Evaluating Limits Involving Infinity

For Problems 1 to 4 use algebraic/analytic methods to find the limit (if it exists). If the limit does not exist, explain why.

1. Find $\lim_{x\to\infty} \sqrt{9x^2+x} - 3x$ or show it does not exist (DNE).

   Solution:

We need to multiply by the conjugate:

$$\lim_{x\to\infty} \sqrt{9x^2+x} - 3x = \lim_{x\to\infty} \sqrt{9x^2+x} - 3x \left(\frac{\sqrt{9x^2+x} + 3x}{\sqrt{9x^2+x} + 3x} \right)$$

Multiply by the Conjugate

$$= \lim_{x\to\infty} \frac{(\sqrt{9x^2+x})^2 - (3x)^2}{\sqrt{9x^2+x} + 3x}$$

Apply difference of squares property

$$= \lim_{x\to\infty} \frac{\cancel{9x^2} + x - \cancel{9x^2}}{\sqrt{9x^2+x} + 3x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x} + 3x}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\sqrt{\frac{9x^2}{x^2} + \frac{x}{x^2}} + \frac{3x}{x}}$$

Divide by highest power term in denominator.

Note: $\sqrt{x^2} = x$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9 + \frac{1}{x}} + 3}$$

$$= \frac{\lim_{x\to\infty} 1}{\sqrt{\lim_{x\to\infty} 9 + \lim_{x\to\infty} \frac{1}{x}} + \lim_{x\to\infty} 3}$$

Apply properties of limits

$$= \frac{1}{\sqrt{9+0} + 3}$$

$$= \frac{1}{\sqrt{9} + 3}$$

$$= \frac{1}{6}$$

2. Evaluate $\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$. Justify each step with property used.

Solution:

Note that in this case the numerator can't be factored.

We will need to divide by highest power term in denominator.

Let $$f(x) = \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$$

$$= \frac{\cancel{x^2}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}{\cancel{x^2}\left(2 + \frac{5}{x} - \frac{8}{x^2}\right)}$$

Dividing by highest power term in denominator is same as factoring out highest power term

$$= \frac{\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}{\left(2 + \frac{5}{x} - \frac{8}{x^2}\right)}$$

Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}{\left(2 + \frac{5}{x} - \frac{8}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} + 4 \lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 2 - 5 \lim_{x\to\infty} \frac{1}{x} + 8 \lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{3 - 0 + 4(0)}{2 - 5(0) + 8(0)}$$

Recall that $\lim_{x\to\infty} \frac{1}{x^n} = 0$

$$= \frac{3}{2}$$

3. $\lim_{x\to\infty} \frac{\sin^2 x}{x^2+1}$

Solution:

We'll use the squeeze theorem:

We know that the range of $\sin^2(x)$ is

$$-1 \le \sin^2(x) \le 1$$

and so

$$\frac{-1}{x^2+1} \le \frac{\sin^2(x)}{x^2+1} \le \frac{1}{x^2+1}$$

then it follows that

$$-\lim_{x\to\infty} \frac{1}{x^2+1} \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le \lim_{x\to\infty} \frac{1}{x^2+1}$$

Since

$$\lim_{x\to\infty} \frac{-1}{x^2+1} = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x^2+1} = 0$$

then $0 \le \lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} \le 0$

Therefore, by the squeeze theorem

$$\lim_{x\to\infty} \frac{\sin^2(x)}{x^2+1} = 0$$

4. $\displaystyle\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

Solution:

Let $f(x) = \dfrac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$$= \frac{\frac{x}{4x}+\frac{4}{4x}}{4+x}$$

$$= \frac{\frac{x+4}{4x}}{4+x}$$

$$= \frac{\cancel{x+4}}{4x(\cancel{4+x})}$$

$$= \frac{1}{4x}$$

Now evaluating our limit:

$$\lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{1}{4x}$$

$$= \frac{1}{-8}$$

5. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

Solution:

$$\lim_{x\to-\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} = \lim_{x\to-\infty} \frac{\sqrt{x^6\left(9 - \frac{1}{x^5}\right)}}{x^3\left(1 + \frac{1}{x^3}\right)}$$

