# Lesson 10

## Indefinite Integration and Area Under Curves

**Sections**
4.9, 5.1

## Indefinite Integration

### Antiderivatives

Up to this point in the course it has been assumed that we already know the function whose rate of change is of interest to us. Unfortunately, such an assumption is rarely true for a vast majority of the applications of calculus. It is typically more practical to measure the rate of change of a parameter rather than the parameter itself.

For instance, while driving your car the speedometer tells you how fast you are going but does not directly tell you when you will get to your destination. A biologist may know the rate at which a bacteria population is growing but not the exact number of bacteria in that population. It is easier for a physicist to measure the velocity of a moving particle than its position at specific times. In these cases, the problem is to find a function $F$ whose derivative satisfies the known information. If such a function $F$ exists, it is referred to as the antiderivative of $f$. The process of finding these functions is referred to as integration.

**Definition 10.1: Antiderivative**

A function $F$ is an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Antiderivatives are not necessarily unique. In other words, $F$ is *an* antiderivative and not *the* antiderivative of $f$. For example the functions

$$F_1(x) = x^4 + 6, \quad F_2(x) = x^4, \quad F_3(x) = x^4 - 17$$

are all antiderivatives of the function $f(x) = 4x^3$.

**Theorem 10.1: Representation of Antiderivatives**

If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on the interval $I$ is of the form

$$F(x) + C$$

for all $x$ in $I$, where $C$ is a constant.

In other words, you can represent the entire family of antiderivatives of a function by adding a constant to a known antiderivative.

For example, the entire family of antiderivatives for $f(x) = 3x^2$ is given by

$$G(x) = x^3 + C$$

where $C$ is some constant. This $C$ is known as the **constant of integration**. The function $G$ can sometimes be referred to as the **general antiderivative** of $f$. Sometimes, it is possible to find a *specific* value for the constant $C$. This is covered later in the section on differential equations.

## Notation for Integrals

The operation of finding an antiderivative can be called **antidifferentiation** but is more commonly referred to as **indefinite integration**.

There is specific notation that goes along with integration. Figure 10.1 breaks this notation. The function being integrated is known as the **integrand** while the variable you are integrating with respect to is the **variable of integration**.

The expression $\int f(x)\,dx$ is read as the "integral of $f$ with respect to $x$".

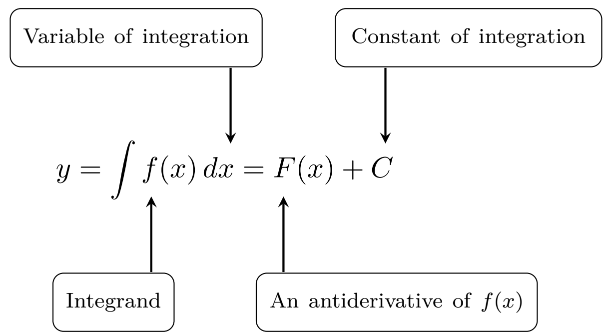


**Fig. 10.1:** Integral Notation

## Basic Integration Rules

Integration can be thought of as the "inverse" or "undoing" of the operation of differentiation. Suppose you take the derivative of a function $f(x)$ and obtain $f'(x)$ then integrating we obtain

$$\int f'(x)\,dx = f(x) + C$$

Due to the "inverse" nature of integration we also have

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

These two equations allow you to obtain integration formulas directly from known differentiation formulas. For example, the most common integration formula you will use is the counterpart to the power rule of differentiation.

**Theorem 10.2: Power Rule for Indefinite Integrals**

Let $p$ be a real number then

- For $p \neq -1$

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$$

- For $p = -1$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

where $C$ is an arbitrary constant.

We can also obtain the equivalent basic rules for integrals that we have for derivatives. These are given in Table 10.1 along with the equivalent differentiation rule.

**Table 10.1:** Basic Integration Rules

**Constant Rule**

$$\frac{d}{dx}[C] = 0 \quad \Longrightarrow \quad \int 0\,dx = C$$

**Constant Multiple Rule**

$$\frac{d}{dx}[kf(x)] = kf'(x) \quad \Longrightarrow \quad \int kf(x)\,dx = k\int f(x)\,dx$$

**Sum & Difference Rule**

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \quad \Longrightarrow \quad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

**Example 10.1: Applying Basic Integration Rules**

Integrate the following

**a.)** $\int 6x\,dx$

**Solution.**

$$\int 6x\,dx = 6\int x\,dx = 6\left(\frac{x^{1+1}}{1+1}\right) + C = \frac{6x^2}{2} + C$$

Note that since $C$ is just an arbitrary constant then $6C$ is also an arbitrary constant. We could call it something different like $K$ but instead just keep using $C$. In other words, multiplying by 6 will not change the fact that it's an arbitrary constant.

