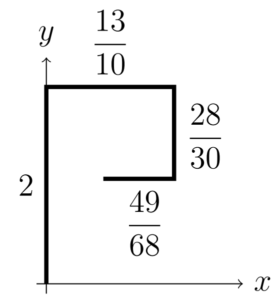
Problem 1: Your task is to compute the length (if it exists) of the following path. The path consists of an infinite number of legs, each being a straight line segment of a certain distance. Here are instructions for the first few legs. Begin at $(0,0)$ and head in the positive y-direction for a distance of 2. Now make a 90 degree right turn and head in the positive x-direction for a distance of 13/10. Make another 90 degree right turn and head in the negative y-direction for a distance of 28/30. This process will continue forever with the subsequent distances traveled given by $\dfrac{3k^2+1}{k^3+k}$ for $k = 1, 2, 3, \ldots$. After travelling the given distance on each leg, a 90 right turn is made and the next distance is travelled. The first 4 legs of the journey (labeled by their lengths) are illustrated below.



Solution:

Let D represent the total distance traveled. The distance traveled during any leg $k=1,2,3,\ldots$ is given by:

$$a_k = \frac{3k^2+1}{k^3+k}$$

Thus, the total distance traveled will be

$$D = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{3k^2+1}{k^3+k}$$

Before we determine what this series converges to lets make sure it does converge.

Lets try the integral test:

$a_k = \dfrac{3k^2+1}{k^3+k}$ so we consider $f(x) = \dfrac{3x^2+1}{x^3+x}$.

Note that:

- $f(x)$ is a rational function so it is continuous everwhere that $x^3+x \neq 0$. Since $x^3+x=0$ only when $x=0$ the function $f(x)$ is continuous on $[1,\infty)$.
- $3x^2+1>0$ for all x and $x^3+x \geq 0$ and $x^3+x \geq 0$ provided $x \geq 0$. Thus, $f(x)>0$ on $[1,\infty)$.
- $f'(x) = -\dfrac{3x^4-1}{(x^3+x)^2} < 0$ for all x $\Rightarrow$ $f(x)$ is decreasing on $[1,\infty)$

Problem 1: (cont'd)

Thus, $f(x)$ satisfies the requirements of the integral test.

We then find that:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{3x^2+1}{x^3+x}\,dx$$
$$= \lim_{b\to\infty} \int_1^b \frac{3x^2+1}{x^3+x}\,dx.$$

Let $u = x^3+x$ then $du = (3x^2+1)dx$

and $u(1) = 1^3+1 = 2$, $u(b) = b^3+b$.

Thus,

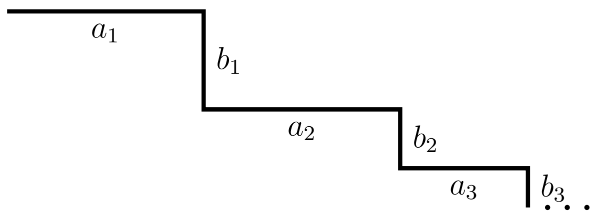
$$\int_1^b \frac{3x^2+1}{x^3+x}\,dx = \int_2^{b^3+b} \frac{1}{u}\,du$$
$$= \ln(|u|)\Big|_{u=2}^{u=b^3+b}$$
$$= \ln(b^3+b) - \ln(2)$$

$$\Rightarrow \int_1^\infty f(x)\,dx = \lim_{b\to\infty} \ln(b^3+b) - \ln(2) = \infty.$$

By the Integral Test, since $\int_1^\infty \frac{3x^2+1}{x^3+x}\,dx$ diverges, the series $\sum_{k=1}^\infty \frac{3k^2+1}{k^3+k}$ diverges.

$\Rightarrow$ D does not exist

Problem 2: Consider the following path as illustrated in the diagram. The path consists of a horizontal piece, followed by a vertical piece, followed by a horizontal piece, and so on. The lengths of the successive horizontal pieces are given by $a_k = \dfrac{2^k}{k!}$, $k = 1, 2, 3, \ldots$. The lengths of the succesive vertical pieces are given by $b_k = \dfrac{\sqrt{k}+2}{k^{5/2}+k+1}$, $k = 1, 2, 3, \ldots$. Express the length of this path as an infinite series and determine whether it has an infinite or finite length.



Solution:

The total length of this path is given by

$$L = \sum_{k=1}^{\infty} a_k + b_k.$$

where $a_k = \dfrac{2^k}{k!}$ and $b_k = \dfrac{\sqrt{k}+2}{k^{5/2}+k+1}$.

