Lesson 7

L'Hôpital's Rule

Sections
4.7

Indeterminate Forms and L'Hôpital's Rule

In previous lessons we saw infinite limits and limits as x goes to infinity. Many of these limits can be difficult to handle using techniques we have learned thus far. Recall that many of these limits can have an **indeterminate form**. As a reminder the types of indeterminate forms are given below.

Definition 7.1: Indeterminate Forms

$$1^\infty \qquad 0^0 \qquad \infty^0 \qquad 0\cdot\infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty-\infty$$

Now we know how to find the derivative of a function we can make use of the following rule.

Theorem 7.1: L'Hôpital's Rule

Suppose f and g are differentiable functions on an open interval (a,b) containing c and that $g'(x) \neq 0$ for all x in (a,b). If $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$ then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

L'Hôpital's Rule can only be used for these two types of indeterminate forms! If a limit does not have form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then it must first be rewritten or simplified (if possible) to give it the necessary form *before* you can apply L'Hôpital's Rule.

Indeterminate Forms $0/0$ and ∞/∞

We start by examining how to apply L'Hôpital's Rule for limits that already have the basic indeterminate forms required by the theorem namely, $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

It is important not to forget the differentiation rules you have just learned. The most common mistake students make when applying L'Hôpital's Rule is calculating derivatives incorrectly.

Since the derivative is now just a part of a calculation, it may be worth doing derivative calculations off to the side so you don't confuse your work. This will also give you additional practice in taking derivatives. Note that the derivative calculations themselves will be skipped in the examples presented in this lesson.

Example 7.1: Indeterminate form $\frac{0}{0}$

Evaluate $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

Solution. Using direct substitution we see that

$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \frac{3^3-2(3)-3}{3-3} = \frac{0}{0}$$

Since this has the indeterminate form $\frac{0}{0}$, we can apply L'Hôpital's Rule:

$$\begin{aligned} \lim_{x\to 3} \frac{x^2-2x-3}{x-3} &\stackrel{\text{H}}{=} \lim_{x\to 3} \frac{\frac{d}{dx}[x^2-2x-3]}{\frac{d}{dx}[x-3]} && \text{Apply L'Hôpital's Rule} \\ &= \lim_{x\to 3} \frac{2x-2}{1} \\ &= \frac{2(3)-2}{1} = 4 && \text{By direct substitution} \end{aligned}$$

The symbol $\stackrel{\text{H}}{=}$ is used to indicate that L'Hôpital's Rule has been applied. It is "polite" in mathematics to indicate when special rules or techniques are applied so that others can follow what you're doing.

Example 7.2: Indeterminate Form $\frac{\infty}{\infty}$

Evaluate the following limits.

(a) $\lim\limits_{x\to\infty} \dfrac{x^3}{x+2}$

Solution. Using direct substitution we have

$$\lim_{x\to\infty} \frac{x^3}{x+2} = \frac{\infty^3}{\infty+2} = \frac{\infty}{\infty}$$

We have the form $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule

$$\begin{aligned} \lim_{x\to\infty} \frac{x^3}{x+2} &\stackrel{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[x^3]}{\frac{d}{dx}[x+2]} && \text{Apply L'Hôpital's Rule} \\ &= \lim_{x\to\infty} \frac{3x^2}{1} \\ &= 3(\infty)^2 = \infty && \text{By direct substitution} \end{aligned}$$

(b) $\lim_{x\to\infty} \dfrac{\ln(x)}{x}$

Solution. Since this has the indeterminate form $\frac{\infty}{\infty}$, we can apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x)}{x} &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x]} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \\
&= \lim_{x\to\infty} \frac{1}{x} && \text{Simplifying} \\
&= \frac{1}{\infty} = 0 && \text{By direct substitution}
\end{aligned}$$

It is possible to apply L'Hôpital's Rule more than once. Each time you apply the rule you need to make sure that the resulting expression still has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

We must also take care that at each step we have $g'(x) \neq 0$. In other words, you should stop when you have a constant in the denominator.

