1. The solid region E is inside the cone $z = \sqrt{x^2 + y^2}$ and under the paraboloid $z = 2 - x^2 - y^2$.

   (a) Convert the cone and paraboloid equations to cylindrical coordinates **and** compute the values of r and z at the intersection of the surfaces.

Solution:

Start with the cone:

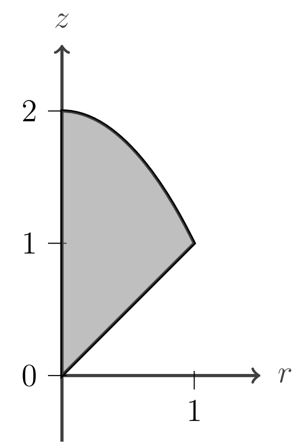
$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ &= \sqrt{r^2} = r. \end{aligned}$$

The paraboloid becomes

$$\begin{aligned} z &= 2 - x^2 - y^2 \\ &= 2 - (x^2 + y^2) \\ &= 2 - r^2. \end{aligned}$$

At the intersection of the surfaces

$$\begin{aligned} &r = 2 - r^2, \text{ or} \\ &r^2 + r - 2 = 0. \end{aligned}$$



Factoring the left side we see

$$(r + 2)(r - 1) = 0.$$

Because we require $r > 0$ there is a single solution, $r = 1$. Use the cone equation to see that $z = r = 1$.

   (b) Using cylindrical coordinates, compute the triple integral $\iiint_E 2z\, dV$.

Solution:
The integral is

$$\begin{aligned} \int_0^{2\pi}\int_0^1\int_r^{2-r^2} 2z\, r\, dz\, dr\, d\theta &= \int_0^{2\pi} d\theta \int_0^1 rz^2\Big|_r^{2-r^2} dr \\ &= 2\pi \int_0^1 r(4 - 5r^2 + r^4)\, dr \\ &= 2\pi \int_0^1 (4r - 5r^3 + r^5)\, dr \\ &= 2\pi\,(2 - \frac{5}{4} + \frac{1}{6}) = \frac{11}{6}\pi. \end{aligned}$$

2. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 2z$ and outside the sphere $x^2 + y^2 + z^2 = 1$.

   (a) Transform the sphere equations to cylindrical coordinates and plot the region of integration in the rz half plane.

Solution:

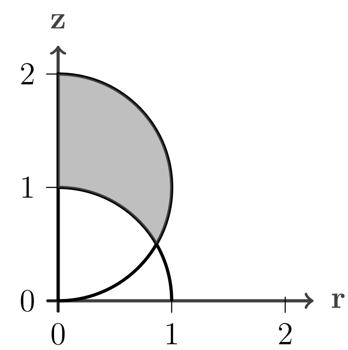
In cylindrical coordinates the sphere $x^2 + y^2 + z^2 = 2z$ becomes

$$r^2 + (z-1)^2 = 1$$

which is a (half) circle of radius 1 centered at 1 on the z-axis. The second sphere is

$$r^2 + z^2 = 1$$

which is a unit circle centered $(0, 0)$.



   (b) Using spherical coordinates, compute the values of ρ and ϕ at the intersection of the two spheres.

Solution: In spherical coordinates the sphere $x^2 + y^2 + z^2 = 2z$ becomes $\rho = 2\cos\phi$. Because the second sphere is $\rho = 1$ we must have $\rho = 1$ at the point of intersection. And because $\rho = 1$ we have

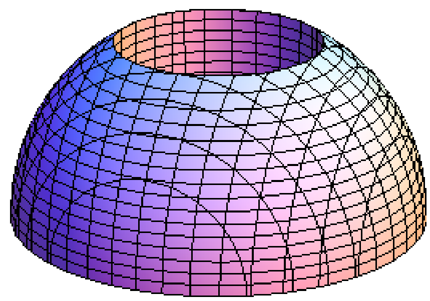
$$2\cos\phi = 1$$
$$\cos\phi = \frac{1}{2}$$
$$\phi = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}.$$

(c) Use a single triple integral in spherical coordinates to compute the volume of the solid.

Solution:

