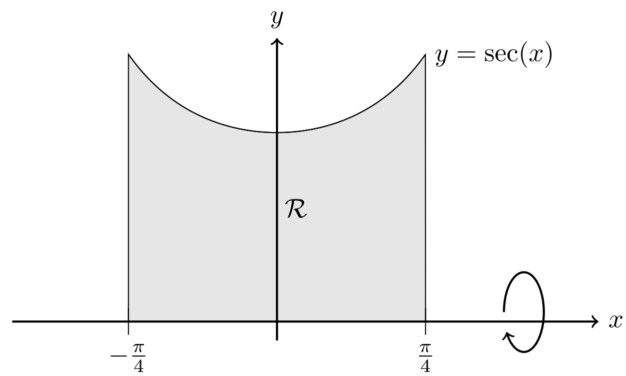
Problem 1: Consider the region $\mathcal{R}$ as given in the diagram. Compute the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.

Solution:



Use washer method

$$V = \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] dx$$

So

$$V = \pi \int_{-\frac{\pi}{4}}^{\pi/4} \sec^2(x)\, dx$$

Note:
$\int \sec^2 x \, dx = \tan(x) + C$

$$= \pi \left[\tan(x) \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \pi \left[\tan\left(\tfrac{\pi}{4}\right) - \tan\left(-\tfrac{\pi}{4}\right) \right]$$

$$= \pi \left[\tan\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right) \right]$$

$$= \pi \left(2\tan\left(\tfrac{\pi}{4}\right) \right)$$

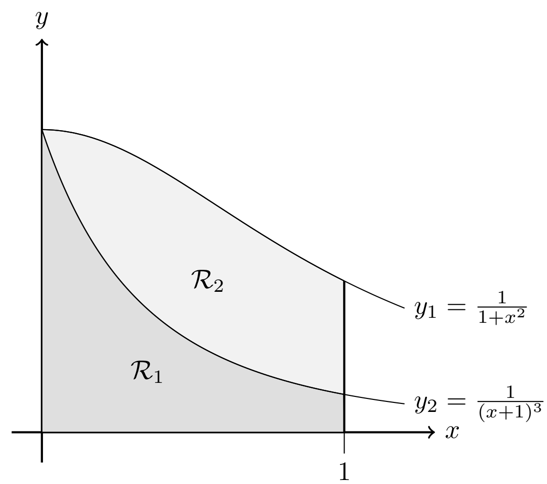
Note:
$$\tan\left(\tfrac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$= \pi \left[2(1) \right]$$

$$= 2\pi$$

Problem 2: Which of the shaded regions $\mathcal{R}_1$ or $\mathcal{R}_2$ has the larger area? The upper curve is $y_1 = \dfrac{1}{1+x^2}$, the lower curve is $y_2 = \dfrac{1}{(x+1)^3}$, and both regions go from $x = 0$ to $x = 1$.

Solution:



Area btwn 2 Curves

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Area of R_1:

w/ $f(x) = \dfrac{1}{1+x^2}$, $g(x) = 0$

$$\Rightarrow A_1 = \int_0^1 \frac{1}{(x+1)^3}\,dx = \int_1^2 \frac{1}{u^3}\,du = \int_1^2 u^{-3}\,du =$$

u-sub:

$u = x+1$, $u(1) = 1+1 = 2$

$du = dx$, $u(0) = 0+1 = 1$

$$= -\frac{1}{2} u^{-2}\Big|_1^2$$

$$= -\frac{1}{2}\left[\frac{1}{(2)^2} - \frac{1}{(1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{3}{4}\right] = \frac{3}{8} \approx 0.375$$

Problem 2: (cont'd)

Area of R_2:

$$A_1 = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_0^1 \left[\frac{1}{1+x^2} - \frac{1}{(x+1)^3}\right] dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{1}{(x+1)^3}\,dx$$

Note: $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$$\Rightarrow A_2 = \arctan(x)\Big|_0^1 - \underset{\text{(from } A_1\text{)}}{\frac{3}{8}}$$

$$= \arctan(1) - \arctan(0) - \frac{3}{8}$$

Note: $\arctan(1) = \theta \Rightarrow \tan\theta = 1$
$\Rightarrow \theta = \frac{\pi}{4}$

