Lesson 9

Optimization, Linear Approximation, and Differentials

Sections
4.4, 4.5

Introduction to Optimization

In the previous lesson we learned how to determine the maximum and minimum values of a function. In this lesson we will now learn how this knowledge can be used in real world applications.

In the real world we often want to find a way minimize or maximize a process. This is a process known as *optimization*. We'll demonstrate how to do this with the derivative through a few of the classic optimization problems in calculus. As in the section on related rates it is recommend that you use the GASCAP method of problem solving or a similar organizational method for your work.

Maximizing Area

Example 9.1: Maximizing Area

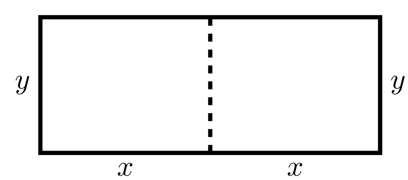
A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum? Also find the area of each pen.

Solution.

Given: 400 ft of fencing and 2 rectangular pens. The equation for the perimeter is

$$4x + 2y = 400 \text{ ft}$$

Sketch:



Asked: We need to find the maximum area of the two pens. The combined area will be given by $A = 2xy$

Compute: The area formula is a function of two variables x and y. Solving our perimeter formula for one of the variables allows us to simpify our area equation. It doesn't really matter which variable we solve for but in general it is good practice to choose the one that will simplify equations in which it is used. This is not always clear initially but becomes easier with more practice. Here we solve for y and obtain

$$400 - 4x = 2y \implies y = 200 - 2x$$

Plugging this into our equation for A we have

$$A(x) = 2x(200 - 2x) = 400x - 4x^2$$

Now to find the maximum area we need to find the maximum value of the function A. We know from Lesson 8 that extreme values exist only at critical points so we first need to find values of x for which $A'(x) = 0$ where $A'(x) = 400 - 8x$

$$400 - 8x = 0 \quad \implies \quad x = 50$$

We need to confirm that this is a maximum value using either the first or second derivative test. Sometimes you will need to use one test over the other but here it does not matter which you use so we demonstrate both.

Using the first derivative test:

Interval	$(-\infty, 50)$	$(50, \infty)$
x	40	60
$f'(x)$	$f'(40) = 80$	$f'(60) = -80$
Sign	$+$	$-$
Behavior	increasing	decreasing

Since $f'(x)$ changes from increasing to decreasing at $x = 50$ by the first derivative test we have that $x = 50$ is a maximum value of A.

Using the second derivative test:
Finding the second derivative we have $f''(x) = -8$. This is a constant function so we will always have $f''(x) < 0$. By the second derivative test we have that $x = 50$ is a maximum. We see that in this case it is much easier to apply the second derivative test. This saves us from needing to determine whether the function is increasing or decreasing when we apply the first derivative test.

Now that we have confirmed that $x = 50$ is a maximum we can find the value of our maximum area as well as our missing dimension y. Using our equation for y we have

$$y = 200 - 2x = 200 - 2(50) = 100$$

and so our maximum area is

$$A = 2x \cdot y = 2(50)(100) = 10000$$

Answer: The dimensions of each pen is $x = 50$ and $y = 100$ where the maximum area of each pen is

$$\frac{1}{2}A = \frac{1}{2}(10000) = 5000\,\text{ft}^2$$

In the previous example the amount of fencing available placed a restriction on the area of the pen that could be created. Values that have this effect are referred to as **constraints**. The equation that produces this value is often referred to as the **constraint equation**.

Maximizing Volume

Example 9.2: Maximizing Volume

A manufacturer wants to design an open box with a square base and a surface area of 108 square inches. What dimensions produce a box with maximum volume?

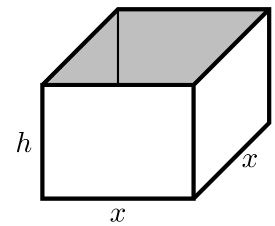
Solution.

Given: We are given that the surface area is $108\,\text{in}^2$. The surface area is given by the equation

$$x^2 + 4xh = 108$$

Where x^2 accounts for the square base and the $4xh$ accounts for the area of the four sides.

Sketch:



Asked: Find x and h that will give the maximum volume which is given by the equation $V = x^2 \cdot h$.

