Lecture # 15: Series; Integral & Divergence Tests Date: Mon. 11/12/18

Series

Def. A series is the sum of the terms of a sequence. Given a sequence

$$\{a_1, a_2, \ldots, a_n\}$$

the corresponding infinite series is

$$a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{\infty} a_k$$

As with sequences, our main question will be regarding convergence or divergence.

With a sequence we can determine if it converges and what the seq. converges to.

However, with series, aside from a few special cases, we can only determine if the series converges / diverges.

Ex. 1 For the seq. $\{1, 1, 1, \ldots\}$ the series is

$$1 + 1 + \ldots = \sum_{k=1}^{\infty} 1$$

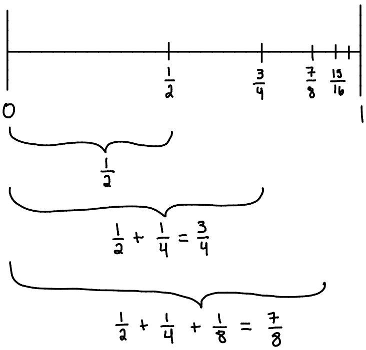
Clearly, this sum will get larger & larger

Lecture #15: Series; Integral & Divergence Tests — Date: Mon. 11/12/18

Ex. 2) Consider the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

Geometrically,



The series seems to be approaching 1.

This idea of progressively adding up the series is important.

Def. Given a series $a_1 + a_2 + \dots = \sum_{k=1}^{\infty} a_k$ the partial sums

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$S_n = a_1 + a_2 + \dots + a_n = \underbrace{\sum_{k=1}^{n} a_k}_{\text{a finite sum!}}$$

Form the sequence of partial sums

$$\{S_1, S_2, \dots, S_n, \dots\}$$

Lecture # 15: Series; Integral & Divergence Tests Date: Mon. 11/12/18

Now we can discuss whether this seq. has a limit. If the limit exists, then the series <u>converges</u> & write

$$\sum_{k=1}^{\infty} a_k = L$$

otherwise, the series diverges.

In other words, the sum of a series is the limit of partial sums.

<u>Ex.3</u> Consider the infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

(a) Find the 1st 4 terms of the partial sums

$$S_1 = a_1 = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{4}{6} + \frac{1}{3(3+1)} = \frac{4}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{9}{12} + \frac{1}{4(4+1)} = \frac{9}{12} + \frac{1}{20} = \frac{4}{5}$$

(b) What is the pattern for S_n?
Does it have a limit?

$$S_n = \frac{n}{(n+1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{H}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

Lecture # 15: Series; Integral & Divergence Tests Date: Mon. 11/12/18

Properties of Infinite Series

Let $\sum a_n$ & $\sum b_n$ be convergent series & let $A, B, c \in \mathbb{R}$ s.t.

$$\sum a_n = A \quad \& \quad \sum b_n = B$$

then

(i) $\sum c a_n = c \sum a_n = cA$

(ii) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$

There are 2 types of series for which we can determine convergence <u>and</u> compute the sum. telescoping series & Geometric series.

Telescoping series

A telescoping series has the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots$$

Since most of the terms cancel, we end up with

$$S_n = b_1 - b_{n+1}$$

This series will converge if

$$\lim_{n\to\infty} (b_1 - b_{n+1}) = b_1 - \lim_{b\to\infty} b_{n+1}$$

is a finite value.

Ex. 4 $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

Use partial fractions to break up the fraction.

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

Find the n^{th} partial sum:

$$S_n = \underbrace{\left(\frac{1}{1} - \frac{1}{2}\right)}_{a_1} + \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_2} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{a_3} + \dots + \underbrace{\left(\frac{1}{n-1} - \frac{1}{n}\right)}_{a_{n-1}} + \underbrace{\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{a_n}$$

$$= 1 - \frac{1}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{H}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

So series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges & $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$

Harmonic Series

<u>Def</u> The Harmonic Series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

The Harmonic series <u>diverges</u>

Lecture #15: Series; Integral & Divergence Tests Date: Mon. 11/12/18

Geometric Series

Def a <u>geometric series</u> is a series with terms from a geometric seq. i.e. a geometric series has form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots + ar^n + \dots$$

for $a \neq 0$

The seq. of partial sums is

$$\{a,\ a+ar,\ a+ar+ar^2,\ \dots\}$$

with

$$S_n = \underbrace{a + ar + \dots + ar^{n-1}}_{n \text{ terms}} = \sum_{k=0}^{n-1} ar^k$$

by using a "tricky trick" we can rewrite

$$S_n = a + ar + \dots + ar^{n-1}$$

$$r\,S_n = ar + ar^2 + \dots + ar^n$$

Subtracting

$$S_n - rS_n = a + ar + \dots + ar^{n-1} - (ar + ar^2 + \dots + ar^n)$$

$$= a - ar^n$$

so

$$S_n - rS_n = a - ar^n$$

Solving for S_n we have

$$S_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}$$

Convergence / divergence depends on whether $\{S_n\}$ converges.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$

$$= \frac{\lim_{n\to\infty} a - a\lim_{n\to\infty} r^n}{\lim_{n\to\infty} 1 - \lim_{n\to\infty} r}$$

$$= \frac{a - a\lim_{n\to\infty} r^n}{1-r}$$ only term w/ n in it!

