Worksheet 16 Solutions

Solve IVPs with Step Functions MATH 2310, Spring 2019

You are permitted to use technology to assist you with factoring and partial fraction decompositions. All other work should be done by hand. Solutions that do not have an appropriate amount of detail will not receive credit!

1. (20 pts) Use the Laplace transform to find the solution of the initial value problem

$$y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t-2\pi), \quad y(0) = 0, \quad y'(0) = 0$$

Solution: Note that the fcn on the RHS is already written in $u_c(t)$ notation & has appropriate shifts represented

Taking the LT of the ODE

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\}$$

By Cor. 6.2.2

$$\begin{aligned}\mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0)\\ &= s^2\mathcal{L}\{y\} - s(0) - (0)\\ &= s^2\mathcal{L}\{y\}\end{aligned}$$

and on the RHS we have

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

$$\mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} = e^{-2\pi s}F(s) = e^{-2\pi s}\left(\frac{1}{s^2+1}\right)$$

where

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

since $f(t-2\pi) = \sin(t-2\pi)$

$\Rightarrow f(t) = \sin(t)$

Then letting $\mathcal{L}\{y\} = Y(s)$ the ODE becomes

$$s^2Y(s) + 4Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\left[\frac{1}{s^2+1}\right]$$

1. (cont'd)

Solving for $Y(s)$ we have

$$(s^2+4)Y(s) = \frac{1}{s^2+1} - e^{-2\pi s}\left[\frac{1}{s^2+1}\right]$$

$$Y(s) = \frac{1}{s^2+4}\left[\frac{1}{s^2+1} - e^{-2\pi s}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{1}{s^2+4}\left[\frac{1}{s^2+1}\left(1 - e^{-2\pi s}\right)\right]$$

$$= \frac{1}{(s^2+4)(s^2+1)}\left(1-e^{-2\pi s}\right)$$

By Partial fraction Decomp.

$$\frac{1}{(s^2+4)(s^2+1)} = \frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{1}{s^2+2^2}\right)$$

$$\Rightarrow Y(s) = \left[\frac{1}{3}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}\left(\frac{1}{s^2+2^2}\right)\right]\left(1-e^{-2\pi s}\right)$$

$$= \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2^2} - e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2^2}\right]\right]$$

We find $y(t)$ by taking the ILT of $Y(s)$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{3}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}\right.$$

$$\left. - \mathcal{L}^{-1}\left\{e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2^2}\right]\right\}\right]$$

1. (cont'd)

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \frac{1}{3}\left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2^2}\right]\right\}\right]$$

where

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t)$$

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s^2+2^2}\right]\right\} = u_{2\pi}(t)\, f(t-2\pi)$$

$$= u_{2\pi}(t)\left[\sin(t-2\pi) - \frac{1}{2}\sin(2(t-2\pi))\right]$$

where

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= \sin(t) - \frac{1}{2}\sin(2t)$$

so $f(t-2\pi) = \sin(t-2\pi) - \frac{1}{2}\sin(2(t-2\pi))$

$$y(t) = \frac{1}{3}\left(\sin(t) - \frac{1}{2}\sin(t) - u_{2\pi}(t)\left[\sin(t-2\pi) - \frac{1}{2}\sin(2(t-2\pi))\right]\right)$$

2. (20 pts) Use the Laplace transform to find the solution of the initial value problem

$$y'' + y = f(t); \quad y(0) = 0; \quad y'(0) = 0; \quad f(t) = \begin{cases} t, & 0 \le t < 2\pi \\ -2t, & 2\pi \le t < \infty \end{cases}$$

Solution:

