Lesson 8

Graphing Functions Using the Derivative

Sections
4.1, 4.2, 4.3, 4.6

Introduction

The graph of a function can reveal important information about the function's behavior. Behavior of a function can include things like when it is increasing, decreasing, reaching a maximum or minimum, etc. We learn a variety of techniques in algebra which allow us to obtain this information.

Now that we know how to differentiate we can obtain this information faster by using the derivative. This lesson will cover how the derivative can be used

Extrema of a Function

The *extrema* of a function refers to the maximum and minimum values a function may have. There are two main types of extrema that we will discuss here. When locating extreme values of a function we may want to talk about values of the function as a whole or over a smaller interval.

Recall that there are two types of intervals; *open* interval (a, b) that do not contain their endpoints, and *closed* intervals $[a, b]$ which do contain their endpoints. The type of interval we are using will determine what kind of maximum or minimum values we can locate.

First we will discuss what happens on a *closed* interval $[a, b]$.

Definition 8.1: Absolute Max/Min

Let f be defined on a *closed* interval $I = [a, b]$ containing c then the value $f(c)$ is the

(i) **absolute minimum of f on I** if $f(c) \leq f(x)$ for all x in I.

(ii) **absolute maximum of f on I** if $f(c) \geq f(x)$ for all x in I.

Do we always have maximum or minimum values of a function? On a closed interval the answer is yes!

Theorem 8.1: Extreme Value Theorem

If f is continuous on a *closed* interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Of course we will not always be dealing with functions on closed intervals. For a function on an open interval (a, b) we refer to these extrema as *local* maximum or minimum.

Definition 8.2: Local Maximum and Minimum Values

Let f be defined on an *open* interval $I = (a, b)$ containing c then,

(i) $f(c)$ is the **local minimum of f on** I if $f(c) \leq f(x)$ for all x in I.

(ii) $f(c)$ is the **local maximum of f on** I if $f(c) \geq f(x)$ for all x in I.

This definition is demonstrated graphically in Figure 8.1.

On the open interval I_a the point $(a, f(a))$ is a local *maximum*. We can see that every value on either side of a is less than $f(a)$. In other words, $f(x) \geq f(a)$ for all points of the function in I_a.

On the interval I_b, the point $(b, f(b))$ is a local *minimum* value. We can see that every value on either side of b is greather than $f(b)$. In other words, $f(b) \leq f(x)$ for all points of the function in I_b.

These are called local minimum and maximum values as they are valid *locally* in the interval you have chosen. A local maximum or minimum value may not be a minimum or maximum at another point on your function!

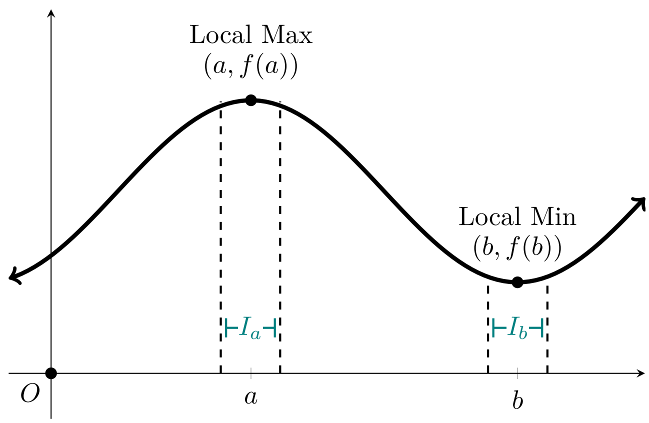


Fig. 8.1: Examples of Local Maximum and Local Minimum Values

Local minimum and maximum values are also referred to as **relative minimum** and **relative maximum** values as they are *relative* to the interval you are dealing with.