If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge then

$$L = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Thus, it suffices to check the convergence of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ seperately.

That is if both series converge then L will be finite.

Problem 2: (cont'd)

Consider: $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{2^k}{k!}$

Let's try the ratio Test

$$a_n = \frac{2^n}{n!} \Rightarrow a_{n+1} = \frac{2^{n+1}}{(n+1)!}$$

Thus, $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$$= \frac{2^n \cdot 2 \cdot n!}{(n+1)n! \cdot 2^n}$$

$$= \frac{2}{n+1}.$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1. \checkmark$$

By the Ratio test, since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$

the series $\sum_{k=1}^{\infty} a_k$ Converges.

That is, $\sum_{k=1}^{\infty} \frac{2^k}{k!} < \infty$.

Problem 2: (Cont'd)

Consider $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{\sqrt{k}+2}{k^{5/2}+k+1}$

Let's try a comparison test:

Consider $\sum_{k=1}^{\infty} c_k$ with $c_k = \frac{k^{1/2}}{k^{5/2}} = \frac{1}{k^2}$.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p>1$, this series converges.

Furthermore, $\frac{b_k}{c_k} = \frac{k^{1/2}+2}{k^{5/2}+k+1} \cdot \frac{k^2}{1}$

$$= \frac{k^{5/2}+2k}{k^{5/2}+k+1}$$

Then,

$$\lim_{k\to\infty} \frac{b_k}{c_k} = \lim_{k\to\infty} \frac{k^{5/2}+2k}{k^{5/2}+k+1} = \lim_{k\to\infty} \frac{k^{5/2}(1+2k^{-3/2})}{k^{5/2}(1+k^{-3/2}+k^{-5/2})}$$

$$= \lim_{k\to\infty} \frac{1+2k^{-3/2}}{1+k^{-3/2}+k^{-5/2}}$$

$$= \frac{1+0}{1+0+0} = 1 < \infty \checkmark$$

Problem 2: (cont'd)

By the limit comparison test, since $b_k > 0$, $c_k > 0$, $\sum_{k=1}^{\infty} c_k$ converges and $\lim_{k \to \infty} \frac{b_k}{c_k} = 1 < \infty$.

The series $\sum_{k=1}^{\infty} b_k$ converges.

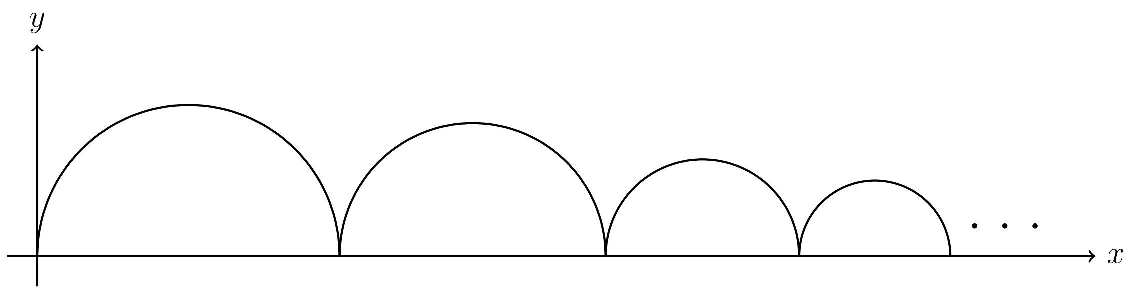
That is, $\sum_{k=1}^{\infty} \frac{\sqrt{k}+2}{k^{5/2}+k+1} < \infty$.

Since $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge we find that

$$L = \sum_{k=1}^{\infty} a_k + b_k = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k < \infty$$

So the length of the path is finite.

Problem 3: Consider an infinite sequence of semicircles with decreasing radius lined up along the x-axis as illustrated in the diagram. The area of each semicircle follows the pattern $\frac{\pi}{2}k^4(k^3+1)^{-2}$ for $k = 1, 2, 3, \ldots$, so that the first semicircle has area $\frac{\pi}{8}$, the second has area $\frac{8\pi}{81}$, and so on. Also consider the path $\mathcal{L}$ that begins at (0,0) and traces the top of each semicircle in succession. Write down two different infinite series, one expressing the combined area of this sequence of semicircles, and another expressing the length of the path $\mathcal{L}$. Then determine whether each of these quantities is finite or infinite.



