

Lecture #09: Directional Derivatives & Gradients

Date: Tue. 10/16/18

Previously we have seen how to find the rate of change in terms of  $x \& y$ .

What about the rate of change in other directions?

Example: say I have a fcn that describes volume of a gas in terms of temperature & pressure.

We want to know how volume changes as both temp. & pressure changes.

### Directional Derivatives

Let  $z = f(x, y)$  be a surface &  $(x_0, y_0)$  is in the domain of  $f$ .

Then the unit vector pointing in the direction of  $\theta$  (measured from pos. x-axis) is

$$\vec{u} = \cos(\theta)\hat{j} + \sin(\theta)\hat{i}$$

Note that  $\|u\| = 1$

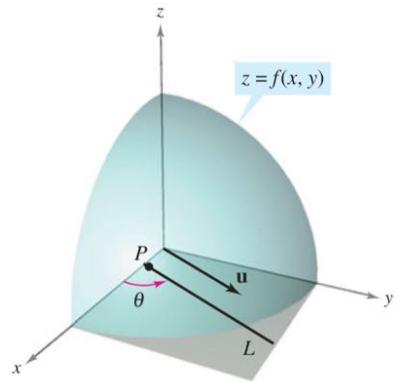
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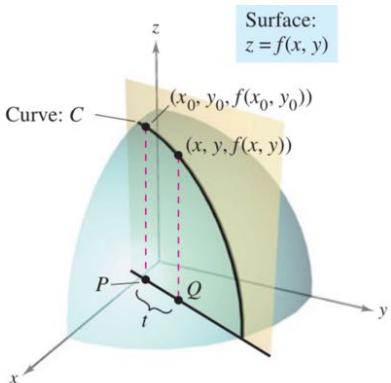
Let  $z = f(x, y)$  be a surface  
 &  $(x_0, y_0)$  is in the domain of  $f$

Then the unit vector pointing in  
 the direction of  $\theta$  (measured  
 from positive  $x$ -axis) is

$$\vec{u} = \cos(\theta) \hat{j} + \sin(\theta) \hat{j}$$



Note that  $\|\vec{u}\| = 1$



Let  $P$  be the plane passing  
 thru  $(x_0, y_0)$  & parallel to  $\vec{u}$

This problem is now reduced  
 to 2 dimensions by slicing  
 the surface

The intersection of the plane  $P$  & the  $xy$  plane  
 is given by

$$x = x_0 + \cos(\theta)t$$

$$y = y_0 + \sin(\theta)t$$

So the point of intersection on the surface  
 is  $(x_0, y_0, f(x_0, y_0))$

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The slope of the secant line thru the pts  $(x_0, y_0, f(x_0, y_0))$  &  $(x, y, f(x, y))$

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}$$

Def**DEFINITION OF DIRECTIONAL DERIVATIVE**

Let  $f$  be a function of two variables  $x$  and  $y$  and let  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  be a unit vector. Then the **directional derivative of  $f$  in the direction of  $\mathbf{u}$** , denoted by  $D_{\mathbf{u}} f$ , is

$$D_{\mathbf{u}} f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.

Of course, as with single variable fcns, we don't usually apply the definition directly.

Thm**THEOREM 13.9 DIRECTIONAL DERIVATIVE**

If  $f$  is a differentiable function of  $x$  and  $y$ , then the directional derivative of  $f$  in the direction of the unit vector  $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$  is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

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Ex. 1] Find the directional derivative of

$$f(x, y) = x^2 \sin(ay)$$

in direction of  $\vec{v} = \langle 3, -4 \rangle$

Soln. Directional derivative is given by

$$D_{\vec{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

the direction vector  $\vec{u}$  is given by

$$\begin{aligned} \vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} \\ &= \cos \theta \hat{i} + \sin \theta \hat{j} \\ \Rightarrow \cos \theta &= \frac{3}{5} \text{ & } \sin \theta = -\frac{4}{5} \end{aligned}$$

Partial Derivatives:

$$\frac{\partial F}{\partial x} = 2x \sin(ay)$$

$$\frac{\partial F}{\partial y} = x^2 \cos(ay)$$

So the directional derivative in direction of  $\vec{u}$  is

$$\begin{aligned} \Rightarrow D_{\vec{u}} f(x, y) &= f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \\ &= 2x \sin(ay) \cos \theta + x^2 \cos(ay) \sin \theta \\ &= 2x \sin(ay) \left(\frac{3}{5}\right) + x^2 \cos(ay) \left(-\frac{4}{5}\right) \\ &= \frac{6}{5}x \sin(ay) - \frac{8}{5}x^2 \cos(ay) \end{aligned}$$

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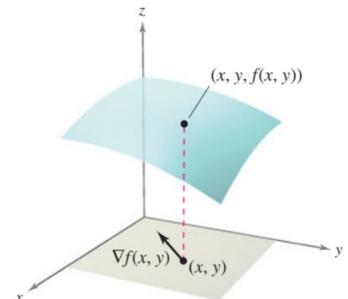
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**Gradients****DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES**

Let  $z = f(x, y)$  be a function of  $x$  and  $y$  such that  $f_x$  and  $f_y$  exist. Then the **gradient of  $f$** , denoted by  $\nabla f(x, y)$ , is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

$\nabla f$  is read as "del  $f$ ." Another notation for the gradient is **grad  $f(x, y)$** . In Figure 13.48, note that for each  $(x, y)$ , the gradient  $\nabla f(x, y)$  is a vector in the plane (not a vector in space).



