

Lecture #06: Tangent & Normal vectors

Date: Thu. 9/27/18

We learned previously that the Velocity vector is given by

$$\vec{v}(t) = \vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j}$$

Note: just as with real-valued fcn's, the Velocity vector gives the direction of motion.

Def Let C be a smooth curve given by \vec{r} on an interval, The unit tangent vector $\hat{T}(t)$ @ t is defined as

$$\hat{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}, \quad \vec{r}'(t) \neq 0$$

Ex. 1] Find the unit tangent vector $T(t)$ of
 $r(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + t\hat{k}$

Soln.

$$r'(t) = -2\sin(t)\hat{i} + 2\cos(t)\hat{j} + \hat{k}$$

$$\begin{aligned} \|\vec{r}'(t)\| &= [(-2\sin(t))^2 + (2\cos(t))^2 + (1)^2]^{1/2} \\ &= \sqrt{5} \end{aligned}$$

then

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \left\langle -\frac{2\sin(t)}{\sqrt{5}}, \frac{2\cos(t)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

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Ex. 1) Find the parametric eqns of the line tangent to

$$\mathbf{r}(t) = 2\cos(t)\hat{i} + 2\sin(t)\hat{j} + t\hat{k}$$

@ the point $t = \frac{\pi}{4}$

Soln. From Ex. 1 we have

$$\mathbf{T}(t) = \left\langle -\frac{2\sin(t)}{\sqrt{5}}, \frac{2\cos(t)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right) = \left\langle -\frac{2\sin\left(\frac{\pi}{4}\right)}{\sqrt{5}}, \frac{2\cos\left(\frac{\pi}{4}\right)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= \left\langle -\frac{2\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{5}}, \frac{2\left(\frac{\sqrt{2}}{2}\right)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= \left\langle \sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

a pt on the line will be given by

$$\begin{aligned}\mathbf{r}\left(\frac{\pi}{4}\right) &= (2\cos\left(\frac{\pi}{4}\right), 2\sin\left(\frac{\pi}{4}\right), \frac{\pi}{4}) \\ &= (\sqrt{2}, \sqrt{2}, \frac{\pi}{4})\end{aligned}$$

so the parametric eqns of the tangent line are

$$x = x_0 + a\delta = \sqrt{2} - \sqrt{\frac{2}{5}}\delta$$

$$y = y_0 + b\delta = \sqrt{2} + \sqrt{\frac{2}{5}}\delta$$

$$z = z_0 + c\delta = \frac{\pi}{4} + \delta$$

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Normalizing $\vec{T}(t)$ we have the following definition

Def Let C be a smooth curve given by \vec{r} on an interval I . If $T'(t) \neq 0$ then the principal unit normal vector @ t is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

Ex] Find $N(t)$ & $N(1)$ for the curve

$$\vec{r}(t) = 3t\hat{i} + 2t^2\hat{j}$$

Soln.

$$\text{Since } N(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{\frac{d}{dt} \left[\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right]}{\left\| \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \right\|}$$

$$\text{Find } \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$\vec{r}'(t) = 3\hat{i} + 4t\hat{j}$$

$$\|\vec{r}'(t)\| = ((3)^2 + (4t)^2)^{1/2} = (9 + 16t^2)^{1/2}$$

then the unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{(9+16t^2)^{1/2}} (3\hat{i} + 4t\hat{j})$$

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Find $T'(t)$:

$$T'(t) = \frac{d}{dt} [\omega(t) \vec{r}(t)] = \omega(t) \vec{r}'(t) + \omega'(t) \vec{r}(t)$$

$$\omega(t) = (9+16t^2)^{-1/2} \quad \omega'(t) = \frac{1}{2} (9+16t^2)^{-3/2} (36t)$$

$$= 16t (9+16t^2)^{-3/2}$$

$$\vec{r}(t) = 3\hat{i} + 4t\hat{j} \quad \vec{r}'(t) = 4\hat{j}$$

$$\Rightarrow T'(t) = (9+16t^2)^{-1/2} (4\hat{j}) + 16t (9+16t^2)^{-3/2} (3\hat{i} + 4\hat{j})$$

$$= \frac{12}{(9+16t^2)^{3/2}} (-4t\hat{i} + 3\hat{j})$$

$$\Rightarrow \|T(t)\| = \frac{12}{9+16t^2}$$

So the principal Unit normal vector

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = (9+16t^2)^{-1/2} (-4t\hat{i} + 3\hat{j})$$

□

Clearly, $N(t)$ takes a lot of work to calculate

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If it's so difficult to find why do we need it?

When velocity of an object varies then the velocity & accel. vectors are not necessarily perpendicular.

Thm IF $\vec{r}(t)$ is a position vector for a smooth curve C & $N(t)$ exists, then the acceleration vector $\vec{a}(t)$ lies in the plane determined by $T(t)$ & $N(t)$

Pf Let $T(t) = T$, etc

$$\text{Since } T = \frac{\vec{r}'}{\|\vec{r}'\|} = \frac{\vec{v}}{\|\vec{v}\|} \text{ then, } \vec{v} = \|\vec{v}\| \vec{T}$$

Differentiating

$$\begin{aligned} \vec{a} &= \vec{v}' = \frac{d}{dt} [\|\vec{v}\|] \vec{T} + \|\vec{v}\| \vec{T}' \\ &= \frac{d}{dt} [\|\vec{v}\|] \vec{T} + \|\vec{v}\| \vec{T}' \left(\frac{\|\vec{T}'\|}{\|\vec{T}\|} \right) \\ &= \frac{d}{dt} [\|\vec{v}\|] \vec{T} + \|\vec{v}\| \|\vec{T}'\| \vec{N} \end{aligned}$$

This is a linear combination of T & N
 $\Rightarrow \vec{a}$ in plane determined by T & N

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The coeffs. of T & N in the proof are defined as the following

THEOREM 12.5 TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

If $\mathbf{r}(t)$ is the position vector for a smooth curve C [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

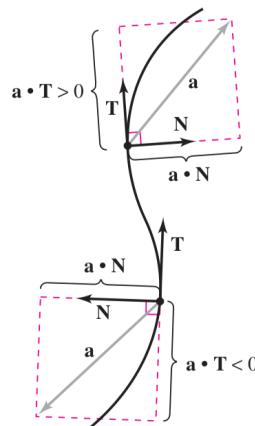
$$a_T = D_t[\|\mathbf{v}\|] = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_N = \|\mathbf{v}\| \|\mathbf{T}'\| = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_T^2}$$

Note that $a_N \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration**.

So the acceleration vector can be written as

$$\mathbf{a}(t) = a_T \mathbf{T}(t) + a_N \mathbf{N}(t).$$



The tangential and normal components of acceleration are obtained by projecting \mathbf{a} onto \mathbf{T} and \mathbf{N} .