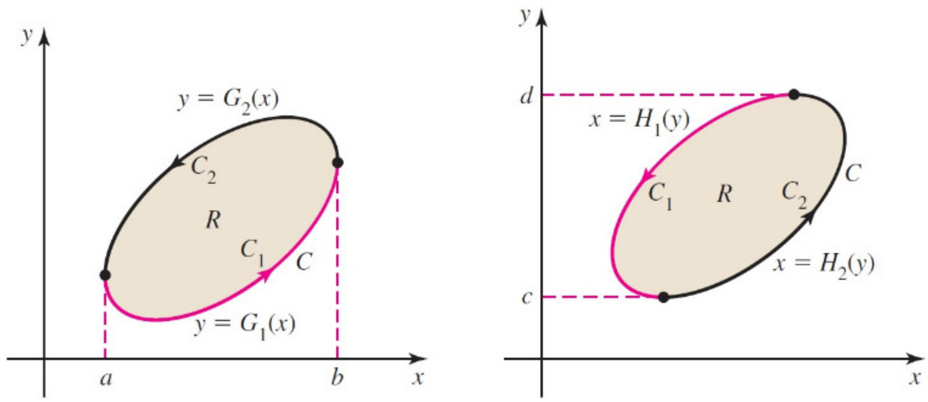
1. Consider the situation below where you have a simple, closed, and smooth curve C that is oriented counter clockwise and defines the boundary of a simply connected region R. Let $\boldsymbol{F} = \langle f, g\rangle$ be a vector field over R that has continuous partial derivatives.



(a) Use the figure on the left and the Fundamental Theorem of Calculus to show that

$$\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A = -\oint_C f\,dx.$$

Solution: Note that you can rewrite the integral $\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A$ by using the first figure as

$$\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A = \int_a^b \int_{G_1(x)}^{G_2(x)} \frac{\partial f}{\partial y}\,\mathrm{d}y\,\mathrm{d}x$$

Then by the Fundamental Theorem of Calculus we find that

$$\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A = \int_a^b \Big(f(x, G_2(x)) - f(x, G_1(x))\Big)\,\mathrm{d}x.$$

Over the interval $a \le x \le b$, the points $(x, G_2(x))$ trace out the upper part of C (labeled C_2) in the negative (clockwise) direction. Similarly, over the interval $a \le x \le b$, the points $(x, G_1(x))$ trace out the lower part of C (labeled C_1) in the positive (counterclockwise) direction.

Therefore,

$$\begin{aligned}\iint_R \frac{\partial f}{\partial y}\, \mathrm{d}A &= \int_a^b \Big(f(x, G_2(x)) - f(x, G_1(x))\Big)\, \mathrm{d}x \\ &= \int_{-C_2} f\, \mathrm{d}x - \int_{C_1} f\, \mathrm{d}x \\ &= -\int_{C_2} f\, \mathrm{d}x - \int_{C_1} f\, \mathrm{d}x \\ &= -\oint_C f\, \mathrm{d}x.\end{aligned}$$

(b) Use the figure on the right and the Fundamental Theorem of Calculus to show that

$$\iint_R \frac{\partial g}{\partial x}\, \mathrm{d}A = \oint_C g dy.$$

Solution: Note that you can rewrite the integral $\displaystyle\iint_R \frac{\partial g}{\partial x}\, \mathrm{d}A$ by using the second figure as

$$\iint_R \frac{\partial g}{\partial x}\, \mathrm{d}A = \int_c^d \int_{H_1(y)}^{H_2(y)} \frac{\partial g}{\partial x}\, \mathrm{d}x\, \mathrm{d}y = \int_c^d \Big(g(H_2(y), y) - g(H_1(y), y)\Big)\, \mathrm{d}y.$$

Over the interval $c \le y \le d$, the points $(H_1(y), y)$ trace out the left most part of C (labeled C_1) in the negative (clockwise) direction. Similarly, over the interval $c \le y \le d$, the points $(H_2(y), y)$ trace out the right most part of C (labeled C_2) in the positive (counterclockwise) direction.

Therefore,

$$\begin{aligned}\iint_R \frac{\partial g}{\partial x}\, \mathrm{d}A &= \Big(g(H_2(y), y) - g(H_1(y), y)\Big)\, \mathrm{d}y \\ &= \int_{C_2} g\, \mathrm{d}y - \int_{-C_1} g\, \mathrm{d}y \\ &= \int_{C_2} g\, \mathrm{d}y + \int_{C_1} g\, \mathrm{d}y \\ &= \oint_C g\, \mathrm{d}y.\end{aligned}$$

(c) Combine the results from part (a) and (b) to show that

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \,\mathrm{d}A = \oint_C f \,\mathrm{d}x + g \,\mathrm{d}y.$$

Solution: From put previous work we know that

$$\iint_R \frac{\partial f}{\partial y} \,\mathrm{d}A = -\oint_C f \,\mathrm{d}x \quad \text{and} \quad \iint_R \frac{\partial g}{\partial x} \,\mathrm{d}A = \oint_C g dy.$$

