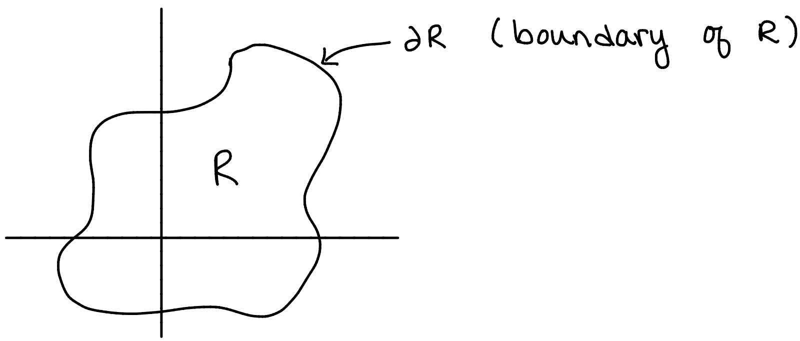
Lecture # 18B : Green's Theorem   Date: Thu. 11/29/18

Motivation for Green's Thm

The Fundamental Theorem of line integrals gives a good tool to calculate line integrals in a Conservative vector fields, but not every vector field is Conservative.

Green's Theorem simplifies line integrals by Combining the ideas of line integrals & double integrals

Green's Theorem states that the value of a double integral over a region R is the same as the value of the line integral on the boundary of R



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Simple & Connected Curves

Consider the curve C given by

$$r(t) = \langle x(t), y(t) \rangle \quad \text{for} \quad a \le t \le b$$

<u>Def</u> a curve is <u>connected</u> if it connects to itself. i.e.

$$\vec{r}(a) = \vec{r}(b)$$

<u>Def</u> a curve is <u>simple</u> if it does not intersect itself. i.e.

$$\vec{r}(c) \neq \vec{r}(d) \qquad \forall\, c,d \in (a,b)$$

<u>Def.</u> A plane region R is <u>simply connected</u> if every simple closed curve in R encloses only pts in R.

Simply connected

Not simply connected

<u>Ex. 1</u> The curve $\vec{r}(t) = \langle t^2, t^3 - t \rangle$ is not simple

since $\vec{r}(-1) = \langle 1, 0 \rangle$ & $\vec{r}(1) = \langle 1, 0 \rangle$

thus $\vec{r}(-1) = r(1)$ but $-1 \neq 1$

Therefore, Curve intersects itself $\Rightarrow$ not simple

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Green's Theorem (Circulation Version)

Theorem: Let C be a simple closed, piecewise smooth, curve, oriented counterclockwise, that encloses a simply connected region R.

Let $\vec{F} = \langle f, g \rangle$ where f & g have continuous partial derivatives, then

$$\oint_C \vec{F} \cdot dr = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

This connects work integrals along closed paths to the rotation of the vector field, ~~or flux~~, or circulation, enclosed by the path.

$$W = \oint_C \vec{F} \cdot dr = \iint_R \nabla \times \vec{F}\, dA.$$

Green's Theorem (Flux Version)

Theorem: Let C be a simple closed, piecewise smooth curve, oriented counterclockwise, that encloses a simply connected region R. Let $\vec{F} = \langle f, g \rangle$ where f & g have continuous partial derivatives, then

$$\oint_C \vec{F} \cdot \vec{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \operatorname{div} \vec{F}\, dA = \iint_R \nabla \cdot \vec{F}\, dA$$

$$= \iint_R \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\, dA$$

where $\vec{n}$ is the unit normal vector to C.

