Problem 1: Provide (non-constant) examples of sequences satisfying the following conditions.

(a) A bounded sequence that diverges

(b) An alternating sequence that converges

(c) An increasing sequence that converges to 0

How you come up with your sequence is not important. A properly written solution will state your sequence, and then demonstrate that the sequence meets the required condition(s). Clearly label each part as (a), (b), (c). Do not question parts in columns!

Solution:

(a) $\{(-1)^n\}_{n=0}^{\infty}$

Bounded: Since $(-1)^n = \begin{cases} -1, & \text{if } n \text{ odd} \\ 1, & \text{if } n \text{ even} \end{cases}$

then we have $|(-1)^n| \leq 1$ for all n.

Diverges For n odd: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} -1 = -1$

For n even: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 = 1$

So $\lim_{n\to\infty} a_n$ does not exist

(b) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

Alternating: $\frac{(-1)^n}{n} = \begin{cases} -\frac{1}{n}, & \text{if } n \text{ odd} \\ \frac{1}{n}, & \text{if } n \text{ even} \end{cases}$

Converges:
For n odd: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n} = 0$

For n even: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

So $\lim_{n\to\infty} a_n = 0$ & sequence converges.

Problem 1: (cont'd)

(c) $\left\{ -\frac{1}{n} \right\}_{n=1}^{\infty}$

Increasing: $\frac{d}{dn}[-n^{-1}] = -(-n^{-2}) = \frac{1}{n^2} > 0$ for $n = 1, 2, 3, \ldots$

Since $-\frac{1}{n}$ has a positive derivative it is increasing.

Alternatively, a seq. is increasing if $a_n < a_{n+1}$. $a_n = \frac{-1}{n}$, $a_{n+1} = \frac{-1}{n+1}$

Since $-n > -(n+1)$ then we have

$$\frac{-1}{n} < \frac{-1}{n+1}$$

Converges to 0:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} -\frac{1}{n} = 0.$$

So seq. converges to zero.

Problem 2: Provide sequences $\{a_n\}$ and $\{b_n\}$ with following properties

(a) $\lim_{n\to\infty} \cos(a_n)$ exists and

(b) $\lim_{n\to\infty} \cos(b_n)$ does not exist.

How you come up with your sequence is not important. A properly written solution will state your sequence, and then demonstrate that the sequence meets the required condition(s). Clearly label each part as (a), (b), (c). Do not question parts in columns!

Solution:

(a) Consider $\{a_n\} = \{\frac{1}{n}\}$

Then

$$\lim_{n\to\infty} \cos(a_n) = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$$

$$= \cos\left(\lim_{n\to\infty} \frac{1}{n}\right)$$ b/c $\cos(x)$ is continuous on $(-\infty, \infty)$

$$= \cos(0)$$

$$= 1$$

Therefore, since $\lim_{n\to\infty} \cos(a_n)$ exists (and so the seq. converges)

(b) Consider $\{b_n\} = \{n\}$

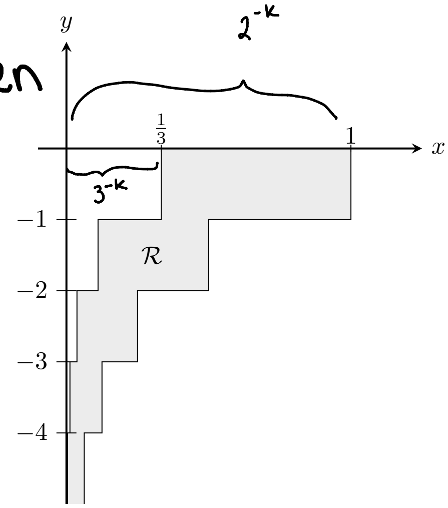
Then

$$\lim_{n\to\infty} \cos(b_n) = \lim_{n\to\infty} \cos(n)$$

$$= \cos\left(\lim_{n\to\infty} n\right)$$

Since $\lim_{n\to\infty} n = \infty$ & $\cos(x)$ oscillates between -1 & 1 on $(-\infty, \infty)$ then $\lim_{n\to\infty} \cos(b_n)$ does not exist

Problem 3: Compute the area of the infinitely long region $\mathcal{R}$ indiated in the picture. The oustide width of each step decreases according to 2^{-k}, $k = 0, 1, 2, 3, \ldots$. The inside width of each step decreases according to 3^{-k}, $k = 1, 2, 3, \ldots$.