Example 7.3: Applying L'Hôpital's Rule multiple times

Evaluate $\lim_{x\to-\infty} \dfrac{x^2}{e^{-x}}$

Solution. Using direct substitution we see that

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{(-\infty)^2}{e^{-(-\infty)}} = \frac{(-\infty)^2}{e^{\infty}} = \frac{\infty}{\infty}$$

Since we have the form $\frac{\infty}{\infty}$, we can apply L'Hôpital's Rule

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This still has the form $\frac{\infty}{\infty}$ so we apply L'Hôpital's Rule a second time.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{2x}{-e^{-x}} &\overset{\text{H}}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{-\frac{d}{dx}[e^{-x}]} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x\to-\infty} \frac{2}{e^{-x}} \\
&= \frac{2}{e^{-(-\infty)}} = \frac{2}{e^{\infty}} = \frac{1}{\infty} = 0 && \text{By direct substitution}
\end{aligned}$$

Indeterminate Form $0 \cdot \infty$

We now proceed with some examples of limits which are *not* initially in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The goal here is to manipulate the expression in the limit so that it has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Before we continue, you should recall the fact that the product of two functions can be written as

$$fg = \frac{f}{\frac{1}{g}} = \frac{g}{\frac{1}{f}}$$

This trick will be very useful when re-writing limits in the form necessary to apply L'Hôpital's Rule.

Example 7.4: Indeterminate form $0 \cdot \infty$

Evaluate $\lim\limits_{x \to 0^+} x \ln(x)$

Solution. By applying limit laws we have

$$\lim_{x \to 0^+} x \cdot \lim_{x \to 0^+} \ln(x)$$

Recall that $\lim\limits_{x \to 0^+} \ln(x) = -\infty$. So this limit has the indeterminate form $0 \cdot \infty$. This is not one of the forms necessary to apply L'Hôpital's rule. We must first re-write this product using the form shown above to obtain

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now the limit has the indeterminate form $\frac{\infty}{\infty}$ so we can now apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}} &\overset{\text{H}}{=} \lim_{x \to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[\frac{1}{x}]} && \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x \to 0^+} \frac{-x^2}{x} && \text{Rewriting} \\
&= \lim_{x \to 0^+} -x && \text{Simplifying} \\
&= 0 && \text{By direct substitution}
\end{aligned}$$

Indeterminate Form $\infty - \infty$

The indeterminate form $\infty - \infty$ can be handled using several different strategies. The main goal is to create a single quotient. We can do this by finding a common denominator, rationalizing, or factoring out common terms.

Example 7.5: Indeterminate Form $\infty - \infty$

Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right)$

Solution. This limit has the indeterminate form $\infty - \infty$. This is not one of the forms necessary to apply L'Hôpital's rule. We must first re-write the limit. In this case, by finding a common denominator.

$$\lim_{x\to 0}\frac{x-\sin(x)}{x\sin(x)}$$

The limit now has form $\frac{0}{0}$ and so we can apply L'Hôpital's rule. Evaluating this limit will involve two applications of L'Hôpital's rule. Take care when taking derivatives. Note that the derivatives in the denominator will involve the product rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{x-\sin(x)}{x\sin(x)} &\overset{\text{H}}{=} \lim_{x\to 0}\frac{\frac{d}{dx}[x-\sin(x)]}{\frac{d}{dx}[x\sin(x)]} && \text{Apply L'Hôpital's rule}\\
&= \lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)+x\cos(x)} && \text{Has indeterminate form } \frac{0}{0}\\
&\overset{\text{H}}{=} \lim_{x\to 0}\frac{\frac{d}{dx}[1-\cos(x)]}{\frac{d}{dx}[\sin(x)+x\cos(x)]} && \text{Apply L'Hôpital's again}\\
&= \lim_{x\to 0}\frac{\sin(x)}{2\cos(x)-x\sin(x)}\\
&= \frac{\sin(0)}{2\cos(0)-(0)\sin(0)} = \frac{0}{2} = 0 && \text{By direct substitution}
\end{aligned}$$

It may not always be immediately evident which strategy you should use. Through practice, you will be able to better recognize when each strategy will work best for a particular limit. Until you have developed this intuition, try out each strategy to see which works best.

