1. (a) $\int te^t \, dt$

Solution:

Use int. by parts:

$$\int u\,dv = uv - \int v\,du$$

$u = t \qquad v = \int e^t dt = e^t$

$du = dt \qquad dv = e^t dt$

$$\int te^t dt = te^t - \int e^t dt$$
$$= te^t - e^t + C$$

(b) $\int_0^\pi x\cos(x)\,dx$

Solution: Use int. by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$u = x \qquad v = \int \cos(x)\,dx = \sin(x)$

$du = dx \qquad dv = \cos(x)\,dx$

$$\Rightarrow \int x\cos(x)\,dx = x\sin(x)\Big|_0^\pi - \int \sin(x)\,dx$$
$$= x\sin(x)\Big|_0^\pi - \Big[-\cos(x)\Big]_0^\pi$$
$$= \Big[x\sin(x) + \cos(x)\Big]_0^\pi$$
$$= \pi\sin(\pi) + \cos(\pi) - \big[0\cdot\sin(0) + \cos(0)\big]$$
$$= -1 - 1 = -2$$

2. $\int x^3 e^x \, dx$

Solution: Use int. by parts:

$$\int u\,dv = uv - \int v\,du$$

$$u = x^3 \qquad v = \int e^x\,dx = e^x$$

$$du = 3x^2\,dx \qquad dv = e^x\,dx$$

$$\Rightarrow \int x^3 e^x\,dx = x^3 e^x - \underbrace{\int 3x^2 e^x\,dx}_{\text{Int. by parts}}$$

$$u = 3x^2 \qquad v = \int e^x\,dx$$

$$du = 6x\,dx \qquad dv = e^x\,dx$$

$$\int 3x^2 e^x\,dx = 3x^2 e^x - 6\underbrace{\int x e^x\,dx}_{\text{Int. by Parts}}$$

$$u = x \qquad v = e^x$$

$$du = dx \qquad dv = e^x\,dx$$

$$\int x e^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x$$

$$= 3x^2e^x - 6[xe^x - e^x]$$

$$= 3x^2e^x - 6xe^x + 6e^x$$

$$\Rightarrow \int x^3 e^x\,dx = x^3 e^x - \int 3x^2 e^x\,dx$$

$$= x^3e^x - [3x^2e^x - 6xe^x + 6e^x] + C$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

3. $\displaystyle\int_1^{e^2} x^2 \ln(x)\, dx$

Solution: Use int. by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = \ln(x) \qquad v = \int x^2 dx = \frac{1}{3}x^3$$

$$du = \frac{1}{x}dx \qquad dv = x^2\,dx$$

$$\Rightarrow \int x^2 \ln(x)\,dx = \frac{1}{3}x^3\ln(x)\Big|_1^{e^2} - \int_1^{e^2} \frac{1}{x}\left(\frac{1}{3}x^3\right)dx$$

$$= \frac{1}{3}x^3\ln(x)\Big|_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx$$

$$= \frac{1}{3}x^3\ln(x)\Big|_1^{e^2} - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^{e^2}$$

$$= \left[\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3\right]_1^{e^2}$$

$$= \left[\frac{x^3}{3}\left(\ln(x) - \frac{1}{3}\right)\right]_1^{e^2}$$

$$= \frac{(e^2)^3}{3}\left(\ln(e^2) - \frac{1}{3}\right) - \frac{1}{3}\left(\ln(1) - \frac{1}{3}\right)\Big]$$

$$= \frac{e^6}{3}\left(2 - \frac{1}{3}\right) + \frac{1}{9}$$

$$= \frac{e^6}{3}\left[\frac{6}{3} - \frac{1}{3}\right] + \frac{1}{9} = \frac{e^5}{9} + \frac{1}{9} = \frac{1}{9}(5e^5 + 1)$$

4. $\displaystyle\int \sin^2(x)\cos^5(x)\,dx$

Solution:

$$\int \sin^2(x)\cos^5(x)\,dx = \int \sin^2(x)\left(\cos^2(x)\right)^2 \cos(x)\,dx$$

Since $\cos^2(x) = 1 - \sin^2(x)$

$$= \int \sin^2(x)\left(1 - \sin^2(x)\right)^2 \cos(x)\,dx$$

Using u-sub with $u = \sin(x)$

$du = \cos(x)\,dx$

$$\Rightarrow \int \sin^2(x)\cos^5(x)\,dx = \int u^2(1-u^2)^2\,du$$

$$= \int u^2(1 - 2u^2 + u^4)\,du$$

$$= \int (u^2 - 2u^4 + u^6)\,du$$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{1}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + \frac{1}{7}\sin^7(x) + C$$

5. Use integration by parts to show that

$$\int \sec^3(x)\ dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\ dx.$$

Solution: $\int \sec^3(x)\,dx = \int \sec^2(x)\sec(x)\,dx$

Int. by Parts

$$\int u\,dv = uv - \int v\,du$$

$u = \sec(x)\,dx$ $\qquad v = \int \sec^2(x)dx = \tan(x)$

$du = \sec(x)\tan(x)$ $\qquad dv = \sec^2(x)dx$

$$\Rightarrow \int \sec^3(x)dx = \sec(x)\tan(x) - \int \tan(x)(\sec(x)\tan(x))\,dx$$

$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx$$

Since $\tan^2(x) = 1 - \sec^2(x)$

$$= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x) - 1)\,dx$$

$$= \sec(x)\tan(x) - \int [\sec^3(x) - \sec(x)]\,dx$$

$$\Rightarrow \int \sec^3(x)dx = \sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)dx$$

$+\int \sec^3(x)dx$ $\qquad +\int \sec^3(x)dx$

$$\Rightarrow 2\int \sec^3(x) = \sec(x)\tan(x) + \int \sec(x)\,dx$$

$$\Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$$