# Lecture #17: Vector Fields & Line Integrals

Date: Tue. 11/20/18

## Vector Fields

**DEFINITION OF VECTOR FIELD**

A **vector field over a plane region *R*** is a function **F** that assigns a vector **F**$(x, y)$ to each point in $R$.

A **vector field over a solid region *Q* in space** is a function **F** that assigns a vector **F**$(x, y, z)$ to each point in $Q$.

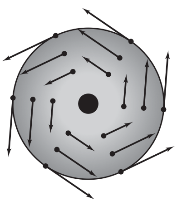
Examples: The gradient

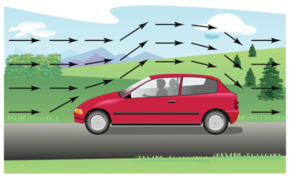
Gravitational Field

Electro magnetic Field

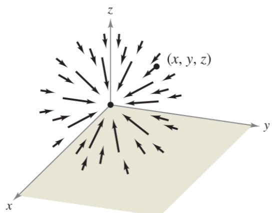
Air Flow

Velocity field





Air flow vector field




$m_1$ is located at $(x, y, z)$.
$m_2$ is located at $(0, 0, 0)$.

Gravitational force field

We can also write this as

$$F(x, y, z) = \underbrace{M(x, y, z)}_{\text{component Fcn}}\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$$

A vector field is <u>continuous</u> @ a pt iff each component fcn is cont. @ that pt.

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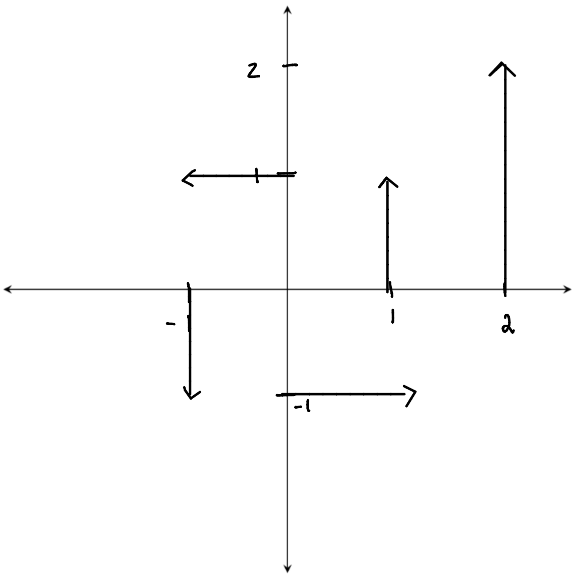
**Ex. 1)** Sketch the vector field

$$F(x,y) = -y\hat{i} + x\hat{j}$$

Select some points:
(1,0), (0,1), (-1,0), (0,-1)



@ $F(1,0) = \langle 0, 1 \rangle$
$F(0,1) = \langle -1, 0 \rangle$
$F(-1,0) = \langle 0, -1 \rangle$
$F(0,-1) = \langle 1, 0 \rangle$

$F(2,0) = \langle 0, 2 \rangle$

## Conservative Vector Fields

**DEFINITION OF CONSERVATIVE VECTOR FIELD**

A vector field **F** is called **conservative** if there exists a differentiable function $f$ such that $\mathbf{F} = \nabla f$. The function $f$ is called the **potential function** for **F**.

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## Line Integrals

**Def** A plane curve C given by

$$r(t) = x(t)\hat{i} + y(t)\hat{j} = \langle x(t), y(t) \rangle$$

is smooth if $\frac{dx}{dt}$ & $\frac{dy}{dt}$ are cont. on $[a,b]$ & not both 0 there.

Equivalenty for curves in space

**Def** A curve is piece-wise smooth if the interval $[a,b]$ can be partioned into a finite # of sub intervals on which C is smooth.

So far we've seen integration as

$$\int_a^b f(x)\,dx \quad \text{Int. over interval } [a,b]$$

$$\iint_R f(x,y)\,dA \quad \text{Int. over region } R$$

Now we see a line integral where we integrate over a curve C:

$$\int_C f(x,y)\,ds \quad \text{Int. over curve } C$$

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**DEFINITION OF LINE INTEGRAL**

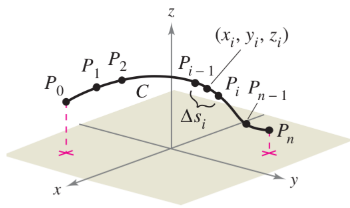
If $f$ is defined in a region containing a smooth curve $C$ of finite length, then the **line integral of $f$ along $C$** is given by

$$\int_C f(x, y)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta s_i \qquad \text{Plane}$$

or

$$\int_C f(x, y, z)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta s_i \qquad \text{Space}$$

provided this limit exists.



Partitioning of curve $C$

To evaluate a line integral we use the fact that

$$ds = \|r'(t)\|\, dt = \left[ (x'(t))^2 + (y'(t))^2 \right]^{1/2} dt$$

**THEOREM 15.4 EVALUATION OF A LINE INTEGRAL AS A DEFINITE INTEGRAL**

Let $f$ be continuous in a region containing a smooth curve $C$. If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, where $a \le t \le b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

If $C$ is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $a \le t \le b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

Note: If $f(x,y,z) = 1$ then

$$\int_C 1\, ds = \int_a^b \|r'(t)\|\, dt$$

gives the arc length of a curve.

