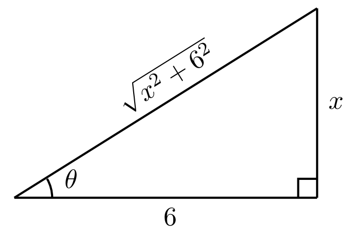
Evaluate the given integrals.

1. $\displaystyle\int \frac{x}{\sqrt{x^2+36}}\,dx$

Solution: We see that the integral has form

$$\sqrt{u^2+a^2}$$



So we let
$$x = a\tan\theta$$
$$= 6\tan\theta$$
$$\Rightarrow dx = 6\sec^2\theta\,d\theta$$

and

$$\sqrt{x^2+36} = \sqrt{(6\tan\theta)^2+6^2}$$
$$= \left[6^2(\tan^2\theta+1)\right]^{1/2}$$
$$= \left[6^2\sec^2\theta\right]^{1/2} = 6\sec\theta$$

Since
$$\sec^2\theta = 1+\tan^2\theta$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+36}}\,dx = \int \frac{6\tan\theta}{6\sec\theta}\left(6\sec^2\theta\,d\theta\right)$$
$$= \int 6\tan\theta\sec\theta\,d\theta$$
$$= 6\sec\theta + C$$

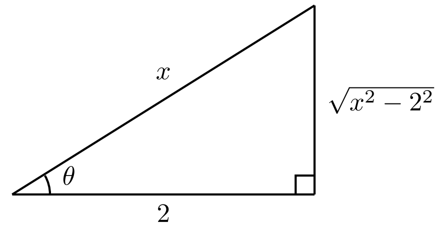
From the figure: $\sec\theta = \dfrac{\text{hyp.}}{\text{adj.}} = \dfrac{\sqrt{x^2+36}}{6}$

$$\Rightarrow \int \frac{x}{\sqrt{x^2+36}}\,dx = 6\sec\theta + C$$
$$= 6\left(\frac{\sqrt{x^2+36}}{6}\right) + C$$
$$= \sqrt{x^2+36} + C$$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$

Solution: We see that the integral has form

$$\sqrt{u^2-a^2} = \sqrt{x^2-2^2}$$



So we let

$$x = a\sec\theta$$
$$= 2\sec\theta$$
$$\Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$$

and

$$\sqrt{x^2-4} = \sqrt{(2\sec\theta)^2-4}$$
$$= \left[2^2(\sec^2\theta-1)\right]^{1/2}$$
$$= \left[2^2\tan^2\theta\right]^{1/2} = 2\tan\theta$$

Since $\sec^2\theta - 1 = \tan^2\theta$

$$\Rightarrow \int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{2\tan\theta}(2\sec\theta\tan\theta\,d\theta)$$
$$= \int \sec\theta\,d\theta$$
$$= \ln(\sec\theta+\tan\theta)+C$$

From the figure: $\sec\theta = \frac{\text{hyp.}}{\text{adj.}} = \frac{x}{2}$

$$\tan\theta = \frac{\text{opp}}{\text{adj.}} = \frac{\sqrt{x^2-2^2}}{2}$$

$$\Rightarrow \int \frac{x}{\sqrt{x^2-4}}\,dx = \ln(\sec\theta+\tan\theta)+C$$
$$= \ln\left(\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\right)+C$$
$$= \ln(x+\sqrt{x^2-4}) - \ln(2) + C$$
$$= \ln(x+\sqrt{x^2-4}) + C$$

3. $\displaystyle\int \frac{x}{\sqrt{36-x^2}}\,dx$

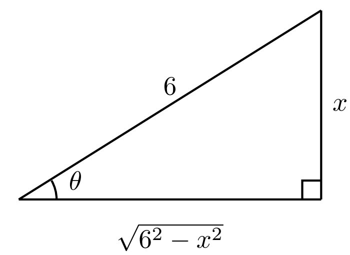
Solution: We see that the integral has form

