Worksheet 10 Solutions

Tangent Planes & Linear Approximation

MATH 2210, Fall 2018

1. Decide if each statement is true or false. Give an appropriate justification for your conclusion.

   (a) The tangent plane to the graph of $f(x, y) = e^{x+2y}$ at the point $(0, 0)$ is $x + 2y - z = -1$.

   > **Solution:** TRUE: because $f(0, 0) = 1$, $f_x(0, 0) = 1$, and $f_y(0, 0) = 2$.

   (b) The planes tangent to the cylinder $x^2 + z^2 = 1$ in $\mathbb{R}^3$ all have the form $ax + bz + c = 0$.

   > **Solution:** TRUE: This is because the function $F(x, y, z) = x^2+z^2$ has $F_y(x, y, z) = 0$.

   (c) The gradient $\nabla F(a, b, c)$ lies on the plane tangent to the surface $F(x, y, z) = 0$ at (a, b, c).

   > **Solution:** FALSE: The gradient $\nabla F(a, b, c)$ is perpendicular to the tangent plane for the surface $F(x, y, z) = 0$ at (a, b, c).

2. Find an equation of the plane tangent to the surface $yze^{xz} - 8 = 0$ at the points $(0, 2, 4)$ and $(0, -8, -1)$.

Solution: Recall that the equation for a plane to a surface $F(x, y, z) = 0$ at the point (a, b, c) is given by

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

Computing the gradient of $F(x, y, z) = yze^{xz} - 8$ we find that

$$\nabla F = \langle yz^2e^{xz}, ze^{xz}, (1 + xz)ye^{xz}\rangle,$$

Note that

$$\nabla F(0, 2, 4) = \langle (2)(4)^2e^{(0)(4)}, (4)e^{(0)(4)}, (1 + (0)(4))(2)e^{(0)(4)}\rangle = \langle 32, 4, 2\rangle$$

It then follows that the tangent plane at $(0, 2, 4)$ has equation

$$32(x - 0) + 4(y - 2) + 2(z - 4) = 0 \quad \text{or} \quad 16x + 2y + z = 8.$$

Also,

$$\begin{aligned}\nabla F(0, -8, -1) &= \langle (-8)(-1)^2e^{(0)(-1)}, (-1)e^{(0)(-1)}, (1 + (0)(-1))(-8)e^{(0)(-1)}\rangle \\ &= \langle -8, -1, -8\rangle\end{aligned}$$

So the tangent plane at $(0, 8, 1)$ has equation

$$-8(x - 0) - 1(y + 8) - 8(z + 1) = 0 \quad \text{or} \quad -8x - y - 8z = 16.$$

3. Use an appropriate linear approximation of the function $f(x, y) = \ln(1 + x + y)$ to approximate the value of $f(0.1, -0.2)$.

Solution: Recall that the linear approximation of a function $f(x, y)$ at the point (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We choose to use the linear approximation of f at $(0, 0)$ since it is reasonbably close to the given point For the given point $(0.1, -0.2)$.

For the given function $f(x, y)$ we have

$$f_x(x, y) = \frac{1}{1 + x + y} \quad \text{and} \quad f_y(x, y) = \frac{1}{1 + x + y}.$$

Then

$$f_x(0, 0) = \frac{1}{1 + 0 + 0} = 1 \quad \text{and} \quad f_y(0, 0) = \frac{1}{1 + 0 + 0} = 1.$$

Since $f(0, 0) = ln(1) = 0$ we find that the linear approximation of f(x,y) at $(0, 0)$ is

$$L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 0 + 1x + 1y = x + y$$

Thus,

$$f(0.1, -0.2) \approx L(0.1, -0.2) = 0.1 + (-0.2) = -0.1.$$

4. The following questions involve an unknown function $f(x, y)$.

   (a) The space curves

   $$\mathbf{r}_1(t) = \langle 1, t, 7 - t^2 \rangle \quad \text{and} \quad \mathbf{r}_2(s) = \langle s, 2, 4 - s^2 \rangle$$

   lie on the surface $z = f(x, y)$ and intersect at $P_0(1, 2, 3)$. Find the equation of the tangent plane to the surface $z = f(x, y)$ at $P_0(1, 2, 3)$. Hint: we use a point and a normal vector to define a plane.

   Solution: To define the tangent plane we use the point $P_0(1, 2, 3)$ and a normal vector.

   Notice that $\mathbf{r}_1(2) = \langle 1, 2, 3 \rangle$ and $\mathbf{r}_2(1) = \langle 1, 2, 3 \rangle$. Then the vectors

   $$\mathbf{r}_1'(2) = \langle 0, 1, -4 \rangle$$
   $$\mathbf{r}_2'(1) = \langle 1, 0, -2 \rangle$$

   must be tangent to the graph of $z = f(x, y)$ at $P_0(1, 2, 3)$.

   Because the normal vector is perpendicular to both $\mathbf{r}_1'(2)$ and $\mathbf{r}_2'(1)$, let

   $$\mathbf{n} = \mathbf{r}_1'(2) \times \mathbf{r}_2'(1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{vmatrix} = \langle -2, -4, -1 \rangle.$$

   The equation of the tangent plane is

   $$z = 3 - 2(x - 1) - 4(y - 2).$$

   (b) Use your answer from part (a) to find a linear approximation to the number $f(1.01, 1.98)$.

   Solution: Using the tangent plane equation, from above, the linear approximation at $(1, 2)$ is

   $$f(x, y) \approx 3 - 2(x - 1) - 4(y - 2).$$

   So

   $$f(1.01, 1.98) \approx 3 - 2(1.01 - 1) - 4(1.98 - 2) = 3 - 0.02 + 0.08 = 3.06.$$