For each of these questions we will consider the integral

$$\int_0^{\pi} \sin(x)\,dx = 2$$

For problems 1 to 3 let $n = 6$ so that

$$\Delta x = \frac{b-a}{n} = \frac{\pi}{6}$$

Round all results to 6 decimal places.

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1. Using the Trapezoid rule (T_6),

   (a) Find the approximation of this integral using the indicated method.

   Solution: Note that for the Trapezoid rule, the node values are given by $x_i = a + i\Delta x$ and so

   $$\begin{aligned} T_6 &= \frac{\Delta x}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(b)\right] \\ &= \frac{\Delta x}{2}\left[f(0) + 2f\left(\tfrac{\pi}{6}\right) + 2f\left(\tfrac{\pi}{3}\right) + 2f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{2\pi}{3}\right) + 2f\left(\tfrac{5\pi}{6}\right) + f(\pi)\right] \\ &= \frac{\Delta x}{2}\left[0 + 2\cdot\tfrac{1}{2} + 2\cdot\tfrac{\sqrt{3}}{2} + 2\cdot 1 + 2\cdot\tfrac{\sqrt{3}}{2} + 2\cdot\tfrac{1}{2} + \cdot 0\right] \\ &= \frac{1}{12}\left(2\sqrt{3} + 4\right)\pi \approx 1.954097 \end{aligned}$$

   (b) Compute the actual error of this approximation.

   Solution: The actual error is given by

   $$E = \text{exact} - \text{approximate} = 2 - 1.954097 \approx 0.0459028$$

   (c) Compute the error bound for this approximation and compare it to the error obtained in part (b).

   Solution: The error bound for the trapezoid rule is given by

   $$|E_T| \le \frac{|f''(x)|(b-a)^3}{12n^2}$$

   Since $f''(x) = -\sin(x)$ and $-1 \le \sin(x) \le 1$ then

   $$|f''(x)| = |-\sin x| \le 1$$

   and so the error bound is given by

   $$|E_M| \le \frac{1(\pi - 0)^3}{12(6)^2} \approx 0.0717738 \le 0.08$$

   The actual error was found to be $E \approx 0.0459028$ and we see that $E \le |E_T|$ so our actual error is within the error bound.

2. Using the Midpoint rule (M_6),

   (a) Find the approximation of this integral using the indicated method.

Solution: Note that for the midpoint rule, the node values are given by $m_i = \dfrac{x_{i-1}+x_i}{2}$ and so

$$\begin{aligned}
M_6 &= \frac{\Delta x}{2}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)+f\left(\frac{x_4+x_5}{2}\right)+f\left(\frac{x_5+x_6}{2}\right)\right]\\
&= \frac{\Delta x}{2}\left[f(0)+2f\left(\tfrac{\pi}{6}\right)+2f\left(\tfrac{\pi}{3}\right)+2f\left(\tfrac{\pi}{2}\right)+2f\left(\tfrac{2\pi}{3}\right)+2f\left(\tfrac{5\pi}{6}\right)+f(\pi)\right]\\
&= \frac{\Delta x}{2}\left[0+2\cdot\tfrac{1}{2}+2\cdot\tfrac{\sqrt{3}}{2}+2\cdot 1+2\cdot\tfrac{\sqrt{3}}{2}+2\cdot\tfrac{1}{2}+\cdot 0\right]\\
&= \frac{1}{12}\left(2\sqrt{3}+4\right)\pi \approx 2.023030
\end{aligned}$$

   (b) Compute the actual error of this approximation.

Solution: The actual error is given by

$$E = \text{exact} - \text{approximate} = 2 - 1.954097 \approx -0.023030$$

   (c) Compute the error bound for this approximation and compare it to the error obtained in part (b).

Solution: The error bound for the trapezoid rule is given by

$$|E_M| \le \frac{|f''(x)|(b-a)^3}{24n^2}$$

Since $f''(x) = -\sin(x)$ and $-1 \le \sin(x) \le 1$ then

$$|f''(x)| = |-\sin x| \le 1$$

and so the error bound is given by

$$|E_M| \le \frac{1(\pi-0)^3}{12(6)^2} \approx 0.035887 \le 0.04$$

The actual error was found to be $|E| \approx 0.023030$ and we see that $E \le |E_M|$ so our actual error is within the error bound.

3. Using the Simpson's rule (S_6),

(a) Find the approximation of this integral using the indicated method.

> **Solution:** Note that for the Trapezoid rule, the node values are given by $x_i = a + i\Delta x$ and so
>
> $$\begin{aligned} S_6 &= \frac{\Delta x}{2}\big[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(b)\big] \\ &= \frac{\Delta x}{2}\left[f(0) + 2f\left(\tfrac{\pi}{6}\right) + 2f\left(\tfrac{\pi}{3}\right) + 2f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{2\pi}{3}\right) + 2f\left(\tfrac{5\pi}{6}\right) + f(\pi)\right] \\ &= \frac{\Delta x}{2}\left[0 + 2\cdot\tfrac{1}{2} + 2\cdot\tfrac{\sqrt{3}}{2} + 2\cdot 1 + 2\cdot\tfrac{\sqrt{3}}{2} + 2\cdot\tfrac{1}{2} + \cdot 0\right] \\ &= \frac{1}{12}\left(2\sqrt{3}+4\right)\pi \approx 2.000863 \end{aligned}$$

(b) Compute the actual error of this approximation.

> **Solution:** The actual error is given by
>
> $$E = \text{exact} - \text{approximate} = 2 - 2.000863 \approx -0.000863$$

(c) Compute the error bound for this approximation and compare it to the error obtained in part (b).

> **Solution:** The error bound for the trapezoid rule is given by
>
> $$|E_S| \leq \frac{|f^{(4)}(x)|(b-a)^5}{180n^4}$$
>
> Since $f^{(4)}(x) = \sin(x)$ and $-1 \leq \sin(x) \leq 1$ then
>
> $$\left|f^{(4)}(x)\right| = |\sin x| \leq 1$$
>
> and so the error bound is given by
>
> $$|E_S| \leq \frac{1(\pi - 0)^5}{180(6)^4} \approx 0.00131181 \leq 0.002$$
>
> The actual error was found to be $|E| \approx 0.000863$ and we see that $E \leq |E_S|$ so our actual error is within the error bound.

4. How large should we choose n in each case so that the integral is accurate to within $0.000001 = 10^{-6}$.

   (a) Trapezoid Rule.

   Solution:

   (b) Midpoint Rule.

   Solution:

   (c) Simpson's Rule

   Solution: