Lecture # 16: Series; Comparison Tests Date: Wed. 11/14/18

Comparison Tests

Rather than relying on whether a series will "act" like an integral or geometric series we develop two direct comparison tests.

<u>Thm</u> (Comparison Test)

Let $\sum a_k$ & $\sum b_k$ be series with positive terms

1) If $0 < a_k \leq b_k$ and $\sum b_k$ converges, then $\sum a_k$ converges.

2) If $0 < b_k \leq a_k$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

In order for the comparison test to be used easily, we'll need a collection of series that we can use for easy comparison.

Lecture # 16: Series; Comparison Tests Date: Wed. 11/14/18

P-series

Thm (p-series Test)

The p-series defined by $\sum_{k=1}^{\infty} \frac{1}{k^p}$, k constant

converges for $p > 1$ & diverges for $p \leq 1$

For $p < 1$: $\sum_{k=1}^{\infty} \frac{1}{k^0} = \sum_{k=1}^{\infty} 1 \Rightarrow$ diverges

$\sum_{k=1}^{\infty} \frac{1}{k^{-p}} = \sum (k)^p \Rightarrow$ diverges

Ex. 1 The Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a p-series w/ $p = 1$

& we know the Harmonic series diverges

Lecture # 16: Series; Comparison Tests

Date: wed. 11/14/18

Ex. 2 $\sum_{k=1}^{\infty} \frac{k^3}{2k^4-1}$

Need to choose a series for comparison

As $k \to \infty$

$$\frac{k^3}{2k^4-1} \sim \frac{k^3}{2k^4} = \frac{1}{2k} \Rightarrow \sum \frac{1}{2k} \text{ Harmonic} \Rightarrow \text{diverges}$$

"similar to"

Is this a good choice for $\sum b_k$?

Need $0 < b_k \le a_k$

$$\frac{1}{2k} < \frac{k^3}{2k^4-1} \Longrightarrow 2k^4 - 1 < 2k^4$$

Test
if $k=0$
$\Rightarrow -1 < 0$ ✓
if $k=1$
$1 < 2$ ✓

So we have $0 < b_k \le a_k$

Let $b_k = \frac{1}{2k}$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2k}$ diverges: p-series w/ $p=1$ (or state it's Harmonic)

By the Comparison test

Since $0 < b_k \le a_k$ & $\sum_{k=1}^{\infty} b_k$ diverges

then the series $\sum_{k=1}^{\infty} a_k$ diverges

Lecture # 16: Series; Comparison Tests

Date: Wed. 11/14/18

Limit Comparison Test

If we are willing to deal with limits then there is an even better comparison test.

<u>Thm</u> (Limit Comparison Test)

Let $\sum a_k$ & $\sum b_k$ be series with positive terms and let

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

1) If $0 < L < \infty$ (i.e. finite), then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

2) If $L = 0$ & $\sum b_k$ converges, then $\sum a_k$ converges

3) If $L = \infty$ & $\sum b_k$ diverges, then $\sum a_k$ diverges

$\boxed{\text{Ex. 3}}$ $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k-\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{2k-k^{1/2}}$

As $k \to \infty$

$$\frac{1}{2k-k^{1/2}} \sim \frac{1}{2k}$$

choose $b_k = \frac{1}{2k} \Rightarrow \sum b_k = \sum \frac{1}{2k}$ Diverges b/c Harmonic

then

$$\frac{a_k}{b_k} = \frac{1}{2k-k^{1/2}} \div \frac{1}{2k} = \frac{1}{2k-k^{1/2}} \cdot \frac{2k}{1} = \frac{2k}{2k-k^{1/2}}$$

$$= \frac{2\cancel{k}}{\cancel{k}\left(2-\frac{1}{k^{1/2}}\right)}$$

$$= \frac{2}{2-\frac{1}{k^{1/2}}}$$

then

$$L = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2}{2-\frac{1}{k^{1/2}}}$$

$$= \frac{\lim_{k\to\infty} 2}{\lim_{k\to\infty} 2 - \lim_{k\to\infty} \frac{1}{k^{-1/2}}}$$

$$= \frac{2}{2-0} = 1$$ ← Finite!

By the limit comparison test

Since $0 < L < \infty$ & $\sum b_k$ diverges

then the series $\sum a_k$ diverges

Lecture # 16: Series; Comparison Tests

Date: wed. 11/14/18

Alternating Series

An alternating series is a series like

$$\sum_{k=1}^{\infty} (-1)^k k = -1 + 2 - 3 + 4 - 5 + \dots$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

We parse these series in the following way

- Signs : $(-1)^k$ or $(-1)^{k+1}$
- magnitude of terms: $1, 2, 3, 4$ or $1, \frac{1}{2}, \frac{1}{3}, \dots$

So a general alternating series has form

$$\sum (-1)^{k+1} a_k \ , \quad a_k > 0$$

Why are alt. series important

- Taylor series (discussed later)
- Can have a large effect on convergence

Lecture # 16: Series; Comparison Tests Date: Wed. 11/14/18

Ex. 4 The alternating Harmonic series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

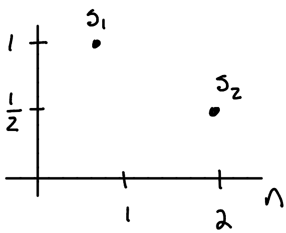
The sequence of partial sums is

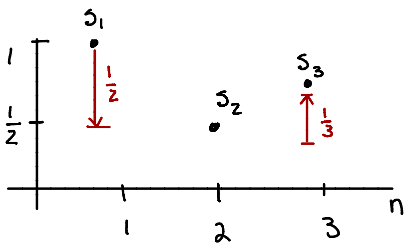
$$S_1 = 1$$

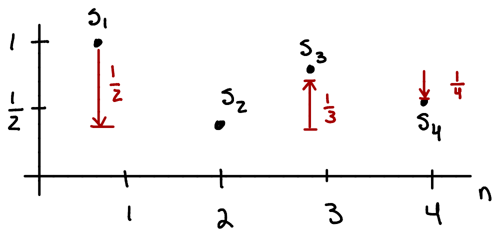
$$S_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Graphically,







We can see that the distance each partial sum "jumps" gets smaller.

This means that as n gets large, this distance will get closer & closer to a particular value i.e. the seq. of partial sums has a limit.

This is the idea behind why the alt. Harmonic series converges.

Lecture # 16: Series; Comparison Tests

Date: Wed. 11/14/18

The prev. example also illustrates how <u>any</u> alt. series converges.

<u>Thm</u> (Alternating Series Test)

The alternating series $\sum (-1)^{k+1} a_k$ converges if

(i) eventually $0 \le a_{k+1} \le a_k$

i.e. terms are decreasing for suff. large k

(ii) $\lim_{k \to \infty} a_k = 0$

<u>Ex. 5</u> $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

Is this an alt. series? yes

Are terms decreasing? $a_k = \frac{1}{k^2}$, yes!

$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k^2} = 0$

By the <u>Alt. Series test</u>

Since <u>series is alt., dec, & $\lim_{k \to \infty} a_k = 0$</u>

then the series <u>Converges</u>

Ex. 6 $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots$

Write this in summation notation:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{k}$$

Is this an alt. series? yes

Are terms decreasing? yes

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k+1}{k} \overset{H}{=} \lim_{k \to \infty} \frac{1}{1} = 1$$

By the Alt. series test

Since $\lim_{k \to \infty} a_k \neq 0$

then the series diverges