Implicit Differentiation

Differentiate the the given curves.

1. $y + x\cos(y) = x^2 y$

Solution:

Differentiate both sides:

$$\frac{d}{dx}[y + x\cos(y)] = \frac{d}{dx}[x^2 y]$$

Let $LHS = \frac{d}{dx}[y + x\cos(y)] = \frac{d}{dx}[y] + \frac{d}{dx}[x\cos(y)]$

$\frac{d}{dx}[y] = y'$

$\frac{d}{dx}[x\cos(y)] = \frac{d}{dx}[fg] = f'g + fg' = \cos(y) - x\sin(y)y'$ $\Longrightarrow$ Recall: $y = y(x)$!

$f = x \qquad g = \cos(y) = \cos(y(x))$

$f' = 1 \qquad g' = -\sin(y(x))\,y'(x) = -\sin(y)y'$ i.e. (need to use Chain Rule!)

$\Rightarrow LHS = y' + \cos(y) - x\sin(y)y'$

Let $RHS = \frac{d}{dx}[x^2 y] = \frac{d}{dx}[p \cdot q] = p'q + pq' = 2xy + x^2 y'$

$p = x^2 \qquad q = y$

$p' = 2x \qquad q' = y'$

Put these back together & Solve for y'

$$y' + \cos(y) - x\sin(y)y' = 2xy + x^2 y'$$

$$y' - x\sin(y)y' - x^2 y' = 2xy - \cos(y)$$

$$y'(1 - x\sin(y) - x^2) = 2xy - \cos(y)$$

$$\Rightarrow y' = \frac{2xy - \cos(y)}{1 - x\sin(y) - x^2}$$

2. $e^y \sin x = x + xy$

Solution:

Differentiate both sides

$$\frac{d}{dx}[e^y \sin(x)] = \frac{d}{dx}[x + xy]$$

Let $LHS = \frac{d}{dx}[e^y \sin(x)] = \frac{d}{dx}[f \cdot g] = f'g + fg'$

$$f = e^y \qquad g = \sin(x)$$
$$f' = y'e^y \qquad g' = \cos(x)$$

$$\Rightarrow LHS = y'e^y \sin(x) + e^y \cos(x)$$

Let $RHS = \frac{d}{dx}[x + xy] = \frac{d}{dx}[x] + \frac{d}{dx}[xy]$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[xy] = \frac{d}{dx}[p \cdot q] = p'q + pq' = y + xy'$$

$$p = x \qquad q = y$$
$$p' = 1 \qquad q' = y'$$

$$\Rightarrow RHS = 1 + y + xy'$$

Putting everything back together & Solve for y'

$$y'e^y \sin(x) + e^y \cos(x) = 1 + y + xy'$$
$$y'e^y \sin(x) - xy' = 1 + y - e^y \cos(x)$$
$$y'(e^y \sin(x) - x) = 1 + y - e^y \cos(x)$$
$$\Rightarrow y' = \frac{1 + y - e^y \cos(x)}{e^y \sin(x) - x}$$

Differentiation of Logarithmic Functions

3. $y = \ln(x\ln(x))$

Solution:

y is a fcn composition $\Rightarrow$ Need Chain Rule

Chain Rule w/ natural log:

$$\frac{d}{dx}[\ln(u(x))] = \frac{u'(x)}{u(x)} = \frac{u'}{u}$$

$$y' = \frac{d}{dx}[\ln(x\ln(x))] = \frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

$u = x\ln(x)$

$$u' = \frac{d}{dx}[x\ln(x)] = \frac{d}{dx}[fg] = f'g + fg'$$

$f = x \qquad g = \ln(x)$

$f' = 1 \qquad g' = \frac{1}{x}$

$$\Rightarrow u' = \ln(x) + x\left(\frac{1}{x}\right)$$

$$= \ln(x) + 1$$

$$\Rightarrow y' = \frac{u'}{u} = \frac{\ln(x)+1}{x\ln(x)}$$

4. $y = \log_5(1 + 2x)$

Solution:

Method I: Use change of base formula to change to natural log:

change of base: $\log_a(x) = \dfrac{\ln(x)}{\ln(b)}$

$$\Rightarrow y = \log_5(1+2x) = \frac{\ln(1+2x)}{\ln(5)} = \frac{1}{\ln(5)} \ln(1+2x)$$

$$y' = \frac{d}{dx}\left[\frac{1}{\ln(5)} \ln(1+2x)\right] = \frac{1}{\ln(5)} \frac{d}{dx}\left[\ln(u)\right] = \frac{1}{\ln(5)} \frac{u'}{u}$$

