Lecture # 14: Sequences

Date: Wed. 11/7/18

Def a sequence is an ordered list of numbers

$$\{a_1, a_2, a_3, a_4, \ldots, a_n\}$$

Which can also be written as

$\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$, n is an integer (i.e. $n \in \mathbb{Z}$) or $n \in \mathbb{N}$

Note: In general, n will start @ 1, but we it is also common to start @ 0 & it is possible to start @ neg. values

Each number in the sequence is called a term

a generic term is referred to as the n^{th} term, a_n

Generally, sequences are described with a formula for this n^{th} term

Ex. 1 Write 1st 4 terms of $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{2^n}$

$$\{a_1, a_2, a_3, a_4\}$$

$$\Rightarrow \left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\right\}$$

Ex. 2 Repeat for $a_n = \frac{(-1)^n \cdot n}{n^2+1}$

$$a_1 = \frac{(-1)^1 \cdot 1}{(1)^2+1} = \frac{-1}{2}$$

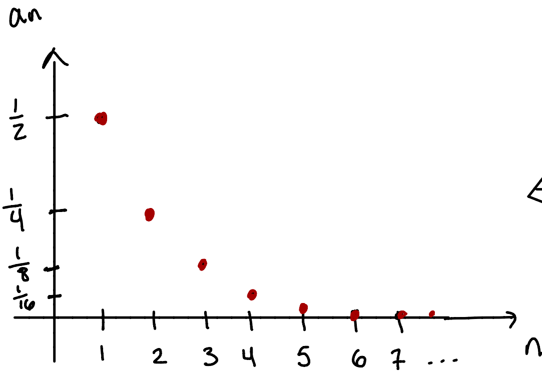
$$a_2 = \frac{(-1)^2 \cdot 2}{(2)^2+1} = \frac{2}{5}$$

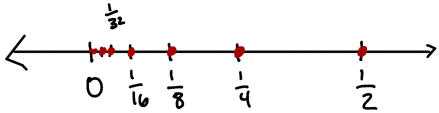
$$a_3 = \frac{(-1)^3 \cdot 3}{(3)^2+1} = \frac{-3}{10}$$

$$a_4 = \frac{(-1)^4 \cdot 4}{(4)^2+1} = \frac{4}{17}$$

We can also rep. these values on a number line or as a graph

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \right\}$$





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There are some sequences that can't be described with an explicit formula

Fibonacci Seq:

$f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$

$\Rightarrow \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

Def A sequence $\{a_n\}$ has the limit L written as:

$$\lim_{n \to \infty} a_n = L$$

if we can take terms a_n as close to L as we want by taking n sufficiently large.

If $\lim\limits_{n \to \infty} a_n$ exists, then the sequence $\{a_n\}_{n=1}^{\infty}$ Converges.

Otherwise, it diverges.

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Ex. 3 Determine if seq. $\{a_n\}_{n=1}^{\infty} = \{3 + (-1)^n\}$ converges or diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (3 + (-1)^n)$$

$$= \lim_{n\to\infty} 3 + \lim_{n\to\infty} (-1)^n$$

$$= 3 + \ ?$$

This limit DNE $\Rightarrow$ seq. diverges

Ex. 4 Determine if seq. $\{b_n\}_{n=1}^{\infty} = \left\{\frac{n}{1-2n}\right\}$ converges or diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{1-2n} \overset{H}{=} \lim_{n\to\infty} \frac{1}{-2} = -\frac{1}{2}$$

or

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(\frac{1}{n} - 2\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n} - 2}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{1}{n} - \lim_{n\to\infty} 2}$$

$$= \frac{1}{-2} = -\frac{1}{2}$$

Limit exists $\Rightarrow \{a_n\}$ converges

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Recall: limits do not exist when

- They oscillate
- Approach $\pm\infty$
- When limit from left does not equal limit from the right i.e.

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

<u>Thm</u> If $\lim\limits_{n\to\infty} f(x) = L$ & $f(n) = a_n$, $n \in \mathbb{Z}$

then $\lim\limits_{n\to\infty} a_n = L$

<u>Thm</u> (squeeze thm for sequences)

If $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} b_n$

& there exists an integer N s.t. $a_n \leq c_n \leq b_n$

for all $n > N$, then $\lim\limits_{n\to\infty} c_n = L$

Ex. 5 $\{c_n\} = \left\{ (-1)^n \frac{1}{n!} \right\}$

Determine convergence by applying the squeeze thm for seq.

Recall: $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$

n factorial

Two possible choices:

$a_n = -\frac{1}{2^n}$ & $b_n = \frac{1}{2^n}$

For $n \geq 4$:

$$-\frac{1}{2^n} \leq (-1)^n \frac{1}{n!} \leq \frac{1}{2^n}$$

So since

$$\lim_{n \to \infty} -\frac{1}{2^n} = \lim_{n \to \infty} \frac{1}{2^n} = 0$$

then

$$\lim_{n \to \infty} (-1)^n \frac{1}{n!} = 0$$

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Thm (Absolute Value Thm)

For the sequence $\{a_n\}$, if

$$\lim_{n\to\infty} |a_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} a_n = 0$$

Additional Terminology:

A seq. $\{a_n\}$ is

- increasing if $a_{n+1} > a_n$

  $\{1, 2, 3, 4, \ldots\}$

- Decreasing if $a_{n+1} < a_n$

  $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\}$

- non increasing if $a_{n+1} \leq a_n$

  $\{0, -1, -1, -2, -2, -3, -3, \ldots\}$

- nondecreasing if $a_{n+1} \geq a_n$

  $\{1, 1, 2, 2, 3, 3, \ldots\}$

Def (monotonic seq.)

A sequence $\{a_n\}$ is <u>monotonic</u> if its terms are nonincreasing or nondecreasing

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Def (Bounded Seq.)

- A sequence $\{a_n\}$ is bounded above if there is a number M s.t.

  $$a_n \leq M \qquad \forall n \geq 1$$

  "For all"

- A seq. $\{a_n\}$ is bounded below if there is a number m s.t.

  $= \exists$ "there exists"

  $$m \leq a_n \qquad \forall n \geq 1$$

If $\{a_n\}$ is bounded above & below then it is a bounded sequence

$\exists!$ "there exists a unique"

Thm Every bounded monotonic sequence converges.

Def (Geometric Sequence)

Let r be a real number then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \text{ i.e. } -1<r<1 \\ 1 & \text{if } r = 1 \\ \text{DNE} & \text{otherwise} \end{cases}$$

The sequence $\{r^n\}$ is known as a Geometric Sequence.

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Ex. 6 Determine convergence of the following:

a) $\{0.2^n\}$

Has form $\{r^n\}$ where $r = 0.2$

Since $|r| = |0.2| < 1 \Rightarrow$ seq. converges since

$$\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0.2^n = 0$$

by def. of geometric seq.

b) $\{1.0001^n\}$

Has form $\{r^n\}$ (geometric seq.)

w/ $|r| = |1.0001| > 1 \Rightarrow$ diverges