Lecture #09: Directional Derivatives & Gradients

Date: Tue. 10/16/18

Previously we have seen how to find the rate of change in terms of x & y.

What about the rate of change in other directions?

Example: say I have a fcn that describes volume of a gas in terms of temperature & pressure.

We want to know how volume changes as <u>both</u> temp. & pressure changes.

Directional Derivatives

Let $z = f(x,y)$ be a surface & (x_0, y_0) is in the domain of f.

Then the unit vector pointing in the direction of θ (measured from pos. x-axis) is

$$\vec{u} = \cos(\theta)\hat{\jmath} + \sin\theta\hat{\jmath}$$

Note that $\|u\| = 1$

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$$\vec{u} = \cos(\theta)\hat{j} + \sin(\theta)\hat{j}$$

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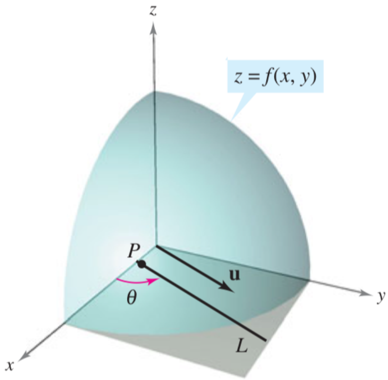

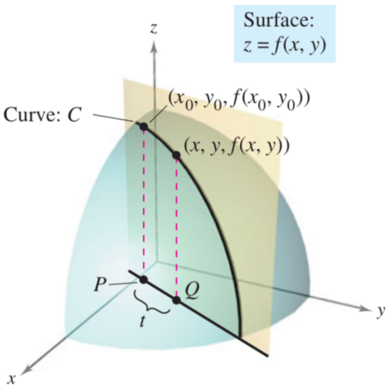

Let P be the plane passing thru (x_0, y_0) & parallel to $\vec{u}$

This problem is now reduced to 2 dimensions by slicing the surface

The intersection of the plane P & the xy plane is given by

$$x = x_0 + \cos(\theta)t$$

$$y = y_0 + \sin(\theta)t$$

So the point of intersection on the surface is $(x_0, y_0, f(x_0, y_0))$

Lecture #09: Directional Derivatives & Gradients

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The slope of the secant line thru the pts $(x_0, y_0, f(x_0,y_0))$ & $(x, y, f(x,y))$

$$\frac{f(x,y) - f(x_0,y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0,y_0)}{t}$$

<u>Def</u>

DEFINITION OF DIRECTIONAL DERIVATIVE

Let f be a function of two variables x and y and let $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ be a unit vector. Then the **directional derivative of f in the direction of u,** denoted by $D_{\mathbf{u}}f$, is

$$D_{\mathbf{u}}f(x,y) = \lim_{t\to 0} \frac{f(x + t\cos\theta, y + t\sin\theta) - f(x,y)}{t}$$

provided this limit exists.

Of course, as with single variable fcns, we don't usually apply the definition directly.

<u>Thm</u>

THEOREM 13.9 DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

Lecture # 09: Directional Derivatives & Gradients

Date: Tue. 10/16/18

Ex. 1) Find the directional derivative of

$$f(x,y) = x^2 \sin(2y)$$

in direction of $\vec{v} = \langle 3, -4 \rangle$

Soln. Directional derivative is given by

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

the direction vector $\vec{u}$ is given by

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\hat{i} - 4\hat{j}}{(3^2+4^2)^{1/2}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$= \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\Rightarrow \cos\theta = \frac{3}{5} \ \& \ \sin\theta = -\frac{4}{5}$$

Partial Derivatives:

$$\frac{\partial f}{\partial x} = 2x\sin(2y)$$

$$\frac{\partial f}{\partial y} = 2x^2\cos(2y)$$

So the directional derivative in direction of $\vec{u}$ is

$$\Rightarrow D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

$$= 2x\sin(2y)\cos\theta + 2x^2\cos(2y)\sin\theta$$

$$= 2x\sin(2y)\left(\frac{3}{5}\right) + 2x^2\cos(2y)\left(\frac{-4}{5}\right)$$

$$= \frac{6}{5}x\sin(2y) - \frac{8}{5}x^2\cos(2y)$$

Lecture # 09: Directional Derivatives & Gradients   Date: Tue. 10/16/18

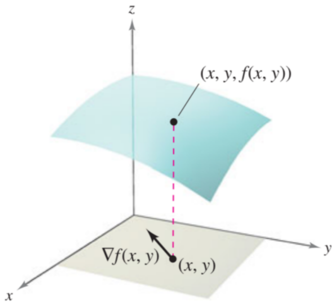
Gradients

DEFINITION OF GRADIENT OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f,** denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}.$$

∇f is read as "del f." Another notation for the gradient is **grad** $f(x, y)$. In Figure 13.48, note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).



