Section Summary

Some Definitions

- A **2nd order linear differential equation** has the general form

$$P(t)y'' + Q(t)y' + R(t)y = g(t)$$

with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

- A **homogeneous** 2nd order linear differential equation is when $g(t) = 0$. i.e.

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

otherwise it is known as an **inhomogeneous** equation.

- A 2nd order ODE with **constant coefficients** is when $P(t)$, $Q(t)$, $R(t)$ are constants and has general form

$$ay'' + by' + cy = 0 \tag{1}$$

- The **characteristic equation** is found by substituting $y'' = r^2$, $y' = r\,y = 1$ and has general form

$$ar^2 + br + c = 0 \tag{2}$$

with roots r_1 and r_2

Now we find solutions to equations with constant coefficients by finding the roots of the characteristic equation. Since the charactersitic equation is a quadratic polynomial, the usual methods for finding roots are used (i.e. factoring, completing the square, quadratic formula, etc).

Recall that there are 3 cases for the roots r_1, r_2 of quadratics. The table below summarizes the general solution for the ode in each case

	Case		In other words the roots are	General Solution
1.	$r_1 \neq r_2,$	$r_1, r_2 \in \mathbb{R}$	"real and different"	$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
2.	$r_1 = r_2,$	$r_1, r_2 \in \mathbb{R}$	"real and repeated"	$y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$
3.	$r_1, r_2 = \lambda \pm i\mu,$	$\lambda, \mu \in \mathbb{R}$	"complex"	$y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$

1. (20 pts) Solve the initial value problem

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Solution: This equation has the general form (1) so it is a 2nd order homogeneous equation with constant coefficients. We assume our solution will have the form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where r_1 and r_2 are the roots of the characteristic equation $ar^2 + br + c = 0$. So

$$\begin{aligned} r^2 + r - 2 &= 0 \\ (r-1)(r+2) &= 0 \\ &\implies r_1 = 1, \quad r_2 = -2 \end{aligned}$$

So our general solution is

$$y = c_1 e^t + c_2 e^{-2t}$$

For 2nd order equations we have two initial conditions. In order to use the second initial condition we will also need y'. Differentiating our general solution y yields

$$y' = c_1 e^t - 2c_2 e^{-2t}$$

Our initial conditions are $y(0) = 1$ and $y'(0) = 1$. Substituting these in to y and y' yields the system of equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - 2c_2 &= 1 \end{aligned}$$

Solving this system we find $c_1 = 1$ and $c_2 = 0$. So our particular solution is

$$y = e^t$$

2. (20 pts) Solve the IVP and describe the solution's behavior as t increases.

$$6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Solution: This equation has the general form (1) so it is a 2nd order homogeneous equation with constant coefficients. We assume our solution will have the form

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where r_1 and r_2 are the roots of the characteristic equation $ar^2 + br + c = 0$. So

$$\begin{aligned} 6r^2 - 5r + 1 &= 0 \\ (3r-1)(2r-1) &= 0 \\ &\implies r_1 = \frac{1}{3}, \quad r_2 = \frac{1}{2} \end{aligned}$$

So our general solution is

$$y = c_1 e^{t/2} + c_2 e^{t/3}$$

For 2nd order equations we have two initial conditions. In order to use the second initial condition we will also need y'. Differentiating our general solution y yields

$$y' = \frac{1}{2} c_1 e^{t/2} + \frac{1}{3} c_2 e^{t/3}$$

Our initial conditions are $y(0) = 4$ and $y'(0) = 0$. Substituting these in to y and y' yields the system of equations

$$\begin{aligned} c_1 + c_2 &= 4 \\ \frac{1}{2}c_1 + \frac{1}{3}c_2 &= 0 \end{aligned}$$

Solving this system we find $c_1 = -8$ and $c_2 = 12$. So our particular solution is

$$y = 12e^{t/3} - 8e^{t/2}$$

To investigate the behavior as t increases we can be precise and examine the limit as $t \to \infty$. We see that

$$\begin{aligned} \lim_{t\to\infty} (12e^{t/3} - 8e^{t/2}) &= \lim_{t\to\infty} -4e^{t/3}(2e^{t/6} - 3) \\ &= \lim_{t\to\infty} -4e^{t/3} \cdot \lim_{t\to\infty} (2e^{t/6} - 3) \end{aligned}$$

From our knowledge of basic limits we know that $\lim\limits_{t\to\infty} e^t = \infty$. So our limit becomes

$$\lim_{t\to\infty} -4e^{t/3} \cdot \lim_{t\to\infty} (2e^{t/6} - 3) = (-\infty) \cdot (\infty) = -\infty$$

So as $t \to \infty$, we have $y \to -\infty$.