Lesson 4

The Rules of Differentiation

Sections
3.3, 3.4, 3.5, 3.7

Basic Rules of Differentiation

The limit definition is of course is cumbersome to use every time you need to calculate a derivative. The next few lectures discuss formulas that allow us to find the derivative directly without bothering with limits.

It is important to note that each of these formulas comes directly from the definition of the derivative. This is why it was so important to learn before. The differentiation rules also tend to tell us nothing about differentiability. Just because we can find a formula using the differentiation rules does not mean that our derivative will exist everywhere!

We will demonstrate some of the proofs for the differentiation rules. If interested, the proofs of the remaining results can easily be found either in your textbook or by doing a quick internet search.

The Constant Rule

The derivative of a constant is the easiest of the differentiation rules to show.

Theorem 4.1: Derivative of a Constant

Let c be a real number number, then

$$\frac{d}{dx}[c] = 0$$

Proof. Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[c] = f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0}\frac{0}{h} = 0\end{aligned}$$

□

This result can be seen in the graph of a constant function. Recall that the slope of a constant function is 0 so any tangent line must also have a slope of zero.

The Power Rule

The next rule is by far the most common derivative rule you will use.

Theorem 4.2: The Power Rule

Let n be a real number then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for all x where x^n and x^{n-1} are both defined.

This rule works for $n < 0$ (i.e. when n is negative), $n = 1$, n is rational (i.e. $n = \frac{a}{b}$ where $a, b \in \mathbb{R}$ and $b \neq 0$), or n is irrational (i.e. a number line π). The textbook presents this rule separately for each case. For simplicity, we present it once here and will also skip the proof of this rule for the same reason.

The main thing to take away at this point is to see the pattern that this rule follows. Essentially, we are multiplying our term by the value of the exponent and then subtracting 1 from the exponent. For example, when we use the power rule on the function $f(x) = x$ we have

$$f'(x) = (1)x^{1-1} = x^0 = 1$$

Example 4.1: Applying the Power Rule

Using the power rule. Find the derivative of the following functions

a.) $f(x) = x^3$

Solution.

$$f'(x) = \frac{d}{dx}[x^3] = 3x^{3-1} = 3x^2$$

b.) $g(x) = \dfrac{1}{x^4}$

Solution. In order to use the Power Rule we want the function to have the form $g(x) = x^n$ so we rewrite g

$$g(x) = \frac{1}{x^4} = x^{-4}$$

Now we can apply the Power Rule to find the derivative

$$g'(x) = \frac{d}{dx}[x^{-4}] = -4x^{-4-1} = -4x^{-5} = -4\left(\frac{1}{x^5}\right) = \frac{-4}{x^5}$$

c.) $h(x) = \sqrt[3]{x^4}$

Solution. Rewriting as a single exponent we have

$$h(x) = \sqrt[3]{x^4} = x^{\frac{4}{3}}$$

Taking the derivative

$$h'(x) = \frac{d}{dx}[x^{\frac{4}{3}}] = \frac{4}{3}x^{\frac{4}{3}-1} = \frac{4}{3}x^{\frac{4}{3}-\frac{3}{3}} = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$$

The Constant Multiple Rule

Theorem 4.3: Constant Multiple Rule

If f is differentiable function and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} && \text{Definition of Derivative}\\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] \\
&= c\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] && \text{Apply Theorem 2.2}\\
&= cf'(x)
\end{aligned}$$

□

Example 4.2: Applying the Constant Multiple Rule

Differentiate the following functions

a.) $f(x) = \dfrac{7}{x}$

Solution. We first rewrite $f(x)$ so that it has an exponent.