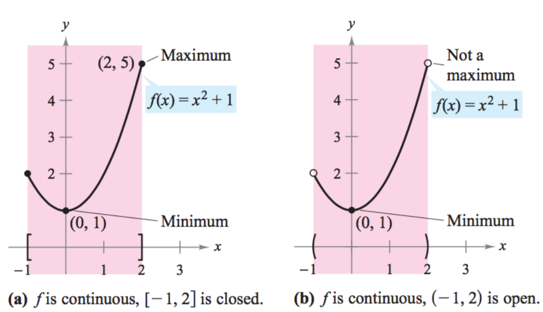


Fig. 8.2: Comparison of absolute extrema and local extrema

What is the difference between absolute and local extrema?

For absolute extrema we have restricted the interval to some finite set of points that includes the endpoints. We will always have a maximum and a minimum value in this case.

For local extrema we may not have a maximum or a minimum value on the interval chosen. See 8.2 for a graphical comparison of the two types of extrema.

Now we know how to find extrema if looking at the graph of a function. The main question now is how do we locate where these maximum and minimum values occur without a graph? The answer is found using the derivative of the function along with a key value known as a **critical point**.

The two types of critical points are represented graphically in Figure 8.3 and Figure 8.4

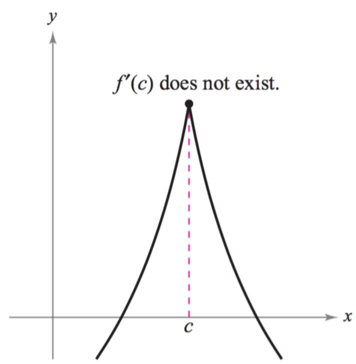


Fig. 8.3: When $f'(c)$ does not exist

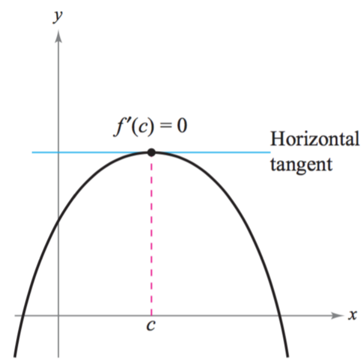


Fig. 8.4: When $f'(c) = 0$

Definition 8.3: Critical Point

A **critical point** of a function f is a point $x = c$ where $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 8.2

If a function f has a local maximum or local minimum at $x = c$ then c is a critical point of f.

This theorem is telling us that local maximum and minimum values can occur *only* at critical points of f. The converse, (i.e. reverse) of this statement is not true. In other words, it is possible that a critical point is *not* a maximum or minimum value.

Sometimes a critical point is also called a *critical value*. This makes a bit more sense since the critical value tells us *where* a critical point is located (i.e. the x value of the point). In order to get the actual point itself, we would need to plug the value of x into our function to obtain the actual y value.

Ultimately, what this means is that for a *closed* interval $[a, b]$ we can follow these steps to find absolute max and min values of a function f.

Steps for finding absolute max and min values

(1) Find the critical points of f in (a, b).

(2) Evaluate f at each critical point.

(3) Evaluate f at a and b, the endpoints of the interval $[a, b]$.

(4) The least of these values is the absolute minimum, the greatest is the absolute max.

Recall that $x = c$ is actually the equation for a vertical line. Note that a critical point is located *at* $x = c$. Meaning that it is located on the line $x = c$. Saying that the critical point *is* $x = c$ is equivalent to saying that the the critical point is the entire line. Language matters!

Example 8.1: Locating Absolute Maximums and Minimums

Find the extrema of $f(x) = \frac{x^3}{3} - 4x$ on the interval $[-3, 3]$.

Solution. First we need the derivative f':

$$f'(x) = x^2 - 4 = (x-2)(x+2)$$

Note that the graph of $f(x)$ is depicted in Figure 8.5 for your reference only. You do not need to graph your results although you are encouraged to do so to check your work.

(1) Since $f'(x)$ is a polynomial it is continuous on $[-3, 3]$. There are no points where $f'(x)$ does not exist. So to find the critical points we set $f'(x) = 0$ and solve for x:

$$f'(x) = (x-2)(x+2) = 0$$

Thus, our critical points are at $x = -2$ and $x = 2$.

