Math 2210

Lecture # 07: Multivariable Fcns & Partial Derivatives

Date: Tue. 10/9/18

Multivariable Fcns

Fcn of 2 Variables:

$$z = f(x,y) = x^2 + xy$$

x & y are the independent variables
z is the dependent variable

Def

DEFINITION OF A FUNCTION OF TWO VARIABLES

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is called a **function of x and y.** The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f.

Fcn of 3 Variables:

$$w = f(x,y,z) = x + 2y - 3z$$

Fcn of n variables:

$$f(x_1, x_2, \dots, x_n)$$

Where independent variables are $x_1, \dots, x_n$
note that pts of an n-variable Fcn are called <u>n-tuples</u>: $(x_1, x_2, \dots, x_n)$

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Def The domain of a multivariable fcn is the set of all pts for which the fcn is defined.

Ex. 1 Domain & Range of

$$f(x,y) = \cos(2x+y)$$

"for all"

Soln Since $2x+y$ is defined $\forall\ x, y$ & cosine defined $\forall$ inputs

$\Rightarrow$ domain is $\mathbb{R}^2$

Since range of cosine is $[-1, 1]$ the range of f will also fall in this range.

Operations for Multivar. Fcns

We have equivalent rules for multivar fcns as we do for single variable fcns.

$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$	Sum or difference
$(fg)(x, y) = f(x, y)g(x, y)$	Product
$\dfrac{f}{g}(x, y) = \dfrac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$	Quotient

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Fcn Composition

We do not have fcn composition for 2 multivar. Fcns. However, we can compose a multivar. Fcn with a single variable fcn.

For a single variable fcn $g(t)$ & a multivar Fcn $f(x,y)$ the composition of g & f is

$$(g \circ f)(x,y) = g(f(x,y))$$

Def A <u>polynomial Fcn</u> of two variables is the sum of terms of the form cx^my^n were c is a real #)

Def A <u>rational Fcn</u> in 2 variables is the quotient of two polynomial Fcns.

These two definitions are equivalent for Fcns of n variables.

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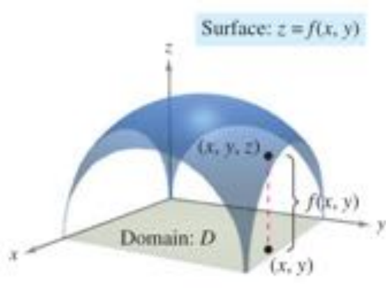
Graph of Fcn in 2 Variables

The graph of a fcn $z = f(x,y)$ is a surface in space.

The Domain of $f(x,y)$ is the projection of the surface onto the x,y plane as seen in the figure.



In other words, to each pt (x,y) in D there corresponds a pt (x,y,z) on the surface.

To sketch a surface by hand it helps to again use the traces of the surface

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Ex.2) Graph $f(x,y) = (16 - 4x^2 - y^2)^{1/2}$

Domain of this fcn is all (x,y) for which

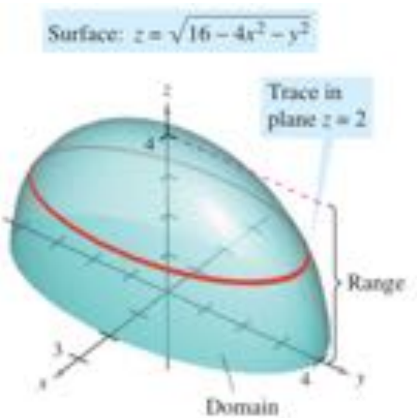
$$16 - 4x^2 - y^2 \geq 0$$

D is then the set of all pts given by

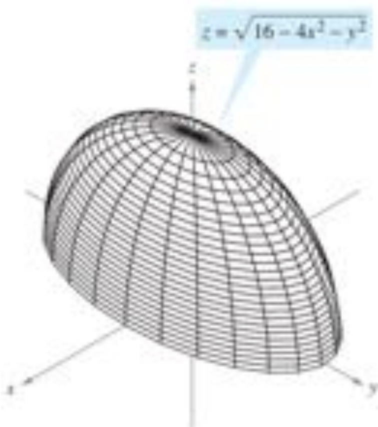
$$16 - 4x^2 - y^2 = 0$$

$$\Rightarrow 16 = 4x^2 + y^2$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$$ an ellipse in the xy-plane



The graph of $f(x,y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.



traces of $f(x,y)$

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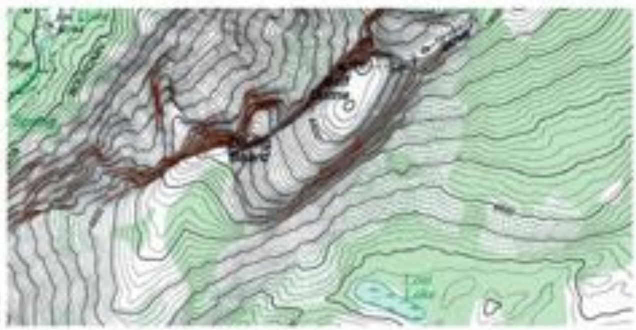
Level Curves

Another way to examine $F(x,y)$ visually is by looking @ it's level curves.

<u>Def</u> a level curve is a curve for which the value of $F(x,y)$ is constant.

The best real world example is that of a topographic map where the level curves rep. the height of the land above sea level.