$$f(x) = \frac{7}{x} = 7\left(\frac{1}{x}\right) = 7x^{-1}$$

Now differentiating we obtain

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[7x^{-1}] = 7\frac{d}{dx}[x^{-1}] && \text{Applying Constant Multiple Rule}\\
&= \frac{1}{7}(-1)x^{-1-1} = \frac{-1}{7}x^{-2} && \text{Applying Power Rule}\\
&= \frac{-1}{7x^2}
\end{aligned}$$

b.) $g(t) = \dfrac{x^3}{2}$

Solution. Differentiating we having

$$\begin{aligned}
g'(t) &= \frac{d}{dx}\left[\frac{x^3}{2}\right] = \frac{1}{2}\frac{d}{dx}\left[x^3\right] && \text{Applying Constant Multiple Rule}\\
&= (3)x^{3-1} = \frac{3}{2}x^2 && \text{Applying Power Rule}
\end{aligned}$$

Sum and Difference Rules

Theorem 4.4: The Sum and Difference Rules

Let f and g be differentiable functions then

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

We show the proof the sum rule. The difference rule is proved similarly.

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}$$

□

Example 4.3: Applying Sum and Difference Rules

Differentiate the function $f(x) = \dfrac{-x^4}{2} + 3x^3 - 2x + 7$

Solution.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{-x^4}{2} + 3x^3 - 2x + 7\right] \\
&= \frac{d}{dx}\left[\frac{-x^4}{2}\right] + \frac{d}{dx}[3x^3] - \frac{d}{dx}[2x] + \frac{d}{dx}[7] && \text{Sum \& Difference Rule} \\
&= \frac{-1}{2}(4)x^{4-1} + 3(3)x^{3-1} - 2x^{1-1} + 0 && \text{Power \& Constant Rules} \\
&= \frac{-4}{2}x^3 + 9x^2 - 2 = -2x^3 + 9x^2 - 2
\end{aligned}$$

Finding Tangent Lines

At this point the most common applications you will see is find the equation for a tangent line. We have already seen how to do this using the definition. We now demonstrate this using our newly learned differentiation rules.

Example 4.4: Find the Equation of Line Tangent to the Graph

Find the line tangent to the curve $f(x) = -3x^2 + 2$ at $x = 1$.

Solution. Recall that the derivative is the slope of the tangent line and that the equation of the tangent line at a point $x = a$ given in Definition 3.8 is

$$y = f(a) + f'(a)(x - a)$$

Now that we have all of these differentiation rules, this derivative is simple to find. We have

$$f'(x) = -6x$$

This is the equation that gives the slope of the line at a given x value. Here we have $x = 1$ so

$$f'(1) = -6(1) = -6$$

We also need to find $f(1)$. We have

$$f(1) = -3(1)^2 + 2 = -3 + 2 = -1$$

We now have $f(1)$ and $f'(1)$ and so the equation of the tangent line is

$$y = -1 - 6(x - 1) \implies y = -6x + 5$$

The Product Rule

The product of two functions f and g looks like $f \cdot g$ or $f(x) \cdot g(x)$. The following rule gives us a means to differentiate functions of this form.

Theorem 4.5: The Product Rule

Let f and g be differentiable functions then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve "tricky tricks" that may appear unmotivated to a reader. This proof involves such a step where we substract and add the same quantity, essentially adding nothing, to manipulate our limit. The added terms are show below in color.

Proof.

$$
\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right]\\
&= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}\right]+\lim_{h\to 0}\left[g(x)\frac{f(x+h)-f(x)}{h}\right]\\
&= \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

□

Remark: Your book does these derivatives in a process I call "brute force". It involves carrying out the derivatives at each step inline as you are doing your work. This process not only looks messy, but it is very difficult to catch your errors on intermediary derivative calculations. I organize my work differently than the book does when calculating derivatives. I will demonstrate what this looks like in class and have done my best to demonstrate this process in the following examples

Example 4.5: Applying the Product Rule

Differentiate the function $y = (2x+7)(4x^2+3x)$

Solution. Although we could expand this and use the power rule we can get the derivative faster by using the product rule.

