Lecture # 8: Homog. Eqns with Complex Roots Date: Mon 3/4/19

Recall a 2^{nd} order linear homog. eqn w/ cst coeffs has form

$$ay'' + by' + cy = 0$$

With characteristic eqn. $ar^2 + br + c = 0$

Complex roots of the char. eqn will have the form $\lambda \pm i\mu$, $\lambda, \mu \in \mathbb{R}$

The fundamental set of solns where

$$r_1 = \lambda + i\mu \quad , \quad r_2 = \lambda - i\mu$$

is

$$\{ e^{(\lambda + i\mu)t} , e^{(\lambda - i\mu)t} \}$$

So soln of ODE can be expressed as

$$\begin{aligned} y(t) &= D_1 e^{(\lambda + i\mu)t} + D_2 e^{(\lambda - i\mu)t} \\ &= D_1 e^{\lambda t} \cdot e^{i\lambda t} + D_2 e^{\lambda t} e^{-i\mu t} \\ &= e^{\lambda t} [D_1 e^{i\mu t} + D_2 e^{-i\mu t}] \end{aligned}$$

Recall <u>Euler's Formula</u>

$$e^{iA} = \cos(A) + i \sin(A)$$

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Using this in $y(t)$:

$$y(t) = e^{\lambda t}\left[D_1 e^{i\mu t} + D_2 e^{-i\mu t}\right]$$

$$= e^{\lambda t}\Big[D_1(\cos(\mu t) + i\sin(\mu t)) + D_2\big(\underbrace{\cos(-\mu t)}_{\substack{=\cos(\mu t)\\ \text{b/c } \cos\theta \text{ is}\\ \text{an } \underline{\text{even}} \text{ fcn}}} + i\underbrace{\sin(-\mu t)}_{\substack{=-\sin(\mu t)\\ \text{b/c } \sin\theta \text{ is}\\ \text{an } \underline{\text{odd}} \text{ fcn}}}\big)\Big]$$

$$= e^{\lambda t}\left[(D_1 + D_2)\cos(\mu t) + (iD_1 - iD_2)\sin(\mu t)\right]$$

Keep in mind that D_1 & D_2 are constants

So

$D_1 + D_2 = C_1$ is just another constant

and

$iD_1 - iD_2 = C_2$ is another constant

So gen. soln when we have complex roots

$$y(t) = e^{\lambda t}\left[C_1\cos(\mu t) + C_2\sin(\mu t)\right]$$

Conclusion: a fund. set of solns is <u>not</u> unique

For one case:

$$\{e^{(\lambda+i\mu)t}, e^{(\lambda-i\mu)t}\} \Rightarrow \{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\}$$

Lecture #8: Homog. Eqns with Complex Roots — Date: Mon 3/4/19

Ex. 1) Solve $y'' + y' + 9.25y = 0$, $y(0) = 2$, $y'(0) = 8$

Char. eqn: $r^2 + r + 9.25 = 0$

w/ complex roots, use quadratic formula

$$r_1 = -\frac{1}{2} + 3i$$
$$r_2 = -\frac{1}{2} - 3i$$
$$\Rightarrow r = -\frac{1}{2} \pm 3i$$

Here $\lambda = -\frac{1}{2}$, $\mu = 3$

So general soln is

$$y(t) = c_1 e^{-\frac{t}{2}} \cos(3t) + c_2 e^{-\frac{t}{2}} \sin(3t)$$

Apply I.C.s

@ $y(0) = 2$

$$2 = y(0) = c_1 \cos(0) + c_2 \sin(0) \Rightarrow c_1 = 2$$

$$\Rightarrow y(t) = e^{-\frac{t}{2}} [2\cos(3t) + c_2 \sin(3t)]$$

$$y'(t) = -\frac{1}{2} e^{-\frac{t}{2}} [2\cos(3t) + c_2 \sin(3t)] + e^{-\frac{t}{2}} [-6\sin(3t) + c_2 \cos(3t)]$$

@ $y'(0) = 8$

$$8 = y'(0) = -\frac{1}{2}[2 + 0] + [0 + 3c_2] \Rightarrow c_2 = 3$$

So particular soln is

$$y(t) = e^{-\frac{t}{2}} [2\cos(3t) + 3\sin(3t)]$$