Indefinite Integration

Antiderivatives

Up to this point in the course it has been assumed that we already know the function whose rate of change is of interest to us. Unfortunately, such an assumption is rarely true for a vast majority of the applications of calculus. It is typically more practical to measure the rate of change of a parameter rather than the parameter itself.

For instance, while driving your car the speedometer tells you how fast you are going but does not directly tell you when you will get to your destination. A biologist may know the rate at which a bacteria population is growing but not the exact number of bacteria in that population. It is easier for a physicist to measure the velocity of a moving particle than its position at specific times. In these cases, the problem is to find a function F whose derivative satisfies the known information. If such a function F exists, it is referred to as the antiderivative of f. The process of finding these functions is referred to as integration.

Definition 1: Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives are not necessarily unique. In other words, F is *an* antiderivative and not *the* antiderivative of f. For example the functions

$$F_1(x) = x^4 + 6, \quad F_2(x) = x^4, \quad F_3(x) = x^4 - 17$$

are all antiderivatives of the function $f(x) = 4x^3$.

Theorem 1: Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on the interval I is of the form

$$F(x) + C$$

for all x in I, where C is a constant.

In other words, you can represent the entire family of antiderivatives of a function by adding a constant to a known antiderivative.

For example, the entire family of antiderivatives for $f(x) = 3x^2$ is given by

$$G(x) = x^3 + C$$

where C is some constant. This C is known as the **constant of integration**. The function G can sometimes be referred to as the **general antiderivative** of f. Sometimes, it is possible to find a *specific* value for the constant C. This is covered later in the section on differential equations.

Notation for Integrals

The operation of finding an antiderivative can be called **antidifferentiation** but is more commonly referred to as **indefinite integration**.

There is specific notation that goes along with integration. Figure 1 breaks this notation. The function being integrated is known as the **integrand** while the variable you are integrating with respect to is the **variable of integration**.

The expression $\int f(x)\,dx$ is read as the "integral of f with respect to x".

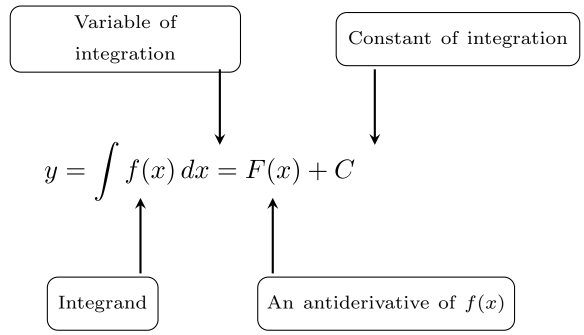

Figure 1: Integral Notation

Basic Integration Rules

Integration can be thought of as the "inverse" or "undoing" of the operation of differentiation. Suppose you take the derivative of a function $f(x)$ and obtain $f'(x)$ then integrating we obtain

$$\int f'(x)\,dx = f(x) + C$$

Due to the "inverse" nature of integration we also have

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

These two equations allow you to obtain integration formulas directly from known differentiation formulas. For example, the most common integration formula you will use is the counterpart to the power rule of differentiation.

Theorem 2: Power Rule for Indefinite Integrals

Let p be a real number then

- For $p \neq -1$

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$$

- For $p = -1$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

where C is an arbitrary constant.

We can also obtain the equivalent basic rules for integrals that we have for derivatives. These are given in Table 1 along with the equivalent differentiation rule.

Table 1: Basic Integration Rules

Constant Rule

$$\frac{d}{dx}[C] = 0 \quad \Longrightarrow \quad \int 0\,dx = C$$

Constant Multiple Rule

$$\frac{d}{dx}[kf(x)] = kf'(x) \quad \Longrightarrow \quad \int kf(x)\,dx = k\int f(x)\,dx$$

Sum & Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \quad \Longrightarrow \quad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 1: Applying Basic Integration Rules

Integrate the following

(a) $\int 6x\,dx$

Solution.

$$\int 6x\,dx = 6\int x\,dx = 6\left(\frac{x^{1+1}}{1+1}\right) + C = \frac{6x^2}{2} + C$$

Note that since C is just an arbitrary constant then $6C$ is also an arbitrary constant. We could call it something different like K but instead just keep using C. In other words, multiplying by 6 will not change the fact that it's an arbitrary constant.

(b) $\int \sqrt{x}\,dx$

Solution.

$$\begin{aligned} \int \sqrt{x}\,dx &= \int x^{1/2}\,dx && \text{Rewrite the integrand} \\ &= \frac{x^{1/2+1}}{1/2+1} + C && \text{Power rule for integrals} \\ &= \frac{2}{3}x^{3/2} + C && \text{Simplify} \end{aligned}$$

(c) $\int \frac{2}{x}\,dx$

Solution.

$$\begin{aligned} \int \frac{2}{x}\,dx &= 2\int \frac{1}{x}\,dx && \text{Constant multiple rule} \\ &= 2\ln x + C && \text{By Theorem 2} \end{aligned}$$

In Table 2 we summarize some of the common integration formulas you will use.

Table 2: Common Integration Formulas

$\int 0\,dx = C$	$\int k\,dx = kx + C$
$\int 1\,dx = x$	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$
$\int kf(x)\,dx = k\int f(x)\,dx$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int e^x\,dx = e^x + C$	$\int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + C$
$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int a^x\,dx = \left(\dfrac{1}{\ln a}\right)a^x + C$

Example 2: Integrating Simple Functions

Integrate the following functions.

