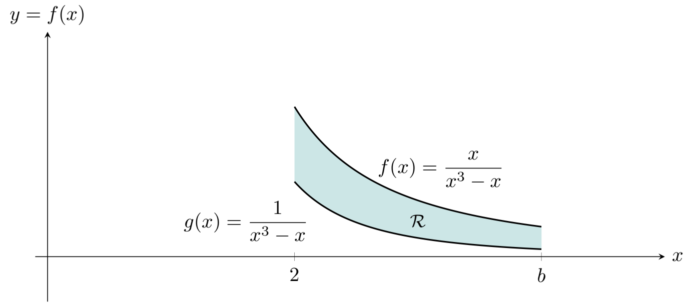
Problem 1: Find the area of the region $\mathcal{R}$ indicated in the picture. Use that result to determine the limit of the area of $\mathcal{R}$ as b goes to infinity.



Solution:

Area btwn 2 Curves

$$A = \int_a^b [f(x) - g(x)]\,dx$$

So

$$A = \int_2^{\infty} \left[\frac{x}{x^3-x} - \frac{1}{x^3-x}\right] dx$$

$$= \int_2^{\infty} \frac{x-1}{x(x-1)(x+1)}\,dx = \int_2^{\infty} \frac{1}{x(x-1)}\,dx$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{x(x+1)}\,dx$$

Note: $\int \frac{1}{x(x+1)}\,dx$ Partial Fractions : $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{1}{x} - \frac{1}{x+1}$

$$\Rightarrow \quad 1 = A(x+1) + Bx$$
$$1 = Ax + A + Bx$$
$$1 = (A+B)x + A$$

Equate Coeff.s

$A + B = 0$ $\quad B = -A = -1$

$A = 1$ $\quad \Rightarrow A = 1$

So

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln(x) - \ln(x+1)$$

$$= \ln\left(\frac{x}{x+1}\right)$$

Problem 1: (cont'd)

$$\Rightarrow \quad A = \lim_{b\to\infty} \int_0^b \frac{1}{x(x+1)}\,dx$$

$$= \lim_{b\to\infty} \ln\left(\frac{x}{x+1}\right)\Big|_2^b$$

$$= \lim_{b\to\infty} \left[\ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{2}{2+1}\right)\right]$$

$$= \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) - \lim_{b\to\infty} \ln\left(\frac{2}{3}\right)$$

$$= \ln\left(\lim_{b\to\infty} \frac{b}{b+1}\right) - \ln\left(\frac{2}{3}\right)$$

Note:

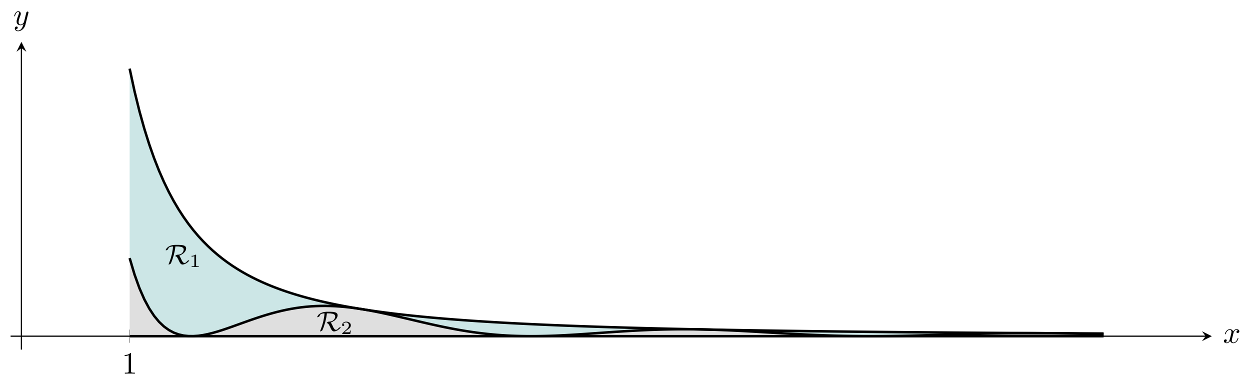
$$\lim_{b\to\infty} \frac{b}{b+1} \overset{H}{=} \lim_{b\to\infty} \frac{1}{1} = 1$$