**b.)** $\int \sqrt{x}\,dx$

**Solution.**

$$\begin{aligned}
\int \sqrt{x}\,dx &= \int x^{1/2}\,dx && \text{Rewrite the integrand} \\
&= \frac{x^{1/2+1}}{1/2+1} + C && \text{Power rule for integrals} \\
&= \frac{2}{3}x^{3/2} + C && \text{Simplify}
\end{aligned}$$

**c.)** $\int \frac{2}{x}\,dx$

**Solution.**

$$\begin{aligned}
\int \frac{2}{x}\,dx &= 2\int \frac{1}{x}\,dx && \text{Constant multiple rule} \\
&= 2\ln x + C && \text{By Theorem 10.2}
\end{aligned}$$

In Table 10.2 we summarize some of the common integration formulas you will use.

**Table 10.2:** Common Integration Formulas

| | |
|---|---|
| $\int 0\, dx = C$ | $\int k\, dx = kx + C$ |
| $\int 1\, dx = x$ | $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ |
| $\int kf(x)\, dx = k \int f(x)\, dx$ | $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$ |
| $\int e^x\, dx = e^x + C$ | $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$ |
| $\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$ | $\int a^x\, dx = \left(\frac{1}{\ln a}\right) a^x + C$ |

**Example 10.2: Integrating Simple Functions**

Integrate the following functions.

**a.)** $\int (x^2 + 4x - 7)\, dx$

**Solution.**

$$\begin{aligned}
\int (x^2 + 4x - 7)\, dx &= \int x^2\, dx + \int 4x\, dx - \int 7\, dx && \text{Sum/Difference rule} \\
&= \int x^2\, dx + 4 \int x\, dx - 7 \int 1\, dx && \text{Constant Multiple rule} \\
&= \frac{1}{2+1} x^{2+1} + 4\left(\frac{1}{1+1} x^{1+1}\right) - 7x + C && \text{Power rule} \\
&= \frac{1}{3} x^3 + 2x^2 - 7x + C && \text{Simplify}
\end{aligned}$$

**b.)** $\int \frac{x^2 + 3x}{\sqrt{x}}\, dx$

**Solution.** First we rewrite and simplify the integrand.

$$\begin{aligned}
\int \frac{x^2 + 3x}{\sqrt{x}}\, dx &= \int \left(\frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}}\right) dx \\
&= \int \left(x^{3/2} + 3x^{1/2}\right) dx \\
&= \int x^{3/2}\, dx + 3 \int x^{1/2}\, dx \\
&= \frac{1}{3/2+1} x^{3/2+1} + 3\left(\frac{1}{1/2+1} x^{1/2+1}\right) + C \\
&= \frac{2}{5} x^{5/2} + 2x^{3/2} + C
\end{aligned}$$

Integration can get complicated pretty quickly. Just as with derivatives, many antiderivatives can only be found be rewriting the integrand into a simpler form that we know how to handle. The following examples are about as complicated as integration will get in this course. Techniques for how to deal with more complicated integrals than these are explored in the second semester of calculus. The one nice thing about integration is that you can always check your work by taking the derivative.

We DO NOT have a direct integration rule for the product or quotient of two functions. Techniques for handling most functions of this form are covered in the second semester of calculus. At this level, you must reduce or simplify your integrand into a form that we can easily find an antiderivative for based on our knowledge of derivatives.

### Integrals of Trigonometric Functions

We also have integration formulas for trigonometric functions. Some of the more common integration formulas are given in Table 10.3.

**Table 10.3:** Integrals of Trigonometric Functions

| | |
|---|---|
| $\int \sin x\, du = -\cos u + C$ | $\int \csc u\, du = -\ln\lvert\csc u + \cot u\rvert + C$ |
| $\int \cos u\, du = \sin u + C$ | $\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert + C$ |
| $\int \tan u\, du = -\ln\lvert\cos u\rvert + C$ | $\int \cot u\, du = \ln\lvert\sin u\rvert + C$ |

**Example 10.3: Integrating a Trigonometric Function**

Integrate $\displaystyle\int \frac{\sin x}{\cos^2 x}\, dx$

**Solution.** First we rewrite the integrand and apply a simple trig identity to get it in a form we recognize.

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\, dx = \int \sec x \tan x\, dx$$

