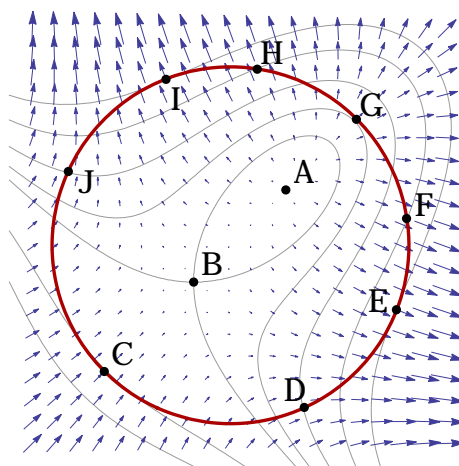


1. This figure shows several level curves, $f(x, y) = k$, and the gradient vector field ∇f . The circle is the graph of a constraint equation $g(x, y) = 0$. Locations of ten points, A, B, C, D, E, F, G, H, I and J are marked.



- (a) List the points on the circle where ∇f is perpendicular to the graph of the constraint equation $g(x, y) = 0$.

Solution: By inspection, points C, E, I, and G.

- (b) List the points where $f(x, y)$ has an absolute minimum, subject to the constraint $g(x, y) = 0$. If there are no such points, write DNE.

Solution: Point C, because of the flow of the gradient vectors and $\nabla f(C)$ being perpendicular to the circle.

- (c) List the points where $f(x, y)$ has an absolute maximum, subject to the constraint $g(x, y) = 0$. If there are no such points, write DNE.

Solution: Points E, I, because of the flow of the gradient vectors and $\nabla f(E)$ and $\nabla f(I)$ being perpendicular to the circle.

2. Find the extreme value of the function $f(x, y, z) = xyz$ subject to the constraint $3x + 3y + 2z = 12$. Is the extreme value a maximum or a minimum?

Solution:

First we solve for the gradients of the objective and constraint functions.

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle \quad \text{and} \quad \nabla g(x, y, z) = \langle 3, 3, 2 \rangle.$$

The four equations we must solve are:

$$yz = 3\lambda$$

$$xz = 3\lambda$$

$$xy = 2\lambda$$

$$3x + 3y + 2z = 12$$

Solving for x we find that $y = x$ and $z = \frac{3}{2}x$, which requires

$$x = \frac{4}{3}, y = \frac{4}{3}, \text{ and } z = 2.$$

If $\lambda = 0$ then the first three equations are satisfied and $x = 0$ or $y = 0$ or $z = 0$.

The maximum value is $f\left(\frac{4}{3}, \frac{4}{3}, 2\right) = \frac{32}{9}$. The minimum value of $f(x, y, z)$ is 0 which occurs whenever x, y or z are zero.

3. Find the extreme value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x + 2y - z = 12$. Is the extreme value a maximum or a minimum?

Solution: First we solve for the gradients of the objective and constraint functions:

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle \quad \text{and} \quad \nabla g(x, y, z) = \langle 1, 2, -1 \rangle.$$

The four equations we must solve are:

$$2x = \lambda$$

$$2y = 2\lambda$$

$$2z = -\lambda$$

$$x + 2y - z = 12$$

Solving we find that $\lambda = 4$, which requires $x = 2$, $y = 4$, and $z = -2$.

If $\lambda = 0$ then the first three equations are satisfied and $x = 0$ or $y = 0$ or $z = 0$.

$f(2, 4, -2) = 24$ is a minimum; at nearby points on the constraint surface we find $f(0, 6, 0) = 36$ and $f(12, 0, 0) = f(0, 0, -12) = 144$.

4. Find the extreme value of $f(x, y) = x^2 + y^2$ on the hyperbola $4x^2 - y^2 = 4$. Is the extreme value a maximum or minimum?

Solution: For the function $g(x, y) = 4x^2 - y^2 - 4$ the constraint equation is $g(x, y) = 0$. Solving for the gradients of the objective and constraint functions we find

$$\nabla f(x, y) = \langle 2x, 2y \rangle \quad \text{and} \quad \nabla g(x, y) = \langle 8x, -2y \rangle$$

To find critical points on the hyperbola, we solve the Lagrange equations

$$\begin{aligned} 8x &= 2\lambda x \\ -2y &= 2\lambda y \\ 4x^2 - y^2 &= 4. \end{aligned}$$

The first equation factors to $2x(4 - \lambda) = 0$, so either $x = 0$ or $\lambda = 4$.

If $x = 0$, then $4x^2 - y^2 = 4$ requires $y^2 = -4$. This is a dead end; $x = 0$ produces no critical points.

If $\lambda = 4$, then $-2y = 2\lambda y$ requires $y = 0$. And if $y = 0$, then $4x^2 - y^2 = 4$ requires $x = \pm 1$. So $\lambda = 4$ produces two critical points: $(1, 0)$ and $(-1, 0)$.

Calculating f at the critical points produces $f(1, 0) = f(-1, 0) = 4$. You can convince yourself that $f(x, y) = 4$ is an absolute minimum by computing $f(x, y)$ at another point on the hyperbola, or by understanding that $f(x, y)$ measures the square of the distance from the origin to the point (x, y) .