Lecture # 2: The Substitution Rule

Date: Thu. 9/13/18

Consider: $\int 10e^{10x}\,dx$

We want to find the antiderivative of $10e^{10x}$

Question: Can we just guess & check?

$\Rightarrow$ of course, but this doesn't seem very efficient!

Question: If we don't guess & check how should we proceed?

$\Rightarrow$ Since the derivative of e^{10x} is found using the chain rule, we are essentially in need of a process that reverses it.

Recall: The Chain Rule is

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$$

Let's integrate each side of this eqn.

$$\int \frac{d}{dx}[f(g(x))]\,dx = \int f'\underbrace{(g(x))}_{\text{composition of functions}}\cdot g'(x)\,dx$$

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Back to our example, we see that our integrand, $10e^{10x}$, is a composition of fcns.

Ex. 1 $\int \cos(2x)\,dx$

Soln. ID the composition:

$f(x) = \cos(x)$

$g(x) = 2x \qquad g'(x) = 2$

Is the derivative of the inside fcn, $g(x)$, present in the integrand (excluding any constants in front.) $\Rightarrow$ yes!

Since it is, we replace the inside fcn with a substitution variable, u.

$\Rightarrow u = 2x$

Find its derivative

$$\Rightarrow \frac{du}{dx} = 2 \qquad \frac{d}{dx}[u] = \frac{du}{dx}$$

Which we write as $du = 2\,dx$

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our goal is to represent

$\int f(g(x))\,dx$ as $\int f(u)\,du$

We found that $du = 2\,dx$

which means that $dx = \frac{1}{2}\,du$

So using $u = 2x$

$dx = \frac{1}{2}\,du$

& substituting them in our integral we have

$$\int \cos(2x)\,dx = \int \cos(u)\left(\tfrac{1}{2}\,du\right)$$

$$= \frac{1}{2}\int \cos(u)\,du$$

$$= \frac{1}{2}\sin(u) + C$$

Replace $u = 2x$

$$= \frac{1}{2}\sin(2x) + C$$

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Ex. 2 $\int (1-2x)^9\,dx$

Let $u = 1-2x$

$\Rightarrow du = -2\,dx \quad \Rightarrow dx = \frac{1}{-2}\,du$

So

$$\int (1-2x)^9\,dx = \int (u)^9 \left(-\frac{1}{2}\,du\right)$$

$$= -\frac{1}{2}\int u^9\,du$$

$$= -\frac{1}{2}\left[\frac{u^{10}}{10}\right] + C$$

$$= \frac{-u^{10}}{20} + C$$

$$= \frac{-(1-2x)^{10}}{20} + C$$

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Ex. 3 $\int \cos^3(x)\sin(x)\,dx = \int (\cos(x))^3 \underline{\sin(x)\,dx}$

$u = \cos(x)$

$du = -\sin(x)\,dx \quad \Rightarrow \quad \underline{\sin(x)\,dx} = -du$

$$\int \cos^3(x)\sin(x)\,dx = \int u^3(-du)$$
$$= -\int u^3\,du$$
$$= -\left[\frac{u^4}{4}\right] + C$$
$$= -\frac{\cos^4(x)}{4} + C$$

What about definite integrals?
We have two choices:

1) Find the indefinite integral & then apply FTOC
2) Apply what is known as a <u>change of variables</u>

Def Substitution Rule for Definite Integrals

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

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Ex. 4 $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3(\theta)}\,d\theta$

method 1:

$u = \cos\theta$

$du = -\sin\theta\,d\theta \implies \sin\theta\,d\theta = -du$

$$\int \frac{\sin\theta}{\cos^3(\theta)}\,d\theta = \int \frac{1}{u^3}(-du)$$

$$\int \frac{1}{\cos^3(\theta)}\sin\theta\,d\theta$$

$$= -\int \frac{1}{u^3}\,du$$

$$= -\int u^{-3}\,du$$

$$= -\frac{u^{-2}}{-2} + C$$

$$= \frac{\cos^{-2}(\theta)}{2} + C$$

$$= \frac{1}{2\cos^2\theta} + C$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta}\,d\theta = \frac{1}{2\cos^2\theta}\Bigg|_0^{\pi/4} = \frac{1}{2\cos^2(\frac{\pi}{4})} - \frac{1}{2\cos^2(0)}$$

$$= \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{2(1)}$$

$$= \ldots$$

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Ex. 5 (cont'd) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3(\theta)}\,d\theta$

method 2:

$u = \cos\theta$

$du = -\sin\theta\,d\theta \implies \sin\theta\,d\theta = -du$

Figure out bounds:

$\theta = \frac{\pi}{4} \qquad u = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$\theta = 0 \qquad u = \cos(0) = 1$

Now integral becomes

$$\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3(\theta)}\,d\theta = -\int_1^{\sqrt{2}/2} \frac{1}{u^3}\,du$$

$$= \frac{1}{2u^2}\Big|_1^{\sqrt{2}/2}$$

$$= \frac{1}{2\left(\frac{\sqrt{2}}{2}\right)^2} - \frac{1}{2(1)^2}$$