Since $\quad y' = \frac{d}{dx}[fg] = f'g + fg'$

Let $\quad f = 2x+7 \quad , \quad g = 4x^2+3x$
$\quad\quad f' = 2 \quad , \quad g' = 8x+3$

So our derivative is

$$
\begin{aligned}
y' &= 2(4x^2+3x)+(2x+7)(8x+3)\\
&= 24x^2+68x+21
\end{aligned}
$$

Take care when simplifying and combining terms! Often, expanding your final answer is messy and does not help in you understanding. If you miss a step in the simplification any subsequent questions answered using the derivative will be incorrect. At a minimum, you should always include the initial application of the rule so that it is easy to go back and check for errors.

The Quotient Rule

Recall that the quotient of two functions f and g has the general form

$$\frac{f}{g} = \frac{f(x)}{g(x)}$$

The next rule demonstrates how to differentiate functions of this form.

Theorem 4.6: The Quotient Rule

Let f and g be differentiable functions then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Example 4.6: Using the Quotient Rule differentiate

$$y = \frac{5x-2}{x^2+1}$$

Solution.

Here we can apply the quotient rule.

Since $\quad y' = \dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{f'g - fg'}{g^2}$

Let $\quad f = 5x - 2 \quad , \quad g = x^2 + 1$

$\quad\quad f' = 5 \quad , \quad g' = 2x$

So our derivative is

$$y' = \frac{5(x^2+1) - (5x-2)(2x)}{(x^2+1)^2}$$

Higher Order Derivatives

So far we have learned an extensive toolbox for finding derivatives of most functions. Better yet, we see that we can avoid the use of the limit definition when calculating most derivatives. It is also possible to find higher order derivatives. A higher order derivative can be thought of as a "derivative of a derivative".

In order to find the nth order derivative you take the derivative of the $n-1$ derivative.

$$\frac{d}{dx}[f^{(n)}(x)] = \frac{d}{dx}[f^{(n-1)}(x)]$$

For example, to find the second derivative, $f''(x)$, we must take the derivative of the first derivative, $f'(x)$, i.e.

$$f''(x) = \frac{d}{dx}[f'(x)]$$

There are a few ways to denote higher order derivatives as seen in the table below.

First Derivative:	y',	$f'(x)$,	$\dfrac{dy}{dx}$,	$\dfrac{d}{dx}[f(x)]$
Second Derivative:	y'',	$f''(x)$,	$\dfrac{d^2y}{dx^2}$,	$\dfrac{d^2}{dx^2}[f(x)]$
Third Derivative:	y''',	$f'''(x)$,	$\dfrac{d^3y}{dx^3}$,	$\dfrac{d^3}{dx^3}[f(x)]$
Fourth Derivative:	$y^{(4)}$,	$f^{(4)(x)}$,	$\dfrac{d^4y}{dx^4}$,	$\dfrac{d^4}{dx^4}[f(x)]$
$\vdots$				
n-th Derivative:	$y^{(n)}$,	$f^{(n)}(x)$,	$\dfrac{d^ny}{dx^n}$,	$\dfrac{d^n}{dx^n}[f(x)]$

Table 4.1: Common Derivative Notation

It is important to note that when parenthesis are used around an exponent that this denotes the "nth order derivative of f".

Example 4.7: Higher order derivatives

Find the indicated derivative of $y = x^2e^{4x}$

a.) Find y'

Solution. Here we will need to apply the Product Rule.

Since $\quad y' = \dfrac{d}{dx}[x^2e^{4x}] = \dfrac{d}{dx}[fg] = f'g + fg'$

Let $\quad f = x^2 \quad , \quad g = e^{4x}$
$\qquad f' = 2x \quad , \quad g' = 4e^{4x}$

So our *first* derivative is

$$y' = 2xe^{4x} + 4x^2e^{4x} = e^{4x}(2x + 4x^2)$$

b.) Find y''

Solution. We first note that $y'' = [y']' = \dfrac{d}{dx}[e^{4x}(2x + 4x^2)]$. We need to apply the Product Rule.

