Worksheet 9 Solutions

Partial Fractions MATH 2205, Fall 2018

1. Integrate $\displaystyle\int \frac{x+1}{x^3+3x^2-18x}\,dx$

Solution: Note: $\dfrac{x+1}{x^3+3x^2-18x} = \dfrac{x+1}{x(x-3)(x+6)}$

Partial Fraction Decomp.

$$\frac{x+1}{x(x+6)(x-3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+6}$$

$$\Rightarrow \quad x+1 = A(x-3)(x+6) + Bx(x+6) + Cx(x-3)$$

$$x+1 = Ax^2 + 3Ax - 18A + Bx^2 + 6Bx + Cx^2 - 3Cx$$

$$= (A+B+C)x^2 + (3A+6B-3C)x - 18A$$

$$\Rightarrow \quad A+B+C=0$$

$$6A+6B+6C=1$$

$$-18A=1$$

Solving system:

$$A = -\frac{1}{18}, \quad B = \frac{4}{27}, \quad C = -\frac{5}{54}$$

So we have:

$$\frac{x+1}{x^3+3x^2+18x} = \frac{-1}{18}\left(\frac{1}{x}\right) + \frac{4}{27}\left(\frac{1}{x-3}\right) - \frac{5}{54}\left(\frac{1}{x+6}\right)$$

Integrating:

$$\int \frac{x+1}{x^3+3x^2+18x}\,dx = \frac{-1}{18}\int \frac{1}{x}\,dx + \frac{4}{27}\int \frac{1}{x-3}\,dx - \frac{5}{54}\int \frac{1}{x+6}\,dx$$

$$= -\frac{1}{18}\ln(x) + \frac{4}{27}\ln(x-3) - \frac{5}{54}\ln(x+6) + C$$

2. Integrate $\int_0^5 \frac{2}{x^2-4x-32}\,dx$

Solution: Note:
$$\frac{2}{(x-8)(x+4)} = \frac{A}{x-8} + \frac{B}{x+4}$$

$$\Rightarrow \quad 2 = A(x+4) + B(x-8)$$
$$= Ax + 4A + Bx - 8B$$
$$= (A+B)x + 4A - 8B$$

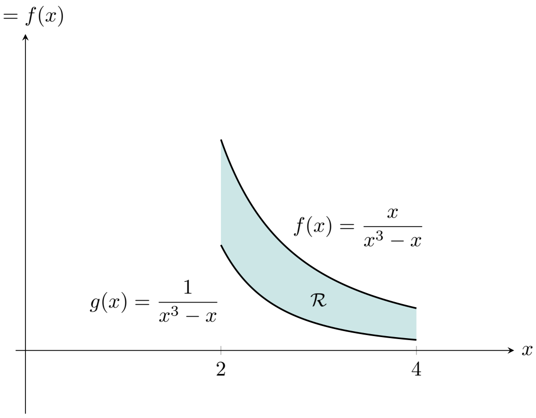
Equate coeff.s:

$$A + B = 0 \qquad\qquad A = -B = \frac{1}{6}$$
$$\Rightarrow$$
$$4A - 8B = 2 \qquad\qquad 4(-B) - 8B = 2$$
$$-12B = 2 \Rightarrow B = \frac{-1}{6}$$

$$\frac{2}{(x-8)(x+4)} = \frac{1}{6}\left(\frac{1}{x-8}\right) - \frac{1}{6}\left(\frac{1}{x+4}\right)$$

$$\Rightarrow \int_0^5 \frac{2}{(x-8)(x+4)}\,dx = \frac{1}{6}\int_0^5 \frac{1}{x-8}\,dx - \frac{1}{6}\int_0^5 \frac{1}{x+4}\,dx$$
$$= \left[\frac{1}{6}\ln(x-8) - \frac{1}{6}\ln(x+4)\right]_0^5$$
$$= \frac{1}{6}\ln\left(\frac{x-8}{x+4}\right)\Big|_0^5$$
$$= \frac{1}{6}\left[\ln\left(\frac{5-8}{5+4}\right) - \ln\left(\frac{0-8}{0+4}\right)\right]$$
$$= \frac{1}{6}\left[\ln\left(-\frac{3}{9}\right) - \ln(-2)\right]$$
$$= \frac{1}{6}\ln\left(\frac{-\frac{1}{3}}{-2}\right) = \frac{1}{6}\ln\left(-\frac{1}{3}\cdot\frac{-1}{2}\right) = \frac{1}{6}\ln\left(\frac{1}{6}\right)$$

