Step Functions

In some applications you will encounter discontinuous forcing functions. These types of functions appear in models involving electrical circuits or a force with an impulse. This is where the **unit step function** (or **Heaviside function**) comes in to play.

Definition: Unit Step Function, $u_c(t)$

The **unit step function** is defined as

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

and is shown in the Fig. 1.

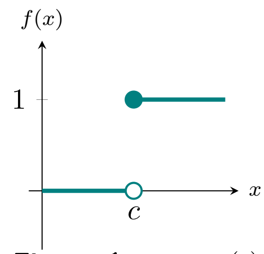


Figure 1: $y = u_c(t)$

Conceptually, think of this as a switch that is "off" until it switches "on" and jumps "up" a (positive) distance of 1 at $t = c$.

There are several alternative notations for the Unit Step function

$$u_c(t) = u(t - c) = H(t - c)$$

There is also the alternative case where the unit step function is "on" until $t = c$ when it jumps "down" a (negative) distance of 1 and turns "off".

Definition: Unit Step Function, $1 - u_c(t)$

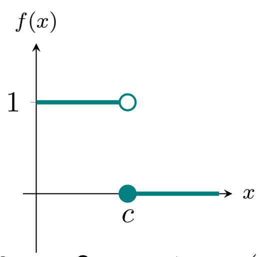


Figure 2: $y = 1 - u_c(t)$

This function is defined as

$$1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$

and is shown in Figure 2.

In general, try to think of the constant (or function) in front of the $u_c(t)$ as the distance "jumped" either in a positive or negative direction (i.e. "up" vs "down"). The time $t = c$ is when the function "jumps".

There are two key Laplace tranforms for these types of functions given in your table.

$$\# \ \mathbf{12} \quad \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \qquad\qquad \# \ \mathbf{13} \quad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

When using # 13 we are taking the Laplace transform of a *shifted* function. We may not see this directly in $f(t)$ so we need to account for the shift in our function (if it isn't there already) i.e. we need to rewrite it so it has the form of $f(t-c)$ before we can use the transform. Notice that we don't have a formula for $u_c(t)f(t)$!

The result is the Laplace transform of the *shifted* function. When we take the inverse Laplace transform of $\mathcal{L}^{-1}\{e^{-cs}F(s)\}$ we will get $\mathcal{L}^{-1}\{F(s)\} = f(t)$ which is the inverse Laplace transform of the *unshifted* function. This means we will need to add the shift back in to get back to our $f(t-c)$.

For shifting trig functions you may find the following identities useful.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \qquad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\cos(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \qquad \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\cos(\beta)$$

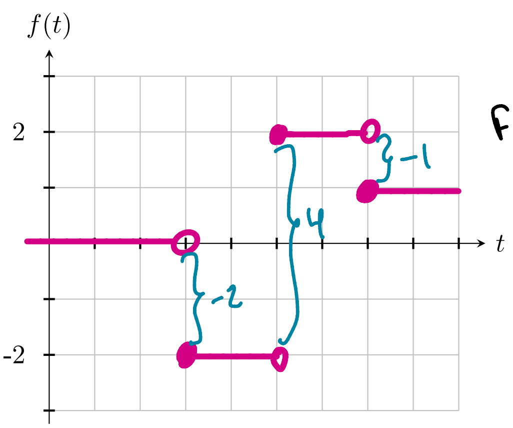
The Paul's Online Math Notes table has the following two identities which allow you to skip the heavy trig work.

$$\textbf{\# 15} \quad \mathcal{L}\{\sin(at+b)\} = \frac{s\sin(b)+a\cos(b)}{s^2+a^2} \qquad\qquad \textbf{\# 16} \quad \mathcal{L}\{\cos(at+b)\} = \frac{s\cos(b)-a\sin(b)}{s^2+a^2}$$

Example 1: Express the function

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

in terms of the unit step function $u_c(t)$ and sketch the graph.

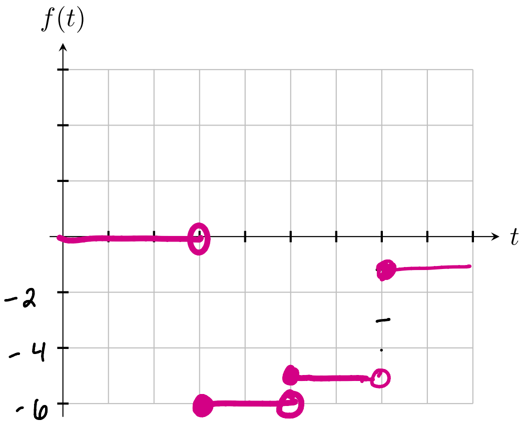


$f(t) = -2u_3(t) + 4u_5(t) - u_7(t)$

Example 2: Convert the function

$$g(t) = -6u_3(t) + u_5(t) + 4u_{7}(t)$$

to piecewise function notation and sketch the graph on the interval $t \geq 0$.



