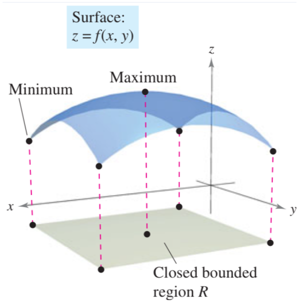
Lecture # 11: Extrema of 2 Variable Fcns Date: Tue. 10/23/18

Just as with single variable fc.s we have the concept of absolute max/min values.



R contains point(s) at which $f(x, y)$ is a minimum and point(s) at which $f(x, y)$ is a maximum.

THEOREM 13.15 EXTREME VALUE THEOREM

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.

We also have relative (or local) max/min values

DEFINITION OF RELATIVE EXTREMA

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

Locate max/min values of f Find critical pts

DEFINITION OF CRITICAL POINT

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

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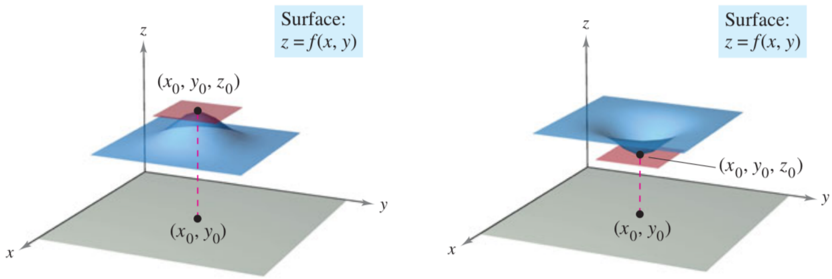
If f is diff'ble and

$$\nabla f(x_0, y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle$$
$$= \langle 0, 0 \rangle \qquad D_{\hat{u}} \nabla f(x,y) \cdot \vec{u} = 0$$

then every directional deriv. $= 0$

$\Rightarrow$ there is a horiz. tangent plane @ (x_0, y_0)



Relative maximum Relative minimum

THEOREM 13.16 RELATIVE EXTREMA OCCUR ONLY AT CRITICAL POINTS

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

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Ex. 1 Find extrema of

$$f(x,y) = 2x^2 + y^2 + 8x - 6y + 20$$

Soln. Find critical pts

$$f_x(x,y) = 4x + 8 \implies 4x + 8 = 0 \implies x = -2$$

$$f_y(x,y) = 2y - 6 \implies 2y - 6 = 0 \implies y = 3$$

So the critical pt is $(-2, 3)$

By completing the square:

$$f(x,y) = 2(x^2 + 4x + 4 - 4) + (y^2 - 6y + 9 - 9) + 20$$

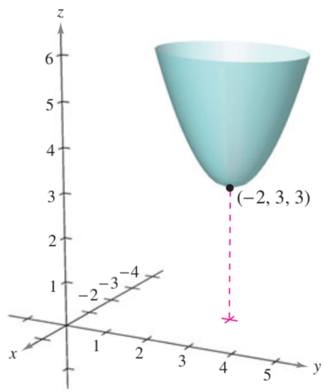
$$= 2(x+2)^2 + (y-3)^2 + 3$$

this is an Elliptic Paraboloid

$\implies f(x,y) > 3$

So $f(-2, 3) = 3$

is a relative minimum



Ex.2) Find extrema of

$$f(x,y) = 1-(x^2+y^2)^{1/3}$$

Soln. Find critical pts:

$$f_x(x,y) = \frac{-2x}{3(x^2+y^2)^{2/3}}$$

$$f_y(x,y) = \frac{-2y}{3(x^2+y^2)^{2/3}}$$

Both partial derivatives do not exist @ (0,0)

Since

$$f(x,y) = 1-(x^2+y^2)^{1/3} < 1$$

then (0,0) is a relative maximum. ∎

Not all critical points yield rel. max/min values

Def. A critical point that does not yield a rel. max/min is a <u>saddle point</u>.

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2nd Partials Test

There will be cases for which we cannot easily determine whether a critical pt. is a rel. max or min value.

Luckily, there is a version of the 2nd derivative test for multivar fcns

THEOREM 13.17 SECOND PARTIALS TEST

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.

However, just as w/single variable fcns this test will not work in all cases.

The 2nd partials test can fail if either partial derivatives do not exist, or if $d = 0$.

The value of d is given by the determinant

$$d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)\,f_{xy}(a,b)$$

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Ex. 3| Find extrema of

$$f(x,y) = -x^3 + 4xy - 2y^2 + 1$$

Soln.

Find critical pts:

$f_x(x,y) = -3x^2 + 4y$

$f_y(x,y) = 4x - 4y \Rightarrow 4x - 4y = 0 \Rightarrow x = y$

$-3y^2 + 4y = 0$

$\Rightarrow y(4 - 3y) = 0 \Rightarrow y = 0, \frac{4}{3}$

Critical pts: $(0,0)$ & $(\frac{4}{3}, \frac{4}{3})$

Find 2^{nd} partials:

$f_{xx} = -6x, \quad f_{yy} = -4, \quad f_{xy} = 4$

Apply 2^{nd} partials test:

For $(0,0)$: $f_{xx}(0,0) = 0, \quad f_{yy}(0,0) = -4, \quad f_{xy}(0,0) = 4$

$\Rightarrow d = f_{xx} \cdot f_{yy} - f_{xy}^2 = (0)(4) - (4)^2 = -16 < 0$

$\Rightarrow d < 0$ so $(0,0,1)$ is a saddle pt.

For $(\frac{4}{3}, \frac{4}{3})$: $f_{xx}(\frac{4}{3}, \frac{4}{3}) = -8, \quad f_{yy}(\frac{4}{3}, \frac{4}{3}) = -4, \quad f_{xy}(\frac{4}{3}, \frac{4}{3}) = 4$

$\Rightarrow d = f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(-4) - 16 = 16 > 0$

Since $d > 0$ & $f_{xx}(\frac{4}{3}, \frac{4}{3}) < 0$ the pt. $(\frac{4}{3}, \frac{4}{3})$ is a rel. max.

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