Recalling the differentiation rule $\frac{d}{dx}[\sec x] = \sec x \tan x$ we find that

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \sec x \tan x\, dx = \sec x + C.$$

## Introduction to Differential Equations

A **differential equation** is an equation involving an (unknown) function and its derivative. The study of differential equations makes up a large portion of applied mathematics as they are used to model much of the phenomena observed in the real world. The simplest form of a differential equation looks like

$$y'(x) = x^2 + x \quad \text{or} \quad \frac{dy}{dx} = x^2 + x$$

While there are many methods used to obtain solutions to a differential equation, the operation underlying most of them is just plain old integration. By integrating the above equation we obtain

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Since we still have a constant of integration this is known as a **general solution** to the differential equation. In order to find the value of $C$ we need to know the value of our function at a single value of $x$. This is what is known as an an **initial condition**. Usually, the initial condition is the value of the function at the time $t = 0$ but this is not always the case.

An equation involving derivatives along with an initial condition is what is known as an **initial value problem**. For example, suppose we know that $y(0) = 1$ for the differential equation given above. Then we have the initial value problem

$$y'(x) = x^2 + x, \quad y(0) = 1$$

When we are able to determine the constant of integration we call this a **particular solution** to the differential equation. For example, the above initial value problem above has the particular solution

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$$

**Example 10.4: Solving an Initial Value Problem**

Find the function $y(t)$ that satisfies the initial value problem

$$y'(t) = \frac{3}{t} + 6, \quad y(1) = 8$$

**Solution.** We know that $\int y'(t)\,dt = y(t) + C$ so to obtain $y(t)$ we first integrate $y'(t)$.

$$y(t) = \int \left(\frac{3}{t} + 6\right) dt = 3\int \frac{1}{t}\,dt + 6\int 1\,dt = 3\ln t + 6t + C$$

This gives us the **general solution** to the differential equation

$$y(t) = 3\ln t + 6t + C$$

To find a particular solution we need to know a value of the function $y(t)$. We are given that $y(1) = 8$ so we plug this into our general solution and solve for $C$.

$$8 = 3\ln(1) + 6(1) + C \quad \Longrightarrow 8 = 0 + 6 + C \quad \Longrightarrow C = 2$$

So our **particular solution** is

$$y(t) = 3\ln t + 6t + 2$$

We often have rate of change information for something but need to know the function that governs the rate of change we are observing. This is why the solution of differential equations is such a key part of applied mathematics, engineering, and other sciences. We will demonstrate the solution of these types of problems through some examples.

The most common type of initial value problem you will see at this level is an application involving velocity, acceleration and position. Previously, we were given a position function $s(t)$ and asked to find its velocity, $v(t)$ or its acceleration $a(t)$. We found these functions by taking the derivative of the position function. In the real world, we actually have the opposite situation. We will have data (and therefore a function) governing the acceleration or velocity and then need to find the position at a particular time.

Why is this? Well, it is much easier to determine how fast an object is going by recording the distance traveled over a certain period of time rather than trying to measure the exact path traveled by the object. Now we will solve a motion problem. In this situation, we will deal with two initial conditions. In general, the number of initial conditions you have for a problem will match the highest order of derivative you have. Recall that $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

**Example 10.5: Solving a Motion Initial Value Problem**

Find the position function given

$$a(t) = 4, \quad v(0) = -3, \quad s(0) = 2$$

**Solution.** Recall that $a(t) = v'(t) = s''(t)$ so we integrate $a(t)$ to find $v(t)$,

$$v(t) = \int a(t)\,dt = \int 4\,dt = 4t + C$$

The value of the constant $C$ is determined by the initial condition $v(0) = -3$,

$$-3 = v(0) = 4(0) + C \qquad \Longrightarrow \; C = -3.$$

So our velocity function is

$$v(t) = 4t - 3.$$

To find the position function we recall that $v(t) = s'(t)$ and so

$$s(t) = \int v(t)\,dt = \int 4t - 3\,dt = 4\left(\frac{1}{2}t^2\right) - 3t + D = 2t^2 - 3t + D.$$

We use the initial condition $s(0) = 2$ to solve for $D$. Note that the use of $D$ for the constant of integration is simply to differentiate it from the previously used $C$. The letter you choose to represent this constant is essentially arbitrary.

$$2 = 2(0)^3 - 3(0) + D \qquad \Longrightarrow \; D = 2.$$

So the position function is

$$s(t) = 2t^2 - 3t + 2$$

You may not always have initial conditions with respect to a derivative. This is demonstrated in the following example.

