Applying the Rules of Differentiation

Problems 1 to 6: Differentiate the given functions.

1. $f(x) = 3x^4 + 7x + 2$

   Solution:

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[3x^4 + 7x + 2\right] \\
&= \frac{d}{dx}[3x^4] + \frac{d}{dx}[7x] + \frac{d}{dx}[2] && \text{Sum Rule} \\
&= 3\frac{d}{dx}[x^4] + 7\frac{d}{dx}[x] + \frac{d}{dx}[2] && \text{Constant multiple Rule} \\
&= 3(4x^{4-1}) + 7(x^{1-1}) + 0 && \text{Power Rule} \\
& && \text{\& Constant Rule} \\
&= 12x^3 + 7x^0 \\
&= 12x^3 + 7
\end{aligned}
$$

2. $y = \sqrt{1 + 2e^{3x}}$

Solution:

Note: $y = \sqrt{1+2e^{3x}} = (1+2e^{3x})^{1/2}$

This is a fcn composition $\Rightarrow$ apply Chain Rule

$$y' = \frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

Let $f(g) = g^{1/2}$ $\qquad g(x) = 1 + 2e^{3x}$

$f'(g) = \frac{1}{2}g^{-1/2}$ $\qquad g'(x) = \frac{d}{dx}[1] + 2\frac{d}{dx}[e^{3x}]$

$$= 0 + 2\frac{d}{dx}[e^u]$$

$$= 2 \cdot e^u u' = 2 \cdot 3e^{3x} = 6e^{3x}$$

$u = 3x$

$u' = 3$

$$\Rightarrow y' = f'(g)\,g'$$

$$= \frac{1}{2}g^{-1/2}\,g'$$

$$= \frac{1}{2}(1+2e^{3x})^{-1/2}(6e^{3x})$$

$$= 3e^{3x}(1+2e^{3x})^{-1/2}$$

3. $y = \cos(x^2)$

Solution:

This is a Fcn Composition $\Longrightarrow$ apply Chain Rule

$$y' = \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

$$f(g) = \cos(g) \qquad g = x^2$$

$$f'(g) = -\sin(g) \qquad g' = 2x$$

$$\Longrightarrow y' = f'(g)g'$$

$$= -\sin(g)\,g'$$

$$= -\sin(x^2)(2x)$$

$$= -2x\sin(x^2)$$

4. $y = \dfrac{t\sin(t)}{1+t}$

Solution:

Has form $y = \dfrac{f(t)}{g(t)}$ $\Rightarrow$ apply Quotient Rule

$$y' = \frac{d}{dt}\left[\frac{f(t)}{g(t)}\right] = \frac{f'g - fg'}{g^2}$$

Let $f = t\sin(t)$

$$f' = \frac{d}{dt}[p \cdot q]$$
$$= p'q + pq'$$

Let $p = t$ $\quad q = \sin(t)$

$p' = 1$ $\quad q' = \cos(t)$

$$\Rightarrow f' = (1)\sin(t) + t \cdot \cos(t)$$
$$= \sin(t) + t\cos(t)$$

$g = 1 + t$

$$g' = \frac{d}{dt}[1] + \frac{d}{dt}[t]$$
$$= 1$$

$$\Rightarrow y' = \frac{f'g - fg'}{g^2}$$
$$= \frac{(\sin(t) + t\cos(t)) - (t\sin(t))(1)}{(1+t)^2}$$

5. $y = \sqrt{x}e^x$

Solution:

$y = f(x) \cdot g(x) \implies$ apply Product Rule

$$y' = \frac{d}{dx}[f \cdot g] = f'g + fg'$$

Let $f = \sqrt{x} = x^{1/2}$ $\qquad g = e^x$

$f' = \frac{1}{2}x^{-1/2}$ $\qquad g' = e^x$

$$\implies y' = f'g + fg'$$
$$= x^{1/2}e^x + \frac{1}{2}x^{-1/2}e^x$$

6. $y = e^x \cos(\sqrt{x^3+2})$

Solution:

