we use implicit differentiation

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[2\ln(x) + \frac{1}{2}\ln(x^2+1)]$$

where

$$\text{LHS} = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$$

and

$$\begin{aligned}\text{RHS} &= \frac{d}{dx}\left[2\ln(x) + \frac{1}{2}\ln(x^2+1)\right] = 2\frac{d}{dx}[\ln(x)] + \frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right]\\ &= 2\left(\frac{1}{x}\right) + \frac{1}{2}\left[\frac{\frac{d}{dx}[x^2+1]}{x^2+1}\right] = \frac{2}{x} + \frac{1}{2}\left[\frac{2x}{x^2+1}\right] = \frac{2}{x} + \frac{x}{x^2+1}\end{aligned}$$

Now we bring our LHS and RHS back together we have

$$\frac{y'}{y} = \frac{2}{x} + \frac{x}{x^2+1}$$

and solving for y' we have

$$y' = y\left(\frac{2}{x} + \frac{x}{x^2+1}\right)$$

Unlike previous examples we have seen in this case we actually started with an explicit function $y = x^2\sqrt{x^2+1}$. So, we must substitute this into the derivative function.

$$y' = y\left(\frac{2}{x} + \frac{x}{x^2+1}\right) = x^2\sqrt{x^2+1}\left(\frac{2}{x} + \frac{x}{x^2+1}\right)$$

Of course we could have found this derivative using the product and chain rules of differentiation and not bothered with using the logarithm. Try calculating the derivative this way and compare your result with the one above.

You may notice that I never use the formula for the derivative of $\log_b(x)$. This is because I can never remember it. Also, we don't really need it! We can always change the base of any logarithm to one we prefer to use. Recall the following definition from algebra.

Definition 5.2: Change of Base for Logarithms

To change from a base b to a base a we use

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Note that since $\ln(x) = \log_e(x)$ we also have

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

I'd rather remember less rules. You'll notice that by doing this you get exactly the same derivative as you would using the formula for base b!

Differentiating Functions of Form $y = f(x)^{g(x)}$

Logarithmic differentiation also gives us a means of differentiating complicated functions of the form $y = f(x)^{g(x)}$. These are sometimes called tower functions.

Example 5.8: Differentiation Functions of Form $y = f(x)^{g(x)}$

Differentiate $y = x^{\cos(x)}$

Solution. As before we take the natural log (ln) of both sides. This allows us to bring the function in our exponent "down" out of the exponent. We do this by taking advantage of the power rule of logarithms.

$$\ln(y) = \ln(x^{\cos(x)}) = \cos(x)\ln(x)$$

Now we have the implicit function

$$\ln(y) = \cos(x)\ln(x)$$

Using implicit differentiation

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\cos(x)\ln(x)]$$

We see that again we have

$$\text{LHS} = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$$

$$\text{RHS} = \frac{d}{dx}[\cos(x)\ln(x)] = \frac{d}{dx}[fg] = f'g + fg'$$

$$\begin{aligned} \text{Let} \quad f &= \cos(x) \quad , \quad g = \ln(x) \\ f' &= -\sin(x) \quad , \quad g' = \frac{1}{x} \end{aligned}$$

So RHS $= -\sin(x)\ln(x) + \cos(x)\left(\frac{1}{x}\right)$.

$$\frac{y'}{y} = \cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x) \implies y' = y(\cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x))$$

Again, we must plug $y = x^{\cos(x)}$ in for y

$$y' = x^{\cos(x)}\left(\cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x)\right)$$

Of course you could use the chain rule and the formula for the derivative of b^x and achieve the same result but I find that it is more complicated and confusing that just using the logarithm rules. For either method you use, remember that it is very important to stay organized!

Derivatives of Inverse Trigonometric Functions

The derivative formulas for the inverse trig functions are obtained using implicit differentiation. The derivation of these formulas is omitted here so refer to you textbook if you are interested in seeing how these are obtained. We summarize these rules below.

Theorem 5.2: Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x then

$$\frac{d}{dx}[\sin^{-1}(u)] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\csc^{-1}(u)] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\cos^{-1}(u)] = \frac{-u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\sec^{-1}(u)] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\tan^{-1}(u)] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\cot^{-1}(u)] = \frac{-u'}{1+u^2}$$

Practice Problems

Section 3.8
5-64, 82-89

Section 3.9
9-30, 34-92

Section 3.10
7-34