$$g(t) = \begin{cases} 0 & t < 3 \\ -6 & 3 \leq t\ 5 \\ -5 & 5 \leq t < 7 \\ -1 & t \geq 7 \end{cases}$$

Example 3: Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 4 \\ t^2 - 8t + 19, & t \geq 4 \end{cases}$

First, need to rewrite using $u_c(t)$

$$f(t) = (t^2 - 8t + 9) \underbrace{\begin{cases} 0 & t < 4 \\ 1 & t \geq 4 \end{cases}}_{u_4(t)}$$

$$= (t^2 - 8t + 19)\, u_4(t)$$

Want to use $\mathcal{L}\{u_c(t) g(t-c)\} = e^{-cs} G(s)$

LT of unshifted fcn

Note: $t^2 - 8t + 19 = t^2 - 8t + 16 + 3$

Guess: $(t-4)^2 + k$ $\quad = (t-4)^2 + 3$

$(t-4)^2 = t^2 - 8t + 16$

$$\Rightarrow f(t) = [\,\overbrace{(t-4)^2 + 3}^{g(t-4)}\,]\, u_4(t)$$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{[(t-4)^2 + 3]\, u_4(t)\}$$

$$= e^{-4s} G(s) = e^{-4s}\left[\frac{2}{s^3} + \frac{3}{s}\right]$$

Note: $g(t-4) = (t-4)^2 + 3$

$\Rightarrow g(t) = t^2 + 3$

$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t^2 + 3\} = \mathcal{L}\{t^2\} + 3\mathcal{L}\{1\}$

$= \frac{2!}{s^{2+1}} + 3\left(\frac{1}{s}\right)$

$= \frac{2}{s^3} + \frac{3}{s}$

So LT of $f(t)$ is

$$F(s) = e^{-4s}\left[\frac{2}{s^3} + \frac{3}{s}\right]$$

Example 4: Find the Laplace transform of $g(t) = \begin{cases} \sin(t), & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

$$g(t) = \sin(t) \underbrace{\begin{cases} 1, & 0 < t < \pi \\ 0, & t \ge \pi \end{cases}}_{1 - u_\pi(t)}$$

$$g(t) = \sin(t)\,(1 - u_\pi(t)) = \sin(t) - \sin(t)\,u_\pi(t)$$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{\sin(t)\,u_\pi(t)\}$$
$$= \mathcal{L}\{\sin(t)\} - \mathcal{L}\{-\sin(t-\pi)\,u_\pi(t)\}$$

Note: $\sin(t)\,u_\pi(t) \Rightarrow$ Need to write in $f(t-c)$ form

$$\Rightarrow \sin(t - \pi + \pi) = \sin([t-\pi] + \pi)$$

By sum/difference Trig ID

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

w/ $\alpha = t - \pi$ & $\beta = \pi$

$$\sin([t-\pi]+\pi) = \sin(t-\pi)\underbrace{\cos(\pi)}_{=-1} + \underbrace{\sin(\pi)}_{=0}\cos(t-\pi)$$
$$= -\sin(t-\pi)$$

$$\Rightarrow G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{\sin(t)\,u_\pi(t)\}$$
$$= G_1(s) + G_2(s)$$

$$G_1(s) = \mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1^2}$$

$$G_2(s) = \mathcal{L}\{\underbrace{\sin(t-\pi)}_{f(t-\pi)}\,u_\pi(t)\} = e^{-\pi s} F(s)$$
$$= e^{-\pi s}\left[\frac{1}{s^2+1^2}\right]$$

Note: $f(t-\pi) = \sin(t-\pi)$
$\Rightarrow f(t) = \sin(t)$

$$\Rightarrow G(s) = G_1(s) - G_2(s) = \left[\frac{1}{s^2+1}\right] + e^{-\pi s}\left[\frac{1}{s^2+1}\right]$$

Example 4: Find the Laplace transform of $g(t) = \begin{cases} \sin(t), & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

$$g(t) = \sin(t)\,(1 - u_\pi(t)) = \sin(t) - \sin(t)\,u_\pi(t)$$
$$= \sin(t) - \sin([t-\pi]+\pi)$$

Taking L.T.

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{\sin(t)u_\pi(t)\}$$

$$\mathcal{L}\{\sin[(t-\pi)+\pi]u_\pi(t)\} = e^{-\pi s} F(s) = e^{-\pi s}\left[\frac{-1}{s^2+1}\right]$$

$$f(t-\pi) = \sin((t-\pi)+\pi)$$
$$\Rightarrow f(t) = \sin(t+\pi)$$

$$\Rightarrow F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t+\pi)\}$$

Using POMN #15, $a = 1$, $b = \pi$

$$= \frac{s\sin(\pi) + (1)\cos(\pi)}{s^2 + (1)^2}$$

$$= \frac{-1}{s^2+1}$$

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{\sin(t)u_\pi(t)\}$$
$$= \frac{1}{s^2+1^2} - e^{-\pi s}\left[\frac{-1}{s^2+1}\right]$$

$$= \frac{1}{s^2+1} - e^{-\pi s}\left[\frac{-1}{s+1}\right]$$