Lesson 11

Definite Integrals and the Fundamental Theorem of Calculus

Sections
5.2, 5.3

Introduction

We saw in the previous lecture that a limit of the form

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

arises when computing the area under the curve of a function f. It turns out that this same limit occurs in a wide variety of situations even when f is not necessarily a positive function. In fact limits of this form arise when finding the length of curves, volumes of solids, centers of mass, and work as well as other quantities of interest. Because this type of limit is so important we give it a special name and notation. In the following section we will define the definite integral with a more general form of this limit. Towards the end of this lecture we will see that the operations of differentiation and integration are deeply connected, through the Fundamental Theorem of Calculus.

Definite Integrals

In the previous lecture we discussed Riemann sums with regular partitions. In reality we can define a Riemann sum for any partition of $[a, b]$.

Definition 11.1: General Reimann Sum

Suppose

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n],$$

are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b.$$

For $k = 1, \cdots, n$ let $\Delta x_k = x_k - x_{k-1}$ be the length of the subinterval $[x_{k-1}, x_k]$ and x_k^* be an arbitrary sample point in $[x_{k-1}, x_k]$. If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum for f on** $[a, b]$.

We have seen examples where it does not matter what type of Riemann sum we use to define the area under the curve. If the area under the curve of a function will be the same no matter what type of Riemann sum we use we say that that function is integrable. The area under the curve is then given by the **Definite Integral**.

Definition 11.2: Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and is unique over all partitions of $[a, b]$ and any choice of sample point x_k*. This limit is called the **definite integral of f from a to b**, which we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Recall tha the symbol $\int$ is called an integral sign. In the notation

$$\int_a^b f(x)dx$$

a is the lower limit of integration, b the upper limit of integration, and $f(x)$ is the integrand, . The symbol dx has no official meaning by itself. However, it does signal the variable that we are integrating with respect to. It may not seem like an important symbol right now, since the functions we work with are of typically of one variable however, in multivariable calculus this is very important. Recall that the procedure of calculating an integral is called **integration**.

Note the definite integral is a number; it does not depend on x. In fact we could use any letter in its place without changing the value of the definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt$$

As we have already mentioned, the area under the curve of a function can be computed with the definite integral. What about when the graph of the curve does not lie above the x-axis? In this case we must talk about net or signed area. The Riemann sums will then approximate the area of the regions that lie above the x-axis *minus* the area of the regions that lie below the x-axis. In such situations it is possible that the area computed with a definite integral will be negative in value. Since our concept of area requires that it be a positive quantity we use the terms net area or signed area when referring to the area computed with definite integrals.

Definition 11.3: Net Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of areas of the parts of R that lie below the x-axis on $[a, b]$.

We can now formally state how to find the area of a region using the definite integral in the following theorem.

Theorem 11.1: Definite Integral as Area of a Region

If f is continuous on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Net Area} = \int_a^b f(x)\,dx$$

At long last we have a formal definition of the definite integral! However, this definition comes with a serious constraint. A function is only integrable if this limit exists and is unique for any type of Riemann sum. Just like it was with differentiation it is important to ask ourselves: "What types of functions can we integrate?". The conditions for a function to be *integrable* is given in the following theorem. The proof of these results is beyond the scope of this course.

Theorem 11.2: Integrable Functions

If a function f is continuous on the closed interval $[a, b]$, or if f is bounded on $[a, b]$ with a finite number of discontinuities then f is integrable on $[a, b]$. That is $\int_a^b f(x)\,dx$ exists.

If we know that a function is integrable we can compute the area under its curve using the definition of the definite integral. This process can be tedious but a familiarity with it will help us appreciate the simplicity of the Fundamental Theorem of Calculus.

Example 11.1: Evaluating Definite Integral as a Limit

Using the definition of the Definite Integral evaluate the value of $\int_0^3 x^3 - 6x\,dx$.

Solution. Since $x^3 - 6x$ is continuous on $[0, 3]$ it is also integrable. Thus, the *limit* of any Riemann sum will yield the same value, regardless of our choice of sample points. For this problem I will choose to use right Riemann sums to evaluate the definite integral. For n evenly spaced subintervals we have

$$\Delta x = \frac{b-1}{n} = 3 - 0n = \frac{3}{n}$$

and

$$x_k = a + k\Delta x = 0 + k\frac{3}{n} = \frac{3k}{n}$$

Using the definition of right Riemann sums we have

$$\begin{aligned}
\int_0^3 x^3 - 6x\,dx &= \lim_{n\to\infty} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{k=1}^{n} \left[\left(\frac{3k}{n}\right)^3 - 6\left(\frac{3k}{n}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\frac{27}{n^3}k^3 - \frac{18}{n}k\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{k=1}^{n} k^3 - \frac{54}{n^2} \sum_{k=1}^{n} k\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{n^4} \cdot \frac{n^2(n+1)^2}{4} - \frac{54}{n^2} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) - \frac{54}{2}\left(1 + \frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 \\
&= \frac{-27}{4}
\end{aligned}
$$

Since the definite integral is defined as the limit of a sum it inherits a lot of the same properties.