First, we write $f(t)$ with $u_c(t)$ notation

$$f(t) = t \underbrace{\begin{cases} 1 & 0 \le t < 2\pi \\ 0 & 2\pi \le t < \infty \end{cases}}_{1 - u_{2\pi}(t)} - 2t \underbrace{\begin{cases} 0 & 0 \le t < 2\pi \\ 1 & 2\pi \le t < \infty \end{cases}}_{u_{2\pi}(t)}$$

$$= t(1 - u_{2\pi}(t)) - 2t\, u_{2\pi}(t)$$

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$$= t - 3t\, u_{2\pi}(t)$$

$$= t - 3[(t-2\pi) + 2\pi]\, u_{2\pi}(t)$$

Take L.T. of ODE

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

By Cor. 6.2.2

$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0)$$

$$= s^2 \mathcal{L}\{y\} - s(0) - (0) = s^2 \mathcal{L}\{y\}$$

and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - 3\mathcal{L}\{[(t-2\pi) + 2\pi]\, u_{2\pi}(t)\}$$

$$= \frac{1}{s^2} - 3e^{-2\pi s} G(s) = \frac{1}{s^2} - 3e^{-2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]$$

Where $G(s) = \mathcal{L}\{g(t)\}$

$$= \mathcal{L}\{t + 2\pi\}$$

$$= \mathcal{L}\{t\} + 2\pi\, \mathcal{L}\{1\}$$

$$= \frac{1}{s^2} + \frac{2\pi}{s}$$

Shifted Fcn:

$$g(t - 2\pi) = (t - 2\pi) + 2\pi$$

Unshifted Fcn:

$$g(t) = t + 2\pi$$

2. (cont'd)

So our ODE is

$$s^2 \mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{1}{s^2} - 3e^{-2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]$$

Let $\mathcal{L}\{y\} = Y(s)$

$$s^2 Y(s) + Y(s) = \frac{1}{s^2} - 3e^{-2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]$$

Solving for $Y(s)$:

$$(s^2+1)Y(s) = \frac{1}{s^2} - 3e^{-2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]$$

$$\begin{aligned}
Y(s) &= \frac{1}{s^2+1}\left[\frac{1}{s^2} - 3e^{-2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]\right] \\
&= \frac{1}{s^2+1}\left(\frac{1}{s^2}\right) - \frac{1}{s^2+1}\left[-3e^{2\pi s}\left[\frac{1}{s^2} + \frac{2\pi}{s}\right]\right] \\
&= \frac{1}{s^2}\left[\frac{1}{s^2+1}\right] + 3e^{-2\pi s}\left[\frac{1}{s^2}\left(\frac{1}{s^2+1}\right) + 2\pi\frac{1}{s}\left(\frac{1}{s^2+1}\right)\right]
\end{aligned}$$

Note: By Partial Fractions

$$\frac{1}{s^2}\left[\frac{1}{s^2+1}\right] = \frac{1}{s^2} - \frac{1}{s^2+1} \quad \& \quad \frac{1}{s}\left[\frac{1}{s^2+1}\right] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{1}{s^2} - \frac{1}{s^2+1} - 3e^{-2\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2\pi}{s} - \frac{2\pi s}{s^2+1}\right]$$

We find $y(t)$ by taking ILT of $Y(s)$

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&\qquad - 3\mathcal{L}^{-1}\left\{e^{-2\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2\pi}{s} - \frac{2\pi s}{s^2+1}\right]\right\}
\end{aligned}$$

2. (cont'd)

Where

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$\mathcal{L}^{-1}\left\{e^{-2\pi s}\left[\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2\pi}{s} - \frac{2\pi s}{s^2+1}\right]\right\} = u_{2\pi}(t)\, g(t-2\pi)$$

Note that

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\{G(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1} + \frac{2\pi}{s} - \frac{2\pi s}{s^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 2\pi\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\pi\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\ &= t - \sin(t) + 2\pi - 2\pi\cos(t) \end{aligned}$$

so

$$\begin{aligned} g(t-2\pi) &= (t-2\pi) - \sin(t) + 2\pi - 2\pi\cos(t-2\pi) \\ &= t - \sin(t-2\pi) - 2\pi\cos(t-2\pi) \end{aligned}$$

and so we have

$$u_{2\pi}(t)\, g(t-2\pi) = u_{2\pi}(t)\left[t - \sin(t-2\pi) + 2\pi\cos(t-2\pi)\right]$$

then the solution to our ODE is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= t - \sin(t) - 3u_{2\pi}(t)\left[t - \sin(t-2\pi) + 2\pi\cos(t-2\pi)\right] \end{aligned}$$