Solution:

Area semicircle $= \frac{1}{2}(\pi r^2) = \frac{\pi}{2} r^2$

Circumference semi circle $= \frac{1}{2}(2\pi r) = \pi r$

Area of k^{th} semicircle:

$$A_k = \frac{\pi}{2} k^4 (k^3+1)^{-2} = \frac{\pi}{2} \frac{k^4}{(k^3+1)^2}$$

Combined Area

$$A = \sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} \frac{\pi}{2} \frac{k^4}{(k^3+1)^2} = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{k^4}{(k^3+1)^2}$$

We'll use the limit comparison test.

Note as $k \to \infty$: $\frac{k^4}{(k^3+1)^2} \sim \frac{k^4}{k^6} = \frac{1}{k^2}$

Let $b_k = \frac{1}{k^2}$. Clearly, $\frac{1}{k^2} > 0$ for $k = 1, 2, 3, \ldots$

so

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k^2} \Rightarrow \text{Has form of P-series } \sum_{k=1}^{\infty} \frac{1}{k^p}$$

By P-series test, since $p = 2 > 1$ the series $\sum_{k=1}^{\infty} b_k$ converges.

Problem 3: (cont'd)

So $a_k = \dfrac{k^4}{(k^3+1)^2} = \dfrac{k^4}{k^6+2k^3+1}$ Note $a_k > 0$ For $k = 1, 2, 3 \ldots$

& $b_k = \dfrac{1}{k^2}$

$$\Rightarrow \frac{a_k}{b_k} = \frac{\frac{k^4}{k^6+2k^3+1}}{\frac{1}{k^2}} = \frac{k^4}{k^6+2k^3+1} \cdot \frac{k^2}{1} = \frac{k^6}{k^6+2k^3+1}$$

$$= \frac{\frac{k^6}{k^6}}{\frac{k^6}{k^6} + \frac{2k^3}{k^6} + \frac{1}{k^6}} = \frac{1}{1 + \frac{2}{k^3} + \frac{1}{k^6}}$$

So

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{1 + \cancel{\frac{2}{k^3}}^{\to 0} + \cancel{\frac{1}{k^6}}^{\to 0}} = 1$$

By the <u>limit comparison test</u>
since <u>$0 < \lim_{k\to\infty} \frac{a_k}{b_k} < \infty$ and $\sum_{k=1}^{\infty} b_k$ converges</u>
the series $\sum_{k=1}^{\infty} a_k$ <u>converges</u>.

Therefore, the area given by
$A = \dfrac{\pi}{2} \sum_{k=1}^{\infty} \dfrac{k^4}{(k^3+1)^2}$ is finite.

Problem 3: (cont'd)

Combined Path Length

Length of k^{th} path given by

$$L_k = \pi r_k$$

so total path length is

$$L = \sum_{k=1}^{\infty} L_k = \sum_{k=1}^{\infty} \pi r_k$$

Since $A_k = \frac{\pi}{2} \frac{k^4}{(k^3+1)^2} = \frac{\pi}{2}\left(\frac{k^2}{k^3+1}\right)^2$

$\Rightarrow$ radius: $r_k = \frac{k^2}{k^3+1}$

So combined length given by

$$L = \sum_{k=1}^{\infty} \pi \frac{k^2}{k^3+1} = \pi \sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$$

Note that as $k \to \infty$ $\frac{k^2}{k^3+1} \sim \frac{k^2}{k^3} = \frac{1}{k}$

Let $b_k = \frac{1}{k} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ This is the harmonic series so diverges

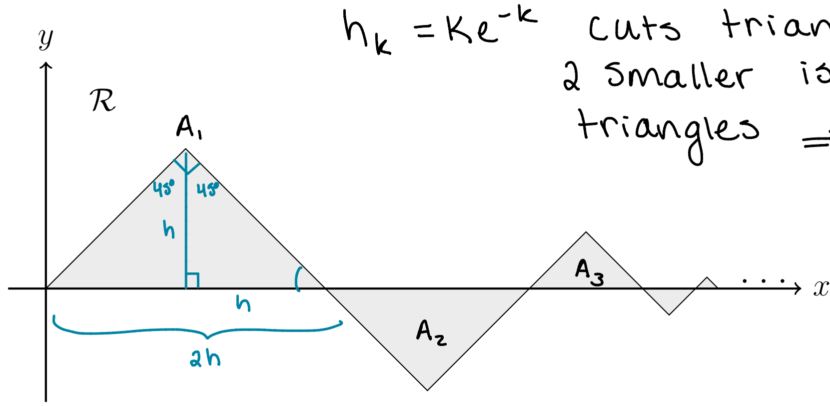
$$\Rightarrow \frac{a_k}{b_k} = \frac{\frac{k^2}{k^3+1}}{\frac{1}{k}} = \frac{k^2}{k^3+1} \cdot \frac{k}{1} = \frac{k^3}{k^3+1} = \frac{\frac{k^3}{k^3}}{\frac{k^3}{k^3} + \frac{1}{k^3}} = \frac{1}{1+\frac{1}{k^3}}$$

$$\text{So } \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{1+\underbrace{\frac{1}{k^3}}_{\to 0}} = 1$$

By the limit comparison test since $\lim_{k\to\infty} \frac{a_k}{b_k} > 0$ and $\sum_{k=1}^{\infty} b_k$ diverges the series diverges.