Since $\quad y' = \dfrac{d}{dx}[e^{4x}(2x + 4x^2)] = \dfrac{d}{dx}[pq] = p'q + pq'$

Let $\quad p = e^{4x} \quad , \quad q = 2x + 4x^2$
$\qquad p' = 4e^{4x} \quad , \quad q' = 2 + 8x$

So our *second* derivative is

$$\begin{aligned} y'' &= 4e^{4x}(2x + 4x^2) + e^{4x}(2 + 8x) \\ &= e^{4x}[4(2x + 4x^2) + (2 + 8x)] \\ &= e^{4x}[16x^2 + 16x + 2] \end{aligned}$$

c.) Find y'''

Solution. We first note that $y''' = [y'']' = \dfrac{d}{dx}[e^{4x}(16x^2 + 16x + 2)]$. We see that we need to apply the Product Rule.

Since $\quad y''' = \dfrac{d}{dx}[e^{4x}(16x^2 + 16x + 2)] = \dfrac{d}{dx}[uv] = u'v + uv'$

$$\text{Let} \quad \begin{array}{lcl} u = e^{4x} & , & v = 16x^2 + 16x + 2 \\ u' = 4e^{4x} & , & v' = 32x + 16 \end{array}$$

So our *third* derivative is

$$y''' = 4e^{4x}[16x^2 + 16x + 2] + e^{4x}(32x + 16)$$

Remark: Note that we did not multiply everything out after finding y' and y''. By factoring our a common term at each step made our next derivative easier to find. Sometimes you may even find that it is not helpful to simplify before finding the next derivative. The main point is to not turn into a simplification robot! Think before you distribute terms or perform other simplifications. Not sure if a certain form will help you or not? Try it out on scratch paper! Don't be afraid to test things. A more complicated series of steps will still get the same answer but it takes longer and you may make mistakes.

The Chain Rule

Function compositions are commonly seen in the study of mathematics. Recall that a function composition is denoted by

$$(f \circ g)(x) = f(g(x))$$

For example the function

$$\sqrt{x^3 + 2x + 5}$$

Is the composition of the functions $f(x) = \sqrt{x}$ and $g(x) = x^3 + 2x + 5$. We have a special rule to find derivatives of these types of functions. This is known as the chain rule.

Theorem 4.7: The Chain Rule

Let $y = f(g(x))$ and $g(x)$ be differentiable functions. Then

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

In other words we take the derivative of the outside function multiplied by the derivative of the inside function.

Example 4.8: Using the Chain Rule

Differentiate the function $y = (x^2 + 1)^4$

Solution. We apply the Chain Rule by recognizing that our "outside" function as $f(x) = x^4$ and our "inside" function as $g(x) = x^2 + 1$

Since $\quad y' = \dfrac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Let $\quad f(g) = g^4 \quad , \quad g(x) = x^2 + 1$

$\qquad f'(g) = 4g^3 \quad , \quad g'(x) = 2x$

So our derivative is

$$\begin{aligned} y' &= 4(g(x))^3 \cdot g'(x) \\ &= 4(x^2 + 1)^3 \cdot (2x) \\ &= 8x(x^2 + 1)^3 \end{aligned}$$

Of course we could have also multiplied everything out and then taken the derivative of each term but this is more tedious than applying the Chain Rule.

The textbook defines a separate rule for applying the power rule with functions. This is really just an application of the chain rule. We find the derivative of the "outside" function using any of the usual rules we have for differentiation. There really is no need for a separate rule to be defined. The rule is defined below in case you have trouble seeing functions of this form.

Theorem 4.8: The General Power Rule

Let $y = [f(x)]^n$ where f is a differentiable function and n is a real number. Then

$$y' = n[f(x)]^{n-1} \cdot f'(x)$$

We are now able to take the derivatives of more complicated functions very quickly simply by rewriting them and applying the chain rule.