We are on a closed interval so we can determine an absolute maximum and minimum value by evaluating $f(x)$ at the critical points and at our endpoints $x = -3$, $x = 3$. We obtain the following values.

(2) Value of f at the critical points:

$$f(-2) = \frac{16}{3} \approx 5.333,$$
$$f(2) = \frac{-16}{3} \approx -5.333$$

(3) Value of f at the endpoints:

$$f(-3) = 3,$$
$$f(3) = -3$$

(4) The largest function value occurs at $x = -2$. This is the **absolute maximum**.

The least function value occurs at $x = 2$. This is the **absolute minimum**.

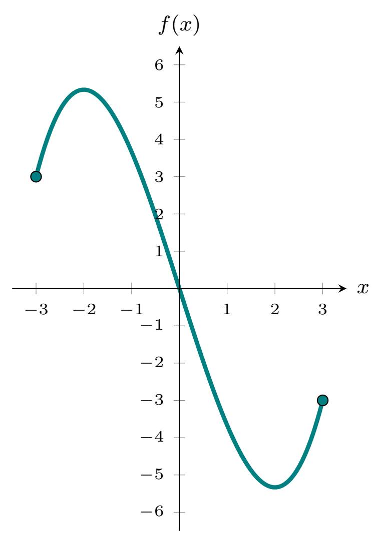


Fig. 8.5: Graph of $f(x)$

As you can see, a smaller critical point value does not always give you a minimum function value. Be very careful that you are evaluating the correct function at each of these steps. It can be easy to confuse the derivative and the original function. This is another example of why notation is very important.

Increasing and Decreasing Functions

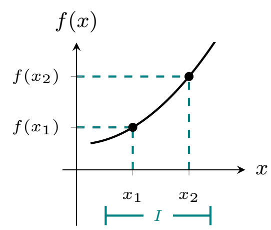
When is a function increasing or decreasing? Graphically, this means that as we move along the graph of a function from left to right if the graph is going up, it's increasing and if the graph is moving down the function is decreasing.

We start with a more formal definition of what it means for a function to be increasing or decreasing.

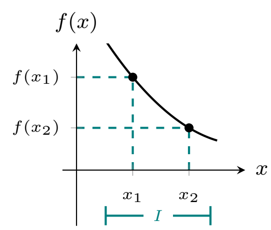
Definition 8.4: Increasing and Decreasing Functions

Let f be a function on an interval I then

f is **increasing** on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.



f is **decreasing** on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.



The first derivative gives us a means to find the intervals on which a function is increasing or decreasing.

Theorem 8.3: Test for Increasing/Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(i) If $f'(x) > 0$ for all x in (a, b), then f is **increasing** on $[a, b]$.

(ii) If $f'(x) < 0$ for all x in (a, b), then f is **decreasing** on $[a, b]$.

(iii) If $f'(x) = 0$ for all x in (a, b), then f is **constant** on $[a, b]$.

Why is this true? Recall that the derivative is the slope of the tangent line. If $f'(x) > 0$ then the tangent line is pointing up and to the right. If $f'(x) < 0$ then the tangent line is pointing down and to the right. We can see this demonstrated in the figures below.

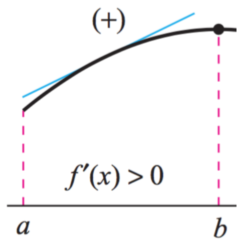


Fig. 8.6: f increasing when $f'(x) > 0$

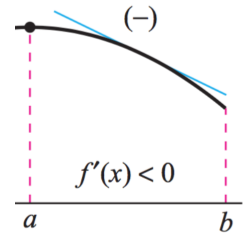


Fig. 8.7: f decreasing when $f'(x) < 0$

In order to find these intervals, recall what the Intermediate Value Theorem tells us: If $f(a) < 0$ and $f(b) > 0$ on an interval $[a, b]$ then there is a point where our function had to equal zero. Applying this to the derivative, we can establish intervals defined by the critical numbers of f i.e. when $f'(x) = 0$. We then test the sign of $f'(x)$ on that interval to determine the behavior.

