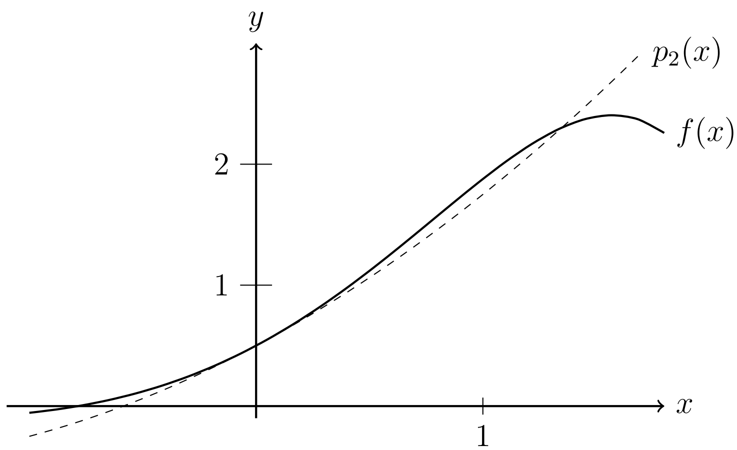
Problem 1: Consider the function $f(x) = \frac{e^x}{2}\big(\sin(x)+\cos(x)\big)$. Use the second degree Taylor polynomial for f centered at $a = 0$ to approximate the value of f at $x = 1$. Use the estimate for the remainder to determine how far your approximation could be from the exact value. For reference, the graphs of f (solid) and p_2 (dashed) are given below.



Solution:

The 2nd deg. Taylor polynomial centered @ a is given by

$$P_2(x) = F(a) + F'(a)(x-a) + F''(a)(x-a)^2$$

for $a=0$ we have

$$P_2(x) = F(0) + F'(0)x + F''(0)x^2$$

where

$f(x) = \frac{e^x}{2}(\sin(x)+\cos(x))$ $\Rightarrow$ $F(0) = \frac{1}{2}$

$F'(x) = e^x\cos(x)$ $\quad F'(0) = 1$

$F''(x) = e^x(\cos(x) - \sin(x))$ $\quad F''(0) = 1$

So

$$P_2(x) = \frac{1}{2} + x + x^2$$

Approx for F @ $x=1$ i.e.

$$F(1) \approx P_2(1) = \frac{1}{2} + (1) + (1)^2 = \frac{5}{2}$$

Problem 1: (cont'd)

The remainder in n^{th} deg. taylor polyn. is

$$R_n(x) = f^{(n+1)}(c)\frac{|x-a|^{n+1}}{(n+1)!}$$

where c is some value btwn x & a.

The maximum value of the remainder is given by

$$|R_n(x)| = |f^{(n+1)}(c)|\ \frac{|x-a|^{n+1}}{(n+1)!}$$

$$\leq M\frac{|x-a|^{n+1}}{|(n+1)!|}$$

i.e. we find the maximum error by finding max value M of $f^{(n+1)}(c)$

For $P_2(x)$ we have

$$|R_2(x)| = |f^{(3)}(c)|\left|\frac{x^3}{3!}\right| = |-2\sin(c)e^c|\left|\frac{x^3}{3!}\right|$$

Now we need to find max value of $f^{(3)}(C)$ btwn x & a:

$$|f'''(c)| = |-2\sin(c)e^c|$$

$\leq |2e^c|$ since max value of $\sin(x) = 1$

$\leq |2e^1|$ since max value btwn 0 & 1 is e^1

So we have

$$|f^{(3)}(c)| \leq \underbrace{2e}_{=M}$$

Problem 1: (Cont'd)

So remainder estimate for $P_2(x)$ is

$$|R_2(x)| \leq 2e\frac{x^3}{3!} = \frac{2}{6}ex^3 = \frac{1}{3}ex^3$$

@ $x=1$:

$$|R_2(1)| \leq \frac{1}{3}e(1)^3 = \frac{e}{3} \approx 0.906\ldots$$

This is the maximum amount that our approximation will be from the exact value.