Compute: To eliminate one of the variables in V we solve the constraint equation for h:

$$x^2 + 4xh = 108 \implies h = \frac{108 - x^2}{4x}$$

Substituting this into our volume equation we obtain

$$V(x) = x^2 \left(\frac{108 - x^2}{4x} \right) = 27x - \frac{x^3}{4}$$

In order to find the maximum value of the volume we need to first find the critical points. Our derivative is $V'(x) = 27 - \frac{3}{4}x^2$. Setting $V' = 0$ and solving for x we have

$$27 - \frac{3x^2}{4} = 0 \qquad \implies \qquad x = \pm 6$$

A negative length does not make sense in this context so we only consider the critical point $x = 6$. We use the second derivative test here where $V''(x) = \frac{-6}{4}x$. Since

$$V''(6) = -\frac{3}{2}(6) = -3(3) = -9 < 0$$

by the second derivative then $x = 6$ is a maximum. Thus, one of our dimensions is $x = 6$. To find h we plug in $x = 6$ to our equation for h and find

$$h = \frac{108 - (6)^2}{4(6)} = 3$$

Answer: The dimensions of the box are $x = 6$ and $h = 3$ with a maximum volume of

$$V = (6)^2 \cdot 3 = 108\,\text{in}^2$$

Minimum Distance

Example 9.3: Find Minimum Distance

Determine which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$.

Solution.
Given: We are given the point $(0, 2)$ and the graph $y = 4 - x^2$.

Asked: Find the minimum distance between the point and the graph of y.Recall that the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this case, we want the distance between $(0, 2)$ and an unknown point (x, y). Since (x, y) is on the graph of y and $y = 4 - x^2$ our second point is actually $(x, 4 - x^2)$. So,

$$d = \sqrt{(x - 0)^2 + (y - 2)^2} = \sqrt{(x - 0)^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

and so the function we aim to minimize is

$$d(x) = \sqrt{x^4 - 3x^2 + 4}$$

Compute: You can find $d'(x)$ directly and then use the usual tests but to simplify things we will actually just consider the fact that $d(x)$ will be smallest when the quantity under the square root is smallest. So instead we will actually minimize $f(x) = x^4 - 3x^2 + 4$. (An explanation why we are doing this is given after the example).

Following the same logic in the first two examples we find the first derivative, locate all critical points, and then apply an appropriate derivative test. We have $f'(x) = 4x^3 - 6x$. Setting this equal to zero we have

$$4x^3 - 6x = 0$$
$$2x(2x^2 - 3) = 0$$

The critical points of f are $x = 0$ and $x = \pm\frac{3}{2}$. Our second derivative is $f''(x) = 12x^2 - 6$. Evaluating the second derivative at each of these points we have

Critical Point	Value of $f''(x)$	Behavior	Conclusion
$x = -\sqrt{\frac{3}{2}}$	$f''(-\sqrt{\frac{3}{2}}) = \frac{\sqrt{12}}{7}$	$f''(x) > 0$	relative minimum
$x = 0$	$f''(0) = -\frac{3}{2}$	$f''(x) < 0$	relative maximum
$x = \sqrt{\frac{3}{2}}$	$f''(\sqrt{\frac{3}{2}}) = \frac{\sqrt{12}}{7}$	$f''(x) > 0$	relative minimum

So by the second derivative test $x = 0$ is a relative maximum and $\pm\sqrt{\frac{3}{2}}$ are relative minimums. So, $f(x)$ is smallest at $x = -\sqrt{\frac{3}{2}}$ and $x = -\sqrt{\frac{3}{2}}$ and so $d(x)$ will also be smallest at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$. To identify these points we evaluate y at these x values and obtain

$$y\left(-\sqrt{\frac{3}{2}}\right) = \frac{5}{2} \quad \text{and} \quad y\left(\sqrt{\frac{3}{2}}\right) = \frac{5}{2}$$

Answer: The points on $y = 4 - x^2$ closest to the point $(0, 2)$ are

$$\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right) \quad \text{and} \quad \left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$$

Now why did we choose to look at the quantity underneath the square root instead of the whole function in the last example? Finding the derivative is the easy part, the challenge then comes with locating critical points. The derivative is

$$d'(x) = \frac{4x^3 - 6x}{2\sqrt{x^4 - 3x^2 + 4}}$$

Locating critical points requires us to find points where $d'(x) = 0$. Since this is a quotient of two functions, $d'(x)$ is zero when the numerator is zero and so we need to look at when $4x^3 - 6x = 0$. As in the example above, we need to factor here. If we were to solve for x using typical algebraic steps we would need to divide each side by x. We run into a problem here since in this case x could be zero. This is why paying attention to the domain of a function is important! As above we locate the same 3 critical points $x = 0$ and $x = \pm\frac{3}{2}$.

