Lecture # 13: Separable Eqns & Population Models Date: Mon. 11/5/18

Separable Equations

<u>Def</u> A separable Equation is a 1^{st} order ODE that can be written in the form

$$\frac{dy}{dx} = M(x)N(y)$$

i.e. as a function of x mult. by a fcn of y

To solve this type of eqn we write it in differential form

$$\frac{1}{N(y)}\,dy = M(x)\,dx$$

We then just integrate each side of this equation

$$\int \frac{1}{N(y)}\,dy = \int M(x)\,dx$$

<u>Ex. 1</u> Solve $\frac{dy}{dt} = x\sqrt{y}$

<u>Soln.</u>

Rewriting in diff. form we have

$$\frac{1}{y^{1/2}}\,dy = x\,dx$$

Integrating,

$$\int \frac{1}{y^{1/2}}\,dy = \int x\,dx$$

$$\frac{y^{1/2}}{1/2} = \frac{x^2}{2} + c$$

So the <u>general solution</u> is

$$y^{1/2} = \frac{x^4}{4} + c$$

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Ex. 1 (Cont'd)

A few things to note:

- We only need 1 constant of integration since

$$2y^{1/2} + C_1 = \frac{x^2}{2} + C_2$$
$$\Rightarrow y^{1/2} = \frac{x^2}{4} + \underbrace{C_2 - C_1}_{= C_3}$$

- This is an implicit soln. These are totally fine to have!
  In this case we could rewrite as an explicit soln as follows:

$$\Rightarrow y(t) = \left(\frac{x^2}{4} + C\right)^2$$

Ex. 2 Solve the IVP

$$\frac{dy}{dx} = xe^y, \quad y(0) = 0$$

Rewriting:

$$e^{-y}\frac{dy}{dx} = xe^y \cdot e^{-y}$$
$$\Rightarrow e^{-y}\,dy = x\,dx$$

Integrating

$$\int e^{-y}\,dy = \int x\,dx \Rightarrow -e^{-y} = \frac{x^2}{2} + C$$

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Ex. 2 (cont'd)

$$-e^{-y} = \frac{x^2}{2} + C$$

$$e^{-y} = \frac{-x^2}{2} + C$$

We don't need to explicitly solve in order to apply the I.C.

Given $y(0)=0$ i.e. $y(x=0)=0$

$$\Rightarrow e^{-(0)} = \frac{-(0)}{2} + C \Rightarrow C = 1$$

So particular (Implicit) soln is

$$e^{-y} = -\frac{x^2}{2} + 1$$

Explicit soln:

$$\ln(e^{-y}) = \ln\left(-\frac{x^2}{2} + 1\right)$$

$$-y = \ln\left(-\frac{x^2}{2} + 1\right)$$

$$\Rightarrow y = -\ln\left(1 - \frac{x^2}{2}\right)$$

$$= -\ln\left(1 - \frac{1}{2}x^2\right)$$

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Exponential (Natural) Growth

Let $p(t)$ be the value of a population @ time t.
If the rate of change w.r.t. t is proportional to it's size @ time t then

$$\frac{dP}{dt} = P'(t) = K\,P(t)$$

This is known as exponential or natural growth.

This is a seperable eqn.

Given an initial condition $p(0) = P_0$ we can find the general soln to the eqn.

Rewriting, $\frac{dP}{dt} = KP \implies \frac{1}{P}\,dP = K\,dt$

Integrating, $\frac{1}{P}\,dP = \int K\,dt$

$$\implies \ln(P) = Kt + C$$

Exponentiating,

$$e^{\ln(P)} = e^{(Kt+C)}$$

$$\implies P(t) = e^{(Kt+C)} = e^{C}e^{Kt} = C\,e^{Kt}$$

For $P(0) = P_0 \implies P_0 = Ce^{K(0)} \implies C = P_0$

So the gen. soln to the exponential growth eqn is

$$P(t) = P_0\,e^{Kt}$$

Where
r is the rate of growth
P_0 is the initial population

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This is also used to model continuously compounding interest.

$$S(t) = S_0 e^{rt}$$

Where

r is the interest rate

S_0 is the initial deposit

This can also be modified to include migration in & out of an area (or deposits / withdrawls in an account)

$$\frac{dP}{dt} = KP - m$$

To solve:

$$\frac{dp}{dt} = kp - m$$

$$\Rightarrow \int \frac{1}{KP-m}\, dp = \int 1\, dt$$

using u sub
$u = KP - m$
$du = K\, dp$
$\Rightarrow \frac{1}{K} du = dp$

$$\Rightarrow \int \frac{1}{u}\left(\frac{1}{k} du\right) = \frac{1}{K}\int \frac{1}{u} du$$
$$= \frac{1}{K}\ln(u)$$
$$= \frac{1}{K}\ln(Kp-m)$$

$$\Rightarrow \frac{1}{K}\ln(Kp-m) = t + C$$

$$\Rightarrow \ln\left((kp-m)^{1/k}\right) = t + C$$

$$\Rightarrow e^{\ln((kp-m)^{1/k})} = e^{t+C}$$

$$\Rightarrow (Kp-m)^{1/k} = Ce^{t}$$

$$Kp - m = (Ce^{t})^{1/k}$$

$$\Rightarrow Kp = Ce^{t/k} + m$$

$$\Rightarrow p = \frac{1}{K}(Ce^{t/k} + m) = Ce^{t/k} + \frac{m}{K}$$

For $p(0) = P_0$

$$\Rightarrow p = \frac{1}{K}e^{0} + \frac{m}{K} = \frac{1}{K} + \frac{m}{K}$$

$$\Rightarrow p = \left(\frac{1}{K} + \frac{m}{K}\right)e^{t/k} + \frac{m}{K} = \frac{1}{K}\left((1+m)e^{t/k} + m\right)$$

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Logistic Growth

For logistic Growth we assume that when the pop. is small

$$\frac{dP}{dt} \approx rP$$

but that it will level off @ some pt due to limited resources.

<u>Def</u> Logistic growth is given by eqns of the form

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

where

K is the <u>carrying capacity</u>

r is the rate of growth

the solution to the logistic growth eqn is given by

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}$$

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Ex. 3 Solve

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right), \quad P(0) = 100$$

What is the carrying capacity?

$K = 1000$

What is the growth rate?

$r = 0.08$

Soln has gen. form

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}$$

$$= \frac{1000}{1 + \left(\frac{1000 - 100}{100}\right)e^{-0.08t}}$$

$$= \frac{1000}{1 + 9e^{-0.08t}}$$