This connects the flux of a vector field along a closed curve C to the divergence of the field inside of C.

$$\underbrace{\oint_C \vec{F} \cdot \vec{n}\, ds}_{\text{flux}} = \underbrace{\iint_R \nabla \cdot \vec{F}\, dA}_{\text{divergence inside of } R.}$$

Lecture # 18B : Green's Theorem

Ex. Evaluate

$$\oint_C (4x^3 + \sin(y^2))\,dy - (4y^3 + \cos(x^2))\,dx$$

where C is circle $x^2 + y^2 = 4$ (oriented counterclkwse)

Soln

$$\vec{F} = \langle 4x^3 + \sin(y^2),\ 4y^3 + \cos(x^2) \rangle$$

$$\nabla \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle \cdot \langle f, g \rangle$$

$$= \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g$$

$$= 12x^2 + 12y^2$$

By Green's Thm:

$$\oint_C f\,dy - g\,dx = \iint_R \nabla \cdot \vec{F}\,dA$$

$$= \iint_R (12x^2 + 12y^2)\,dA$$

$$= 12 \iint_R (x^2 + y^2)\,dA$$

C is $x^2 + y^2 = 4$

Convert to polar! $r = 2 \Rightarrow 0 \le r \le 2$

$0 \le \theta \le 2\pi$

Ex. (cont'd)

$$= 12 \iint_R (x^2 + y^2)\, dA$$

$r = 2 \implies 0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$= 12 \int_0^{2\pi} \int_0^2 r^2 r \, dr\, d\theta$$

$$= 12 \int_0^{2\pi} \int_0^2 r^3 \, dr\, d\theta$$

$$= 12 \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta$$

$$= 12 \int_0^{2\pi} 4 \, d\theta = 12 \left[4\theta \right]_0^{2\pi}$$

$$= 96\pi$$

Find Area w/ Green's Thm

Green's Theorem can connect line integrals over vector fields to double integrals over regions and visa versa.

- The double integral

$$\iint_R f(x,y)\,dA$$ gives the area of R if $f(x,y) = 1$

By Green's theorem

$$\oint_C \vec{F}\cdot dr = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

if we choose $\vec{F}$ in such a way that

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 \text{ in } R\text{, then}$$

$$\oint_C \vec{F}\cdot dr = \iint_R dA = \text{Area of } R.$$

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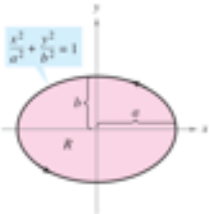

Figure 15.32

EXAMPLE 5 Finding Area by a Line Integral

Use a line integral to find the area of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Solution Using Figure 15.32, you can induce a counterclockwise orientation to the elliptical path by letting

$$x = a\cos t \quad \text{and} \quad y = b\sin t, \quad 0 \le t \le 2\pi.$$

So, the area is

$$\begin{aligned} A = \frac{1}{2}\int_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt] \\ &= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt \\ &= \frac{ab}{2}\Big[t\Big]_0^{2\pi} \\ &= \pi ab. \end{aligned}$$

Green's Theorem can be extended to cover some regions that are not simply connected. This is demonstrated in the next example.

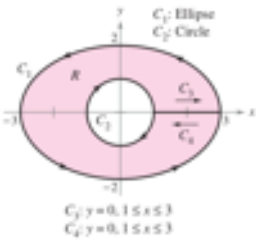

Figure 15.33

EXAMPLE 6 Green's Theorem Extended to a Region with a Hole

Let R be the region inside the ellipse $(x^2/9) + (y^2/4) = 1$ and outside the circle $x^2 + y^2 = 1$. Evaluate the line integral

$$\int_C 2xy\,dx + (x^2 + 2x)\,dy$$

where $C = C_1 + C_2$ is the boundary of R, as shown in Figure 15.33.

Solution To begin, you can introduce the line segments C_3 and C_4, as shown in Figure 15.33. Note that because the curves C_3 and C_4 have opposite orientations, the line integrals over them cancel. Furthermore, you can apply Green's Theorem to the region R using the boundary $C_1 + C_4 + C_2 + C_3$ to obtain

$$\begin{aligned} \int_C 2xy\,dx + (x^2 + 2x)\,dy &= \int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\ &= \int_R\int (2x + 2 - 2x)\,dA \\ &= 2\int_R\int dA \\ &= 2(\text{area of } R) \\ &= 2(\pi ab - \pi r^2) \\ &= 2[\pi(3)(2) - \pi(1^2)] \\ &= 10\pi. \end{aligned}$$