Lecture # 18: Power Series

Date: Mon. 12/3/18

Power series

Recall the basics of a geometric series

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad \text{for } |r|<1$$

This is a power series if we replace r with x. i.e.

$$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}, \quad \text{for } |x|<1$$

In this case we say the power series

$$\sum_{n=0}^{\infty} ax^n$$

Converges for x in the interval $(-1,1)$

This is a power series centered @ 0.

A general power series is centered @ a by replacing x with $x-a$

<u>Def</u> A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

is a <u>power series</u> centered @ $x=a$

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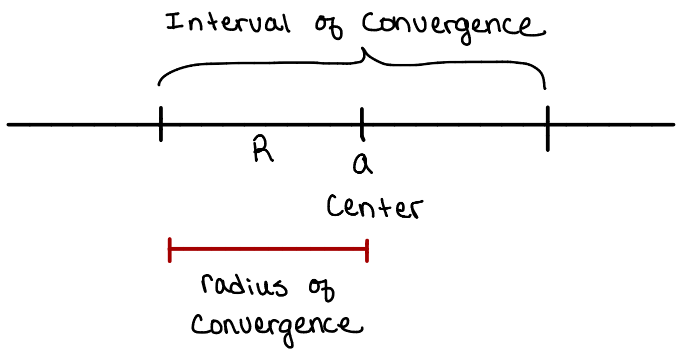
Not every power series is geometric, but there will be x values for which the series will converge.

$\boxed{\text{Convergence of a power series}}$

<u>Def</u> a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has coeff.s c_n & <u>center</u> a. The set of values of x for which the series converges is the <u>interval of convergence</u>.

The <u>radius of convergence</u> written R, is the distance from the center to the edge of the interval of convergence



The points on the edges of the interval of convergence may or may not be in the interval of convergence. This must be checked in each case.

How do we find the int. of convergence?

Generally, we need the ratio or root test most of the time i.e. testing for absolute convergence.

<u>Thm</u> a power series $\sum c_k(x-a)^k$ can converge in one of 3 ways:

(i) The series converges for all x
The int. of conv. is $(-\infty, \infty)$; $R=\infty$

(ii) The series converges on a finite interval $(a-R, a+R)$ for $0 < R < \infty$, where R is the radius of convergence. The pts $a-R$ & $a+R$ need to be checked individually.

(iii) The series only converges @ the center $x=a$. The radius of conv. is $R=0$.

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Ex. 1) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

Ratio Test:

$$a_k = \frac{x^k}{k!}$$

$$a_{k+1} = \frac{x^{k+1}}{(k+1)!}$$

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x \cdot \cancel{x^k} \ \cancel{k!}}{(k+1)\cancel{k!} \ \cancel{x^k}} = \frac{x}{k+1}$$

$$\Rightarrow \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{x}{k+1}\right| = \left|x \lim_{k\to\infty} \frac{1}{k+1}\right| = 0$$

By the Ratio Test, since $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ the series converges.

So $\sum \left|\frac{x^k}{k!}\right|$ converges $\Rightarrow \sum \frac{x^k}{k!}$ converges for all x

Radius of Convergence: $R = \infty$

Interval of Convergence: $(-\infty, \infty)$

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Ex. 2 $\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k}$

Root Test:

$$a_k = \frac{(-1)^k (x-2)^k}{4^k} = \left(-\frac{(x-2)}{4}\right)^k$$

$$\sqrt[k]{|a_k|} = \left[\left(\left|-\frac{(x-2)}{4}\right|\right)^k\right]^{1/k} = \left|-\frac{(x-2)}{4}\right| = \frac{|x-2|}{4}$$

$$\Rightarrow \rho = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{|x-2|}{4} = \frac{|x-2|}{4}$$

For series to converge need $\rho < 1$

$$\Rightarrow \frac{|x-2|}{4} < 1 \Rightarrow |x-2| < 4$$

Note: $|x-2| = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$

$$\Rightarrow x-2 < 4 \quad \& \quad -(x-2) < 4$$

$$x < 6 \qquad x-2 > -4$$

$$x > -2$$

So series converges for $-2 < x < 6$

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Ex. 2 (cont'd)

Need to check $x = -2$ & $x = 6$

For $x = -2$:

$$\sum_{k=0}^{\infty} \frac{(-1)^k(-2-2)^k}{4^k} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{-4}{4}\right)^k = \sum_{k=0}^{\infty} (-1)^k (-1)^k$$
$$= \sum_{k=0}^{\infty} 1^k$$

By Divergence test since
$\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} 1 = 1 \neq 0$ this series diverges

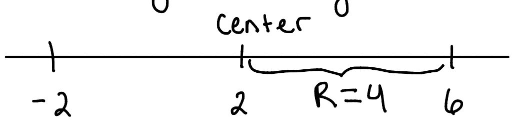
i.e. $x = -2$ is not included in interval of convergence

