1. Decide if each statement is true or false. If the statement is true, explain why. If the statement is false, provide a counter example. Solutions without appropriate justification will receive no credit.

   (a) If $\hat{\mathbf{u}}$ is tangent to the level curve of f that passes through $(1, 2)$ then $D_{\hat{\mathbf{u}}}f(1, 2) = 0$.

   Solution: TRUE: The statement is true because the gradient vector ∇f is perpendicular to level curves like $f(x, y) = k$. So $D_{\hat{\mathbf{u}}}f(1, 2) = \nabla f(1, 2) \cdot \hat{\mathbf{u}} = 0$ if $\hat{\mathbf{u}}$ is perpendicular to ∇f.

   (b) If $f_x(1, 2) = -3$ and $f_y(1, 2) = 4$ then $-3 \leq D_{\hat{\mathbf{u}}}f(1, 2) \leq 4$.

   Solution: FALSE: The statement is false because if $\hat{\mathbf{u}} = -\hat{\jmath}$ then
   $$D_{\hat{\mathbf{u}}}f(1, 2) = -4.$$
   Of course there are many other possible counter examples and reasons.

2. Calculate the directional derivative of $f(x, y) = \sqrt{\sin(x) + y^2}$ at the point $P(0, 2)$ in the direction of the origin.

   Solution: Note that
   $$\nabla f = \left\langle \frac{\cos(x)}{2\sqrt{\sin(x) + y^2}}, \frac{y}{\sqrt{\sin(x) + y^2}} \right\rangle,$$
   Since $\overrightarrow{PO} = \langle 0, -2\rangle$, has length $|\overrightarrow{PO}| = \sqrt{0^2 + (-2)^2} = 2$ the unit vector in the direction of $\overrightarrow{PO}$ is
   $$\hat{\mathbf{u}} = \frac{\overrightarrow{PO}}{|\overrightarrow{PO}|} = \frac{\langle 0, -2\rangle}{2} = \langle 0, -1\rangle.$$
   Thus, the directional derivative is
   $$D_{\langle 0,-1\rangle}f(0, 2) = \langle \frac{1}{4}, 1\rangle \cdot \langle 0, -1\rangle = -1.$$

3. Find the points on the surface $xy+z^2 = 1$ where the tangent plane is parallel to the plane $3x + y + 2z = 5$. Hint: the surface $xy + z^2 = 1$ is a level surface for $f(x, y, z) = xy + z^2$.

Solution: Because ∇f is perpendicular to level surfaces, we look for points on the surface where $\nabla f = \langle y, x, 2z\rangle$ is parallel to the plane's normal vector $\langle 3, 1, 2\rangle$. If two vectors are parallel they are multiples of one another:

$$\langle y, x, 2z\rangle = \lambda\langle 3, 1, 2\rangle.$$

Solving for x, y and z in terms of λ we find that

$$x = \lambda, y = 3\lambda, \quad \text{and } z = \lambda$$

Noting that the point (x, y, z) lies on the surface, $xy + z^2 = 1$. We require that

$$\begin{aligned}(\lambda)(3\lambda) + (\lambda)^2 &= 1\\ 4\lambda^2 &= 1\\ \lambda &= \pm\sqrt{\frac{1}{4}}\\ \lambda &= \pm\frac{1}{2}.\end{aligned}$$

If $\lambda = \frac{1}{2}$ then $(x, y, z) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$ and if $\lambda = -\frac{1}{2}$ then $(x, y, z) = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$.

4. Find parametric equations for the line that is normal to the hyperboloid

$$x^2 + y^2 - z^2 = -4$$

at the point $P_0(1, 2, 3)$ Hint: the hyperboloid is a level surface for $f(x, y, z) = x^2+y^2-z^2$.

Solution: To define a line we need a point and a direction vector. The point is obviously $P_0(1, 2, 3)$ and the direction vector must be perpendicular to the hyperboloid at P_0.

$P_0(1, 2, 3)$ is on the hyperboloid and the hyperboloid is a level surface for the function $f(x, y, z) = x^2 + y^2 - z^2$. So,

$$\nabla f(1, 2, 3) = \langle 2, 4, -6\rangle = 2\langle 1, 2, -3\rangle$$

is perpendicular to the hyperboloid.

The parametric equations of the normal line are

$$\begin{aligned} x &= 1 + t \\ y &= 2 + 2t \\ z &= 3 - 3t \end{aligned}$$