

Lecture #09: Directional Derivatives & Gradients

Date: Tue. 10/16/18

Previously we have seen how to find the rate of change in terms of x & y .

What about the rate of change in other directions?

Example: say I have a fcn that describes volume of a gas in terms of temperature & pressure.

We want to know how volume changes as both temp. & pressure changes.

Directional Derivatives

Let $z = f(x, y)$ be a surface & (x_0, y_0) is in the domain of f .

Then the unit vector pointing in the direction of θ (measured from pos. x -axis) is

$$\vec{u} = \cos(\theta)\hat{j} + \sin(\theta)\hat{j}$$

Note that $\|\vec{u}\| = 1$

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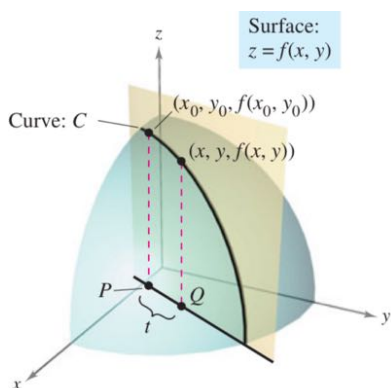
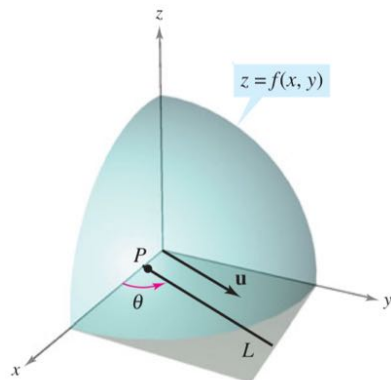
Let $z = f(x, y)$ be a surface

& (x_0, y_0) is in the domain of f

Then the unit vector pointing in the direction of θ (measured from positive x -axis) is

$$\vec{u} = \cos(\theta)\hat{j} + \sin(\theta)\hat{j}$$

Note that $\|\vec{u}\| = 1$



Let P be the plane passing thru (x_0, y_0) & parallel to \vec{u}

This problem is now reduced to 2 dimensions by slicing the surface

The intersection of the plane P & the xy plane is given by

$$x = x_0 + \cos(\theta)t$$

$$y = y_0 + \sin(\theta)t$$

So the point of intersection on the surface is $(x_0, y_0, f(x_0, y_0))$

The slope of the secant line thru the pts $(x_0, y_0, f(x_0, y_0))$ & $(x, y, f(x, y))$

$$\frac{f(x, y) - f(x_0, y_0)}{t} = \frac{f(x_0 + t \cos \theta, y_0 + t \sin \theta) - f(x_0, y_0)}{t}$$

Def

DEFINITION OF DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of \mathbf{u}** , denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x, y) = \lim_{t \rightarrow 0} \frac{f(x + t \cos \theta, y + t \sin \theta) - f(x, y)}{t}$$

provided this limit exists.

Of course, as with single variable fns, we don't usually apply the definition directly.

Thm

THEOREM 13.9 DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ is

$$D_{\mathbf{u}}f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

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Ex. 1 Find the directional derivative of

$$f(x, y) = x^2 \sin(2y)$$

in direction of $\vec{v} = \langle 3, -4 \rangle$ Soln. Directional derivative is given by

$$D_{\vec{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

the direction vector \vec{u} is given by

$$\begin{aligned} \vec{u} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\hat{i} - 4\hat{j}}{(3^2 + 4^2)^{1/2}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} \\ &= \cos \theta \hat{i} + \sin \theta \hat{j} \end{aligned}$$

$$\Rightarrow \cos \theta = \frac{3}{5} \text{ \& \; } \sin \theta = -\frac{4}{5}$$

Partial Derivatives:

$$\frac{\partial f}{\partial x} = 2x \sin(2y)$$

$$\frac{\partial f}{\partial y} = 2x^2 \cos(2y)$$

So the directional derivative in direction of \vec{u} is

$$\begin{aligned} \Rightarrow D_{\vec{u}} f(x, y) &= f_x(x, y) \cos \theta + f_y(x, y) \sin \theta \\ &= 2x \sin(2y) \cos \theta + 2x^2 \cos(2y) \sin \theta \\ &= 2x \sin(2y) \left(\frac{3}{5}\right) + 2x^2 \cos(2y) \left(-\frac{4}{5}\right) \\ &= \frac{6}{5} x \sin(2y) - \frac{8}{5} x^2 \cos(2y) \end{aligned}$$

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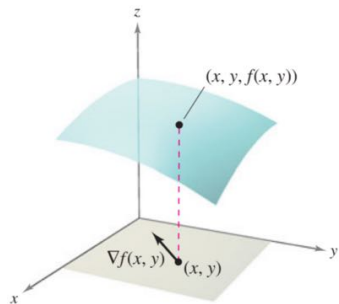
Gradients

DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f** , denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del f ." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y) , the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).