(a) $\int (x^2 + 4x - 7)\,dx$

Solution.

$$\begin{aligned}
\int (x^2 + 4x - 7)\,dx &= \int x^2\,dx + \int 4x\,dx - \int 7\,dx && \text{Sum/Difference rule} \\
&= \int x^2\,dx + 4\int x\,dx - 7\int 1\,dx && \text{Constant Multiple rule} \\
&= \frac{1}{2+1}x^{2+1} + 4\left(\frac{1}{1+1}x^{1+1}\right) - 7x + C && \text{Power rule} \\
&= \frac{1}{3}x^3 + 2x^2 - 7x + C && \text{Simplify}
\end{aligned}$$

(b) $\int \dfrac{x^2 + 3x}{\sqrt{x}}\,dx$

Solution. First we rewrite and simplify the integrand.

$$\begin{aligned}
\int \frac{x^2 + 3x}{\sqrt{x}}\,dx &= \int \left(\frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}}\right)\,dx \\
&= \int \left(x^{3/2} + 3x^{1/2}\right)\,dx \\
&= \int x^{3/2}\,dx + 3\int x^{1/2}\,dx \\
&= \frac{1}{3/2+1}x^{3/2+1} + 3\left(\frac{1}{1/2+1}x^{1/2+1}\right) + C \\
&= \frac{2}{5}x^{5/2} + 2x^{3/2} + C
\end{aligned}$$

Integration can get complicated pretty quickly. Just as with derivatives, many antiderivatives can only be found be rewriting the integrand into a simpler form that we know how to handle. The following examples are about as complicated as integration will get in this course. Techniques for how to deal with more complicated integrals than these are explored in the second semester of calculus. The one nice thing about integration is that you can always check your work by taking the derivative. **Note:** We DO NOT have a direct integration rule for the product or quotient of two functions. Techniques for handling most functions of this form are covered in the second semester of calculus. At this level, you must reduce or simplify your integrand into a form that we can easily find an antiderivative for based on our knowledge of derivatives.

Integrals of Trigonometric Functions

We also have integration formulas for trigonometric functions. Some of the more common integration formulas are given in Table 3.

Table 3: Integrals of Trigonometric Functions

$\int \sin x\, du = -\cos u + C$	$\int \csc u\, du = -\ln\lvert\csc u + \cot u\rvert + C$
$\int \cos u\, du = \sin u + C$	$\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert + C$
$\int \tan u\, du = -\ln\lvert\cos u\rvert + C$	$\int \cot u\, du = \ln\lvert\sin u\rvert + C$

Example 3: Integrating a Trigonometric Function

Integrate $\int \frac{\sin x}{\cos^2 x}\, dx$

Solution. First we rewrite the integrand and apply a simple trig identity to get it in a form we recognize.

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\, dx = \int \sec x \tan x\, dx$$

Recalling the differentiation rule $\frac{d}{dx}[\sec x] = \sec x \tan x$ we find that

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \sec x \tan x\, dx = \sec x + C.$$

Introduction to Differential Equations

A **differential equation** is an equation involving an (unknown) function and its derivative. The study of differential equations makes up a large portion of applied mathematics as they are used to model much of the phenomena observed in the real world. The simplest form of a differential equation looks like

$$y'(x) = x^2 + x \quad \text{or} \quad \frac{dy}{dx} = x^2 + x$$

While there are many methods used to obtain solutions to a differential equation, the operation underlying most of them is just plain old integration. By integrating the above equation we obtain

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Since we still have a constant of integration this is known as a **general solution** to the differential equation. In order to find the value of C we need to know the value of our function at a single value of x. This is what is known as an an **initial condition**. Usually, the initial condition is the value of the function at the time $t = 0$ but this is not always the case.

An equation involving derivatives along with an initial condition is what is known as an **initial value problem**. For example, suppose we know that $y(0) = 1$ for the differential equation given above. Then we have the initial value problem

$$y'(x) = x^2 + x, \quad y(0) = 1$$

When we are able to determine the constant of integration we call this a **particular solution** to the differential equation. For example, the above initial value problem above has the particular solution

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$$

Example 4: Solving an Initial Value Problem

Find the function $y(t)$ that satisfies the initial value problem

$$y'(t) = \frac{3}{t} + 6, \quad y(1) = 8$$

Solution. We know that $\int y'(t)\,dt = y(t) + C$ so to obtain $y(t)$ we first integrate $y'(t)$.

$$y(t) = \int \left(\frac{3}{t} + 6\right)\,dt = 3\int \frac{1}{t}\,dt + 6\int 1\,dt = 3\ln t + 6t + C$$

This gives us the **general solution** to the differential equation

$$y(t) = 3\ln t + 6t + C$$

To find a particular solution we need to know a value of the function $y(t)$. We are given that $y(1) = 8$ so we plug this into our general solution and solve for C.

$$8 = 3\ln(1) + 6(1) + C \quad \Longrightarrow 8 = 0 + 6 + C \quad \Longrightarrow C = 2$$

So our **particular solution** is

$$y(t) = 3\ln t + 6t + 2$$

We often have rate of change information for something but need to know the function that governs the rate of change we are observing. This is why the solution of differential equations is such a key part of applied mathematics, engineering, and other sciences. We will demonstrate the solution of these types of problems through some examples.

