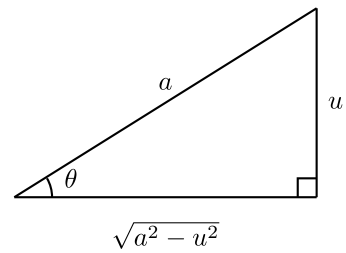
Lecture 08: Trigonometric Substitution

Math 2205: Calculus II Fall 2018

These are the common trigonometric substitutions.

For Form $\sqrt{a^2 - u^2}$



Let

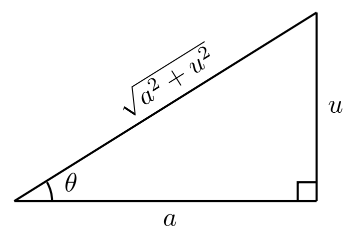
$$u = a\sin\theta$$

then

$$\sqrt{a^2 - u^2} = a\cos\theta$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

For Form $\sqrt{a^2 + u^2}$



Let

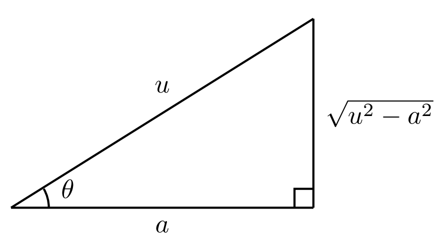
$$u = a\tan\theta$$

then

$$\sqrt{a^2 + u^2} = a\sec\theta$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

For Form $\sqrt{u^2 - a^2}$



Let

$$u = a\sec\theta$$

then

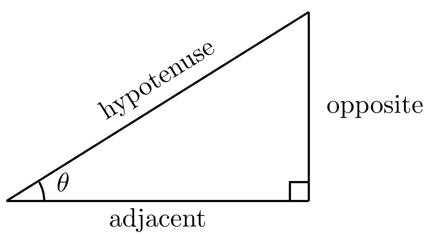
$$\sqrt{u^2 - a^2} = \begin{cases} a\tan\theta, & \text{if } u > a, \text{ where } 0 \leq \theta \leq \frac{\pi}{2} \\ -a\tan\theta, & \text{if } u < -a, \text{ where } \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Additional Helpful Identities

Reciprocal	**Pythagorean**	**Double Angle**
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\sin^2\theta + \cos^2\theta = 1$	$\sin 2\theta = 2\sin\theta\cos\theta$
$\csc\theta = \dfrac{1}{\sin\theta}$	$1 + \cot^2\theta = \csc^2\theta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\tan^2\theta + 1 = \sec^2\theta$	**Half Angle**
$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$		$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
		$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

It is also helpful to recall that for an angle θ the trigonometric functions are given by



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

When coverting back to functions of x you may also need the following identities for inverse trig functions

Inverse Trig Functions

$y = \sin^{-1}(x)$	is equivalent to	$x = \sin(y)$
$y = \cos^{-1}(x)$	is equivalent to	$x = \cos(y)$
$y = \tan^{-1}(x)$	is equivalent to	$x = \tan(y)$

Alternative Notation

$$y = \sin^{-1}(x) = \arcsin(x)$$
$$y = \cos^{-1}(x) = \arccos(x)$$
$$y = \tan^{-1}(x) = \arctan(x)$$

Examples

Example 1: Integrate $\displaystyle\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$

Rewriting we have $\displaystyle\int \frac{1}{x^2\sqrt{3^2-x^2}}\,dx$

This has form $\sqrt{a^2-u^2}$

Use substitution $u = a\sin\theta$

$\Rightarrow x = 3\sin\theta \qquad \Rightarrow x^2 = (3\sin\theta)^2 = 9\sin^2\theta$

$\Rightarrow dx = 3\cos\theta\, d\theta$

Integral becomes

$$\int \frac{1}{9\sin^2\theta\sqrt{9-9\sin^2\theta}}\,(3\cos\theta\, d\theta)$$

Note: $\sqrt{9-9\sin\theta} = \sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9\cos^2\theta}$ Since $1-\sin^2\theta = \cos^2\theta$

$= \sqrt{9}\sqrt{\cos^2\theta}$

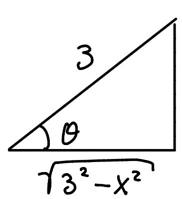
$= 3\cos\theta$

$$\Rightarrow \int \frac{\cancel{3\cos\theta}}{9\sin^2\theta\,(\cancel{3\cos\theta})}\,d\theta = \frac{1}{9}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

Now we need to rewrite the result in terms of x



$$\cot\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3^2-x^2}}{x}$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{9-x^2}}\,dx = -\frac{1}{9}\left(\frac{\sqrt{3^2-x^2}}{x}\right) + C$$

Example 2: Integrate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

This has form $\sqrt{u^2 - a^2}$

Use substitution $u = a\sec\theta$

$\Rightarrow x = \sqrt{3}\sec\theta \quad \Rightarrow x^2 = (\sqrt{3}\sec\theta)^2 = 3\sec^2\theta$

$\Rightarrow dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$

We also need to change our bounds in terms of θ

$x = 2 \Rightarrow \sqrt{3}\sec\theta = 2$

$\Rightarrow \sec\theta = \frac{2}{\sqrt{3}}$

Since $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$

$x = \sqrt{3} \Rightarrow \sqrt{3}\sec\theta = \sqrt{3}$

$\Rightarrow \sec\theta = \frac{\sqrt{3}}{\sqrt{3}} = 1$

$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{1} = 1$

$\Rightarrow \theta = 0$

Integral becomes

$$\int_0^{\frac{\pi}{6}} \frac{\sqrt{3\sec^2\theta - 3}\,(\cancel{\sqrt{3}\sec\theta}\tan\theta\,d\theta)}{\cancel{\sqrt{3}\sec\theta}}$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{3(\sec^2\theta - 1)}\tan\theta\,d\theta = \int_0^{\frac{\pi}{6}} \sqrt{3}\sqrt{\tan^2\theta}\tan\theta\,d\theta$$

$$= \sqrt{3}\int_0^{\frac{\pi}{6}} \tan^2\theta\,d\theta$$

$$= \sqrt{3}\int_0^{\frac{\pi}{6}} (\sec^2\theta - 1)\,d\theta$$

$$= \sqrt{3}\left[\int_0^{\frac{\pi}{6}} \sec^2\theta\,d\theta - \int_0^{\frac{\pi}{6}} 1\,d\theta\right] = \sqrt{3}\Big[\tan\theta - \theta\Big]_0^{\frac{\pi}{6}}$$

$$= \sqrt{3}\left[\tan\left(\tfrac{\pi}{6}\right) - \tfrac{\pi}{6} - (\tan(0) - 0)\right] = \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right] = 1 - \frac{\pi\sqrt{3}}{6}$$