Worksheet 10 Solutions

Undetermined Coefficients

MATH 2310, Spring 2019

Section Summary

The general form of a non-homogeneous equation is

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

with a general solution of the form

$$y(t) = y_c(t) + Y(t) \tag{2}$$

where $y_c(t)$ is the solution to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$ (also known as the complimentary solution) and $Y(t)$ is the particular solution found by using the *method of undetermined coefficients*.

The Method of Undetermined Coefficients assumes that $Y(t)$ should have a form similar to $g(t)$. A good guess for a solution to $Y(t)$ is found using generic coefficients $A, B, C \ldots$, etc. To determine the coefficients we plug $Y(t)$, $Y'(t)$, $Y''(t)$ into the differential equation, simplify, and equate coefficients (similar to the process in partial fraction decomposition). Recall that this is exactly what we do to check if an equation is a solution to an ODE. In this case, we are doing it when our coefficients are to be determined. In other words, they are *undetermined coefficients*.

Procedure for Undetermined Coefficients

1. **Homogeneous Solution** $Y_c(t)$ Recall to find the general homogeneous solution we
   - Determine the characteristic equation.
   - Find it's roots and determine if they are real and distinct, repeated, or complex.
   - Write out the general solution of $y_c(t)$
2. **Particular Solution** $Y(t)$: Now find the particular solution $Y(t)$ using *the Method of Undetermined Coefficients*.
   - Determine a good "guess", $y_p(t)$, that the form $Y(t)$ should have based on what our $g(t)$ looks like. (See problem 2 for a reference table and examples.)
   - Let $y_p(t) = Y(t)$ and find $Y''(t)$ and $Y'(t)$.
   - Plug these equations into the differential equation (1) and simplify the right hand side of the equation.
   - Equate coefficients on the right hand side with those on the left hand side to obtain a system of equations, then solve to get your coefficients.
   - Plug coefficients into $y_p(t)$ to obtain the $Y(t)$ for your solution.
3. **General Solution** $y(t)$**:** Plug your results for $y_c(t)$ and $Y(t)$ in to (2) to obtain your general solution $y(t)$.
4. **Particular Solution to ODE:** If given initial conditions, now use the same procedure you did previously with homogeneous equations to determine the values of c_1 and c_2.

1. (8 pts) Determine the "guess" for $g(t)$ for the equation $y''+p(t)y'+q(t)y = g(t)$. Assume the solution is not repeated in the homogeneous equation.

(a) $g(t) = 4\cos(6t) - 9\sin(6t)$

Solution: Just look at the table above and insert the relevant terms.

$$Y_p(t) = A\cos(6t) + B\sin(6t)$$

(b) $g(t) = 6t^2 - 7\sin(3t) + 9$

Solution: First rewrite the equation so that it's easier to see what we need

$$g(t) = \underbrace{6t^2 + 9}_{\text{2nd deg. polyn.}} \quad - \underbrace{7\sin(3t)}_{\text{trig fcn}}$$

So $6t^2 + 9$ implies we use $At^2 + Bt + C$ and $\sin(3t)$ implies we use $D\cos(3t) + E\sin(3t)$ and we have

$$Y_p(t) = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$$

we include the 9 in our polynomial because solving for 5 coefficients is easier than solving for 6!

(c) $g(t) = 10e^t - 5te^{-8t} + 2e^{-8t}$

Solution: Again, rewrite this first by combining like terms

$$g(t) = 10e^t + (-5t + 2)e^{-8t}$$

So $10e^t$ implies we use Ae^t and $(-5t + 2)e^{-8t}$ implies we use $(Bt + C)e^{-8t}$ and we have

$$Y_p(t) = Ae^t + (Bt + C)e^{-8t}$$

We don't need to worry about adding a constant in front of e^{-8t}. This would produce too many constants to solve for which would be combined at the end of our process anyway. Consider the polynomial $(Bt+C)$ as accounting for this constant.

(d) $g(t) = e^{7t} + 6$

Solution: Put a constant in front of our exponential and add a constant to account for the 6.

$$Y_p(t) = Ae^{7t} + B$$

2. (16 pts) Find the solution of the initial value problem

$$y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$$

Solution: This is a nonhomogeneous equation with constant coefficients. Our solution will have the form

$$y(t) = y_c(t) + y_p(t)$$

Homogeneous Solution $y_c(t)$:

Using the methods for homogeneous equations with constant coefficients we need to solve

$$y'' + y' - 2y = 0$$

Characteristic equation: $r^2 - 2r - 3 = 0$

$$(r-1)(r+2) = 0 \implies r_1 = -2,\, r_2 = 1$$

These are real, distinct roots, so our general solution for $y_c(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is

$$y_c(t) = c_1 e^{-2t} + c_2 e^{t}$$

Particular Solution $Y(t)$:

First we need a guess based on $g(t) = 2t$. We choose

$$y_p(t) = At + B$$

Now we let $y_p(t) = Y(t)$, and see what this looks like if it is a solution to our equation. In other words, we need to calculate some derivatives and plug them into our differential equation.

