Lecture #10: Tangent planes & Normal lines

Date: Thu. 10/18/18

Alt. Form of Eqn of a surface

Previously, a surface in space has been given by

$$z = f(x,y)$$

We can also write this in a more general form as

$$F(x,y,z) = f(x,y) - z$$

Since $f(x,y) - z = 0$ then the level surface of F is

$$F(x,y,z) = 0$$

Ex. 1)

For $F(x,y,z) = x^2 + y^2 + z^2 - 4$

the level surface given by $F(x,y,z) = 0$ is

$$x^2 + y^2 + z^2 - 4 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 = 4$$

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Tangent Planes

For a surface given by $F(x,y,z)=0$

Let P be a pt (x_0, y_0, z_0).

Let a curve C on the surface given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

then for all t

$$F(x(t), y(t), z(t)) = 0$$

If F is diff'ble & $x'(t), y'(t), z'(t)$ all exist then

$$0 = F'(t)$$

$$= F_x(x,y,z)\,x'(t) + F_y(x,y,z)\,y'(t) + F_z(x,y,z)\,z'(t)$$

$$= \langle F_x, F_y, F_z \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle$$

$$= \nabla F(x,y,z) \cdot \vec{r}\,'(t)$$

Then @ the pt (x_0, y_0, z_0)

$$0 = \nabla F(x_0, y_0, z_0) \cdot r'(t_0)$$

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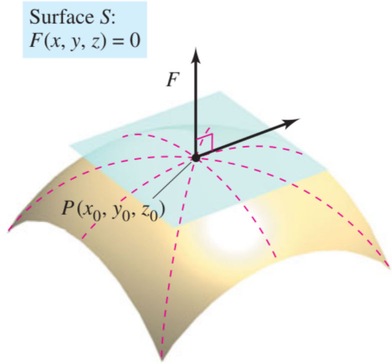
The gradient @ (x_0, y_0, z_0) is orthogonal to the tangent vector of every curve on the surface thru the pt (x_0, y_0, z_0)



In other words, all tangent lines thru this pt lie in a plane tangent to the surface & normal to $\nabla F(x_0, y_0, z_0)$

Def. Let F be diff'ble @ pt (x_0, y_0, z_0) on the surface given by $F(x,y,z)=0$ s.t. $\nabla F(x_0, y_0, z_0) \neq 0$

The plane thru pt (x_0, y_0, z_0) that is normal to $\nabla F(x_0, y_0, z_0)$ is the tangent plane to a surface at that pt.

Thm

THEOREM 13.13 EQUATION OF TANGENT PLANE

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

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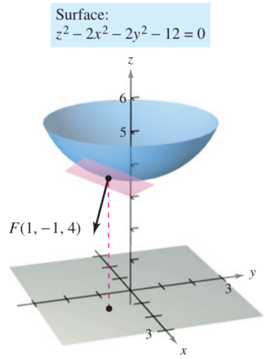
Ex. 2 Find an eqn of the tangent plane to

$$z^2 - 2x^2 - 2y^2 = 12$$

@ pt $(1, -1, 4)$



Tangent plane to surface

Soln Rewriting as

$$\underbrace{z^2 - 2x^2 - 2y^2 - 12}_{F(x,y,z)} = 0$$

Need $F_x = -4x$, $F_y = -4y$, $F_z = 2z$

@ $(1,-1,4)$ $F_x = -4$, $F_y = 4$, $F_z = 8$

Eqn of tangent plane:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$\Rightarrow -4(x-1) + 4(y-1) + 8(z-4) = 0$$

■

To find an eqn of a tangent plane to $z = f(x,y)$ define

$$F(x,y,z) = f(x,y) - z$$

then the eqn of the tangent plane is

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

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Ex. 3) Find eqn of tangent plane to

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

@ pt. $(1, 1, \frac{1}{2})$

Soln.

$$f_x(x,y) = -\frac{1}{10}(2x) = -\frac{1}{5}x \implies f_x(1,1) = -\frac{1}{5}$$

$$f_y(x,y) = -\frac{1}{10}(8y) = -\frac{4}{5}y \implies f_y(1,1) = -\frac{4}{5}$$

Eqn. of tangent plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0) - (z - z_0) = 0$$

$$\implies -\frac{1}{5}(x-1) - \frac{4}{5}(y-1) - (z - \frac{1}{2}) = 0$$

$\blacksquare$

Normal Lines

Def. Let F be diff'ble @ pt (x_0, y_0, z_0) on the surface given by $F(x,y,z)=0$ s.t. $\nabla F(x_0, y_0, z_0) \neq 0$

The line thru pt. (x_0, y_0, z_0) that has the same direction as $\nabla F(x_0, y_0, z_0)$ is called the normal line to a surface at a pt.