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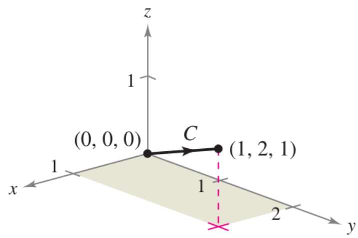
We will need to parameterize line segments

A line segment that starts @ $r_0$ & ends @ $\vec{r}_1$ is given by

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \qquad 0 \le t \le 1$$

Ex.) Evaluate $\displaystyle\int_C (x^2 - y + 3z)\, ds$

where C is shown in Figure




First we need to write parametric form of the line segment.

C: $\langle 1, 2, 1 \rangle$

$\Rightarrow \vec{r}(t) = \langle t, 2t, t \rangle = \langle x(t), y(t), z(t) \rangle$

Need: $x'(t) = 1, \quad y'(t) = 2, \quad z'(t) = 1$

$$\|r'(t)\| = \left( (x'(t))^2 + (y'(t))^2 + (z'(t))^2 \right)^{1/2}$$

$$= \left( (1)^2 + (2)^2 + (1)^2 \right)^{1/2} = \sqrt{6}$$

Line Integral:

$$\int_C (x^2 - y + 3z)\, ds = \int_0^1 f(x(t), y(t), z(t))\, \|\vec{r}'(t)\|\, dt$$

$$= \int_0^1 (t^2 - 2t + 3t)\sqrt{6}\, dt = \frac{5\sqrt{6}}{6}$$

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### Line Integrals over Vector fields

An important application of line integrals is finding the work done on an object moving in a vector field

To see this consider an object moving along a path $C$, in a force field $F$, to det. work done by $F$ we need only the part of the force field acting in same direction the object is moving.

This means for each pt on $C$ we want the projection $F \cdot T$ of the force field $F$ onto the unit vector tangent to $C$, $T$ on a small arc of $C$, $\Delta s_i$, work is

$$\Delta w_i = \text{force} \times \text{distance}$$
$$\approx (F \cdot T)\,\Delta s_i$$

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So the <u>work</u> is given by

$$W = \int_C F \cdot T \, ds$$

**DEFINITION OF THE LINE INTEGRAL OF A VECTOR FIELD**

Let **F** be a continuous vector field defined on a smooth curve $C$ given by $\mathbf{r}(t)$, $a \le t \le b$. The **line integral** of **F** on $C$ is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F}(x(t), \ y(t), z(t)) \cdot \mathbf{r}'(t) \, dt.$$

<u>Note:</u>

$$F \cdot T \, ds = F \cdot \frac{r'(t)}{\|r'(t)\|} \|r'(t)\| \, dt$$

$$= F \cdot \underbrace{r'(t) \, dt}_{dr}$$

$$= F \cdot dr$$

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Ex. | Find the work done by the force field

$$F = \langle -\tfrac{1}{2}x, -\tfrac{1}{2}y, \tfrac{1}{4} \rangle$$

on the particle as it moves along the helix given by $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

From the pt $(1, 0, 0)$ to $(-1, 0, 3\pi)$

Soln

Param. C $\rightarrow$ done!

Find bds: lower bds

$$\langle \cos(t), \sin(t), t \rangle = \langle 1, 0, 0 \rangle$$
$$\Rightarrow t = 0$$

upper bds

$$\langle \cos(t), \sin(t), t \rangle = \langle -1, 0, 3\pi \rangle$$
$$\Rightarrow t = 3\pi$$

Find $\vec{r}'(t)$

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

Find $F(x(t), y(t), z(t))$

$$F = \langle -\tfrac{1}{2}\cos(t), -\tfrac{1}{2}\sin(t), \tfrac{1}{4} \rangle$$

Find $F \cdot \vec{r}'(t)$

$$F \cdot \vec{r}'(t) = \tfrac{1}{2}\cos(t)\sin(t) - \tfrac{1}{2}\cos(t)\sin(t) + \tfrac{1}{4} = \tfrac{1}{4}$$

$$W = \int_C F \cdot dr = \int_a^b F \cdot r'(t)\,dt = \int_0^{3\pi} \tfrac{1}{4}\,dt = \frac{3\pi}{4}$$

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Note: orientation of C is important!

if the orientation is reversed direction
T is changed to −T

$$\int_{-C} F \cdot dr = -\int_{C} F \cdot dr$$

For line integrals in space let

$$F = \langle f(x,y,z),\ g(x,y,z),\ h(x,y,z) \rangle$$

and

$$C = \vec{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle, \quad a \le t \le b$$

then

$$\int_a^b F \cdot r'(t)\, dt = \int_a^b (f(t)\,x'(t) + g(t)\,y'(t) + h(t)\,z'(t))\, dt$$

$$= \int_C f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$$

In 2-D: $F = \langle f, g \rangle$, $C = \vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_a^b F \cdot r'(t)\, dt = \int_a^b [f(t)\,x'(t) + g(t)\,y'(t)]\, dt$$

$$= \int_C f\,dx + g\,dy \quad \text{(differential form)}$$