Solution: Let A_T be region given by the rectangles of width 2^{-k} & height 1. Then the total area is given by

$$A_T = \sum_{k=0}^{\infty} 2^{-k}$$

The area of the inside region is then

$$A_I = \sum_{k=1}^{\infty} 3^{-k}$$

Then the area of R will be given by

$$A_R = A_T - A_I = \sum_{k=0}^{\infty} 2^{-k} - \sum_{k=1}^{\infty} 3^{-k}$$

We can only subtract these two areas if both series Converge.

$$A_T = \sum_{k=0}^{\infty} 2^{-k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

This is a geometric series w/ $a = 1$ & $r = \frac{1}{2}$

By the geometric series test since $|r| < 1$ then the series converges.

Since A_T is a Convergent geometric series

$$A_T = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

Problem 3: (Cont'd)

$$A_I = \sum_{k=1}^{\infty} 3^{-k} = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{-k} = \sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{k}$$

This is a geometric series w/ $a = \frac{1}{3}$ & $r = \frac{1}{3}$
By the geometric series test since $|r| < 1$
then the series converges.

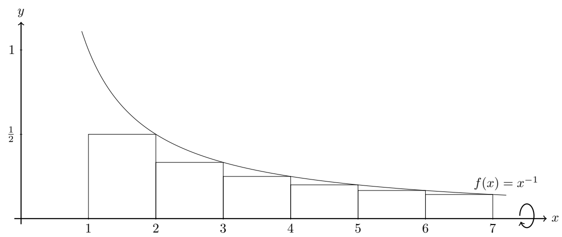
Since A_I is a convergent geometric series

$$\sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{k} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$$

Both A_T & A_I converge area of R is given by

$$A_R = A_T - A_I = 2 - \frac{1}{2} = \frac{3}{2}$$

Problem 4: The following picture shows an infinite sequence of rectangles with heights given by $\left\{\dfrac{1}{n}\right\}_{n=2}^{\infty}$ and the curve $f(x) = x^{-1}$.



You are to compare the volume of the solid obtained by revolving the rectangles about the x-axis with that of solid obtained by revolving the region under the curve f about x-axis.

- The volume of the rectangles should be expressed as an infinite series (Hint: This solid will look like a bunch of stacked discs.)
- The volume of the solid obtained by revolving the curve $f(x) = x^{-1}$ about the x-axis from $[1, \infty)$ should be computed with an appropriate integral.

What relationship do you see concerning these two volumes? What can you conclude based on this relationship?

Solution:

volume as a series:

Volume of a cylinder : $\pi r^2 \cdot h$

h : width of rectangles $\to$ Fixed @ 1

r : height of each rectangle $\to \frac{1}{n}$

$$\Rightarrow \pi\left(\tfrac{1}{2}\right)^2 + \pi\left(\tfrac{1}{3}\right)^2 + \pi\left(\tfrac{1}{4}\right)^2 + \dots = \sum_{k=2}^{\infty} \pi\left(\tfrac{1}{n}\right)^2$$

volume as an integral: Use disk method $V = \int_a^b \pi (f(x))^2 dx$

$$V = \int_1^{\infty} \pi \left(\tfrac{1}{x}\right)^2 dx = \lim_{b\to\infty} \int_1^b \pi x^{-2} dx = \pi \lim_{b\to\infty} \left[-x^{-1}\right]_1^b$$

$$= -\pi \lim_{b\to\infty} \left[b^{-1} - 1^{-1}\right]$$

$$= -\pi \left[\underbrace{\lim_{b\to\infty} \frac{1}{b}}_{=0} - \lim_{b\to\infty} 1\right] = -\pi[-1] = \pi$$

Problem 4: (cont'd)

we can easily see that the volume obtained by revolving rectangles is less than the total volume of the solid. i.e.

$$\sum_{K=2}^{\infty}\left(\frac{1}{n}\right)^2 < \pi$$