$$\begin{aligned} Y(t) &= At + B \\ Y'(t) &= A \\ Y''(t) &= 0 \end{aligned}$$

We plug these into the differential equation

$$Y''(t) + Y'(t) - 2Y(t) = 2t$$

and this gives us this algebraic nightmare

$$(0) + (A) - 2(At + B) = 2t$$

Ultimately, it reduces down to

$$A - 2At - 2B = 2t$$

You should organize your coefficients in the following way

$$(-2A)t + (A - 2B) = 2t$$

as it makes it easier to see which coefficients you need to equate. Equating our coefficients yields the system

$$\begin{aligned} -2A &= 2 \\ A - 2B &= 0 \end{aligned}$$

Solving this system we have $A = -1$ and $B = -\frac{1}{2}$. We plug these into our guess $y_p(t)$ and we get

$$Y(t) = -t - \frac{1}{2}$$

General Solution to ODE $y(t)$:

Recall that our general solution is $y(t) = y_c(t) + Y(t)$ and so we have

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= c_1 e^{-2t} + c_2 e^t - t - \frac{1}{2} \end{aligned}$$

Particular Solution to ODE $y(t)$:

Differentiating our general solution y yields

$$y' = -2c_1 e^{-2t} + c_2 e^t - 1$$

Our initial conditions are $y(0) = 4$ and $y'(0) = 0$. Substituting these in to y and y' yields the system of equations

$$\begin{aligned} c_1 + c_2 - \frac{1}{2} &= 0 \\ -2c_1 + c_2 - 1 &= 1 \end{aligned}$$

Solving this system we find $c_1 = -\frac{1}{2}$ and $c_2 = 1$. So our particular solution is

$$y = -\frac{1}{2}e^{-2t} + e^t - t - \frac{1}{2}$$

If you were given initial conditions you would proceed similarly to the process used for homogeneous equations with constant coefficients.

3. (16 pts) Find the general solution of

$$y'' - 2y' - 3y = -3te^{-t}$$

Solution:

Homogeneous Solution $y_c(t)$:

Using the methods for homogeneous equations with constant coefficients we need to solve

$$y'' - 2y - 3y = 0$$

Characteristic equation: $r^2 - 2r - 3 = 0$

$$(r-3)(r+1) = 0 \implies r_1 = 3,\, r_2 = -1$$

These are real, distinct roots, so our general solution for $y_c(t) = c_1e^{r_1t} + c_2e^{r_2t}$ is

$$y_c(t) = c_1e^{3t} + c_2e^{-t}$$

Particular Solution $Y(t)$:

First we need a guess based on $g(t) = -3te^{-t}$. The first instinct might be to use $y_p(t) = (At + B)e^{-t}$. However, we will run into problems later on since an e^{-t} term already appears in our $y_c(t)$.

To solve this problem we need a multiple of that solution (think of what we needed to do for our second solution when solving homogeneous equations with repeated roots). So our guess is actually

$$y_p(t) = (At^2 + Bt)e^{-t}$$

which is just our initial guess multiplied by t. Now we let $y_p(t) = Y(t)$, and see what this looks like if it is a solution to our equation. In other words, we need to calculate some derivatives and plug them into our differential equation.

$$\begin{aligned} Y(t) &= (At^2 + Bt)e^{-t} \\ Y'(t) &= (B + 2At - Bt - At^2)e^{-t} \\ Y''(t) &= (2A - 2B - 4At + Bt + At^2)e^{-t} \end{aligned}$$

We plug these into the differential equation

$$Y''(t) - 2Y'(t) - 3Y(t) = -3te^{-t}$$

and this gives us this algebraic nightmare

$$(2A - 2B - 4At + Bt + At^2)e^{-t} - 2(B + 2At - Bt - At^2)e^{-t} - 3(At^2 + Bt)e^{-t} = -3te^{-t}$$

I recommend walking through this process at least once so you know how to do it (or more than once if you need more algebra practice) but after that use technology. You can easily drive yourself crazy with these problems. This is a relatively simple example and it can get even crazier if you've got some trig functions mixed in there as well. Ultimately, it reduces down to

$$(2A - 4B - 8At)e^{-t} = -3te^{-t} \implies 2Ae^{-t} - 4Be^{-t} - 8Ate^{-t} = -3te^{-t}$$

You should organize your coefficients in the following way

$$(2A - 4B)e^{-t} + (-8At)e^{-t} = -3te^{-t}$$

as it makes it easier to see which coefficients you need to equate. Equating our coefficients yields the system

$$\begin{aligned} -8A &= -3 \\ 2A - 4B &= 0 \end{aligned}$$

Solving this system we have $A = \dfrac{3}{8}$ and $B = \dfrac{3}{16}$. We plug these into our guess $y_p(t)$ and we get

$$Y(t) = e^{-t}\left(\frac{3}{8}t^2 + \frac{3}{16}t\right)$$

General Solution to ODE $y(t)$:

Recall that our general solution is $y(t) = y_c(t) + Y(t)$ and so we have

$$y(t) = c_1e^{3t} + c_2e^{-t} + e^{-t}\left(\frac{3}{8}t^2 + \frac{3}{16}t\right)$$

If you were given initial conditions you would proceed similarly to the process used for homogeneous equations with constant coefficients.