Lecture # 2 : The Dot & Cross Product

Date: Thu. 9/13/18

The Dot Product

Def The dot product of 2 vectors $\vec{u} = \langle u_1, u_2 \rangle$ & $\vec{v} = \langle v_1, v_2 \rangle$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

Note:

If $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^3$ then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

If $\vec{u}, \vec{v}$ are vectors in $\mathbb{R}^n$ then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$= \sum_{i=1}^{n} u_i v_i$$

Ex Find the dot product of $\vec{u} = \langle 6, -2, 3 \rangle$ & $\vec{v} = \langle 2, 5, -1 \rangle$

$$\vec{u} \cdot \vec{v} = (6)(2) + (-2)(5) + (3)(-1)$$

$$= -1$$

Note

- the dot product will always result in a scalar quantity.
- the dot product is also known as an inner product.

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Properties of the dot product

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ Commutative Property
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ Distributive Property
3. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{v} = 0$
5. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

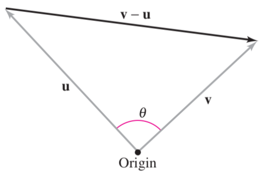
Proof of 5:

Since $\|V\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ then

$$\|V\|^2 = V_1^2 + V_2^2 + V_3^2 = \vec{V} \cdot \vec{V}$$

Angle Btwn two vectors

The angle btwn 2 nonzero vectors is the angle θ such that $0 \le \theta \le \pi$ btwn their respective std. position vectors.



We can find this angle using the dot product.

Thm The angle θ btwn 2 nonzero vectors $\vec{u}$ & $\vec{v}$ is given by

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

If we already know this angle then we have an alternative form of the dot product is

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

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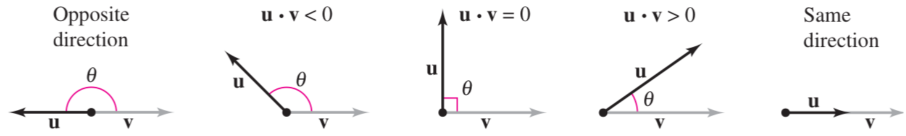
we can have the following possible orientations for the 2 vectors



$\theta = \pi$	$\pi/2 < \theta < \pi$	$\theta = \pi/2$	$0 < \theta < \pi/2$	$\theta = 0$
$\cos\theta = -1$	$-1 < \cos\theta < 0$	$\cos\theta = 0$	$0 < \cos\theta < 1$	$\cos\theta = 1$

Note that when 2 vectors meet @ a right angle the dot product is zero.

<u>Def</u> Two vectors $\vec{u}$ & $\vec{v}$ are <u>orthogonal</u> if $\vec{u}\cdot\vec{v} = 0$

It follows that $\vec{0}$ is orthogonal to every vector since $\vec{0}\cdot\vec{u} = 0$

Direction Angles & Cosines

In 2D (i.e. in the plane) we measure direction in terms of the angle counter clockwise from the positive x-axis.

In 3D (i.e. in space) we measure direction in terms of the angles btwn the vector & each of the 3 unit vectors i, j, k

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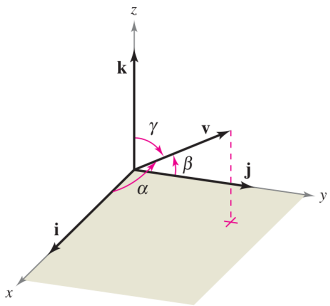

In the figure α, β, γ are the direction angles of $\vec{v}$ and $\cos(\alpha), \cos(\beta), \cos(\gamma)$ are the direction cosines of $\vec{v}$

For example: $\vec{v}\cdot\vec{i} = \|\vec{v}\|\,\|\vec{i}\| \cos\alpha = \|\vec{v}\|\cos\alpha$

