Lecture # 08: Multivariable Chain Rule

Date: Thu. 10/11/18

Differentiability

Def

DEFINITION OF DIFFERENTIABILITY

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. The function f is **differentiable in a region R** if it is differentiable at each point in R.

Note: This definition is different than that for single variable fcns

For a fcn of 2 variables both f_x & f_y can exist @ a pt but $f(x,y)$ is not diff'ble there.

Ex. 1 | Show that $f(x,y) = x^2 + 3y$ is diff'ble

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= (x+\Delta x)^2 + 3(y+\Delta y) - (x^2+3y)$$

$$= 2x\Delta x + \Delta x^2 + 3\Delta y$$

Note that $f_x = 2x$ & $f_y = 3$

$$= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \underbrace{\Delta x}_{\varepsilon_1}(\Delta x) + \underbrace{0}_{\varepsilon_2}(\Delta y)$$

Since $(\Delta x, 0) \to 0$ as $(\Delta x, \Delta y) \to 0$

so $f(x,y)$ is diff'ble @ every pt.

Lecture #08: Multivariable Chain Rule

Date: Thu. 10/11/18

The following theorem gives us one way to determine if a 2 variable fcn is diff'ble.

THEOREM 13.4 SUFFICIENT CONDITION FOR DIFFERENTIABILITY

If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Note that the term "sufficient" means that if the partial derivatives are not cont. then we can't say either way if the fcn is diff'ble or not!

Luckily, we have the following property for 2 variable fcns that we have for single variable fcns. Remember: This is not a reversible statement!

THEOREM 13.5 DIFFERENTIABILITY IMPLIES CONTINUITY

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

The contra positive of this statement is true i.e.

<u>Thm</u> If $f(x, y)$ is <u>not</u> continuous @ (x_0, y_0) then it is <u>not</u> diff'ble there.

Lecture #08: Multivariable Chain Rule

Ex. 2) Show that the fcn

$$f(x,y) = \begin{cases} \dfrac{-3xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

has partial derivatives that exist @ $(0,0)$ but the fcn is <u>not</u> differentiable @ $(0,0)$.

By definition of partial derivatives we have

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

so both partial derivatives exist @ $(0,0)$.

To show that $f(x,y)$ is not diff'ble @ $(0,0)$ it is enough to show that $f(x,y)$ is not continuous @ $(0,0)$.

When dealing w/ single variable fcns, we only need to worry about left & right derivatives. For multivar. fcns we can come @ a single pt from <u>any</u> direction. We only need to find 2 for which $f(x,y)$ will fail to be continuous.

Lecture #08: Multivariable Chain Rule Date: Thu. 10/11/18

Ex. 2 (cont'd)

To show $f(x,y)$ is not cont. @ $(0,0)$ we approach the pt. $(0,0)$ from two different directions.

Along the line $y=x$:

$$\lim_{(x,x)\to(0,0)} f(x,y) = \lim_{(x,x)\to(0,0)} \frac{-3x}{x^2+(x)^2} = \lim_{(x,x)\to(0,0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

Along the line $y=-x$

$$\lim_{(x,-x)\to(0,0)} f(x,y) = \lim_{(x,-x)\to(0,0)} \frac{-3(-x)}{(-x)^2+(-x)^2} = \lim_{(x,x)\to(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}$$

So the limit as $(x,y)\to(0,0)$ does not exist

Therefore, $f(x,y)$ is not Continuous @ $(0,0)$
So it is <u>not</u> differentiable there.

Lecture #08: Multivariable Chain Rule Date: Thu. 10/11/18

Differentials

Recall that for a single variable $y = f(x)$ the differential of y is defined as

$$dy = f'(x)\,dx$$

The differential is used as an approx. of the change in y i.e. $\Delta y \approx dy$ & $\Delta y \approx f'(x)dx$

We have similar terminology for 2 var. fcns.

<u>Def</u> For a 2 variable fcn $z = f(x,y)$

Δx is the <u>increment of x</u>

Δy is the <u>increment of y</u>

& the <u>increment of z</u> is

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$$

DEFINITION OF TOTAL DIFFERENTIAL

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

This can be extended to 3 var. fcns

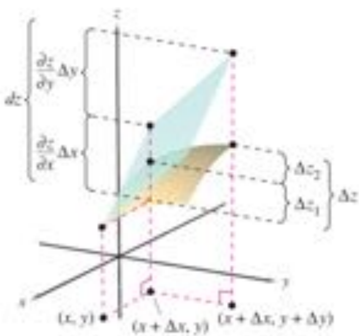
Lecture #08: Multivariable Chain Rule
Date: Thu. 10/11/18

Ex. 3) Find the total differential of

$$z = 2x\sin(y) - 3x^2y^2$$

total differential is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$
$$= (2\sin(y) - 6xy^2)dx + (2x\cos(y) - 6x^2y)\,dy$$



The exact change in z is Δz. This change can be approximated by the differential dz.

