Lecture #06: Applications of Integration

Date: Thu. 9/27/18

Work

Work is a physical concept. It measures the change in energy as the result of a force.

Forces can be a constant or a variable. If a force is constant (like the force of gravity close to the Earth's gravity) Then work is calculated as

$$\text{Work} = \text{Force} \cdot \text{distance}$$

If the force is variable (changing with time or location for example), then this formula isn't accurate.

First, an example of a force that varies based on the position of the object.

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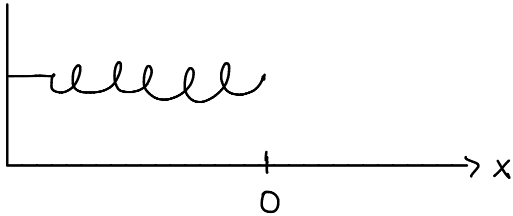
Springs & Hooke's Law

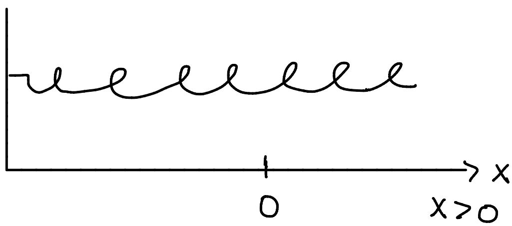
The force a spring exerts (push or pull) depends on how compressed or stretched it is.

No Force

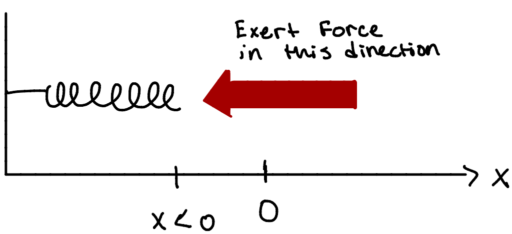


a relaxed spring the position $x=0$ is the <u>equilibrium</u> (or natural) <u>position</u> of the spring.

Force



a stretched spring



a compressed spring

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Hooke's Law

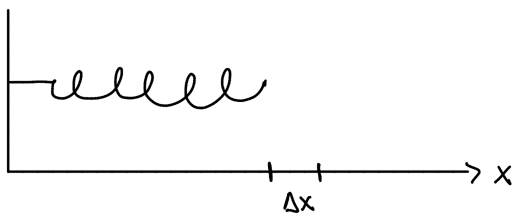
Tells us (approximately) how the force is related to how far we want to stretch or compress the spring.

$$F(x) = kx$$

where k is the spring constant (related to stiffness).

<u>Question</u>: How do we calculate work with such a force?

Imagine a "little piece"



If Δx is small, then the work is $F(x)\Delta dx$

or $kx\,\Delta x$

Any displacement can be broken up into tiny steps so the total work is the sum of the work from all of these tiny steps.

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In terms of Calculus, the amt of work to displace the spring in an infinitesimal amt from its current location @ x is

$$F(x)\,dx \qquad \text{or} \qquad Kx\,dx$$

<u>Def</u> (work)

the total work to go from $x=a$ to $x=b$ is

$$W = \int_a^b F(x)\,dx = \int_a^b kx\,dx$$

<u>Ex. 1</u> (Compressing a spring)

Suppose a force of 10N is required to stretch a spring from 0.1 from its equilibrium position & hold it in that position

(a) Find the spring constant K

Hooke's law tells us that $F(x) = Kx$.

In this case, it takes 10 N to hold @ 0.1 m so $F = 10N$ when $x = 0.1\,m$

$$\Rightarrow \quad 10N = K(0.1m) \quad \Rightarrow \quad K = 100 \ N/m$$

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Ex. 1 (cont'd)

(b) How much work is needed to compress the spring 0.5m from equilibrium?

$F(x) = 100x, \quad a = 0, \quad b = -0.5$ (negative $\Rightarrow$ compress)

So

$$W = \int_0^{-0.5} 100x\,dx = 50x^2 \Big|_0^{-0.5} = \frac{25}{2} \text{ N}\cdot\text{m}$$

(joules)

(c) How much work is needed to stretch the spring 0.25m from equilibrium?

$F(x) = 100x, \quad a = 0, \quad b = 0.25$

So

$$W = \int_0^{0.25} 100x\,dx = 50x^2 \Big|_0^{0.25} = \frac{50}{16} = \frac{25}{8} \text{ J}$$

(d) How much additional work is required to stretch the spring 0.25m if it has already been stretched 0.1m from equilibrium?