$$\sqrt{a^2-u^2} = \sqrt{6^2-x^2}$$

So we let $x = a\sin\theta$

$= 6\sin\theta$

$\Rightarrow dx = 6\cos\theta\, d\theta$



and

$$\sqrt{36-x^2} = \sqrt{6^2-(6\sin\theta)^2}$$

$$= \left[6^2(1-\sin^2\theta)\right]^{1/2}$$

$$= \left[6^2\cos^2\theta\right]^{1/2} = 6\cos\theta$$

Since $\sin^2\theta = 1-\cos^2\theta$

$$\Rightarrow \int \frac{x}{\sqrt{36-x^2}}\,dx = \int \frac{6\sin\theta}{6\cos\theta}(6\cos\theta\, d\theta)$$

$$= 6\int \sin\theta\, d\theta$$

$$= -6\cos\theta + C$$

From the figure: $\cos\theta = \frac{\text{adj.}}{\text{hyp.}} = \frac{\sqrt{36-x^2}}{6}$

$$\Rightarrow \int \frac{x}{\sqrt{36-x^2}}\,dx = -6\cos\theta + C$$

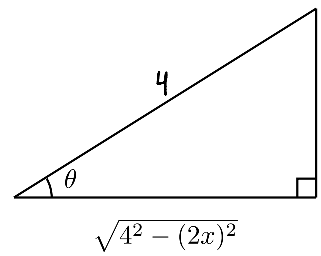
$$= -6\left(\frac{\sqrt{36-x^2}}{6}\right) + C$$

$$= -\sqrt{36-x^2} + C$$

4. $\displaystyle\int x\sqrt{16-4x^2}\,dx$

Solution: We see that the integral has form

$$\sqrt{a^2-u^2} = \sqrt{4^2-(2x)^2}$$



So we let $2x = a\sin\theta$

$$x = \frac{4}{2}\sin\theta = 2\sin\theta$$

$$\Rightarrow dx = 2\cos\theta\, d\theta$$

and

$$\sqrt{4^2-(2x)^2} = \sqrt{4^2-(2\sin\theta)^2}$$

$$= \left[4^2(1-\sin^2\theta)\right]^{1/2}$$

$$= \left[4^2\cos^2\theta\right]^{1/2} = 4\cos\theta$$

Since $\sin^2\theta = 1-\cos^2\theta$

$$\Rightarrow \int x\sqrt{4^2-(2x)^2}\,dx = \int 2\sin\theta\,(4\cos\theta)(2\cos\theta\,d\theta)$$

$$= 16\int \sin\theta\cos^2\theta$$

$$= -16\int u^2\,du$$

$$= -16\,\frac{u^3}{3} + C$$

$$= -\frac{16}{3}\cos^3\theta + C$$

Let $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$\Rightarrow -du = \sin\theta\, d\theta$

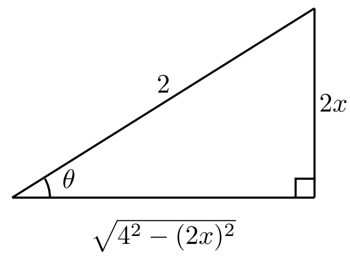
From the figure: $\cos\theta = \dfrac{\text{adj.}}{\text{hyp.}} = \dfrac{\sqrt{16-4x^2}}{4}$

$$\Rightarrow \int x\sqrt{4^2-(2x)^2}\,dx = -\frac{16}{3}\cos^3\theta + C$$

$$= -\frac{16}{3}\left(\frac{\sqrt{16-4x^2}}{4}\right)^3 + C = -\frac{2}{3}(4-x^2)^{3/2} + C$$

4. $\displaystyle\int x\sqrt{16-4x^2}\,dx$

Solution:



$$= -\frac{16}{3}\left(\frac{\sqrt{16-4x^2}}{4}\right)^3 + C$$

$$= -\frac{16}{3}\left[\frac{(4^2-4x^2)^{1/2}}{4}\right]^3$$

$$= -\frac{16}{3}\left[\frac{(4(4-x^2))^{1/2}}{4}\right]^3$$

$$= -\frac{16}{3}\,\frac{2^3(4-x^2)}{4^3}$$

$$= \frac{-4^2}{3}\,\frac{2^3(4-x^2)^{3/2}}{4^3}$$

$$= -\frac{2^3}{3}\,\frac{(4-x^2)^{3/2}}{4}$$

$$= -\frac{2\cdot\cancel{4}}{3}\,\frac{(4-x^2)^{3/2}}{\cancel{4}}$$

$$= -\frac{2}{3}(4-x^2)^{3/2}$$