Theorem 1: (3.2.1) Existence and Uniqueness

Consider the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \tag{1}$$

where p, q, g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

In other words, if p, q, g are all continuous on some interval that contains t_0 then there is a single solution of the ODE in that particular interval. Now we will focus on the 2nd order linear *homogeneous* equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \tag{2}$$

First we need a term that is not very well defined in your textbook.

Definition

A **linear combination** is an expression that is constructed from a set of terms by multiplying each term by a constant and adding the results.

The following theorem allows us to generate an infinite *family* of solutions given two solutions y_1 and y_2.

Theorem 2: (3.2.2) Principle of Superposition

Suppose that y_1 and y_2 are solutions of (2) then the linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{3}$$

is also a solution for any values of c_1, c_2.

Some Linear Algebra Background

The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{4}$$

can be represented as a matrix in the following way

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solving this system requires finding a solution to the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

Where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

A solution can be found by finding and inverse of the matrix A: A^{-1} so that

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}$$

This requires that the matrix A actually be **invertible**.

This brings us to an important definition in linear algebra known as the **determinant**.

Definition: The Determinant

The **determinant** of the 2 by 2 matrix A is given by

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

For 2 by 2 matrices we can find the inverse pretty easily with the following formula

Definition: Inverse of a 2 by 2 Matrix

The inverse of the 2 by 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is found by the following calculation

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Since the determinant appears in the denominator we see that in order for an inverse to be found we must have that $\det(A) \neq 0$. This condition is also necessary for larger matrices. The key point to take away here is

For any n by n system of equations to have a solution we must have that $\det(A) \neq 0$.

The Wronskian

For the 2nd order linear homogeneous equation (2) we have the initial conditions

$$y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0$$

We know from the principle of superposition that given two solutions y_1 and y_2 to (2) that

$$y = c_1 y_1(t) + c_2 y_2(t)$$

is also a solution to (2) where c_1 and c_2 must satisify the equations

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= y_0' \end{aligned} \tag{5}$$

Converting this to a matrix representation we have

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

We saw previously that for this system to have a solution we must have that the determinant is nonzero. The determinant of this matrix has a special name given in the next definition.

Definition

The determinant of the system of equations (5) is called the Wronskian determinant or just the **Wronskian**. The Wronskian is given by

$$W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1 y_2' - y_1' y_2 \tag{6}$$

Theorem 3: (3.2.3)

Suppose that y_1, y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{2}$$

then it is possible to choose constants c_1, c_2 so that

$$y = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

will satisfy the ode (2) if and only if the Wronskian

$$W(y_1, y_2)(t_0) = y_1 y_2' - y_1' y_2 \tag{6}$$

is not zero at t_0.

The following theorem states is similar to the previous theorem, but now justifies our use of the term "general solution" for (3).

Theorem 4: (3.2.4)

Suppose that y_1, y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{2}$$

then the family of solutions given by

$$y = c_1 y_1(t) + c_2 y_2(t) \tag{3}$$

with arbitrary c_1, c_2 includes every solution of (2) if and only if there is a point t_0 where the Wronskian of y_1, y_2

$$W = y_1 y_2' - y_1' y_2 \tag{6}$$

is not zero.

Particularly this theorem tells us

> If and only if the Wronskian of y_1 and y_2 is not everywhere zero then the linear combination $c_1 y_1 + c_2 y_2$ contains *all* solutions of (2).

Definition

The expression

$$y = c_1 y_1 + c_2 y_2$$

is referred to as the **general solution** of (2) where y_1, y_2 form a **fundamental set of solutions** (if and only if $W \neq 0$).

Now that we've seen some requirements for the fundamental set of solutions we should ask ourselves if a fundamental set of solutions always exists for (2).

Theorem 5: (3.2.5)

Consider the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{2}$$

whose coefficients p and q are continuous on some open interval I. Choosing some point $t_0 \in I$ we let y_1 be a solution to (2) such that

$$y(t_0) = 1 \quad \text{and} \quad y'(t_0) = 0,$$

and we let y_2 be a solution to (2) such that

$$y(t_0) = 0 \quad \text{and} \quad y'(t_0) = 1,$$

Then y_1 and y_2 form a fundamental set of solutions of (2).

The following theorem gives us an explicit expression to find the Wronskian.

Theorem 6: (3.2.6) Abel's Theorem

If y_1, y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{2}$$

where p, q are continuous on an open interval I, then the Wronskian is given by

$$W(y_1, y_2)(t_0) = c \exp\left[-\int p(t)\, dt\right]$$

where c is a constant that depends on y_1 and y_2 (but not on t!). If $c = 0$, then W is always zero for all $t \in I$ or never zero for $c \neq 0$ for all $t \in I$.

Some things to note as a result of this theorem

- The Wronskians of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant.
- The Wronskian of any fundamental set of solutions can be determined up to a multiplicative constant without solving the differential equation.
- You can determine if the Wronskian is always zero or never zero by evaluating at any easy to use value of t.