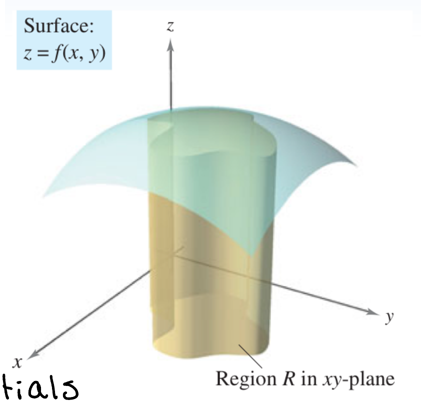
Lecture # 14: Surface Area & Triple Integrals

Date: Thu. 11/8/18

Surface Area



Our goal is to find the surface area of a surface

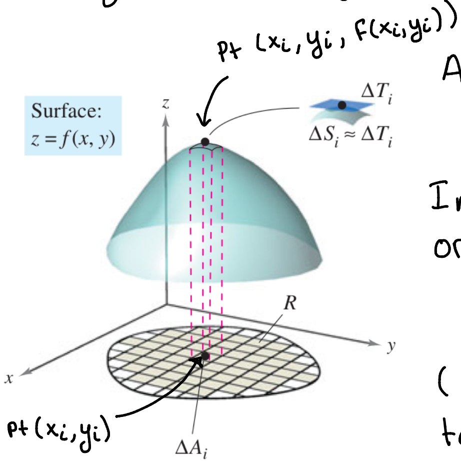
$$z = f(x,y)$$

defined on some region R.

Assumptions: R is closed & bdd

f has cont. 1^{st} partials

To do this, we construct an inner partition of R using n rectangles.



Area of i^{th} rectangle R_i:

$$\Delta A_i = \Delta x_i \Delta y_i$$

In each R_i let pt closest to origin be (x_i, y_i). Then @ the pt

$$(x_i, y_i, z_i) = (x_i, y_i, f(x_i, y_i))$$

(on the surface) Construct a tangent plane T_i.

Since a tangent plane serves as an approx. to the surface near $(x_i, y_i, f(x_i,y_i))$ then $\Delta S_i \approx \Delta T_i$

So surface area will be given by

$$A_s = \sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \Delta T_i$$

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Since ΔT_i is a parallelogram its area will be given by $\|\vec{u} \times \vec{v}\|$ where vectors are:

$$\vec{u} = \Delta x_i \hat{\imath} + f_x(x_i, y_i)\Delta x_i \hat{k}$$

&

$$\vec{v} = \Delta y_i \hat{\imath} + f_y(x_i, y_i)\Delta y_i \hat{k}$$

Then area of each ΔT_i is given by

$$A_{\Delta T_i} = \|\vec{u} \times \vec{v}\|$$

Where

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x_i & 0 & f_x(x_i, y_i)\Delta x_i \\ 0 & \Delta y_i & f_y(x_i, y_i)\Delta y_i \end{vmatrix}$$

$$= -f_x(x_i, y_i)\Delta x_i \Delta y_i \hat{\imath} - f_y(x_i, y_i)\Delta x_i \Delta y_i \hat{\jmath} - \Delta x_i \Delta y_i \hat{k}$$

$$= \left(-f_x(x_i, y_i)\hat{\imath} - f_y(x_i, y_i)\hat{\jmath} - \hat{k}\right)\underbrace{\Delta x_i \Delta y_i}_{= \Delta A_i}$$

$$= \left(-f_x(x_i, y_i)\hat{\imath} - f_y(x_i, y_i)\hat{\jmath} - \hat{k}\right)\Delta A_i$$

and so

$$A_{\Delta T_i} = \|\vec{u} \times \vec{v}\| = \left[(f_x(x_i, y_i))^2 + (f_y(x_i, y_i))^2 + 1\right]^{1/2} \cdot \Delta A_i$$

$\Rightarrow$ Surface Area:

$$A_f \approx \sum_{i=1}^{n} \Delta S_i$$

$$\approx \sum_{i=1}^{n} \left(1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2\right]^{1/2} \Delta A_i$$

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All this work leads us to the definition surface area

DEFINITION OF SURFACE AREA

If f and its first partial derivatives are continuous on the closed region R in the xy-plane, then the **area of the surface S** given by $z = f(x, y)$ over R is defined as

$$\text{Surface area} = \iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA.$$

Compare to arc length:

Length on x-axis: $\int_a^b dx$

Arc length in xy-plane: $\int_a^b ds = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

Area in xy-plane: $\iint_R dA$

Surface area in space: $\iint_R dS = \iint_R \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2}\, dA$

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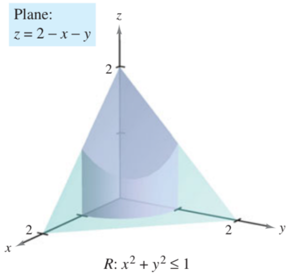
Ex. 1) Find the area of the portion of the plane

$$z = 2 - x - y$$

which lies above the circle

$$x^2 + y^2 \leq 1$$

in the 1^{st} Quadrant.



