Lecture #16 : Change of variables in Triple Integrals   Date: Thu. 11/15/18

Change of variables

This works similar to how it works for double integrals.

Def   Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a one to one transformation that maps a closed region S in uvw - space to a region R in xyz - space.

$$T: \quad x = g(u,v,w) \quad , \quad y = h(u,v,w) \quad , \quad z = p(u,v,w)$$

if $f(x,y,z)$ is cont. then

$$\left| \iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \underbrace{\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right|}_{J(u,v,w)} du\, dv\, dw \right.$$

Def (Jacobian in $\mathbb{R}^3$)

The __Jacobian__ in $\mathbb{R}^3$ is defined as

$$J(u,v,w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} x_u & x_v & x_w \\[2mm] y_u & y_v & y_w \\[2mm] z_u & z_v & z_w \end{vmatrix}$$

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Ex.1) Evaluate $\iiint\limits_{D} xz \, dv$,

where D is region bdd by

$y = x$ $y = x+3$ $\Rightarrow$ $y - x = 0$, $y - x = 3$

$z = x$ $z = x+3$ $\Rightarrow$ $z - x = 0$, $z - x = 3$

$z = 0$ $z = 4$

using the C.O.U. $u = y-x$, $v = z-x$, $w = z$

    w/bds: $0 \le u \le 3$, $0 \le v \le 3$, $0 \le w \le 4$

Solving for x, y, z in terms of u, v, w

$$z = w$$

$$v = z - x = w - x \qquad \Rightarrow \quad x = w - v$$

$$u = y - x = y - (w - v)$$
$$= y - w + v \quad \Rightarrow \quad y = u + v - w$$

Need to calculate Jacobian:

   partial deriv.s of $x = w - v$
                         $y = u + v - w$
                         $z = w$

$$x_u = 0 \; , \quad x_v = -1 \; , \quad x_w = 1$$

$$y_u = 1 \; , \quad y_v = 1 \; , \quad y_w = -1$$

$$z_u = 0 \; , \quad z_v = 0 \; , \quad z_w = 1$$

Ex.1) (cont'd)

So

$$
J(u,v,w) = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}
$$

$$
= 0 \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}
$$

$$
= 0 + (1) + 0 = 1
$$

So $J(u,v,w) = 1$

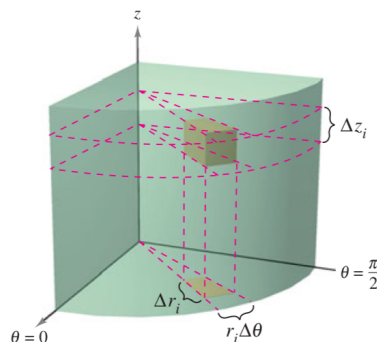$f(x,y,z) = xz \implies f(u,v,w) = (w-v)w$

$$= w^2 - vw$$

$$
\implies \iiint\limits_{D} xz \, dV = \int_0^4 \int_0^3 \int_0^2 (w^2 - vw) \, du \, dv \, dw
$$

$$
= \int_0^4 \int_0^3 \Big[(w^2 - vw)u\Big]_0^2 \, dv \, dw
$$

⋮   (left as an exercise)

$$= 56$$

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Cylindrical Coord.s

The basic idea is using polar
coord's along the z-axis

z

$\Big\}\Delta z_i$

$\theta = \frac{\pi}{2}$

Δr_i

$\theta = 0$

$r_i \Delta \theta$

Volume of cylindrical block:
$\Delta V_i = r_i \Delta r_i \Delta \theta_i \Delta z_i$

The transf. from cartesian to cylindrical coords
given by

$$T: \quad x = r\cos\theta , \quad y = r\sin\theta , \quad z = z$$

$$\text{w/ Jacobian} : \quad J(r, \theta, z) = r$$

$$\Rightarrow dx\,dy\,dz = r\,dr\,d\theta\,dz$$

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Ex. 2) Evaluate $v = \iiint\limits_{Q} dv$

where   Q is given by

$$4x^2 + 4y^2 + z^2 = 16 \qquad (*)$$

Need to find our bds.