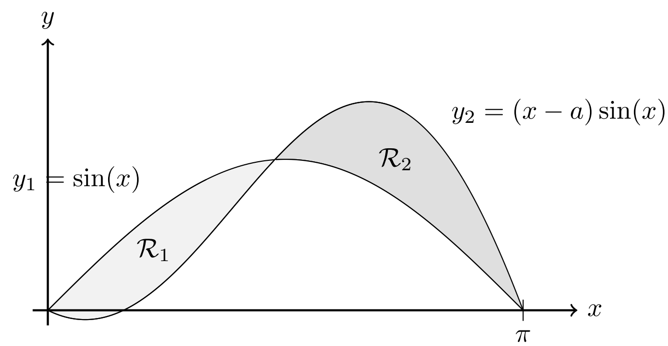
$$= \frac{\pi}{4} - 0 - \frac{3}{8}$$

$$= \frac{2\pi - 3}{8} \approx 0.410$$

So we see that Area R_2 > Area R_1

Problem 3: Find the value of the parameter a between 0 and 2 for which the areas of the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are equal. The two curves involved are $y_1 = \sin(x)$ and $y_2 = (x-a)\sin(x)$.

Solution:



Area btwn 2 Curves

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Find pt of intersection

$$\sin(x) = (x-a)\sin(x)$$

$$\Rightarrow \sin(x) - (x-a)\sin(x) = 0$$

$$\sin(x)(1-(x-a)) = 0$$

$$\sin(x)(1-x+a) = 0$$

$\sin(x) = 0$ $\quad$ $1 - x + a = 0$

$x = 0$ $\quad$ $x = a+1$

$x = \pi$

$\Rightarrow$ Pts of intersection: $x = 0,\ a+1,\ \pi$

$\Rightarrow$ R_1 exists on $[0, a+1]$

R_2 exists on $[a+1, \pi]$

Area of R_1:

$$A_1 = \int_0^{a+1} [\sin(x) - (x-a)\sin(x)]\,dx$$

$$= \int_0^{a+1} \sin(x)[1-x+a]\,dx$$

Problem 3: (cont'd)

Note: $\int \sin(x)[1-x+a]\,dx$

use int. by parts: $\int u\,dv = uv - \int v\,du$

$$u = 1-x+a \qquad v = -\cos(x)$$
$$du = -dx \qquad dv = \sin(x)\,dx$$

$$\int \sin(x)[1-x+a]\,dx = -(1-x+a)\cos(x) + \int \cos(x)\,dx$$
$$= -(1-x+a)\cos(x) - \sin(x)$$

$$\Rightarrow A_1 = \int_0^{a+1} \sin(x)[1-x+a]\,dx$$

$$= \Big[-(1-x+a)\cos(x) - \sin(x)\Big]_0^{a+1}$$

$$= [-(1-(a+1)+a)\cos(a+1) - \sin(a+1)$$
$$-[-(1-(0)+a)\cos(0) - \sin(0)]$$

$$= -(1-a-1+a)\cos(a+1) - \sin(a+1) + (1+a)$$

$$= -\sin(a+1) + 1 + a$$

$$= 1 + a - \sin(a+1)$$

Area of R_2:

$$A_2 = \int_{a+1}^{\pi} [(x-a)\sin(x) - \sin(x)]\,dx$$

$$= \int_{a+1}^{\pi} [x-a-1]\sin(x)\,dx$$

Problem 3: (cont'd)

Note: $\int [x-a-1]\sin(x)\,dx$

Use int. by parts: $\int u\,dv = uv - \int v\,du$

$u = x-a-1 \qquad v = -\cos(x)$

$du = dx \qquad dv = \sin(x)\,dx$

$$\int [x-a-1]\sin(x)\,dx = -(x-a-1)\cos(x) + \int \cos(x)\,dx$$
$$= -(x-a-1)\cos(x) + \sin(x)dx$$

$$\Rightarrow A_2 = \int_{a+1}^{\pi} [x-a-1]\sin(x)\,dx$$
$$= \Big[-(x-a-1)\cos(x) + \sin(x)\Big]_{a+1}^{\pi}$$
$$= \Big[-(\pi-a-1)\underbrace{\cos(\pi)}_{=-1} + \underbrace{\sin(\pi)}_{=0}\Big]$$
$$- \Big[-(\underbrace{(a+1)-a-1}_{=0})\cos(a+1) + \sin(a+1)\Big]$$
$$= \pi - a - 1 - \sin(a+1)$$

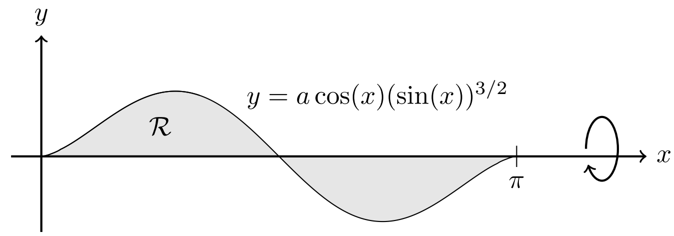
Now set $A_1 = A_2$ & solve for a

$$1 + a - \sin(a+1) = \pi - a - 1 - \sin(a+1)$$
$$1 + a = \pi - a - 1$$
$$2a = \pi - 2$$
$$\Rightarrow a = \frac{\pi - 2}{2} = \frac{\pi}{2} - 1$$

So value of a that gives area of R_1 equal to area of R_2 is $a = \frac{\pi}{2} - 1$

Problem 4: Find the range of values of the parameter $a > 0$ for which the volume of the solid generated by revolving the gray region $\mathcal{R}$ about the x-axis is between $\frac{\pi}{15}$ and 15π.

