Lecture #15: Change of variables & polar coord.s Date: Tue. 11/13/18

In single variable integration, we use a u-sub. to change variables

$$\Rightarrow \int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ & $dx = g'(u)\,du$

with $a = g(c)$, $b = g(d)$

This same idea can be extended to multiple integrals.

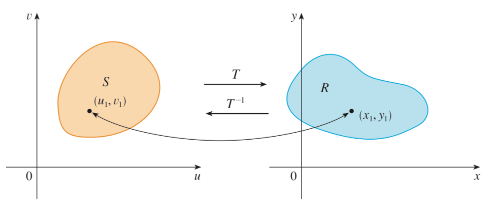
One to one transformations

A change of variables is defined as a 1-1 transformation or mapping.

This mapping takes a region S in a uv-plane and transforms it into a region R in the xy plane by

$$T(u,v) = (x,y) = (g(u,v), h(u,v))$$

Note: $T: \mathbb{R}^2 \to \mathbb{R}^2$



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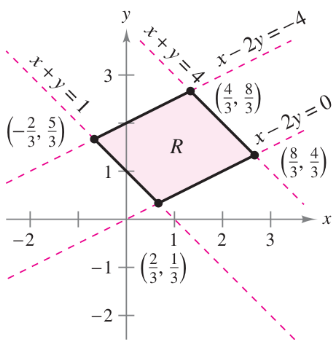
Ex. 1) Let R be the region bdd by

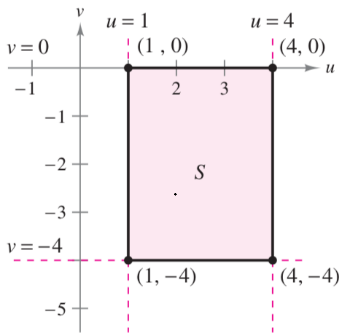
$$x - 2y = 0, \quad x + y = 4$$
$$x - 2y = -4 \quad x + y = 1$$

Let $u = x + y$ & $v = x - 2y$

then our bounds become
$$\begin{array}{lcl} x+y=4 & & u=4 \\ x+y=1 & \Rightarrow & u=1 \\ x-2y=0 & & v=0 \\ x-2y=-4 & & v=-4 \end{array}$$



$\Longrightarrow$



To define the mapping $T(u,v)$ we solve the eqns for x & y

$$\begin{array}{l} u = x + y \\ v = x - 2y \end{array} \Longrightarrow \begin{array}{l} y = \frac{1}{3}(u - v) \\ x = \frac{1}{3}(2u + v) \end{array}$$

So $T(u,v) = (g(u,v), h(u,v)) = \left(\frac{1}{3}(2u+v), \frac{1}{3}(u-v)\right)$

As a check you can eval vertices in uv-space.

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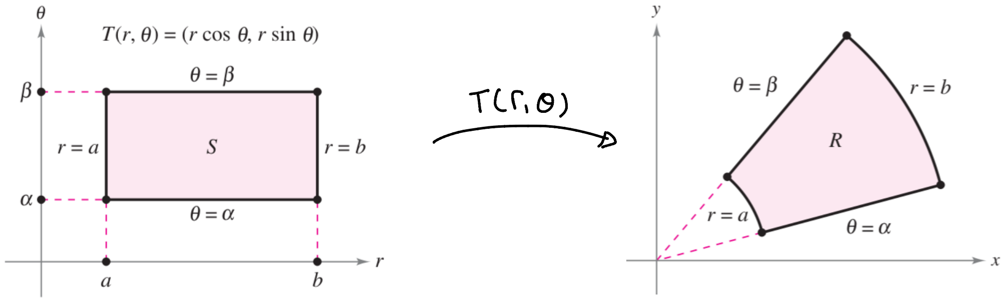
Polar Coordinates

The most common change of variables is using polar coord.s

We take a pt p in xy & maps it to the length of the vector from the origin to p & the angle θ btwn vector & the x-axis.