$y = f(x) \cdot g(x) \implies$ apply Product Rule

$y' = \frac{d}{dx}[f \cdot g] = f'g + fg'$

Let $f = e^x$ $\quad g = \cos(\sqrt{x^3+2})$

$f' = e^x$ $\quad g' = \frac{d}{dx}[p(q)] = p'(q)\,q'$

Let $p(q) = \cos(q)$ $\quad q = (x^3+2)^{1/2}$

$p'(q) = -\sin(q)$ $\quad q' = \frac{d}{dx}[v(u)] = v'(u)\,u'$

$v(u) = u^{1/2}$, $u = x^3+2$

$v'(u) = \frac{1}{2}u^{-1/2}$, $u' = 3x^2$

$\implies q' = v'(u)u'$

$= \frac{1}{2}u^{-1/2}\,u'$

$= \frac{1}{2}(x^3+2)^{-1/2}(3x^2)$

$= \frac{3}{2}x^2(x^3+2)^{-1/2}$

$\implies g' = p'(q)\,q'$

$= -\sin(q)\,q'$

$= -\sin\left((x^3+2)^{1/2}\right)\left(\frac{3}{2}x^2(x^3+2)^{-1/2}\right)$

$= -\frac{3}{2}x^2\sin(\sqrt{x^3+2})(x^3+2)^{-1/2}$

$\implies y' = f'g + fg'$

$= e^x\cos(\sqrt{x^3+2}) + e^x\left(-\frac{3}{2}x^2\sin(\sqrt{x^3+2})(x^3+2)^{-1/2}\right)$

$= e^x\cos(\sqrt{x^3+2}) - \frac{3}{2}x^2e^x\sin(\sqrt{x^3+2})(x^3+2)^{-1/2}$

7. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, $g'(2) = 7$. Find $h'(2)$ for each of the following.

   (a) $h(x) = 5f(x) - 4g(x)$

   Solution:

$$h'(x) = \frac{d}{dx}[5f(x)] - \frac{d}{dx}[4g(x)]$$
$$= 5\frac{d}{dx}[f(x)] - 4\frac{d}{dx}[g(x)]$$
$$= 5f'(x) - 4g'(x)$$
$$\Longrightarrow \quad h'(2) = 5f'(2) - 4g'(2)$$
$$= 5(-2) - 4(7) = -10 - 28 = -38$$

   (b) $h(x) = f(x)g(x)$

   Solution: $h'(x) = \frac{d}{dx}[f(x)g(x)]$

$$= f'(x)g(x) + f(x)g'(x)$$
$$\Rightarrow h'(2) = f'(2)g(2) + f(2)g'(2)$$
$$= (-2)(4) + (-3)(7) = -8 - 21 = -29$$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   Solution: $h'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$
$$h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \frac{(-2)(4)-(-3)(7)}{(4)} = \frac{-8-(-21)}{16} = \frac{13}{16}$$

   (d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

   Solution: $h'(x) = \frac{d}{dx}\left[\frac{g(x)}{u(x)}\right] = \frac{g'(x)u(x) - g(x)u'(x)}{(u(x))^2}$

Let $u(x) = 1 + f(x)$

$$\Longrightarrow u'(x) = \frac{d}{dx}[1] + \frac{d}{dx}[f'(x)] = f'(x)$$
$$\Rightarrow h'(x) = \frac{g'(x)(1+f(x)) - g(x)f'(x)}{(1+f(x))^2}$$
$$\Rightarrow h'(2) = \frac{g'(2)(1+f(2)) - g(2)f'(2)}{(1+f(2))^2}$$
$$= \frac{(7)(1+(-3)) - (4)(-2)}{(1+(-3))^2} = \frac{-14+8}{(-2)^2} = \frac{6}{4} = \frac{3}{2}$$

8. Find an equation of the line tangent to the curve $y = 2x\sin(x)$ at the point $\left(\frac{\pi}{2}, \pi\right)$.

Solution:

The eqn of the tangent line @ the pt $(a, y(a))$ is

$$L(x) = y(a) + y'(a)(x-a)$$

Note: Use of L (or some other letter besides y) is necessary here because y is already the name of our given function! Watch out for this!

First we need to find $y'(x)$

$$y'(x) = \frac{d}{dx}[f \cdot g] = f'g + fg'$$

$f = 2x$ $\quad g = \sin(x)$

$f' = 2$ $\quad g' = \cos(x)$

$$\Rightarrow y'(x) = 2\sin(x) + 2x\cos(x)$$

So given the pt $(\frac{\pi}{2}, \pi)$ we know

$a = \frac{\pi}{2}$

$y(a) = y(\frac{\pi}{2}) = \pi$

Slope of tangent line @ $(\frac{\pi}{2}, \pi)$

$$y'(a) = y'(\tfrac{\pi}{2}) = 2\underbrace{\sin(\tfrac{\pi}{2})}_{=1} + 2(\tfrac{\pi}{2})\underbrace{\cos(\tfrac{\pi}{2})}_{=0}$$
$$= 2(1)$$
$$= 2$$

So the equation of the tangent line is

$$L(x) = y(\tfrac{\pi}{2}) + y'(\tfrac{\pi}{2})(x - \tfrac{\pi}{2})$$
$$= \pi + 2(x - \tfrac{\pi}{2}) \leftarrow$$

its okay to leave it like this. Don't waste time doing unnecessary simplification. It's too easy to make mistakes here!

$$= \pi + 2x - 2\tfrac{\pi}{2}$$
$$= \cancel{\pi} + 2x \cancel{-\pi}$$
$$= 2x$$