Solution:



Volume of Region about x-axis given by the disk method

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

Note that

$$y = a\cos(x)(\sin(x))^{3/2}$$

Crosses x-axis @ $a\cos(x)(\sin(x))^{3/2} = 0$

$$\Rightarrow \cos(x) = 0 \quad @ \quad x = \frac{\pi}{2}$$

$$\sin(x) = 0 \quad @ \quad x = 0, \pi$$

on $[0, \frac{\pi}{2}]$ $y \geq$ x-axis & on $[\frac{\pi}{2}, 0]$ $y \leq$ x-axis

Let V_1 be volume on $[0, \frac{\pi}{2}]$

$$V_1 = \int_0^{\pi/2} \pi\left[(a\cos(x)\sin(x)^{3/2})^2 - 0^2\right]dx$$

$$= \pi\int_0^{\pi/2} a^2\cos^2(x)\sin^3(x)\,dx$$

$$= \pi a^2\int_0^{\pi/2}\cos^2(x)\underbrace{\sin^2(x)}_{=1-\cos^2(x)\text{ by Pyth. Id.}}\sin(x)\,dx$$

$$= \pi a^2\int_0^{\pi/2}\cos^2(x)\left[1-\cos^2(x)\right]\sin(x)\,dx$$

Problem 4: (cont'd)

$$V_1 = \pi a^2 \int_0^{\pi/2} \cos^2(x)[1-\cos^2(x)]\sin(x)\,dx$$

Note:

$$\int_a^b \cos^2(x)[1-\cos^2(x)]\sin(x)\,dx = -\int_{u(a)}^{u(b)} [u^2-u^4]\,du$$

$u = \cos(x)$

$du = -\sin(x)dx$

$\Rightarrow -du = \sin(x)dx$

$$= -\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{u(a)}^{u(b)}$$

$$= -\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{\cos(a)}^{\cos(b)}$$

So

$$V_1 = -\pi a^2 \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{\cos(0)}^{\cos(\pi/2)}$$

$$= -\pi a^2 \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_1^0 = -\pi a^2\left[(0) - \left(\frac{(1)^3}{3} - \frac{(1)^5}{5}\right)\right]$$

$$= \pi a^2\left[\frac{5}{15} - \frac{3}{15}\right] = \frac{2\pi}{15}a^2$$

Let V_2 be volume on $[\frac{\pi}{2}, \pi]$

$$V = \int_{\frac{\pi}{2}}^{\pi} \pi\left[0^2 - \left(a\cos(x)\sin(x)^{3/2}\right)^2\right]dx$$

$$= -a^2\pi \int_{\frac{\pi}{2}}^{\pi} \cos^2(x)\sin^3(x)\,dx$$

$$= -a^2\pi\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_{\cos(\pi/2)}^{\cos(\pi)} = -a^2\pi\left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^{-1}$$

$$= -a^2\pi\left[\left[\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right] - 0\right] = -a^2\pi^2\left[-\frac{5}{15} - \frac{3}{15}\right]$$

$$= -a^2\pi^2\left[-\frac{2}{15}\right] = \frac{2\pi}{15}a^2$$

Problem 4: (cont'd)

So $V = V_1 + V_2$

$$= \frac{2\pi}{15}a^2 + \frac{2\pi}{15}a^2 = \frac{4\pi}{15}a^2$$

When $V = \frac{\pi}{15}$: $\frac{4\cancel{\pi}}{\cancel{15}}a^2 = \frac{\cancel{\pi}}{\cancel{15}} \Rightarrow 4a^2 = 1$

$$\Rightarrow a^2 = \frac{1}{4}$$

$$a = \frac{1}{2}$$

When $V = 15\pi$: $\frac{4\cancel{\pi}}{15}a^2 = 15\cancel{\pi} \Rightarrow \frac{4}{15}a^2 = 15$

$$\Rightarrow 4a^2 = 15^2$$

$$a^2 = \frac{15^2}{4^2}$$

$$\Rightarrow a = \sqrt{\frac{15^2}{4^2}} = \frac{15}{2}$$

So for $\frac{\pi}{15} < V < 15\pi$

we must have $\frac{1}{2} < a < \frac{15}{2}$