**Example 10.6: Solving an Initial Value Problem**

Consider the problem addressed in Example 10.5 where $a(t) = 4$ but now with the initial conditions $s(0) = 2$ and $s(2) = 4$.

**Solution.** In Example 10.5 we found that

$$v(t) = 4t + C.$$

Since we have no other information about $v(t)$ we integrate again to obtain the position function

$$s(t) = \int v(t)\,dt = \int 4t + C\,dt = 2t^2 + Ct + D$$

Now we use the initial conditions $s(0) = 2$ and $s(2) = 4$ to solve for $C$ and $D$.

$$\begin{aligned} 2 &= s(0) = 2(0)^2 + C(0) + D \\ 4 &= s(2) = 2(2)^2 + C(2) + D \end{aligned}$$

The first equation tells us that $D = 2$ so we plug this result into the second equation to find

$$2C + 2 = -4 \quad \implies 2C = -6 \quad \implies C = -3.$$

Plugging in $D = 2$ and $C = -3$ we obtain the particular solution

$$s(t) = 2t^2 - 3t + 2$$

## Sigma Notation and Basic Summation Rules

When we started our discussion on the derivative we used the tangent line to provide a geometric meaning to its definition. In much the same way, we will use area problems to formulate the idea of definite integration. At first glance, the concept of the indefinite integral and definite integration may seem unrelated to each other, but both of these processes are closely related through the Fundamental Theorem of Calculus.

Before we can define the definite integral we need to pause for a moment to introduce a concise notation for sums. To be clear, integration is not the only time that you will see summation notation as students! It is used in the study of series which you will see in the second semester of calculus in addition to the solution of differential equations, complexity reduction in nonlinear systems, approximation theory and a variety of other applications.

### Sigma Notation

Adding up a bunch of terms can be cumbersome when there are a large number of terms. Luckily, we have notation that will make a sum much easier to handle. This is known as **sigma notation**. The notation is represented by the upper case version of the Greek letter sigma. Although it looks like the letter $E$ in English, it is the equivalent of the letter $S$ in the Greek alphabet to represent the word sum. This notation is a valuable tool as it allows us to express sums in a compact way.

**Definition 10.2: Sigma Notation**

The sum of $n$ terms $a_1, a_2, a_3 \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where $i$ is the **index of summation** $a_i$ is the $i$**th term** of the sum, the **upper bound of summation** is $n$ and the **lower bound of summation** is 1. Note that $i$ and $n$ are integers.

Essentially, this notation allows us to represent the a sum of many terms in a repeatable pattern. The index $i$ can be interpreted as the "count" of the term and $n$ the number of terms you want to add up. Both of these values must of course be integers. Also note that the lower bound doesn't have to be 1. In fact, any integer less than or equal to the upper bound is valid.

The index (for example $i$) in a sum is a dummy variable that serves as a place holder for an actual integer value to be used in that term. Any letter can be used as the index of summation. The letters $i, j$ and $k$ are often used. It is important to recognize that the index of summation does not appear in the terms of the expanded sum.

**Example 10.7: Evaluating Basic Summations**

Evaluate the following sums

**a.)** $\displaystyle\sum_{k=1}^{5} k$

**Solution.** In words this notation tells us "add up the integers from 1 to 5". Note that there is no $k$ in our final result.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

**b.)** $\displaystyle\sum_{j=1}^{3} (1 + j^2)$

**Solution.**

$$\sum_{j=1}^{3} (1 + j^2) = (1 + (1)^2) + (1 + (2)^2) + (1 + (3)^2) = 2 + 5 + 10 = 17$$

**c.)** $\displaystyle\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right)$

**Solution.**

$$\begin{aligned}\sum_{n=0}^{4}\sin\left(\frac{n\pi}{2}\right) &= \sin\left(\frac{0\pi}{2}\right)+\sin\left(\frac{1\pi}{2}\right)+\sin\left(\frac{2\pi}{2}\right)+\sin\left(\frac{3\pi}{2}\right)+\sin\left(\frac{4\pi}{2}\right)\\ &= \sin(0)+\sin\left(\frac{\pi}{2}\right)+\sin(\pi)+\sin\left(\frac{3\pi}{2}\right)+\sin(2\pi)\\ &= 0+1+0-1+0=0\end{aligned}$$

## Basic Summation Rules

Just as with derivatives and integrals we also have basic rules for a summation. These rules of course deal with multiplying a sum by a constant and the sum and difference of two summations.