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Ex. 4 Find the eqn of the normal line to the surface

$$xyz = 12$$

@ pt $(2,-2,3)$

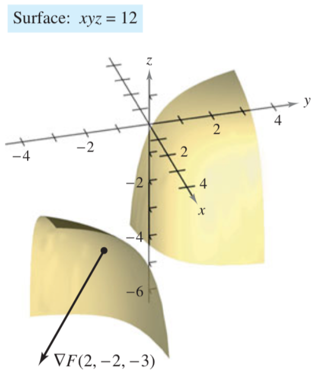


Figure 13.60

Soln. Rewriting:

$$F(x,y,z) = xyz - 12$$

$$\nabla F(x,y,z) = \langle F_x, F_y, F_z \rangle$$
$$= \langle yz, xz, xy \rangle$$

Evaluating @ pt $(2,-2,3)$

$$\nabla F(2,-2,3) = \langle (-2)(3), (2)(3), (2)(-2) \rangle$$
$$= \langle -6, 6, -4 \rangle$$

Eqn. of Normal line:

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\Rightarrow \frac{x-2}{-6} = \frac{y+2}{6} = \frac{z-3}{-4}$$

Ex. 5) Describe the eqn of the tangent line to the curve of intersection of the surfaces

$$x^2 + 2y^2 + 2z^2 = 20 \quad \text{(Ellipsoid)}$$

$$x^2 + y^2 + z = 4 \quad \text{(paraboloid)}$$

@ pt. $(0, 1, 3)$

$F(x,y,z) = x^2 + 2y^2 + 2z^2 - 20$

$\nabla F(x,y,z) = \langle 2x, 4y, 4z \rangle$

$\nabla F(0,1,3) = \langle 0, 4, 12 \rangle$

$G(x,y,z) = x^2 + y^2 + z - 4$

$\nabla G(x,y,z) = \langle 2x, 2y, 1 \rangle$

$\nabla G(0,1,3) = \langle 0, 2, 1 \rangle$

Direction Vector for tangent line given by

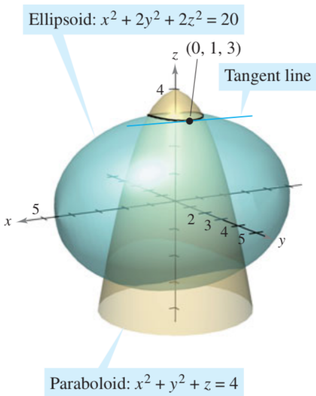
$$\nabla F(0,1,3) \times \nabla G(0,1,3) = \begin{vmatrix} i & j & k \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = \langle -20, 0, 0 \rangle$$

Eqn. of tangent line is

$$x = x_0 + at = 0 - 20t = -20t$$

$$y = y_0 + bt = 1 + 0t = 1$$

$$z = z_0 + ct = 3 + 0t = 3$$



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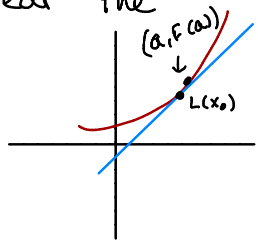
Linear Approximation

In one variable:

the linear approx of a fcn f near the pt $(a, f(a))$ is given by

$$L(x) = f(a) + f'(a)(x-a)$$



In 2 variables:

<u>Def</u> Let f be diff'ble @ (x_0, y_0). The linear approximation to the surface $z = f(x,y)$ @ pt $(x_0, y_0, f(x_0,y_0))$ is the tangent plane @ that pt given by

$$L(x,y) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0)$$