The most common type of initial value problem you will see at this level is an application involving velocity, acceleration and position. Previously, we were given a position function $s(t)$ and asked to find its velocity, $v(t)$ or its acceleration $a(t)$. We found these functions by taking the derivative of the position function. In the real world, we actually have the opposite situation. We will have data (and therefore a function) governing the acceleration or velocity and then need to find the position at a particular time.

Why is this? Well, it is much easier to determine how fast an object is going by recording the distance traveled over a certain period of time rather than trying to measure the exact path traveled by the object. Now we will solve a motion problem. In this situation, we will deal with two initial conditions. In general, the number of initial conditions you have for a problem will match the highest order of derivative you have. Recall that $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

Example 5: Solving a Motion Initial Value Problem

Find the position function given

$$a(t) = 4, \quad v(0) = -3, \quad s(0) = 2$$

Solution. Recall that $a(t) = v'(t) = s''(t)$ so we integrate $a(t)$ to find $v(t)$,

$$v(t) = \int a(t)\,dt = \int 4\,dt = 4t + C$$

The value of the constant C is determined by the initial condition $v(0) = -3$,

$$-3 = v(0) = 4(0) + C \qquad \Longrightarrow C = -3.$$

So our velocity function is

$$v(t) = 4t - 3.$$

To find the position function we recall that $v(t) = s'(t)$ and so

$$s(t) = \int v(t)\,dt = \int 4t - 3\,dt = 4\left(\frac{1}{2}t^2\right) - 3t + D = 2t^2 - 3t + D.$$

We use the initial condition $s(0) = 2$ to solve for D. Note that the use of D for the constant of integration is simply to differentiate it from the previously used C. The letter you choose to represent this constant is essentially arbitrary.

$$2 = 2(0)^3 - 3(0) + D \qquad \Longrightarrow D = 2.$$

So the position function is

$$s(t) = 2t^2 - 3t + 2$$

You may not always have initial conditions with respect to a derivative. This is demonstrated in the following example.

Example 6: Solving an Initial Value Problem

Consider the problem addressed in Example 5 where $a(t) = 4$ but now with the initial conditions $s(0) = 2$ and $s(2) = 4$.

Solution. In Example 5 we found that

$$v(t) = 4t + C.$$

Since we have no other information about $v(t)$ we integrate again to obtain the position function

$$s(t) = \int v(t)\,dt = \int 4t + C\,dt = 2t^2 + Ct + D$$

Now we use the initial conditions $s(0) = 2$ and $s(2) = 4$ to solve for C and D.

$$2 = s(0) = 2(0)^2 + C(0) + D$$
$$4 = s(2) = 2(2)^2 + C(2) + D$$

The first equation tells us that $D = 2$ so we plug this result into the second equation to find

$$2C + 2 = -4 \implies 2C = -6 \implies C = -3.$$

Plugging in $D = 2$ and $C = -3$ we obtain the particular solution

$$s(t) = 2t^2 - 3t + 2$$

Sigma Notation and Basic Summation Rules

When we started our discussion on the derivative we used the tangent line to provide a geometric meaning to its definition. In much the same way, we will use area problems to formulate the idea of definite integration. At first glance, the concept of the indefinite integral and definite integration may seem unrelated to each other, but both of these processes are closely related through the Fundamental Theorem of Calculus.

Before we can define the definite integral we need to pause for a moment to introduce a concise notation for sums. To be clear, integration is not the only time that you will see summation notation as students! It is used in the study of series which you will see in the second semester of calculus in addition to the solution of differential equations, complexity reduction in nonlinear systems, approximation theory and a variety of other applications.

Sigma Notation

Adding up a bunch of terms can be cumbersome when there are a large number of terms. Luckily, we have notation that will make a sum much easier to handle. This is known as **sigma notation**. The notation is represented by the upper case version of the Greek letter sigma. Although it looks like the letter E in English, it is the equivalent of the letter S in the Greek alphabet to represent the word sum. This notation is a valuable tool as it allows us to express sums in a compact way.

Definition 2: Sigma Notation

The sum of n terms $a_1, a_2, a_3 \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation** a_i is the i**th term** of the sum, the **upper bound of summation** is n and the **lower bound of summation** is 1. Note that i and n are integers.

Essentially, this notation allows us to represent the a sum of many terms in a repeatable pattern. The index i can be interpreted as the "count" of the term and n the number of terms you want to add up. Both of these values must of course be integers. Also note that the lower bound doesn't have to be 1. In fact, any integer less than or equal to the upper bound is valid.

The index (for example i) in a sum is a dummy variable that serves as a place holder for an actual integer value to be used in that term. Any letter can be used as the index of summation. The letters i, j and k are often used. It is important to recognize that the index of summation does not appear in the terms of the expanded sum.

Example 7: Evaluating Basic Summations

Evaluate the following sums

(a) $\displaystyle\sum_{k=1}^{5} k$

Solution. In words this notation tells us "add up the integers from 1 to 5". Note that there is no k in our final result.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

(b) $\displaystyle\sum_{j=1}^{3} (1 + j^2)$

Solution.

$$\sum_{j=1}^{3} (1 + j^2) = (1 + (1)^2) + (1 + (2)^2) + (1 + (3)^2) = 2 + 5 + 10 = 17$$

(c) $\displaystyle\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right)$

Solution.

$$\begin{aligned}\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) &= \sin\left(\frac{0\pi}{2}\right) + \sin\left(\frac{1\pi}{2}\right) + \sin\left(\frac{2\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{4\pi}{2}\right) \\ &= \sin(0) + \sin\left(\frac{\pi}{2}\right) + \sin(\pi) + \sin\left(\frac{3\pi}{2}\right) + \sin(2\pi) \\ &= 0 + 1 + 0 - 1 + 0 = 0\end{aligned}$$

Basic Summation Rules

Just as with derivatives and integrals we also have basic rules for a summation. These rules of course deal with multiplying a sum by a constant and the sum and difference of two summations.