We must also consider when $d'(x)$ does not exist. This will occur when the denominator is zero and since we also have a square root in the numerator we must also look at when $x^4 - 3x^2 + 4$ is less than zero. By examining a graph of this function you will see that this function is never zero and always positive. So there are no points where $d'(x)$ does not exist.

The main point here is to make sure you are looking at the functions you are maximizing very carefully. Full use of your algebra knowledge will be required for these problems. For these problems the "possible" step of GASCAP is maybe the most important.

Linear Approximation

We know that the slope of a tangent line at a point $(a, f(a))$ is given by $f'(a)$. The general equation of the equation of a tangent line for any function $f(x)$ at $x = a$ can be defined as follows.

Definition 9.1: Equation of a Tangent Line at a Point

The **equation of the tangent line** at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

Let's take another look at the graph of some function f and its tangent line at $x = a$. If we were to walk along the curve of f we can see that points on the tangent line are very close to points on f as we get closer and closer to $x = a$ on either side.

This observation leads us to another key application of the derivative known as **linear approximation**.

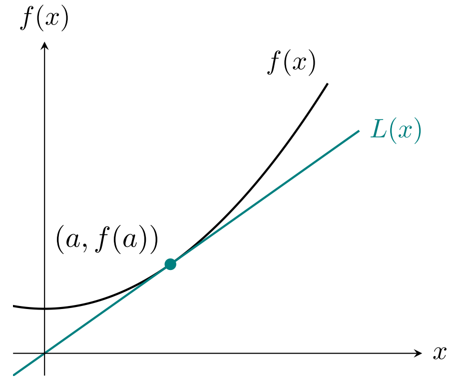


Fig. 9.1: Linear Approximation of f

Definition 9.2: Linear Approximation

The **linear approximation** of f near $x = a$ is given by $f(x) \approx L(x)$ where

$$L(x) = f(a) + f'(a)(x - a)$$

In other words, the derivative allows us to approximate the value of a function using a line. Why would we want to do this? Don't we have calculators that can evaluate functions for us so we can get exact values? While this is true, there are some functions that can be difficult to evaluate at certain points or may not be defined there. This allows us to approximate the value without evaluating the function itself.

Example 9.4: Using Linear Approximation

Use a linear approximation to estimate the value of $\sqrt{99.8}$.

Solution. To approximate the value $\sqrt{99.8}$ we recognize that this is the function $f(x) = \sqrt{x}$ evaluated at the point $x = 99.8$. We then choose $a = 100$ and find that $f(100) = \sqrt{100} = 10$ where our derivative is $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

Evaluating at $a = 100$ we have

$$f'(a) = f'(100) = \frac{1}{2}(100)^{-\frac{1}{2}}$$

So our linear approximation is the function

$$\begin{aligned} L(x) &= 10 + \frac{1}{20}(x - 100) \\ &= 10 + \frac{x}{20} - 5 \\ &= \frac{1}{20}x + 5 \end{aligned}$$

We evaluate $L(x)$ at $x = 99.8$ and obtain

$$L(99.8) = \frac{1}{20}(99.8) + 5 = 9.99$$

This is where $\approx$ versus $=$ is very important! $L(x)$ is the approximation of $f(x)$ which means $L(x) \approx f(x)$. Thus the linear approximation of $f(x)$ at $x = 99.8$ is

$$f(99.8) \approx L(99.8) = 9.99$$

In other words, $\sqrt{99.8} \approx 9.99$

Differentials

Consider again the equation of tangent line

$$y = f(a) + f'(a)(x - a)$$

when we use this to approximate the function f. The term $(x - a)$ is referred to as the **change in x**, denoted by Δx. If this value is small then the **change in y** denoted by Δy is given as

$$\Delta y = f(a + \Delta x) - f(a)$$

This can be approximated by

$$\Delta y \approx f'(a)\Delta x$$

Definition 9.3: Differentials

Let $y = f(x)$ be a differentiable function on an open interval I that contains x. The **differential of x**, denoted dx, is any non-zero real number and the **differential of y**, denoted dy, is given by

$$dy = f'(x)\,dx$$

In other words we have

$$\Delta y \approx dy \implies f(x + dx) - f(x) \approx dy = f'(x)\,dx$$

Example 9.5: Approximating Changes using Differentials

Approximate the change in the volume of a sphere when its radius changes from $r = 5\,\text{ft}$ to $r = 5.1\,\text{ft}$.

