Determine if the given series converges.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2+7}$

Solution:

Note: As $K \to \infty$

$$\frac{K^{1/2}}{K^2+7} \sim \frac{K^{1/2}}{K^2} = \frac{1}{K^{3/2}}$$

Let $b_k = \frac{1}{K^{3/2}} \Rightarrow \sum b_k = \sum \frac{1}{K^{3/2}}$

By p-series test, since $p = \frac{3}{2} > 1$ the series converges

then

$$a_k = \frac{K^{1/2}}{K^2+7}$$
$$b_k = \frac{1}{K^{3/2}}$$
$$\Rightarrow \frac{a_k}{b_k} = \frac{K^{1/2}}{K^2+7} \cdot \frac{K^{3/2}}{1} = \frac{K^2}{K^2+7} = \frac{1}{1+\frac{7}{K^2}}$$

$$\Rightarrow \lim_{K\to\infty} \frac{a_k}{b_k} = \lim_{K\to\infty} \frac{1}{1+\frac{7}{K^2}} = \frac{\lim_{K\to\infty} 1}{\lim_{K\to\infty} 1 + 7\lim_{K\to\infty} \frac{1}{K^2}} = 1$$

By the limit comparison test, since $a_k > 0$, $b_k > 0$, $\sum b_k$ converges and $\lim_{K\to\infty} \frac{a_k}{b_k} < \infty$, then the series $\sum a_k$ converges.

2. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\sin\left(\frac{1}{k+1}\right)$

Solution:

This is an alt. series: $\sum (-1)^{k+1} a_k$

Use alt. series Test w/ $a_k = \sin\left(\frac{1}{k+1}\right)$

Need to check that a_k is decreasing

$\Rightarrow f(x) = \sin\left(\frac{1}{1+x}\right)$

$\Rightarrow f'(x) = \cos\left(\frac{1}{k+1}\right)\left(\frac{-1}{(x+1)^2}\right) = -\frac{1}{(x+1)^2}\cos\left(\frac{1}{x+1}\right)$

For $x \geq 1$ we have

$$\frac{1}{x+1} \leq \frac{1}{2} \Rightarrow \cos\left(\frac{1}{x+1}\right) > 0$$

so $f'(x) < 0$ for $x > 1$.

Therefore, a_k is decreasing.

$\Rightarrow \lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} \sin\left(\frac{1}{k+1}\right)$

$$= \sin\left(\lim_{k\to\infty}\frac{1}{k+1}\right) = \sin(0) = 0$$

By the Alt. series test since a_k is decreasing & $\lim\limits_{k\to\infty} a_k = 0$, then the series Converges.

3. $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{k}{\ln(k)}$

Solution: Divergence Test w/ $a_k = (-1)^k \dfrac{k}{\ln(k)}$

$$\lim_{k\to\infty}(-1)^k\frac{k}{\ln(k)} = \begin{cases} -\displaystyle\lim_{k\to\infty}\frac{k}{\ln(k)}, & \text{for } k \text{ odd} \\ \displaystyle\lim_{k\to\infty}\frac{k}{\ln(k)}, & \text{for } k \text{ even} \end{cases}$$

$$\lim_{k\to\infty}\frac{k}{\ln(k)} \overset{H}{=} \lim_{k\to\infty}\frac{1}{\frac{1}{k}} = \lim_{k\to\infty} k = \infty$$

By the divergence test since $\displaystyle\lim_{k\to\infty} a_k \neq 0$ then the series diverges

4. $\displaystyle\sum_{k=1}^{\infty} \frac{2k^3+k}{k^4+k^2+2}$

Solution:

As $K \to \infty$

$$\frac{2K^3+K}{K^4+K^2+2} \sim \frac{2K^3}{K^4} = \frac{2}{K}$$

Let $b_k = \frac{2}{K} \Rightarrow \sum b_k = \sum \frac{2}{K}$ Harmonic series so it diverges

Need $b_k \leq a_k$

$$\Rightarrow \frac{2}{K} \leq \frac{2K^3+K}{K^4+K^2+2} \Rightarrow 2(K^4+K^2+2) \overset{?}{\leq} K(2k^3+k)$$

$$\Rightarrow 2K^4+2K^2+4 \overset{?}{\leq} 2K^4+K$$

Not true!

Instead we choose

$b_k = \frac{1}{K} \Rightarrow \sum b_k = \frac{1}{K}$ Harmonic series so diverges

$$\Rightarrow \frac{1}{K} \leq \frac{2k^3+k}{K^4+K^2+2} \Rightarrow K^4+K^2+2 \leq K(2k^3+k)$$

$$\Rightarrow \begin{array}{rl} K^4+K^2+2 & \leq 2k^4+K^2 \\ -K^4 \;\; -K^2 \qquad & \;\; -K^4 \;\; -K^2 \end{array}$$

$$\Rightarrow 2 \leq K^4 \quad \text{True for } K \geq 2$$

By the Comparison Test since $b_k < a_k$ & $\sum b_k$ diverges, then the series $\sum a_k$ diverges

Note: Could also have used the limit Comp test