

Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Alt. Form of Eqn of a surface

Previously, a surface in space has been given by

$$z = f(x, y)$$

We can also write this in a more general form as

$$F(x, y, z) = f(x, y) - z$$

Since $F(x, y, z) = 0$ then the level surface of F is

$$F(x, y, z) = 0$$

Ex. 1

For $F(x, y, z) = x^2 + y^2 + z^2 - 4$

the level surface given by $F(x, y, z) = 0$ is

$$x^2 + y^2 + z^2 - 4 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 = 4$$

Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Tangent Planes

For a surface given by $F(x, y, z) = 0$

Let P be a pt (x_0, y_0, z_0) .

Let a curve C on the surface given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

then for all t

$$F(x(t), y(t), z(t)) = 0$$

If F is diff'ble & $x'(t), y'(t), z'(t)$ all exist then

$$0 = F'(t)$$

$$= F_x(x, y, z)x'(t) + F_y(x, y, z)y'(t) + F_z(x, y, z)z'(t)$$

$$= \langle F_x, F_y, F_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle$$

$$= \nabla F(x, y, z) \cdot \vec{r}'(t)$$

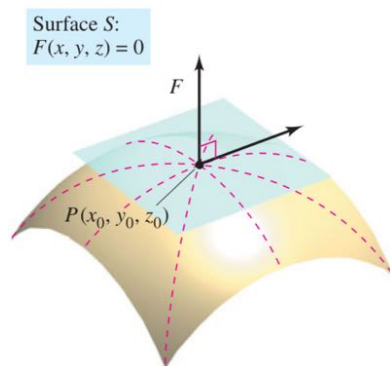
Then @ the pt (x_0, y_0, z_0)

$$0 = \nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0)$$

Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

The gradient @ (x_0, y_0, z_0) is orthogonal to the tangent vector of every curve on the surface thru the pt (x_0, y_0, z_0)



In other words, all tangent lines thru this pt lie in a plane tangent to the surface & normal to $\nabla F(x_0, y_0, z_0)$

Def. Let F be diff'ble @ pt (x_0, y_0, z_0) on the surface given by $F(x, y, z) = 0$ s.t. $\nabla F(x_0, y_0, z_0) \neq 0$

The plane thru pt (x_0, y_0, z_0) that is normal to $\nabla F(x_0, y_0, z_0)$ is the tangent plane to a surface at that pt.

Thm

THEOREM 13.13 EQUATION OF TANGENT PLANE

If F is differentiable at (x_0, y_0, z_0) , then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Ex. 2 | Find an eqn of the tangent plane to

$$z^2 - 2x^2 - 2y^2 = 12$$

@ pt $(1, -1, 4)$

Soln Rewriting as

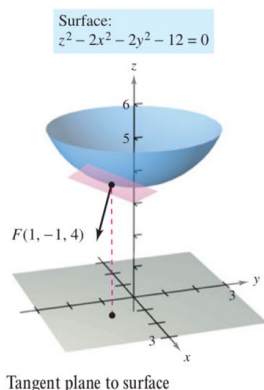
$$\underbrace{z^2 - 2x^2 - 2y^2 - 12}_{F(x,y,z)} = 0$$

Need $F_x = -4x$, $F_y = -4y$, $F_z = 2z$ @ $(1, -1, 4)$ $F_x = -4$, $F_y = 4$, $F_z = 8$

Eqn of tangent plane:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$\Rightarrow -4(x-1) + 4(y-1) + 8(z-4) = 0$$

To find an eqn of a tangent plane to $z = F(x, y)$ define

$$F(x, y, z) = F(x, y) - z$$

then the eqn of the tangent plane is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Lecture # 10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Ex. 3 Find eqn of tangent plane to

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

@ pt. $(1, 1, \frac{1}{2})$ Soln.

$$f_x(x, y) = -\frac{1}{10}(2x) = -\frac{1}{5}x \quad \Rightarrow \quad f_x(1, 1) = -\frac{1}{5}$$

$$f_y(x, y) = -\frac{1}{10}(8y) = -\frac{4}{5}y \quad \Rightarrow \quad f_y(1, 1) = -\frac{4}{5}$$

Eqn. of tangent plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

$$\Rightarrow -\frac{1}{5}(x-1) - \frac{4}{5}(y-1) - (z - \frac{1}{2}) = 0$$

Normal Lines

Def. Let F be diff'ble @ pt (x_0, y_0, z_0) on the surface given by $F(x, y, z) = 0$ s.t. $\nabla F(x_0, y_0, z_0) \neq 0$

The line thru pt. (x_0, y_0, z_0) that has the same direction as $\nabla F(x_0, y_0, z_0)$ is called the normal line to a surface at a pt.