Theorem 3: Basic Summation Rules

Suppose that $\{a_1, a_2 \ldots, a_k\}$ and $\{b_1, b_2 \ldots, b_k\}$ are two sets of real numbers and that c is also a real number.

- Constant Multiple Rule: $\displaystyle\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$
- Sum and Difference Rule: $\displaystyle\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Similar to the process of differentiation and integration it is often easier to use the basic summation rules to simplify the process of evaluating a sum. We demonstrate this process in the following examples.

Example 8: Using Basic Summation Rules

Evaluate the following summations.

(a) $\displaystyle\sum_{k=1}^{5} 3k$

Solution.

$$\sum_{k=1}^{5} 3k = 3\sum_{k=1}^{5} k = 3(15) = 45$$

(b) $\displaystyle\sum_{j=1}^{3} (j + j^2)$

Solution.

$$\begin{aligned}\sum_{j=1}^{3} (j + j^2) &= \sum_{j=1}^{3} j + \sum_{j=1}^{3} j^2 \\ &= (1+2+3) + \left((1)^2 + (2)^2 + (3)^2\right) \\ &= 20\end{aligned}$$

(c) $\displaystyle\sum_{n=1}^{3} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right)$

Solution. In a previous example we found that $\displaystyle\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) = 0$. The Sum and Difference rule lets us use this prior knowledge quickly compute this sum.

$$\begin{aligned}\sum_{n=0}^{4} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right) &= \sum_{n=0}^{4} \cos(n\pi) - \sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) \\ &= \sum_{n=0}^{4} \cos(n\pi) - 0 \\ &= \cos(0\pi) + \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) \\ &= 1 - 1 + 1 - 1 \\ &= 0\end{aligned}$$

What about upper bounds of summation that are very large? It can get tedious very quickly if calculating an even slightly more involved summation. Luckily, for some expressions we have patterns that allow us to develop simple formulas for evaluating certain summations.

Theorem 4: Summation Formulas

1. $\displaystyle\sum_{i=1}^{n} c = cn$
2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

As we will see in the following examples, the process of evaluating a sum is made significantly easier by combining these known sum formulas with the basic summation rules.

Example 9: Evaluating Summations Using Formulas

Evaluate the following summations.

(a) $\displaystyle\sum_{p=1}^{105} p$

Solution.

$$\sum_{p=1}^{105} p = \frac{(105)\Big((105)+1\Big)}{2} = 5565$$

(b) $\displaystyle\sum_{k=1}^{35} (1+k^2)$

Solution.

$$\begin{aligned}\sum_{k=1}^{35}(1+k^2) &= \sum_{k=1}^{35} 1 + \sum_{k=1}^{35} k^2 \\ &= 1(35) + \frac{(35)\Big((35)+1\Big)\Big(2(35)+1\Big)}{6} = 14945\end{aligned}$$

(c) $\displaystyle\sum_{m=1}^{72} (2m+m^3)$

Solution.

$$\begin{aligned}\sum_{m=1}^{12}(2m+m^3) &= 2\sum_{m=1}^{12} m + \sum_{m=1}^{12} m^3 \\ &= 2\left(\frac{(12)\Big((12)+1\Big)}{2}\right) + \left(\frac{(12)^2\Big((12)+1\Big)^2}{4}\right) \\ &= 2\,(78) + (6084) = 6240\end{aligned}$$

Typically it is rare that formulas like those presented in 4 can be obtained for a given series. There is one series for which we do have a formula for obtaining its value. This is the Geometric series.

Definition 3: Geometric Series

For real numbers $a \neq 0$ and r, a series of the form

$$\sum_{k=0}^{n} ar^k$$

is called a **Geometric Series**. If and only if $|r| \leq 1$ then the general formula for the value of a geometric series is

$$\sum_{k=0}^{n} ar^k = a\frac{1-r^n}{1-r}$$

What happens when the upper bound of a summation gets arbitrarily large? The upper bound of a summation does not need to be a finite number. Now that we have some general formulas for sums we might be curious to know the behavior of this sum as more and more terms are added. To accomplish this task we can use our old friend: Limits!

Example 10: Evaluating Summations Using Formulas

Evaluate the following summations.

(a) $\displaystyle\sum_{k=1}^{\infty} k$

Solution.

$$\sum_{k=1}^{\infty} k = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

(b) $\displaystyle\sum_{k=1}^{\infty} 2^k$

Solution.

$$\begin{aligned}\sum_{k=1}^{\infty} 2^k &= \lim_{n\to\infty} \sum_{k=1}^{n} 2^k = \lim_{n\to\infty} \frac{1-2^n}{1-2} \\ &= \lim_{n\to\infty} (2^n - 1) \\ &= \lim_{n\to\infty} 2^n - \lim_{n\to\infty} 1 \\ &= \infty - 1 = \infty\end{aligned}$$

(c) $\displaystyle\sum_{k=1}^{\infty} 2^{-k}$

Solution.

