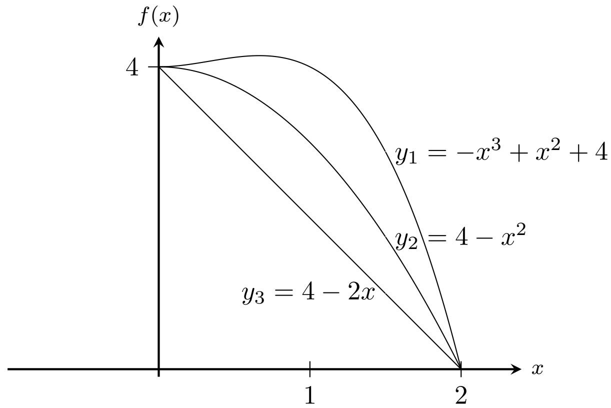
Written Homework 1 Solutions
MATH 2205: Calculus II, Fall 2018

Problem 1: Here is a picture containing the graphs of three functions.



w/ $f(x) \geq g(x)$

Which is larger, the area between the curves $y = -x^3 + x^2 + 4$ and $y = 4 - x^2$, or the area between the curves $y = 4 - x^2$ and $y = 4 - 2x$?

Solution: Area btween 2 curves $f(x)$ & $g(x)$ on $[a,b]$ where $f(x) \geq g(x)$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Points of intersection for both sets of curves by looking @ graph are @ $x = 0$ & $x = 2$.

Let A_1 be area btwn y_1 & y_2

$$A_1 = \int_0^4 [y_1(x) - y_2(x)]\,dx$$

$$= \int_0^1 [(-x^3 + x + 4) - (4 - x^2)]\,dx$$

$$= \int_0^1 [-x^3 + x + 4 - 4 + x^2]\,dx$$

$$= \int_0^2 [-x^3 + 2x^2]\,dx = \left[\frac{-x^4}{4} + \frac{2x^3}{3}\right]_0^2$$

$$= \left[-\frac{1}{4}(2)^4 + \frac{2}{3}(2)^3\right] - [0] = -4 + \frac{16}{3} = -\frac{12}{3} + \frac{16}{3} = \frac{4}{3}$$

Problem 1: (Cont'd)

Let A_2 be area btwn y_2 & y_3

$$A_2 = \int_0^2 [y_2 - y_3]\, dx$$

$$= \int_0^2 [(4-x^2) - (4-2x)]\, dx$$

$$= \int_0^2 [4 - x^2 - 4 + 2x]\, dx$$

$$= \int_0^2 [-x^2 + 2x]\, dx$$

$$= \left[-\frac{x^3}{3} + \frac{2x^2}{2}\right]_0^2$$

$$= \left[-\frac{(2)^3}{3} + (2)^2\right] - [0]$$

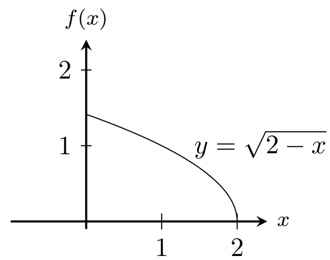
$$= -\frac{8}{3} + 4$$

$$= -\frac{8}{3} + \frac{12}{3}$$

$$= \frac{4}{3}$$

Since $A_1 = A_2$, the areas of the two regions are equal

Problem 2: Here is a graph of the function $y = \sqrt{2 - x}$.



The region between this curve and the coordinate axes in the first quadrant is used to create a solid of revolution by revolving about the x-axis. Assuming that x and y are being measured in centimeters (cm), if someone intends to manufacture this solid using cherry wood (density $= 0.5\ g/cm^3$), what is the mass (in grams, g) of the solid.

Solution: The region is bdd by $y = \sqrt{2-x}$, x-axis & lines $x = 0$ & $x = 2$

Since we're revolving around the x-axis we use the disk/washer method so volume is

$$V = \int_a^b \pi \left(f(x)^2\right) dx$$

Here $f(x) = \sqrt{2-x}$, $a = 0$, $b = 2$

$$V = \int_0^2 \pi (f(x))^2 \, dx$$

$$= \pi \int_0^2 (\sqrt{2-x})^2 \, dx$$

$$= \pi \int_0^2 (2-x) \, dx$$

$$= \pi \left[2x - \frac{x^2}{2} \right]_0^2$$

$$= \pi \left[\left(2(2) - \frac{(2)^2}{2} \right) - (0) \right] = \pi [4-2] = 2\pi$$

Problem 2: (cont'd)

Since x & y are measured in cm

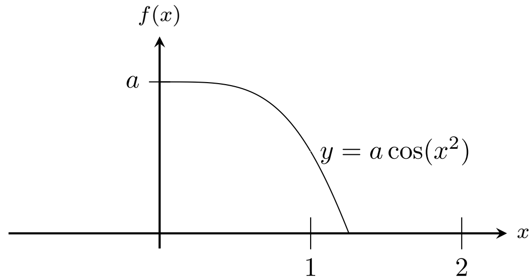
$$V = 2\pi \text{ cm}^3$$

Density of wood: $0.5 \frac{g}{cm^3}$

So mass of the solid is

$$V = 2\pi \text{ cm}^3 \cdot \frac{0.5 g}{cm^3} = 2\pi(0.5) g = \pi g$$

Problem 3: Here is a graph of the function $y = a\cos(x^2)$, where $a > 0$ is a parameter.