**Theorem 10.3: Basic Summation Rules**

Suppose that $\{a_1, a_2 \dots, a_k\}$ and $\{b_1, b_2 \dots, b_k\}$ are two sets of real numbers and that $c$ is also a real number.

- Constant Multiple Rule: $\displaystyle\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$
- Sum and Difference Rule: $\displaystyle\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Similar to the process of differentiation and integration it is often easier to use the basic summation rules to simplify the process of evaluating a sum. We demonstrate this process in the following examples.

**Example 10.8: Using Basic Summation Rules**

Evaluate the following summations.

**a.)** $\displaystyle\sum_{k=1}^{5} 3k$

**Solution.**

$$\sum_{k=1}^{5} 3k = 3\sum_{k=1}^{5} k = 3(15) = 45$$

**b.)** $\displaystyle\sum_{j=1}^{3}(j+j^2)$

**Solution.**

$$\begin{aligned}\sum_{j=1}^{3}(j+j^2) &= \sum_{j=1}^{3} j + \sum_{j=1}^{3} j^2\\ &= (1+2+3)+\left((1)^2+(2)^2+(3)^2\right)\\ &= 20\end{aligned}$$

**c.)** $\displaystyle\sum_{n=1}^{3} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right)$

**Solution.** In a previous example we found that $\displaystyle\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) = 0$. The Sum and Difference rule lets us use this prior knowledge quickly compute this sum.

$$\begin{aligned}
\sum_{n=0}^{4} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right) &= \sum_{n=0}^{4} \cos(n\pi) - \sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) \\
&= \sum_{n=0}^{4} \cos(n\pi) - 0 \\
&= \cos(0\pi) + \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) \\
&= 1 - 1 + 1 - 1 \\
&= 0
\end{aligned}$$

What about upper bounds of summation that are very large? It can get tedious very quickly if calculating an even slightly more involved summation. Luckily, for some expressions we have patterns that allow us to develop simple formulas for evaluating certain summations.

**Theorem 10.4: Summation Formulas**

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

As we will see in the following examples, the process of evaluating a sum is made significantly easier by combining these known sum formulas with the basic summation rules.

**Example 10.9: Evaluating Summations Using Formulas**

Evaluate the following summations.

**a.)** $\displaystyle\sum_{p=1}^{105} p$

**Solution.**

$$\sum_{p=1}^{105} p = \frac{(105)\Big((105)+1\Big)}{2} = 5565$$

**b.)** $\displaystyle\sum_{k=1}^{35} (1+k^2)$

**Solution.**

$$\begin{aligned}\sum_{k=1}^{35}(1+k^2) &= \sum_{k=1}^{35} 1 + \sum_{k=1}^{35} k^2 \\ &= 1(35) + \frac{(35)\Big((35)+1\Big)\Big(2(35)+1\Big)}{6} = 14945\end{aligned}$$

**c.)** $\displaystyle\sum_{m=1}^{72}(2m+m^3)$

**Solution.**

$$\begin{aligned}\sum_{m=1}^{12}(2m+m^3) &= 2\sum_{m=1}^{12} m + \sum_{m=1}^{12} m^3 \\ &= 2\left(\frac{(12)\Big((12)+1\Big)}{2}\right) + \left(\frac{(12)^2\Big((12)+1\Big)^2}{4}\right) \\ &= 2\,(78) + (6084) = 6240\end{aligned}$$

Typically it is rare that formulas like those presented in 10.4 can be obtained for a given series. There is one series for which we do have a formula for obtaining its value. This is the Geometric series.

**Definition 10.3: Geometric Series**

For real numbers $a \neq 0$ and $r$, a series of the form

$$\sum_{k=0}^{n} ar^k$$

is called a **Geometric Series**. If and only if $|r| \leq 1$ then the general formula for the value of a geometric series is

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^n}{1-r}$$

What happens when the upper bound of a summation gets arbitrarily large? The upper bound of a summation does not need to be a finite number. Now that we have some general formulas for sums we might be curious to know the behavior of this sum as more and more terms are added. To accomplish this task we can use our old friend: Limits!

**Example 10.10: Evaluating Summations Using Formulas**

Evaluate the following summations.