We know what $\lim_{n\to\infty} r^n$ is for different values of r.

If $|r|<1 \Longrightarrow \lim_{n\to\infty} r^n = 0$ then $\lim_{n\to\infty} S_n = \frac{a}{1-r}$

So for $|r|<1$, $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

Lecture # 15: Series; Integral & Divergence Tests   Date: Mon. 11/12/18

If $r = 1$, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} 1^n = 1$

then $\lim_{n\to\infty} S_n = \frac{a(1-1)}{1-r} = \frac{0}{0}$ An ind. form!

i.e.

$$\sum_{k=0}^{\infty} ar^k = a + a + a + \dots$$

this series does <u>not</u> converge.

<u>Thm</u> (Geometric Series Test)

Let $a \neq 0$ & r be a real number.

If $|r|<1$, then $\sum_{k=0}^{\infty} ar^k$ converges &

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If $|r| \geq 1$, then thes series diverges.

<u>Ex. 5</u> Does $\sum_{k=0}^{\infty} e^{-k}$ converge?

$a =$ <u>1</u> $r =$ <u>e^{-1}</u> $= \frac{1}{e}$

By the <u>Geometric Series test</u> since

<u>$|r| = |\frac{1}{e}| < 1$</u>, the series <u>converges</u>

$$\sum_{k=0}^{\infty} e^{-k} = \frac{1}{1-\frac{1}{e}}$$

Lecture # 15: Series; Integral & Divergence Tests Date: Mon. 11/12/18

Series Tests

For many series, it is not possible (or easy) to determine an expression for the n^{th} partial sum. This means we wont be able to det. the actual value of the series.

It will still be necessary to determine if a series converges or diverges.

There are a variety of tests we can use to do this. While many of these tests involve some calculations, showing calculations alone will not be enough.

You will need to ID the test used, explicitly ID conditions that were met & clearly state your conclusion.

You will be required to end <u>every</u> use of a particular series test with the following statement.

Lecture # 15: Series; Integral & Divergence Tests

Date: Mon. 11/12/18

Series "Mad Libs"

By the ______________________
(state test used)

Since ______________________
(condition(s) that were met)

then the series ______________
(converges/diverges)

Divergence Test

<u>Thm</u> If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Note that the reverse of this Thm is <u>not</u> true.

<u>Ex. 6</u> $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ but we know that the Harmonic Series Diverges!

In other words, we can't det. anything about convergence when $\lim_{n\to\infty} a_n = 0$!

Lecture # 15: Series; Integral & Divergence Tests   Date: Mon. 11/12/18

Instead, taking the contra-positive is true:

<u>Thm</u> (Divergence Test)

If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges

<u>Ex. 8</u> $\sum\limits_{k=0}^{\infty} \frac{n}{2n+3}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+3} \overset{H}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

Conclusion:

By the <u>Divergence test</u>
(state test used)

Since <u>$\lim\limits_{n\to\infty} a_n \neq 0$</u>
(condition(s) that were met)

then the series <u>diverges</u>
(converges/diverges)

Lecture # 15: Series; Integral & Divergence Tests   Date: Mon. 11/12/18

Integral Test

Thm If f is positive, continuous, & decreasing for $x \geq 1$ & $a_n = f(n)$ then

$$\sum_{n=1}^{\infty} a_n \quad \& \quad \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.

Ex. 9 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$a_n = \dfrac{n}{n^2+1} \Rightarrow$ Let $f(x) = \dfrac{x}{x^2+1}$

For $x \geq 1$, $f(x)$ is cont,

$f'(x) < 0$ for $x \geq 1 \Rightarrow f(x)$ dec.

To apply the integral test, need to eval

$$\int_1^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_1^{b} \frac{x}{x^2+1}\,dx$$

Note $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln(u)$

$= \frac{1}{2}\ln(x^2+1)$

u-sub

$u = x^2+1$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

Ex. 10 (cont'd)

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$$

$$\Rightarrow \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1} = \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(b^2+1) + \frac{1}{2}\ln(1^2+1)\right]$$

$$= \lim_{b\to\infty}\frac{1}{2}\ln(b^2+1) + \lim_{b\to\infty}\frac{1}{2}\ln(2)$$

$$= \infty + \frac{1}{2}\ln(2)$$

$$= \infty$$ Integral Diverges

By the integral test since $\int_1^\infty f(x)\,dx$ diverges then the series diverges