1. Use the Chain Rule to find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for

$$w = xyz, \quad x = s + t, \quad y = s - t, \quad z = st^2$$

Solution: Note that

$$\begin{aligned}
&\frac{\partial w}{\partial x} = yz \qquad &&\frac{\partial w}{\partial y} = xz \qquad &&\frac{\partial w}{\partial z} = xy \\
&\frac{\partial x}{\partial s} = 1 &&\frac{\partial y}{\partial s} = 1 &&\frac{\partial z}{\partial s} = t^2 \\
&\frac{\partial x}{\partial t} = 1 &&\frac{\partial y}{\partial t} = -1 &&\frac{\partial z}{\partial t} = 2st
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (yz)(1) + (xz)(1) + (xy)(t^2) \\
&= yz + xz + t^2xy
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} \\
&= (yz)(1) + (xz)(-1) + (xy)(2st) \\
&= yz - xz + 2stxy
\end{aligned}$$

2. Differentiate implicitly to find the first partial derivatives of z for

$$e^{xz} + xy = 0$$

Solution: Since $f(x, y, z) = e^{xz} + xy = 0$ and

$$\begin{aligned} F_x(x, y, z) &= e^{xz} + y \\ F_y(x, y, z) &= x \\ F_z(x, y, z) &= e^{xz} \end{aligned}$$

we note that $F_z(x, y, z) = e^{xz} \neq 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{e^{xz} + y}{e^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = -\frac{x}{e^{xz}}$$

3. If $z = 5x^2 + y^2$ and (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$, compare the values of Δz and dz.

Solution: Let $(x, y) = (1, 2)$ then $(x + \Delta x, y + \Delta y) = (1.05, 2.1)$. This means that

$$dx = \Delta x = 0.05 \qquad dy = \Delta y = 0.1$$

so the differential is given by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = 10x\, dx + 2y\, dy$$

when $(x, y) = (1, 2)$ then

$$\Delta z \approx 10(1)(0.05) + 2(2)(0.1) = 0.9$$

for the actual change in z we have

$$\begin{aligned} \Delta z &= f(1.05, 2.1) - f(1, 2) \\ &= 5(1.05)^2 + (2.1)^2 - (5(1)^2 + (2)^2) \\ &= 9.9225 - 9 = 0.9225 \end{aligned}$$

4. Compute $\frac{\partial g}{\partial s}$ at $(t, s) = (1, 2)$ if

$$g(x, y) = x^2 - y^2, \quad x = t^2 + s^2, \quad y = t^3 - 2s$$

Solution: Since x and y are functions of t and s, then $g(x, y)$ is a function of two independent variables, t and s i.e $g(x(t, s), y(t, s))$. The appropriate version of the chain rule is

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

Since

$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = -2y$$

$$\frac{\partial x}{\partial s} = 2s, \qquad \frac{\partial y}{\partial s} = -2$$

and so

$$\frac{\partial g}{\partial s} = 4xs + 4y$$

When $(t, s) = (1, 2)$ we see that

$$x(1, 2) = (1)^2 + (2)^2 = 5 \qquad \text{and} \qquad y(1, 2) = (1)^3 - 2(2) = -3$$

then

$$\begin{aligned}\frac{\partial g}{\partial s}(1, 2) &= 4(5)(2) + 4(-3)\\ &= 40 - 12\\ &= 28.\end{aligned}$$