Note: Choice of M is not necessarily unique! We could easily have said

$$2e < 2\cdot 3 = 6$$

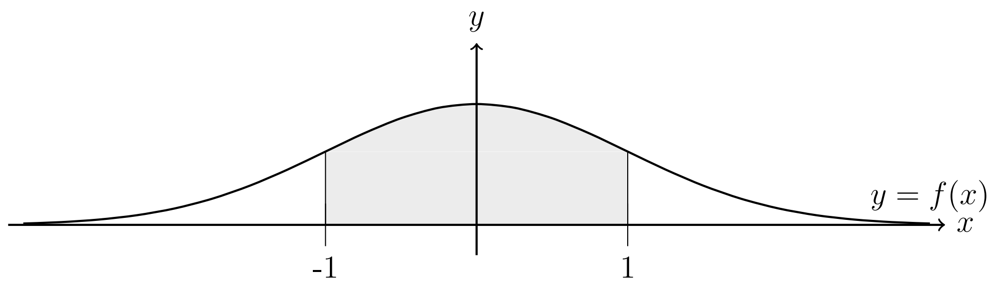
& chose $M=6$. It gives <u>a</u> max value, not <u>the</u> max value. Also, this is <u>not</u> necessarily the best max.

How good do we want this max to be depends on what you're using the approximation $P_2(x)$ for. Different applications require different tolerances for errors (think building a bridge v.s. microscale chemistry for example). Sometimes we can afford for error to be larger & sometimes we can't.

Problem 2: In probability, the normal distribution is used to calculate the probabilities that certain observations could occur, e.g., that a randomly chosen person will be of a certain height. In such a scenario, the observations have numerical values (e.g., height) and the probability that any particular observation has a value lying in a given range (e.g., the height is between 65 and 69 inches) is given by the area under a curve from 65 to 69. To make things more precise, one such curve is given by the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. This curve has a mean of 0 and a standard deviation of 1. Let's consider one example. We want to know the probability that the randomly obtained observation will lie within 1 standard deviation of the mean. The corresponding area is shaded in the picture. In symbols, this probability will be equal to

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx,$$

but we have no way of computing this using the Fundamental Theorem of Calculus. Use the second degree Taylor polynomial for f centered at $a = 0$ to approximate this probability. It may be useful to know that $\frac{1}{\sqrt{2\pi}} \approx 0.4$. For reference, the commonly used value is 0.6827.



Solution:

In this problem we are asked to find an approximation of

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx.$$

Using a second degree Taylor polynomial. Centered at $a=0$.

$$\Rightarrow f(x) \approx P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

Problem 2: (cont'd)

For $f(x) = e^{-x^2/2}$ $\quad f(0) = e^{-0^2/2} = 1$

$f'(x) = -xe^{-x^2/2}$ $\quad f'(0) = -0e^{-0^2/2} = 0$

$f''(x) = -e^{-x^2/2} + x^2e^{-x^2/2}$ $\quad f''(0) = -e^{-0^2/2} + 0^2e^{-0^2/2} = -1.$

Thus,

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$
$$= 1 + 0x - \frac{1}{2}x^2$$
$$= 1 - \frac{x^2}{2}$$

Note: From Table 4.9 of Briggs we know that

$$e^u = 1 + u + \frac{u^2}{2} + \cdots$$

So for $u = -\frac{x^2}{2}$.

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \left(-\frac{x^2}{2}\right)^2\left(\frac{1}{2}\right) + \cdots$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{4} + \cdots$$

Dropping any terms with a higher degree than 2. we get:

$$f(x)\ e^{-x^2/2} \approx 1 - \frac{x^2}{2} = P_2(x)$$

Problem 2: (cont'd)

Using the 2nd degree Taylor polynomial of $e^{-x^2/2}$ we can approximate the value of the integral of interest:

$$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}\,dx \approx \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} 1-\frac{x^2}{2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[x-\frac{x^3}{6}\right]_{x=-1}^{x=1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(1-\frac{1}{6}\right)-\left(-1+\frac{1}{6}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{5}{6}+\frac{5}{6}\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{5}{3}\right).$$

$$\approx 0.4(1.66).$$

$$= 0.6667$$

Comparing to the reference this approximation has an relative error of:

$$100\,\frac{|0.6667-0.6827|}{|0.6827|} = 2.34\%$$

Problem 3: Consider the number $\sqrt{e}$. We can approximate the decimal expansion of this number using the Taylor series for $e^{x/2}$ centered at 0. How many terms of this Taylor series does one need to be certain that $\sqrt{e} > 1.6$? You will need to justify your claim using remainder estimates.