Figure 13.7



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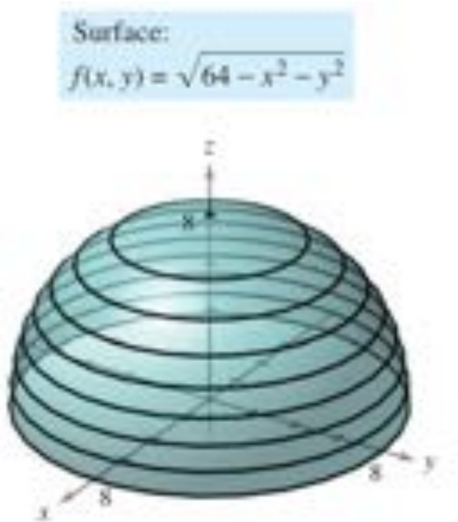
Ex.3 Sketch level curves of $f(x,y) = (64 - x^2 - y^2)^{1/2}$

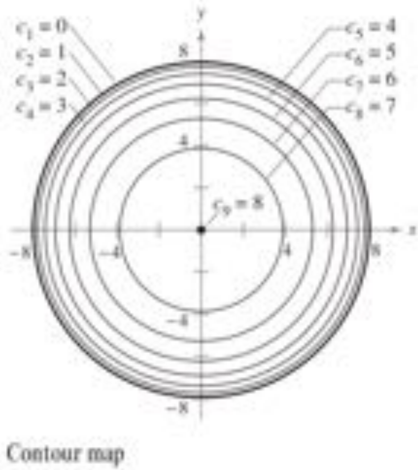
We just graph the eqns given by solving $f(x,y) = c$ for several values of c.

For example:

When $c = 0$ we have $x^2 + y^2 = 64$

a circle w/ radius 8





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Continuity

We won't get two deep into the ideas behind limits of multivar. Fcns. However, before discussing differentiation it is important to understand continuity.

Def

DEFINITION OF CONTINUITY OF A FUNCTION OF TWO VARIABLES

A function f of two variables is **continuous at a point (x_0, y_0)** in an open region R if $f(x_0, y_0)$ is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0). That is,

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Thm

THEOREM 13.1 CONTINUOUS FUNCTIONS OF TWO VARIABLES

If k is a real number and f and g are continuous at (x_0, y_0), then the following functions are continuous at (x_0, y_0).

1. Scalar multiple: kf
2. Sum and difference: $f \pm g$
3. Product: fg
4. Quotient: f/g, if $g(x_0, y_0) \neq 0$

Def

DEFINITION OF CONTINUITY OF A FUNCTION OF THREE VARIABLES

A function f of three variables is **continuous at a point (x_0, y_0, z_0)** in an open region R if $f(x_0, y_0, z_0)$ is defined and is equal to the limit of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0). That is,

$$\lim_{(x, y, z)\to(x_0, y_0, z_0)} f(x, y, z) = f(x_0, y_0, z_0).$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

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Partial Derivatives

We now have multiple variables to deal with when we want to investigate the rate of change of a variable, we must do so independent of the other variables.

Since we're only looking at the derivative with respect to a part of the fcn f we call this the partial derivative.

Def

DEFINITION OF PARTIAL DERIVATIVES OF A FUNCTION OF TWO VARIABLES

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

We now have some new notation for partial derivatives. Just as with single variable fcns there are several different acceptable notations

NOTATION FOR FIRST PARTIAL DERIVATIVES

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a, b).$$

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To find a partial derivative with respect to a variable we treat all other variables as if they were constants.

Ex.4) $g(x,t) = t^5 - 3tx$

$$\frac{\partial g}{\partial x} = g_x = -3t$$

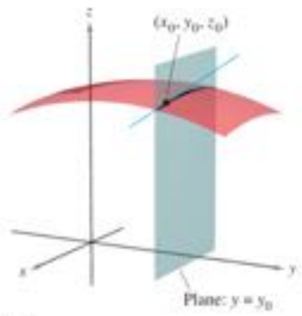
$$\frac{\partial g}{\partial t} = g_t = 5t^4 - 3x$$

$\boxed{\text{Slopes of Surfaces}}$

Geometrically, for 2 variable fcns if $y = y_0$ then

$$z = f(x, y_0)$$

is the curve formed by intersecting the surface $z = f(x,y)$ with the plane $y = y_0$.



$\frac{\partial f}{\partial x}$ = slope in x-direction

This means that

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

gives the slope of this curve @ the pt $(x_0, y_0, f(x_0, y_0))$

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This will work similarly for $x = x_0$

Informally, the values of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ give the slope of the surface in the x & y directions.

Higher Order Partial Derivatives

We can also take higher order derivatives for partial derivatives, they are taken in the order differentiation occurs.

For a fcn $f(x,y)$:

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives.**

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Note: For mixed partials

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$ Right to left order

$$(f_x)_y = f_{xy}$$ Left to right order

Using either notation take derivative in order of which variable is "closest" to f

$\boxed{\text{Mixed Partial Derivatives}}$

THEOREM 13.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Ex. 5

$$f(x,y) = x^2 y^2$$

$$f_x = 2x y^2$$

$$f_y = 2y x^2$$

$$f_{xy} = 4xy$$

$$f_{yx} = 4xy$$

$$\Rightarrow f_{xy} = f_{yx}$$