Math 2210

Lecture # 13: Double Integrals

Date: Tue. 11/6/18

Area over Rectangles

In single variable calculus what did we use integration for?

$\Rightarrow$ Area under a curve

$\Rightarrow$ Area btwn 2 curves

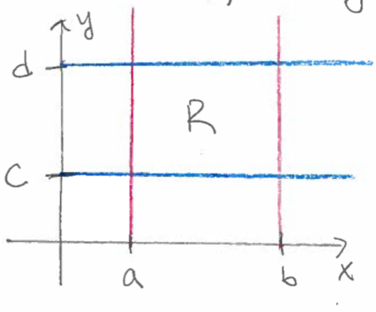
$\Rightarrow$ Vol. of surface of Revolution

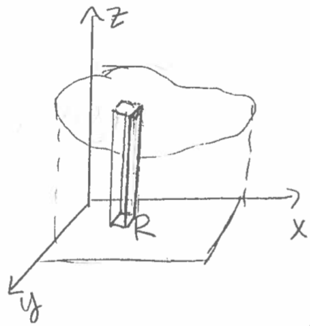
$\Rightarrow$ Arc Length

$\Rightarrow$ Surface Area.

Sum of rectangles

Consider a fcn $z = f(x,y)$ on a closed Region R where R is the rectangle in the xy plane $a \le x \le b$, $c \le y \le d$ or $R: [a,b] \times [c,d]$





$$R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$$

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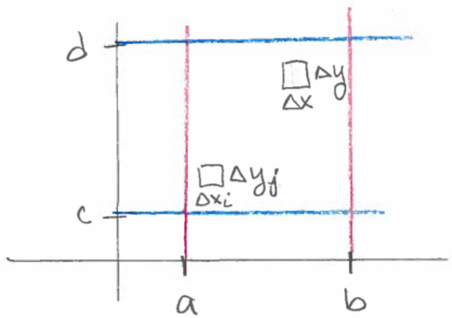
Let S be the solid generated by $z=f(x,y)$ & R.

To approx value of S we can break R into small rectangles of length Δx & width Δy.

This a <u>partition</u> of R



Let R_{ij} be a rectangle of my partition:

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

If height inside of R_{ij} of my rectangle is some sample pt $f(\tilde{x}_i, \tilde{y}_j)$ then approx vol. of my surface is

$$V \approx \sum_{i=1}^{n} \sum_{j=1} f(\tilde{x}_i, \tilde{y}_j) \Delta x \Delta y$$

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Double Integrals

Taking limit as $n \to \infty$ $(\Delta x, \Delta y \to 0)$ then this gives the <u>double integral</u>

<u>Def</u> The double integral of $F(x,y)$ over region R is

$$\iint_R F(x,y)\,dx\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1} F(\tilde{x}_i, \tilde{y}_j)\,\Delta x \Delta y$$

provided the limit exists.

Properties of Double Integrals

1) $\iint_R cF(x,y)\,dx\,dy = c\iint_R F(x,y)\,dx\,dy$

   where $c \in \mathbb{R}$

2) $\iint_R (F(x,y) + g(x,y))\,dx\,dy = \iint_R F(x,y)\,dx\,dy + \iint_R g(x,y)\,dx\,dy$

3) If $F(x,y) \geq g(x,y)$ $\forall x,y \in R$ then

$$\iint_R F(x,y)\,dx\,dy \geq \iint_R g(x,y)\,dx\,dy$$

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Partial Integration

Recall from Calc I/II:

$$\int F'(x)\,dx = F(x) + C$$ ← cst of integration

For partial derivatives:

$$\int f_x(x,y)\,dx = f(x,y) + C(y)$$

where $C(y)$ is cst. wrt x

Ex| If $f_x(x,y) = 2xy$

then
$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx \\ &= \int 2xy\,dx \\ &= 2y\int x\,dx \\ &= 2y\left[\frac{x^2}{2}\right] + C(y) \\ &= yx^2 + C(y) \end{aligned}$$

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$C(y)$ & $C(x)$ are csts of int. so FTOC still holds.

From Calc I/II:

$$\int_a^b f'(x)\,dx = F(b) - F(a)$$

For dble Int's:

$$\int_{g_1(y)}^{g_2(y)} F_x(x,y)\,dx = F(g_2(y),y) - F(g_1(y),y)$$

equiv. for int. wrt y.