Solution. We are given that r changes from $r = 5\,\text{ft}$ to $r = 5.1\,\text{ft}$ which means that

$$\Delta r = 5.1\,\text{ft} - 5\,\text{ft} = 0.1\,\text{ft}$$

volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$. By Definition 9.3 we have

$$\Delta V \approx V'(r)\,\Delta r = 4\pi r^2 \cdot \Delta r$$

We choose $r = 5$ and so along with $\Delta r = 0.1$ we have

$$\Delta V \approx 4\pi(5)^2 \cdot (0.1) = 10\pi \approx 31.416\,\text{ft}^3$$

So we find that a change of 0.1 ft in the radius results in a change in volume of the sphere of approximately $31.416\,\text{ft}^3$.

Calculating Error

Error in Approximation

Now our error associated with any approximation is in general just the difference between the actual function and our approximating function $L(x)$.

Definition 9.4: Error in Approximation

Let $L(x)$ be a function approximating the function $f(x)$. Then the **error in approximation** is given by

$$\text{Error} = f(x) - L(x)$$

Often, the direct error in approximation does not give a clear idea of the error in an approximation. In many applications it makes more sense to examine the **relative error**.

Definition 9.5: Relative Error

The **relative error** is given by

$$\text{Relative Error} = \frac{|\text{exact} - \text{approximate}|}{|\text{exact}|}$$

This can also be expressed as $\dfrac{\Delta f}{f} \approx \dfrac{df}{f}$

Multiplying the relative error by 100 gives the **percent error**

$$\text{Percent Error} = \frac{|\text{exact} - \text{approximate}|}{|\text{exact}|} \cdot 100$$

In Example 9.4 we found that $\sqrt{99.8} \approx 9.99$. Using Mathematica the value is given by 9.98999 which gives a relative error of -1.001×10^{-6}. This is a pretty good estimate considering we used a simple line to approximate this value! Note that different calculators and computer programs can sometimes output slightly different values as they may have different methods for rounding or truncating values.

Error Propagation

In real world situations there is a certain amount of error that can result from the tools used to take measurements. In this context x will represent the measured value of a variable and $x + \Delta x$

the actual value. Then Δx is the **error in measurement**.

Definition 9.6: Propagated Error

Let x be the measured value of a variable and $x + \Delta x$ the exact value. If x is used to compute $f(x)$ then the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error** Δy:

$$\Delta y = f(x + \Delta x) - f(x)$$

In other words, $f(x + \Delta x)$ is the exact value of the function, $f(x)$ is the measured value and Δy represents the propagated error.

Example 9.6: Estimation of Error

The measured radius of a ball bearing is 0.7 in. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. Our measured radius is $r = 0.7$. The possible error is in the interval

$$-0.01 \leq \Delta r \leq 0.01$$

To approximate the propagated error in the volume, differentiate V to obtain $\frac{dV}{dr} = 4\pi r^2$ we want to approximate ΔV by dV so we have

$$\begin{aligned} \Delta V \approx dV &= 4\pi r^2 dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr \\ &\approx \pm 0.06158\,\text{in}^3 \end{aligned}$$

So, the volume has a propagated error of about 0.06 cubic inch.
The relative error is found by comparing V and dV. In other words we have the ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} = \frac{3\,dr}{r} \\ &= \frac{3}{0.7}(\pm 0.01) = \pm 0.0429 \end{aligned}$$

From this we have a percent error of 4.29%.

Differential Forms

Differentiation rules can be written in **differential form**. This is the form used to develop some of the integration rules you learn in the next semester of calculus. This comes up in more advanced courses but it is worth mentioning here.

Suppose u and v are differentiable functions of x then by the definition of differentials we have

$$du = u'\,dx \quad \text{and} \quad dv = v'\,dx$$

For example we can write the product rule as

$$d[uv] = \frac{d}{dx}[uv]\,dx = [vu' + uv']\,dx = u'\,dx + uv'\,dx = u\,dv + v\,du$$

The same derivative formulas we learned before can be represented in this form.

Definition 9.7: Differential Formulas

Let u and v be differentiable functions of x.

Constant Multiple: $d[cu] = c\,du$

Sum/Difference: $d[u \pm v] = du \pm dv$

Product $d[uv] = u\,dv + v\,du$

Quotient: $d\left[\dfrac{u}{v}\right] = \dfrac{u\,dv + v\,du}{v^2}$

Practice Problems

Section 4.4
5-23

Section 4.5
13-30, 35-40, 41-45