Determining Continuity at a Point

1. Determine whether the following functions are continuous at the given value a. Use the continuity checklist to justify your answer.

(a) $f(x) = \begin{cases} \dfrac{x^2-4x+3}{x-3}, & \text{if } x \neq 3 \\ 2, & \text{if } x = 3 \end{cases}; \quad a = 3$

Solution:

1. $f(a)$ exists: since $f(3) = 2$ then $f(3)$ exists

2. $\lim\limits_{x\to a} f(x)$ exists:

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-4x+3}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-1)}{\cancel{x-3}} = \lim_{x\to 3} x-1 = 3-1 = 2$$

so $\lim\limits_{x\to 3} f(x)$ exists

3. $\lim\limits_{x\to a} f(x) = f(a)$

since $\lim\limits_{x\to 3} f(x) = 2$ & $f(3) = 2$ then $\lim\limits_{x\to 3} f(x) = f(3)$

Since $f(x)$ meets all 3 parts of the definition then $f(x)$ is continuous at $x = 2$.

(b) $g(x) = \begin{cases} \dfrac{x^2+x}{x+1}, & \text{if } x \neq -1 \\ 2, & \text{if } x = -1 \end{cases}; \quad a = -1$

Solution:

1. $g(a)$ exists: since $g(-1) = 2$ then $g(-1)$ exists

2. $\lim\limits_{x\to a} g(x)$ exists:

$$\lim_{x\to -1} g(x) = \lim_{x\to -1} \frac{x^2+x}{x+1} = \lim_{x\to -1} \frac{x\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x\to -1} x = -1$$

so $\lim\limits_{x\to -1} g(x)$ exists

3. $\lim\limits_{x\to a} g(x) = g(a)$ X Fails!

since $\lim\limits_{x\to -1} g(x) = -1$ & $g(-1) = 2$ then $\lim\limits_{x\to -1} g(x) \neq g(-1)$

Since $g(x)$ fails Part 3 of the definition $g(x)$ is Not continuous at $x = -1$

Limit of a Function Composition

2. $\lim_{x \to 0} e^{-1/x^2}$

   Solution:

$$\lim_{x \to 0} e^{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{1}{e^{x^2}}$$

$$= \frac{1}{e^{\lim_{x \to 0} x^2}}$$

$$= \frac{1}{e^0} \qquad \text{Since } \lim_{x \to 0} x^2 = 0$$

$$= \frac{1}{1}$$

$$= 1$$

Applying the Intermediate Value Theorem

3. Use the Intermediate Value Theorem to show that the equation

$$x^3 - 5x^2 + 2x = -1$$

has a solution on the interval $(-1, 5)$.

Solution:

We want to find when

$$x^3 - 5x^2 + 2x + 1 = 0$$

Let $f(x) = x^3 - 5x^2 + 2x + 1$. We want to find a value c in $(-1, 5)$ such that $f(c) = 0$.

Since $f(x)$ is a polynomial, we know its continuous on the interval $(-1, 5)$

Evaluating f at the endpoints we have

$$f(-1) = (-1)^3 - 5(-1)^2 + 2(-1) + 1 = -7$$

and $f(5) = (5)^3 - 5(5)^2 + 2(5) + 1 = 11$

Since $f(-1) < 0$ and $f(5) > 0$ then by the Intermediate value theorem there exists a value c in the interval $(-1,5)$ such that $f(c) = 0$.

Since $f(x) = 0$ at at least one value in $(-1,5)$ then the equation

$$x^3 - 5x^2 + 2x = -1$$

must have at least one solution in the interval $(-1,5)$

Using the Definition of the Derivative

4. Consider $f(x) = 3x^2 - x$. Find $f'(1)$ using the version of the definition of the derivative given in **Definition 3.7** of the lesson notes. Then use the value of $f'(1)$ to find the equation of the line tangent to the curve $y = 3x^2 - x^3$ at the point $(1, 2)$.

