1. Decide if each statement is true or false. Give an appropriate justification for your conclusion.

   (a) T F If $f(x, y)$ has a local maximum (with $D > 0$) at the point $(0, 0)$ then $g(x, y) = f(x, y) + x^4 - y^4$ also has a local maximum at $(0, 0)$.

   Solution: TRUE: The statement is true. Differentiating g we find that

   $$g_x(x, y) = f_x(x, y) + 4x^3 \quad \text{and} \quad g_y(x, y) = f_y(x, y) - 4y^3.$$

   Then,

   $$g_{xx}(x, y) = f_{xx}(x, y)+12x^2, \quad g_{yy}(x, y) = f_{yy}(x, y)-12y^2, \quad \text{and} \quad g_{xy}(x, y) = f_{xy}(x, y)$$

   Then,

   $$\begin{aligned} D(x, y) &= g_{xx}(x, y)g_{yy}(x, y) - (g_{xy}(x, y))^2 \\ &= (f_{xx}(x, y) + 12x^2)(f_{yy}(x, y) - 12x^2) - (f_{xy}(x, y))^2 \end{aligned}$$

   At the point $(0, 0)$ we find that

   $$D(0, 0) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$$

   Since $f(x, y)$ has a local maximum at the point $(0, 0)$ we conclude from the second derivative test that $D(0, 0) > 0$. Consequently, $g(x, y) = f(x, y) + x^4 - y^4$ also has a local maximum at $(0, 0)$

   (b) T F The function $f(x, y) = x^2 - y^2$ has an absolute minimum.

   Solution: FALSE: This statement is false because f has a single critical point at $(0, 0)$. Using the second derivative test we find that $D(0, 0) < 0$, so f has a saddle point at this critical point.

   (c) T F If $f(x, y)$ has a critical point at $(1, 2)$ then $g(x, y) = e^{f(x,y)}$ also has a critical point at $(1, 2)$.

   Solution: TRUE: This statement is true. Since $f(x, y)$ has a critical point at $(1, 2)$ we find that

   $$f_x(1, 2) = 0, \quad \text{and} \quad f_y(1, 2) = 0.$$

   By the chain rule of differentiation we find that

   $$\begin{aligned} g_x(x, y) &= e^{f(x,y)}f_x(x, y) = 0 \quad \Rightarrow \quad g_x(1, 2) = e^{f(1,2)}f_x(1, 2) = 0 \\ g_y(x, y) &= e^{f(x,y)}f_y(x, y) = 0 \quad \Rightarrow \quad g_y(1, 2) = e^{f(1,2)}f_y(1, 2) = 0. \end{aligned}$$

   Thus, $\nabla g(1, 2) = \mathbf{0}$ so $(1, 2)$ is a critical point of $g(x, y)$.

2. Locate and classify the critical points of

$$f(x, y) = 3y^2 + 2y^3 - 3x^2 - 6xy.$$

Solution: The first derivatives of f are

$$\begin{aligned} f_x(x, y) &= -6x - 6y \\ f_y(x, y) &= 6y + 6y^2 - 6x. \end{aligned}$$

At the critical points these derivatives are both equal to zero:

$$\begin{aligned} -6x - 6y &= 0 \\ 6y + 6y^2 - 6x &= 0. \end{aligned}$$

The first equation requires $y = -x$. Substituting for y in the second equation we find

$$-6x + 6x^2 - 6x = 6x^2 - 12x = 6x(x - 2) = 0,$$

so either $x = 0$ or $x = 2$. Because $y = -x$, the critical points are $(0, 0)$ and $(2, -2)$.

The second derivatives of f are

$$\begin{aligned} f_{xx}(x, y) &= -6 \\ f_{xy}(x, y) &= -6 \\ f_{yy}(x, y) &= 6 + 12y. \end{aligned}$$

Using the second derivatives test to classify each critical point we find

critical point	$f_{xx}(x, y)$	$D(x, y)$	classification
$(0, 0)$	-6	-72	saddle point
$(2, -2)$	-6	72	local maximum

3. If possible find the absolute maximum and minimum values of the function $f(x, y) = 2e^{-x-y}$ on the region $R = \{(x, y) : x \geq 0, y \geq 0\}$.

Solution: Observe that $f(0, 0) = 2$ and $f(x, y) \leq 2$ for all points $(x, y) \in R$; hence, the absolute maximum value of f on R is 2. The function f on R takes on all values in the interval $(0, 2]$; therefore, f has no absolute minimum on R.

4. What point on the plane $x - y + z = 2$ is closest to the point $(1, 1, 1)$?

Solution: The distance from a point (x_0, y_0, z_0) to a point (x, y, z) satisfies

$$d(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Which shares extremal points with the easier to handle function

$$D(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

Rearranging the equation of the plane we have $z = 2 - x + y$, so the distance between $(x, y, 2x + y)$, a point on the plane, to the point $(1, 1, 1)$ is given by

$$f(x, y) = (x - 1)^2 + (y - 1)^2 + (x - y - 1)^2$$

It suffices to minimize the function $f(x, y) = (x - 1)^2 + (y - 1)^2 + (x - y - 1)^2$ on $\mathbb{R}^2$. We have

$$f_x(x, y) = 2(x - 1 + x - y - 1) = 2(2x - y - 2) \quad \text{and} \quad f_y(x, y) = 2(y - 1 + y - x + 1) = 2(-x + 2y)$$

so the critical point of f satisfies

$$2x - y = 2 \quad \text{and} \quad x - 2y = 0$$

which gives $x = \dfrac{4}{3}, y = \dfrac{2}{3}$. The corresponding point on the plane $z = 2 - x + y$ is $\left(\dfrac{4}{3}, \dfrac{2}{3}, \dfrac{4}{3}\right)$. Because there is only one critical point and their exists a point on the plane closest to the point $(1, 1, 1)$, this must be the point we found.