The Dot Product

1. Determine whether **u** and **v** are parallel, orthogonal or neither.

   (a) $\mathbf{u} = \langle 4, 0 \rangle$, $\mathbf{v} = \langle 1, 1 \rangle$

   Solution:

   The angle btwn 2 vectors is given by $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

   We have

   $$\vec{u}\cdot\vec{v} = \langle 4,0\rangle \cdot \langle 1,1\rangle = (4)(1) + (0)(1) = 4$$

   $$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2} = ((4)^2 + (0)^2)^{1/2} = \sqrt{4} = 2$$

   $$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = ((1)^2 + (1)^2)^{1/2} = \sqrt{2}$$

   So

   $$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

   These vectors are neither parallel or orthogonal

   (b) $\mathbf{u} = \langle 2, 18 \rangle$, $\mathbf{v} = \langle \frac{3}{2}, -\frac{1}{6} \rangle$

   Solution:

   We have

   $$\vec{u}\cdot\vec{v} = \langle 2, 18\rangle \cdot \langle \tfrac{3}{2}, -\tfrac{1}{6}\rangle$$
   $$= (2)\left(\frac{3}{2}\right) + (18)\left(-\frac{1}{6}\right)$$
   $$= 3 + (-3)$$
   $$= 0$$

   Since $\vec{u}\cdot\vec{v} = 0$ these vectors are orthogonal.

2. Consider the vectors $\mathbf{u} = \langle 8, 2, 0\rangle$ and $\mathbf{v} = \langle 2, 1, -1\rangle$.

   (a) Find the projection of $\mathbf{u}$ onto $\mathbf{v}$.

   Solution:

   The projection of $\vec{u}$ onto $\vec{v}$ is the vector given by

   $$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{v}$$

   We have

   $$\vec{u}.\vec{v} = \langle 8,2,0\rangle.\langle 2,1,-1\rangle$$
   $$= 8(2) + (2)(1) + (0)(-1)$$
   $$= 18$$

   & $\|\vec{v}\| = \sqrt{v_1^2+v_2^2+v_3^2} = \left((2) + (1)^2 + (-1)^2\right)^{1/2} = \sqrt{6}$

   so

   $$\text{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{v} = \frac{18}{(\sqrt{6})^2}\langle 2,1,-1\rangle = 3\langle 2,1,-1\rangle$$
   $$= \langle 6,3,-3\rangle$$

   (b) Find the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

   Solution:

   The vector component of $\vec{u}$ orthogonal to $\vec{v}$ is

   $$\vec{w} = \vec{u} - \text{Proj}_{\vec{v}}\vec{u}$$
   $$= \langle 8,2,0\rangle - \langle 6,3,-3\rangle$$
   $$= \langle 2,-1,-3\rangle$$

The Cross Product

3. Given $\mathbf{u} = \langle 2, -3, 1\rangle$ and $\mathbf{v} = \langle 1, -1, 1\rangle$

   (a) Find $\mathbf{u} \times \mathbf{v}$.

   Solution:

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -1 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} \hat{k}$$

$$= \big((-3)(1) - (-1)(1)\big)\hat{i} - \big((2)(1) - (1)(1)\big)\hat{j} + \big((2)(-1) - (-3)(1)\big)\hat{k}$$

$$= -2\hat{i} - \hat{j} + \hat{k}$$

   (b) Show that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   Solution:

$(\hat{u} \times \hat{v})$ is orthogonal to $\hat{u}$ if $(u \times v) \cdot u = 0$

$$\Rightarrow (\hat{u} \times \hat{v}) \cdot \hat{u} = (-2)(2) + (-1)(-3) + (1)(1)$$
$$= -4 + 3 + 1 = 0 \checkmark$$

So $\hat{u} \times \hat{v}$ is orthogonal to $\hat{u}$

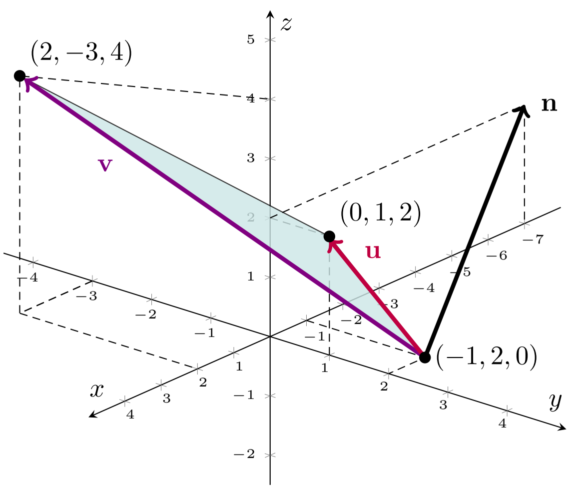
$(\hat{u} \times \hat{v})$ is orthogonal to $\hat{v}$ if $(\hat{u} \times \hat{v}) \cdot \hat{v} = 0$

$$\Rightarrow (\hat{u} \times \hat{v}) \cdot \hat{v} = (-2)(1) + (-1)(-1) + (1)(1)$$
$$= -2 + 1 + 1 = 0 \checkmark$$

So $\hat{u} \times \hat{v}$ is orthogonal to $\hat{v}$

4. Find the area of the triangle with the vertices $(2,-3,4), (0,1,2), (-1,2,0)$. Note that area of a triangle is given by $\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$.

Solution:



Vector btwn $(2,-3,4)$ & $(-1,2,0)$

$$\Rightarrow \vec{v} = \langle 2-(-1), -3-2, 4-0\rangle$$
$$= \langle 3, -5, 4\rangle$$

Vector btwn $(0,1,2)$ & $(-1,2,0)$

$$\Rightarrow \vec{u} = \langle 0-(-1), 1-2, 2-0\rangle$$
$$= \langle 1, -1, 2\rangle$$

We note that

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -5 & 4 \\ 1 & -1 & 2 \end{vmatrix} = \langle 6, 2, -2\rangle$$

and so

$$\|u\times v\| = ((6)^2 + (2)^2 + (-2)^2)^{1/2} = 2\sqrt{11}$$

Therefore the area of the triangle is given by

$$\text{Area} = \frac{1}{2}\|u\times v\| = \frac{1}{2}(2\sqrt{11}) = \sqrt{11}$$