$$= \ln(1) - \ln\left(\frac{2}{3}\right)$$

$$= \ln\left(\frac{1}{\frac{2}{3}}\right)$$

$$= \ln\left(\frac{3}{2}\right)$$

Problem 2: Let $\mathcal{R}_1$ be the infinite region between the x-axis and the curve $y = x^{-2}$ on the interval $[1, \infty)$. Let $\mathcal{R}_2$ be the infinite region between the x-axis and the curve $y = x^{-2}\cos^2(x)$ on the interval $[1, \infty)$. What can you conclude about the area of $\mathcal{R}_2$ based on its relationship to the area of $\mathcal{R}_1$? (Note: you do not need to calculate the area of $\mathcal{R}_2$ explicitly, just establish the relationship between $\mathcal{R}_1$ and $\mathcal{R}_2$).



Solution:

Area btwn 2 curves given by

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Area of R_1 Given by

$$A_1 = \int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = -\lim_{b\to\infty}\left[b^{-1} - 1^{-1}\right]$$

$$= -\left[\lim_{b\to\infty} \frac{1}{b} - \lim_{b\to\infty} 1\right]$$

$$= -\left[0 - 1\right] = 1$$

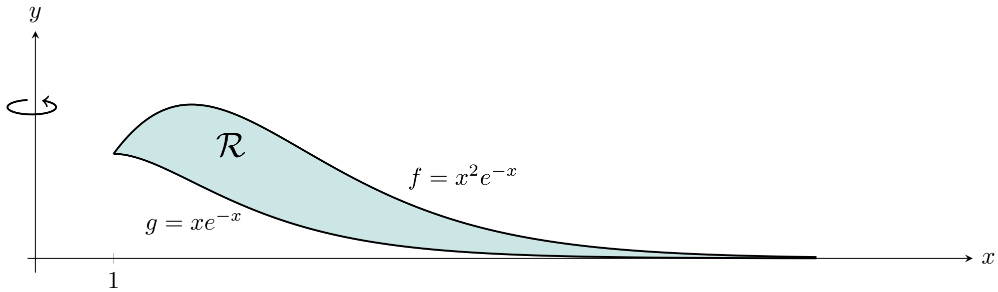
Since $A_1 = 1$ then we conclude that

$$0 \le \text{area } R_2 \le 1$$

In other words,

$$0 \le \int_0^\infty x^{-2}\cos^2(x)\,dx \le 1$$

Problem 3: Compute the area of the region $\mathcal{R}$ bounded by the curves $f = x^2e^{-x}$ and $g = xe^{-x}$ on the interval $[1, \infty)$.



Solution:

Area of region btwn 2 curves

$$A = \int_a^b [f(x) - g(x)]\,dx$$

So area of Region is

$$A = \int_1^\infty [x^2e^{-x} - xe^{-x}]\,dx = \int_1^\infty e^{-x}(x^2 - x)\,dx$$

$$= \lim_{b\to\infty} \int_1^b e^{-x}(x^2-x)\,dx$$

Note: $\int e^{-x}(x^2 - x)\,dx$

Int. by Parts $\quad u = x^2 - x \quad v = -e^{-x}$

$\int u\,dv = uv - \int v\,du \quad du = (2x-1)dx \quad dv = e^{-x}\,dx$

$$\Rightarrow \int e^{-x}(x^2-x)\,dx = -e^{-x}(x^2-x) + \underbrace{\int e^{-x}(2x-1)\,dx}_{\text{Int. by Parts}}$$

