Lecture # 03: Lines & planes in Space

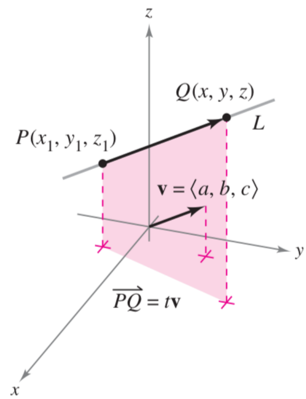
Date: Tue. 9/18/18

Lines in Space

In 2D (i.e. in the plane) slope is used to find the equation of a line.

In space (i.e. in $\mathbb{R}^3$) we instead use vectors.

Consider the line L thru the point $P(x_1, y_1, z_1)$ & parallel to the vector $\vec{v} = \langle a, b, c \rangle$.



Line L and its direction vector $\mathbf{v}$

$\vec{v}$ is the <u>direction vector</u> for L

a, b, c are the <u>direction numbers.</u>

The line L consists of all points $P(x_1, y_1, z_1)$ & $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\vec{v}$

i.e. $\overrightarrow{PQ}$ is a scalar multiple of $\vec{v}$. i.e.

$$\overrightarrow{PQ} = t\vec{v}, \quad t \text{ is a scalar}$$

$$\Rightarrow \overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle$$

$$= \langle at, bt, ct \rangle$$

$$= t\vec{v}$$

Equating components:

$$x - x_1 = at \quad \Rightarrow \quad x = x_1 + at$$
$$y - y_1 = bt \qquad\qquad y = y_1 + bt$$
$$z - z_1 = ct \qquad\qquad z = z_1 + ct$$

<u>parametric Equations</u>

If we instead solve for t:

$$t = \frac{x - x_1}{a} \qquad t = \frac{y - y_1}{b} \qquad t = \frac{z - z_1}{c}$$

which give the symmetric equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \quad \text{for } a, b, c \neq 0$$

<u>Ex1</u> Find the parametric & symmetric eqns of the line L thru $(1, -2, 4)$ & parallel to $\vec{v} = \langle 2, 4, -4 \rangle$

x_1, y_1, z_1 $\qquad$ a, b, c

$x_1 = 1 \qquad a = 2$

$y_1 = -2 \qquad b = 4$

$z_1 = 4 \qquad c = -4$

parametric Eqns: $x = x_1 + at = 1 + 2t$

$y = y_1 + bt = -2 + 4t$

$z = z_1 + ct = 4 - 4t$

Symmetric Eqns: (since a, b, c are nonzero)

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}$$

□

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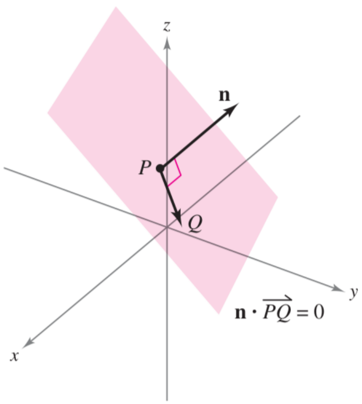
Planes in Space

The equation of a plane in space can be found with a point in the plane and a vector normal to it.

<u>Def</u> a vector $\vec{n}$ is <u>normal</u> to a vector $\vec{v}$ if $\vec{n}$ is perpendicular to $\vec{v}$.



The normal vector **n** is orthogonal to each vector $\overrightarrow{PQ}$ in the plane.

The plane that contains $P(x_1, y_1, z_1)$ w/ nonzero normal vector $\vec{n}$.

This plane will contain all points $Q(x, y, z)$ for which $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$

Using the dot product:

$$\vec{n} \cdot \overrightarrow{PQ} = 0$$

$$\langle a, b, c\rangle \cdot \langle x - x_1, y - y_1, z - z_1\rangle = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

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Def The plane containing pt. (x_1, y_1, z_1) with normal vector $\vec{n} = \langle a, b, c \rangle$ in

<u>standard Form</u>

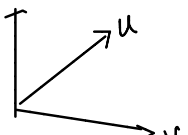
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

<u>General Form</u>

$$ax + by + cz + d = 0$$

Ex The plane that contains the point $(2, 1, 1)$ and the vectors

$\vec{n} = \vec{u} \times \vec{v}$



$$\vec{u} = \langle -2, 3, 0 \rangle \quad \text{and}$$
$$\vec{v} = \langle -4, 0, 3 \rangle$$

Use the cross product to find the normal vector $\vec{n}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix} = 9i + 6j + 12k = \langle 9, 6, 12 \rangle$$

Eqn of a plane is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$\Rightarrow 9(x - 2) + 6(y - 1) + 12(z - 1) = 0$$
$$\text{or} \quad 3x + 2y + 4z - 12 = 0$$

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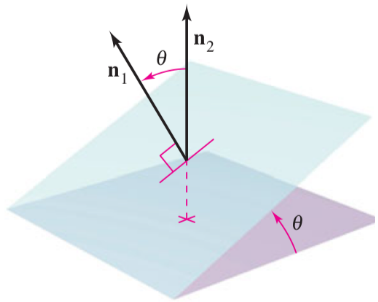
Two distinct planes in $\mathbb{R}^3$ are either parallel or intersect in a line.



