Lecture # 18A: Curl, Divergence & Fund. Thm of Line Integrals Date: Thu. 11/29/18

Circulation & Flux

Circulation is a measure of how much of the vector field points in the direction of C

__Def.__ Let F be continuous vector field on a region R and C a smooth curve in R, then the circulation on C is

$\int_C F \cdot T\,ds$, where T is the unit tangent vector for C.

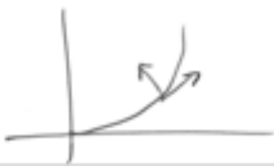
Flux is a measure of how much the vector field points orthogonal to C.

__Def.__ : Let F be continuous vector field on a region R, and C a smooth curve in R, then the flux across C is

$$\int_C F \cdot n\,ds = \int_a^b (f(t)y'(t) - g(t)x'(t))\,dt$$

where $\vec{n}$ is the unit normal vector

$$\vec{n} = \frac{T'}{\|T'\|} = T \times \hat{k}$$



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We've seen that the gradient of a scalar valued function is close to the idea of a derivative in multidimensions.

$f(x,y,z)$ is scalar valued function.

$\nabla f(x,y,z)$ is vector valued function

How can we define a derivative of vector field?

The Del Operator

So far we've looked at alot of mappings.

$f: \mathbb{R}^n \to \mathbb{R}$, surface, scalar valued function , $f(x,y,z) \mapsto k$

$\vec{r}: \mathbb{R} \to \mathbb{R}^n$ vector valued function curves , $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}(t) \mapsto (x,y,z)$

$\vec{r}(u,v): \mathbb{R}^2 \to \mathbb{R}^3$ parametric surfaces.

$T: \mathbb{R}^n \to \mathbb{R}^n$ transforms

All of these map spaces to spaces, or versions of the real line to other versions of the real line.

Question: What about mappings that don't only send real numbers to other real numbers.

Consider the derivative as a operator.

$$\frac{d}{dx}\left[f(x)\right] = f'(x)$$

if $f(x) = x^2$, then $\frac{d}{dx}$ as ~~an~~ a mapping sends x^2 to $2x$, which are both functions!

So the derivative is a mapping which ~~sends~~ sends functions to functions.

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Def: The Del Operator, (the gradient as an operator)
The operator ∇ on $\mathbb{R}^n$ is the vector of partial derivatives on $\mathbb{R}^n$.

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ on $\mathbb{R}^2$

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ on $\mathbb{R}^3$

if we apply the ∇ operator on a scalar valued function, $f(x,y,z)$, we get a vector valued function

$\nabla(x^2 - y + z) = \langle 2x, -1, 1 \rangle$

Divergence & Curl are ways to apply ∇ to vector fields.

Divergence Operator

Def: The divergence of a vector field $\vec{F} = \langle f, g, h \rangle$ that is differentiable on a Region of $\mathbb{R}^3$ is

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle f, g, h \rangle$$
$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

Ex: $\vec{F} = \langle -y, x, z \rangle$, then

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle -y, x, z \rangle$$
$$= \frac{\partial}{\partial x}[-y] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[z]$$
$$= 0 + 0 + 1$$
$$= 1$$

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Note: Divergence produces a scalar valued function.
If $\nabla \cdot \vec{F} = 0$ everywhere, then $\vec{F}$ is said to be source free.

Ex: if $\vec{v}$ is a velocity vector field for a fluid, then the statement $\nabla \cdot \vec{v} = 0$ means no fluid is added or removed from the region.

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Meaning: The "mathematical" meaning of divergence is a measure of how the magnitudes of vectors in the vector field are changing.

The physical interpretation is a measure of net flow out of a point, or region; negative divergence is flow into a point or region.

in 2-D if $\vec{F} = \langle f, g \rangle$ then

$$\operatorname{div} \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}.$$

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Curl Operator

DEFINITION OF CURL OF A VECTOR FIELD

The curl of $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is

$$\text{curl } \mathbf{F}(x, y, z) = \nabla \times \mathbf{F}(x, y, z)$$
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

Where $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$

Ex.) if $\vec{F} = xz\hat{i} + xyz\hat{j} - y^2\hat{k}$ then

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= (-2y - xy)\hat{i} + (x - 0)\hat{j} + (yz - 0)\hat{k}$$

$$= \langle -2y - xy, x, yz \rangle.$$

Note: the curl produces a vector in $\mathbb{R}^3$

if $\nabla \times \vec{F} = 0$ then $\vec{F}$ is said to irrotational.

Meaning: The "mathematical" meaning of curl is a measure in the change of direction of vectors in the vector field.

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The physical interpretation is measure of rotation of the vector field.

$$\nabla \times \vec{F} = \langle \tilde{f}, \tilde{g}, \tilde{h} \rangle$$

Where

$\tilde{f}$ is the component measuring rotation in the yz-plane, or about the x-axis

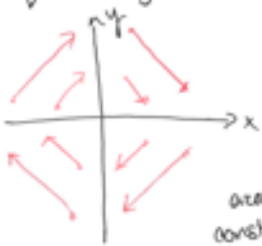
$\tilde{g}$ is the component measuring rotation in the xz-plane, or about the y-axis

$\tilde{h}$ is the component measuring rotation in the xy-plane, or about the z-axis

Note in 2-D if $\vec{F} = \langle f, g \rangle$ then we need to extend to $\mathbb{R}^3$ for $\nabla \times \vec{F}$ to be computable.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ f & g & 0 \end{vmatrix} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}$$

Ex. 1 if $\vec{F} = \langle y, -x \rangle$.



