Worksheet 20 Solutions

Divergence Theorem

MATH 2210, Fall 2018

1. If $\mathbf{F}$ is an electrostatic field then the scalar function $\operatorname{div}\mathbf{F}$ is proportional to the charge density at a point (x, y, z). Consider the field

$$\mathbf{F}(x, y, z) = \frac{\kappa\langle x, y, 0\rangle}{x^2 + y^2}$$

where κ is a constant. Show that $\operatorname{div}\mathbf{F} = 0$ if $x^2 + y^2 > 0$. What can you conclude about the location of the electrical charges responsible for this field?

Solution: Compute $\operatorname{div}\mathbf{F}$:

$$\begin{aligned}\nabla\cdot\mathbf{F} &= K\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial z}0\right] \\ &= K\left[\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} + 0\right] = 0\end{aligned}$$

at any point (x, y, z) where $x^2 + y^2 > 0$. Therefore the electrical charges causing the field must lie on the z-axis.

2. For the vector function

$$\mathbf{F}(x, y, z) = \langle x^3 - x^2, y^3 + xy, z^3 + xz\rangle.$$

use the Divergence Theorem to evaluate the flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Computing the Divergence of $\mathbf{F}$ we find that

$$\nabla\cdot\mathbf{F} = (3x^2 - 2x) + (3y^2 + x) + (3z^2 + x) = 3(x^2 + y^2 + z^2) = 3\rho^2.$$

By the Divergence theorem

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3\rho^2 dV$$

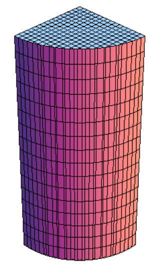
where E is the solid region surrounded by S. So,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 3\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4\sin\phi\, d\rho\, d\phi\, d\theta = \frac{12}{5}\pi.$$

3. Consider the vector function

$$\mathbf{F} = \langle x^2y^2, \sin(xz), -xy^2z\rangle$$

and let E be the solid region in the first octant bounded by the cylinder $x^2 + y^2 = 1$, the plane $z = 3$, and the coordinate planes.



Compute the outward flux of $\mathbf{F}$ across the S, the boundary of the solid region E.

Solution: Using the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\operatorname{div} \mathbf{F})\, dV = \iiint_E xy^2\, dV.$$

where E is the solid region in the first octant bounded by the cylinder $x^2 + y^2 = 1$ and the plane $z = 3$. Make a change of variables to cylindrical coordinates, where $xy^2 = r^3 \cos\theta \sin^2\theta$ and $dV = r\, dz\, dr\, d\theta$:

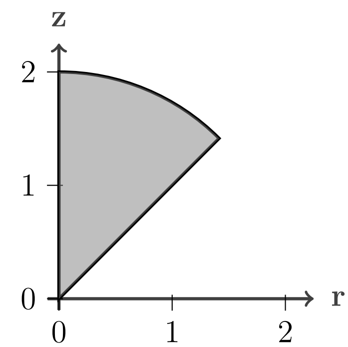
$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\pi/2}\int_0^1\int_0^3 (r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta \\ &= \int_0^{\pi/2} \cos\theta\sin^2\theta\, d\theta \int_0^2 r^4\, dr \int_0^3 dz \\ &= \frac{3}{5}\int_0^1 u^2\, du = \frac{1}{5}.\end{aligned}$$

If you don't want to use cylindrical coordinates, try the integration order $dz\, dx\, dy$.

4. E is the solid region in the first octant that lies inside the the sphere $x^2+y^2+z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$. The closed surface S is the boundary of E. *Use the divergence theorem* to calculate the outward oriented flux

$$\iint_S \langle y^2 + 3x, 2x^2 - 2y, 3z^2 - z\rangle \cdot d\mathbf{S}.$$

The graph shows a cross section of E.



Solution: Start by finding the divergence of $\mathbf{F} = \langle y^2 + 3x, 2x^2 - 2y, 3z^2 - z \rangle$:

$$\operatorname{div} \mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle y^2 + 3x, 2x^2 - 2y, 3z^2 - z \rangle = 3 - 2 + 6z - 1 = 6z.$$

In spherical coordinates, the sphere is $\rho = 2$ and the cone is $\phi = \tan^{-1}(1) = \pi/4$.

Using the divergence theorem, the flux integral

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\operatorname{div} \mathbf{F})\, dV &= 6 \iiint_E z\, dV \\
&= 6 \int_0^{\pi/2} \int_0^{\pi/4} \int_0^2 \rho \cos(\phi)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= 6 \int_0^{\pi/2} d\theta \int_0^{\pi/4} \cos(\phi) \sin(\phi)\, d\phi \int_0^2 \rho^3\, d\rho \\
&= 3\pi \int_0^{\sqrt{2}/2} u\, du \quad (\frac{\rho^4}{4}\Big|_0^2) \\
&= 12\pi\ (\frac{u^2}{2}\Big|_0^{\sqrt{2}/2}) \\
&= 12\pi\ (\frac{2}{8} - 0) = 3\pi.
\end{aligned}$$

If you choose cylindrical coordinates, the limits are more complicated but the integral is easier to solve:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 6 \int_0^{\pi/2} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} z\, r\, dz\, dr\, d\theta = 3\pi.$$