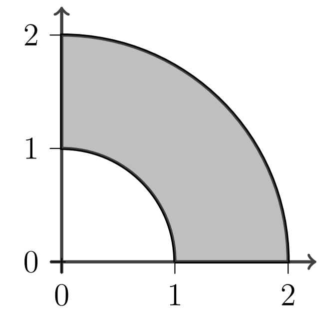
Worksheet 15 Solutions

Change of Variables & Polar Coordinates

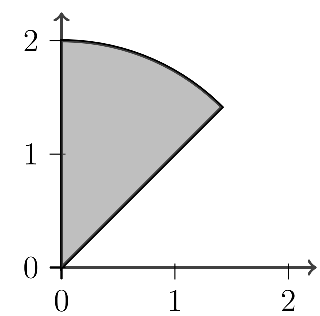
1. (a) Give the limits of integration for this region in polar coordinates.

Solution: $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$



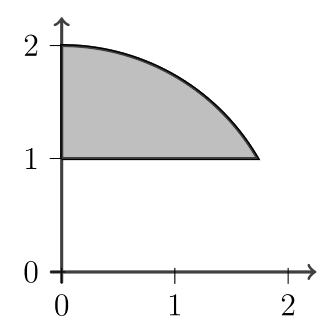
(b) Give the limits of integration for this region in both Cartesian and polar coordinates. The arc is part of the circle $x^2 + y^2 = 4$.

Solution:
$0 \leq x \leq \sqrt{2}$ and $x \leq y \leq \sqrt{4 - x^2}$
$0 \leq r \leq 2$ and $\pi/4 \leq \theta \leq \pi/2$



(c) Give the limits of integration for this region in both Cartesian and polar coordinates. The arc is part of the circle $x^2 + y^2 = 4$.

Solution:
$0 \leq x \leq \sqrt{3}$ and $1 \leq y \leq \sqrt{4 - x^2}$
$0 \leq x \leq \sqrt{4 - y^2}$ and $1 \leq y \leq 2$
$\dfrac{1}{\sin\theta} \leq r \leq 2$ and $\pi/6 \leq \theta \leq \pi/2$



2. Compute the Jacobian $J(u, v)$ for the the transformation

$$T : x = u\cos(\pi v) \quad \text{and} \quad y = u\sin(\pi v).$$

Solution: First we compute the required partials

$$\frac{\partial x}{\partial u} = \cos(\pi v), \qquad \frac{\partial y}{\partial u} = \sin(\pi v),$$
$$\frac{\partial x}{\partial v} = -\pi u\sin(\pi v), \qquad \frac{\partial y}{\partial v} = \pi u\cos(\pi v).$$

Then,

$$\begin{aligned} J(u, v) &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \\ &= (\cos(\pi v))\,(\pi u\cos(\pi v)) - (-\pi u\sin(\pi v))\,(\sin(\pi v)) \\ &= \pi u(\cos^2(\pi v) + \sin^2(\pi v)) \\ &= \pi u \end{aligned}$$

3. Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

Solution: The volume of the solid is given by

$$V = \iint_R \left(2 - x^2 - y^2\right) - \left(x^2 + y^2\right) \, dA = 2 \iint_R \left(1 - x^2 - y^2\right) \, dA$$

Where R is the region of the xy-plane that is contained inside the curve of intersection of these two parabloids. Finding this curve we set the equations for these parabloids equalto eachother and solve

$$x^2 + y^2 = 2 - x^2 - y^2 \quad \Rightarrow \quad 2x^2 + 2y^2 = 2 \quad \Rightarrow x^2 + y^2 = 1.$$

Noting that this curve is a the unit circle a conversion to polar coordinates is worthwhile:

$$\begin{aligned}
\iint_R \left(2 - 2x^2 - 2y^2\right) \, dA &= 2 \int_0^{2\pi} \int_0^1 \left(1 - r^2\right) \, r dr d\theta \\
&= 2 \int_0^{2\pi} 1 d\theta \int_0^1 \left(r - r^3\right) dr \\
&= 2 \cdot \quad (\theta) \Big|_{\theta=0}^{\theta=2\pi} \cdot \quad \left(\frac{r^2}{2} - \frac{r^4}{4}\right) \Big|_{r=0}^{r=1} \\
&= 2(2\pi)\left(\frac{1}{4}\right) \\
&= \pi
\end{aligned}$$

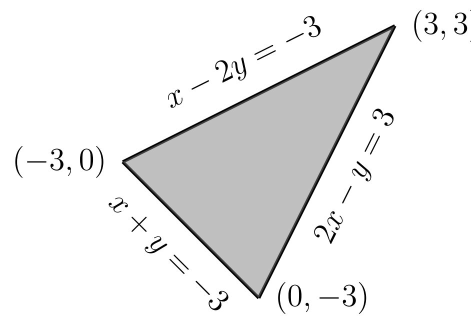
4. A region R in the xy-plane is shown here. Use the substitutions

$$x = 2u + v \text{ and } \; y = u - v$$

to express the double integral

$$\iint_R (x + y)\, dA$$

in terms of u and v and evaluate it.



Solution: Start with the integrand:

$$x + y = 2u + v + u - v = 3u.$$

Next, compute the Jacobian determinant:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

Then the area element $dA = dx\, dy = |-3|\, du\, dv = 3\, du\, dv$.

The boundaries of R become

$$\begin{aligned} x + y &= 2u + v + u - v = 3u = -3 \;\; \text{Doh! We did } x + y \text{ earlier.} \\ x - 2y &= 2u + v - 2(u - v) = 3v = -3 \\ 2x - y &= 2(2u + v) - (u - v) = 3(u + v) = 3 \end{aligned}$$

Simplifying, the boundaries are $u = -1$, $v = -1$, and $u + v = 1$. To understand the limits of integration, sketch the region of integration in the uv-plane.

$$\iint_R (x + y)\, dA = 3 \int_{-1}^{2} \int_{-1}^{1-v} 3u\, du\, dv = 3 \int_{-1}^{2} \int_{-1}^{1-u} 3u\, dv\, du.$$