Soln

Let $f(x,y) = 2 - x - y$

then $f_x(x,y) = -1$

$f_y(x,y) = -1$

$$[1 + f_x^2 + f_y^2]^{1/2} = [1 + (-1)^2 + (-1)^2]^{1/2} = \sqrt{3}$$

then surface area is

$$S_A = \iint_R [1 + f_x^2 + f_y^2]^{1/2}\, dx\,dy$$

$$= \iint_R \sqrt{3}\, dA = \sqrt{3} \underbrace{\iint_R dA}_{\text{area of } R:\ \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1) = \frac{\pi}{4}}$$

$$= \sqrt{3}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi\sqrt{3}}{4}$$

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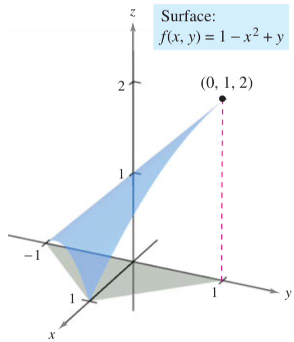
Ex. 2 Find the area of the surface

$$f(x,y) = 1 - x^2 + y$$

that lies above the triangle region w/ vertices

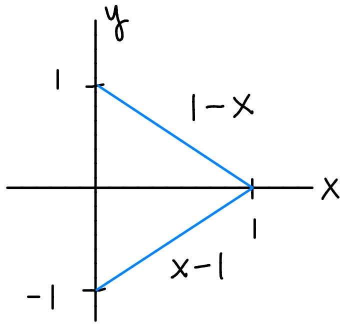
$$(1,0,0),\ (0,-1,0),\ (0,1,0)$$



Soln First lets determine bds for our region R

By looking @ figure we see that in the xy plane we have

$$x-1 \le y \le 1-x$$
$$0 \le x \le 1$$



Next we det. the Integrand:

Since $f(x,y) = 1 - x^2 + y$

then $f_x(x,y) = -2x$

$f_y(x,y) = 1$

$$[1 + f_x^2 + f_y^2]^{1/2} = [1 + (-2x)^2 + (1)^2]^{1/2} = [2 + 4x^2]^{1/2}$$

then surface area is

$$S_A = \iint_R [1 + f_x^2 + f_y^2]^{1/2}\, dx\,dy$$

$$= \int_0^1 \int_{x-1}^{1-x} [2 + 4x^2]^{1/2}$$

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Triple Integrals

Triple integrals work similar to double integrals.

DEFINITION OF TRIPLE INTEGRAL

If f is continuous over a bounded solid region Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, \Delta V_i$$

provided the limit exists. The **volume** of the solid region Q is given by

$$\text{Volume of } Q = \iiint_Q dV.$$

We have some of the same properties for triple integrals

Properties of Triple Integrals

1. $\iiint_Q cf(x, y, z)\, dV = c \iiint_Q f(x, y, z)\, dV$

2. $\iiint_Q [f(x, y, z) \pm g(x, y, z)]\, dV = \iiint_Q f(x, y, z)\, dV \pm \iiint_Q g(x, y, z)\, dV$

3. $\iiint_Q f(x, y, z)\, dV = \iiint_{Q_1} f(x, y, z)\, dV + \iiint_{Q_2} f(x, y, z)\, dV$

To actually evaluate a triple integral, we proceed similar to double integrals

THEOREM 14.4 EVALUATION BY ITERATED INTEGRALS

Let f be continuous on a solid region Q defined by

$$a \le x \le b, \quad h_1(x) \le y \le h_2(x), \quad g_1(x, y) \le z \le g_2(x, y)$$

where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z)\, dz\, dy\, dx.$$

Determining the bounds is the most challenging part of setting up the iterated integral

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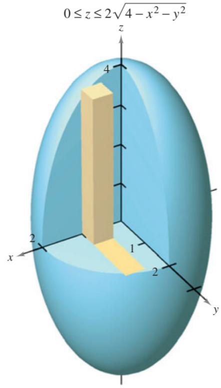
Ex. 3 Find the volume of the ellipsoid

$$4x^2 + 4y^2 + z^2 = 16$$

Soln We'll use the order $dz\,dy\,dx$.

To simplify our integration, we'll only consider the volume of the ellipsoid in the 1^{st} octant & mult. by 8.



Ellipsoid: $4x^2 + 4y^2 + z^2 = 16$

Bds for z: (in top octant)

Solving eqn. for ellipsoid for z:

$$\Rightarrow \quad z = 2\sqrt{4 - x^2 - y^2}$$

Since we're in the 1^{st} octant:

$$0 \le z \le 2\sqrt{4 - x^2 - y^2}$$

Bds for y:

Letting $z = 0$ we solve for y

$$4x^2 + 4y^2 = 16$$

$$x^2 + y^2 = 4 \quad \text{(eqn of a circle)}$$

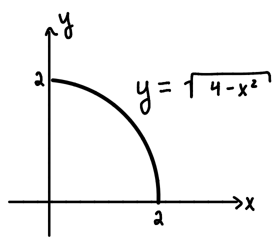
$$\Rightarrow \quad y = \pm\sqrt{4 - x^2}$$

$$0 \le y \le \sqrt{4 - x^2}$$

Bds for x:

Since the region in the xy plane is a circle we have

$$0 \le x \le 2$$



Our iterated integral becomes

$$V = 8\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

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Ex. 4 Evaluate $\displaystyle\int_0^{\sqrt{\frac{\pi}{2}}}\int_x^{\sqrt{\frac{\pi}{2}}}\int_1^3 \sin(y^2)\,dz\,dx\,dy$

Soln

To simplify the integral we rewrite with the order
$dz\,dy\,dz$

By sketching the region we see that the integral becomes

$$\int_{y=0}^{y=\sqrt{\frac{\pi}{2}}}\int_{x=0}^{x=y}\int_{x=0}^{z=3} \sin(y^2)\,dz\,dx\,dy$$