Worksheet 17 Solutions
Line Integrals

1. The closed curve C is the intersection of the hyperboloid $x^2 + y^2 - z^2 = 3$ with the plane $z = 1$. Compute the integral

$$\oint_C (x - z)\, ds.$$

Solution: An appropriate parametrization for C is $x = 2\cos t$, $y = 2\sin t$, and $z = 1$ where $0 \leq t \leq 2\pi$. Because $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$, $ds = |\mathbf{r}'(t)|\, dt = 2\, dt$. Then

$$\oint_C (x - z)\, ds = \int_0^{2\pi} (2\cos t - 1)\, 2\, dt$$

$$= 4\int_0^{2\pi} \cos t\, dt - 2\int_0^{2\pi} dt$$

$$= 0 - 4\pi = -4\pi.$$

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2. In these problems C consists of the arc of the circle $x^2 + y^2 = 4$ from $(2, 0)$ counterclockwise to $(0, 2)$.

   (a) Give a parametrization for C. Specify the domain for the parametrization.

> **Solution:** In polar coordinates the circle $x^2 + y^2 = 4$ becomes $r = 2$. Remembering the transformation equations $x = r \cos \theta$ and $y = r \sin \theta$ we have the parametric equations $x = 2 \cos \theta$ and $y = 2 \sin \theta$, where $0 \leq \theta \leq \dfrac{\pi}{2}$. To avoid scaring anyone, set $\theta = t$. The position vector becomes the usual
>
> $$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t \rangle.$$
>
> The domain for this parametrization is $0 \leq t \leq \dfrac{\pi}{2}$ because $\mathbf{r}(0) = \langle 2, 0 \rangle$ and $\mathbf{r}(\pi/2) = \langle 0, 2 \rangle$.

   (b) A mass moves along C while under the influence of the force field $\mathbf{F}(x, y) = \langle y, 1 \rangle$. Determine the amount of work done by $\mathbf{F}$ on the mass by computing an appropriate line integral.

> **Solution:**
>
> Use the parametric equations for C to make a change of variables from x and y to t:
>
> $$\mathbf{F}(x, y) = \langle y, 1 \rangle = \langle 2 \sin t, 1 \rangle = \mathbf{F}(t),$$
> $$d\mathbf{r} = \langle -2 \sin t, 2 \cos t \rangle \, dt, \text{ and}$$
> $$\mathbf{F}(x, y) \cdot d\mathbf{r} = \langle 2 \sin t, 1 \rangle \cdot \langle -2 \sin t, 2 \cos t \rangle \, dt = (-4 \sin^2 t + 2 \cos t) \, dt.$$
>
> Then the work is
>
> $$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_0^{\pi/2} (-4 \sin^2 t + 2 \cos t) \, dt$$
> $$= -4 \int_0^{\pi/2} \sin^2 t \, dt + 2 \int_0^{\pi/2} \cos t \, dt$$
> $$= -4 \cdot \frac{\pi}{4} + 2 \cdot 1 = 2 - \pi.$$
>
> The kinetic energy of the mass decreases by $\pi - 2$.

3. A mass moves in the xy-plane while under the influence of a force. The work done by the force is

$$W = \int_C (x^2 - y^2)\, dx + (1 + 4xy)\, dy.$$

The positively oriented curve C that the mass travels along is a triangle formed by the lines $y = 0$, $y - x = 0$, and $y + x = 4$. Compute the work done by this force by breaking this integral into three pieces.
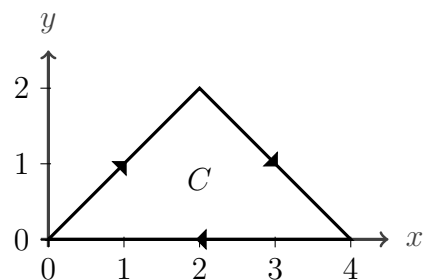
4. A mass moves along a curve C, the portion of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$, while under the influence of the force field $\mathbf{F}(x, y) = \langle y, -x \rangle$. Determine the amount of work done by $\mathbf{F}$ on the mass by computing the line integral

$$\text{Work} = \int_C y \, dx - x \, dy.$$

Solution:

To parametrize C start by setting $x = t$. Then $y = t^2$ and $-1 \leq t \leq 1$. The position vector is $\mathbf{r}(t) = \langle t, t^2 \rangle$. Use the parametric equations to make a change of variables from x and y to t:

$$d\mathbf{r} = \langle dx, dy \rangle = \langle 1, 2t \rangle \, dt, \text{ and}$$
$$y \, dx - x \, dy = t^2 \, dt - t \, 2t \, dt = -t^2 \, dt.$$

Then the work is

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \int_C y \, dx - x \, dy = -\int_{-1}^1 t^2 \, dt = -\frac{2}{3}$$

The kinetic energy of the mass decreases by $\dfrac{2}{3}$.