Lecture 09: Partial Fraction Decomposition

Math 2205: Calculus II   Fall 2018

How do we integrate rational functions? i.e. expressions of the form

$$\frac{\text{polynomial}}{\text{polynomial}}$$

For example,

$$\frac{3x}{x^2 - 2x - 8} = \frac{3x}{(x-2)(x+4)}$$

wouldn't it be nice if

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+4)}$$

Finding the coefficients that make this statement is a process called **partial fraction decomposition**.

Table of Partial Fraction Decomposition Forms

You may find the following helpful in determining how to break down your partial fraction decompositions.

Factor	Term in decomposition
$\dfrac{1}{ax+b}$	$\dfrac{A}{ax+b}$
$\dfrac{1}{(ax+b)^k}$	$\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$
$\dfrac{1}{ax^2+bx+c}$	$\dfrac{Ax+B}{ax^2+bx+c}$
$\dfrac{1}{(ax^2+bx+c)^k}$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

We'll demonstrate this process through an example.

Examples

A Partial Fraction Decomposition

Ex. 1)

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+4)}$$

$$\frac{3x}{(x-2)(x+4)} = \frac{A}{(x-2)}\left(\frac{x+4}{x+4}\right) + \frac{B}{(x+4)}\left(\frac{x-2}{x-2}\right)$$

$$\frac{3x}{(x-2)(x+4)} = \frac{A(x+4) + B(x-2)}{(x-2)(x+4)}$$

$$\Rightarrow \quad 3x = A(x+4) + B(x-2)$$

$$3x = Ax + 4A + Bx - 2B$$

Combine like terms on RHS:

$$3x + 0 = (A+B)x + (4A-2B)$$

Equate coefficients:

$$3 = A+B \quad \Big\} \text{ Solve system}$$
$$0 = 4A-2B$$

$$\Rightarrow 4A = 2B \Rightarrow B = 2A$$

Sub into $3 = A+B$

$$3 = A + (2A) \Rightarrow 3 = 3A$$
$$A = 1$$
$$\Rightarrow B = 2(1) = 2$$

So

$$\frac{3x}{(x-2)(x+4)} = \frac{1}{(x-2)} + \frac{2}{(x+4)}$$

Integration using Partial Fraction Decomposition

The next example demonstrates how we use this to aid in integration

Ex.2

$$\int \frac{3x}{(x-2)(x+4)}\, dx = \int \left[\frac{1}{x-2} + \frac{2}{x+4} \right] dx$$

$$= \int \frac{1}{x-2}\, dx + 2\int \frac{1}{x+4}\, dx$$

Note:

$$\int \frac{1}{x-2}\, dx = \int \frac{1}{u}\, du = \ln(u) = \ln(x-2)$$

u-sub.

$$u = x-2$$
$$du = dx$$

$$\int \frac{1}{x+4}\, dx = \int \frac{1}{u}\, du = \ln(u) = \ln(x+4)$$

u-sub

$$u = x+4$$
$$du = dx$$

$$\int \frac{3x}{(x-2)(x+4)}\, dx = \int \frac{1}{x-2}\, dx + 2\int \frac{1}{x+4}\, dx$$

$$= \ln(x-2) + 2\ln(x+4) + C$$