Lecture # 05: Arc Length & Surface Area

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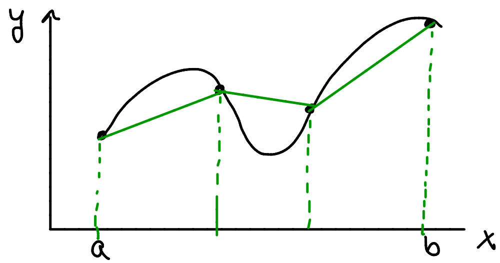
Length of Curves

Basic idea:

We can approx the length of the curve using a Polygonal path i.e one made up of straight line segments.



Because we know how to find the length of a straight line, this gives us a way of breaking the curve into pieces — each of which we know how to handle — & then sum the contributions.

This is just another variation of the approach we have already been using.

<u>Question</u> What would make our polygonal approx better?

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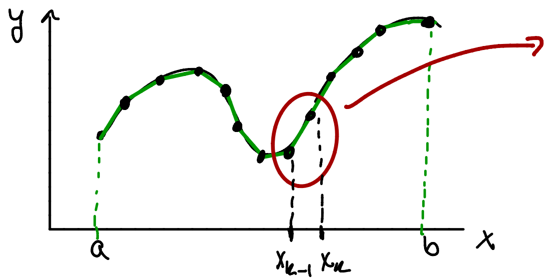
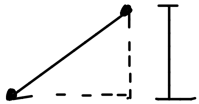
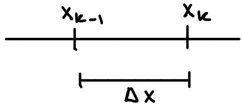




$\Delta y = y_2 - y_1$

Pts on each interval:

$(x_{k-1}, f(x_{k-1})), (x_k, f(x_k))$



$$\Longrightarrow \text{Distance} = \sqrt{(\Delta y)^2 + (\Delta x)^2}$$

$$= \sqrt{(f(x_k) - f(x_{k-1}))^2 + (x_k - x_{k-1})^2}$$

Factoring out a Δx:

$$\text{Distance} = \Delta x \sqrt{\left(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)^2 + 1}$$

If we imagine (like slices or shells) that these line segments are infinitesimally small, then

$$\frac{\Delta y}{\Delta x} \to f'(x)$$

$$\Delta x \longrightarrow dx \qquad \& \qquad \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \to f'(x)$$

So the length of an infinitesimally line segment is

$$\sqrt{1 + (f'(x))^2}\, dx$$

To get the total length, we sum (or integrate) all these contributions.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

Ex. 1) Find the length of the curve

$$f(x) = 2e^x + \frac{1}{8}e^{-x}$$

on the interval $[0, \ln(2)]$

Soln We need to calculate $f'(x)$

$$f'(x) = 2e^x - \frac{1}{8}e^{-x}$$

We also need $a = 0$ & $b = \ln(2)$

This yields the integral

$$L = \int_0^{\ln(2)} \sqrt{1 + (2e^x - \frac{1}{8}e^{-x})^2}\, dx$$

Ex 1 (cont'd)

Note that

$$1 + \left(2e^{x} - \frac{1}{8}e^{-x}\right)^2 = 1 + \left(4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}\right)$$

$$= 4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}$$

This is a perfect square

$$= \left(2e^{x} + \frac{1}{8}e^{-x}\right)^2$$

Putting this back in we have

$$L = \int_0^{\ln(2)} \sqrt{1 + \left(2e^{x} - \frac{1}{8}e^{-x}\right)^2}\, dx$$

$$= \int_0^{\ln(2)} \sqrt{\left(2e^{x} + \frac{1}{8}e^{-x}\right)^2}\, dx$$

$$= \int_0^{\ln(2)} \left(2e^{x} + \frac{1}{8}e^{-x}\right) dx$$

$$= \left[2e^{x} - \frac{1}{8}e^{-x}\right]_0^{\ln(2)}$$

$$= \frac{33}{16}$$

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We can also change perspectives & calculate lengths wrt y instead of x.