$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}[-x] - \frac{\partial}{\partial y}[y] \right) \hat{k}$$

$$= -1 - 1 = -2\hat{k}$$

thus the vector field rotates around the z-axis in a constant pattern, and rotates clockwise.

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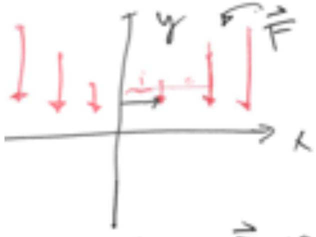
Ex. $F = \langle 0, -x^2 \rangle$ then $\nabla \times \vec{F} = (-2x)\hat{k}$



Note if $\nabla \times \vec{F} < 0$, rotating clockwise
$\nabla \times \vec{F} > 0$, rotating counterclockwise

Divergence of Curl of $\vec{F}$

Theorem: Suppose $\vec{F} = \langle f, g, h \rangle$, where the second partials of $\vec{F}$ are continuous, then

$$\nabla \cdot (\nabla \times \vec{F}) = 0.$$

Product Rule for divergence.

if u is a scalar valued function $(u = u(x,y,z))$
$\vec{F}$ is a vector field then,

$$\nabla \cdot (u\vec{F}) = \nabla u \cdot \vec{F} + u(\nabla \cdot \vec{F}).$$

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Laplace Operator

if u is a scalar valued function, then we note
∇u is a vector valued function, and
$\nabla \cdot (\nabla u)$ is scalar valued function
written
$$\nabla^2 = \Delta = \nabla \cdot \nabla$$
if u represents a potential (heat, pressure, ~~force~~ energy) then
$\nabla^2 u$ describes the diffusion of this property.
There are some variation of the Laplace operator.
Let K represent permeability of porous media, let
p be a fluid pressure of a resident fluid, then
$-\nabla \cdot (K \nabla p)$ gives the spatial diffusion of the fluid.

Conservative Vector Fields

<u>Def.</u> A vector field is <u>Conservative</u> if there exists a differentiable Fcn φ s.t.

$$\vec{F} = \nabla \varphi$$

The Fcn φ is called the potential function of $\vec{F}$

Many vector fields are conservative
Ex: gravity, electro-mag, etc.

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Ex. 1

For $\varphi = \frac{1}{4}x^2y$ then $\nabla\varphi = \langle \frac{1}{2}xy, \frac{1}{4}x^2 \rangle = \vec{F}$

A vector field is conservative if φ exists such that

Let $\vec{F} = \langle f, g, h \rangle$,

$\vec{F} = \nabla\varphi \Rightarrow$

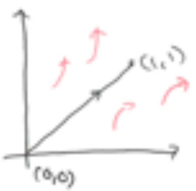
$f = \varphi_x$, $g = \varphi_y$, $h = \varphi_z$

than curl of $\vec{F}$

Independence of Path

Consider finding the work done by the Force field $\vec{F} = \langle \frac{1}{2}xy, \frac{1}{4}x^2 \rangle$ on a particle moving from $(0,0)$ to $(1,1)$ along different paths.

C_1

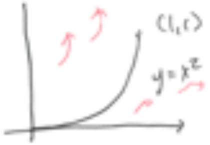


$\vec{r}(t) = \langle t, t \rangle \quad 0 \le t \le 1$

$r'(t) = \langle 1, 1 \rangle \Rightarrow F(x(t), y(t)) = \langle \frac{1}{2}t^2, \frac{1}{4}t^2 \rangle$

$$W = \int_{C_1} \vec{F} \cdot dr = \int_0^1 \frac{3}{4}t^2\,dt = \frac{1}{4}$$

C_2



$\vec{r}(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$

$\vec{r}'(t) = \langle 1, 2t \rangle \quad F = \langle \frac{1}{2}t^3, \frac{1}{4}t^2 \rangle$

$$\int_{C_2} F \cdot \vec{r}'(t)\,dt = \int_0^1 t^3\,dt = \frac{1}{4}$$

So the work to move the particle from point A to point B was the same, for different paths.

It turns out that any path gives the same work. The work is path independent.

When is this true?

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THEOREM 15.6 INDEPENDENCE OF PATH AND CONSERVATIVE VECTOR FIELDS

If $\mathbf{F}$ is continuous on an open connected region, then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path if and only if $\mathbf{F}$ is conservative.

Equivalent Conditions

THEOREM 15.7 EQUIVALENT CONDITIONS

Let $\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The following conditions are equivalent.

1. $\mathbf{F}$ is conservative. That is, $\mathbf{F} = \nabla f$ for some function f.
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve C in R.

Fundamental Thm of Line Integrals

This all leads us to an easier way to evaluate line integrals (within a conservative vec. field)

THEOREM 15.5 FUNDAMENTAL THEOREM OF LINE INTEGRALS

Let C be a piecewise smooth curve lying in an open region R and given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \le t \le b.$$

If $\mathbf{F}(x, y) = M\mathbf{i} + N\mathbf{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is a potential function of $\mathbf{F}$. That is, $\mathbf{F}(x, y) = \nabla f(x, y)$.