Lecture # 07: Integration by Parts & Trig. Integral Date: Mon. 10/8/18

Integration by Parts

When we use substitution, we seek a fcn u & it's derivative dx. We do this because we know we're looking @ the derivative of a fcn composition

In other words, we know we're trying to undo the chain rule.

Question: Can we use this same strategy for other differentiation rules?

Let's take a look @ the product rule

$$\frac{d}{dx}[uv] = u'v + uv'$$

Integrating we have

$$\int \frac{d}{dx}[uv]\,dx = \int u'v\,dx + \int uv'\,dx$$

Using differential notation we have

$$du = u'dx \quad \& \quad dv = v'dx$$

then

$$\int \frac{d}{dx}[uv]\,dx = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int v\,du + \int u\,dv$$

Rearranging some terms & we have

$$\int u\,dv = uv - \int v\,du$$

This is known as <u>integration by Parts</u>

<u>Ex.1</u> Find $\int x\sin(x)\,dx$

To use integration by parts we will need to choose what u & v are.

We can use

$u = x \qquad v = \int \sin(x)\,dx = -\cos(x)$

$du = 1\,dx \qquad dv = \sin(x)\,dx$

Our integral becomes

$$\int x\sin(x)\,dx = x(-\cos(x)) - \int(-\cos(x))(1\,dx)$$

$$= -x\cos(x) + \int \cos(x)\,dx$$

$$= -x\cos(x) + \sin(x) + C$$

What if we went the other way with our choice of u & v?

$$u = \sin(x) \qquad v = \int x\,dx = \frac{x^2}{2}$$

$$du = \cos(x)\,dx \qquad dv = x\,dx$$

So

$$\int x\sin(x)\,dx = \frac{1}{2}x^2\sin(x) - \frac{1}{2}\int x^2\cos(x)\,dx$$

This integral is more complicated than our first choice!

While both are mathematically valid, only one of these choices actually makes our integral easier.

When choosing u & v, remember that we will be differentiating & integrating. We want these to not only be easier to do, but also to lead to a more simple integral.

Ex. 2) $\int t^2 e^{-t}\, dt$

Integration by Parts:

$$\int u\, dv = uv - \int v\, du$$

Where $u = t^2$ $\quad v = \int e^{-t} dt = -e^{-t}$

$du = 2t\, dt$ $\quad dv = e^{-t} dt$

$$\Rightarrow \int t^2 e^{-t} dt = -e^{-t} t^2 - \int (-e^{-t})(2t\, dt)$$

$$= -e^{-t} t^2 + 2 \underbrace{\int e^{-t}\, t\, dt}_{\text{Int. by Parts}}$$

Note:

$$\int t e^{-t} dt = -te^{-t} + \int e^{-t} dt = -te^{-t} - e^{-t}$$

$u = t$ $\quad v = -e^{-t}$

$du = dt$ $\quad dv = e^{-t} dt$

So

$$\Rightarrow \int t^2 e^{-t} dt = -e^{-t} t^2 + 2\int e^{-t}\, t\, dt$$

$$= -e^{-t} t^2 + 2\left[-te^{-t} - e^{-t}\right] + C$$

$$= -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + C$$

Int. by Parts for Definite Integrals

For definite integrals we have

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex. 3) $\displaystyle\int_0^{\pi/2} x\cos(2x)\,dx$

Int. by parts: $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

where $u = x$ $\quad v = \int \cos(2x)\,dx = \frac{1}{2}\sin(2x)$

$du = dx$ $\quad dv = \cos(2x)\,dx$

Note: $\displaystyle\int \cos(2x)\,dx = \int \cos(u)\left(\tfrac{1}{2}du\right) = \tfrac{1}{2}\sin(u) = \tfrac{1}{2}\sin(2x)$

u-sub w/ $u = 2x$

$du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

So

$$\int_0^{\pi/2} x\cos(2x)\,dx = \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}\sin(2x)\,dx$$

Note: $\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$

$$= \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} - \frac{1}{2}\Big[-\frac{1}{2}\cos(2x)\Big]_0^{\pi/2}$$

$$= \frac{1}{2}\Big[x\sin(2x) + \frac{1}{2}\cos(2x)\Big]_0^{\pi/2}$$

Ex. 3) (cont'd)

$$\int_0^{\pi/2} x\cos(2x)\,dx = \frac{1}{2}\left[x\sin(2x) + \frac{1}{2}\cos(2x)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\sin\left(2\left(\frac{\pi}{2}\right)\right) + \frac{1}{2}\cos\left(2\left(\frac{\pi}{2}\right)\right) - \left(0 + \frac{1}{2}\cos(0)\right)\right]\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}(0) + \frac{1}{2}(-1) - \frac{1}{2}(1)\right]$$

$$= \frac{1}{2}[-1] = -\frac{1}{2}$$

Trigonometric Integrals

These are integrals that only involve trig. fcns
For example:

$$\int \cos^4(2x)\,dx, \quad \int \sin^3(x)\cos^2(x)\,dx$$

We'll need some trig. identities. These are the most common we will use.

$$\left.\begin{aligned} \cos^2(x) + \sin^2(x) &= 1 \\ 1 + \tan^2(x) &= \sec^2(x) \end{aligned}\right\} \text{Pythagorean Indetities}$$

$$\left.\begin{aligned} \sin^2(x) &= \frac{1-\cos(2x)}{2} \\ \cos^2(x) &= \frac{1+\cos(2x)}{2} \end{aligned}\right\} \text{Half angle identities}$$

Lecture # 07: Integration by Parts & Trig. Integral Date: Mon. 10/8/18

Ex. 4) $\int \sin^4(x)dx$

Note: $\int \sin^4(x)\,dx = \int \sin^2(x)\sin^2(x)\,dx = \int (\sin^2(x))^2 dx$

by $\frac{1}{2}$ angle I.D.: $(\sin^2(x))^2 = \left(\frac{1-\cos(2x)}{2}\right)^2$

$$\int \sin^4(x)dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{4}\int (1-\cos(2x))^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos(2x) + \cos^2(2x))\,dx$$

$$= \frac{1}{4}\left[\int 1\,dx - 2\int \cos(2x)dx + \int \cos^2(x)\,dx\right]$$

Note:

$$\int \cos^2(2x)dx = \int \frac{1+\cos(4x)}{2}\,dx$$

$$= \frac{1}{2}\int (1+\cos(4x))dx$$

$$= \frac{1}{2}\left[\int 1\,dx + \int \cos(4x)\,dx\right.$$

$$= \frac{1}{2}\left[x + \frac{1}{4}\sin(4x)\right]$$

$$= \frac{1}{2}x + \frac{1}{8}\sin(4x)$$

$$\Rightarrow \int \sin^4(x)dx = \frac{1}{4}\left[\int 1\,dx - 2\int \cos(2x)dx + \int \cos^2(x)\,dx\right]$$

$$= \frac{1}{4}\left[x - \frac{2}{2}\sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x)\right] + C$$

$$= \frac{3}{8} - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

Ex. 4) $\int \cos^5(x)dx$

Note: $\int \cos^5(x)\,dx = \int (\cos^2(x))^2 \cos(x)\,dx$

By Pyth. ID: $\cos^2(x) = 1 - \sin^2(x)dx$

$\Rightarrow \int \cos^5(x)dx = \int (1 - \sin^2(x))^2 \cos(x)\,dx$

$= \int (1 - 2\sin^2(x) + \sin^4(x)) \cos(x)dx$

using u-sub: $u = \sin(x)$

$du = \cos(x)dx$

$= \int (1 - 2u^2 + u^4)\,du$

$= u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$

$= \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$