**a.)** $\displaystyle\sum_{k=1}^{\infty} k$

**Solution.**

$$\sum_{k=1}^{\infty} k = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

**b.)** $\displaystyle\sum_{k=1}^{\infty} 2^k$

**Solution.**

$$\begin{aligned}\sum_{k=1}^{\infty} 2^k &= \lim_{n\to\infty} \sum_{k=1}^{n} 2^k = \lim_{n\to\infty} \frac{1-2^n}{1-2}\\ &= \lim_{n\to\infty} (2^n - 1)\\ &= \lim_{n\to\infty} 2^n - \lim_{n\to\infty} 1\\ &= \infty - 1 = \infty\end{aligned}$$

**c.)** $\displaystyle\sum_{k=1}^{\infty} 2^{-k}$

**Solution.**

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-k} &= \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = \lim_{n\to\infty} \left(2 - 2\left(\frac{1}{2}\right)^n\right)\\ &= \lim_{n\to\infty} 2 - 2\lim_{n\to\infty} \left(\frac{1}{2}\right)^n\\ &= 2 - 0 = 2\end{aligned}$$

## Area Under Curves

We already know several formulas for computing the area of a region in Euclidean geometry. For example, the area of a rectangle is given by $A = bh$ where $b$ and $h$ represent the length of the rectangles base and height respectively. In fact, it is possible to compute the area of any polygon using the area formulas for triangles and rectangles. That is, the area of a polygon can be found by dividing it into triangles and rectangles and adding their areas.

What do we do if we have a shape with curved with curved sides, like a circle? Here is where we would run into problems.

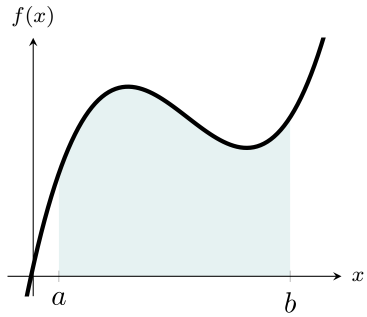


**Fig. 10.2:** Region under a curve

Our goal in this section is to find a solution to the *Area Problem*: Find the area of the region $R$ that lies under the curve $y = f(x)$ between the boundaries formed from the vertical lines $x = a$ to $x = b$. This is pictured in Figure 10.2.

Recall how we defined the tangent line. We used secant lines to approximate the value of the slope of the tangent line and then took the limit of these approximations. To approximate the area under a curve we will use a similar idea.

We first approximate the area of the region using rectangles. We can get a more accurate approximation with the more rectangles that we use. The exact value for the area under the curve will then be obtained by taking the number of rectangles to infinity via a limit. Unfortunately, this process can be extremely confusing without an appropriate mathematical framework and notation.

## Approximating Area by Riemann Sums

In order to approximate an area using rectangles we must first decide how we will divide the interval from $[a, b]$ into *subintervals*. Each subinterval will define the base of each rectangle used in our approximation. For simplicity we will use subintervals of equal length. This is referred to as a regular partition for the interval $[a, b]$.

**Definition 10.4: Regular Partition**

A **regular partition** is the division of a closed interval $[a, b]$ into $n$ subintervals of equal length where

- The **subintervals** are given by

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots\ ,\ [x_{n-1}, x_n]$$

- The **grid points** are the endpoints of the subintervals given by $x_0$, $x_1$, …, $x_n$ where the $k$th grid point is given by

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, \cdots, n$$

- The **length** of each subinterval is given by $\Delta x = \frac{b-a}{n}$

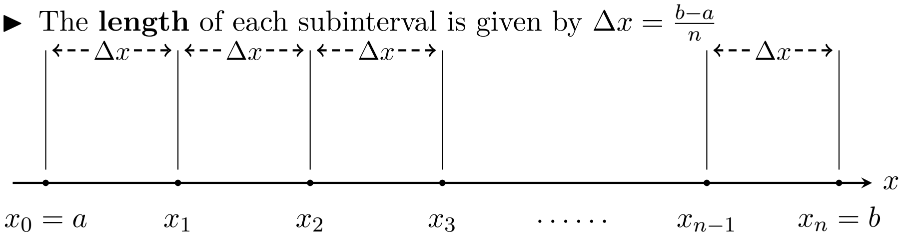


**Fig. 10.3:** A Regular Partion of $[a, b]$

Each subinterval of the chosen partition serves as a base for a rectangle used to approximate the area under the curve. That is, we can approximate the area under the curve on each subinterval with the area of a rectangle. The use of a a regular partition is convenient for us since the length of each subinterval will be the same (i.e., each rectangle used in our approximation will have the same base length).

