1. Find the general solution to the differential equation $\dfrac{dy}{dx} = 3x^2y^2$

Solution:

Write in differential form:

$$y^{-2}\,dy = 3x^2\,dx$$

Integrate

$$\int y^{-2}\,dy = \int 3x^2\,dx$$

$$-y^{-1} = \frac{3x^3}{3} + c$$

$$-\frac{1}{y} = x^3 + c$$

$$-\frac{1}{y} = x^3 + c$$

Gen. implicit soln is $-\dfrac{1}{y} = x^3 + c$

Gen explicit soln: $y = \dfrac{-1}{x^3+c}$

2. Find the solution to the initial value problem

$$y' = \frac{2x}{y + x^2 y}, \quad y(0) = -2$$

Solution:

Write in diff. form:

$$\frac{dy}{dx} = \frac{2x}{y+x^2y} = \frac{2x}{y(1+x^2)} \Rightarrow y\,dy = \frac{2x}{1+x^2}\,dx$$

Integrate

$$\int y\,dy = \int \frac{2x}{1+x^2}\,dx$$

Note:

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{1+u}\,du = \int \frac{1}{v}\,dv = \ln(1+u) + c$$

use u-sub $u = x^2$, $du = 2x\,dx$

use v sub $v = 1+u$, $dv = du$

$$= \ln(1+x^2) + c$$

$$\Rightarrow \frac{y^2}{2} = \ln(1+x^2) + c$$

Gen. implicit soln is

$$\Rightarrow y^2 = 2\ln(1+x^2) + c$$

Gen explicit soln:

$$\Rightarrow y = \pm\left(2\ln(1+x^2) + c\right)^{1/2}$$

3. Consider

$$\frac{dP}{dt} = 0.04P\left(1 - \frac{P}{1200}\right), \qquad p(0) = 60$$

(a) What is the carrying capacity? What is the value of k?

(b) Write the solution of the equation.

(c) What is the population after 10 weeks?

Solution:

(a) Has gen. Form $\dfrac{dP}{dt} = rP\left(1 - \dfrac{P}{K}\right)$

Where $r = 0.04$, $K = 1200$

Carrying capacity is 1200

Rate of growth ($k = r$) is 0.04

(b) Soln to logistic Eqn: $P(t) = \dfrac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}$

So $P(t) = \dfrac{1200}{1 + \left(\frac{1200 - 60}{60}\right)e^{0.04t}} = \dfrac{1200}{1 + 19e^{0.04t}}$

(c) After 10 weeks:

$$P(10) = \frac{1200}{1 + 19e^{0.04(10)}} \approx 87.361 \Rightarrow 88$$

4. Consider
$$\frac{dP}{dt} = 0.02P - 0.0004P^2, \qquad p(0) = 40$$
   (a) What is the carrying capacity? What is the value of k?
   (b) Write the solution of the equation.
   (c) What is the population after 10 weeks?

Solution: Rewriting eqn:

$$\frac{dP}{dt} = 0.02P(1 - 0.002P) = 0.02P\left(1 - \frac{P}{500}\right)$$

(a) Has gen. Form $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$

Where $r = 0.02$, $K = 500$

Carrying capacity is 500
Rate of growth ($k = r$) is 0.02

(b) Soln to logistic eqn: $P(t) = \dfrac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}$

So $P(t) = \dfrac{500}{1 + \left(\frac{500 - 40}{40}\right)e^{-0.02t}} = \dfrac{500}{1 + \frac{23}{2}e^{-0.02t}}$

(c) After 10 weeks:

$$P(10) = \frac{500}{1 + \frac{23}{2}e^{-0.02(10)}} \approx 48.005 \Rightarrow 49$$