Note: $\int e^{-x}(2x-1)\,dx$

Int. by Parts $\quad u = 2x-1 \quad v = -e^{-x}$

$\int u\,dv = uv - \int v\,du \quad du = 2dx \quad dv = e^{-x}\,dx$

$$\Rightarrow \int e^{-x}(2x-1)\,dx = -e^{-x}(2x+1) + 2\int e^{-x}\,dx = -e^{-x}(2x+1) - 2e^{-x}$$

Problem 3: (cont'd)

$$\Rightarrow \int e^{-x}(x^2-x)\,dx = -e^{-x}(x^2-x) + \int e^{-x}(2x-1)\,dx$$

$$= -e^{-x}(x^2-x) - e^{-x}(2x+1) - 2e^{-x}$$

$$= -e^{-x}\left[x^2 - x + (2x+1) + 2\right]$$

$$= -e^{-x}\left[x^2 + x + 1\right]$$

So

$$A = \lim_{b\to\infty} \int_1^b e^{-x}(x^2-x)\,dx$$

$$= \lim_{b\to\infty} \Big[-e^{-x}(x^2+x+1)\Big]_1^b$$

$$= \lim_{b\to\infty} \Big[-e^{-b}(b^2+b+1) - (-e^{-b}(1+1+1))\Big]$$

$$= \lim_{b\to\infty} \Big[-e^{-b}(b^2+b+1) + \frac{3}{e}\Big]$$

$$= \lim_{b\to\infty} -e^{-b}(b^2+b+1) + \lim_{b\to\infty} \frac{3}{e}$$

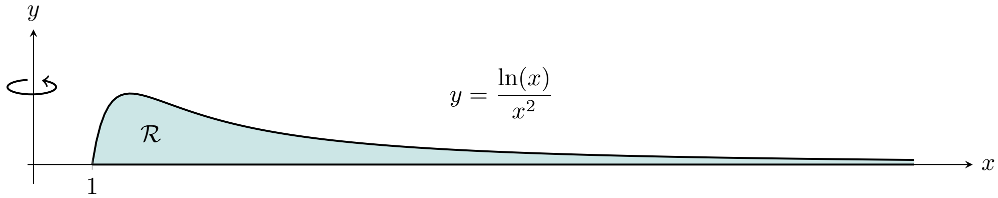
Note:

$$\lim_{b\to\infty} \frac{b^2+b+1}{e^b} \overset{H}{=} \lim_{b\to\infty} \frac{2b+1}{e^b} \overset{H}{=} \lim_{b\to\infty} \frac{2}{e^b} = 0$$

$$\Rightarrow \quad A = \lim_{b\to\infty} \Big[-e^{-b}(b^2+b+1) + \frac{3}{e}\Big]$$

$$= 0 + \frac{3}{e} = \frac{3}{e}$$

Problem 4: Consider the infinite region $\mathcal{R}$ in the first quadrant bounded by the x-axis and the curve $y = \dfrac{\ln(x)}{x^2}$ on the interval $[1, \infty)$. Compute the volume of the solid obtained when $\mathcal{R}$ is revolved about the y-axis.



Solution: Volume of solid revolved around the y-axis given by shell method

$$V = 2\pi \int_a^b x\,(f(x) - g(x))\,dx$$

So $$V = 2\pi \int_1^\infty x\left(\frac{\ln(x)}{x^2}\right)dx = 2\pi \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$$

Note:

$$\int \frac{\ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2} + c = \frac{(\ln(x))^2}{2} + c$$

u-sub: $u = \ln(x) \Rightarrow du = \frac{1}{x}\,dx$

So

$$V = 2\pi \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x}\,dx$$

$$= 2\pi \lim_{b\to\infty} \left[\frac{(\ln(x))^2}{2}\right]_1^b$$

$$= \pi \lim_{b\to\infty} \left[\ln(b)^2 - \ln(1)\right]$$

$$= \pi \lim_{b\to\infty} (\ln(b))^2 = \pi(\infty) = \infty$$

The value of the volume diverges.