The gradient of f is a vector in the xy-plane.

Note that the gradient can also be written as

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$
$$= \langle f_x, f_y \rangle$$

The gradient is also found for 3 variables (& for n-variables)

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

Ex. 2) Find the gradient of

$$f(x,y) = y\ln(x) + xy^2$$

Soln.

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

$$= \langle \frac{y}{x} + y^2, \ln(x) + 2xy \rangle$$

Lecture # 09: Directional Derivatives & Gradients Date: Tue. 10/16/18

Using the gradient

$$D_{\vec{u}} f(x,y) = f_x u_1 + f_y u_2$$
$$= \langle f_x, f_y \rangle \cdot \langle u_1, u_2 \rangle$$
$$= \nabla f(x,y) \cdot \vec{u}$$

This gives us another form for our directional derivative.

Thm.

THEOREM 13.10 ALTERNATIVE FORM OF THE DIRECTIONAL DERIVATIVE

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

Note: This is a scalar value when evaluated @ a pt (a,b)

Ex.3 Find the directional derivative of

$$f(x,y) = 3x^2 - 2y^2$$

@ pt $(-\frac{3}{4}, 0)$ in direction $\vec{v} = \langle \frac{3}{4}, 1 \rangle$

Soln. $\nabla f(x,y) = \langle 6x, -4y \rangle$ & $\nabla f(-\frac{3}{4}, 0) = \langle -\frac{9}{2}, 0 \rangle$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

$$\Rightarrow D_{\vec{u}} f(x,y) = \nabla f(-\tfrac{3}{4}, 0) \cdot \vec{u}$$
$$= \langle -\frac{9}{2}, 0 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle$$
$$= -27$$

Lecture # 09: Directional Derivatives & Gradients Date: Tue. 10/16/18

Properties of the gradient

THEOREM 13.11 PROPERTIES OF THE GRADIENT

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y) = 0$ for all $\mathbf{u}$.
2. The direction of *maximum* increase of f is given by $\nabla f(x, y)$. The maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$.
3. The direction of *minimum* increase of f is given by $-\nabla f(x, y)$. The minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$.

Skier going down mountain $\Rightarrow$ altitude given by $f(x,y)$

Conceptually, $-\nabla f(x,y)$ is the fastest path down the mountain (or to dec. volume of a gas)

Ex. 4) The temp. in °C on surface of a metal plate is

$$T(x,y) = 20 - 4x^2 - y^2$$

where x & y are measured in cm in what direction from pt $(2,-3)$ does temp increase most rapidly?

Soln. $\nabla T(x,y) = \langle T_x, T_y \rangle = \langle -8x, -2y \rangle$

Direction of max increase: $\nabla T(2,-3) = \langle -16, 6 \rangle$

max rate increase is

$$\|\nabla T(2,-3)\| = (292)^{1/2} \approx 17.09°/\text{cm}$$

Lecture # 09: Directional Derivatives & Gradients Date: Tue. 10/16/18

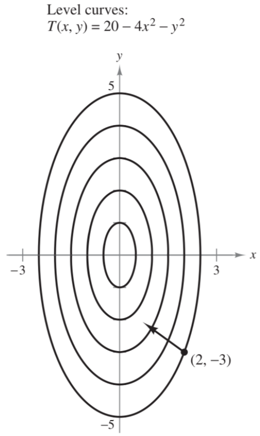
Thm.

THEOREM 13.12 GRADIENT IS NORMAL TO LEVEL CURVES

If f is differentiable at (x_0, y_0) and $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $\nabla f(x_0, y_0)$ is normal to the level curve through (x_0, y_0).

Ex. 5| Consider the last example.

we can see that @ (2,-3) that $\nabla f(2,-3)$ is normal to the level curve thru (2,-3)



Gradients For Fcns of 3 or more Variables

DIRECTIONAL DERIVATIVE AND GRADIENT FOR THREE VARIABLES

Let f be a function of x, y, and z, with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is given by

$$D_{\mathbf{u}} f(x, y, z) = af_x(x, y, z) + bf_y(x, y, z) + cf_z(x, y, z).$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}.$$

Properties of the gradient are as follows.

1. $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
2. If $\nabla f(x, y, z) = \mathbf{0}$, then $D_{\mathbf{u}} f(x, y, z) = 0$ for all $\mathbf{u}$.
3. The direction of *maximum* increase of f is given by $\nabla f(x, y, z)$. The maximum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $\|\nabla f(x, y, z)\|$. Maximum value of $D_{\mathbf{u}} f(x, y, z)$

4. The direction of *minimum* increase of f is given by $-\nabla f(x, y, z)$. The minimum value of $D_{\mathbf{u}} f(x, y, z)$ is

   $-\|\nabla f(x, y, z)\|$. Minimum value of $D_{\mathbf{u}} f(x, y, z)$