1. Find the arc length of the curve $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ on the interval $[1, 2]$.

Solution:

Arc length of a curve $f(x)$ on interval $[a,b]$ is

$$L = \int_a^b \sqrt{1 + f'(x)}\, dx$$

For $f(x) = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$

We have $f'(x) = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$

$$L = \int_1^2 \sqrt{1 + \left(\tfrac{1}{2}x^3 - \tfrac{1}{2}x^{-3}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{1}{2} + \frac{1}{4}x^{-6} + \frac{1}{4}x^6\right) dx$$

$$= \int_1^2 \left[\frac{(1+x^6)^2}{4x^6}\right]^{1/2} dx = \int_1^2 \left[\left(\frac{(1+x^6)}{2x^3}\right)^2\right]^{1/2} dx$$

$$= \int_1^2 \frac{1+x^6}{2x^3}\, dx$$

$$= \frac{1}{2}\int_1^2 (x^{-3} + x^3)\, dx$$

$$= \frac{1}{2}\left[\frac{x^{-2}}{-2} + \frac{x^4}{4}\right]_1^2$$

$$= \frac{33}{16}$$

2. Find the arc length of the curve $y = \frac{1}{3}(x^2+2)^{3/2}$ on the interval $[0,1]$

Solution:

Arc length of a curve $f(x)$ on interval $[a,b]$ is

$$L = \int_a^b \sqrt{1 + f'(x)}\, dx$$

For $f(x) = \frac{1}{3}(x^2+2)^{3/2}$

We have $f'(x) = x\sqrt{2+x^2}$ (using Mathematica)

$$L = \int_1^2 \left[1 + (x\sqrt{2+x^2})^2\right]^{1/2} dx$$

$$= \int_1^2 \left[1 + x^2(2+x^2)\right]^{1/2} dx$$

$$= \int_1^2 \left[(1 + 2x^2 + x^4)\right]^{1/2} dx$$

$$= \int_1^2 \left[(1+x^2)^2\right]^{1/2} dx$$

$$= \int_1^2 (1+x^2)\, dx$$

$$= \left[x + \frac{x^3}{3}\right]_1^2$$

$$= \frac{10}{3}$$

3. Find the surface area of the solid generated by the curve $y = \frac{x^2}{4}$ revolved around the y-axis on the interval $2 \leq x \leq 4$.

Solution: Surface area for solid revolved around the y axis is

$$S = \int_c^d 2\pi g(y) \left[1 + (g'(y))^2\right]^{1/2} dy$$

Given $y = \frac{x^2}{4} \Rightarrow 4y = x^2 \Rightarrow x = \sqrt{4y} = 2\sqrt{y}$

So $x'(y) = y^{-1/2}$

Bounds will be given by

$$\begin{matrix} x = 4 \\ x = 2 \end{matrix} \Longrightarrow \begin{matrix} y = \frac{(4)^2}{4} = \frac{16}{4} = 4 \\ y = \frac{(2)^2}{4} = 1 \end{matrix}$$

Then surface area is

$$S = \int_1^4 2\pi (2y^{1/2}) (1 + (y^{-1/2})^2)^{1/2} dy$$

$$= 4\pi \int_1^4 y^{1/2} (1 + y^{-1})^{1/2} dy$$

$$= \int_1^4 (y+1)^{1/2} dy$$

Using u-sub: Let $u = y+1$

$\Rightarrow du = dy$

W/ Bounds

$$\begin{matrix} y = 4 \\ y = 1 \end{matrix} \Rightarrow \begin{matrix} u = 5 \\ u = 2 \end{matrix}$$

$$\Rightarrow S = \int_2^5 u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_2^5$$

$$= 4\pi \left(\frac{2}{3}\left[u^{3/2}\right]_2^5\right) = \frac{8\pi}{3}\left(5\sqrt{5} - 2\sqrt{2}\right)$$

4. Find the surface area of the solid generated by the curve $y = \dfrac{x^4}{8} + \dfrac{1}{4x^2}$ revolved around the x-axis on the interval $[1, 2]$.

Solution: Surface area for solid revolved around x axis is

$$S = 2\pi \int_a^b f(x) \left[1 + (f'(x))^2\right]^{1/2} dx$$

Given $f(x) = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$

$$f'(x) = -\frac{1}{2}x^{-3} + \frac{1}{2}x^3$$

Surface area is

$$S = 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right) \underbrace{\left[1 + \left(-\frac{1}{2}x^{-3} + \frac{1}{2}x^3\right)^2\right]^{1/2}}_{} dx$$

$$= \frac{1+x^6}{2x^3} \quad \text{from \#1}$$

$$= \frac{1}{2}\left(x^{-3} + x^3\right)$$

$$= 2\pi \int_1^2 \left(\frac{1}{8}x^4 + \frac{1}{4}x^{-2}\right)\left[\frac{1}{2}\left(x^{-3} + x^3\right)\right] dx$$

$$= \pi \int_1^2 \left[\frac{1}{8}x^{-5} + \frac{3}{16}x + \frac{1}{16}x^7\right] dx$$

$$= \pi \left[-\frac{1}{32}x^{-4} + \frac{3}{32}x^2 + \frac{1}{128}x^8\right]_1^2$$

$$= \frac{\pi}{16}\left[-\frac{1}{2}x^{-4} + \frac{3}{2}x^2 + \frac{x^8}{8}\right]_1^2$$

$$= \frac{1179\pi}{256}$$