Lecture 16: Solving IVPs with Step Functions

Math 2310-360: Differential Equations Spring 2019

Solving IVPs with Step Functions

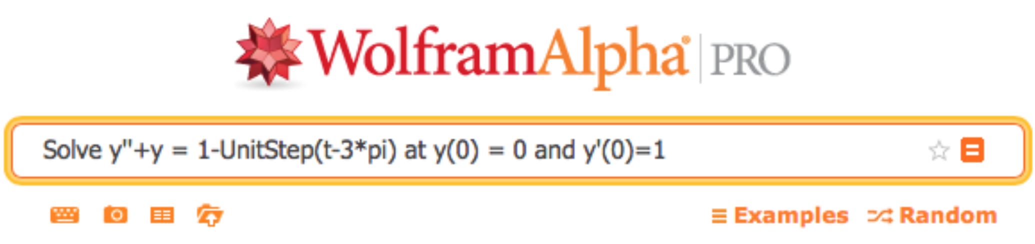
Solving IVPs with step functions can be complicated. It's easy to mess up the step function part of the equation, partial fractions can be messy, and then taking inverse Laplace transforms of the step function portions can be easy to confuse.

For partial fractions don't bother going through the details. You know how to do partial fractions at this point. It's okay to use technology. Don't irritate yourself by doing tedious algebra that you already know how to do! Here are some tools you can use:

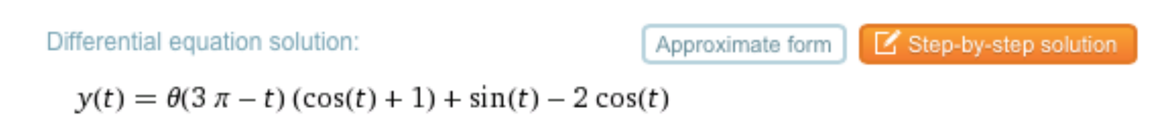
- Wolfram Alpha: Just type 'partial fractions' and then your equation.
- Mathematica: Use 'Apart[$< eqn >$]'.
- TI-89/nspire: Use 'expand' command in the Algebra menu.

You can enter step functions into Wolfram Alpha and have it solve these IVPs for you. Only use this to check your work or find where you might have made an error! Using Wolfram Alpha to do your homework for you is considered academic dishonesty. Also, you get better at math by practicing. Getting all your answers from the internet may help you improve your homework score but it does you no favors on exam day!

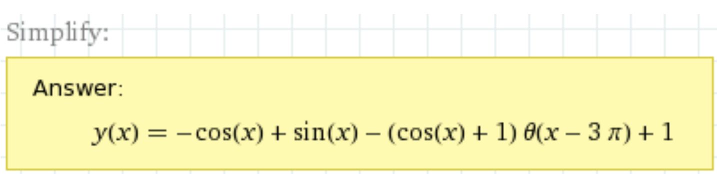
To enter step function IVPs in Wolfram Alpha follow the example shown below:



This gives the following output where $\theta(3\pi - t)$ is the same as $u_{3\pi}(t)$.



Keep in mind that Wolfram Alpha simplifies answers in ways other than a human would. By looking at the step-by-step solution we have a form we would obtain ourselves:



Example 1: Find the solution of the inital value problem

$$y'' + y = f(t); \quad y(0) = 0; \quad y'(0) = 1; \quad f(t) = \underbrace{\begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}}_{= 1 - u_c(t)} \Rightarrow t \ge 3\pi$$

First write $f(t)$ in terms of unit step fcn $u_c(t)$

$$f(t) = 1 - u_{3\pi}(t)$$

So our ODE is

$$y'' + y = 1 - u_{3\pi}(t)$$

Take L.T. of ODE

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1 - u_{3\pi}(t)\}$$

By Thm 6.2.2

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) \\ &= s^2\mathcal{L}\{y\} - s(0) - 1 \\ &= s^2\mathcal{L}\{y\} - 1 \end{aligned}$$

& $$\begin{aligned} \mathcal{L}\{1 - u_{3\pi}(t)\} &= \mathcal{L}\{1\} - \mathcal{L}\{u_{3\pi}(t)\} \\ &= \frac{1}{s} - \frac{e^{-3\pi s}}{s} \end{aligned}$$

So we have

$$\left(s^2\mathcal{L}\{y\} - 1\right) + \mathcal{L}\{y\} = \frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right)$$

Let $\mathcal{L}\{y\} = Y(s)$

$$s^2 Y(s) - 1 + Y(s) = \frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right)$$

Combine like terms & solve for $Y(s)$

$$(s^2 + 1)Y(s) - 1 = \frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right)$$

$$+1 \qquad\qquad +1$$

Example 1: (continued)

$$(s^2+1)Y(s) = \frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right) + 1$$

$$\Rightarrow \quad Y(s) = \left[\frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right) + 1\right]\left[\frac{1}{s^2+1}\right]$$

Need to take ILT to get back a fcn $y(t)$.
First need it in "nice" form

$$Y(s) = \left[\frac{1}{s} - e^{-3\pi s}\left(\frac{1}{s}\right) + 1\right]\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{s}\left[\frac{1}{s^2+1}\right] - e^{-3\pi s}\left(\frac{1}{s}\right)\left(\frac{1}{s^2+1}\right) + \frac{1}{s^2+1}$$

$$= \left[1 - e^{-3\pi s}\right]\underbrace{\left[\frac{1}{s(s^2+1)}\right]}_{\text{Partial Fractions}} + \frac{1}{s^2+1}$$

Partial Fraction Decomp

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \qquad \text{(using TI n'spire)}$$

$$Y(s) = \left[1 - e^{-3\pi s}\right]\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{1}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} - e^{-3\pi s}\left[\frac{1}{s} - \frac{s}{s^2+1}\right] + \frac{1}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1} - e^{-3\pi s}\left(\frac{1}{s}\right) - e^{-3\pi s}\left[\frac{s}{s^2+1}\right] + \frac{1}{s^2+1}$$

Example 1: (continued)

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+1} - e^{-3\pi s}\left(\frac{1}{s}\right) - e^{-3\pi s}\left[\frac{s}{s^2+1}\right] + \frac{1}{s^2+1}$$

Take ILT to get $y(t)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{e^{-3\pi s}\left(\frac{1}{s}\right)\right\} - \mathcal{L}^{-1}\left\{e^{-3\pi s}\left(\frac{s}{s^2+1}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

Where

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$\mathcal{L}^{-1}\left\{e^{-3\pi s}\left(\frac{1}{s}\right)\right\} = u_{3\pi}(t)$$

Note

$$\mathcal{L}\{u_c(t)\underbrace{f(t-c)}_{\text{shifted}}\} = e^{-cs}\underbrace{F(s)}_{\text{LT of unshif}}$$

$$\mathcal{L}^{-1}\left\{e^{-3\pi s}\left(\frac{s}{s^2+1}\right)\right\} = u_{3\pi}(t)\,f(t-3\pi) = u_{3\pi}(t)\cos(t-3\pi)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

$f(t) = \cos(t) \Rightarrow f(t-3\pi) = \cos(t-3\pi)$

So soln is

$$y(t) = 1 - \cos(t) + \sin(t) - u_{3\pi}(t) - u_{3\pi}(t)\cos(t-3\pi)$$