Section Summary

Recall that the general form of a 2nd order homogeneous equation with constant coefficients is

$$ay'' + by' + cy = 0 \tag{1}$$

To find a solution we examine the characteristic equation

$$ar^2 + br + c = 0$$

For the case of repeated roots we have the case that $r_1 = r_2$. Our general solution has the form

$$y(t) = c_1 e^{rt} + c_2 \cdot te^{rt} \tag{2}$$

Where did this t in $y_2(t)$ come from? In the case of a repeated root we have the problem that our $y_1(t)$ is the same as our $y_2(t)$ in our fundamental set of solutions. Multiplying by some multiple of t will differentiate our solutions from each other so that they form a true fundamental set of solutions. Why a multiple of t? This is derived using the process of *reduction of order* on the general form of a second order linear equation.

Reduction of Order

Recall that a homogeneous 2nd order linear differential equation with *non-constant* coefficients has the general form

$$P(t)y'' + Q(t)y' + R(t)y = 0 \tag{3}$$

Given one of the solutions to (3) we can find a second solution $y_2(t)$ using using the method of *reduction of order*. The general procedure for this method is to assume that $y_2(t) = y_1(t) \cdot v(t)$ is a solution to (3) where $v(t)$ is some unknown function. If $y_2(t)$ is a solution to (3) then it must satisfy the ODE. By plugging in $y_2(t)$, $y_2'(t)$, $y_2''(t)$ into the ODE we obtain the second order equation

$$y_1 v'' + (2y_1' + p(t)y_1)v' = 0$$

We then use the substitution $v''(t) = w'(t)$ and $w(t) = v'(t)$ to transform this ODE into a first order equation which we can then solve for $w(t)$. To obtain $v(t)$ we simply integrate $w(t)$ and this $v(t)$ will then give us our $y_2(t)$. This process is best demonstrated by problem 2.

1. (10 pts) Find the solution of the initial value problem

$$9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

Solution:

Characteristic Equation:

$$\begin{aligned} 9r^2 - 12r + 4 &= 0 \\ (3r-2)^2 &= 0 \qquad \Longrightarrow r_1 = r_2 = \frac{2}{3} \end{aligned}$$

General Solution:
We have repeated roots and so our general solution is

$$y(t) = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}$$

Particular Solution:
To find our particular solution we evaluate our first initial condition $y(0) = 2$ to obtain

$$c_1 e^0 + c_2(0)e^0 = 2 \qquad \Longrightarrow c_1 = 2$$

To use our second initial condition we need $y'(t)$

$$y'(t) = c_1 \left(\frac{2}{3}\right) e^{\frac{2}{3}t} + c_2 \left[e^{\frac{2}{3}t} + \left(\frac{2}{3}\right) e^{\frac{2}{3}t} \right]$$

Evaluating at $y'(0) = -1$ we have

$$\begin{aligned} 2\left(\frac{2}{3}\right) e^0 + c_2 \left[e^0 + \left(\frac{2}{3}\right) e^0 \right] &= -1 \\ \frac{4}{3} + c_2 &= -1 \qquad \Longrightarrow c_2 = \frac{-7}{3} \end{aligned}$$

and so our particular solution is

$$y(t) = 2e^{\left(\frac{2t}{3}\right)} - \tfrac{7}{3} t e^{\left(\frac{2t}{3}\right)}$$

2. (10 pts) Use the method of reduction of order to find a second solution of the initial value problem

$$ty'' - y' + 4t^3 y = 0, \quad t > 0, \quad y_1(t) = \sin(t^2)$$

Solution:

$$\text{Let} \quad y_2(t) = v(t)\, y_1(t) = v(t) \sin(t^2)$$

$$\text{then} \quad y_2'(t) = 2t\cos(t^2)\, v + \sin(t^2)\, v'$$

$$y_2''(t) = 2\cos(t^2)\, v - 4t^2 \sin(t^2)\, v + 4\cos(t^2) v' + \sin(t^2) v''$$

sub into ODE

$$t\left[2\cos(t^2)\, v - 4t^2\sin(t^2)\, v + 4\cos(t^2)v' + \sin(t^2)v''\right] - \left[2t\cos(t^2)\, v + \sin(t^2)\, v'\right] + 4t^3\left[v\sin(t^2)\right] = 0$$

$$\cancel{2t\cos(t^2)\, v} - \cancel{4t^3\sin(t^2)\, v} + 4t\cos(t^2)v' + t\sin(t^2)v'' - \cancel{2t\cos(t^2)\, v} - \sin(t^2)\, v' + \cancel{4t^3\sin(t^2)\, v} = 0$$

Re ordering terms

$$t\sin(t^2)\, v'' + \left[4t\cos(t^2) - \sin(t^2)\right] v' = 0$$

$$\Rightarrow \quad v'' + \left[\frac{4t\cos(t^2)}{t\sin(t^2)} - \frac{\sin(t^2)}{t\sin(t^2)}\right] v' = 0$$

$$\Rightarrow \quad v'' + \left[4\frac{\cos(t^2)}{\sin(t^2)} - \frac{1}{t}\right] v' = 0$$

2. (cont'd)

$$\Rightarrow v'' + \left[4\frac{\cos(t^2)}{\sin(t^2)} - \frac{1}{t}\right]v' = 0$$

Since $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$

$$\Rightarrow v'' + \left[4\cot(t^2) - \frac{1}{t}\right]v' = 0$$

Reducing order by letting $w = v' \Rightarrow w' = v''$

So our 2^{nd} order eqn becomes

$$w' + [4\cot(t^2) - t^{-1}]w = 0$$

This is a separable Eqn:

$$\frac{dw}{dt} = -[4\cot(t^2) - t^{-1}]w$$

$$\Rightarrow \frac{1}{w}dw = [-4\cot(t^2) + t^{-1}]dt$$

Integrating

$$\int \frac{1}{w}dw = \int [-4\cot(t^2) + t^{-1}]dt$$

$$\ln(w) = \ln(t) - 2\ln(\sin(t^2))$$

$$= \ln(t) - \ln(\sin^2(t^2))$$

$$\Rightarrow \ln(w) = \ln\left[\frac{t}{\sin^2(t^2)}\right]$$

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2. (cont'd)

$$e^{\ln(w)} = e^{\ln\left[\frac{t}{\sin^2(t^2)}\right]}$$

$$\Rightarrow \quad w = \frac{t}{\sin^2(t^2)}$$

Then we find $v(t)$

$$v = \int v' \, dt = \int w \, dt = \int \frac{t}{\sin^2(t^2)} \, dt = -\frac{1}{2}\cot(t^2)$$

So $v(t) = -\frac{1}{2}\cot(t^2)$

then
$$\begin{aligned} y_2(t) &= v(t)\sin(t^2) \\ &= -\frac{1}{2}\cot(t^2)\sin(t^2) \\ &= -\frac{1}{2}\frac{\cos(t^2)}{\sin(t^2)}\sin(t^2) \\ &= -\frac{1}{2}\cos(t^2) \end{aligned}$$