Indeterminate Forms 1^∞, 0^∞, and 0^0

There are a few different ways to think about limits that involve the indeterminate forms 1^∞, 0^∞, and 0^0. The method presented in these notes differs from the one in the text but may be easier to remember since it is similar to how we carried out logarithmic differentiation.

In general, limits with these indeterminate forms are functions of the form

$$y = [f(x)]^{g(x)}$$

Just as in logarithmic differentiation, we will use the natural logarithm. This allows us to use the properties of logarithms to bring the function in our exponent "down" a level.

Example 7.6: Indeterminate Form 0^0

Evaluate $\lim_{x\to 0^+} x^x$

Solution. Since we have a function in our exponent let $y = x^x$ and apply the natural logarithm to both sides

$$\ln(y) = \ln[x^x] \implies \ln(y) = x\ln(x)$$

Now we take the limit of each side:

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0+} x\ln(x)$$

Focusing on the right hand side we re-write this product as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now the limit has the indeterminate form $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} &\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[\frac{1}{x}]} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x && \text{Rewriting \& Simplifying} \\
&= 0 && \text{By direct substitution}
\end{aligned}$$

So we have

$$\lim_{x\to 0^+} x\ln(x) = 0$$

Since

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x\ln(x) = 0$$

Then we have

$$\lim_{x\to 0^+} \ln(y) = 0$$

We "undo" the $\ln(y)$ by exponentiating each side:

$$\lim_{x\to 0^+} e^{\ln(y)} = e^0 \implies \lim_{x\to 0^+} y = 1$$

Recall that we said that $y = x^x$ and so we have

$$\lim_{x\to 0^+} x^x = 1$$

The cases for 1^∞ and 0^∞ are handled similarly.

In the example above you should note that we used the fact that

$$e^{\left(\lim\limits_{x\to 0^+} \ln(y)\right)} = \lim_{x\to 0^+} e^{\ln(y)} = \lim_{x\to 0^+} y$$

This comes from the fact that we have the limit of a function composition.

> Keep in mind that there are cases in which no amount of manipulation will give us the form we need to use L'Hôpital's Rule. When this occurs, we must use alternative means to find the limit. It is also not appropriate to use L'Hôpital's Rule when using the definition of the derivative. Since L'Hôpital's Rule requires use of the derivative, it makes no sense to use a derivative to evaluate the limit that defines the derivative. For both of these reasons it is still important to learn techniques for evaluating limits that we learned at the beginning of the course.

A word about the textbook

The textbook and online system both say to write

$$[f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$$

Where does this come from? Recall that

$$\ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x))$$

and by expontiating each side we obtain

$$e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln(f(x))} \qquad \Longrightarrow \qquad [f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$$

It is prefered that you not skip steps and use memorized formulas to carry out work that can be found just by applying basic properties of logarithms. This property assumes we know what we need to do at the end of the process before we even start. It is not recommended to do things this way. It doesn't help to learn the "tricky tricks" before you even understand how the process itself even works.

The strategy for functions of the form $[f(x)]^{g(x)}$ is the same as for logarithmic differentiation. So it is recommended to just use the same strategy we've seen before and disregard the explanation in the textbook and in the on-line system. If for some reason, the explanation in the text makes more sense, of course use it. However, always state this property if you are applying it directly!

Your textbook has limits organized by type in the exercises portion of section 4.7. It is recommend to work through problems from each section so you can see the variety within each type of indeterminate form. It is then recommended that you attempt a random assortment of problems so you can practice recognizing different forms of these limits and applying the relevant strategy without being directed.

Practice Problems

Section 4.7

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