As a transformation polar coord.s are given by

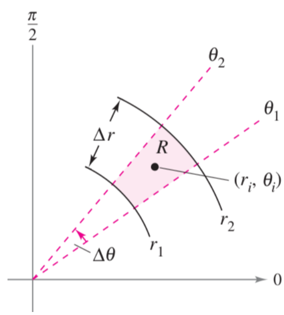
$$T(r,\theta) = (x,y) = (r\cos\theta,\ r\sin\theta)$$

$$\Rightarrow \boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Also:

$$\boxed{r^2 = x^2 + y^2}$$

$$\boxed{\tan\theta = \frac{y}{x}}$$



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The Jacobian

DEFINITION OF THE JACOBIAN

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. = J(u,v)$$

Ex. 2) For the transformation

$$T: \; x = r\cos\theta \quad \& \quad y = r\sin\theta$$

we have

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = \cos\theta$$

so

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Note: The determinant of a 2 by 2 system is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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Change of Variables (for double Integrals)

Thm Let $T: x = g(u,v)$ & $y = h(u,v)$ be a 1-1 transformation. If f is cont. then

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v))\,\underbrace{|J(u,v)|}_{\text{Jacobian}}\,du\,dv$$

Ex. 3| Evaluate $\iint_R 3xy\,dA$

where R is the region bdd by $x - 2y = 0$, $x+y=4$
$x - 2y = 4$, $x+y=1$

For this region (from Ex. 1) we have that

$$T: x = \frac{1}{3}(2u+v) \quad \& \quad y = \frac{1}{3}(u-v)$$

To rewrite the integral, we first need to find the Jacobian

Partial Deriv.s: $\frac{\partial x}{\partial u} = \frac{2}{3}$, $\frac{\partial y}{\partial u} = \frac{1}{3}$

$\frac{\partial x}{\partial v} = \frac{1}{3}$, $\frac{\partial y}{\partial v} = -\frac{1}{3}$

Jacobian:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(-\frac{1}{3}\right) = -\frac{1}{3}$$

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Ex. 3 (cont'd)

For $f(x,y) = 3xy$: $f(g(u,v), h(u,v)) = f\left(\frac{1}{3}(2u+v), \frac{1}{3}(u-v)\right)$

$$= 3\left[\tfrac{1}{3}(2u+v)\right]\left[\tfrac{1}{3}(u-v)\right]$$

$$= \tfrac{1}{3}(2u^2 - uv - v^2)$$

The change of variables is given by

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v), h(u,v))\,|J(u,v)|\,du\,dv$$

So our integral becomes

$$\iint_R 3xy\,dx\,dy = \int_1^4 \int_{-4}^0 \tfrac{1}{3}(2u^2 - uv - v^2)\left|-\tfrac{1}{3}\right|\,du\,dv$$

Evaluating the integral:

$$\iint_R 3xy\,dx\,dy = \int_1^4 \int_{-4}^0 \tfrac{1}{3}(2u^2 - uv - v^2)\left|-\tfrac{1}{3}\right|\,du\,dv$$

$$= \frac{1}{9}\int_1^4 \left[2u^2v - \frac{u}{2}v^2 - \frac{v^3}{3}\right]_{v=-4}^{v=0} du$$

$$= \frac{1}{9}\int_1^4 \left(8u^2 + 8u - \frac{64}{3}\right) du$$

$$= \frac{1}{9}\left[\frac{8}{3}u^3 + 4u^2 - \frac{64}{3}u\right]_{u=1}^{u=-4}$$

$$= \frac{164}{9}$$

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Change of Variables (using Polar coord.s)

Thm. Let f be cont. on region of xy space

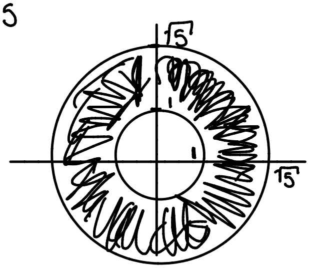
$$R = \{(r,\theta) : a \le r \le b \ \& \ \alpha \le \theta \le \beta\}$$

where $\beta - \alpha \le 2\pi$. Then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r,\theta)\, r\, dr\, d\theta$$

where T: $x = r\cos\theta$ & $y = r\sin\theta$ & $J(r,\theta) = r$

Ex. 4| Let R be annular region btwn circles $x^2 + y^2 = 1$ & $x^2 + y^2 = 5$



Evaluate $\iint_R (x^2 + y)\,dA$

Soln use Polar coord.s

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Bounds: $1 \le r \le \sqrt{5}$ & $0 \le \theta \le 2\pi$

For $f(x,y) = x^2 + y$

$$\Rightarrow f(r\cos\theta, r\sin\theta) = (r\cos\theta)^2 + r\sin\theta = r^2\cos^2\theta + r\sin\theta$$

so the integral becomes

$$\iint_R (x^2 + y)\,dA = \int_0^{2\pi} \int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta)\, r\, dr\, d\theta$$

$= $ (left as an exercise)

$= 6\pi$

Math

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Ex. Let R be region bdd w/ vertices $(0,1), (1,2), (2,1), (1,0)$

Evaluate $\displaystyle\iint_R (x+y)^2 \sin^2(x-y)\,dA$