# Lecture 10: Numerical Integration

MATH 2205: Calculus II — Fall 2018

## Area Under Curves

In this course, our main view of integration is that of finding area under a curve. Previously, we have learned about several different ways to approximate this area, mainly with the use of rectangles. When we first see the concept of a definite integral, we are presented with approximations using left endpoints, right endpoints, and midpoints. We will also learn of two additional methods of approximating this area.

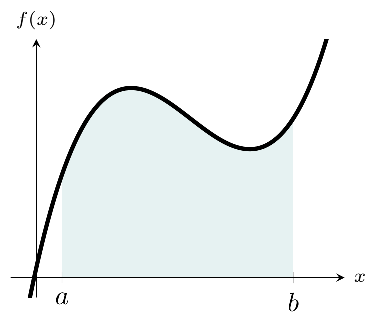


**Figure 1:** Region under a curve

While this process can be tedious to do by hand, these methods are the foundation of how any computer or calculator numerically calculates the value of a definite integral. As with any approximation, each method will have an associated error. The choice of method not only depends on accuracy, but also on computational efficiency.

### Approximating Area by Riemann Sums

In order to approximate an area using rectangles we must first decide how we will divide the interval from $[a, b]$ into *subintervals*. Each subinterval will define the base of each rectangle used in our approximation. For simplicity we will use subintervals of equal length. This is referred to as a regular partition for the interval $[a, b]$.

**Definition 1: Regular Partition**

A **regular partition** is the division of a closed interval $[a, b]$ into $n$ subintervals of equal length where

- The **subintervals** are given by

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots\ ,\ [x_{n-1}, x_n]$$

- The **grid points** are the endpoints of the subintervals given by $x_0$, $x_1$, …, $x_n$ where the $k$th grid point is given by

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, \cdots, n$$

- The **length** of each subinterval is given by $\Delta x = \frac{b-a}{n}$

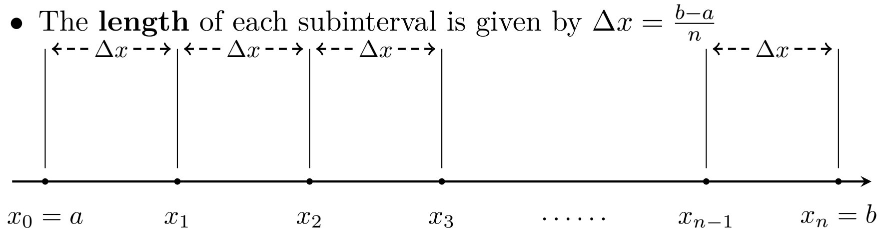


**Figure 2:** A Regular Partion of $[a, b]$

Each subinterval of the chosen partition serves as a base for the shape used to approximate the area under the curve. That is, we can approximate the area under the curve on each subinterval with the area of a particular shape. The use of a regular partition is convenient for us since the length of each subinterval will be the same (i.e., each rectangle used in our approximation will have the same base length).

The other ingredient we need to define most shapes is height. In the $k$th subinterval $[x_{k-1}, x_k]$, we choose a **sample point** $x_k^*$ and build a shape whose height is $f(x_k^*)$, the value of $f$ at $x_k^*$. For example given $b = \Delta x$ and $h = f(x_k^*)$ then the area of the rectangle on the $k$th subinterval is given by

$$b \cdot h = \Delta x f(x_k^*)$$

The choice of sample point will vary depending on the method we are using. We can see that summing the areas of all of these rectangles gives an approximation of the area under the curve.

### Definition 2: Riemann Sum on a Regular Partition

Let $f$ be continuous and non-negative on the interval $[a, b]$. For a regular partition of $[a, b]$ into $n$ subintervals a **area approximation** of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x, \quad x_{i-1} \leq x_i^* \leq x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

This type of area approximation is referred to as a *Riemann sum*. Notice that $x_k^*$ is an arbitrary point on the subinterval $[x_{k-1}, x_k]$. The choice of $x_k^*$ determines the type of Riemann sum being used. Three common choices of $x_k^*$ are $x_{k-1}$, $x_k$ and $\frac{x_{k-1}+x_k}{2}$. Riemann sums constructed using these three choices are referred to as left, right and midpoint Riemann sums respectively.