$u = 1+2x$

$u' = 2$

$$\Rightarrow y' = \frac{1}{\ln(5)} \frac{u'}{u} = \frac{1}{\ln(5)}\left(\frac{2}{1+2x}\right) = \frac{2}{\ln(5)(1+2x)}$$

Method II

Using differentiation Rule for logs of base b:

$$\frac{d}{dx}\left[\log_b(u)\right] = \frac{u'}{u \cdot \ln(b)}$$

$$\Rightarrow y' = \frac{d}{dx}\left[\log_5(1+2x)\right] = \frac{d}{dx}\left[\log_5(u)\right] = \frac{u'}{\ln(5) \cdot u}$$

$u = 1 + 2x$

$u' = 2$

$$\Rightarrow y' = \frac{u'}{\ln(5)\, u} = \frac{2}{\ln(5)(1+2x)}$$

Either method will get the same result!

5. $y = \ln(\sin^2 x)$

Solution:

$$y' = \frac{d}{dx}[\ln(\sin^2(x))] = \frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

$$u = \sin^2(x) = (\sin(x))^2$$

$$u' = \frac{d}{dx}[(\sin(x))^2] = \frac{d}{dx}[f(g)] = f'(g)g'$$

$$f(g) = g^2 \qquad g = \sin(x)$$

$$f'(g) = 2g \qquad g' = \cos(x)$$

$$\Rightarrow u' = 2gg'$$

$$= 2\sin(x)\cos(x)$$

$$\Rightarrow y' = \frac{u'}{u} = \frac{2\sin(x)\cos(x)}{\sin^2(x)} = \frac{2\cos(x)}{\sin(x)}$$

6. $y = e^{\cos(x)} + \cos(e^x)$

Solution:

$$y' = \frac{d}{dx}\left[e^{\cos(x)} + \cos(e^x)\right]$$

$$= \frac{d}{dx}\left[e^{\cos(x)}\right] + \frac{d}{dx}\left[\cos(e^x)\right]$$

<u>Note</u>

$$\frac{d}{dx}\left[e^{\cos(x)}\right] = \frac{d}{dx}\left[e^u\right] = u'e^u = -\sin(x)e^{\cos(x)}$$

$$u = \cos(x)$$
$$u' = -\sin(x)$$

$$\frac{d}{dx}\left[\cos(e^x)\right] = \frac{d}{dx}\left[\cos(g)\right] = -g'\sin(g) = -e^x\sin(e^x)$$

$$g = e^x$$
$$g' = e^x$$

$$\Longrightarrow y' = -\sin(x)e^{\cos(x)} - e^x\sin(e^x)$$

Logarithmic Differentiation

Apply properties of logarithms to differentiate the following.

7. $y = x^2 \cos x$.

Solution:

First we apply properties of logarithms by taking the natural log of each side

$$\ln(y) = \ln(x^2 \cos(x))$$

$$= \ln(x^2) + \ln(\cos(x)) \qquad \text{Product Rule of Logs: } \ln(xy) = \ln(x) + \ln(y)$$

$$= 2\ln(x^2) + \ln(\cos(x)) \qquad \text{Apply Power Rule of Logs: } \ln(x^p) = p\ln(x^p)$$

So we have

$$\ln(y) = 2\ln(x) + \ln(\cos(x))$$

This is now an implicit fcn

Differentiate each side

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[2\ln(x) + \ln(\cos(x))]$$

Let $\text{LHS} = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$

Let $\text{RHS} = \frac{d}{dx}[2\ln(x) + \ln(\cos(x))] = 2\frac{d}{dx}[\ln(x)] + \frac{d}{dx}[\ln(\cos(x))]$

<u>Note:</u> $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ and $\frac{d}{dx}[\ln(\cos(x))] = \frac{d}{dx}[\ln(u)] = \frac{u'}{u} = -\frac{\sin(x)}{\cos(x)}$

$u = \cos(x)$

$u' = -\sin(x)$

$$\Rightarrow \text{RHS} = 2\left(\frac{1}{x}\right) - \frac{\sin(x)}{\cos(x)}$$

Putting everything back together:

$$\frac{y'}{y} = \frac{2}{x} - \frac{\sin(x)}{\cos(x)}$$

Solve for y': $y' = y\left[\frac{2}{x} - \frac{\sin(x)}{\cos(x)}\right]$

In this case, we know what y equals since its the function we started with!

$$\Rightarrow y' = x^2\cos(x)\left[\frac{2}{x} - \frac{\sin(x)}{\cos(x)}\right]$$

This method may seem like more work because you do a lot before even taking a derivative. However, none of the actual derivatives we took were that complicated. This method can significantly reduce the complexity in taking derivatives of some functions!