Solution:

Definition 3.7: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Here $a = 1$

$f(x) = 3x^2 - x$

$f(1) = 3(1)^2 - (1) = 2$

So

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{x \to 1} \frac{3x^2 - x - 2}{x - 1}$$
$$= \lim_{x \to 1} \frac{(3x+2)(x-1)}{x - 1}$$
$$= \lim_{x \to 1} 3x + 2$$
$$= 3(1) + 2$$
$$= 5$$

So $f'(1) = 5$

The equation of the tangent line is given by

$$y = f(a) + f'(a)(x - a)$$

So equation of tangent line at the point $(1, 2)$ is

$$y = f(1) + f'(1)(x - 1)$$
$$= 2 + 5(x - 1)$$

5. Using **Definition 3.6** of the lesson notes, find the derivatives of the given functions.

   (a) $f(t) = 5t - 9t^2$

   Solution:

$$\begin{aligned}
f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\
&= \lim_{h\to 0} \frac{[5(t+h) - 9(t+h)^2] - [5t - 9t^2]}{h} \\
&= \lim_{h\to 0} \frac{[5(t+h) - 9(t^2 + 2th + h^2)] - [5t - 9t^2]}{h} \\
&= \lim_{h\to 0} \frac{[5t + 5h - 9t^2 - 18th - 9h^2] - [5t - 9t^2]}{h} \\
&= \lim_{h\to 0} \frac{\cancel{5t} + 5h - \cancel{9t^2} - 18th - 9h^2 - \cancel{5t} + \cancel{9t^2}}{h} \\
&= \lim_{h\to 0} \frac{5h - 18th - 9h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(5 - 18t - 9h)}{\cancel{h}} \\
&= \lim_{h\to 0} (5 - 18t - 9h) \\
&= 5 - 18t - 9(0) \\
&= 5 - 18t
\end{aligned}$$

   So $f'(t) = 5 - 18t$

(b) $g(x) = \sqrt{9-x}$

Solution:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h} \cdot \frac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{9-(x+h)}\right)^2-\left(\sqrt{9-x}\right)^2}{h\left(\sqrt{9-(x+h)}+\sqrt{9-x}\right)} \\
&= \lim_{h\to 0} \frac{9-(x+h)-(9-x)}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{\cancel{9}-\cancel{x}-h-\cancel{9}+\cancel{x}}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} \\
&= \frac{-1}{\sqrt{9-(x+(0))}+\sqrt{9-x}} \\
&= \frac{-1}{\sqrt{9-x}+\sqrt{9-x}} \\
&= \frac{-1}{2\sqrt{9-x}}
\end{aligned}$$

So $g'(x) = \dfrac{-1}{2\sqrt{9-x}}$

Determining Differentiability

6. Determine whether the function is differentiable at the given value of x. (Hint: use Definition 3.7 of the *lesson notes*).

(a) $f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \geq 2 \end{cases}$

Note

$f(2) = \sqrt{2(2)} = \sqrt{4} = 2$

Solution:

Need to examine left & right derivatives

Derivative on left.

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{\frac{1}{2}x + 1 - 2}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{\frac{1}{2}x - 1}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{\frac{1}{2}\cancel{(x-2)}}{\cancel{x-2}}$$

$$= \lim_{x \to 2^-} \frac{1}{2}$$

$$= \frac{1}{2}$$

Derivative on right:

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{\sqrt{2x} - 2}{x - 2} \left(\frac{\sqrt{2x} + 2}{\sqrt{2x} + 2} \right)$$

$$= \lim_{x \to 2^+} \frac{(\sqrt{2x})^2 - 2^2}{(x-2)(\sqrt{2x} + 2)}$$

$$= \lim_{x \to 2^+} \frac{2x - 2}{(x-2)(\sqrt{2x} + 2)}$$

$$= \lim_{x \to 2^+} \frac{2\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{2x} + 2)}$$

$$= \lim_{x \to 2^+} \frac{2}{\sqrt{2x} + 2}$$

$$= \frac{2}{\sqrt{2(2)} + 2}$$

$$= \frac{1}{2}$$

Since the derivative from the left equals the derivative from the right

i.e. $\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$

Then $f(x)$ is differentiable at $x = 2$.

(b) $g(x) = \begin{cases} x, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

Note
$g(1) = (1) = 1$

Solution:

Need to examine left & right derivatives

Derivative on left.

$$\lim_{x \to 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^-} \frac{\cancel{x-1}}{\cancel{x-1}}$$

$$= \lim_{x \to 1^-} 1$$

$$= 1$$

Derivative on right:

$$\lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= \lim_{x \to 1^+} x + 1$$

$$= (1) + 1$$

$$= 2$$

Since the derivative from the left Does not equal the derivative from the right

i.e. $\lim_{x \to 1^-} \frac{g(x) - g(1)}{x - 1} \neq \lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1}$

Then $g(x)$ is NOT differentiable at $x = 1$.