The angle θ between two planes

If two planes intersect the angle btwn the normal vectors is the same as the angle btwn the two planes. This angle is given by

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$

Note: two planes w/ normal vectors $\vec{n}_1$ & $\vec{n}_2$ are

- perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$
- parallel if $\vec{n}_1$ is a scalar multiple of $\vec{n}_2$

Ex 1 Find the line of intersection of 2 planes given by

$$x - 2y + z = 0$$
$$2x + 3y - 2z = 0$$

We need to find the set of points that simultaneously satisfy both equations.

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Ex (cont'd)

To find the line of intersection we solve the system for x, y, z

$$x - 2y + z = 0 \quad (1)$$
$$2x + 3y - 2z = 0 \quad (2)$$

multiplying Eqn (1) & add to (2)

$$2x + 3y - 2z = 0$$
$$-2x + 4y - 2z = 0$$
$$7y - 4z = 0$$

Solving for y: $\Rightarrow y = \frac{4z}{7}$

Solving for x: $x - 2\left(\frac{4}{7}z\right) + z = 0$

$$x - \frac{8}{7}z + z = 0$$
$$x - \frac{1}{7}z = 0$$
$$\Rightarrow x = \frac{1}{7}z$$

Since both x & y can be found in terms of z then we let $z = t$

Parametric Eqns are

$$x = \frac{1}{7}t, \quad y = \frac{4}{7}t, \quad z = t$$

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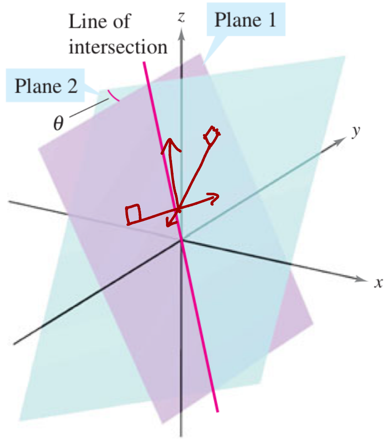
Note:
The cross product of the normal vectors of two intersecting planes is parallel to their line of intersection.



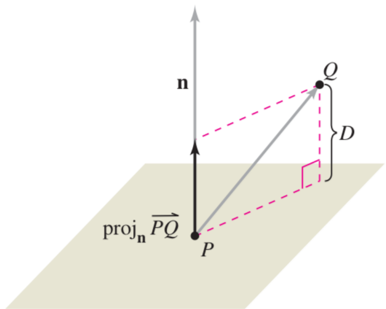
Distances Btwn Pts, Planes & Lines

THEOREM 11.13 DISTANCE BETWEEN A POINT AND A PLANE

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ}\cdot\mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.



$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\|$

The distance between a point and a plane

The distance btwn the pt $Q(x_0, y_0, z_0)$ & the plane $ax + by + cz + d = 0$ is also given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

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Ex | Find the distance btwn the pt $Q(1,5,-4)$ & the plane

$$3x - y + 2z = 6$$

Vector normal to the plane $\vec{n} = \langle 3,-1,2 \rangle$

Find a pt in the plane:

Let $y = 0$ & $x = 0$

$\Rightarrow 3x - 0 + 2(0) = 6$

$\Rightarrow x = 2$

So pt $(2, 0, 0)$ is in the plane

Find vector $\vec{PQ} = \langle 1-2, 5-0, -4-0 \rangle$
$= \langle -1, 5, -4 \rangle$

$$D = \frac{|\vec{PQ} \cdot \vec{n}|}{\|n\|}$$

$$= \frac{|\langle -1, 5, -4 \rangle \cdot \langle 3,-1,2 \rangle|}{\sqrt{(3)^2 + (-1)^2 + (2)^2}}$$

$$= \frac{|-3 - 5 - 8|}{\sqrt{14}}$$

$$= \frac{16}{\sqrt{14}}$$

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Recall:

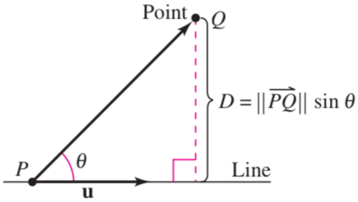
$$\|u \times v\| = \|u\| \|v\| \sin\theta$$

THEOREM 11.14 DISTANCE BETWEEN A POINT AND A LINE IN SPACE

The distance between a point Q and a line in space is given by

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\|\overrightarrow{PQ}\| \cancel{\|u\|} \sin\theta}{\cancel{\|u\|}} = \|\overrightarrow{PQ}\| \sin\theta$$

where $\mathbf{u}$ is a direction vector for the line and P is a point on the line.



The distance between a point and a line

Ex) Find the distance btwn pt $Q(3,-1,4)$ & the line given by

$$x = -2 + 3t$$
$$y = -2t$$
$$z = 1 + 4t$$

Direction vector: $\vec{v} = \langle 3, -2, 4 \rangle$

For $t = 0$: Pt is $P(-2, 0, 1)$

$\overrightarrow{PQ} = \langle 3+2,\ 0-1,\ 4-1 \rangle = \langle 5, -1, 3 \rangle$

$$\overrightarrow{PQ} \times \vec{v} = \begin{vmatrix} i & j & k \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = \langle 2, -11, -7 \rangle$$

$$D = \frac{\|\overrightarrow{PQ} \times \vec{v}\|}{\|\vec{v}\|} = \sqrt{6}$$