Lecture 13: Definition of the Laplace Transform

We have seen that procedures for solving second order non-homogeneous equations and homogeneous and non-homogeneous higher order equations can be difficult and tedious. The Laplace transform is a type of ***integral transform***. It will allow us to *transform* an ordinary differential equation into a form that will be much easier to solve.

Integral Transforms

Definition

An **integral transform** is defined by the relation

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt$$

where the function $K(s,t)$ known as the ***kernel***, is a given function.

An integral transform allows us to take a function $f(t)$ and transform it into a function $F(s)$. First, recall the definition of a piecewise continuous function.

Definition

A function f is **piecewise continuous** on an interval $a \leq t \leq b$ if the interval can be partitioned into a finite number of subintervals such that

(a) f is continuous on each subinterval.

(b) f has finite limits at the endpoints of each subinterval (i.e. none of them go to infinity).

Now we are ready for the definition of the Laplace transform.

The Laplace Transform

Definition

Suppose that $f(t)$ is piecewise continuous function.
The **Laplace transform** of $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

In the textbook this is denoted as $\mathcal{L}\{f(t)\} = F(s)$.

The Laplace transform is a ***linear operator***. The textbook does not do a very good job of explicitly stating what this means.

Definition

A **linear operator** is a linear mapping (i.e. an operation) with the following properties

(a) $\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$ for some constant c.

(b) $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$.

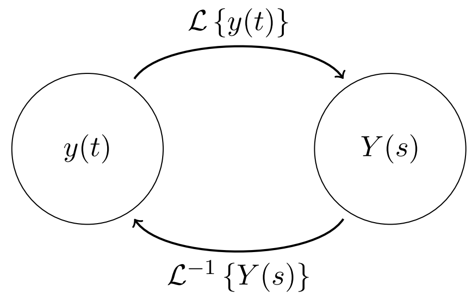
In other words, a linear operator preserves addition and scalar multiplication.

The Laplace Transform and ODEs

It may not be very clear at this point as to how this will help us solve an ODE.

The idea is to take a differential equation, which involves derivatives of a function $y(t)$, and transform it into a function $Y(s)$. This function $Y(s)$ will be manipulated with simple algebra.

We then take the inverse of the transform $\mathcal{L}^{-1}\{Y(s)\}$ to get back to the desired function $y(t)$, and thus our solution to the ode.



We can see that the definition requires evaluation of an improper integral. Recall the following facts before proceeding.

Definition

An **improper integral** is an integral over an unbounded interval. It is defined by

$$\int_0^\infty f(t)\,dt = \lim_{A\to\infty}\int_0^A f(t)\,dt$$

where A is a real constant and provided that the limit exists and is finite.

If we have that

- The integral exists for each $A > a$ and
- the limit as $A \to \infty$ exists

then the integral ***converges***, otherwise it ***diverges***. You should also take note that

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s} \quad \text{and} \quad \lim_{t\to\infty} e^{-t} = 0$$

The following theorem illustrates when we know that the Laplace transform will actually exist.

Theorem 1: (6.1.2)

Suppose that

- f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive A.
- $|f(t)| \leq Ke^{at}$ when $t \geq M$. In this inequality K, a and M are real constants, K, M necessarily positive.

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$, defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

exists for $s > a$.

Note that the requirement for $s > a$ is required in most cases to ensure that the improper integral given by the definition will converge for the function you are transforming ($f(t)$).

Since the Laplace transform is a linear operator we will be able to combine functions with addition and scalar multiplication to obtain transforms of other functions.

Finding the Laplace transform using the definition requires all of your integration skills. This can be a tedious process and for some functions, it may not even be possible to evaluate the integral. Eventually, we will use a table of Laplace transforms. This is why it's important to be comfortable with the general forms of a functions and how to integrate them.

Why use the definition?

If we can get a function to have a similar form to a function for which we already know the Laplace transform then we can skip the integration. This means, that after finding the general forms of Laplace transforms for general funcitons, we can actually use a table instead of integrating directly. As with differentiation and integration, it can seem tedious to find the Laplace transform from the definition. It is important, and helpful, to see how these common transforms are derived, if for no other reason to appreciate what you will have in your table.