### Definition 3: Left Riemann Sum

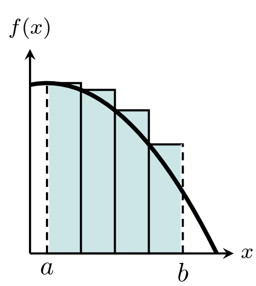


**Fig. 3**
Left Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Left Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots f(x_{n-1})\Delta x$$

### Definition 4: Right Riemann Sum

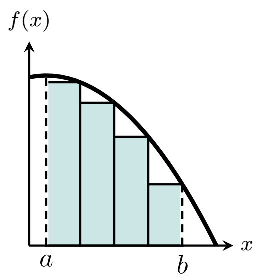

**Fig. 4**
Right Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Right Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x$$

### Definition 5: Midpoint Riemann Sum

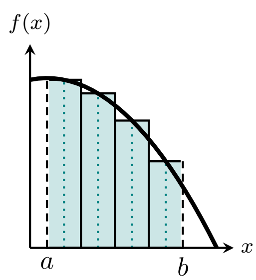

**Fig. 5**
Midpoint Riemann Sum

Let $f$ be continuous and non-negative on the interval $[a, b]$, which is divided into $n$ subintervals of equal length $\Delta x$.

A **Midpoint Riemann sum** for $f$ on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x$$

where $x_i^* = \dfrac{x_{i-1} + x_i}{2}$

Without knowing the shape of the function (or it's exact value) there is no "best" way to choose the point $x_k^*$. That is, you don't necessarily know which type of Riemann sum is more accurate or easier to use than the others.

### Example 1: Approximating Area Under a Curve

Use left, right and midpoint Riemann sums with $n = 5$ subintervals to estimate the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the $x$-axis, between $x = 0$ and $x = 2$.

Compare the value of each approximation to the exact value of the area, $\dfrac{22}{3}$.

**Solution.** For this example we will use the definition of a regular partition with $a = 0$, $b = 2$ and $n = 5$. Thus, the length of each subinterval is given by

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{5} = \frac{2}{5} = 0.4.$$

The value of the $k$th grid point is

$$x_k = a + k\Delta x, = 0 + k\left(\frac{2}{5}\right) = k\left(\frac{2}{5}\right) \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

In the example we find that

$$
\begin{aligned}
x_0 &= 0 + 0\left(\frac{2}{5}\right) = 0, & f(x_0) &= -(0)^2 + 5 = 5 \\
x_1 &= 0 + 1\left(\frac{2}{5}\right) = \frac{2}{5}, & f(x_1) &= -\left(\frac{2}{5}\right)^2 + 5 = 4.84 \\
x_2 &= 0 + 2\left(\frac{2}{5}\right) = \frac{4}{5}, & f(x_2) &= -\left(\frac{4}{5}\right)^2 + 5 = 4.36 \\
x_3 &= 0 + 3\left(\frac{2}{5}\right) = \frac{6}{5}, & f(x_3) &= -\left(\frac{6}{5}\right)^2 + 5 = 3.56 \\
x_4 &= 0 + 4\left(\frac{2}{5}\right) = \frac{8}{5}, & f(x_4) &= -\left(\frac{8}{5}\right)^2 + 5 = 2.44 \\
x_5 &= 0 + 5\left(\frac{2}{5}\right) = 2, & f(x_5) &= -\left(2\right)^2 + 5 = 1.
\end{aligned}
$$

The left Riemann sum is given by

$$
\begin{aligned}
\sum_{i=1}^{5} f(x_{i-1})\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= 5(0.4) + 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) \\
&= 8.08.
\end{aligned}
$$

The right Riemann sum is given by

$$
\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&= 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) + 1(0.4) \\
&= 6.48.
\end{aligned}
$$

For the function $f(x) = f(x) = -x^2 + 5$ the left Riemann sum was found to be an overestimate of the the area with about an 10% relative error. The right Riemann sum gave an underestimate of the area with a relative error of about 12%.