If $x = g(y)$, then

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

gives lengths of the curve $x = g(y)$ from $y = c$ to $y = d$.

Ex. 2) Find the arc length of the curve

$$y = \ln(x - \sqrt{x^2 - 1}) \quad \text{for } 1 \leq x \leq 2$$

Soln Lets try this wrt x first

Since $$f'(x) = \frac{1 - x(x^2-1)^{-1/2}}{x - (x^2-1)^{1/2}}$$

then

$$L = \int_1^{\sqrt{2}} \left[1 + \left(\frac{1 - x(x^2-1)^{-1/2}}{x - (x^2-1)^{1/2}}\right)^2\right]^{1/2} dx$$

Gross! This may simplify further but lets see what this looks like wrt y

Ex. 2 (cont'd)

using y instead: we need $x = g(y)$

$$y = \ln\left(x - (x^2-1)^{1/2}\right)$$

$$e^y = x - (x^2-1)^{1/2}$$

$$(x^2-1)^{1/2} = x - e^y$$

$$x^2 - 1 = (x - e^y)^2$$

$$x^2 - 1 = x^2 - 2xe^y + e^{2y}$$

$$2xe^y = 1 + e^{2y}$$

$$x = \tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^y$$

Now we need $g'(y) = -\frac{1}{2}e^{-y} + \frac{1}{2}e^y$

and $c = f(1) = 0$

$d = f(\sqrt{2}) = \ln(\sqrt{2} - 1)$

So

$$L = \int_0^{\ln(\sqrt{2}-1)} \left(1 + \left(-\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^y\right)^2\right)^{1/2} dy$$

Note:

$$1 + \left(-\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^y\right)^2 = 1 + \tfrac{1}{4}e^{-2y} - \tfrac{1}{2} + \tfrac{1}{4}e^{2y}$$

$$= \tfrac{1}{4}e^{-2y} + \tfrac{1}{2} + \tfrac{1}{4}e^{2y}$$

$$= \left(\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^y\right)^2$$

Ex. 2 (cont'd)

So

$$L = \int_0^{\ln(\sqrt{2}-1)} \left(1 + \left(-\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^{y}\right)^2\right)^{1/2} dy$$

$$= \int_0^{\ln(\sqrt{2}-1)} \left(\left(\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^{y}\right)^2\right)^{1/2} dy$$

$$= \int_0^{\ln(\sqrt{2}-1)} \left(\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^{y}\right) dy$$

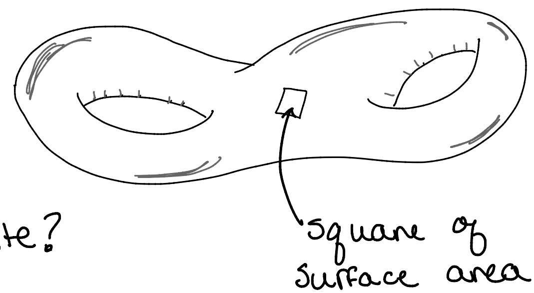
$$= \left[-\tfrac{1}{2}e^{-y} + \tfrac{1}{2}e^{y}\right]_0^{\ln(\sqrt{2}-1)}$$

$$= -1$$

$\boxed{\text{Surface Area}}$

In general, the surface area of a surface is not easy to calculate

How do we break the surface into small pieces that we can handle & then integrate?



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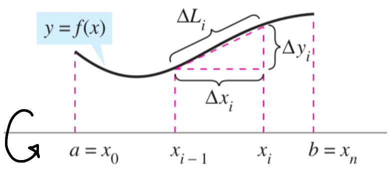
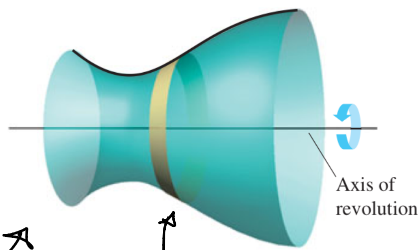
We will stick to surfaces of revolution, since their structure greatly simplifies the problem.

