Lecture # 11: Improper Integrals

Date: Mon. 10/22/18

Infinite Intervals

Let's consider a fcn $f(x) = e^{-3x}$.

We can find the area under this curve from $x=0$ to any positive number b by eval'ing

$$\int_0^b e^{-3x}\,dx = -\frac{1}{3}e^{-3x}\Big|_0^b = -\frac{1}{3}\left[e^{-3b} - e^0\right]$$

$$= -\frac{1}{3}e^{-3b}$$

What happens as b gets larger? i.e.

$$\lim_{b\to\infty}\int_0^b e^{-3x}\,dx = \lim_{b\to\infty}\left[\frac{1}{3} - \frac{e^{-3b}}{3}\right] = \lim_{b\to\infty}\frac{1}{3} - \lim_{b\to\infty}\frac{1}{3e^{3b}} = \frac{1}{3}$$

In other words, the integral of $f(x)e^{-3x}$ on the infinite interval $[0,\infty)$ is

$$\int_0^\infty e^{-3x}\,dx = \lim_{b\to\infty}\int_0^b e^{-3x}\,dx = \frac{1}{3}$$

this is what is known as an <u>improper integral</u>.

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We can also integrate on an interval $(-\infty, a]$ or $(-\infty, \infty)$.

The 3 cases for infinite intervals is represented below.

DEFINITION OF IMPROPER INTEGRALS WITH INFINITE INTEGRATION LIMITS

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number (see Exercise 120).

We must represent each of these cases with appropriate limits. Why?

Recall that ∞ is not a number. This means we cannot plug in infinity as a value when applying FTOC.

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An improper integral exists if it <u>converges</u> i.e. is equal to a finite value. Otherwise, it is said to <u>diverge</u>.

<u>Ex. 1)</u> $\int_1^\infty \frac{1}{x}\,dx$

<u>Soln.</u>
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \Big[\ln(x)\Big]_1^b$$
$$= \lim_{b\to\infty} \Big[\ln(b) - \ln(1)\Big]$$
$$= \lim_{b\to\infty} \ln(b)$$
$$= \infty$$

Since $\int_1^\infty \frac{1}{x}\,dx = \infty$ this integral <u>diverges</u>

Warning

When consulting outside sources you may see ∞ being used as a limit of integration as if it were a number. This is <u>not</u> mathematically correct!
i.e. expect major pt deductions for doing so!

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Ex. 2) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Soln. For this case we must break up the integral in two pieces.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

Note: $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \Big[\arctan(x)\Big]_{a}^{0} + \lim_{b\to\infty} \Big[\arctan(x)\Big]_{0}^{b}$$

$$= \lim_{a\to-\infty} \Big[\underbrace{\arctan(0)}_{=0} - \arctan(a)\Big]$$

$$+ \lim_{b\to\infty} \Big[\arctan(b) - \underbrace{\arctan(0)}_{=0}\Big]$$

$$= -\lim_{a\to-\infty} \arctan(a) + \lim_{b\to\infty} \arctan(b)$$

$$= -\left[-\frac{\pi}{2}\right] + \left[\frac{\pi}{2}\right] = \pi$$

Note: If one of the pieces of the original integral diverges then the integral diverges!

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Since these types of integrals involve use of limits you must also use proper limit notation & apply limit properties & rules correctly. No shortcuts!

Recall the following theorem from Calc I.

Thm (L'Hôpital's Rule)

Let f & g be differentiable fcns on (a,b). For $g'(x) \neq 0$ on (a,b) if the limit has indeterminate forms $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Ex. 3) $\int_1^\infty (1-x)e^{-x}\,dx$

Soln. $\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty} \int_1^b (1-x)e^{-x}\,dx$

Note: $\int (1-x)e^{-x}\,dx$ use int. by parts

$$\int u\,dv = uv - \int v\,du$$

$$u = 1-x \qquad v = -e^{-x}$$

$$du = -1 \qquad dv = e^{-x}\,dx$$

$$\Rightarrow \int (1-x)e^{-x}\,dx = -e^{-x}(1-x) - \int e^{-x}\,dx$$

$$= -e^{-x}(1-x) + e^{-x} + C$$

$$= -e^{-x} + xe^{-x} + e^{-x} + C$$

$$= xe^{-x} + C$$

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Ex. 3 (cont'd)

So we now have

$$\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx$$

$$= \lim_{b\to\infty}\Big[xe^{-x}\Big]_1^b$$

$$= \lim_{b\to\infty}\Big[be^{-b} - e^{-1}\Big]$$

$$= \lim_{b\to\infty}\frac{b}{e^b} - \lim_{b\to\infty}\frac{1}{e}$$

Note:

$\lim_{b\to\infty}\frac{b}{e^b}$ has indet. form $\frac{\infty}{\infty}$ so we can apply L'Hopital's rule

$$\Rightarrow \lim_{b\to\infty}\frac{b}{e^b} \overset{H}{=} \lim_{b\to\infty}\frac{1}{e^b} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

So,

$$\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty}\frac{b}{e^b} - \lim_{b\to\infty}\frac{1}{e}$$

$$= 0 - \frac{1}{e} = \frac{-1}{e}$$

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Ex. 4 Find the volume of the solid when the region bdd by $f(x) = \frac{1}{x}$ & the x-axis is revolved around the x-axis on $[1, \infty)$.

Soln. Using disk method:

$$V = \int_a^b \pi (f(x))^2 \, dx = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \lim_{b \to \infty} \int_1^b x^{-2} \, dx$$

$$= \pi \lim_{b \to \infty} \left[-x^{-1}\right]_1^b$$

$$= \pi \lim_{b \to \infty} \left[\frac{1}{b} - 1\right]$$

$$= \pi \left[\lim_{b \to \infty} \frac{1}{b} + \lim_{b \to \infty} 1\right]$$

$$= \pi [0 + 1] = \pi$$

Unbounded Integrands

We can also have the situation where an integrand is unbounded on one or both interval endpts or contains a discontinuity on the interval.

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DEFINITION OF IMPROPER INTEGRALS WITH INFINITE DISCONTINUITIES

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex. $\displaystyle\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$

Soln We can see that the integrand is undefined @ $x = -3, 3$.

To evaluate we need to split up the integral just as we did for integrals on $(-\infty, \infty)$:

$$\int_{-3}^{3} \frac{1}{\sqrt{9-x^2}}\,dx = \int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}\,dx + \int_{0}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \lim_{x\to -3^+} \int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}\,dx + \lim_{x\to 3^-} \int_{0}^{3} \frac{1}{\sqrt{9-x^2}}\,dx$$

= (left as an exercise)

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Comparison of Integrals

Consider $\int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx$

We can see that $\sqrt{x^4} \le \sqrt{x^4+1}$ then we can say the same about their integrals:

$$\int_1^\infty \frac{1}{x^3}\sqrt{x^4}\,dx \le \int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx$$

Note:

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\sqrt{x^4}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\sqrt{x^4}}{x^3}\,dx \\ &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln(x)\Big|_1^b \\ &= \lim_{b\to\infty}\left[\ln(b)-\ln(1)\right] \\ &= \lim_{b\to\infty}\ln(b) = \infty\end{aligned}$$

then

$$\infty \le \int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx$$

Which implies that $\int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx = \infty$

This is known as a comparison test & will come up again later on.