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-k} &= \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = \lim_{n\to\infty} \left(2 - 2\left(\frac{1}{2}\right)^n\right) \\ &= \lim_{n\to\infty} 2 - 2\lim_{n\to\infty} \left(\frac{1}{2}\right)^n \\ &= 2 - 0 = 2\end{aligned}$$

Area Under Curves

We already know several formulas for computing the area of a region in Euclidean geometry. For example, the area of a rectangle is given by $A = bh$ where b and h represent the length of the rectangles base and height respectively. In fact, it is possible to compute the area of any polygon using the area formulas for triangles and rectangles. That is, the area of a polygon can be found by dividing it into triangles and rectangles and adding their areas.

What do we do if we have a shape with curved with curved sides, like a circle? Here is where we would run into problems.

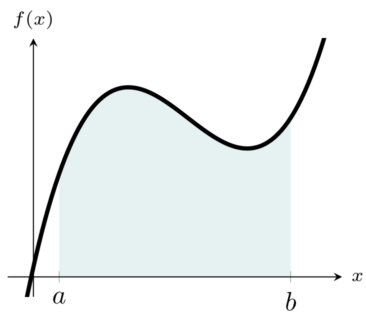

Figure 2: Region under a curve

Our goal in this section is to find a solution to the *Area Problem*: Find the area of the region R that lies under the curve $y = f(x)$ between the boundaries formed from the vertical lines $x = a$ to $x = b$. This is pictured in Figure 2.

Recall how we defined the tangent line. We used secant lines to approximate the value of the slope of the tangent line and then took the limit of these approximations. To approximate the area under a curve we will use a similar idea.

We first approximate the area of the region using rectangles. We can get a more accurate approximation with the more rectangles that we use. The exact value for the area under the curve will then be obtained by taking the number of rectangles to infinity via a limit. Unfortunately, this process can be extremely confusing without an appropriate mathematical framework and notation.

Approximating Area by Riemann Sums

In order to approximate an area using rectangles we must first decide how we will divide the interval from $[a, b]$ into *subintervals*. Each subinterval will define the base of each rectangle used in our approximation. For simplicity we will use subintervals of equal length. This is referred to as a regular partition for the interval $[a, b]$.

Definition 4: Regular Partition

A **regular partition** is the division of a closed interval $[a, b]$ into n subintervals of equal length where

- The **subintervals** are given by

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots\ ,\ [x_{n-1}, x_n]$$

- The **grid points** are the endpoints of the subintervals given by x_0, x_1, …, x_n where the kth grid point is given by

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, \cdots, n$$

- The **length** of each subinterval is given by $\Delta x = \frac{b-a}{n}$

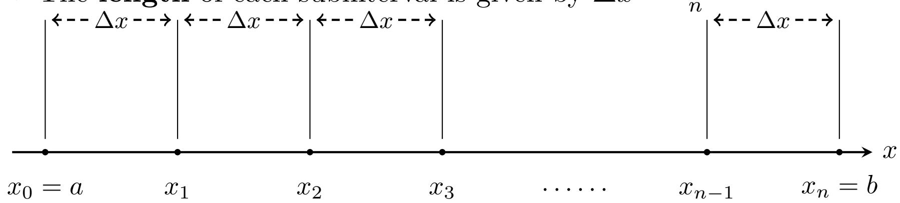

Figure 3: A Regular Partion of $[a, b]$

Each subinterval of the chosen partition serves as a base for a rectangle used to approximate the area under the curve. That is, we can approximate the area under the curve on each subinterval with the area of a rectangle. The use of a a regular partition is convenient for us since the length of each subinterval will be the same (i.e., each rectangle used in our approximation will have the same base length).

The other ingredient we need to define a rectangle is its height. In the kth subinterval $[x_{k-1}, x_k]$, we choose a **sample point** x_k^* and build a rectangle whose height is $f(x_k^*)$, the value of f at x_k^*. So for $b = \Delta x$ and $h = f(x_k^*)$ then the area of the rectangle on the kth subinterval is given by

$$b \cdot h = \Delta x f(x_k^*)$$

The choice of sample point will vary depending on the method we are using. We can see that summing the areas of all of these rectangles gives an approximation of the area under the curve.

Definition 5: Riemann Sum on a Regular Partition

Let f be continuous and non-negative on the interval $[a, b]$. For a regular partition of $[a, b]$ into n subintervals a **area approximation** of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x, \quad x_{i-1} \leq x_i^* \leq x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

This type of area approximation is referred to as a *Riemann sum*. Notice that x_k^* is an arbitrary point on the subinterval $[x_{k-1}, x_k]$. The choice of x_k^* determines the type of Riemann sum being used. Three common choices of x_k^* are x_{k-1}, x_k and $\frac{x_{k-1}+x_k}{2}$. Riemann sums constructed using these three choices are referred to as left, right and midpoint Riemann sums respectively.

Definition 6: Left Riemann Sum

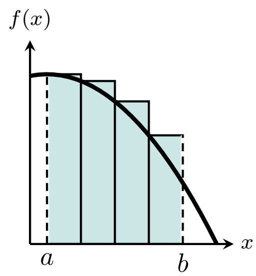

Fig. 4
Left Riemann Sum

Let f be continuous and non-negative on the interval $[a,b]$, which is divided into n subintervals of equal length Δx.

