1. Give a parametric description of the form $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ for the following surfaces. Specify the required rectangle in the uv-plane.

   (a) The cap of the sphere $x^2 + y^2 + z^2 = 16$ for $\sqrt{8} \leq z \leq 4$.

Solution: Recall that the parametric description of a sphere of radius a is

$$\mathbf{r}(u,v) = \langle a\cos(u)\sin(v), a\sin(u)\sin(v), a\cos(u)\rangle.$$

For the given sphere we have a radius $a = 4$ so it is parameterized by

$$\mathbf{r}(u,v) = \langle 4\cos(u)\sin(v), 4\sin(u)\sin(v), 4\cos(u)\rangle.$$

In this parameterization the angle v revolves around the xy-plane. Since we are interested intthe cap of the sphere we will make one full revolution so our bounds will be $0 \leq v \leq 2\pi$.
In this parametrization we have that $z = 4\cos(u)$. Setting z equal to the given upper and lower bounds we find that

$$\begin{aligned} 4\cos(u) = 2\sqrt{2} \quad &\Rightarrow \quad \cos(u) = \frac{\sqrt{2}}{2} \quad \Rightarrow u = \frac{\pi}{4} \\ 4\cos(u) = 4 \quad &\Rightarrow \quad \cos(u) = 1 \quad \Rightarrow u = 0. \end{aligned}$$

Thus, the bounds on u are $0 \leq u \leq \dfrac{\pi}{4}$.

   (b) The cylinder $y^2 + z^2 = 36$, for $0 \leq x \leq 9$.

Solution: This cylinder is obtained by extending a circle in the yz-plane in the direction of x. Thus, it is parameterized by

$$\mathbf{r}(u,v) = \langle u, 6\cos(v), 6\sin(v)\rangle$$

with $0 \leq u \leq 9$ and $0 \leq v \leq 2\pi$.

2. Find the area of the surface S that lies in the plane $z = 12 - 4x - 3y$ directly above the region R bounded by the ellipse $\frac{x^2}{4} + y^2 = 1$.

Solution: The plane is an explicitly defined surface of the form $z = f(x, y) = 12 - 4x - 3y$. Computing the needed partial derivatives we find

$$f_x(x, y) = -4 \quad \text{and} \quad f_y(x, y) = -3.$$

Then

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + (-4)^2 + (-3)^2} = \sqrt{1 + 16 + 9} = \sqrt{26}.$$

Consequently, the surface area is given by the integral

$$\iint_S 1dS = \iint_R \sqrt{26}dA = \sqrt{26} \iint_R dA$$

where R is the region bouned by the ellipse $\frac{x^2}{4} + y^2 = 1$. This integral is easiest to compute using a change of variables.

For $0 \leq u$. we can set

$$T : x = 2u\cos(v) \quad \text{and} \quad y = u\sin(v).$$

Then we find that

$$\frac{x^2}{4} + y^2 = u^2(\cos^2(v) + \sin^2(v)) = u^2$$

So our bounds will be $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

Solving for the required partial derivatives we find

$$\begin{aligned} \frac{\partial x}{\partial u} &= 2\cos(v), & \frac{\partial y}{\partial u} &= \sin(v), \\ \frac{\partial x}{\partial v} &= -2u\sin(v), & \frac{\partial y}{\partial v} &= u\cos(v). \end{aligned}$$

Then,

$$\begin{aligned} J(u, v) &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (2\cos(v))(u\cos(v)) - (-2u\sin(v))(\sin(v)) \\ &= 2u(\cos^2(v) + \sin^2(v)) \\ &= 2u \end{aligned}$$

Then applying the change of variables we find that

$$\sqrt{26}\iint_R 1 \text{ d}A = \int_0^{2\pi}\int_0^1 2u \; dudv = 2\sqrt{26}\int_0^{2\pi} u\Big|_{u=0}^{u=1} dv = 2\sqrt{26}\int_0^{2\pi} dv = 4\pi\sqrt{26}$$

3. In the following problem the surface S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane, the boundary of S is the circle $x^2 + y^2 = 4$ in the xy-plane, and $\mathbf{F} = \langle y, -x, xy \rangle$.