Example 8.2: Determining Intervals of Increase and Decrease

Find the intervals on which $f(x) = \frac{x^3}{3} - 4x$ is increasing and decreasing.

Solution. From Example 8.1 we know that the critical numbers of f are $x = -2$ and $x = 2$. The easiest way to organize your tests is using a table:

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Test value (x)	$x = -3$	$x = 1$	$x = 3$
Value of $f'(x)$	$f'(-3) = 5$	$f'(1) = -3$	$f'(3) = 5$
Sign of $f'(x)$	$+$	$-$	$+$
Behavior	increasing	decreasing	increasing

So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 2)$.

Note that when you are filling in your table, the actual value of $f'(x)$ doesn't really matter. We only really care about the sign.

The First Derivative Test

From our work in Example 8.1 we know that an absolute maximum occurs at $x = -2$. If we consider this function now on an *open* interval, say $(-\infty, \infty)$, then we know that at $x = -2$ we have a *local* maximum value. Likewise, we have a local minimum at $x = 2$.

Now think about what we saw in Example 8.2. From our test for increasing and decreasing we know that at the local maximum (located at $x = -2$) that f went from increasing to decreasing. We also saw that at the local minimum (located at $x = 2$) that f went from decreasing to increasing.

In other words, the first derivative gave us the means to determine whether a critical point gives us a local maximum or minimum value! This is known as the **first derivative test** and is formalized in the following theorem.

Theorem 8.4: First Derivative Test

Let c be a critical point of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval (except possibly at c) then $f(c)$ is classified as follows.

(i) If $f'(x)$ changes from negative to positive at c, then f has a *local minimum* at $(c, f(c))$.

(ii) If $f'(x)$ changes from positive to negative at c, then f has a *local maximum* at $(c, f(c))$.

(iii) If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.

We can see this demonstrated in the figures below.

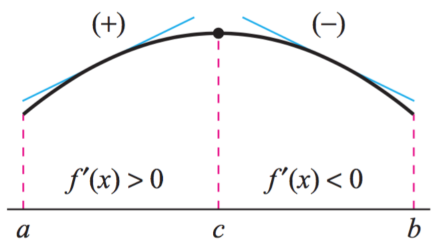


Fig. 8.8: Local Maximum

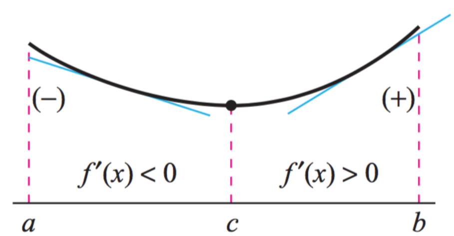


Fig. 8.9: Local Minimum

If the first derivative does not change sign then the derivative does not have a relative maximum or minimum value. See the figures below for why this is true.

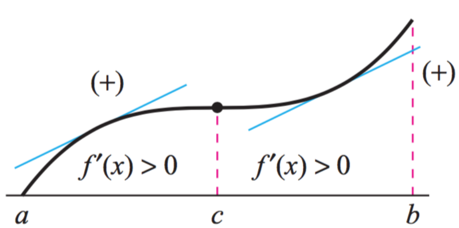


Fig. 8.10: $f'(x)$ is always positive.

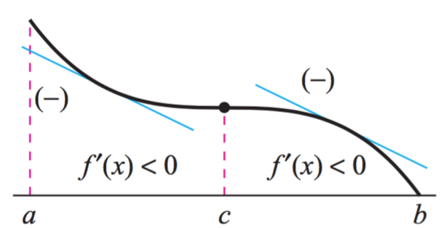


Fig. 8.11: $f'(x)$ is always negative.

It is highly recommended to check your work via graphing. If you find yourself confused about the material you should be graphing the function, first derivative, and second derivative for each of these examples. Verify that your calculations match what you see in the graph. Many students completely mess up their calculations with the derivatives and never bother to check what the graph actually looks like! Remember that you still need to show your work on an exam or homework assignment. This means graphing is only used to verify your results.