A **Left Riemann sum** for f on $[a,b]$ is given by

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots f(x_{n-1})\Delta x$$

Definition 7: Right Riemann Sum

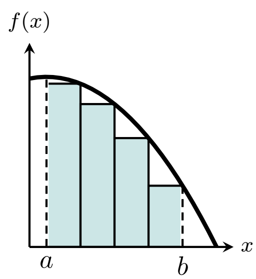

Fig. 5
Right Riemann Sum

Let f be continuous and non-negative on the interval $[a,b]$, which is divided into n subintervals of equal length Δx.

A **Right Riemann sum** for f on $[a,b]$ is given by

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x$$

Definition 8: Midpoint Riemann Sum

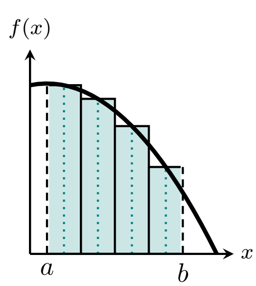

Fig. 6
Midpoint Riemann Sum

Let f be continuous and non-negative on the interval $[a,b]$, which is divided into n subintervals of equal length Δx.

A **Midpoint Riemann sum** for f on $[a,b]$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x$$

where $x_i^* = \dfrac{x_{i-1}+x_i}{2}$

Without knowing the shape of the function (or it's exact value) there is no "best" way to choose the point x_k^*. That is, you don't necessarily know which type of Riemann sum is more accurate or easier to use than the others.

Example 11: Approximating Area Under a Curve

Use left, right and midpoint Riemann sums with $n = 5$ subintervals to estimate the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis, between $x = 0$ and $x = 2$.
Compare the value of each approximation to the exact value of the area, $\dfrac{22}{3}$.

Solution. For this example we will use the definition of a regular partition with $a = 0$, $b = 2$ and $n = 5$. Thus, the length of each subinterval is given by

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{5} = \frac{2}{5} = 0.4.$$

The value of the kth grid point is

$$x_k = a + k\Delta x, = 0 + k\left(\frac{2}{5}\right) = k\left(\frac{2}{5}\right) \qquad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

In the example we find that

$$\begin{aligned}
x_0 &= 0 + 0\left(\frac{2}{5}\right) = 0, & f(x_0) &= -(0)^2 + 5 = 5\\
x_1 &= 0 + 1\left(\frac{2}{5}\right) = \frac{2}{5}, & f(x_1) &= -\left(\frac{2}{5}\right)^2 + 5 = 4.84\\
x_2 &= 0 + 2\left(\frac{2}{5}\right) = \frac{4}{5}, & f(x_2) &= -\left(\frac{4}{5}\right)^2 + 5 = 4.36\\
x_3 &= 0 + 3\left(\frac{2}{5}\right) = \frac{6}{5}, & f(x_3) &= -\left(\frac{6}{5}\right)^2 + 5 = 3.56\\
x_4 &= 0 + 4\left(\frac{2}{5}\right) = \frac{8}{5}, & f(x_4) &= -\left(\frac{8}{5}\right)^2 + 5 = 2.44\\
x_5 &= 0 + 5\left(\frac{2}{5}\right) = 2, & f(x_5) &= -(2)^2 + 5 = 1.
\end{aligned}$$

The left Riemann sum is given by

$$\begin{aligned}
\sum_{i=1}^{5} f(x_{i-1})\Delta x &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x\\
&= 5(0.4) + 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4)\\
&= 8.08.
\end{aligned}$$

The right Riemann sum is given by

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x\\
&= 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) + 1(0.4)\\
&= 6.48.
\end{aligned}$$

For the function $f(x) = f(x) = -x^2 + 5$ the left Riemann sum was found to be an overestimate of the the area with about an 10% relative error. The right Riemann sum gave an underestimate of the area with a relative error of about 12%.

Note that we could have done this exercise with the midpoint Riemann sum as well. In fact the midpoint Riemann sum would give an approximate value of 7.36. Try it for yourself!

In the previous example we saw that the resulting approximations where fairly inaccurate. Depending on the setting an approximation with a 10% error could be unacceptable. The primary source of error in that approximation comes from the small number of rectangles used ($n = 5$).

If we wanted a more precise answer we would have needed to use more subintervals to define our partition. However, as we increase the value of n the amount of work required to get the approximation also increases.

If the value of n is large it is helpful to use the known summation formulas for basic sums to simplify the work. In the following examples we will use some of the rules presented in 4 to evaluate some Riemann sums with a large value of n.

Example 12: Evaluating Riemann Sums When n is Large

Evaluate the left Riemann sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $a = 0$ and $b = 2$ using $n = 40$ subintervals.

Solution. With $n = 50$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{40} = \frac{1}{20} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of x_k^* in the kth grid point will is

$$x_k^* = a + (k-1)\Delta x = 0 + (k-1)0.05 = 0.05k - 0.05 \quad \text{for} \quad k = 1, 2, \cdots, 40.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{40} f(0.05(k-1))0.05 \\
&= \sum_{i=1}^{40} (0.05(k-1))^2\, 0.05 \\
&= (0.05)^3 \sum_{i=1}^{40}(k^2 - 2k + 1) \\
&= (0.05)^3 \left(\frac{40(40+1)(2(40)+1)}{6} - 2\left(\frac{40(40+1)}{2}\right) + 40 \right) \\
&= (0.05)^3(20540) \\
&= 2.5675
\end{aligned}$$

Determining Exact Area Under a Curve

In the next section we will see how Riemann sums are used to define the definite integral. For now we close our discussion about approximate area with the following remark. The accuracy of the area approximation using Riemann sums can be improved by increasing the value of n, the number of intervals used to define the partition of $[a, b]$. It then stands to reason that we could determine the exact value of the area under the curve if an infinite number of rectangles are used in the Riemann sum. We summarize this result in the following definition.