$\vec{v}\cdot\vec{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$

$$\Rightarrow \quad \|\vec{v}\|\cos\alpha = v_1$$

$$\Rightarrow \quad \cos\alpha = \frac{v_1}{\|\vec{v}\|}$$

We can use similar reasoning for $\vec{j}$ & $\vec{k}$ to obtain

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|} \qquad \cos\beta = \frac{v_2}{\|\vec{v}\|} \qquad \cos\gamma = \frac{v_3}{\|\vec{v}\|}$$

Ex) Find the direction angles for the vector $v = 2\vec{i} + 3\vec{j} + 4\vec{k}$

Note: $\|\vec{v}\| = \left((2)^2 + (3)^2 + (4)^2\right)^{1/2} = \sqrt{29}$

$\cos\alpha = \dfrac{v_1}{\|\vec{v}\|} = \dfrac{2}{\sqrt{29}}$ so $\alpha \approx 68.2°$

$\cos\beta = \dfrac{v_2}{\|\vec{v}\|} = \dfrac{3}{\sqrt{29}}$ $\beta \approx 56.1°$

$\cos\gamma = \dfrac{v_3}{\|\vec{v}\|} = \dfrac{4}{\sqrt{29}}$ $\gamma \approx 42.0°$

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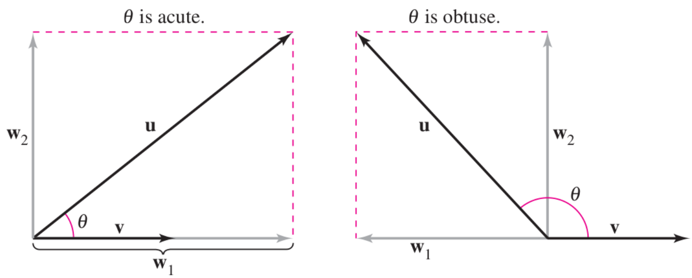
DEFINITIONS OF PROJECTION AND VECTOR COMPONENTS

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v**, and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} =$ projection of $\mathbf{u}$ onto $\mathbf{v}$ = vector component of $\mathbf{u}$ along $\mathbf{v}$
$\mathbf{w}_2 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

<u>Def</u> If $\vec{u}$ & $\vec{v}$ are nonzero vectors, then the projection of $\vec{u}$ onto $\vec{v}$ is given by

$$\text{proj}_v u = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

This can also be written as

$$\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\frac{\vec{v}}{\|\vec{v}\|^2} = (k)\frac{\vec{v}}{\|\vec{v}\|}$$

$$\Rightarrow k = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|} = \|\vec{u}\|\cos\theta \quad \text{where } k \text{ is some constant}$$

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The Cross Product

The cross product gives us the means to find a vector that is orthogonal to two vectors.

Def Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

The cross product of $\vec{u}$ & $\vec{v}$ is

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}$$

Unlike the dot product, the cross product produces a new vector (as opposed to a scalar)

Note: the cross product is only defined for 3D vectors

Calculating the Cross Product

A nice way to calculate the cross product is to use the determinant form.

For those that have had linear algebra note that this is not a true determinant, it is just calculated using a similar process

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Using the determinant Form:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \begin{matrix} \longleftarrow \text{Put "}\mathbf{u}\text{" in Row 2.} \\ \longleftarrow \text{Put "}\mathbf{v}\text{" in Row 3.} \end{matrix}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

We obtain the last line from the fact that each 2x2 determinant is calculated using the following pattern

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

<u>Ex</u> Given $\vec{w} = \langle 4, 3, -2 \rangle$ & $\vec{r} = \langle 2, -1, 1 \rangle$

Calculate $\vec{w} \times \vec{r}$

$$\begin{vmatrix} i & j & k \\ 4 & 3 & -2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}\vec{k}$$

$$= \big((3)(1) - (-2)(-1)\big)\vec{i}$$

$$- \big((4)(1) - (-2)(2)\big)\vec{j}$$

$$+ \big((4)(-1) - (3)(2)\big)\vec{k}$$

$$= \vec{i} - 8\vec{j} - 10\vec{k}$$

$$= \langle 1, -8, -10 \rangle$$

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We can verify that $\vec{w} \times \vec{r}$ is orthogonal to both $\vec{w}$ & $\vec{r}$ by calculating the dot product.