   (a) Directly compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

Solution: We begin by computing the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \langle x, -y, -2 \rangle$$

The paraboloid can be parametrized by $\mathbf{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle$ where $x^2 + y^2 \leq 4$. The partial derivatives of $\mathbf{r}(x, y)$ are

$$\mathbf{r}_x = \langle 1, 0, -2x \rangle, \text{ and } \mathbf{r}_y = \langle 0, 1, -2y \rangle.$$

And the cross product

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle$$

is oriented upward (it's z component is positive). So $d\mathbf{S} = \langle 2x, 2y, 1 \rangle \, dx \, dy$. Then simplifying the integrand we find that

$$\nabla \times \mathbf{F} \cdot d\mathbf{S} = \langle x, -y, -2 \rangle \cdot \langle 2x, 2y, 1 \rangle \, dx \, dy = (2x^2 - 2y^2 - 2) \, dx \, dy.$$

The region of integration is the disk (call it D) $x^2 + y^2 \leq 4$ in the xy-plane. If we break up the integral on the $-$ signs then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 2 \iint_D x^2 \, dx \, dy - 2 \iint_D y^2 \, dx \, dy - 2 \iint_D dx \, dy$$

If you think about how the region D is a disk centered at the origin it will be clear that

$$\iint_D x^2 \, dx \, dy = \iint_D y^2 \, dx \, dy.$$

If you would rather compute the integrals, switch to polar coordinates:

$$\iint_D x^2 \, dx \, dy = \int_0^{2\pi} \int_0^2 r^3 \cos^2 \theta \, dr \, d\theta = \int_0^{2\pi} \cos^2 \theta \, d\theta \int_0^2 r^3 \, dr = 4\pi$$
$$\iint_D y^2 \, dx \, dy = \int_0^{2\pi} \int_0^2 r^3 \sin^2 \theta \, dr \, d\theta = \int_0^{2\pi} \sin^2 \theta \, d\theta \int_0^2 r^3 \, dr = 4\pi$$

Regardless of your approach,

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot d\mathbf{S} = -2 \iint_D dx \, dy = -2 \, \text{Area}(D) = -8\pi.$$

(b) Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ using Stoke's Theorem.

Solution: The curve $x^2 + y^2 = 4$ can be parametrized in $\mathbb{R}^3$ by

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), 0\rangle\rangle \quad \text{for } 0 \le t \le 2\pi.$$

For this parameterization we have

$$d\mathbf{r} = \mathbf{r}'(t)dt = \langle -2\sin(t), 2\cos(t), 0\rangle dt$$

Then by Stokes Theorem

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \langle 2\sin(t), -2\cos(t), 4\cos(t)\sin(t)\rangle \cdot \langle -2\sin(t), 2\cos(t), 0\rangle dt \\
&= \int_0^{2\pi} -4\sin^2(t) - 4\cos^2(t)\ dt \\
&= \int_0^{2\pi} -4\ dt \\
&= -8\pi.
\end{aligned}$$

4. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(2,0,0)$, $(0,2,0)$, and $(0,0,2)$ is oriented counterclockwise as viewed from above and $\mathbf{F} = \langle y^2, z^2, x^2 \rangle$.

Solution: One possible parametrization of the surface of this triangle is

$$\mathbf{r}(u,v) = \langle 2,0,0 \rangle + u \langle -2,2,0 \rangle + v \langle -2,0,2 \rangle$$

where $0 \le u \le 1$ and $0 \le v \le 1-u$. For this parameterization we find that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2,2,0 \rangle \times \langle -2,0,2 \rangle = \langle 4,4,4 \rangle$$

so $d\mathbf{S} = \langle 4,4,-4 \rangle \, du \, dv$ and $dS = 4\sqrt{3} \, du \, dv$.
Computing the required curl we find that

$$\nabla \times \mathbf{F} = \langle -2z, -2x, -2y \rangle.$$

By Stokes' theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = -8 \int_0^1 \int_0^{1-u} (x+y+z) \, du \, dv$$

and, since $x+y+z=2$ on this plane,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -16 \int_0^1 \int_0^{1-u} dv \, du = -8.$$