For $x = 6$:

$$\sum \frac{(-1)^k(6-2)^k}{4^k} = \sum (-1)^k \frac{4^k}{4^k} = \sum (-1)^k$$

Again, diverges by divergence test
$\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} (-1)^k = \text{DNE}$

So interval of Convergence is $(-2, 6)$



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Ex. 3) $\sum_{k=1}^{\infty} k!\, x^k$

Ratio Test:

$a_k = k!\, x^k$

$a_{k+1} = (k+1)!\, x^{k+1}$

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!\, x^{k+1}}{k!\, x^k} = \frac{(k+1)\cancel{k!}\, x \cdot \cancel{x^k}}{\cancel{k!}\, \cancel{x^k}} = x(k+1)$$

So

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} |x(k+1)| = \infty$$

By ratio test since $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$

Series diverges

Since center @ $x=0$ & no other value of x will make $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ finite

$\Rightarrow$ interval of convergence: $x=0$

Radius of convergence: $R=0$

Ex. 4 | Find the interval & radius of Conv. for $\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$

Ratio Test:

$$a_k = \frac{(x-2)^k}{\sqrt{k}}$$

$$a_{k+1} = \frac{(x-2)^{k+1}}{\sqrt{k+1}}$$

$$\frac{a_{k+1}}{a_k} = \frac{(x-2)^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{(x-2)^k} = \frac{(x-2)\sqrt{k}}{\sqrt{k+1}} = (x-2)\frac{\sqrt{k}}{\sqrt{k+1}}$$

$$= (x-2)\left(\frac{k}{k+1}\right)^{1/2}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|(x-2)\left(\frac{k}{k+1}\right)^{1/2}\right| = \left|(x-2)\left(\lim_{k\to\infty}\frac{k}{k+1}\right)^{1/2}\right|$$

$$\overset{H}{=} \left|(x-2)\left(\lim_{k\to\infty}\frac{1}{1}\right)^{1/2}\right|$$

$$= |x-2|$$

So series converges if $|x-2| < 1 \Rightarrow 1 < x < 3$

For $x = 1$

$$\sum_{k=1}^{\infty}\frac{(1-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{\sqrt{k}}\right)$$

By Alt. series test, Since

$\lim_{k\to\infty} a_k = \lim_{k\to\infty}\frac{1}{\sqrt{k}} = 0$ so series converges

Ex. 4 (cont'd)

For $x = 3$

$$\sum_{k=1}^{\infty} \frac{(3-2)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$$

By p-series test since $p<1$ series Diverges.

$\Rightarrow$ interval of convergence : $[1, 3)$
Radius of convergence : $R = 1$

Representing Functions as Power Series

We can create new power series thru mult. & Composition. This is best illustrated w/ examples.

Ex. 5 $f(x) = \frac{x^5}{1-x}$ Write this as a power series

$$f(x) = \frac{x^5}{1-x} = x^5 \underbrace{\left(\frac{1}{1-x}\right)}_{\text{Has form } \frac{1}{1-r}}$$

$$= x^5 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^5 x^n$$

$$= \sum_{n=0}^{\infty} x^{n+5}, \quad |x| < 1$$

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Ex. 6 $g(x) = \frac{1}{1-2x}$

$$g(x) = \frac{1}{1-2x} = \sum_{k=0}^{\infty} (2x)^k \quad \text{for } |2x| < 1$$

$$\Rightarrow |x| < \frac{1}{2}$$

Ex. 7 $h(x) = \frac{1}{1+x^2}$

$$h(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |-x^2| < 1$$

$$|x^2| < 1$$

$$|x^2| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$-x^2 < 1$ $\qquad$ $x^2 > 1$

$x^2 > -1$

$x > \sqrt{-1}$ $\qquad$ $x > \pm 1$

$\times$ $\qquad$ $x > -1$ & $x < 1$

$|x| < 1$

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Operations on Power series

One of the key reasons why power series are so useful is that if

$$f(x) = \sum c_k (x-a)^k, \quad |x-a| < R$$

i.e. we have a power series rep. for a f, then we get term-by-term

Differentiation:

$$f'(x) = \sum k c_k (x-a)^{k-1}, \quad |x-a| < R$$

Integration:

constant of integration

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$

i.e. we treat power series like a polyn.

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Ex. 8 $f(x) = \frac{1}{1-x} \Rightarrow f(x) = \sum_{k=0}^{\infty} x^k \;, \; |x|<1$

Find $f'(x)$.

$$f'(x) = \sum_{k=0}^{\infty} k\,x^{k-1} \;, \quad |x|<1$$

Ex. 9 Integrate $g(x) = \frac{1}{1+x}$ using a power series

Rep. as power series:

$$g(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k, \quad |x|<1$$

Integrate:

$$\int \frac{1}{1+x}\,dx = C + \sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k+1} \;, \quad |x|<1$$