Example 4.9: More Applications of the Chain Rule

Differentiate the following functions.

a.) $y = \sqrt[3]{(x^2 + 1)^4}$.

Solution. First we rewrite the function with an exponent.

$$y = \sqrt[3]{(x^2 + 1)^4} = (x^2 + 1)^{\frac{4}{3}}$$

Now we can use the chain rule. Here I will be using u for my inside function. The letter u is often used as a substitution variable.

Since $\quad y' = \dfrac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$

Let $f(u) = u^{\frac{4}{3}} \quad , \quad u(x) = x^2 + 1$

$f'(u) = \frac{4}{3}u^{\frac{1}{3}} \quad , \quad u'(x) = 2x$

Note that in our definition of f we treat g as a variable to simplify our derivatives and to ensure we have the correct derivative of the "outside" function. So our derivative is

$$\begin{aligned} y' &= \frac{4}{3}(u(x))^{\frac{1}{3}} \cdot u'(x) \\ &= \frac{4}{3}(x^2+1)^{\frac{1}{3}}(2x) \\ &= \frac{8x}{3}(x^2+1)^{\frac{1}{3}} \text{ or } \frac{8x}{3}\sqrt[3]{x^2+1} \end{aligned}$$

b.) $y = \dfrac{-7}{(2t-3)^2}$

Solution. Re-write with an exponent.

$$y = \frac{-7}{(2t-3)^2} = -7(2t-3)^{-2}$$

Now we apply Chain Rule

Since $\quad y' = \dfrac{d}{dx}[f(u)] = f'(u(t)) \cdot u'(t)$

Let $f(u) = -7u^{-2} \quad , \quad u(t) = 2t - 3$

$f'(u) = 14u^{-3} \quad , \quad u'(t) = 2$

So our derivative is

$$\begin{aligned} y' &= 14(u(t))^{-3} \cdot u'(t) \\ &= 14(2t-3)^{-3}(2) \\ &= 28(2t-3)^{-3} \text{ or } \frac{28}{(2t-3)^3} \end{aligned}$$

Remark: Since the denominator of any rational function can be written with a negative exponent it is possible to write any rational function as a product of functions. In some cases this results in a simpler calculation using the product rule (instead of the quotient rule). It is very beneficial to improve your ability to rewrite functions in new ways as it often drastically simplifies your calculations.

Derivatives of Trigonometric Functions

Now we can more accurately represent the differentiation formulas for trigonometric, exponential and logarithmic functions. Often these rules involve the use of the chain rule so it is more helpful to be familiar with these versions over the more basic ones.

Theorem 4.9: Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin u] = (\cos u)u' \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$
$$\frac{d}{dx}[\cos u] = -(\sin u)u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$
$$\frac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

Example 4.10: Differentiating Trigonometric Functions

a.) $y = x\sin(x)$

Solution. Here we need to apply the Product Rule

Since $\quad y' = \frac{d}{dx}[fg] = f'g + fg'$

Let $\quad f = x \quad , \quad g = \sin(x)$

$\quad f' = 1 \quad , \quad g' = \cos(x)$

So our derivative is

$$y' = \sin(x) + x\cos(x)$$

b.) $y = \cos(3x)$

Solution. Here we need to apply the Chain Rule

Since $\quad y' = \frac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$

Let $\quad f(u) = \cos(u) \quad , \quad u(x) = 3x$

$\quad f'(u) = -\sin(u) \quad , \quad u'(x) = 3$

So our derivative is

$$\begin{aligned} y' &= -\sin(u(x)) \cdot u'(x) \\ &= -\sin(3x)(3) \\ &= -3\sin(3x) \end{aligned}$$

Derivatives of the Exponential and Natural Logarithms

The last two rules covered here are derivatives of exponential function and the natural logarithm. Derivatives of logarithmic and exponential functions with bases other than e will be covered in a later lesson.