Example 1

Find the Laplace Transform of $f(t) = t$

Solution. Using the definition we have

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} e^{-st} \cdot t\,dt = \lim_{A\to\infty} \int_0^A e^{-st} \cdot t\,dt$$

Now we integrate by parts to obtain

$$\begin{aligned}
\lim_{A\to\infty} \int_0^A e^{-st} \cdot t\,dt &= \lim_{A\to\infty} \left[t\left(-\frac{1}{s}e^{-st}\right)\Big|_0^A + \frac{1}{s}\underbrace{\int_0^A e^{-st}\,dt}_{=-\frac{1}{s}e^{-st}\big|_0^A} \right] \\
&= \lim_{A\to\infty} \left[-\frac{t}{s}e^{-st}\Big|_0^A - \frac{1}{s^2}e^{-st}\Big|_0^A \right] \\
&= \lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} - \frac{1}{s^2}\left(e^{-sA} - 1\right) \right] \\
&= \underbrace{\lim_{A\to\infty} \left[-\frac{A}{s}e^{-sA} \right]}_{=0} - \frac{1}{s^2} \underbrace{\lim_{A\to\infty} \left[e^{-sA} \right]}_{=0} + \frac{1}{s^2} \lim_{A\to\infty} 1 \\
&= \frac{1}{s^2}
\end{aligned}$$

So we have $F(s) = \mathcal{L}\left\{t\right\} = \dfrac{1}{s^2}$

This next example is a little more complicated but demonstrates a key strategy for evaluating these integrals.

Example 2

Find the Laplace Transform of $f(t) = \sin at$, where a is a real constant.

Solution. Using the definition we have

$$F(s) = \mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st} \cdot \sin(at)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} \cdot \sin(at)\,dt$$

Now we integrate by parts to obtain

$$\begin{aligned}
\lim_{A\to\infty} \int_0^A e^{-st} \cdot \sin(at)\,dt &= \lim_{A\to\infty} \left[-\frac{1}{a} e^{-st} \cos(at)\Big|_0^A - \frac{s}{a} \int_0^A e^{-st} \cdot \cos(at)\,dt \right] \\
&= \lim_{A\to\infty} \left[-\frac{1}{a} e^{-sA} \cos(aA) + \frac{1}{a} - \frac{s}{a} \int_0^A e^{-st} \cdot \cos(at)\,dt \right] \\
&= \underbrace{\lim_{A\to\infty} \left[-\frac{1}{a} e^{-sA} \cos(aA) \right]}_{=0} + \frac{1}{a} \lim_{A\to\infty} 1 - \frac{s}{a} \lim_{A\to\infty} \int_0^A e^{-st} \cdot \cos(at)\,dt \\
&= \frac{1}{a} - \frac{s}{a} \lim_{A\to\infty} \int_0^A e^{-st} \cdot \cos(at)\,dt
\end{aligned}$$

Now we need to integrate by parts again and we obtain

$$\begin{aligned}
\lim_{A\to\infty} \int_0^A e^{-st} \cdot \cos(at)\,dt &= \lim_{A\to\infty} \left[\frac{1}{a} e^{-st} \sin(at)\Big|_0^A + \frac{s}{a} \int_0^A e^{-st} \sin(at)\,dt \right] \\
&= \underbrace{\lim_{A\to\infty} \left[-\frac{1}{a} e^{-sA} \sin(aA) \right]}_{=0} + \frac{s}{a} \lim_{A\to\infty} \int_0^A e^{-st} \sin(at)\,dt \\
&= \frac{s}{a} \int_0^\infty e^{-st} \sin(at)\,dt
\end{aligned}$$

and so we have

$$\int_0^\infty e^{-st} \cdot \cos(at)\,dt = \frac{1}{a} - \frac{s}{a} \left[\frac{s}{a} \int_0^\infty e^{-st} \cdot \sin(at)\,dt \right]$$

Since we defined $F(s) = \int_0^\infty e^{-st} \cdot \sin(at)\,dt$ we have

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

Now we solve for $F(s)$:

$$F(s) + \frac{s^2}{a^2} F(s) = \frac{1}{a} \implies F(s) \left(\frac{a^2 + s^2}{a^2} \right) = \frac{1}{a}$$

$$F(s) = \frac{1}{a} \left(\frac{a^2}{a^2 + s^2} \right) = \frac{a}{a^2 + s^2}$$

So we have

$$F(s) = \mathcal{L}\{\sin(at)\} = \frac{a}{a^2 + s^2}$$