$F(x) = 100x, \quad a = 0.1, \quad b = 0.35$

So

$$W = \int_{0.1}^{0.35} 100x\,dx = 50x^2 \Big|_{0.1}^{0.35} = 5.625 \text{ J}$$

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Constant Forces with Extended Objects

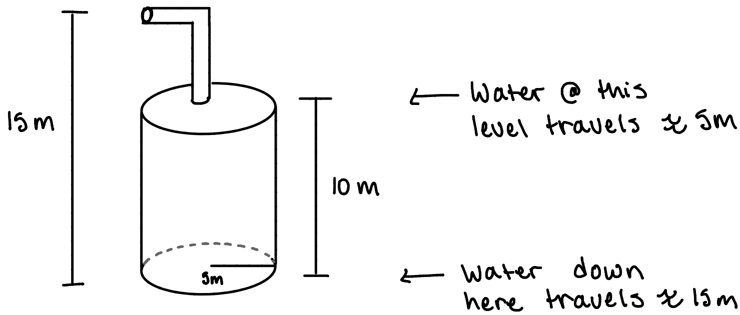
Lets consider gravity near the Surface of the Earth) It's constant so work = force * distance

However, if we are dealing with an object that isn't entirely in one place, the work calculation is more difficult.

Ex. 2 (Pumping water)

How much work is needed to pump all the water out of a Cylindrical tank w/ a height of 10m & a radius of 5m?

The water is pumped to an outflow pipe 15m above the bottom of the tank



The amt of work depends on where the water is.

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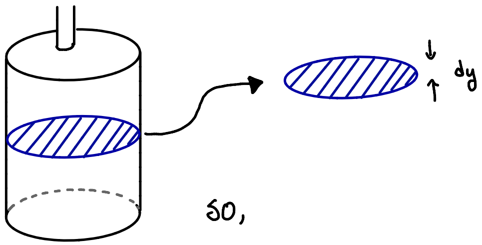
Ex. 2 (cont'd)

We need to break this up into "pieces"



This little layer of water is y meters below the outflow

so,

$$\text{work} = F_g \cdot \text{distance} = F_g \cdot y$$

so we need the mass of the water. That is,

$$\text{mass} = \text{density} \cdot \text{volume}$$

Density of water: $1000 \text{ kg}/\text{m}^3$

so $\quad \text{Volume} = \pi r^2 \, dy = 25\pi \, dy \text{ m}^3$

and $\quad \text{mass} = m = (1000 \text{ kg}/\text{m}^3)(25\pi \, dy \text{ m}^3)$

$$= \pi (25{,}000 \text{ kg}) \, dy$$

the work to pump this infinitesimal piece is

$$W_i = mgy = \pi (25{,}000\text{kg})(9.8 \text{ m}/\text{s}^2) \, y \, dy$$

The first layer is @ $y = -5\text{m}$ & last @ $y = -15\text{m}$

Ex. 2 (cont'd)

So total work is given by

$$W = \int_{-5}^{-15} \pi(25,000)(9.8)\, y\, dy$$

$$= \frac{\pi(25,000)(9.8)}{2}\, y^2 \Big|_{-5}^{-15}$$

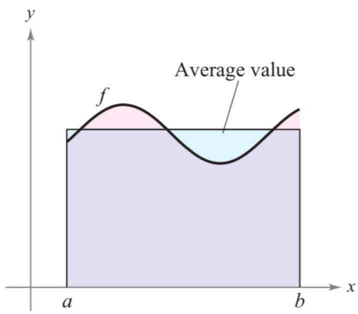
$$= \frac{\pi(25,000)(9.8)}{2}\left[(15)^2 - (5)^2\right]$$

$$= 80,000,000 \text{ Joules}$$

Average Value of a Function

Def (Average value of a Fcn)
If f is integrable on the closed interval $[a, b]$, then the <u>average value</u> of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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Suppose we divide $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$

If c_i is any pt in the ith subinterval the arithmetic mean of the fcn values @ the c_i's given by

$$a_n = \frac{1}{n}\left[f(c_1) + f(c_2) + \dots f(c_n)\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} f(c_i)$$

$$= \frac{1}{n} \sum_{i=1}^{n} f(c_i) \left(\frac{b-a}{b-a}\right)$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i) \left(\frac{b-a}{n}\right)$$

$$= \frac{1}{b-a} \sum_{i=1}^{n} f(c_i)\, \Delta x$$

Taking the limit as $n \to \infty$:

$$\Longrightarrow \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\, \Delta x = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Ex. 3

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$

$$\text{Avg value} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{1}{4-1} \int_1^4 (3x^2 - 2x)\,dx$$

$$= \frac{1}{3} \left[x^3 - x^2 \right]_1^4$$

$$= \frac{1}{3} \left[\left((4)^3 - (4)^2 \right) - \left(1^3 - 1^2 \right) \right]$$

$$= \frac{1}{3}(48)$$

$$= 16$$

There are many other applications that use integration. If you are interested you can read more in your textbook.