Example 8.3: Applying First Derivative Test

Find local extrema of $g(x) = (x^2 - 4)^{\frac{2}{3}}$

Solution. First we find the derivative

$$g'(x) = \frac{2}{3}(x^2 - 4)^{-\frac{1}{3}}(2x) = \frac{4x}{3(x^2 - 4)^{\frac{1}{3}}}$$

We see that $g'(x) = 0$ when $x = 0$ and $g'(x)$ does not exist when $x = \pm 2$.

So the critical numbers are located at $x = -2, 0, 2$. Now we can set up our table.

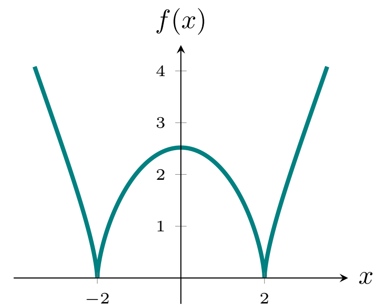


Fig. 8.12: Graph of $g(x)$

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
x	-3	-1	1	3
$g'(x)$	$g'(-3) \approx -2.339$	$g'(-1) \approx 0.942$	$g'(1) \approx -0.942$	$g'(3) \approx 2.339$
Sign	$-$	$+$	$-$	$+$
Behavior	decreasing	increasing	decreasing	increasing

Applying the First Derivative Test we can conclude the following:

Critical Value	Behavior at Critical Value	Conclusion
$x = -2$	$g(x)$ is decreasing then increasing	local minimum
$x = 0$	$g(x)$ is increasing then decreasing	local maximum
$x = 2$	$g(x)$ is decreasing then increasing	local minimum

The graph of g is pictured in Figure 8.12 for your reference.

It is recommended that you use the function table feature in your graphing calculator to assist you in evaluating your derivatives and functions. This will eliminate error and hopefully speed up your work. You do not need to show your arithmetic, simply report your values. Even if your values themselves are incorrect, it is still important to see that you can make correct conclusions from your wrong results.

Concavity

Concavity of a function is when the graph curves upward or downward. We obtain this information from whether the *first* derivative is increasing or decreasing.

Definition 8.5: Concavity

Let f be differentiable on an open interval I. The graph of f is

(i) **concave upward** on I if f' is increasing on the interval

(ii) **concave down** on I if f' is decreasing on the interval.

It is easier to visualize concavity using the figures below.

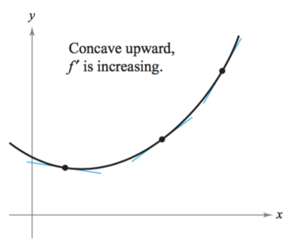


Fig. 8.13: Concave up function

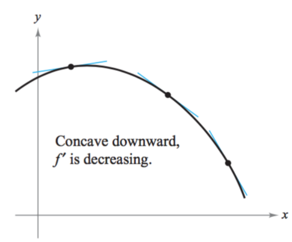


Fig. 8.14: Concave down function

In the figures above we can see that when f is concave up that the graph of f lies *above* its tangent lines. This means the graph is concave up when f' is increasing, in other words when $f''(x) > 0$. When f is concave down the graph of f lies *below* its tangent lines. This means that the graph is concave down when f' is decreasing, in other words when $f''(x) < 0$.

In summary, in order to determine concavity, we will need to apply the test for increasing and decreasing functions to the derivative of f, $f'(x)$. This means we'll need the derivative of the derivative, i.e. the *second* derivative of f, $f''(x)$. This test is summarized in the following theorem.

Theorem 8.5: Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

(i) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(ii) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Note that common language for each of these cases is different in more advanced mathematics courses. Equivalent language is saying *convex* to mean concave upward, and *concave* to mean concave downward. You may have also seen these terms before in a geometry or trigonometry course.

Example 8.4: Applying Test for Concavity

Find the intervals for which $f(x) = \frac{x^3}{3} - 4x$ is concave up and concave down.