If both f_x & f_y are cont. then we can choose Δx & Δy small enough so that

$$\Delta z \approx dz$$

Since $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ give slopes of the surface in x & y directions then $dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$ is the change in height of the plane tangent to the surface @ $(x, y, f(x,y))$

A plane in space is linear wrt x, y, z then approximating Δz by dz is known as a <u>linear approximation</u>

Lecture #08: Multivariable Chain Rule Date: Thu. 10/11/18

Ex. 4 Approximate the change in

$$z = (4 - x^2 - y^2)^{1/2}$$

as (x,y) moves from the pt $(1,1)$ to $(1.01, 0.97)$

Soln.

Let $(x,y) = (1,1)$ & $(x+\Delta x, y+\Delta y) = (1.01, 0.97)$

$$\Rightarrow \Delta x = 0.01 = dx \quad \& \quad \Delta y = -0.03 = dy$$

The change in z is approx'd by

$$\Delta z \approx dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \frac{-x}{(4-x^2-y^2)^{1/2}} \Delta x + \frac{-y}{(4-x^2-y^2)^{1/2}} \Delta y$$

$$= \frac{-x}{(4-x^2-y^2)^{1/2}} \Delta x + \frac{-y}{(4-x^2-y^2)^{1/2}} \Delta y$$

when $x=1$ & $y=1$:

$$\Rightarrow dz \approx = -\frac{1}{\sqrt{2}}(0.01) + -\frac{1}{\sqrt{2}}(-0.03) \approx 0.0141$$

Actual change in z:

$$\Delta z = f(x_2, y_2) - f(x_1, y_2)$$

$$= f(1.01, 0.97) - f(1,1)$$

$$\approx 0.0137$$

Lecture #08: Multivariable Chain Rule Date: Thu. 10/11/18

The Chain Rule

The chain rule for multivar. fcns will be needed for fcns $F(x(t), y(t))$, i.e. when x & y are fcns of an independent variable.

THEOREM 13.6 CHAIN RULE: ONE INDEPENDENT VARIABLE

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$ See Figure 13.39.

Ex. 5 | Find $\frac{dw}{dt}$ for $w = x^2 y - y^2$

where $x = \sin(t)$ & $y = e^t$

Soln. By Multivariable Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

where $\frac{\partial w}{\partial x} = 2xy$ $\qquad \frac{dx}{dt} = \cos(t)$

$\frac{\partial w}{\partial y} = x^2 - 2y$ $\qquad \frac{dy}{dt} = e^t$

$$\Rightarrow \frac{dw}{dt} = 2xy\cos(t) + (x^2 - 2y)e^t$$

Lecture #08: Multivariable Chain Rule

We may also have the case where x & y are fcns of 2 independent variables

THEOREM 13.7 CHAIN RULE: TWO INDEPENDENT VARIABLES

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

Ex. 6) Find $\frac{\partial w}{\partial s}$ & $\frac{\partial w}{\partial t}$ for $w = 2xy$

where $x = s^2 + t^2$ & $y = \frac{s}{t}$

By chain rule:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

where $\frac{\partial w}{\partial x} = 2y$ $\qquad$ $\frac{\partial x}{\partial s} = 2s$

$\frac{\partial w}{\partial y} = 2x$ $\qquad$ $\frac{\partial y}{\partial s} = \frac{1}{t}$

$$\Rightarrow \frac{\partial w}{\partial s} = 2y\,2s + 2x\left(\frac{1}{t}\right)$$

$\frac{\partial w}{\partial t}$ is left as an exercise

Lecture #08: Multivariable Chain Rule

Date: Thu. 10/11/18

Implicit Partial Differentiation

THEOREM 13.8 CHAIN RULE: IMPLICIT DIFFERENTIATION

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Ex. 7) Find $\frac{dy}{dx}$ for $y^3 + y^2 - 5y - x^2 + 4 = 0$

Soln. Let $F(x,y) = y^3 + y^2 - 5y - x^2 + 4$

$$\frac{dy}{dz} = -\frac{F_x(x,y)}{F_y(x,y)}$$

where $F_x = -2x$
$F_y = 3y^2 + 2y - 5$

$$= \frac{-(-2x)}{3y^2+2y-5}$$

$$= \frac{2x}{3y^2+2y-5}$$

Lecture #08: Multivariable Chain Rule

Date: Thu. 10/11/18

Ex. 8 Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ for

$$3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$$

Soln.

Let $F(x,y,z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5$

where
$$f_x(x,y,z) = 6xz - 2xy^3$$
$$f_y(x,y,z) = -2x^2y + 3z$$
$$f_z(x,y,z) = 3x^2 + 6z^2 + 3y$$

So implicit partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{6xz - 2xy^3}{3x^2 + 6z^2 + 3y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-2x^2y + 3z}{3x^2 + 6z^2 + 3y}$$