Evaluating Definite Integrals

1. Integrate the following definite integrals

   (a) $\displaystyle\int_{1/2}^{1} \left(4x^3 - 2x^2 - 7\right)\, dx$

   Solution: $\displaystyle\int_{1/2}^{1} (4x^3 - 2x^2 - 7)\, dx$

   $$= \left[\frac{4x^4}{4} - \frac{2x^3}{3} - 7x\right]_{1/2}^{1}$$

   $$= \left[x^4 - \frac{2}{3}x^3 - 7x\right]_{1/2}^{1}$$

   $$= \left[(1)^4 - \frac{2}{3}(1)^3 - 7(1)\right] - \left[\left(\frac{1}{2}\right)^4 - \frac{2}{3}\left(\frac{1}{2}\right)^3 - 7\left(\frac{1}{2}\right)\right]$$

   $$= \left[-\frac{20}{3}\right] - \left[-\frac{169}{48}\right] = -\frac{151}{48}$$

   (b) $\displaystyle\int_{1}^{2} \frac{3}{t}\, dt$

   Solution:

   $$3\int_{1}^{2} \frac{1}{t}\, dt = 3\ln(t)\Big|_{1}^{2}$$

   $$= 3\left[\ln(2) - \ln(1)\right]$$

   $$= 3\ln(2)$$

   (c) $\displaystyle\int_{0}^{1} (v - \sqrt{v})\, dv$

   Solution: $\displaystyle\int_{0}^{1} (v - v^{1/2})\, dv = \left[\frac{v^2}{2} - \frac{v^{3/2}}{\frac{3}{2}}\right]_{0}^{1}$

   $$= \left[\frac{v^2}{2} - \frac{2}{3}v^{3/2}\right]_{0}^{1}$$

   $$= \left[\frac{(1)^2}{2} - \frac{2}{3}(1)^{3/2}\right] - [0]$$

   $$= \frac{1}{2} - \frac{2}{3}$$

   $$= \frac{3}{6} - \frac{4}{6}$$

   $$= -\frac{1}{6}$$

Applying Properties of Definite Integrals

2. Write the following as a single integral of the form $\int_a^b f(x)\,dx$

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

Solution:

$$\underbrace{\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx} - \int_{-2}^{-1} f(x)\,dx$$

$$= \int_{-2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

$$= \int_{-2}^{5} f(x)\,dx - \left(- \int_{-1}^{-2} f(x)\,dx\right)$$

$$= \int_{-2}^{5} f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$$

$$= \int_{-1}^{5} f(x)\,dx$$

3. Given that

$$\int_1^4 f(x)\,dx = 6, \quad \int_4^6 f(x)\,dx = 3, \quad \int_1^4 g(x)\,dx = 4, \quad \int_2^4 g(x)\,dx = 1,$$

evaluate the following.

(a) $\displaystyle\int_1^4 \big(3f(x) - 2g(x)\big)\,dx$

Solution:

$$\begin{aligned}\int_1^4 (3f(x) - 2g(x))\,dx &= 3\int_1^4 f(x)\,dx - 2\int_1^4 g(x)\,dx \\ &= 3(6) - 2(4) \\ &= 18 - 8 \\ &= 10\end{aligned}$$

(b) $\displaystyle\int_1^6 f(x)\,dx$

Solution:

$$\int_1^6 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^6 f(x)\,dx$$

$$= 6 + 3$$

$$= 9$$

(c) $\displaystyle\int_1^2 f(x)\,dx - \int_2^4 (g(x) - f(x))\,dx$

Solution:

$$\begin{aligned}\int_1^2 f(x)\,dx - \int_2^4 (g(x) - f(x))\,dx &= \int_1^2 f(x)\,dx - \int_2^4 g(x)\,dx + \int_2^4 f(x)\,dx \\ &= \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx - \int_2^4 g(x)\,dx \\ &= \int_1^4 f(x)\,dx - \int_2^4 g(x)\,dx \\ &= 6 - 1 \\ &= 5\end{aligned}$$

Area Under a Curve

4. Find the exact area of the region bounded by the function $f(x) = x^2 + 2$ on the interval $[0, 2]$ and the x-axis. Compare this result with the results you obtained in Written Homework 10, Problem 10.

Solution:

Exact area under a curve is given by the definite integral.

$$\int_0^2 (x^2+2)\,dx = \left[\frac{x^3}{3} + 2x\right]_0^2$$

$$= \left[\frac{(2)^3}{3} + 2(2)\right] - [0]$$

$$= \frac{8}{3} + 4$$

$$= \frac{20}{3}$$