Theorem 11.3: Basic Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

$$\int_a^b kf\,dx = k\int_a^b f\,dx \quad \text{and} \quad \int_a^b f \pm g\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx.$$

Recall that the the definite integral was defined with the assumption that $a < b$. There are however occasions when it is necessary to reverse the limits of integration ($b < a$), or integrate over a single point ($b = a$). Sometimes it may be necessary to integrate the function separately over parts of the interval ($a < c < b$). This is common for absolute and piecewise defined functions. To handle these cases we have the following theorem.

Theorem 11.4: Special Properties of Definite Integrals

(i) $\displaystyle\int_a^a f(x)\,dx = 0$

(ii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The Fundamental Theorem of Calculus

We have computed definite integral using the definition as a limit of Riemann sums. We saw that this procedure is sometimes long and difficult. In fact, it is usually not possible or practical.

Fortunately, there is a powerful and practical method for evaluating definite integrals. However, in order to understand this method we will need a better understanding of the inverse relationship between differentiation and integration. The bridge we seek to connect these two ideas is known as the Fundamental Theorem of Calculus (FTOC). Since the proof of the FTOC is somewhat complicated, most textbooks present the FTOC in two separate parts. The first part is basically a useful property that can be used to validate the more useful second part.

Theorem 11.5: The Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a, b]$ then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b$$

is continuous on $[a, b]$ and differentiable of (a, b). The area function satisfies

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Since A is an antiderivative of f on $[a, b]$ it is one short step to a powerful method for evaluating definite integrals. Remember that any two antiderivatives of f differ by a constant. Assuming that F is any antiderivative of f on $[a, b]$ we have

$$F(x) = A(x) + C, \quad \text{for } a \le x \le b$$

Noting that if $x = a$ then we have

$$A(a) = \int_a^a f(t)\,dt = 0$$

and so $A(a) = 0$. It follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b)$$

Writing $A(b)$ in terms of the definite integral we have the result that

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

The result is essentially the second part of the Fundamental Theorem of Calculus, sometimes referred to as the Evaluation Theorem.

Theorem 11.6: The Fundamental Theorem of Calculus (Part 2)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We now have a way to compute definite integrals using our knowledge of derivatives. Are you excited? You should be excited! At this point we are free of the dreaded tyranny that is the limit definition of the definite integral. Now if we know the antiderivative of a given function we can easily compute the area under its curve.

Lets pause for a moment to discuss what the fundamental theorem of calculus really means. Part one says that

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$$

and part two says that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are an inverse processes i.e. they each undo each other. This realization is essential for our mastery of the process of integration.

Guidelines for Using the Fundamental Theorem of Calculus

- Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
- When applying the Fundamental Theorem of Calculus the following notation is convenient

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

- It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = (F(x) + C)\Big|_a^b = \Big[(F(b) + C) - (F(a) + C)\Big] = F(b) - F(a)$$

With the help of the Fundemental Theorem of Calculus we are now capable of evaluating a wide range of definite integrals. We demonstrate this process in the following examples.

Example 11.2: Evaluating a Definite Integral

Evaluate each definite integral.

a.) $\displaystyle\int_1^2 (x^2 - 3)\,dx$

Solution.

$$\int_1^2 (x^2 - 3)\,dx = \left(\frac{1}{3}x^3 - 3x\right)\Bigg|_1^2 = \left(\frac{1}{3}(2)^3 - 3(2)\right) - \left(\frac{1}{3}(1)^3 - 3(1)\right) = \frac{-2}{3}$$

b.) $\displaystyle\int_1^4 3\sqrt{x}\,dx$

Solution.

$$\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 \sqrt{x}\,dx = 3\left(\frac{2}{3}x^{3/2}\right)\Bigg|_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$$

c.) $\displaystyle\int_0^{\pi/4} \sec^2 x\, dx$

Solution.

$$\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = \tan(\pi/4) - \tan(0) = 1$$

It is worth noting that we have only scratched the surface of the study of integration. Just like with differentiation there exists several rules and techniques that are needed to evaluate more complicated integrals. A more complete picture of integration is typically presented in a second semester calculus course.

Practice Problems

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