Lecture # 9: Repeated Roots & Reduction of Order Date: Wed. 3/6/19

Rep. Roots of Characteristic Eqn.

If our characteristic eqn has repeated roots then

$$b^2 - ac = 0 \implies r_1, r_2 = \frac{-b}{2a}$$

Ex.) Solve $4y'' + 12y' + 9y = 0$

Char. eqn: $4r^2 + 12r + 9 = 0$

$$\implies (3 + 2r)^2 = 0$$

$$\implies r_{1,2} = \frac{-12}{2(4)} = -\frac{3}{2}$$

Can we have

$$y = c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{3}{2}t} \quad ?$$

$$= (c_1 + c_2) e^{-\frac{3}{2}t}$$

$$= D e^{-\frac{3}{2}t}$$

$\implies$ we only get one soln this way.

How can we get a 2^{nd} soln?

To come up with our $y_2(t)$ we'll look at a general procedure that will work for any 2^{nd} order linear homog. eqn. (i.e. not just for eqns w/ constant coeff.s)

Reduction of Order

Q: What if there is only one soln available for

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

Say $y_1(t)$

A: Need to find the other soln $y_2(t)$ to form the fund. set of solns

How?

Assume $y_2(t) = z(t) \cdot y_1(t)$

for $z(t)$ to be determined

Want to avoid $z(t) \equiv$ constant b/c in this case $y_2(t)$ is just a constant mult. of $y_1(t)$ (so it's the same as $y_1(t)$)

Need to make $y_2(t)$ satisfy the ODE

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \qquad (*)$$

Taking derivatives:

$$y_2'(t) = z'(t)\,y_1(t) + z(t)\,y_1'(t)$$
$$= z'y_1 + zy'$$
$$y_2''(t) = (z''(t)\,y_1(t) + z'(t)\,y_1'(t)) + (z'(t)\,y_1'(t) + z(t)\,y_1''(t))$$
$$= z''(t)\,y_1(t) + 2z'(t)\,y_1'(t) + z(t)\,y_1''(t)$$
$$= z''y_1 + 2z'y_1' + zy_1''$$

Subbing these into $(*)$

$$z''y_1 + 2z'y_1' + zy_1'' + p(t)(z'y_1 + zy') + q(t)y(t) = 0$$

Rearranging terms

$$z\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{=0} + y_1z'' + (2y_1' + p(t)y_1)z' = 0$$

This is $(*)$ w/ soln y_1

$$\Rightarrow y_1z'' + (2y_1' + p(t)y_1)z' = 0$$

Dividing by y_1 (since $y_1 \neq 0$)

$$z'' + \left(\frac{2y_1' + p(t)y_1}{y}\right)z' = 0$$

This is a 2^{nd} order eqn. but we can turn it into a 1^{st} order eqn. This is a <u>reduction of order</u>

$$z'' + \left(\frac{2y_1' + p(t)y_1}{y}\right)z' = 0$$

Assume $w(t) = z'(t) \implies w'(t) = \frac{d}{dt}[z'(t)] = z''(t)$

So our ODE becomes

$$w' + \left(\frac{2y_1' + p(t)y_1}{y}\right)w = 0$$

Which is now a 1^{st} order eqn we can solve as a sep. eqn.

Solving this eqn we obtain w. Then

$$w(t) = z'(t) \implies z(t) = \int w(t)\,dt$$

Substituting this back into our $y_2(t)$ we will have our 2^{nd} soln.

Lecture 9

Ex.) Solve
$$4y'' + 12y' + 9y = 0$$

Char. Eqn: $4r^2 + 12r + 9 = 0$
$$(3+2r)^2 = 0$$
$$r_1, r_2 = -\frac{3}{2}$$

We know $y_1(t) = De^{-\frac{3}{2}t}$

Assume $y_2(t) = z(t)e^{-\frac{3}{2}t}$

Take deriv's
$$y_2'(t) = z'(t)e^{-\frac{3}{2}t} - \frac{3}{2}z(t)e^{-\frac{3}{2}t}$$
$$y_2''(t) = z''(t)e^{-\frac{3}{2}t} - \frac{3}{2}z'(t)e^{-\frac{3}{2}t}$$
$$- \frac{3}{2}z'(t)e^{-\frac{3}{2}t} + \frac{9}{4}z(t)e^{-\frac{3}{2}t}$$
$$= z''e^{-\frac{3}{2}t} - \frac{6}{2}z'e^{-\frac{3}{2}t} + \frac{9}{4}ze^{-\frac{3}{2}t}$$
$$= e^{-\frac{3}{2}t}\left(z'' - 3z' + \frac{9}{4}z\right)$$

Plug into ODE
$$4\left[e^{-3/2t}\left(z'' - 3z' + \frac{9}{4}z\right)\right] + 12\left[e^{-\frac{3}{2}t}\left(z' - \frac{3}{2}z\right)\right] + 9ze^{-\frac{3}{2}t} = 0$$
$$e^{-\frac{3}{2}t}\left[4\left(z'' - 3z' + \frac{9}{4}z\right) + 12\left(z' - \frac{3}{2}z\right) + 9z\right] = 0$$
$$e^{-\frac{3}{2}t}\left[4z'' - \cancel{12z'} + \cancel{9z} + \cancel{12z'} - \cancel{18z} + \cancel{9z}\right] = 0$$
$$4e^{-\frac{3}{2}t}z'' = 0$$

Lecture 9

Ex. 1 (cont'd)

Need to find $z(t)$ so need to solve

$$4\, z''(t)\, e^{-\frac{3}{2}t} = 0$$

$$\Rightarrow z''(t)\, e^{-\frac{3}{2}t} = 0$$

$$\Rightarrow z''(t) = 0$$

Solve this ODE just by integrating.

$$\int z''(t)\, dt = \int 0\, dt = C_1 \Rightarrow z'(t) = C_1$$

$$z(t) = \int z'(t)\, dt = \int C_1\, dt = C_1 t + C_2$$

So $y_2(t) = z(t)\, e^{-\frac{3}{2}t}$

$$= (C_1 t + C_2)\, e^{-\frac{3}{2}t}$$

$$= C_1 t e^{-\frac{3}{2}t} + \underbrace{C_2 e^{-\frac{3}{2}t}}$$

just cst mult of y_1 so we don't use it

$$\Rightarrow y_2(t) = C_1 t e^{-\frac{3}{2}t}$$

So Gen soln is

$$y(t) = C_1 t e^{-\frac{3}{2}t} + D e^{-\frac{3}{2}t}$$