When roots are involved I find it easier to factor out the highest power term in the denominator

$$= \lim_{x\to-\infty} \frac{\sqrt{x^6}\sqrt{\left(9 - \frac{1}{x^5}\right)}}{x^3\left(1 + \frac{1}{x^3}\right)}$$

By ~~Properties~~ of square roots i.e. $\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$

$$= \lim_{x\to-\infty} \frac{-\cancel{x^3}\sqrt{9 + \frac{1}{x^5}}}{\cancel{x^3}\left(1 + \frac{1}{x^3}\right)}$$

Since $x \to -\infty$ then $\sqrt{x^6} = -x^3$

$$= \lim_{x\to\infty} \frac{-\sqrt{9 + \frac{1}{x^5}}}{1 + \frac{1}{x^3}}$$

$$= \frac{-\sqrt{\lim_{x\to-\infty} 9 + \lim_{x\to-\infty} \frac{1}{x^5}}}{\lim_{x\to-\infty} 1 + \lim_{x\to-\infty} \frac{1}{x^3}}$$

Applying limit laws

$$= \frac{-\sqrt{9+0}}{1}$$

Direct substitution

$$= -3$$

Horizontal & Vertical Asymptotes

6. Consider $f(x) = \dfrac{2x^2 - x - 1}{x^2 + x - 2}$. Using your knowledge of limits determine the following.

   (a) The vertical asymptotes or holes of $f(x)$.

   Solution:

Determine points where f(x) DNE

Factoring f(x) we find $f(x) = \dfrac{(2x+1)(x-1)}{(x-1)(x+2)}$

We see that f(x) DNE @ $x = 1$ & $x = -2$ so need to evaluate

$$\lim_{x\to 1} f(x) \quad \text{and} \quad \lim_{x\to -2} f(x)$$

Simplify f(x)

$$f(x) = \frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}(x+2)} = \frac{2x+1}{x+2}$$

Evaluating limits

For $x = 1$:

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{2x+1}{x+2} = \frac{2(1)+1}{1+2} = \frac{3}{3} = 1$$

Conclusion:
Since $\lim_{x\to 1} f(x)$ is a finite value then there is a hole @ $x = 1$

For $x = -2$:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{2x+1}{x+2} = \frac{-3}{-\text{small}\ \#} = +\infty$$

As $x \to -2^-$ we have that
$2x+1 \to -3$
& $x+2 \to -$ small #

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{2x+1}{x+2} = \frac{-3}{+\text{small}\ \#} = -\infty$$

As $x \to -2^-$ we have that
$2x+1 \to -3$
& $x+2 \to -$ small #

Since $\lim_{x\to -2^-} f(x) = \infty$ (and/or since $\lim_{x\to -2^+} f(x) = -\infty$)
Then f(x) has a vertical Asymptote @ $x = -2$

6. (Continued)

(b) The horizontal asymptotes of $f(x)$.

Solution: To find horizontal Asymptotes we need to evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to -\infty} f(x)$

If <u>either</u> of these limits is finite then we have a horizontal Asymptote and so both must be checked to confirm if there are Horizontal Asymptotes or not.

First we get $f(x)$ in form we want by dividing by highest power term in the denominator.

$$f(x) = \frac{2x^2 - x - 1}{x^2 + x - 2} = \frac{\frac{x^2}{x^2} - \frac{x}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{2}{x^2}} = \frac{2 - \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$$

then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2 - \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$$

$$= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{1}{x} - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{1}{x} - 2\lim\limits_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{2 - 0 - 0}{1 + 0 - 2(0)}$$

$$= 2$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2 - \frac{1}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$$

$$= \frac{\lim\limits_{x\to-\infty} 2 - \lim\limits_{x\to-\infty} \frac{1}{x} - \lim\limits_{x\to-\infty} \frac{1}{x^2}}{\lim\limits_{x\to-\infty} 1 + \lim\limits_{x\to-\infty} \frac{1}{x} - 2\lim\limits_{x\to-\infty} \frac{1}{x^2}}$$

$$= \frac{2 - 0 - 0}{1 + 0 - 2(0)}$$

$$= 2$$

Since $\lim\limits_{x\to\infty} f(x) = 2$, (which is a finite value) then there is a horizontal Asymptotes @ $y = 2$

(Note that we also have this from the fact that $\lim\limits_{x\to-\infty} f(x) = 2$)