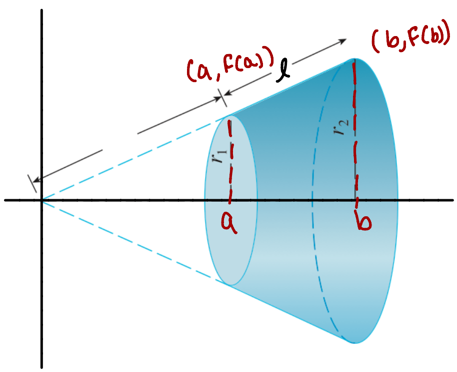
We start with a non-negative fcn:



Revolve about the x-axis to get a surface



we need surface area of this piece



This is the frustrum of a cone. Suface area of this will be our slice.

surface area given by

$$S = \pi\,(f(a) + f(b))\cdot \ell$$

radii

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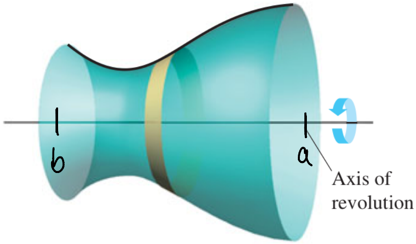
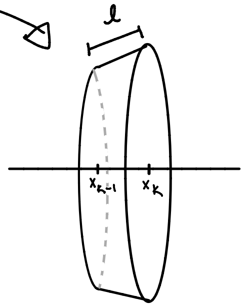
Back to the surface of revolution



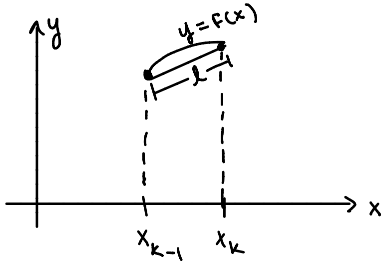
Surface



Slice

$$S = \pi\left(f(x_{k-1}) + f(x_k)\right)\cdot \ell$$

We need to find ℓ:



$$\ell = \left((\Delta x)^2 + (\Delta y)^2\right)^{1/2}$$

$$= \Delta x \left(1 + \left(\frac{\Delta y}{\Delta x}\right)\right)^{1/2}$$

So

$$S = \pi\left(f(x_{k-1}) + f(x_k)\right)\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)^{1/2} \Delta x$$

Let's make $\Delta x \to dx$ then $\dfrac{\Delta y}{\Delta x} \to f'(x)$

and $f(x_{k-1}) + f(x_k) \longrightarrow 2f(x)$

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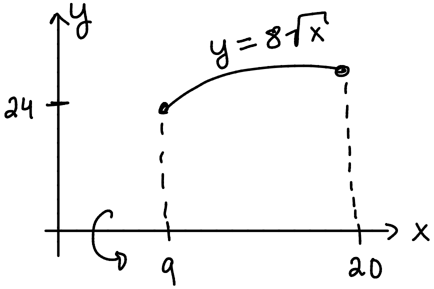
So the surface area of an infinitesimal frustrum is

$$2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Then, to get the total surface area we integrate

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex. 3) Find the area of a surface generated by $y = 8\sqrt{x}$ on $[9, 20]$ revolved about the x-axis.



So $f(x) = 8\sqrt{x}$

$f'(x) = 4x^{-1/2}$

$$S = \int_9^{20} 2\pi(8x^{1/2})(1 + (4x^{-1/2})^2)^{1/2}\,dx$$

$$= 16\pi\int_9^{20} x^{1/2}\left(1+\frac{16}{x}\right)^{1/2} dx$$

$$= 16\pi\int_9^{20} x^{1/2}\left(\frac{x+16}{x}\right)^{1/2} dx$$

$$= 16\pi\int_9^{20} (x+16)^{1/2}\,dx$$

$$= 16\pi\left[\frac{2}{3}(x+16)^{3/2}\right]_9^{20} = \frac{2912\pi}{3}$$