Definition 9: Definition of Area Under the Curve

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** of a the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

In the following example we demonstrate how Riemann sums can be used to exactly compute the area under a curve.

Example 13: Evaluating Summations Using Formulas

Use left Riemann sums to determine the exact area of the region bounded by the graph of $f(x) = x^2$ and the x-axis between $a = 0$ and $b = 2$.

Solution. To begin, partition $[0, 2]$ into n subintervals each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of x_k^* in the kth grid point will is

$$x_k^* = a + (k-1)\Delta x = \frac{2(k-1)}{n} \quad \text{for} \quad k = 1, 2, \cdots, n.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{i=1}^{n} f(x_k^*)\Delta x &= \sum_{i=1}^{n} f\left(\frac{2(k-1)}{n}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(k-1)}{n}\right)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(k-1)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n} k^2 - 2k + 1 \\
&= \left(\frac{2}{n}\right)^3 \left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right) \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

By taking the limit as $n \to \infty$ of the left Riemann sum we obtain the exact area

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty} \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} = \frac{8}{3}.$$

Note that the type of Riemann sums used to find the area under the curve in the previous example does not matter. Instead of left Riemann sums we could have used right or midpoint Riemann sums and arrived at the same answer. Try it! Unfortunately, this property is not true for every function. When the area under the curve of a function is the same regardless of the type of Riemann sum used we say that the function is integrable. In the following lecture we will define the definite integral as the limit of general Riemann sums.

Introduction

We saw in the previous lecture that a limit of the form

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

arises when computing the area under the curve of a function f. It turns out that this same limit occurs in a wide variety of situations even when f is not necessarily a positive function. In fact limits of this form arise when finding the length of curves, volumes of solids, centers of mass, and work as well as other quantities of interest. Because this type of limit is so important we give it a special name and notation. In the following section we will define the definite integral with a more general form of this limit. Towards the end of this lecture we will see that the operations of differentiation and integration are deeply connected, through the Fundamental Theorem of Calculus.

Definite Integrals

In the previous lecture we discussed Riemann sums with regular partitions. In reality we can define a Riemann sum for any partition of $[a, b]$.

Definition 10: General Reimann Sum

Suppose
$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n],$$
are subintervals of $[a, b]$ with
$$a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b.$$
For $k = 1, \cdots, n$ let $\Delta x_k = x_k - x_{k-1}$ be the length of the subinterval $[x_{k-1}, x_k]$ and x_k^* be an arbitrary sample point in $[x_{k-1}, x_k]$. If f is defined on $[a, b]$, the sum
$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$
is called a **general Riemann sum for f on** $[a, b]$.

We have seen examples where it does not matter what type of Riemann sum we use to define the area under the curve. If the area under the curve of a function will be the same no matter what type of Riemann sum we use we say that that function is integrable. The area under the curve is then given by the **Definite Integral**.

Definition 11: Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$
exists and is unique over all partitions of $[a, b]$ and any choice of sample point x_k*. This limit is called the **definite integral of f from a to b**, which we write
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Recall tha the symbol $\int$ is called an integral sign. In the notation
$$\int_a^b f(x)dx$$
a is the lower limit of integration, b the upper limit of integration, and $f(x)$ is the integrand. The symbol dx has no official meaning by itself. However, it does signal the variable that we are integrating with respect to. It may not seem like an important symbol right now, since the functions we work with are of typically of one variable however, in multivariable calculus this is very important. Recall that the procedure of calculating an integral is called **integration**.

Note the definite integral is a number; it does not depend on x. In fact we could use any letter in its place without changing the value of the definite integral
$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt$$

As we have already mentioned, the area under the curve of a function can be computed with the definite integral. What about when the graph of the curve does not lie above the x-axis? In this case we must talk about net or signed area. The Riemann sums will then approximate the area of the regions that lie above the x-axis *minus* the area of the regions that lie below the x-axis. In such situations it is possible that the area computed with a definite integral will be negative in value. Since our concept of area requires that it be a positive quantity we use the terms net area or signed area when referring to the area computed with definite integrals.

Definition 12: Net Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of areas of the parts of R that lie below the x-axis on $[a, b]$.

We can now formally state how to find the area of a region using the definite integral in the following theorem.

Theorem 5: Definite Integral as Area of a Region

If f is continuous on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Net Area} = \int_a^b f(x)\,dx$$

At long last we have a formal definition of the definite integral! However, this definition comes with a serious constraint. A function is only integrable if this limit exists and is unique for any type of Riemann sum. Just like it was with differentiation it is important to ask ourselves: "What types of functions can we integrate?". The conditions for a function to be *integrable* is given in the following theorem. The proof of these results is beyond the scope of this course.

Theorem 6: Integrable Functions

If a function f is continuous on the closed interval $[a, b]$, or if f is bounded on $[a, b]$ with a finite number of discontinuities then f is integrable on $[a, b]$. That is $\displaystyle\int_a^b f(x)\,dx$ exists.

If we know that a function is integrable we can compute the area under its curve using the definition of the definite integral. This process can be tedious but a familiarity with it will help us appreciate the simplicity of the Fundamental Theorem of Calculus.

Example 14: Evaluating Definite Integral as a Limit

Using the definition of the Definite Integral evaluate the value of $\int_0^3 x^3 - 6x\,dx$.