8. $y = \dfrac{x^6 \sin^5(2x)}{\sqrt{2x-3}}$.

Note that $\sqrt{2x-3} = (2x-3)^{1/2}$
and $\sin^5(2x) = (\sin(2x))^5$

Solution:
First we apply properties of logarithms by taking the natural log of each side

$$\ln(y) = \ln\left(\frac{x^6 \sin^5(2x)}{\sqrt{2x-3}}\right)$$

$$= \underbrace{\ln\left(x^6(\sin(2x))^5\right)}_{\text{Product Rule}} - \underbrace{\ln\left((2x-3)^{1/2}\right)}_{\text{Power Rule}}$$

Apply Quotient Rule of Logs: $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$$= \left[\ln(x^6) + \ln\left((\sin(2x))^5\right)\right] - \left[\frac{1}{2}\ln(2x-3)\right]$$

Apply Power Rule of Logs: $\ln(x^p) = p\ln(x^p)$
& Product Rule of Logs: $\ln(xy) = \ln(x) + \ln(y)$

$$= 6\ln(x) + 5\ln(\sin(2x)) - \frac{1}{2}\ln(2x-3)$$

Apply Power Rule of Logs

We then differentiate each side $\Rightarrow$ Now this is just implicit differentiation!

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}\left[6\ln(x) + 5\ln(\sin(2x)) - \frac{1}{2}\ln(2x-3)\right]$$

Let LHS $= \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$

&

Let RHS $= 6\frac{d}{dx}[\ln(x)] + 5\frac{d}{dx}[\ln(\sin(2x))] - \frac{1}{2}\frac{d}{dx}[\ln(2x-3)]$

<u>Note:</u> $\frac{d}{dx}[\ln(\sin(2x))] = \frac{d}{dx}[\ln(u)] = \frac{u'}{u} = \frac{2\cos(2x)}{\sin(2x)}$

$u = \sin(2x)$
$u' = 2\cos(2x)$

$\frac{d}{dx}[\ln(2x-3)] = \frac{d}{dx}[\ln(v)] = \frac{v'}{v} = \frac{2}{2x-3}$

$v = 2x-3$
$v' = 2$

$$\Rightarrow \text{RHS} = 6\left(\frac{1}{x}\right) + 5\left(\frac{2\cos(2x)}{\sin(2x)}\right) - \frac{1}{2}\left(\frac{2}{2x-3}\right) = \frac{6}{x} + \frac{10\cos(2x)}{\sin(2x)} - \frac{1}{2x-3}$$

Putting everything back together: $\frac{y'}{y} = \frac{6}{x} + \frac{10\cos(2x)}{\sin(2x)} - \frac{1}{2x-3}$

Solve for y': $y' = y\left[\frac{6}{x} + \frac{10\cos(2x)}{\sin(2x)} - \frac{1}{2x-3}\right]$

Plug in y: $y' = \frac{x^6\sin^5(2x)}{\sqrt{2x-3}}\left[\frac{6}{x} + \frac{10\cos(2x)}{\sin(2x)} - \frac{1}{2x-3}\right]$

This method may seem like more work because you do a lot before even taking a derivative. If we were to do it the other way we wouldn't end up with such a nice simple form!

9. $y = 3^{x\ln(x)}$

Solution:

First take natural log of each side

$$\ln(y) = \ln(3^{x\ln(x)})$$
$$= (x\ln(x))\ln(3)$$ by applying power rule of logs

Differentiate each side:

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\ln(3)\, x\ln(x)]$$

Let $LHS = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$

Let $RHS = \frac{d}{dx}[\ln(3)\, x\ln(x)] = \ln(3)\frac{d}{dx}[x\ln(x)]$

$$\frac{d}{dx}[x\ln(x)] = \frac{d}{dx}[f\cdot g] = f'g + fg'$$

$f = x$ $\qquad g = \ln(x)$

$f' = 1$ $\qquad g' = \frac{1}{x}$

$$\Rightarrow RHS = \ln(3)\left(\ln(x) + x\left(\tfrac{1}{x}\right)\right)$$
$$= \ln(3)(\ln(x) + 1)$$

Put everything back together & solve for y'

$$\Rightarrow \frac{y'}{y} = \ln(3)(\ln(x)+1)$$

$$y' = y[\ln(3)(\ln(x)+1))]$$
$$= 3^{x\ln(x)}[\ln(3)(\ln(x)+1))]$$