Solution. From Example 8.1 we know the first derivative. To use the test for concavity we need to find $f''(x)$ so we have

$$f'(x) = x^2 - 4 \quad \text{and} \quad f''(x) = 2x$$

Just as with the test for increasing and decreasing, we need intervals where f'' potentially changes sign. In other words, when $f''(x) = 0$. Clearly, $f''(x) = 0$ when $x = 0$. We use $x = 0$ to define the intervals we will test. We then set up our table below.

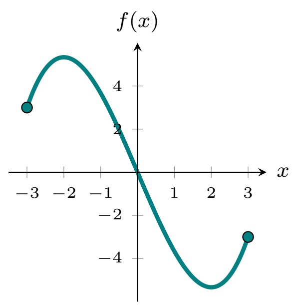


Fig. 8.15: Graph of $f(x)$

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$f''(x)$	$f''(-1) = -2$	$f''(1) = 2$
Sign	$-$	$+$
Behavior	concave down	concave up

So f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

When testing for concavity you must include any points for which the function itself does not exist. It can seriously mess up your results if you do not.

Points of Inflection

Definition 8.6: Point of Inflection

A **point of inflection** occurs when concavity of a function changes from concave up to concave down or from concave down to concave up.

To find *possible* points of inflection we use the following theorem.

Theorem 8.6: Points of Inflection

If $(c, f(c))$ is an inflection point then either $f''(c) = 0$ or f'' does not exist at $x = c$.

An important fact to note is that the converse (i.e. the reverse) of this statement is not true. In other words, it is possible that $x = c$ is not an inflection point when $f''(c) = 0$ or does not exist.

Example 8.5: Locating Points of Inflection

Locate the points of inflection for the following functions.

a.) $f(x) = \frac{x^3}{3} - 4x$

Solution. From example 8.4 we already have our table of second derivative information. We can see that at $x = 0$ the concavity changes from concave down to concave up. Thus, there is an inflection point at $x = 0$.

b.) $g(x) = x^4$

Solution. Finding the first and second derivative we have

$$g'(x) = 4x^3 \quad \text{and} \quad g''(x) = 12x^2$$

Now $g''(x) = 0$ when $x = 0$. We set up our table to test the intervals defined by this value below.

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$g''(x)$	12	12
Sign	$+$	$+$
Conclusion	concave up	concave up

In this case we see that concavity does not change. So even though $g''(x) = 0$ at $x = 0$ there is *not* an inflection point at $x = 0$.

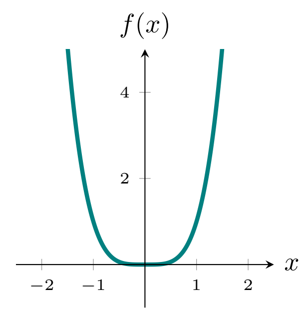


Fig. 8.16: Graph of $g(x)$

c.) $f(x) = \dfrac{x^2+1}{x^2-4}$

Solution. Finding the first and second derivative we have

$$f'(x) = -\frac{10x}{(x^2-4)^2} \quad \text{and} \quad f''(x) = \frac{10(3x^2+4)}{(x^2-4)^3}$$

In this case there are no points where $f''(x) = 0$. We do have that $f''(x)$ does not exist at $x = \pm 2$ (note that f also does not exist at these points). We then test the intervals defined by these values as seen in the table below.

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
x	-3	0	3
$f''(x)$	$f''(-3) \approx 2.48$	$f''(0) \approx -0.625$	$f''(3) \approx 2.48$
Sign	$+$	$-$	$+$
Conclusion	concave up	concave down	concave up

We see that concavity changes at both $x = -2$ and at $x = 2$. Thus, there are points of inflection at $x = 2$ and $x = -2$.

To find points of inflection

(1) Find when $f''(c) = 0$ or $f''(c) = 0$ does not exist.

(2) Test concavity on intervals defined by these points.

(3) If concavity changes then $x = c$ is an inflection point.

The Second Derivative Test

In some cases we can also obtain maximum and minimum information using the second derivative. By examining Figures 8.13 and 8.14 we see that when a function is concave up or down, then the function had to change from decreasing to increasing (or vice versa) at some point. In other words, at a maximum or minimum value of the function. We know that maximum and minimum values must occur at a point c where $f'(c) = 0$.