Solution. Since $x^3 - 6x$ is continuous on $[0, 3]$ it is also integrable. Thus, the *limit* of any Riemann sum will yield the same value, regardless of our choice of sample points. For this problem I will choose to use right Riemann sums to evaluate the definite integral. For n evenly spaced subintervals we have

$$\Delta x = \frac{b-1}{n} = 3 - 0n = \frac{3}{n}$$

and

$$x_k = a + k\Delta x = 0 + k\frac{3}{n} = \frac{3k}{n}$$

Using the definition of right Riemann sums we have

$$\begin{aligned}
\int_0^3 x^3 - 6x\,dx &= \lim_{n\to\infty} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{k=1}^{n}\left[\left(\frac{3k}{n}\right)^3 - 6\left(\frac{3k}{n}\right)\right] \\
&= \lim_{n\to\infty} \frac{3}{n}\sum_{k=1}^{n}\left[\frac{27}{n^3}k^3 - \frac{18}{n}k\right] \\
&= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{k=1}^{n}k^3 - \frac{54}{n^2}\sum_{k=1}^{n}k\right] \\
&= \lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{54}{n^2}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty}\left[\frac{81}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) - \frac{54}{2}\left(1 + \frac{1}{n}\right)\right] \\
&= \frac{81}{4} - 27 \\
&= \frac{-27}{4}
\end{aligned}$$

Since the definite integral is defined as the limit of a sum it inherits a lot of the same properties.

Theorem 7: Basic Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

$$\int_a^b kf\,dx = k\int_a^b f\,dx \quad \text{and} \quad \int_a^b f \pm g\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx.$$

Recall that the the definite integral was defined with the assumption that $a < b$. There are however occasions when it is necessary to reverse the limits of integration ($b < a$), or integrate over a single point ($b = a$). Sometimes it may be necessary to integrate the function separately over parts of the interval ($a < c < b$). This is common for absolute and piecewise defined functions. To handle these cases we have the following theorem.

Theorem 8: Special Properties of Definite Integrals

(i) $\displaystyle\int_a^a f(x)\,dx = 0$

(ii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The Fundamental Theorem of Calculus

We have computed definite integral using the definition as a limit of Riemann sums. We saw that this procedure is sometimes long and difficult. In fact, it is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals. However, in order to understand this method we will need a better understanding of the inverse relationship between differentiation and integration. The bridge we seek to connect these two ideas is known as the Fundamental Theorem of Calculus (FTOC). Since the proof of the FTOC is somewhat complicated, most textbooks present the FTOC in two separate parts. The first part is basically a useful property that can be used to validate the more useful second part.

Theorem 9: The Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a, b]$ then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b$$

is continuous on $[a, b]$ and differentiable of (a, b). The area function satisfies

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Since A is an antiderivative of f on $[a, b]$ it is one short step to a powerful method for evaluating definite integrals. Remember that any two antiderivatives of f differ by a constant. Assuming that F is any antiderivative of f on $[a, b]$ we have

$$F(x) = A(x) + C, \quad \text{for } a \le x \le b$$

Noting that if $x = a$ then we have

$$A(a) = \int_a^a f(t)\,dt = 0$$

and so $A(a) = 0$. It follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b)$$

Writing $A(b)$ in terms of the definite integral we have the result that

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

The result is essentially the second part of the Fundamental Theorem of Calculus, sometimes referred to as the Evaluation Theorem.

Theorem 10: The Fundamental Theorem of Calculus (Part 2)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We now have a way to compute definite integrals using our knowledge of derivatives. Are you excited? You should be excited! At this point we are free of the dreaded tyranny that is the limit definition of the definite integral. Now if we know the antiderivative of a given function we can easily compute the area under its curve.

Lets pause for a moment to discuss what the fundamental theorem of calculus really means. Part one says that

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$$

and part two says that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are an inverse processes i.e. they each undo each other. This realization is essential for our mastery of the process of integration.

Guidelines for Using the Fundamental Theorem of Calculus

- Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
- When applying the Fundamental Theorem of Calculus the following notation is convenient

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

- It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = (F(x) + C)\Big|_a^b = \Big[(F(b) + C) - (F(a) + C)\Big] = F(b) - F(a)$$

With the help of the Fundemental Theorem of Calculus we are now capable of evaluating a wide range of definite integrals. We demonstrate this process in the following examples.

Example 15: Evaluating a Definite Integral

Evaluate each definite integral.

(a) $\displaystyle\int_1^2 (x^2 - 3)\, dx$

Solution.

$$\int_1^2 (x^2 - 3)\, dx = \left(\frac{1}{3}x^3 - 3x\right)\Bigg|_1^2 = \left(\frac{1}{3}(2)^3 - 3(2)\right) - \left(\frac{1}{3}(1)^3 - 3(1)\right) = \frac{-2}{3}$$

(b) $\displaystyle\int_1^4 3\sqrt{x}\, dx$

Solution.

$$\int_1^4 3\sqrt{x}\, dx = 3\int_1^4 \sqrt{x}\, dx = 3\left(\frac{2}{3}x^{3/2}\right)\Bigg|_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$$

(c) $\displaystyle\int_0^{\pi/4} \sec^2 x\, dx$

Solution.

$$\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = \tan(\pi/4) - \tan(0) = 1$$

It is worth noting that we have only scratched the surface of the study of integration. Just like with differentiation there exists several rules and techniques that are needed to evaluate more complicated integrals. A more complete picture of integration is typically presented in a second semester calculus course.