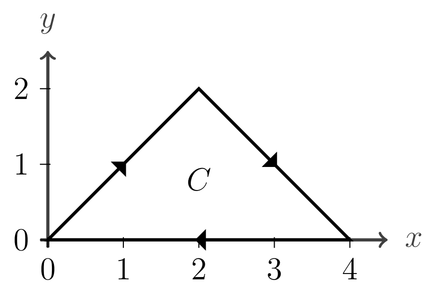
It then follows that

$$\begin{aligned} \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \,\mathrm{d}A &= \iint_R \frac{\partial g}{\partial x} \,\mathrm{d}A - \iint_R \frac{\partial f}{\partial y} \,\mathrm{d}A \\ &= \oint_C g \,\mathrm{d}y - \left(-\oint_C f \,\mathrm{d}x \right) \\ &= \oint_C f \,\mathrm{d}x + g \,\mathrm{d}y \end{aligned}$$

2. A mass moves in the xy-plane while under the influence of a force. The work done by the force is

$$\mathrm{W} = \int_C (x^2 - y^2)\,dx + (1 + 4xy)\,dy.$$

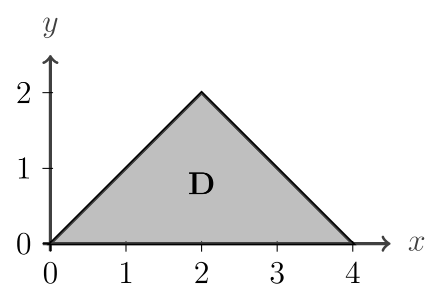
The negatively oriented curve C that the mass travels along is a triangle formed by the lines $y = 0$, $y - x = 0$, and $y + x = 4$. C is a simple closed curve, so use Green's theorem to compute W.



Solution: The force acting on the mass is $\mathbf{F}(x, y) = \langle x^2 - y^2, 1 + 4xy\rangle$. Start by computing the integrand of the double integral:

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 4y + 2y = 6y.$$

The graph shows the region enclosed by C. The region is described by $D = [y, 4 - y] \times [0, 2]$.



By Green's theorem we find that the line integral that represents the work done along the (postively oriented curve) can be rewritten as a double integral over D

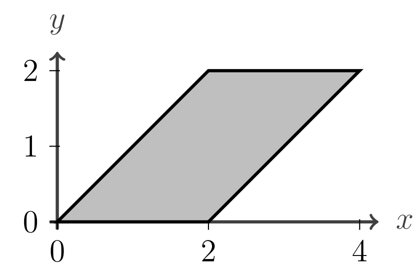
$$\begin{aligned}
\oint_C y^2\,dx + (1+4xy)\,dy &= 6\int_0^2\int_y^{4-y} y\,dx\,dy \\
&= 6\int_0^2 y\,(4-2y)\,dy \\
&= 12\int_0^2 (2y-y^2)\,dy \\
&= 4\,(3y^2-y^3)\Big|_0^2 = 16.
\end{aligned}$$

The work done by the force in along the negatively oriented curve is then -16.

3. Evaluate the flux integral

$$\oint_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}r,$$

where $\mathbf{F} = \langle y^2 - 2xy, x^2 + 2xy \rangle$ and C is the counterclockwise oriented boundary of the parallelogram shown here.



Solution: Because

$$f = y^2 - 2xy \text{ and } g = x^2 + 2xy,$$

we have

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (-2y) + (2x) = 2(x - y).$$

The lines bounding the parallelogram are $y = 0$, $y = x$, $y = 2$, and $y = x - 2$. It's best to integrate in the x direction first.

By Green's theorem

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n} \; \mathrm{d}r &= 2 \int_0^2 \int_y^{y+2} x - y \, dx \, dy \\
&= 2 \int_0^2 [\frac{x^2}{2} - xy]_{x=y}^{y+2} \, dy \\
&= 2 \int_0^2 \frac{1}{2}(y+2)^2 - (y+2)y - \frac{y^2}{2} + y^2 \, dy \\
&= 2 \int_0^2 2 \, dy \\
&= 4y\Big|_{y=0}^{y=2} \\
&= 4(2 - 0) \\
&= 8
\end{aligned}$$

4. Decide if the following expressions are defined. If they are defined, state whether the result is a scalar of or a vector. Assume that $\mathbf{F}$ is a sufficiently differentiable vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and φ is a differentiable scalar valued function.

   (a) $\nabla \cdot (\nabla\varphi)$

   Solution: Yes, this is the divergence of the gradient of φ and is thus a scalar function.

   (b) $\nabla \times (\nabla \cdot \mathbf{F})$

   Solution: No, since $\nabla \cdot \mathbf{F}$ is a scalar valued function.

   (c) $\nabla \times (\nabla\varphi)$

   Solution: Yes, this is the curl of the gradient vector field of φ and is thus a vector field.

5. (Bonus 10 Points) The vector function $\operatorname{curl}\mathbf{F}$ measures the rotation of the vector field $\mathbf{F}$ at a point (x, y, z). If $\operatorname{curl}\mathbf{F} = \mathbf{0}$ everywhere, then $\mathbf{F}$ is said to be irrotational. Show that conservative vector fields, $\mathbf{F} = \nabla f = \langle f_x, f_y, f_z\rangle$, are irrotational if $f(x, y, z)$ is a smooth function with continuous derivatives.

Solution: Because $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F}$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}\rangle = \langle 0, 0, 0\rangle.$$