Note that we could have done this exercise with the midpoint Riemann sum as well. In fact the midpoint Riemann sum would give an approximate value of 7.36. Try it for yourself!

In the previous example we saw that the resulting approximations where fairly inaccurate. Depending on the setting an approximation with a 10% error could be unacceptable. The primary source of error in that approximation comes from the small number of rectangles used ($n = 5$).

If we wanted a more precise answer we would have needed to use more subintervals to define our partition. However, as we increase the value of $n$ the amount of work required to get the approximation also increases.

**Example 2: Evaluating Riemann Sums When $n$ is Large**

Evaluate the left Riemann sums for the region bounded by the graph of $f(x) = x^2$ and the $x$-axis between $a = 0$ and $b = 2$ using $n = 40$ subintervals.

**Solution.** With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{40} = \frac{1}{20} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of $x_k^*$ in the $k$th grid point will is

$$x_k^* = a + (k-1)\Delta x = 0 + (k-1)0.05 = 0.05k - 0.05 \quad \text{for} \quad k = 1, 2, \cdots, 40.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{40} f(0.05(k-1))0.05 \\
&= \sum_{i=1}^{40} (0.05(k-1))^2\, 0.05 \\
&= (0.05)^3 \sum_{i=1}^{40} (k^2 - 2k + 1) \\
&= (0.05)^3 \left( \frac{40(40+1)(2(40)+1)}{6} - 2\left(\frac{40(40+1)}{2}\right) + 40 \right) \\
&= (0.05)^3 (20540) \\
&= 2.5675
\end{aligned}$$

## Determining Exact Area Under a Curve

In the next section we will see how Riemann sums are used to define the definite integral. For now we close our discussion about approximate area with the following remark. The accuracy of the area approximation using Riemann sums can be improved by increasing the value of $n$, the number of intervals used to define the partition of $[a, b]$. It then stands to reason that we could determine the exact value of the area under the curve if an infinite number of rectangles are used in the Riemann sum. We summarize this result in the following definition.

**Definition 6: Definition of Area Under the Curve**

Let $f$ be continuous and nonnegative on the interval $[a, b]$. The **area** of a the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

In the following example we demonstrate how Riemann sums can be used to exactly compute the area under a curve.

**Example 3: Evaluating Summations Using Formulas**

Use left Riemann sums to determine the exact area of the region bounded by the graph of $f(x) = x^2$ and the $x$-axis between $a = 0$ and $b = 2$.

**Solution.** To begin, partition $[0, 2]$ into $n$ subintervals each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of $x_k^*$ in the $k$th grid point will is

$$x_k^* = a + (k-1)\Delta x = \frac{2(k-1)}{n} \quad \text{for} \quad k = 1, 2, \cdots, n.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2(k-1)}{n}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(k-1)}{n}\right)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(k-1)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n} k^2 - 2k + 1 \\
&= \left(\frac{2}{n}\right)^3 \left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right) \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

By taking the limit as $n \to \infty$ of the left Riemann sum we obtain the exact area

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty}\frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} = \frac{8}{3}.$$

Note that the type of Riemann sums used to find the area under the curve in the previous example does not matter. Instead of left Riemann sums we could have used right or midpoint Riemann sums and arrived at the same answer. Try it! Unfortunately, this property is not true for every function. When the area under the curve of a function is the same regardless of the type of Riemann sum used we say that the function is integrable. In the following lecture we will define the definite integral as the limit of general Riemann sums.