Therefore, path length is infinite (Weird right?!)

Problem 4: Consider the region $\mathcal{R}$ defined by an infinite sequence of isoceles triangles (each peak is a right angle) as illustrated in the diagram. The heights of the triangles decrease according to the pattern ke^{-k} for $k = 1, 2, 3, \ldots$. Express the area of $\mathcal{R}$ as an infinite series and determine if it is finite or infinite. Express the *net* area of $\mathcal{R}$ as an infinite series and determine if it is finite or infinite.



$h_k = Ke^{-k}$ cuts triangle into 2 smaller isosceles right triangles $\Rightarrow$ Area $= \frac{b \cdot h}{2}$

Solution:

Area of K^{th} triangle:

$$A_K = h_K^2 = (Ke^{-K})^2 = K^2 e^{-2K} = \frac{K^2}{e^{2K}}$$

Total Area:

$$A_{total} = \sum_{K=1}^{\infty} A_K = \sum_{K=1}^{\infty} \frac{K^2}{e^{2K}}$$

$a_K > 0$ since $K^2 > 0$ & $e^{2K} > 0$ for $K = 1, 2, 3, \ldots$

apply ratio test:

$$a_K = \frac{K^2}{e^{2K}} \quad \& \quad a_{K+1} = \frac{(K+1)^2}{e^{2(K+1)}} = \frac{(K+1)^2}{e^2 e^{2K}}$$

$$\Rightarrow \frac{a_{K+1}}{a_K} = \frac{\frac{(K+1)^2}{e^2 e^{2K}}}{\frac{K^2}{e^{2K}}} = \frac{(K+1)^2}{e^2 \cancel{e^{2K}}} \cdot \frac{\cancel{e^{2K}}}{K^2} = \frac{(K+1)^2}{e^2 K^2}$$

So,

$$\lim_{K\to\infty} \frac{a_{K+1}}{a_K} = \lim_{K\to\infty} \frac{(K+1)^2}{e^2 K^2} \overset{L'H}{=} \lim_{K\to\infty} \frac{\cancel{2}(K+1)}{\cancel{2} e^2 K} \overset{L'H}{=} \lim_{K\to\infty} \frac{1}{e^2} = \frac{1}{e^2}$$

Solution: (Cont'd)

By the ratio test since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{e^2} < 1$ then the series $\sum_{k=1}^{\infty} \frac{k^2}{e^{2k}}$ converges.

Therefore, the total combined area is finite.

Net Area

Net area is area above x-axis minus area below x-axis. i.e.

$$A_{Net} = A_1 - A_2 + A_3 - A_4 + \dots$$
$$= \sum_{k=1}^{\infty} (-1)^{k+1} A_k$$
$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^{2k}}$$

By Thm 8.21 absolute convergence implies convergence i.e.

If $\sum_{k=1}^{\infty} |b_k|$ converges then $\sum_{k=1} b_k$ converges.

So for $b_k = (-1)^{k+1} \frac{k^2}{e^{2k}}$

$$\Rightarrow |b_k| = \left| (-1)^{k+1} \frac{k^2}{e^{2k}} \right| = \frac{k^2}{e^{2k}}$$

Note that

$$\sum_{k=1}^{\infty} |b_k| = \sum_{k=1}^{\infty} \frac{k^2}{e^{2k}} = A_{total}$$

we know from our work above that A_{total} is finite (i.e. series converges)

Solution: (Cont'd)

By the Absolute Convergence test since $\sum_{k=1}^{\infty} \left|(-1)^{k+1} \frac{k^2}{e^{2k}}\right|$ converges then

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^{2k}}$ converges.

Therefore, the net area

$$A_{net} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{e^{2k}}$$

is finite.

Note: you can use the integral test to show convergence. Since the integral involves 2 applications of integration by parts we went with a test that involved more simple calculations work smarter, not harder! (at least not harder than is necessary!)