Theorem 4.10: Derivative of Exponential & Natural Logarithm

$$\frac{d}{dx}[e^u] = e^u u' \qquad \frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

Example 4.11: Derivative of Exponential Function

a.) Differentiate $y = e^{2x}$

Solution. Here we need to apply the Chain Rule

Since $\quad y' = \dfrac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$

Let $\quad f(u) = e^u \quad , \quad u(x) = 2x$

$\qquad f'(u) = e^u \quad , \quad u'(x) = 2$

So our derivative is $y' = 2e^{2x}$

b.) Differentiate $y = \ln(7x)$

Solution. Here we need to apply the Chain Rule. This can look a little different because of the nature of the derivative of the natural logarithm.

Since $\quad \dfrac{d}{dx}[\ln(u)] = \dfrac{u'(x)}{u(x)}$

Let $\quad u(x) = 7x$

$\qquad u'(x) = 2$

So our derivative is

$$y' = \frac{u'(x)}{u(x)} = \frac{7}{7x} = \frac{1}{x}$$

Why always the Chain Rule?

For each of these formulas if $u = x$ then $u' = 1$. So, for example

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Your textbook first presents the basic versions of these rules and then later shows them using the chain rule. I have found that presenting them like that makes it seem like you need to know a whole new set of rules. Presenting these rules using *only* the chain rule helps your brain remember that you are always technically using the Chain Rule when differentiating transcendental functions.

Derivatives of Functions Involving Multiple Rules

There is a reason why derivatives have been organized in a particular way in these notes. First, it keeps things organized, and second it makes it easy to calculate derivatives when multiple rules are involved. This process is demonstrated in the next example.

I recommend using this or a process similar to this. Not only does it result in less errors but it makes it easy to find and correct any errors you may have made along the way. Not to mention it makes it much easier to award partial credit on assignments and exams!

Example 4.12: A Multi-step Derivative

Differentiate

$$y = \frac{x^3}{(x^2+2)^2}$$

Solution. For no particular reason other than to demonstrate it can be done, we rewrite our function so we can use the Product Rule instead of the Quotient Rule

$$y = \frac{x^3}{(x^2+2)^2} = x^3(x^2+2)^{-2}$$

To find the derivative we have a Product rule and in carrying out that Product Rule we will also have to use the Chain Rule.

Since $\quad y' = \dfrac{d}{dx}[fg] = f'g + fg'$

$$\begin{array}{llll}
\text{Let} & f = x^3 & , & g = (x^2+2)^{-2} \\
 & f' = 3x^2 & , & g' = \underbrace{\dfrac{d}{dx}[q(u)]}_{\Downarrow} = q'(u)u' = -2(x^2+2)^{-3}\cdot 2x \\
 & & & q = u^{-2} \quad , \quad u = x^2+2 \\
 & & & q' = -2u^{-3} \quad , \quad u' = 2x
\end{array}$$

So our derivative is

$$\begin{aligned}
y' &= 3x(x^2+2)^{-2} + x^3(-4x(x^2+2)^{-3}) \\
&= 3x(x^2+2)^{-2} - 4x^4(x^2+2)^{-3} \\
&= \frac{3x}{(x^2+2)^2} - \frac{4x^4}{(x^2+2)^3}
\end{aligned}$$

Now in finding g' I used the letters q and u to define the functions used in the application of the chain rule at that step. I did this only so that my derivative pieces didn't get confused. Any letters can be used as long as you clearly label which is which. Writing out the general rule each time you do a derivative not only shows someone what you're doing but also makes these rules easier to remember.

Practice Problems

Section 3.3
7-38, 44-48

Section 3.4
7-50, 57-60, 66-72

Section 3.5
17-28, 32-48, 56-65

Section 3.7
19-34, 41-68, 70-73