# Lecture # 19: Taylor Series & Polynomials

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A power series is basically a polynomial of infinite degree.

An $n^{th}$ degree poly is

$$c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$$

A <u>power series</u> is like this, only the powers never stop going up.

$$\sum c_k x^k = c_0 + c_1x + c_2x^2 + \ldots$$

The $c_k$ are the coeff.s of the power series. This is a power series centered @ 0

We can change the <u>center</u> to $a$ by replacing $x$ by $x-a$

$$\sum_{k=0}^{\infty} c_k(x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

As with numerical series, an infinite sum of powers of $x$ can only be understood by looking @ what happens as we progressively add terms. In this sense, our "sequence of partial sums" is a sequence of polynomials.

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We have

$c_0$ - Constant

$c_0 + c_1 x$ - linear

$c_0 + c_1 x + c_2 x^2$ - quadratic

$c_0 + c_1 x + c_2 x^2 + c_3 x^3$ - Cubic

$\vdots$

etc

Another type of series that builds on this idea is the Taylor series.

## Taylor Series

Def If $f$ has derivatives of all orders @ $x=a$ then the Taylor series for the fcn $f$ @ $a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: when $a=0$, the Taylor series is also called a Maclaurin series.

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Ex. 1 Find Maclaurin series for $f(x) = \cos(x)$

$f(x) = \cos(x)$ $\quad f(0) = \cos(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = -\sin(0) = 0$

$f''(x) = -\cos(x)$ $\quad f''(0) = -\cos(0) = -1$

$f'''(x) = \sin(x)$ $\quad f'''(0) = \sin(0) = 0$

$\vdots$

$$f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \dots$$

$$\Rightarrow \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

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Question: How would one of these things arise?

Let's consider an example

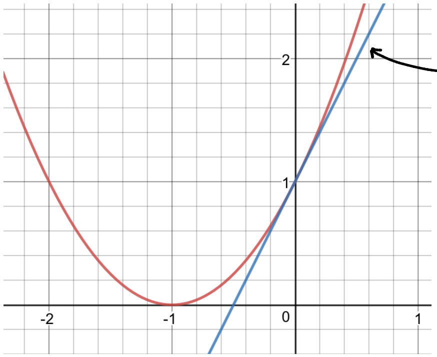
$$F(x) = x^2 + 2x + 1$$

$F'(x) = 2x + 2$

$F'(0) = 2$

$\Rightarrow L(x) = F(0) + F'(0)(x-0)$

$= 1 + 2x$




$y = 2x+1$
tangent line @ $x=0$

The blue line is the linear approx. to $F(x)$ @ $x=0$.

Is there a quadratic Approx.?

To get a quadratic approx. we need a quadratic

$$P_2(x) = C_0 + C_1 x + C_2 x^2$$

We still want $C_0 = F(0)$ & $C_1 = F'(0)$ What should $C_2$ be?

We need the 2nd derivatives to match

$$\left.\begin{aligned} F''(0) &= 2 \\ P_2''(0) &= 2C_2 \end{aligned}\right\} \Rightarrow C_2 = 1$$

So our quadratic approx is

$$P_2(x) = 1 + 2x + x^2$$

Note that we get the original Fcn again!

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Ex. (a) Find linear approx of $f(x) = \ln(x)$ @ $x = 1$

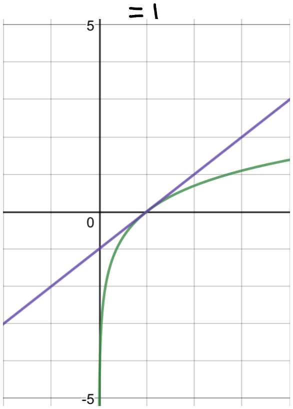
$$f'(x) = \frac{1}{x}$$

$$f(1) = \ln(1) = 0$$

$$f'(1) = \frac{1}{1} = 1$$

So $p_1(x) = f(1) + f'(1)(x-1)$

$= x - 1$



(b) Quadratic approximation?