What does this mean? Well, if a function is concave up or down at a critical point, then we have a minimum or maximum value at that point. This means that we can use the second derivative to actually determine whether a critical point is a local maximum or minimum values. This is known as the Second Derivative Test.

Theorem 8.7: Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

(i) If $f''(c) > 0$, then $f(c)$ is a **relative minimum**.

(ii) If $f''(c) < 0$, then $f(c)$ is a **relative maximum**.

(iii) If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you must use the first derivative test.

Example 8.6: Applying the Second Derivative Test

Use the second derivative test to find the relative extrema for $f(x) = \dfrac{x^3}{3} - 4x$.

Solution. From Example 8.2 we know that the critical numbers are $x = -2$ and $x = 2$. We also know that $f''(x) = 2x$.
Applying the Second Derivative Test we have the following information.

Critical Point	Value of $f''(x)$	Behavior	Conclusion
$x = -2$	$f''(-2) = -4$	$f''(x) > 0$	relative minimum
$x = 2$	$f''(2) = 4$	$f''(x) < 0$	relative maximum

It may seem easier to always use the second derivative test to determine whether you have a local maximum or minimum value at a certain point rather than bothering to determine where a function is increasing or decreasing. However, recall that in Theorem 8.7 part (iii) we have a case when the second derivative test can fail, i.e. when $f''(c) = 0$.

Example 8.7: When the Second Derivative Test Fails

Use the second derivative test to find the relative extrema for the function $h(x) = -3x^5 + 5x^3$

Solution. We need to find the first derivative, second derivative, and the critical points. We have

$$h'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) \quad \text{and} \quad h''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

The critical points are $x = -1, 0, 1$. Applying the second derivative test we have

Critical Point	Value of $h''(x)$	Behavior	Conclusion
$x = -1$	$h''(-1) = 30$	$f''(x) > 0$	relative minimum
$x = 1$	$h''(1) = -30$	$f''(x) < 0$	relative maximum
$x = 0$	$h''(0) = 0$	$f''(x) = 0$	test fails

Since the test fails at $x = 0$ you must use the first derivative test to determine if there is a relative maximum or minimum located at $x = 0$.

You should also keep in mind if the function f itself (and not just the derivatives of f) is actually defined at $f(c)$. Just as in algebra class (and in the rest of mathematics), it is important to always take into account the domain and range of a function. It makes no sense to determine information about points where a function is not even defined!

⚠ Students constantly confuse the first and second derivative tests with the tests for increasing/decreasing and the test for concavity. Applying the first and second derivative tests is the point when you are determining if a critical point is a maximum or minimum value. This involves using information from tests for increasing/decreasing or concavity. Be sure you understand the difference!

The Mean Value Theorem & Other Theorems

The Mean Value Theorem, Rolle's Theorem, and a few other important theorems are discussed in Section 4.6 (pg. 290). Please read over the section and skim through the proofs of the respective theorems. Since you have now seen several examples, the more theoretical explanation of why we can use these approaches will hopefully make more sense.

The first four theorems are the more important theorems in the section and are merely stated below. The last theorem simply restates what we already know about intervals of increase and decrease but provides a proof of why we know this to be true.

Theorem 8.8: Rolle's Theorem

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 8.9: Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Theorem 8.10: Zero Derivative Implies Constant Function

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

The following theorem will be important when we discuss how to "undo" differentiation.

Theorem 8.11: Functions with Equal Derivatives Differ by a Constant

If two functions f and g have the property that

$$f'(x) = g'(x)$$

for all x of an interval I, then

$$f(x) - g(x) = C$$

on I, where C is a constant. In other words, f and g differ only by a constant.

Practice Problems

Section 4.1
11-18, 23-50, 56-63, 68-73, 76-78

Section 4.2
17-38, 39-52, 57-70, 71-82, 101-103

Section 4.3
9-36

Section 4.6
7-14, 17-24, 32-35