Lecture #22: Repeated Eigenvalues

Date: Mon. 5/6/19

When we have repeated eigenvalues (in a 2x2 system) we will only obtain 1 eigenvalue

In order to find our soln we need 2 linearly independent vectors.

How can we find this 2nd soln?

As we did w/ 2nd order eqns suppose we assume our 2nd soln has form

$$\vec{x}^{(2)} = t e^{\lambda t} \vec{\xi}$$

Now we check if this is a soln by plugging into our system $\frac{d\vec{x}}{dt} = A\vec{x}$

$$\Rightarrow \underbrace{\vec{\xi} e^{\lambda t} + \lambda \vec{\xi} t e^{\lambda t}}_{\frac{d\vec{x}}{dt}} = A \underbrace{\vec{\xi} t e^{\lambda t}}_{\vec{x}}$$

For $\vec{x}$ to be the right choice we must have

$$\underbrace{\vec{\xi} e^{\lambda t}}_{=0} + \underbrace{\lambda \vec{\xi}}_{=A\vec{\xi}} t e^{\lambda t} = A \vec{\xi} t e^{\lambda t}$$

This means we must have

$$\lambda \vec{\xi} = A\vec{\xi} \quad \& \quad \vec{\xi} = 0$$

This just yields e-vecs we already have

can't be true since e-vecs $\neq \vec{0}$

This can't be our soln.

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Instead we try

$$\vec{x} = \vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t}$$, $\vec{\eta}$ is some constant vector

Substituting this into our system we have

$$\vec{\xi} e^{\lambda t} + \lambda \vec{\xi} t e^{\lambda t} + \lambda \vec{\eta} e^{\lambda t} = A(\vec{\xi} t e^{\lambda t} + \vec{\eta} e^{\lambda t})$$

$$\Rightarrow \underline{(\vec{\xi} + \lambda \vec{\eta}) e^{\lambda t}} + \underline{(\lambda \vec{\xi}) t e^{\lambda t}} = \underline{(A\vec{\xi}) t e^{\lambda t}} + \underline{(A\vec{\eta}) e^{\lambda t}}$$

This implies we must have

$$\lambda \vec{\xi} = A\vec{\xi}$$ which just gives e-vecs we already have

and

$$\vec{\xi} + \lambda \vec{\eta} = A\vec{\eta} \Rightarrow (A - \lambda I)\vec{\eta} = \vec{\xi}$$

i.e. $\vec{\eta}$ must be a soln to this eqn

For λ with multiplicity 2 this eqn will always have a soln.

Our 2nd soln to the system will have form

$$\vec{x}^{(2)} = \vec{\xi}^{(1)} t e^{\lambda t} + \vec{\eta} e^{\lambda t}$$

Note: the vector $\vec{\eta}$ is referred to as a <u>generalized</u> <u>eigenvector</u> corr. to λ.

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Ex. 1 $\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$

Find eigenvalues: $\lambda_{1,2} = 1$
(details left as an exercise)

Find eigenvector for $\lambda = 1$: $\vec{\xi}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
(details left as an exercise)

To find 2^{nd} soln we need to solve the system $(A - \lambda I)\vec{\eta} = \vec{\xi}^{(1)}$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}}_{A-(1)I} \underbrace{\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}}_{\vec{\eta}} = \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{\xi}^{(1)}} \Rightarrow \left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right] \underset{\text{rref}}{\Rightarrow} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \eta_1 - 2\eta_2 = 1$$

$$\Rightarrow \eta_1 = 1 + 2\eta_2$$

Let $\eta_2 = K$, an arbitrary constant, then

$\eta_1 = 1 + 2K$
$\eta_2 = K$

$$\Rightarrow \vec{\eta} = \begin{bmatrix} 1+2K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + K\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So
$$\vec{x}^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \underbrace{K\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t}$$

multiple of $x^{(1)}(t)$
so let $K = 0$
i.e. $= K\vec{\xi}^{(1)}e^{\lambda t} = \vec{x}^{(1)}(t)$

Gen. soln to system is

$$\Rightarrow \vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

$$\vec{x}(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t\right]$$

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Phase Portrait

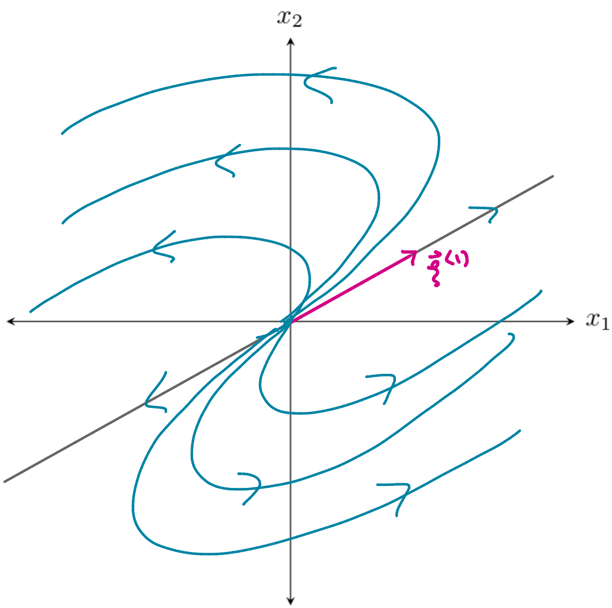
Since the case for rep. e vals has only one true e-vec. trajectories follow along a single line based on the evec.

In this case the equilibrium soln is known as an <u>improper node</u>.

If $\lambda < 0$ the equil. soln is a <u>stable improper node</u>

If $\lambda > 0$ the equil. soln is an <u>unstable improper node</u>

Ex. 2 Phase portrait for Example 1



$$\vec{\xi}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Since $\lambda_{1,2} = 1 > 0$ the equil. soln is an unstable node