Solution:

Taylor series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \ldots$$

$$\Rightarrow e^{x/2} = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!\,2^k} = 1 + \frac{x}{2} + \frac{x^2}{4 \cdot 2!} + \frac{x^3}{8 \cdot 3!} + \ldots$$

so we have

$$e^{1/2} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \ldots$$

Note that $1 + \frac{1}{2} + \frac{1}{8} = \frac{13}{8} \approx 1.625$

So going out at least 3 terms in Taylor series guarantees $\sqrt{e} > 1.6$

In other words we are using the 2nd deg. Taylor Polynomial:

$$P_2(x) = 1 + \frac{x}{2} + \frac{x^2}{8}$$

The remainder estimate is given by

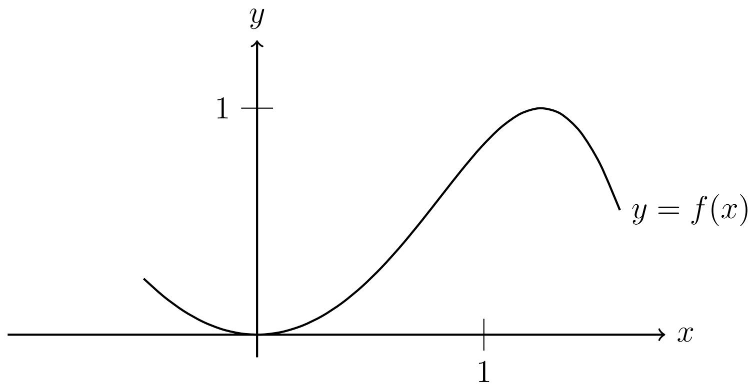
$$|R_2(x)| = |f^{(3)}(c)| \left|\frac{x^3}{3!}\right| = \left|\frac{e^{c/2}}{8}\right| \frac{x^3}{3!}$$

Btwn $a=0$ & $x=1$:

$$|R_2(1)| = \left|\frac{e^{c/2}}{8}\right| \frac{(1)^3}{3!} \leq \frac{e^{1/2}}{3!\,8} = \frac{e^{1/2}}{48} \approx 0.034$$

Since $1.625 - 1.6 = 0.025$ & $0.025 < 0.034$ our approximation using 3 terms is justified.

Problem 4: Approximate the area between the curve $f(x) = \sin(x^2)$ and the x-axis from $x = 0$ to $x = 1$ with an error less than 10^{-3}. For reference, the graph of f is given below.



Solution:

The area between the curve $f(x) = \sin(x^2)$ and the x-axis from $x=0$ to $x=1$ is given by

$$\int_0^1 \sin(x^2)\,dx.$$

We will approximate this area using a Taylor series for $\sin(x^2)$.

Recall: (from Table 9.5)

$$\sin(u) = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{(2k+1)!} \quad \text{for } |u| < \infty.$$

For $u = x^2$ we find that

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}$$

for $|x^2| < \infty$.

Problem 4: (Cont'd)

Since this series is convergent for $|x| < \infty$. it is convergent on the interval $[0,1]$. Thus, we are free to integrate this series!

$$\Rightarrow \int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^1 x^{4k+2}\,dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{1}{4k+3} x^{4k+3}\right)\Bigg|_{x=0}^{x=1}$$

$$= \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{(4k+3)\cdot(2k+1)!}\right)$$

$$= \sum_{k=0}^{\infty} (-1)^k a_k \quad \text{for } a_k = \frac{1}{(4k+3)(2k+1)!}$$

Note that $0 < a_{k+1} < a_k$ and $\lim_{k\to\infty} a_k = 0$.
So this is a convergent alternating series!
We know that the remainder of a convergent alternating series is bounded above by the next term. That is:

$$|R_n| < a_{n+1}.$$

Problem 4: (cont'd)

We want to know the number of terms of this series we require to estimate this area with an error less than 10^{-3}.

This will happen when we can say that $|R_n| < 10^{-3}$. Computing the first couple values of a_n we find that

$$a_0 = \frac{1}{(4(0)+3)(2(0)+1)!} = \frac{1}{3 \cdot 1!} = \frac{1}{3}$$

$$a_1 = \frac{1}{(4(1)+3)(2(1)+1)!} = \frac{1}{7(3!)} = \frac{1}{7 \cdot 6} = \frac{1}{42} \approx 0.024$$

$$a_2 = \frac{1}{(4(2)+3)(2(2)+1)!} = \frac{1}{11 \cdot 5!} = \frac{1}{1320} \approx 7.6 \times 10^{-4}$$

Since $a_2 < 10^{-3}$ we will use $n=1$ in our approximation of this area:

$$A = \int_0^1 \sin(x^2)\,dx = \sum_{k=0}^{1} (-1)^k a_k + R_n$$

$$= (a_0 + a_1) \pm a_3$$

$\Rightarrow$ The approximate area with an error less than 10^{-3} is:

$$A \approx 0.357$$