1. Consider the vector valued function $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}$.

   (a) Find the unit tangent vector $\mathbf{T}(t)$.

Solution: The unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

We have

$$\begin{aligned} \mathbf{r}'(t) &= 2t\mathbf{i} + \mathbf{j} \\ \|\mathbf{r}'(t)\| &= ((2t)^2 + (1)^2)^{1/2} = (4t^2+1)^{1/2} \end{aligned}$$

so the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t\mathbf{i} + \mathbf{j}}{(4t^2+1)^{1/2}}$$

   (b) Find the set of parametric equations for the line tangent to the space curve $\mathbf{r}(t)$ at the point $P(1, 1, \frac{4}{3})$

Solution: The point $P(1, 1, \frac{4}{3})$ is given by $\mathbf{r}(1)$ which implies that $t = 1$ so at this point we have

$$\mathbf{T}(1) = \frac{2(1)\mathbf{i} + \mathbf{j}}{(4(1)^2+1)^{1/2}} = \frac{1}{\sqrt{5}(2\mathbf{i} + \mathbf{j})}$$

The direction vector for the tangent line is then given by

$$\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\rangle \quad \text{or} \quad \langle 2, 1, 0\rangle$$

then the parametric equations for the tangent line are given by

$$\begin{aligned} x &= x_0 + at = 1 + 2t \\ y &= y_0 + bt = 1 + t \\ z &= z_0 + ct = \frac{4}{3} \end{aligned}$$

2. Consider the curve $r(t) = \pi\cos(t)\mathbf{i} + \pi\sin(t)\mathbf{j}$.

   (a) Find the unit tangent vector $\mathbf{T}(t)$.

Solution: The unit tangent vector is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

We have

$$\begin{aligned}
\mathbf{r}'(t) &= -\pi\sin(t)\mathbf{i} + \pi\cos(t)\mathbf{j} \\
\|\mathbf{r}'(t)\| &= [(-\pi\sin(t))^2 + (\pi\cos(t))^2]^{1/2} = \pi \\
\mathbf{T}(t) &= \frac{1}{\pi}(-\pi\sin(t)\mathbf{i} + \pi\cos(t)\mathbf{j}) \\
&= -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}
\end{aligned}$$

so the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t\mathbf{i} + \mathbf{j}}{(4t^2+1)^{1/2}}$$

   (b) Find the unit normal vector $N(t)$ for the value $t = \frac{\pi}{6}$.

Solution: The unit normal vector is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

we have

$$\begin{aligned}
\mathbf{T}'(t) &= -\cos(t)\mathbf{i} + \sin(t)\mathbf{j} \\
\|\mathbf{T}'(t)\| &= (\cos^2(t) + \sin^2(t))^{1/2} = 1
\end{aligned}$$

so the unit normal vector is

$$\mathbf{N}(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$$

at the value $t = \pi 6$ we have

$$\mathbf{N}\left(\frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)\mathbf{i} - \sin\left(\frac{\pi}{6}\right)\mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$