1. $\displaystyle\int_1^\infty \frac{1}{z^2}\sin\left(\frac{\pi}{z}\right)\,dz$

**Solution:**

Improper integral for $f(x)$ cont. on $[a,\infty)$

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b \frac{1}{z^2}\sin\left(\frac{\pi}{z}\right)dz$$

Note that

$$\int \frac{1}{z^2}\sin\left(\frac{\pi}{z}\right)dz = -\frac{1}{\pi}\int \sin(u)\,du = \frac{1}{\pi}\cos(u)+c$$

$$= \frac{1}{\pi}\cos\left(\frac{\pi}{z}\right)+c$$

u-sub:

$u = \pi z^{-1}$

$du = -\pi z^{-2} \Rightarrow -\frac{1}{\pi}du = z^{-2}dz$

then we have

$$\lim_{b\to\infty}\int_a^b \frac{1}{z^2}\sin\left(\frac{\pi}{z}\right)dz = \lim_{b\to\infty}\left[\frac{1}{\pi}\cos\left(\frac{\pi}{z}\right)\right]_1^b$$

$$= \frac{1}{\pi}\lim\left[\cos\left(\frac{\pi}{b}\right)-\cos(\pi)\right]$$

$$= \frac{1}{\pi}\left[\lim_{b\to\infty}\cos\left(\frac{\pi}{b}\right) - \lim_{b\to\infty}(-1)\right]$$

$$= \frac{1}{\pi}\left[\cos\left(\pi\lim_{b\to\infty}\frac{1}{b}\right)+1\right]$$

$$= \frac{1}{\pi}\left[\cos(0)+1\right]$$

$$= \frac{1}{\pi}\left[1+1\right] = \frac{2}{\pi}$$

2. $\displaystyle\int_1^\infty \frac{1}{(x-1)^{1/3}}\,dx$

**Solution:**

Since integrand is undefined @ $x=1$ so this integral is improper @ both ends

We need to split the integral

$$\int_1^\infty \frac{1}{(x-1)^{1/3}}\,dx = \int_1^2 \frac{1}{(x-1)^{1/3}}\,dx + \int_2^\infty \frac{1}{(x-1)^{1/3}}\,dx$$

$$= \underbrace{\lim_{a\to 1^+}\int_a^2 \frac{1}{(x-1)^{1/3}}\,dx}_{I_1} + \underbrace{\lim_{b\to\infty}\int_2^b \frac{1}{(x-1)^{1/3}}\,dx}_{I_2}$$

Note:

$$\int \frac{1}{(x-1)^{1/3}}\,dx = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + c = \frac{3}{2}(x-1)^{2/3} + c$$

u-sub: $u = x-1$
$du = 1\,dx$

For $I_1$: $$\lim_{a\to 1^+}\int_a^2 \frac{1}{(x-1)^{1/3}}\,dx = \lim_{a\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_a^2$$

$$= \frac{3}{2}\lim_{a\to 1^+}\left[(2-1)^{2/3} - (a-1)^{2/3}\right]$$

$$= \frac{3}{2}\Big[\underbrace{\lim_{a\to 1^+} 1^{2/3}}_{=1} - \underbrace{\lim_{a\to 1^+}(a-1)^{2/3}}_{=0}\Big] = \frac{3}{2}$$

For $I_2$: $$\lim_{b\to\infty}\int_2^b \frac{1}{(x-1)^{1/3}}\,dx = \lim_{b\to\infty}\left[\frac{3}{2}(x-1)^{3/2}\right]_2^b$$

$$= \frac{3}{2}\lim_{b\to\infty}\left[(b-1)^{3/2} - (2-1)^{3/2}\right]$$

$$= \frac{3}{2}\Big[\underbrace{\lim_{b\to\infty}(b-1)^{3/2}}_{=\infty} - \lim_{b\to\infty} 1^{3/2}\Big] = \infty$$

Since $I_2 = \infty$ the integral $I_1 + I_2$ diverges

3. $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx$

**Solution:**

Note: undef. @ $x=0$ so we split the interval

$$\int_{-2}^{3} x^{-4}\,dx = \int_{-2}^{0} x^{-4}\,dx + \int_{0}^{3} x^{-4}\,dx$$

$$= \underbrace{\lim_{b\to 0^-}\int_{-2}^{b} x^{-4}\,dx}_{I_1} + \underbrace{\lim_{c\to 0^+}\int_{c}^{3} x^{-4}\,dx}_{I_2}$$

$$= \lim_{b\to 0^-}\left[\frac{x^{-3}}{-3}\right]_{-2}^{b} + \lim_{c\to 0^+}\left[\frac{x^{-3}}{-3}\right]_{c}^{3}$$

$$= -\frac{1}{3}\left[\lim_{b\to 0^-}\left[b^{-3}-(-2)^{-3}\right] + \lim_{c\to 0^+}\left[(3)^{-3} - c^{-3}\right]\right]$$

$$= -\frac{1}{3}\left[\underbrace{\lim_{b\to 0^-} b^{-3}}_{=-\infty} + \frac{1}{8} + \frac{1}{27} - \underbrace{\lim_{c\to 0^+} c^{-3}}_{=\infty}\right]$$

Since $I_1$ & $I_2$ each diverge, then $I_1 + I_2$ also diverges.

4. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

**Solution:**

Split integral

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$$

$$= \lim_{b \to -\infty} \int_{b}^{0} x e^{-x^2} dx + \lim_{c \to \infty} \int_{0}^{c} x e^{-x^2} dx$$

Note:

$$\int x e^{-x^2} dx = -\frac{1}{2} \int e^{u} du = -\frac{1}{2} e^{u} + C = -\frac{1}{2} e^{-x^2} + C$$

$u = -x^2$

$du = -2x\,dx \implies -\frac{1}{2} du = dx$

$$= \lim_{b \to -\infty} \left[-\frac{1}{2} e^{-x^2}\right]_{b}^{0} + \lim_{c \to \infty} \left[-\frac{1}{2} e^{-x^2}\right]_{0}^{c}$$

$$= -\frac{1}{2} \lim_{b \to -\infty} \left[1 - \frac{1}{e^{b^2}}\right] - \frac{1}{2} \lim_{c \to \infty} \left[\frac{1}{e^{c^2}} - 1\right]$$

$$= -\frac{1}{2} \left[\lim_{b \to -\infty} 1 - \lim_{b \to -\infty} \frac{1}{e^{b^2}} - \lim_{c \to \infty} \frac{1}{e^{c^2}} - \lim_{c \to \infty} 1\right]$$

$$= -\frac{1}{2} \left[1 - 0 - 0 - 1\right] = 0$$