Determine if the following series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$

Solution:

$$a_k = k\left(\frac{2}{3}\right)^k$$

$$a_{k+1} = (k+1)\left(\frac{2}{3}\right)^{k+1}$$

$$\frac{a_{k+1}}{a_k} = (k+1)\left(\frac{2}{3}\right)^{k+1} \cdot \frac{1}{k\left(\frac{2}{3}\right)^k} = \frac{(k+1)\left(\frac{2}{3}\right)\cancel{\left(\frac{2}{3}\right)^k}}{k\cancel{\left(\frac{2}{3}\right)^k}}$$

$$= \frac{(k+1)\left(\frac{2}{3}\right)}{k}$$

$$= \frac{2}{3}\left(\frac{k+1}{k}\right)$$

$$r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{2}{3}\left(\frac{k+1}{k}\right) = \frac{2}{3}\lim_{k\to\infty}\frac{k+1}{k}$$

$$\overset{H}{=} \frac{2}{3}\lim_{k\to\infty}\frac{1}{1} = \frac{2}{3}$$

By the ratio test, since $r<1$, the series converges.

2. $\displaystyle\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$

Solution:

$$a_k = \frac{(k!)^2}{(2k)!} \quad \& \quad a_{k+1} = \frac{((k+1)!)^2}{(2(k+1))!}$$

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2}$$

$$= \frac{((k+1)!)^2 (2k)!}{(2k+2)! \; (k!)^2}$$

$$= \frac{(k+1)! \, (k+1)! \, (2k)!}{(2k+2)(2k+1)(2k)! \, k! \, k!}$$

$$= \frac{(k+1)(k+1)\cancel{k!}\,\cancel{k!}\,\cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}\,\cancel{k!}\,\cancel{k!}}$$

$$= \frac{(k+1)(k+1)}{(2k+2)(2k+1)} = \frac{k^2+2k+1}{4k^2+6k+2} = \frac{\cancel{k^2}\left(1+\frac{2}{k}+\frac{1}{k^2}\right)}{\cancel{k^2}\left(4+\frac{6}{k}+\frac{2}{k^2}\right)} = \frac{1+\frac{2}{k}+\frac{1}{k^2}}{4+\frac{6}{k}+\frac{2}{k^2}}$$

So we have

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1+\frac{2}{k}+\frac{1}{k^2}}{4+\frac{6}{k}+\frac{2}{k^2}}$$

$$= \frac{\lim_{k\to\infty} 1 + \lim_{k\to\infty} \frac{2}{k} + \lim_{k\to\infty} \frac{1}{k^2}}{\lim_{k\to\infty} 4 + \lim_{k\to\infty} \frac{6}{k} + \lim_{k\to\infty} \frac{2}{k^2}} = \frac{1}{4}$$

By the Ratio test since $r < 1$ the series Converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

Solution:

$$a_n = \frac{(-2)^n}{n^n}$$

$$\sqrt[n]{a_n} = (a_n)^{1/n} = \left(\frac{(-2)^n}{n^n}\right)^{1/n} = \frac{(-2)}{n}$$

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{-2}{n} = 0$$

By the root test since $\rho < 1$ the series converges.

4. Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is absolutely convergent, conditionally convergent, or diverges.

Solution:

Determine if $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges:

Alt. series w/ $a_n = \frac{1}{\ln(n)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

By Alt. series test, since $\lim a_n = 0$, series converges.

Determine if $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln(n)}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ converges:

$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ use comparison test.

Since $\ln(n) < n \Rightarrow \frac{1}{n} < \frac{1}{\ln(n)}$

Choose $b_k = \frac{1}{n} \Rightarrow \sum_{n=2}^{\infty} b_k = \sum_{n=2}^{\infty} \frac{1}{n}$ Harmonic series so series diverges

By Comparison test since $b_k < a_k$ & $\sum b_k$ diverges, then $\sum \frac{1}{\ln(n)}$ diverges

Since $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges, but $\sum_{n=2} \left|\frac{(-1)^n}{\ln(n)}\right|$ diverges, series is conditionally convergent.