The gradient of f is a vector in the xy -plane.

Note that the gradient can also be written as

$$\begin{aligned}\nabla f(x, y) &= \langle f_x(x, y), f_y(x, y) \rangle \\ &= \langle f_x, f_y \rangle\end{aligned}$$

The gradient is also found for 3 variables (& for n -variables)

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

Ex. 2) Find the gradient of

$$f(x, y) = y \ln(x) + xy^2$$

Soln.

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \left\langle \frac{y}{x} + y^2, \ln(x) + 2xy \right\rangle$$

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Using the gradient

$$\begin{aligned}
 D_{\vec{u}} f(x,y) &= f_x u_1 + f_y u_2 \\
 &= \langle f_x, f_y \rangle \cdot \langle u_1, u_2 \rangle \\
 &= \nabla f(x,y) \cdot \vec{u}
 \end{aligned}$$

This gives us another form for our directional derivative.

Thm.**THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE**

If f is a differentiable function of x and y , then the directional derivative of f in the direction of the unit vector \mathbf{u} is

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

Note: This is a scalar value when evaluated @ a pt (a,b)

Ex.3 Find the directional derivative of $f(x,y) = 3x^2 - 2y^2$

@ pt $(-\frac{3}{4}, 0)$ in direction $\vec{v} = \langle \frac{3}{4}, 1 \rangle$

Soln. $\nabla f(x,y) = \langle 6x, -4y \rangle$ & $\nabla f(-\frac{3}{4}, 0) = \langle -\frac{9}{2}, 0 \rangle$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$\begin{aligned}
 \Rightarrow D_{\vec{u}} f(x,y) &= \nabla f(-\frac{3}{4}, 0) \cdot \vec{u} \\
 &= \langle -\frac{9}{2}, 0 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle \\
 &= -27
 \end{aligned}$$

Properties of the gradient

THEOREM 13.11 PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y) .

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y) = 0$ for all \mathbf{u} .
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}}f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}}f(x, y)$ is $-\|\nabla f(x, y)\|$.

skier going down mountain \Rightarrow altitude given by $F(x, y)$

Conceptually, $-\nabla F(x, y)$ is the fastest path down the mountain (or to dec. volume of a gas)

Ex. 4) The temp. in $^{\circ}\text{C}$ on surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x & y are measured in cm in what direction from pt $(2, -3)$ does temp increase most rapidly?

Soln. $\nabla T(x, y) = \langle T_x, T_y \rangle = \langle -8x, -2y \rangle$

Direction of max increase: $\nabla T(2, -3) = \langle -16, 6 \rangle$

max rate increase is

$$\|\nabla T(2, -3)\| = (292)^{1/2} \approx 17.09^{\circ}/\text{cm}$$

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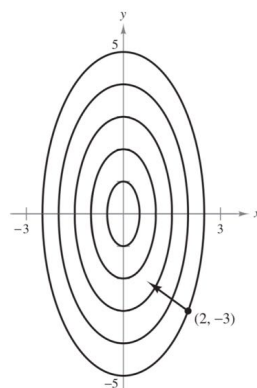
Thm.**THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES**

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0) .

Ex. 5 Consider the last example.

we can see that @ $(2, -3)$
that $\nabla f(2, -3)$ is normal to
the level curve thru $(2, -3)$

Level curves:
 $T(x, y) = 20 - 4x^2 - y^2$



Gradients For FCns of 3 or more Variables

DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES

Let f be a function of x , y , and z , with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}}f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}}f(x, y, z) = 0$ for all \mathbf{u} .
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$\|\nabla f(x, y, z)\|. \quad \text{Maximum value of } D_{\mathbf{u}}f(x, y, z)$$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}}f(x, y, z)$ is

$$-\|\nabla f(x, y, z)\|. \quad \text{Minimum value of } D_{\mathbf{u}}f(x, y, z)$$