Differences btwn Linear and Nonlinear Eqns

Theorem 1

If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \Phi(t)$ that satisfies the differential equation

$$y'(t) + p(t)y(t) = g(t)$$

for each $t \in I$. and that also satisfies the initial condition

$$y(t_0) = y_0$$

for any arbitrary $y_0 \in \mathbb{R}$.

What is this theorem saying?

- The IVP has a solution
- The solution is unique

This known as the *existence* and *uniqueness* of the solution of the IVP.

Proof. First lets mult. the eqn by $\mu(t)$

$$\mu(t)y'(t) + \mu(t)p(t)y(t) = \mu(t)g(t)$$

Recall the product rule gives us

$$[\mu(t)y(t)]' = \mu(t)y'(t) + \mu'(t)y(t)$$

Choosing $\mu(t)$ so that it satisfies $\mu'(t) = \mu(t)p(t)$ will be we obtain the integrating factor

$$\mu(t) = \exp\left(\int (p(t)\, dt\right)$$

and so we have

$$[\mu(t)y(t)]' = \mu(t)g(t)$$

both $\mu(t)$ and $g(t)$ are continuous $\implies$ $\mu(t)g(t)$ is integrable. Choosing the lower limit of integration to be t_0 then

$$\int_{t_0}^{\tau} [\mu(t)y(t)]'\, dt = \int_{t_0}^{\tau} \mu(t)g(t)\, dt, \quad \tau \in (\alpha, \beta)$$

then

$$\begin{aligned}
\mu(t)y(t)\Big|_{t_0}^{\tau} &= \int_{t_0}^{\tau} \mu(t)g(t)\, dt \\
\mu(\tau)y(\tau) - \mu(t_0)y(t_0) &= \int_{t_0}^{\tau} \mu(t)g(t)\, dt \\
\implies y(\tau) &= \frac{1}{\mu(\tau)}\left[\mu(t_0)\underbrace{y(t_0)}_{=y_0} + \int_{t_0}^{\tau} \mu(t)g(t)\, dt\right] \\
&= \frac{1}{\mu(\tau)}\left[\mu(t_0)y_0 + \int_{t_0}^{\tau} \mu(t)g(t)\, dt\right]
\end{aligned}$$

$\square$

Example 1

Consider $ty' + 2y = 4t^2, \quad y(1) = 2.$

Solution. Dividing by t to put the equation in standard form we have

$$y' + \frac{2}{t}y = 4t$$

Where here we see that $p(t) = \dfrac{2}{t}, \;\; g(t) = 4t, \;\; t_0 = 0. \implies t = 0$ must be avoided
By the theorem, $p(t)$ and $g(t)$ are both cont. in the interval $(0, \infty)$
$\implies$ where $y(t)$ will have a soln (*existence*) and it will be unique.

Example 2

Consider $(t-3)y' + \ln(t)y = 2t, \quad y(1) = 2.$

Solution. Dividing by $t-3$ to put the equation in standard form we have

$$y' + \frac{\ln(t)}{t-3}y = \frac{2t}{t-3}$$

Where here we see that $p(t) = \dfrac{\ln(t)}{t-3}, \;\; g(t) = 2t, \;\; t_0 = 0. \implies t = 0, t = 3$ must be avoided.
By the theorem, $p(t)$ and $g(t)$ are both cont. in the interval $(0, 3)$
$\implies$ where $y(t)$ will have a soln (*existence*) and it will be unique.

Theorem 2

Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \Phi(t)$ of the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0$$