Worksheet 15 Solutions

1. Determine if the following geometric series converge or diverge. If the series converges, compute its sum.

(a) $\sum_{k=0}^{\infty} 7(0.3)^{k+2}$

Solution:

$$= \sum_{k=0}^{\infty} 7(0.3)^2 (0.3)^k$$

$$= \sum_{k=0}^{\infty} 0.63 (0.3)^k$$

By the geometric series test,

Since $|r| = |0.3| < 1$, then the series converges.

So

$$\sum_{k=0}^{\infty} 0.63(0.3)^k = \frac{a}{1-r} = \frac{0.63}{1-0.3} = \frac{0.63}{0.7} = 0.9$$

(b) $\sum_{k=-1}^{\infty} \frac{4}{2^k}$

Solution: Geometric series must start @ $k=0$

rewriting we have

$$\sum_{k=-1}^{\infty} \frac{4}{2^k} = \sum_{k=0}^{\infty} \frac{4}{2^{k-1}} = \sum_{k=0}^{\infty} \frac{4}{2^{-1}2^k} = \sum_{k=0}^{\infty} 8\left(\frac{1}{2}\right)^k$$

By the geometric series test,

Since $|r| = |\frac{1}{2}| < 1$, then the series converges.

So

$$\sum_{k=0}^{\infty} 8\left(\frac{1}{2}\right)^k = \frac{a}{1-r} = \frac{8}{1-\frac{1}{2}} = \frac{8}{\frac{1}{2}} = 16$$

2. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-1}$ converges or diverges.

Solution:

Partial Fractions:

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

Telescoping Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2-1} = \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2-1} - \frac{1}{2+1}\right) + \left(\frac{1}{3-1} - \frac{1}{3+1}\right) + \left(\frac{1}{4-1} - \frac{1}{4+1}\right) + \ldots + \left(\frac{1}{(n-1)-1} - \frac{1}{(n-1)+1}\right)\right.$$

$$= \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)\right.$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$$

The n^{th} partial sum is then $S_n = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$

The series converges if $\lim\limits_{n\to\infty} S_n$ exists

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{3}{2} - \frac{1}{n-1} - \frac{1}{n}\right]$$

$$= \frac{3}{2} + \lim_{n\to\infty}\frac{1}{n-1} - \lim_{n\to\infty}\frac{1}{n}$$

$$= \frac{3}{2}$$

So the series converges.

3. Determine if $\sum_{m=1}^{\infty} \cos\left(\frac{1}{m}\right)$ converges or diverges.

Solution:

Divergence Test:

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} \cos\left(\frac{1}{m}\right)$$
$$= \cos\left(\lim_{m\to\infty} \frac{1}{m}\right)$$
$$= \cos(0)$$
$$= 1$$

By the Divergence Test since $\lim_{n\to\infty} a_n \neq 0$ then the series diverges.

4. Determine if $\displaystyle\sum_{j=1}^{\infty} \frac{j}{e^j}$ converges or diverges.

Solution:

Integral Test: Let $f(x) = \dfrac{x}{e^x}$ Since f is pos. & cont. for $x \geq 1$ then need to show its decreasing.

$$f'(x) = \frac{e^x - xe^x}{(e^x)^2} = e^{-x}(1-x)$$

$f'(x) < 0$ for $x \geq 1$ so $f(x)$ decreasing

Then we need to calculate

$$\int_1^\infty xe^{-x}\,dx = \lim_{b\to\infty} \int_1^b xe^{-x}\,dx$$

Note:

$\int xe^{-x}\,dx$ int. by parts $\int u\,dv = uv - \int v\,du$

$u = x$ $\quad v = -e^{-x}$

$du = dx$ $\quad dv = e^{-x}\,dx$

$$\Rightarrow \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} = -e^{-x}(1+x)$$