Lecture # 17: Ratio & Root Tests   Date: Mon. 11/19/18

Ratio Test

Imagine we have a series like $\sum_{k=1}^{\infty} a_k$ & we observe that the ratio of successive terms $\frac{a_{k+1}}{a_k}$ is getting close to a number r.

This means the series

$$\sum a_k = a_1 + a_2 + a_3 + \ldots\ldots$$

"looks like"

$$a_1 + a_2 + a_3 + \ldots + \underbrace{ar^N + ar^{N+1} + ar^{N+2}}_{\text{in the tail}} \ldots$$

In other words, the series looks geometric in the tail.

This is the basis behind the Ratio test

Thm (Ratio Test)

Let $\sum a_k$ be an infinite series with positive terms and let

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

1) If $0 \le r < 1$, the series Converges
2) If $r > 1$, the series diverges
3) If $r = 1$, the test is inconclusive
   i.e. you need to use a different test!

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Factorials

"Recall":

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5\cdot 4\cdot 3! = 5\cdot 4!$$

$$4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3!$$

$$3! = 3\cdot 2\cdot 1 = 3\cdot 2!$$

$$2! = 2\cdot 1$$

$$1! = 1$$

i.e. whenever we want to stop expanding we can account for the remaining terms by using the factorial of the last term. (i.e. stick a ! on the end)

For general K

$$K! = K\cdot(K-1)(K-2)(K-3)(K-4)\dots$$
$$= K(K-1)!$$

&

$$(K+1)! = (K+1)(K+1-1)(K-1)(K-2)\dots$$
$$= (K+1)(K)(K-1)(K-2)(K-3)\dots$$
$$= (K+1)K!$$

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Ex. 1 $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

To apply Ratio Test:

$$a_k = \frac{2^k}{k!}, \qquad a_{k+1} = \frac{2^{k+1}}{(k+1)!}$$

$$\frac{a_{k+1}}{a_k} = \frac{2^{k+1}}{(k+1)!} \div \frac{2^k}{k!}$$

$$= \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k}$$

$$= \frac{2^{k+1} \cdot \cancel{k!}}{2^k (k+1)\cancel{k!}} = \frac{2^{k+1}}{2^k(k+1)} = \frac{2 \cdot \cancel{2^k}}{\cancel{2^k}(k+1)}$$

$$= \frac{2}{k+1}$$

So

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2}{k+1} = 0$$

By the ratio test

Since $0 \le r < 1$ then

the series converges

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Root Test

Another test that helps with tricky series like $\sum_{k=1}^{\infty}\left(\frac{4k^2-3}{7k^2+6}\right)^k$ is the <u>Root Test</u>

This test works on the same principle as the ratio test.
i.e. the tail looks geometric

$$a_1 + a_2 + a_3 + \dots + \underbrace{ar^N + ar^{N+1} + ar^{N+2}}_{\text{in the tail}} \dots$$

While the ratio test looks at $\frac{a_{k+1}}{a_k}$ to get r

we can also get r by taking the N^{th} root of r^N

<u>Thm</u> (Root Test)

Let $\sum a_k$ be an infinite series with nonnegative terms & let

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} (a_k)^{1/k}$$

1) If $0 \le \rho < 1$, the series <u>Converges</u>
2) If $\rho > 1$, the series <u>diverges</u>
3) If $\rho = 1$, the test is inconclusive
i.e. you need to use a different test!

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Ex. 2 $\displaystyle\sum_{k=1}^{\infty}\left(\frac{4k^2-3}{7k^2+6}\right)^k$

To apply root test we need to find

$$\rho = \lim_{k\to\infty} \sqrt[k]{a_k}$$

$$\Rightarrow \sqrt[k]{a_k} = \left[\left(\frac{4k^2-3}{7k^2+6}\right)^k\right]^{\frac{1}{k}} = \frac{4k^2-3}{7k^2+6}$$

$$= \frac{\cancel{k^2}\left(4-\frac{3}{k^2}\right)}{\cancel{k^2}\left(7+\frac{6}{k^2}\right)} = \frac{4-\frac{3}{k^2}}{7+\frac{6}{k^2}}$$

So,

$$\lim_{k\to\infty}\sqrt[k]{a_k} = \lim_{k\to\infty}\frac{4-\frac{3}{k^2}}{7+\frac{6}{k^2}}$$

$$= \frac{\lim_{k\to\infty} 4 - 3\lim_{k\to\infty}\frac{1}{k^2}}{\lim_{k\to\infty} 7 + 6\lim_{k\to\infty}\frac{1}{k^2}} = \frac{4}{7}$$

By the Root Test

Since $\rho = \frac{4}{7}$ & $0 \le \rho < 1$

the series <u>Converges</u>.

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Absolute & Conditional Convergence

Recall:

Harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

but

Alt. Harmonic Series: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Converges

Consider

$\sum \frac{1}{k^2}$ Converges : p-series w/ $p = 2 > 1$

and

$$\sum \frac{(-1)^{k+1}}{k^2} = \sum (-1)^{k+1} \underbrace{\left(\frac{1}{k^2}\right)}_{a_k}$$

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^2} = 0 \Rightarrow$ Converges by Alt. Series test

For Harmonic Series the alt. signs had an effect on Convergence.

For the second example alt. signs had <u>no</u> effect on Convergence.

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Def (Absolute Convergence)

When both

$$\sum (-1)^{k+1} a_k \quad \& \quad \sum a_k$$

converge, the series converges absolutely.

Def (Conditional Convergence)

When $\sum a_k$ diverges

but $\sum (-1)^{k+1} a_k$ converges

the series converges conditionally

This is also true for general series

If $\sum |a_k|$ converges, then $\sum a_k$ converges absolutely

If $\sum |a_k|$ diverges but $\sum a_k$ converges, then $\sum a_k$ converges conditionally.