The other ingredient we need to define a rectangle is its height. In the $k$th subinterval $[x_{k-1}, x_k]$, we choose a **sample point** $x_k^*$ and build a rectangle whose height is $f(x_k^*)$, the value of $f$ at $x_k^*$. So for $b = \Delta x$ and $h = f(x_k^*)$ then the area of the rectangle on the $k$th subinterval is given by

$$b \cdot h = \Delta x f(x_k^*)$$

The choice of sample point will vary depending on the method we are using. We can see that summing the areas of all of these rectangles gives an approximation of the area under the curve.

**Definition 10.5: Riemann Sum on a Regular Partition**

Let $f$ be continuous and non-negative on the interval $[a, b]$. For a regular partition of $[a, b]$ into $n$ subintervals a **area approximation** of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x, \quad x_{i-1} \leq x_i^* \leq x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

This type of area approximation is referred to as a *Riemann sum*. Notice that $x_k^*$ is an arbitrary point on the subinterval $[x_{k-1}, x_k]$. The choice of $x_k^*$ determines the type of Riemann sum being used. Three common choices of $x_k^*$ are $x_{k-1}$, $x_k$ and $\frac{x_{k-1}+x_k}{2}$. Riemann sums constructed using these three choices are referred to as left, right and midpoint Riemann sums respectively.

**Definition 10.6: Left Riemann Sum**

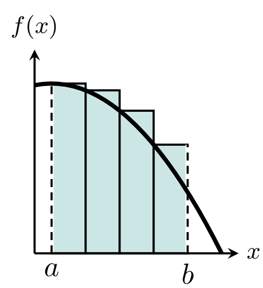


**Fig. 10.4**
Left Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Left Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots f(x_{n-1})\Delta x$$

**Definition 10.7: Right Riemann Sum**

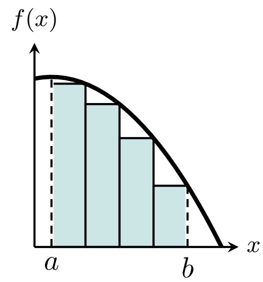


**Fig. 10.5**
Right Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Right Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x$$

**Definition 10.8: Midpoint Riemann Sum**

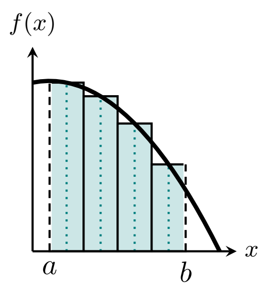


**Fig. 10.6**
Midpoint Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Midpoint Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x$$

where $x_i^* = \dfrac{x_{i-1} + x_i}{2}$

Without knowing the shape of the function (or it's exact value) there is no "best" way to choose the point $x_k^*$. That is, you don't necessarily know which type of Riemann sum is more accurate or easier to use than the others.

**Example 10.11: Approximating Area Under a Curve**

Use left, right and midpoint Riemann sums with $n = 5$ subintervals to estimate the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the $x$-axis, between $x = 0$ and $x = 2$.
Compare the value of each approximation to the exact value of the area, $\dfrac{22}{3}$.

**Solution.** For this example we will use the definition of a regular partition with $a = 0$, $b = 2$ and $n = 5$. Thus, the length of each subinterval is given by

$$\Delta x = \frac{b - a}{n} = \frac{2 - 0}{5} = \frac{2}{5} = 0.4.$$

The value of the $k$th grid point is

$$x_k = a + k\Delta x, = 0 + k\left(\frac{2}{5}\right) = k\left(\frac{2}{5}\right) \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

In the example we find that

$$\begin{aligned}
x_0 &= 0 + 0\left(\frac{2}{5}\right) = 0, & f(x_0) &= -(0)^2 + 5 = 5 \\
x_1 &= 0 + 1\left(\frac{2}{5}\right) = \frac{2}{5}, & f(x_1) &= -\left(\frac{2}{5}\right)^2 + 5 = 4.84 \\
x_2 &= 0 + 2\left(\frac{2}{5}\right) = \frac{4}{5}, & f(x_2) &= -\left(\frac{4}{5}\right)^2 + 5 = 4.36 \\
x_3 &= 0 + 3\left(\frac{2}{5}\right) = \frac{6}{5}, & f(x_3) &= -\left(\frac{6}{5}\right)^2 + 5 = 3.56 \\
x_4 &= 0 + 4\left(\frac{2}{5}\right) = \frac{8}{5}, & f(x_4) &= -\left(\frac{8}{5}\right)^2 + 5 = 2.44 \\
x_5 &= 0 + 5\left(\frac{2}{5}\right) = 2, & f(x_5) &= -(2)^2 + 5 = 1.
\end{aligned}$$