Note:

If we integrate a fcn $f(x,y)$ wrt y, we get a fcn only of x

$$A(x) = \int_c^d f(x,y)\,dy$$

then integrating wrt x we get a constant:

$$\int_a^b A(x)\,dx = \underbrace{\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx}_{\text{iterated integral}}$$

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Special Cases

(a) If bds of iterated integral are constant we can (usually) change the order of integration

<u>Thm</u> If f is cont. on rectangular region R; $R = [a,b] \times [c,d]$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ex.

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_1^2 y\left[\int_0^3 x^2\,dx\right]dy$$
$$= \int_1^2 y\left[\frac{x^3}{3}\Big|_0^3\right]dy$$
$$= \int_1^2 y\,(9)\,dy$$
$$= 9\left[\frac{y^2}{2}\right]_1^2 = 9\left(\frac{3}{2}\right) = \frac{27}{2}$$

(b) If $f(x,y) = g(x)h(y)$ then

$$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

Ex.

$$\int_1^2 \int_0^3 x^2 y\,dx\,dy = \int_0^3 x^2\,dx \int_2^1 y\,dy$$

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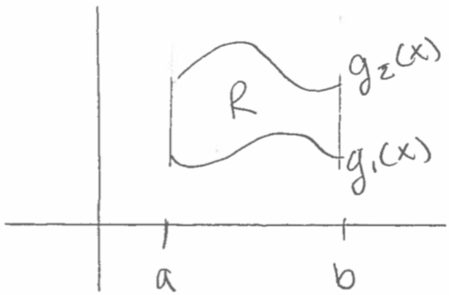
Double Integrals over General Regions

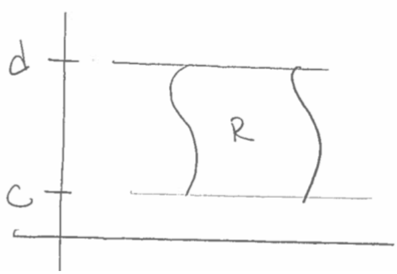
Thm Let R be a region bdd above & below by graphs of 2 cont. fcns $y=g_2(x)$ & $y=g_1(x)$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



or if bdd on left & right by fcns $x=h_1(y)$, $x=h_2(y)$ & lines c & d then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



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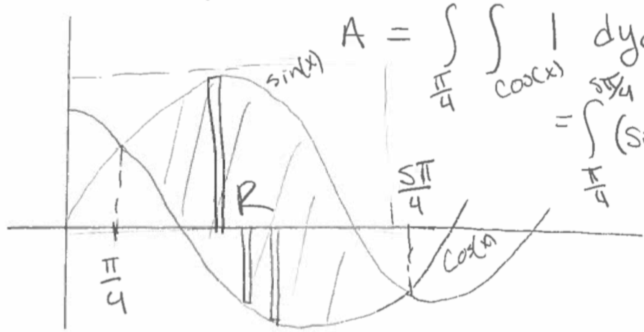
Ex) Find area of region bdd by

$f(x) = \sin(x)$, $g(x) = \cos(x)$

on $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$



$$A = \int_{\pi/4}^{5\pi/4} \int_{\cos(x)}^{\sin(x)} 1 \, dy\,dx$$

$$= \int_{\pi/4}^{5\pi/4} (\sin(x) - \cos(x))\,dx$$

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Volume of Region btwn 2 Surfaces

Another application of double integrals is finding the volume between two surfaces.

Volume of Region btwn 2 Surfaces

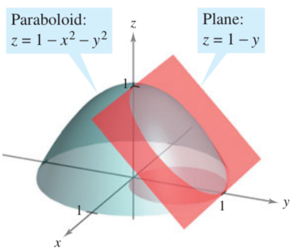
$$V = \iint_R [f(x,y) - g(x,y)]\,dA$$

where $z = f(x,y)$ & $z = g(x,y)$ defines 2 surfaces

Ex. Find the volume of the solid region bdd above by the paraboloid $z = 1 - x^2 - y^2$ & below by the plane $z = 1 - y$

Soln We find the intersection of the 2 surfaces:

$$\begin{aligned} z &= 1 - x^2 - y^2 \\ z &= 1 - y \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 1 - y &= 1 - x^2 - y^2 \\ x^2 &= y - y^2 \\ x &= \pm\sqrt{y - y^2} \end{aligned}$$



R: $0 \le y \le 1$
$-\sqrt{y - y^2} \le x \le \sqrt{y - y^2}$

Volume of region given by

$$V = \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} [1 - x^2 - y^2] - [1 - y]\,dx\,dy$$

$$= \frac{\pi}{32}$$