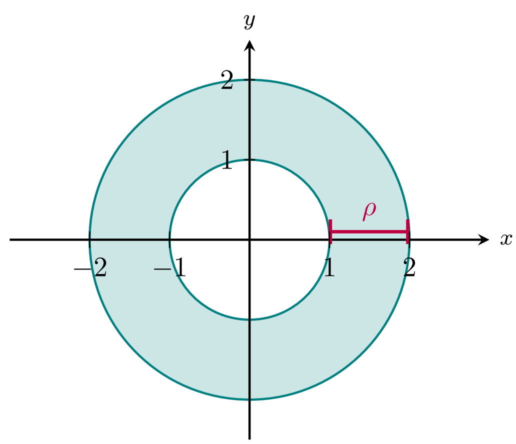
$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi/3}\int_1^{2\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta &= \int_0^{2\pi} 1\, d\theta \int_0^{\pi/3} \sin(\phi)\Big(\int_1^{2\cos(\phi)} \rho^2\, d\rho\Big)\, d\phi \\
&= 2\pi \int_0^{\pi/3} \sin(\phi)\, \frac{1}{3}(8\cos^3(\phi) - 1)\, d\phi \\
&= \frac{16\pi}{3}\int_0^{\pi/3} \sin(\phi)\, \cos^3(\phi)\, d\phi - \frac{2\pi}{3}\int_0^{\pi/3} \sin(\phi)\, d\phi \\
&= -\frac{16\pi}{3}\int_1^{1/2} u^3\, du + \frac{2\pi}{3}\big[\cos(\phi)\big|_0^{\pi/3}\big] \\
&= -\frac{16\pi}{3}\Big(\frac{1}{64} - \frac{1}{4}\Big) + \frac{2\pi}{3}\Big(\frac{1}{2} - 1\Big) = \frac{11}{12}\pi.
\end{aligned}$$

3. Express the integral $\iiint_E (x^2+y^2)\,dV$ in **spherical coordinates** and evaluate it. E is the solid located inside the sphere $x^2+y^2+z^2=4$, outside the cylinder $x^2+y^2=1$ and above the xy-plane. Use the integration order $d\rho\,d\phi\,d\theta$.

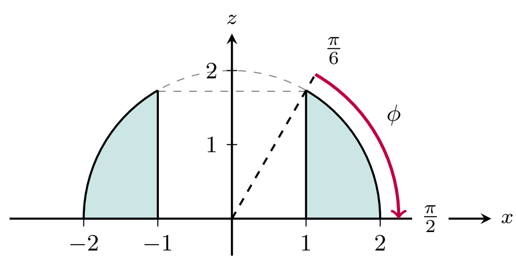


Solution:

In spherical coordinates $x^2+y^2 = \rho^2\sin^2\phi$. Both surfaces are axisymmetric with respect to the z-axis, because x and y appear only in the form x^2+y^2, so $0 \le \theta \le 2\pi$.



The sphere is $\rho = 2$ and the cylinder is $\rho\sin(\phi) = 1$. To find ϕ at the intersection of the cylinder and sphere solve the simultaneous equations for ϕ. Because $\phi = \sin^{-1}(1/2)$, we have $\phi = \pi/6$ when $z \ge 0$.



In spherical coordinates

$$\iiint_E (x^2+y^2)\,dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_{\frac{1}{\sin(\phi)}}^{2} \rho^4\sin^3(\phi)\,d\rho\,d\phi\,d\theta$$

4. Match each iterated integral with the appropriate graph by placing a letter (A, B, C, D, E, or F) in the box next to the graph.

A. $\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

B. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

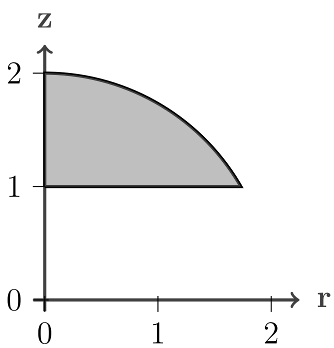
C. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{1/\cos\phi}^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

D. $\displaystyle\int_0^{2\pi}\int_0^{2}\int_r^2 r \, dz \, dr \, d\theta$

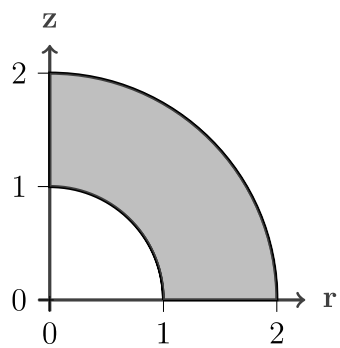
E. $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{2}}\int_z^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta$

F. $\displaystyle\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$

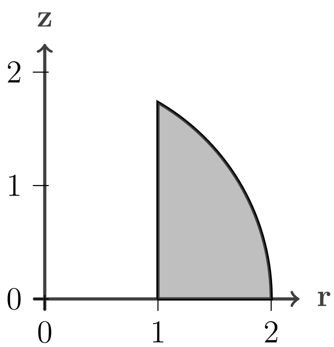
C



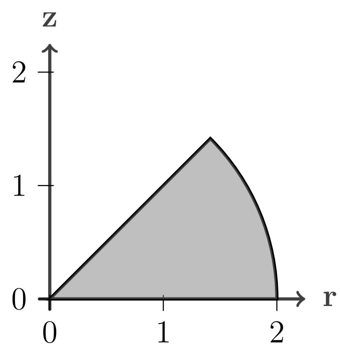
A



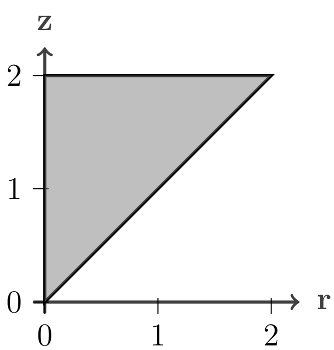
F



E



D



B