Here is where the center matters

We are not centered @ $x = 0$ so we use $x-1$ (as in the tangent line case)

So

$$p_2(x) = f(1) + f'(1)(x-1) + c_2(x-2)^2$$

Again, we want $f''(1) = p_2''(1)$

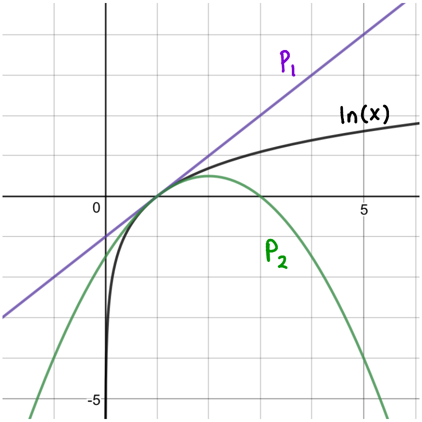
So $f''(x) = -\frac{1}{x^2}$

&

$p_2''(x) = 2c_2$

$\Rightarrow -1 = 2c_2 \Rightarrow c_2 = -\frac{1}{2}$

So $p_2(x) = (x-1) - \frac{1}{2}(x-2)^2$



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These polynomial generalizations of the tangent line are known as <u>Taylor polynomials</u>.

The idea is to pick a pt (i.e. the center)

Then the $n^{th}$ order polynomial

$$p_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$$

For this to be a Taylor Polynomial, we require

$$\left.\begin{aligned} f(a) &= p(a) \\ f'(a) &= p'(a) \\ f''(a) &= p''(a) \\ &\vdots \\ f^{(n)}(a) &= p^{(n)}(a) \end{aligned}\right\}$$

i.e. Taylor poly & $1^{st}$ $n$ derivatives @ $a$ match those of $f$ @ $a$.

We can calculate

$$p(a) = c_0 \implies c_0 = f(a)$$

$$p'(a) = c_1 \implies c_1 = f'(a)$$

$$p''(a) = 2c_2 \implies c_2 = \frac{1}{2}f''(a)$$

$$p'''(a) = 3\cdot 2\cdot c_3 \implies c_3 = \frac{1}{3!}f'''(a)$$

$$p^{(4)}(a) = 4\cdot 3\cdot 2\cdot c_4 \implies c_4 = \frac{1}{4!}f^{(4)}(a)$$

$$\vdots$$

$$p^{(n)}(a) = n!\,c_n \implies c_n = \frac{1}{n!}f^{(n)}(a)$$

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**Def** The $n^{th}$ degree Taylor Polynomial is given by

$$P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor polynomial is the $n^{th}$ partial sum of a Taylor series.

So we've seen that we can approx. a fcn using a Taylor Polynomial. How accurate can this approx. be? How many terms will we need to achieve a high accuracy?

**Theorem 6.7: Taylor's Theorem with Remainder**

Let $f$ be a function that can be differentiated $n+1$ times on an interval $I$ containing the real number $a$. Let $p_n$ be the $n$th Taylor polynomial of $f$ at $a$ and let

$$R_n(x) = f(x) - p_n(x)$$

be the $n$th remainder. Then for each $x$ in the interval $I$, there exists a real number $c$ between $a$ and $x$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

If there exists a real number $M$ such that $\left|f^{(n+1)}(x)\right| \le M$ for all $x \in I$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all $x$ in $I$.

The remainder is $R_n(x)$. Note that this is a fcn!

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Ex. Find a bound for the magnitude of the remainder for the Taylor Polynomial of $f(x) = \cos(x)$ centered @ $a=0$

All derivatives of $\cos(x)$ have form of $\pm\cos(x)$ or $\pm\sin(x)$, either way the value of the derivative is bdd by 1

$\Rightarrow |f^{(n+1)}(c)| \leq 1 = M$ for all $x$

So for our remainder we have

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1}$$

$$\leq \frac{1}{(n+1)!}|x|^{n+1}$$

For $x = 0.01$ (a value close to 0) we have for $n=4$

$$|R_4(0.01)| = |\cos(0.01) - P_4(0.01)|$$

$$\leq \frac{|0.01|^5}{5!} = 8.33 \times 10^{-13}$$

$\Rightarrow \cos(0.01)$ & $P_4(0.01)$ are very close.

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Ex. 1 (cont'd)

What if we choose $x = 1$?

$$|R_n(1)| = |\cos(1) - P_n(1)| \leq \frac{|1|^{n+1}}{(n+1)!}$$

Want

$$\frac{1}{(n+1)!} \leq 1 \times 10^{-12}$$

$$\Rightarrow \quad (n+1)! \geq 1 \times 10^{12}$$

By testing values of $n$ we have

$$15! = 1,307,674,368,000$$

So for $n = 14$ $|R_n(1)|$ will be less than the desired level of accuracy

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## Derivatives & Integrals w/ Taylor Series

Ex.

a. Express $\int e^{-x^2}\,dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2}\,dx$ to within an error of $0.01$.

### Solution

a. The Maclaurin series for $e^{-x^2}$ is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int e^{-x^2}\,dx &= \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots\right) dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots. \end{aligned}$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots.$$

The sum of the first four terms is approximately $0.74$. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$.