$$\vec{w} \times \vec{r} = \langle 1, -8, -10 \rangle$$

$$\vec{w} = \langle 4, 3, -2 \rangle$$

$$\vec{r} = \langle 2, -1, 1 \rangle$$

$$\vec{w} \cdot (\vec{w} \times \vec{r}) = (4)(1) + (-8)(3) + (-10)(-2)$$

$$= 4 - 24 + 20 = 0 \checkmark$$

$$\vec{r} \cdot (\vec{w} \times \hat{r}) = (2)(1) + (-1)(-8) + (1)(-10)$$

$$= 2 + 8 - 10 = 0 \checkmark$$

So $\vec{w} \times \vec{r}$ is orthogonal to both $\vec{w}$ & $\vec{r}$

Properties of the Cross Product

Thm

Let **u**, **v**, and **w** be vectors in space, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ ⇒ cross product is <u>not</u> commutative
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

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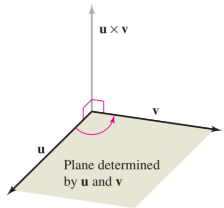
Geometric Properties of the Cross Product

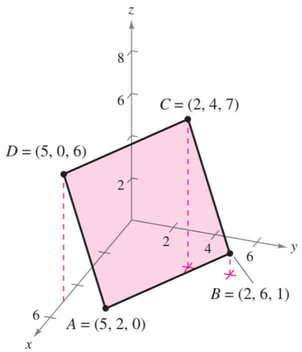
Thm

Let **u** and **v** be nonzero vectors in space, and let θ be the angle between **u** and **v**.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having **u** and **v** as adjacent sides.



Ex Show that the quadrilateral pictured in the Figure is a parallelogram & find its area.



$\vec{AB} = \langle 2-5,\ 6-2,\ 1-0 \rangle = \langle -3, 4, 1 \rangle$

$\vec{AD} = \langle 5-5,\ 0-2,\ 6-0 \rangle = \langle 0, -2, 6 \rangle$

$\vec{CD} = \langle 5-2,\ 0-4,\ 6-7 \rangle = \langle 3, -4, -1 \rangle$

$\vec{CB} = \langle 2-2,\ 6-4,\ 1-7 \rangle = \langle 0, 2, -6 \rangle$

Since $\vec{CD} = -\vec{AB}$

& $\vec{CB} = -\vec{AD}$

This quadrilateral is a parallelogram

To find area we must calculate $\|\vec{AB} \times \vec{AD}\|$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26i + 18j + 6k$$

$$\Rightarrow \text{area} = \|\vec{AB} \times \vec{AD}\| = \left((26)^2 + (18)^2 + (6)^2 \right)^{1/2}$$

$$= (1036)^{1/2}$$

$$\approx 32.19$$

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Triple Scalar Product

<u>Def</u> Given 3 Vectors in space : $\vec{u}, \vec{v}, \vec{w}$ the <u>triple scalar product</u> is defined as

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

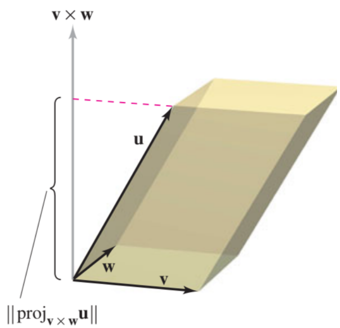
Note that the following are equivalent

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \vec{v} \cdot (\vec{w} \times \vec{u}) = \vec{w} \cdot (\vec{u} \times \vec{v})$$

A key use of the triple scalar product is to find the volume of a parallelepiped

<u>Thm</u> The volume of a parallelepiped w/ vectors $\vec{u}, \vec{v}, \vec{w}$ as adjacent edges is given by

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$$



Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$