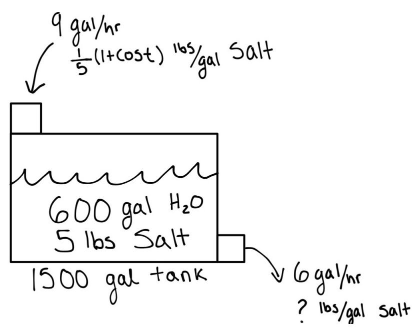
1. (10 pts) A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1+\cos(t))$ lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Solution:



Our basic format for tank problems is

$$Q'(t) = (\text{rate in})\cdot(\text{ conc. in }) - (\text{rate out})\cdot(\text{ conc. out })$$

where $Q'(t)$ is our rate of change in our concentration over time. Our goal is to solve this equation in order to gain an equation $Q(t)$ which will give us the concentration of salt in the tank at any time t.

Now we plug in our given information. Note that units should cancel here to leave you units that make sense for $Q'(t)$ (i.e. concentration/time).

So we have

$$Q'(t) = (9\,{}^{\text{gal}}/_{\text{hr}}) \cdot \left(\frac{1}{5}(1+\cos t)^{\text{lbs}}/_{\text{gal}}\right) - (6\,{}^{\text{gal}}/_{\text{hr}}) \cdot (?^{\text{lbs}}/_{\text{gal}})$$

We assume that the solution in the tank is well mixed. So at any time t the amount of salt is given by $Q(t)$. The volume at any time t is given by and the volume of the tank is given by

$$\text{Vol. of tank} = 600\,\text{gal} + (9\,\text{gal} - 6\,\text{gal})t = 600 + 3t$$

We combine these facts to get

$$\text{Conc. out} = \frac{\text{amount of salt in tank}}{\text{volume of } H_2O} = \frac{Q(t)}{600+3t}$$

So our IVP with initial condition $Q(0) = 5$ is

$$\begin{aligned} Q'(t) &= 9\left(\frac{1}{5}(1+\cos t)\right) - 6\left(\frac{Q(t)}{600+3t}\right) \\ &= \frac{9}{5}(1+\cos t) - \frac{2Q(t)}{200+t} \end{aligned}$$

Using the method of integrating factors we find

$$Q(t) = \frac{9}{5}\left(\frac{1}{3}(200+t) + \sin t + \frac{2\cos t}{200+t} - \frac{2\sin t}{(200+t)^2}\right) - \frac{-4600720}{(200+t)^2}$$

The step by step solution was skipped here since its a little complicated. Now the tank overflows when

$$600 + 3t > 1500 \implies t = 300$$

and we find that the concentration is

$$Q(300) = 279.797\,\text{lbs}$$

2. Suppose that a sum S_0 is invested at an annual rate of return r compounded continuously.
   (a) (4 pts) Find the time T required for the original sum to double in value as a function of r.

Solution: Since interest is compounded continuously we have

$$S'(t) = rS$$

where r is the interest rate and S is the current value of the investment. Our initial condition is given by $S(0) = S_0$ where S_0 is the initial amount invested. This is a basic separable linear equation and so we know the solution has the form

$$S(t) = S_0 e^{rt}$$

(I recommend just remembering this fact for exams. It will save you time rather than going through the solution process.) We want the time when the initial investment, S_0, doubles in value i.e. when $S(t) = 2S_0$ so we solve

$$2S_0 = S_0 e^{rt}$$

for t and find that $T = \dfrac{\ln(2)}{r}$ years.

   (b) (4 pts) Determine T if $r = 7\%$.

Solution: From part (a) we have that $T = \frac{\ln(2)}{r}$ so for $r = 7\% = 0.07$ we have

$$T = \frac{\ln(2)}{r} = \frac{\ln(2)}{0.07} = 9.90\,\text{years}$$

   (c) (4 pts) Find the return rate that must be achieved if the initial investment is to double in 8 years.

Solution: Again from (a) we have

$$T = \frac{\ln(2)}{r} \implies r = \frac{\ln(2)}{T} = \frac{\ln(2)}{8} = 0.0866 \implies 8.66\%$$

3. (8 pts) A population of zombies in a certain county will grow at a rate that is proportional to their current population. In the absence of any outside factors the population will triple in two weeks time. On any given day there are 15 zombies that migrate into the county. There are 16 zombies everyday that are killed by rifle shots or getting run over by cars and 7 die of starvation due to loss of body parts necessary for consumption of brains. If there are initially 100 zombies will the population survive? If not, when do they die out?

Solution: Our basic equation for population questions involving a population, $P(t)$ in a given region at any time t is

$$P'(t) = \text{Rate } P(t) \text{ enters region} - \text{Rate } P(t) \text{ exits region}$$

It's given that the zombie population grows proportionally to its current population at some positive growth rate r, which is written as rP. Now we write our IVP using the given information and we have

$$P'(t) = (rP + 15) - (16 + 7) = rP - 8$$

where $P(0) = 100$. In order to solve this IVP we first must find the rate r. We use the fact that the zombie popluation will triple in 2 weeks in the absence of outside factors (i.e. no new zombies move into the county and none die) so we use the more basic equation

$$P'(t) = rP \implies P(t) = Ce^{rt}$$

Our initial condition $P(0) = 100$ implies that $C = 100$. We now have the IVP

$$P'(t) = 100e^{rt}, \quad P(14) = 300$$

solving for r we find that $r = \dfrac{\ln(3)}{14}$. Now we use this in our original IVP:

$$P'(t) = (rP + 15) - (16 + 7) = \frac{\ln(3)}{14}P - 8$$

We use integrating factors to solve the resulting IVP with initial condition $P(0) = 100$ to get

$$P(t) = \frac{112}{\ln(3)} + \left(100 + \frac{112}{\ln(3)}\right)e^{\frac{\ln(3)}{14}t} = \frac{112}{\ln(3)} - 1.94679e^{\frac{\ln(3)}{14}t}$$

Now to answer our questions. Will the population survive? The answer is no. Even though the exponential will increase as $t \to \infty$ it has a negative coefficient and is therefore decreasing so eventually the population will die out. We find the time t at which this occurs by solving

$$\frac{112}{\ln(3)} - 1.94679e^{\frac{\ln(3)}{14}t} = 0$$

And find that the zombie population will die out at $t = 50.4415$ days.

4. (10 pts) Find a solution to population model governed by the logistic equation

$$\frac{dP}{dt} = 0.0004P(150 - P)$$

with initial condition $P(0) = 20$.

Solution: We rewrite this equation as

$$\frac{dP}{dt} = 0.06P\left(1 - \frac{P}{K}\right)$$

This is the logistic equation with carryng capacity $K = 150$ and rate of growth $r = 0.06$. We know that this will have general solution

$$P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}}$$

(See page 83 equation (11)) Since our initial condition is $P(0) = 20$ we have $P_0 = 20$ and so our solution is

$$P(t) = \frac{20(150)}{20 + (150 - 20)e^{-0.6t}} = \frac{3000}{20 + 130e^{-0.6t}}$$