3. Find the area of the region $\mathcal{R}$ indicated in the figure.



Solution: Area btwn 2 curves

$$A = \int_a^b [f(x) - g(x)]\, dx$$

then area of R is

$$A = \int_2^4 \left[\frac{x}{x^3 - x} - \frac{1}{x^3 - x} \right] dx$$

Note: $\dfrac{x}{x^3 - x} - \dfrac{1}{x^3 - x} = \dfrac{x - 1}{x(x^2 - 1)} = \dfrac{1}{x(x+1)}$

Partial Fraction Decomp.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$\Rightarrow 1 = A(x+1) + Bx$

$\Rightarrow 1 = Ax + A + Bx$

$\Rightarrow 1 = (A+B)x + A$

Equate Coeff.s

$A + B = 0 \Rightarrow B = -A = -1$

$A = 1 \Rightarrow A = 1$

So we have

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

3. (cont'd)

Integrating:

$$\Rightarrow \int_2^4 \frac{x-1}{x(x-1)(x+1)}\,dx = \int_2^4 \frac{1}{x}\,dx - \int_2^4 \frac{1}{x+1}\,dx$$

$$= \Big[\ln(x) - \ln(x+1)\Big]_2^4$$

$$= \left[\ln\left(\frac{x}{x+1}\right)\right]_2^4$$

$$= \ln\left(\frac{4}{4+1}\right) - \ln\left(\frac{2}{2+1}\right)$$

$$= \ln\left(\frac{4}{5}\right) - \ln\left(\frac{2}{3}\right)$$

$$= \ln\left(\frac{\frac{4}{5}}{\frac{2}{3}}\right) = \ln\left(\frac{4}{5}\cdot\frac{3}{2}\right)$$

$$= \ln\left(\frac{12}{10}\right) = \ln\left(\frac{6}{5}\right)$$

4. Use the table of integrals to evaluate the following. State the number and the general form of the integration formula you use in each case.

(a) $\displaystyle\int \frac{\cos(x)}{\sin^2(x) - 9}\,dx$

Solution:

Let $u = \sin(x)$

$du = \cos(x)\,dx$

$$\Rightarrow \int \frac{\cos(x)}{\sin^2(x) - 3^2}\,dx = \int \frac{1}{u^2 - 3^2}\,du$$

$$= \frac{1}{2(3)} \ln\left(\frac{\sin(x) - 3}{\sin(x) + 3}\right) + C$$

Using # 24

$$\int \frac{1}{u^2 - a^2}\,du = \frac{1}{2a} \ln\left(\frac{u-a}{u+a}\right) + C$$

$a = 3$, $u = \sin(x)$

(b) $\displaystyle\int \frac{1}{x^2\sqrt{4x^2+9}}\,dx$ $\quad 4x^2 = 2^2x^2 = \frac{1}{4}u^2$

Solution:

Let $u = 2x \Rightarrow x^2 = \frac{1}{4}u^2$

$du = 2\,dx \Rightarrow \frac{1}{2}du = dx$

$$\Longrightarrow \int \frac{1}{x^2\sqrt{4x^2+9}}\,dx = \frac{1}{2}\int \frac{1}{\frac{1}{4}u^2\sqrt{u^2+3^2}}\,du = 2\int \frac{1}{u^2\sqrt{u^2+3^2}}\,du$$

use # 35

$$\int \frac{1}{u^2\sqrt{u^2 \pm a^2}}\,du = \mp \frac{\sqrt{u^2 \pm a^2}}{a^2 u} + C$$

$u = 2x$, $a = 3$

$$= 2\left(-\frac{((2x)^2 + 3^2)^{1/2}}{3^2(2x)}\right) + C$$

$$= -\frac{(4x^2+9)^{1/2}}{9x} + C$$

(c) $\displaystyle\int \frac{e^x}{3 - e^{2x}}\,dx$

Solution:

Let $u = e^x$

$du = e^x\,dx$

$$\Rightarrow \int \frac{e^x}{3 - e^{2x}}\,dx = \int \frac{1}{3 - u^2}\,du = -\int \frac{-1}{(\sqrt{3})^2 - u^2}\,du$$

Using # 24

$$-\int \frac{1}{a^2 - u^2}\,du = \frac{1}{2a} \ln\left(\frac{u-a}{u+a}\right) + C$$

w/ $a = \sqrt{3}$, $u = e^x$

$$= -\left[\frac{1}{2(\sqrt{3})} \ln\left(\frac{e^x - \sqrt{3}}{e^x + \sqrt{3}}\right)\right] + C$$