## Midpoint Rule

### Definition 7: Midpoint Rule

Suppose $f$ is defined and integrable on $[a, b]$. The **midpoint rule** approximation on $n$ equally spaced subintervals is

$$M_n \approx \int_a^b f(x)\,dx = \Delta x[f(m_1) + f(m_2) + \cdots + f(m_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

where $m_i$ is the midpoint of each subinterval given by

$$m_i = \frac{x_{i-1} + x_i}{2}, \quad k = 1, \ldots, n$$

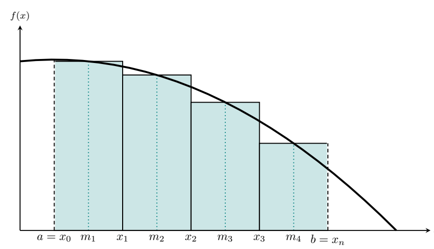


**Figure 6**

## Trapezoid Rule

**Definition 8: Trapezoid Rule**

Suppose $f$ is defined and integrable on $[a, b]$. The **midpoint rule** approximation on $n$ equally spaced subintervals is

$$T_n \approx \int_a^b f(x)\,dx = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)]$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x, \quad \text{for } k = 0, 1, \ldots, n$$

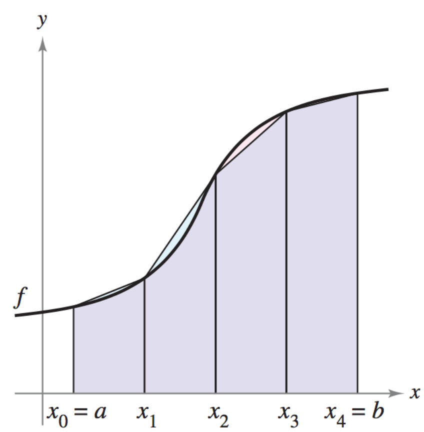


**Figure 7**

## Simpson's Rule

### Definition 9: Simpson's Rule

Suppose $f$ is defined and integrable on $[a, b]$. The **midpoint rule** approximation on $n$ equally spaced subintervals is

$$S_n \approx \int_a^b f(x)\,dx = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\cdots+4f(x_{n-1})+f(x_n)]$$

where

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x, \quad \text{for } k = 0, 1, \ldots, n$$

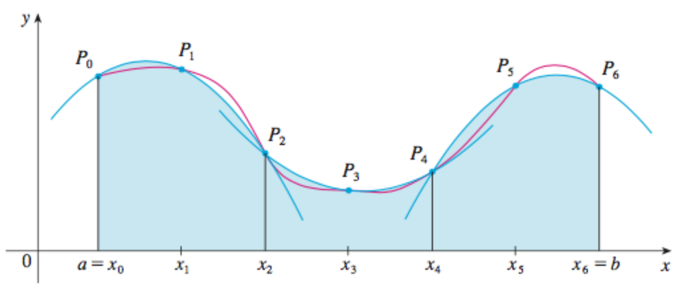


**Figure 8**

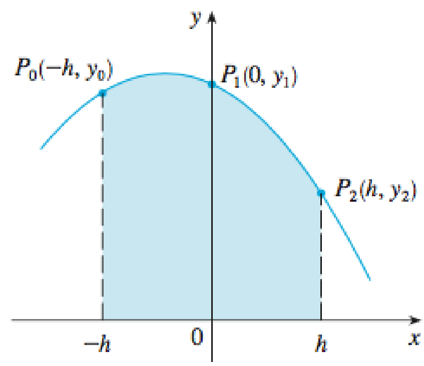


**Figure 9**

## Error Analysis

**Definition 10: Error Bound for Trapezoid Rule**

Suppose $|f''(x)| \leq M$ for $a \leq k \leq b$. If $E_T$ is the error in the Trapezoid rule then a bound on the error is given by

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}$$

**Definition 11: Error Bound for Midpoint Rule**

Suppose $|f''(x)| \leq M$ for $a \leq k \leq b$. If $E_M$ is the error in the Midpoint rule then a bound on the error is given by

$$|E_M| \leq \frac{M(b-a)^3}{24n^2}$$

**Definition 12: Error Bound for Simpson's Rule**

Suppose $\left|f^{(4)}(x)\right| \leq M$ for $a \leq k \leq b$. If $E_S$ is the error in Simpson's rule then a bound on the error is given by

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$