Lecture # 10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Ex. 4 Find the eqn of the normal line to the surface

$$xyz = 12$$

@ pt $(2, -2, 3)$

Soln. Rewriting:

$$F(x, y, z) = xyz - 12$$

$$\begin{aligned}\nabla F(x, y, z) &= \langle F_x, F_y, F_z \rangle \\ &= \langle yz, xz, xy \rangle\end{aligned}$$

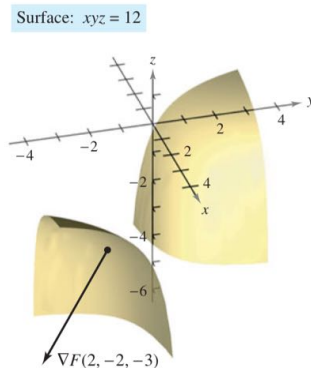


Figure 13.60

Evaluating @ pt $(2, -2, 3)$

$$\begin{aligned}\nabla F(2, -2, 3) &= \langle (-2)(3), (2)(3), (2)(-2) \rangle \\ &= \langle -6, 6, -4 \rangle\end{aligned}$$

Eqn. of Normal line:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\Rightarrow \frac{x-2}{-6} = \frac{y+2}{6} = \frac{z-3}{-4}$$

Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Ex. 5) Describe the eqn of the tangent line to the curve of intersection of the Surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \quad (\text{Ellipsoid})$$

$$x^2 + y^2 + z = 4 \quad (\text{paraboloid})$$

@ pt. $(0, 1, 3)$

$$F(x, y, z) = x^2 + 2y^2 + 2z^2 - 20$$

$$G(x, y, z) = x^2 + y^2 + z - 4$$

$$\nabla F(x, y, z) = \langle 2x, 4y, 4z \rangle$$

$$\nabla G(x, y, z) = \langle 2x, 2y, 1 \rangle$$

$$\nabla F(0, 1, 3) = \langle 0, 4, 12 \rangle$$

$$\nabla G(0, 1, 3) = \langle 0, 2, 1 \rangle$$

Direction Vector for tangent line given by

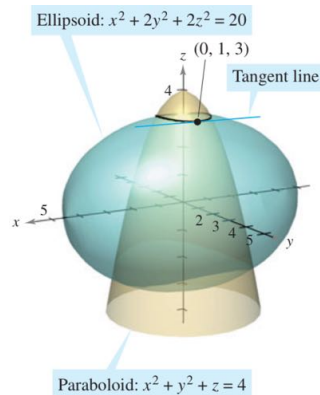
$$\nabla F(0, 1, 3) \times \nabla G(0, 1, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = \langle -20, 0, 0 \rangle$$

Eqn. of tangent line is

$$x = x_0 + at = 0 - 20t = -20t$$

$$y = y_0 + bt = 1 + 0t = 1$$

$$z = z_0 + ct = 3 + 0t = 3$$



Lecture #10: Tangent planes & Normal lines

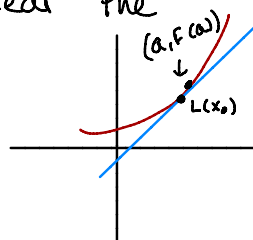
Date: Thu. 10/18/18

Linear Approximation

In one variable:

the linear approx of a fcn F near the pt $(a, F(a))$ is given by

$$L(x) = F(a) + F'(a)(x-a)$$



In 2 variables:

Def Let F be diff'ble @ (x_0, y_0) . The linear approximation to the surface $z = F(x, y)$ @ pt $(x_0, y_0, F(x_0, y_0))$ is the tangent plane @ that pt given by

$$L(x, y) = F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + F(x_0, y_0)$$