The left Riemann sum is given by

$$\begin{aligned}
\sum_{i=1}^{5} f(x_{i-1})\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= 5(0.4) + 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) \\
&= 8.08.
\end{aligned}$$

The right Riemann sum is given by

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&= 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) + 1(0.4) \\
&= 6.48.
\end{aligned}$$

For the function $f(x) = f(x) = -x^2 + 5$ the left Riemann sum was found to be an overestimate of the the area with about an 10% relative error. The right Riemann sum gave an underestimate of the area with a relative error of about 12%.

Note that we could have done this exercise with the midpoint Riemann sum as well. In fact the midpoint Riemann sum would give an approximate value of 7.36. Try it for yourself!

In the previous example we saw that the resulting approximations where fairly inaccurate. Depending on the setting an approximation with a 10% error could be unacceptable. The primary source of error in that approximation comes from the small number of rectangles used ($n = 5$).

If we wanted a more precise answer we would have needed to use more subintervals to define our partition. However, as we increase the value of $n$ the amount of work required to get the approximation also increases.

If the value of $n$ is large it is helpful to use the known summation formulas for basic sums to simplify the work. In the following examples we will use some of the rules presented in 10.4 to evaluate some Riemann sums with a large value of $n$.

**Example 10.12: Evaluating Riemann Sums When $n$ is Large**

Evaluate the left Riemann sums for the region bounded by the graph of $f(x) = x^2$ and the $x$-axis between $a = 0$ and $b = 2$ using $n = 40$ subintervals.

**Solution.** With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{40} = \frac{1}{20} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of $x_k^*$ in the $k$th grid point will is

$$x_k^* = a + (k-1)\Delta x = 0 + (k-1)0.05 = 0.05k - 0.05 \quad \text{for} \quad k = 1, 2, \cdots, 40.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{40} f(0.05(k-1))0.05 \\
&= \sum_{i=1}^{40} (0.05(k-1))^2\, 0.05 \\
&= (0.05)^3 \sum_{i=1}^{40} (k^2 - 2k + 1) \\
&= (0.05)^3 \left( \frac{40(40+1)(2(40)+1)}{6} - 2\left(\frac{40(40+1)}{2}\right) + 40 \right) \\
&= (0.05)^3 (20540) \\
&= 2.5675
\end{aligned}$$

## Determining Exact Area Under a Curve

In the next section we will see how Riemann sums are used to define the definite integral. For now we close our discussion about approximate area with the following remark. The accuracy of the area approximation using Riemann sums can be improved by increasing the value of $n$, the number of intervals used to define the partition of $[a, b]$. It then stands to reason that we could determine the exact value of the area under the curve if an infinite number of rectangles are used in the Riemann sum. We summarize this result in the following definition.

**Definition 10.9: Definition of Area Under the Curve**

Let $f$ be continuous and nonnegative on the interval $[a,b]$. The **area** of a the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

In the following example we demonstrate how Riemann sums can be used to exactly compute the area under a curve.

**Example 10.13: Evaluating Summations Using Formulas**

Use left Riemann sums to determine the exact area of the region bounded by the graph of $f(x) = x^2$ and the $x$-axis between $a = 0$ and $b = 2$.

**Solution.** To begin, partition $[0,2]$ into $n$ subintervals each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of $x_k^*$ in the $k$th grid point will is

$$x_k^* = a + (k-1)\Delta x = \frac{2(k-1)}{n} \quad \text{for} \quad k = 1, 2, \cdots, n.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2(k-1)}{n}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(k-1)}{n}\right)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(k-1)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n} k^2 - 2k + 1 \\
&= \left(\frac{2}{n}\right)^3 \left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right) \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

By taking the limit as $n \to \infty$ of the left Riemann sum we obtain the exact area

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty} \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} = \frac{8}{3}.$$

Note that the type of Riemann sums used to find the area under the curve in the previous example does not matter. Instead of left Riemann sums we could have used right or midpoint Riemann sums and arrived at the same answer. Try it! Unfortunately, this property is not true for every function. When the area under the curve of a function is the same regardless of the type of Riemann sum used we say that the function is integrable. In the following lecture we will define the definite integral as the limit of general Riemann sums.

**Practice Problems**

**Section 4.9**
15-52, 71-79

**Section 5.1**
19-28, 29-34, 55-58