# Lecture # 05: Vector Valued Functions

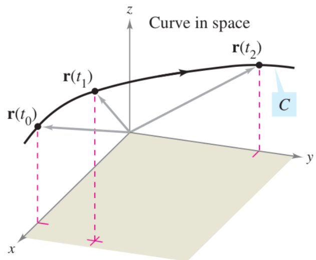
Date: Tue. 9/25/18

**Def** A vector valued function is a function of the form

$$\hat{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

where the components of the vector are functions of the parameter $t$.

A vector valued function gives the graph of a curve in space




Curve $C$ is traced out by the terminal point of position vector $\mathbf{r}(t)$.

Two different fcns can give the same curve in space, but they will differ in their orientation

The domain of a vector valued function is given by the intersection of the domains of the component functions.

The range is the set of all vectors defined by $\hat{r}(t)$ on its domain.

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**Ex)** Domain of $\hat{r}(t) = \ln(t)\hat{i} + \sqrt{1-t}\,\hat{j} + t\hat{k}$ is $(0, 1]$

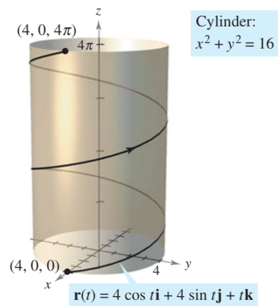
b/c
D: $\ln(t)$ is $(0, \infty)$
D: $\sqrt{1-t}$ is $(-\infty, 1]$
D: $t$ is $(-\infty, \infty)$

$$(0, \infty) \cap (-\infty, 1] \cap (-\infty, \infty) = (0, 1]$$

$\cap$ "intersection"

**Ex)** Sketching a vector valued function

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4\pi.$$




As $t$ increases from 0 to $4\pi$, two spirals on the helix are traced out.

To sketch this curve we can rewrite in "rectangular form" as

$$x^2 + y^2 = 16$$

so that as $t$ increases from $0$ to $4\pi$ we see that for $z = t$ our curve follows the path of a helix.

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many of the definitions & rules we know for functions also apply to vector valued fcns with the only difference being that they are applied component wise.

Ex) sum & difference of $\vec{r}_1(t)$ & $\vec{r}_2(t)$ $\in \mathbb{R}^2$

$$\hat{r}_1(t) \pm \hat{r}_2(t) = (f_1(t)\hat{i} \pm g_1(t)\hat{j}) \pm (f_2(t)\hat{i} \pm g_2(t)\hat{j})$$
$$= (f_1(t) \pm f_2(t))\hat{i} \pm (g_1(t) \pm g_2(t))\hat{j}$$

## Limits & Continuity

**DEFINITION OF THE LIMIT OF A VECTOR-VALUED FUNCTION**

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j}$$ Plane

provided $f$ and $g$ have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$ Space

provided $f$, $g$, and $h$ have limits as $t \to a$.

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**DEFINITION OF CONTINUITY OF A VECTOR-VALUED FUNCTION**

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** $I$ if it is continuous at every point in the interval.

Ex) Is $\vec{r}(t) = t\,\hat{i} + a\,\hat{j} + (a^2 - t^2)\,\hat{k}$ continuous @ $t = 0$?

$$\Rightarrow \lim_{t \to 0} \vec{r}(t) = \left[\lim_{t \to 0} t\right]\hat{i} + \left[\lim_{t \to 0} a\right]\hat{j} + \left[\lim_{t \to 0} (a^2 - t^2)\right]\hat{k}$$

$$= 0\,\hat{i} + a\,\hat{j} + a^2\,\hat{k} = a\hat{j} + a^2\hat{k}$$

Since $\vec{r}(0) = (0)\,\hat{i} + a\,\hat{j} + a^2\,\hat{k} = a\hat{j} + a^2\hat{k}$

Then $\lim_{t \to 0} \vec{r}(t) = \vec{r}(0)$ & $\vec{r}(t)$ is Cont. @ $t = 0$

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## Differentiation

The definition of the derivative is equivalent to the one we know for real valued fcns

**DEFINITION OF THE DERIVATIVE OF A VECTOR-VALUED FUNCTION**

The **derivative of a vector-valued function r** is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all $t$ for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is **differentiable at *t*.** If $\mathbf{r}'(t)$ exists for all $t$ in an open interval $I$, then $\mathbf{r}$ is **differentiable on the interval *I*.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

The basic differentiation rules also apply but are now done component wise.

**THEOREM 12.1 DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS**

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $f$ and $g$ are differentiable functions of $t$, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Plane

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions of $t$, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. Space

Ex) For $\vec{r}(t) = t^2\,\hat{i} - 4\hat{j}$ Find $\vec{r}\,'(t)$

$$r'(t) = \frac{d}{dt}[t^2]\,\hat{i} - \frac{d}{dt}[4]\,\hat{j}$$

$$= 2t\,\hat{i} - 0\,\hat{j}$$

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Since multiplication does not work the same way for vectors we will have the following properties for $\vec{r}'(t)$

**THEOREM 12.2 PROPERTIES OF THE DERIVATIVE**

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of $t$, let $w$ be a differentiable real-valued function of $t$, and let $c$ be a scalar.

1. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $D_t[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$
4. $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $D_t[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

## Integration

**DEFINITION OF INTEGRATION OF VECTOR-VALUED FUNCTIONS**

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where $f$ and $g$ are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}$$

Plane

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}$$

Space

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

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## Velocity & Acceleration

Recall that for real valued fcns that for a Object in motion with Position Fcn $s(t)$ we have

$$\text{Velocity}: \quad v(t) = s'(t)$$

$$\text{acceleration}: \quad a(t) = v'(t) = s''(t)$$

For vector valued fcns these are defined Similarly

**DEFINITIONS OF VELOCITY AND ACCELERATION**

If $x$ and $y$ are twice-differentiable functions of $t$, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time $t$ are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$

$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$