Start by solving $(*)$ for z :

$$z = \pm (16 - 4x^2 - 4y^2)^{1/2}$$
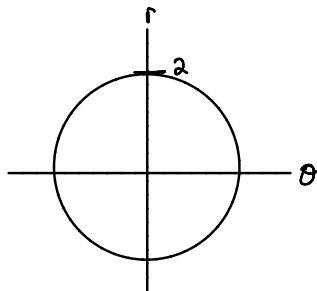
using polar coord.s for x & y

$$X = r\cos\theta$$
$$y = r\sin\theta$$

Writing z in terms of polar coords

$$\Rightarrow z = \pm (16 - 4r^2\cos^2\theta - 4r^2\sin^2\theta)^{1/2}$$
$$= \pm (16 - 4r^2(\cos^2\theta + \sin^2\theta))^{1/2}$$
$$= \pm (16 - 4r^2)^{1/2}$$

Find Bds for r & θ

Since $4x^2 + 4y^2 + z^2 = 16$

Let $z = 0 \Rightarrow 4x^2 + 4y^2 = 16$

$$x^2 + y^2 = 4$$
$$x^2 + y^2 = 2^2$$
$$r = 2$$

Bds are $0 \le r \le 2$ & $0 \le \theta \le 2\pi$

Bds are $0 \le r \le 2$ & $0 \le \theta \le 2\pi$

& $-(16-4r^2)^{1/2} \le z \le (16-4r^2)^{1/2}$

$\Rightarrow$ $V = \iiint\limits_{Q} dV = \int_{0}^{2\pi} \int_{0}^{2} \int_{-(16-4r^2)^{1/2}}^{(16-4r^2)^{1/2}} r \, dz \, dr \, d\theta$

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┌─────────────────────────┐
│ Spherical Coord.s │
└─────────────────────────┘

Def Transf. from Cartesian to Spherical
    Coord.s is
    $T: x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi.$$

$\left.\begin{array}{c} \\ \\ \end{array}\right\}$ Type set

$\begin{array}{c}\Phi \\ \phi\end{array}$ > phi

rho   phi   theta

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\underbrace{\qquad\qquad\qquad}_{J(\rho, \phi, \theta)}$

Jacobian: $J(\rho, \phi, \theta) = \rho^2 \sin\phi$

$\Rightarrow dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\,d\phi\,d\theta$

Note:

ρ is distance from origin

θ is angle in xy plane
    from positive x-axis

ϕ is angle down from
    positive z-axis

$\rho_i \sin\phi_i \Delta\theta_i$

$\Delta\rho_i$

$\rho_i \Delta\phi_i$

Spherical block:
$\Delta V_i \approx \rho_i^2 \sin\phi_i \Delta\rho_i \Delta\phi_i \Delta\theta_i$

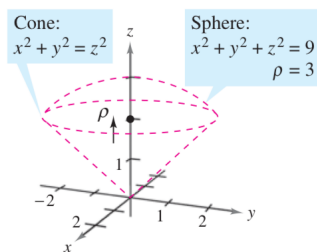Lecture #16: Change of variables in Triple Integrals   Date: Thu. 11/15/18

Ex. 3 | Find volume of region Q which is

bdd below by the cone: $z^2 = x^2 + y^2$
&

bdd above by the sphere: $x^2 + y^2 + z^2 = 9$

Soln First we need to determine our bds

Find bds for ρ:

Cone: Sphere:
$x^2 + y^2 = z^2$ $x^2 + y^2 + z^2 = 9$
                        $\rho = 3$

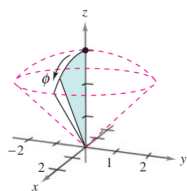ρ varies from 0 to 3 with ϕ and θ held
constant.

$$x^2 + y^2 + z^2 = \rho^2$$

$$\Rightarrow \rho = 3$$

So bds for ρ are

$$0 \leq \rho \leq 3$$

Find bds for ϕ:

ϕ varies from 0 to $\pi/4$
with θ held constant.

Since $z^2 = x^2 + y^2$

Using this in eqn for sphere

$$\Rightarrow (x^2 + y^2) + z^2 = 9$$
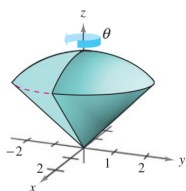
$$z^2 + z^2 = 9 \Rightarrow 2z^2 = 9$$

$$\Rightarrow z = \frac{3}{\sqrt{2}}$$

The change of variables is $z = \rho \cos\phi$, w/ $\rho = 3$

$$\Rightarrow \frac{3}{\sqrt{2}} = 3\cos\phi \Rightarrow \cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$$

So bds for ϕ are : $0 \leq \phi \leq \frac{\pi}{4}$

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<u>Ex. 3</u> (cont'd)

Find bds for θ:

Since the "slice" as we vary z
is a full circle. Bds for θ are

$$0 \leq \theta \leq 2\pi$$

θ varies from 0 to 2π.

To summarize, our bds for region Q are then

$$0 \leq \rho \leq 3$$
$$0 \leq \phi \leq \frac{\pi}{4}$$
$$0 \leq \theta \leq 2\pi$$

Then the volume of region Q can be found
using the triple integral (in spherical coord.s)

$$V = \iiint_Q dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$