Find the value of the parameter a for which the solid generated by revolving the region bounded by $y = a\cos(x^2)$, the x-axis, $x = 0$, and $x = \sqrt{\frac{\pi}{2}}$ about the y-axis has a volume equal to 5.

Solution: Revolving about y-axis so we use shell method: for $f(x) \geq g(x)$ on $[a,b]$

$$V = \int_a^b 2\pi x\,[f(x) - g(x)]\,dx$$

So volume is

$$V = 2\pi \int_0^{\sqrt{\frac{\pi}{2}}} x\,(a\cos(x^2))\,dx$$

$$= 2\pi a \int_0^{\sqrt{\frac{\pi}{2}}} x\cos(x^2)\,dx$$

Note: $\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos(u)\,du = \frac{1}{2}\sin(u) + C$

using u-sub:

$u = x^2$ $\qquad\qquad = \frac{1}{2}\sin(x^2) + C$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$\Rightarrow V = 2\pi a \int_0^{\sqrt{\frac{\pi}{2}}} x\cos(x^2)\,dx = 2\pi a\left[\frac{1}{2}\sin(x^2)\right]_0^{\sqrt{\frac{\pi}{2}}}$$

$$= \pi a\left[\sin(x^2)\right]_0^{\sqrt{\frac{\pi}{2}}}$$

$$= \pi a\left[\sin\left(\left(\sqrt{\tfrac{\pi}{2}}\right)^2\right) - \sin(0)\right]$$

$$= a\pi\sin\left(\tfrac{\pi}{2}\right) = a\pi$$

Problem 3: (cont'd)

Since $V = a\pi$

To find when volume is 5

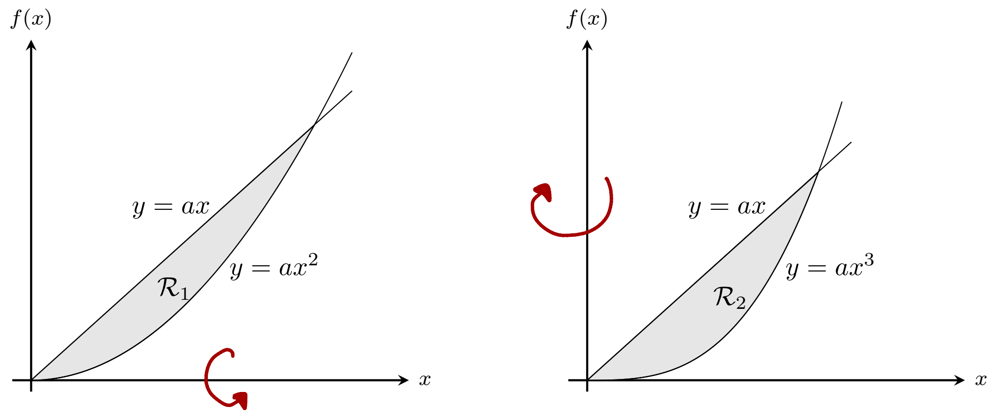
set $V=5$ & solve for a

$$\Rightarrow 5 = a\pi$$

$$a = \frac{5}{\pi}$$

So a value of $a = \frac{5}{\pi}$ will give a solid with a volume of 5.

Problem 4: Consider the following graphs. The symbols $\mathcal{R}_1$ and $\mathcal{R}_2$ refer to the shaded regions indicated in the pictures.



Determine the value of the parameter $a > 0$ for which the volume of the solid obtained when $\mathcal{R}_1$ is revolved about the x-axis is equal to the volume of the solid obtained when $\mathcal{R}_2$ is revolved about the y-axis.

Solution: For region R_1: Use washer method

$$V = \int_c^d \pi \left(f(x)^2 - g(x)^2 \right) dx, \quad f(x) \geq g(x)$$

Pts of intersection of R_1:

$$ax = ax^2 \Rightarrow x = x^2 \Rightarrow 0 = x^2 - x$$
$$= x(x-1) \Rightarrow x = 0, \ x = 1$$

$$V_1 = \pi \int_0^1 \left[(ax)^2 - (ax)^2 \right] dx$$

$$= \pi a^2 \int_0^1 \left[x^2 - x^4 \right] dx$$

For region R_2: Use shell method

$$V = \int_c^d 2\pi x \left(f(x) - g(x) \right) dx, \quad f(x) \geq g(x)$$

Pt of intersection for R_2:

$$ax = ax^3 \Rightarrow x = x^3 \Rightarrow 0 = x^3 - x = x(x^2 - 1)$$
$$\Rightarrow x = \pm 1$$

Problem 4:

$$\Rightarrow V_2 = 2\pi \int_0^1 x(ax - ax^3)\,dx$$

$$= 2\pi a \int_0^1 [x^2 - x^4]\,dx$$

Want to find when $V_1 = V_2$:

$$\pi a^2 \cancel{\int_0^1 [x^2 - x^4]\,dx} = 2\pi a \cancel{\int_0^1 [x^2 - x^4]\,dx}$$

$$\Rightarrow \cancel{\pi} a^{\cancel{2}} = 2\cancel{\pi}\cancel{a}$$

$$a = 2$$

So when $a=2$ the volume of R_1 will be equal to volume of R_2