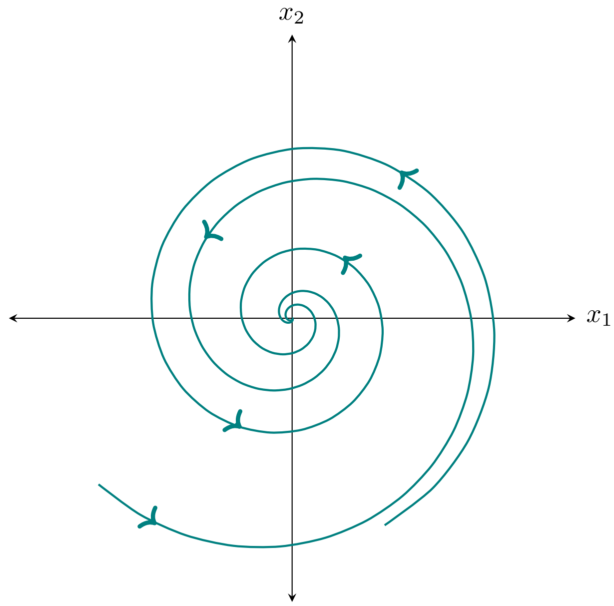
1. (20 pts) Find the general solution of the system of equations

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$$

and draw a phase portrait for the system on the axes provided to the right.



Solution: Since $\lambda_{1,2} = -1 \pm 2i$ which has the form $a \pm bi$. Since $a < 0$, i.e. the real part is negative, the equilibrium solution is a stable spiral.

Solution: First we note the coefficient matrix is $A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$.

Find Eigenvalues

To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - (1)(-4) = \lambda^2 + 2\lambda + 5 = 0$$

Using the quadratic formula we find that the eigenvalues are

$$\lambda_1 = -1 - 2i, \quad \lambda_2 = -1 + 2i$$

Now we find the eigenvectors corresponding to each eigenvalue.

Find Eigenvectors

For $\lambda_1 = -1 - 2i$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We note that

$$A-(-1-2i)I = \begin{bmatrix} 2-(-1-2i) & -1 \\ 3 & -2-(-1-2i) \end{bmatrix} = \begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \left[\begin{array}{cc:c} 2i & -4 & 0 \\ 1 & 2i & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{cc:c} 1 & 2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1 + 2i\xi_2 = 0 \qquad \Longrightarrow \xi_1 = -2i\xi_2$$

So the eigenvector corresponding to λ_1 is

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 2i \\ -1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\xi}^{(1)} = \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{\alpha}} + i \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\boldsymbol{\beta}}$$

Since $\lambda_{1,2} = -1 \pm 2i$ then we know that $\boldsymbol{\xi}^{(2)} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$ We divide the eigenvector(s) into their real and imaginary parts. In other words,

$$\boldsymbol{\xi}^{(1,2)} = \begin{bmatrix} \pm 2i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{\alpha}} \pm i \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\boldsymbol{\beta}}$$

So our eigenvalue is of the form $a + ib$ and our eigenvector has real and imaginary parts $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

The general solution to the system will have the form

$$\mathbf{x}(t) = c_1\mathbf{u}(t) + c_2\mathbf{u}(t)$$

where

$$\mathbf{u}(t) = e^{at}\left(\boldsymbol{\alpha}\cos(bt) - \boldsymbol{\beta}\sin(bt)\right)$$
$$\mathbf{v}(t) = e^{at}\left(\boldsymbol{\alpha}\cos(bt) + \boldsymbol{\beta}\sin(bt)\right)$$

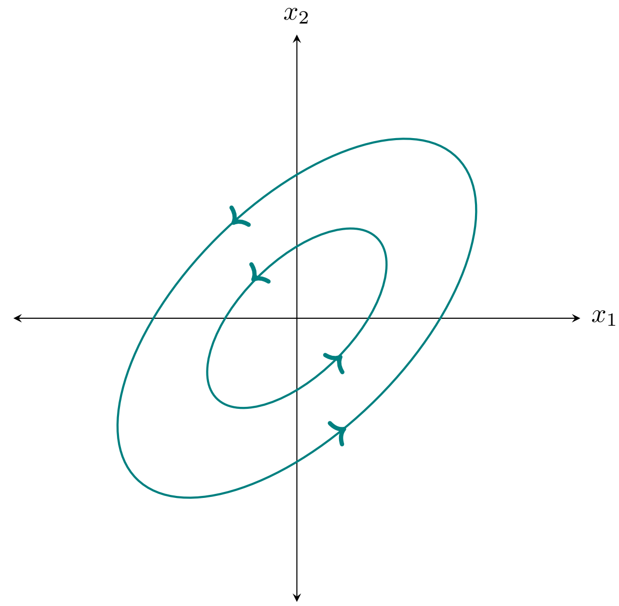
So the general solution to the system is

$$\mathbf{x}(t) = c_1\left(e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right) + c_2\left(e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(2t)\right)$$
$$= e^{-t}\left(c_1\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \end{bmatrix} + c_2\begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}\right)$$

2. (20 pts) Find the general solution of the system of equations

$$\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}$$

and draw a phase portrait for the system on the axes provided to the right.



Solution: Since our eigenvalues are $\lambda_{1,2} = \pm i$ they are purely imaginary. So our equilibrium point is a center. We determine the direction of the trajectories by choosing a point and plugging it into our system. Choosing $\mathbf{x} = [0, 1]^\top$

$$\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

Since this vector points downward, the trajectories must move in the counter-clockwise direction.