The gradient of  $f$  is a vector in the  $xy$ -plane.

Note that the gradient can also be written as

$$\begin{aligned}\nabla f(x, y) &= \langle f_x(x, y), f_y(x, y) \rangle \\ &= \langle f_x, f_y \rangle\end{aligned}$$

The gradient is also found for 3 variables (& for  $n$ -variables)

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

Ex. 2) Find the gradient of

$$f(x, y) = y \ln(x) + xy^2$$

Soln.

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \left\langle \frac{y}{x} + y^2, \ln(x) + 2xy \right\rangle$$

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Using the gradient

$$\begin{aligned} D_{\vec{u}} f(x, y) &= f_x u_1 + f_y u_2 \\ &= \langle f_x, f_y \rangle \cdot \langle u_1, u_2 \rangle \\ &= \nabla f(x, y) \cdot \vec{u} \end{aligned}$$

This gives us another form for our directional derivative.

Thm.THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If  $f$  is a differentiable function of  $x$  and  $y$ , then the directional derivative of  $f$  in the direction of the unit vector  $\mathbf{u}$  is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Note: This is a scalar value when evaluated @ a pt  $(a, b)$

Ex.3 Find the directional derivative of

$$f(x, y) = 3x^2 - 2y^2$$

@ pt  $(-\frac{3}{4}, 0)$  in direction  $\vec{v} = \langle \frac{3}{4}, 1 \rangle$

$$\text{Sln. } \nabla f(x, y) = \langle 6x, -4y \rangle \quad \& \quad \nabla f(-\frac{3}{4}, 0) = \langle -\frac{9}{2}, 0 \rangle$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\begin{aligned} \Rightarrow D_{\vec{u}} f(x, y) &= \nabla f(-\frac{3}{4}, 0) \cdot \vec{u} \\ &= \left\langle -\frac{9}{2}, 0 \right\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\ &= -27 \end{aligned}$$

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Properties of the gradient**THEOREM 13.11 PROPERTIES OF THE GRADIENT**

Let  $f$  be differentiable at the point  $(x, y)$ .

1. If  $\nabla f(x, y) = \mathbf{0}$ , then  $D_{\mathbf{u}} f(x, y) = 0$  for all  $\mathbf{u}$ .
2. The direction of *maximum* increase of  $f$  is given by  $\nabla f(x, y)$ . The maximum value of  $D_{\mathbf{u}} f(x, y)$  is  $\|\nabla f(x, y)\|$ .
3. The direction of *minimum* increase of  $f$  is given by  $-\nabla f(x, y)$ . The minimum value of  $D_{\mathbf{u}} f(x, y)$  is  $-\|\nabla f(x, y)\|$ .

Skier going down mountain  $\Rightarrow$  altitude given by  $f(x, y)$

Conceptually,  $-\nabla f(x, y)$  is the fastest path down the mountain (or to dec. volume of a gas)

Ex. 4) The temp. in  $^{\circ}\text{C}$  on surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where  $x$  &  $y$  are measured in cm in what direction from pt  $(2, -3)$  does temp increase most rapidly?

Soln.  $\nabla T(x, y) = \langle T_x, T_y \rangle = \langle -8x, -2y \rangle$

Direction of max increase:  $\nabla T(2, -3) = \langle -16, 6 \rangle$

max rate increase is

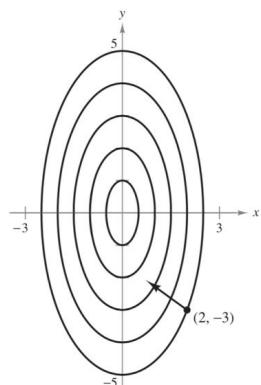
$$\|\nabla T(2, -3)\| = (292)^{1/2} \approx 17.09/\text{cm}$$

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Thm.**THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES**

If  $f$  is differentiable at  $(x_0, y_0)$  and  $\nabla f(x_0, y_0) \neq \mathbf{0}$ , then  $\nabla f(x_0, y_0)$  is normal to the level curve through  $(x_0, y_0)$ .

Ex. 5 Consider the last example.we can see that @  $(2, -3)$ that  $\nabla f(2, -3)$  is normal tothe level curve thru  $(2, -3)$ Level curves:  
 $T(x, y) = 20 - 4x^2 - y^2$ **Gradients for fcns of 3 or more variables****DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES**

Let  $f$  be a function of  $x$ ,  $y$ , and  $z$ , with continuous first partial derivatives. The **directional derivative of  $f$**  in the direction of a unit vector  $\mathbf{u} = ai + bj + ck$  is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of  $f$**  is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1.  $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If  $\nabla f(x, y, z) = \mathbf{0}$ , then  $D_{\mathbf{u}}f(x, y, z) = 0$  for all  $\mathbf{u}$ .
3. The direction of *maximum* increase of  $f$  is given by  $\nabla f(x, y, z)$ . The maximum value of  $D_{\mathbf{u}}f(x, y, z)$  is

$$\|\nabla f(x, y, z)\|. \quad \text{Maximum value of } D_{\mathbf{u}}f(x, y, z)$$

4. The direction of *minimum* increase of  $f$  is given by  $-\nabla f(x, y, z)$ . The minimum value of  $D_{\mathbf{u}}f(x, y, z)$  is

$$-\|\nabla f(x, y, z)\|. \quad \text{Minimum value of } D_{\mathbf{u}}f(x, y, z)$$