Area Between Curves

1. Find the area of the shaded region in Figure 1.

Solution:

Due to symmetry we can focus on the area of the region for $x \geq 0$.

We need to find the pts of intersection of the two curves

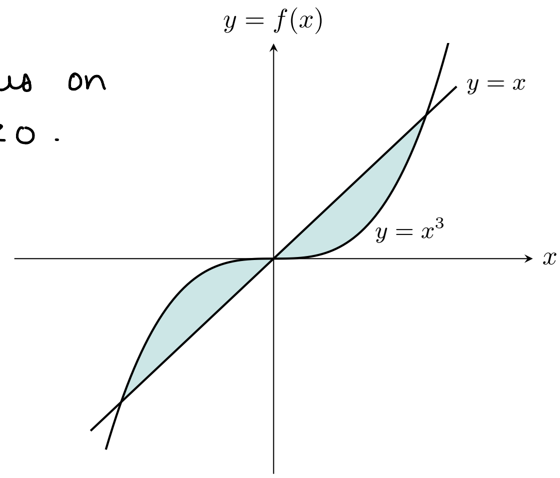


Figure 1

Setting 2 curves equal & solving for x

$$\Rightarrow \quad x = x^3$$
$$\Rightarrow \quad 0 = x^3 - x = x(x^2-1)$$
$$\Rightarrow x = 0, \; x = \pm 1$$

The points of intersection located @ $x=0$, $x=1$ & $x=-1$

The area btwn two curves is given by

$$\text{Area} = 2\int_a^b (f(x) - g(x))\,dx$$
$$= 2\int_0^1 (x - x^3)\,dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$
$$= 2\left(\left[\frac{(1)^2}{2} - \frac{(1)^2}{4}\right] - [\,0\,]\right) = 2\left(\frac{1}{4}\right) = \frac{1}{2}$$

2. Find the area of the shaded region in Figure 2.

Solution:

To find the area btwn 2 curves we consider the region with respect to y.

Our curves are

$$x = \frac{1}{3}(y-2)^2$$

$$x = 8 - y$$

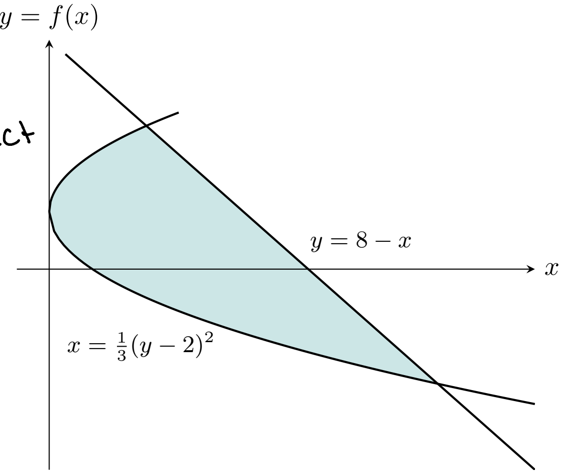


Figure 2

The pt of intersection of these 2 curves is

$$\frac{1}{3}(y-2)^2 = 8 - y$$

$$y^2 - 4y + 4 = 24 - 3y$$

$$y^2 - y - 20 = 0$$

$$(y-5)(y+4) = 0 \implies$$ Points of Intersection @ $y = -4$ & $y = 5$

Then area of region given by

$$\text{Area} = \int_{-4}^{5} \left((8-y) - \frac{1}{3}(y-2)^2\right) dy$$

$$= \int_{-4}^{5} \left(8 - y - \frac{1}{3}(y^2 - 4y + 4)\right) dy$$

$$= \frac{1}{3}\int_{-4}^{5} (20 + y - y^2)\, dy$$

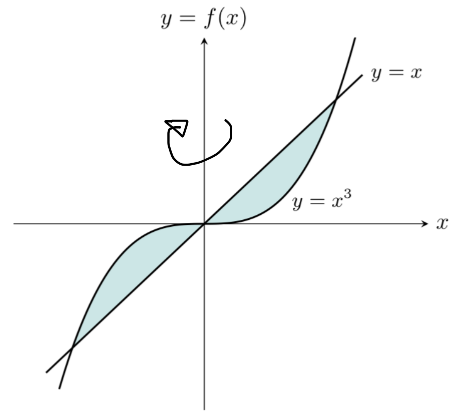
$$= \frac{1}{3}\left[20y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-4}^{5}$$

$$= \frac{81}{2}$$

Volume by Slicing

3. Find the volume of the solid obtained when the shaded region in Figure 1 is revolved about the y-axis.

Solution:



Again, because of symmetry we consider the region where $x \geq 0$.

We already know the pts of intersection are @ $x = 0, 1$

With respect to y: $x = y$, $x = y^{1/3}$ Bds: $x = 1 \Rightarrow y = 1$, $x = 0$, $y = 0$

Using the washer method the volume is given by

$$V = 2\pi \int_b^a \left[(f(y))^2 - (g(y))^2 \right] dy$$

$$= 2\pi \int_0^1 \left[(y^{1/3})^2 - (y)^2 \right) dy$$

$$= 2\pi \int_0^1 \left[y^{2/3} - y^2 \right] dy$$

$$= 2\pi \left[\frac{3}{5} y^{5/3} - \frac{y^3}{3} \right]_0^1$$

$$= 2\pi \left[\frac{3}{5} - \frac{1}{3} \right]$$

$$= 2\pi \left(\frac{4}{15} \right) = \frac{8\pi}{15}$$

4. Find the volume of the solid obtained when the shaded region is revolved about the y-axis in Figure 3.

Solution:

Revolving about the y-axis the volume using washers is given by

$$V = \int_a^b \pi \left((f(x))^2 - (g(x))^2 \right) dy$$

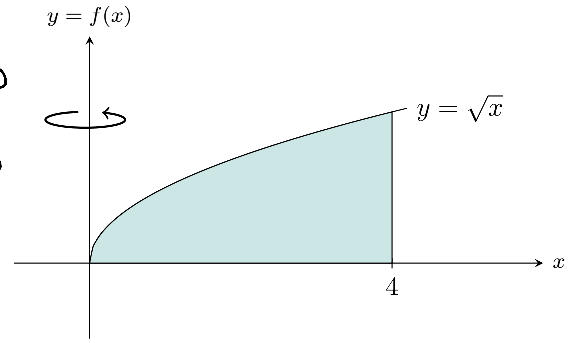


Figure 3

Curves wrt y are $x = y^2$ & $x = 4$

with pt of intersection: $4 = y^2 \Rightarrow y = \pm 2$

$\Rightarrow y = 2$

$$V = \pi \int_0^2 \left((4)^2 - (y^2)^2 \right) dy$$

$$V = \pi \int_0^2 (16 - y^4)\, dy$$

$$= \pi \left[16y - \frac{y^5}{5} \right]_0^2$$

$$= \pi \left[\left(16(2) - \frac{(2)^5}{5} \right) - (0) \right]$$

$$= \frac{128}{5} \pi$$