2.1 Integrating Factors

The general form needed to use integrating factors is

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

and the integrating factor is given by

$$\mu(t) = \exp\left(\int p(t)\,dt\right) = e^{\int p(t)\,dt} \tag{2}$$

It is worth examining the derivation of this formula on page 36 of the textbook. The main point is that we choose $\mu(t)$ so that we end up with the product rule on the left hand side of our equation which makes it easier to solve using integration.

2.2 Separable Equations

A **separable equation** has general form

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{3}$$

or general form

$$M(x)\,dx + N(y)\,dy = 0 \tag{4}$$

We solve this type of equation by integrating the functions M and N *separately*.

1. (20 pts) Solve the initial value problem

$$t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0, \quad t < 0$$

Solution: First, we need to get this equation in the correct form. In other words we need it to "match" the general case given in (1). This means we need to have a coefficient of 1 for y'. So rewriting the equation (by dividing by t^3) we have

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$

This has the form in (1). We know we can use an integrating factor so for $p(t) = \dfrac{4}{t}$ we have So we use an integrating factor

$$\int p(t)dt = 4\int \frac{1}{t}\,dt = 4\ln|t| = \ln(t^4) = \ln(t^4)$$

So our integrating factor given by (2) is

$$\mu(t) = e^{\int p(t)dt} = e^{\ln(t^4)} = t^4$$

Now, multiply both sides of the equation by $\mu(t)$

$$\begin{aligned}
\mu(t)\left[y' + \frac{4}{t}y\right] &= \mu(t)\frac{e^{-t}}{t^3} \\
t^4\left[y; + \frac{4}{t}y\right] &= t^4\frac{e^{-t}}{t^3} \\
\underbrace{t^4y' + 4t^3y}_{\text{Product Rule}} &= te^{-t} \\
[t^4y]' &= te^{-t}
\end{aligned}$$

Now integrate both sides with respect to t

$$\int \left[t^4y\right]'\,dt = \int te^{-t}\,dt$$

To find $\int te^{-t}\,dt$ use Integration by Parts: $\int u\,dv = uv - \int v\,du$ where

$$\begin{aligned}
u &= t & v &= -e^{-t} \\
du &= 1\,dt & dv &= e^{-t}\,dt & \implies \int te^{-t}\,dt = -te^{-t} + \int e^{-t}\,dt
\end{aligned}$$

Using a u substitution we have $\int e^{-t}\,dt = -e^{-t} + c$ and so

$$\begin{aligned}\int \left[t^4 y\right]'\,dt &= \int te^{-t}\,dt \\ t^4 y &= (-t-1)e^{-t} + c\end{aligned}$$

> **Note:** In general, I will skip the integration step or other very tedious algebra reduction/simplifications. I use Mathematica to do most of this for me to ensure I get correct results. Check out my Mathematica notebooks on the WyoGroup to see how to do this!

Now we solve for y we have our **general solution**

$$\begin{aligned}t^4 y &= (-t-1)e^{-t} + C \\ t^{-4}t^4 y &= \left((-t-1)e^{-t} + C\right)t^{-4} \\ y &= ((-t-1)e^{-t} + C)t^{-4}\end{aligned}$$

> **Note:** It is way easier (in my opinion) in most of these cases to think about writing terms with negative exponents rather that as a fraction, especially when it comes to canceling terms! It's a lot easier to add exponents than to substract them as you do when you divide. If you often lose negative signs as I do, I highly recommend writing terms in this way.

So our **general solution** is

$$y = ((-t-1)e^{-t} + C)t^{-4}$$

To find our **particular solution** at $y(-1) = 0$ we have

$$0 = ((-(-1)-1)e^{-(-1)} + C)(-1)^{-4} \implies C = 0$$

Thus, the particular solution is

$$y = ((-t-1)e^{-t})t^{-4} = \frac{(-t-1)e^{-t}}{t^4}, \quad t < 0$$

Note that it is mathematically correct (and important) to specify your condition on t here!

2. (20 pts) Solve the initial value problem

$$x\,dx + ye^{-x}\,dy = 0, \quad y(0) = 1$$

Solution: We can see that this equation has form given in (4) so it's separable:

$$\begin{aligned} x\,dx + ye^{-x}\,dy &= 0 \\ ye^{-x}\,dy &= -x\,dx \\ y\,dy &= -xe^{x}\,dx \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int y\,dy &= -\int xe^{x}\,dx \\ \frac{y^2}{2} &= e^{x}(1-x) + C \\ y^2 &= 2e^{x}(1-x) + C \end{aligned}$$

This is our **general solution**. Note that it is currently an *implicit* function. To find our c we use our initial condition $y(0) = 1$:

$$\begin{aligned} (1)^2 &= 2e^{0}(1-(0)) + C \\ 1 &= 2 + c \\ \implies c &= -1 \end{aligned}$$

So our *implicit* **Particular Solution** is

$$y^2 = 2e^{x}(1-x) - 1$$

To find our explicit solution we simply take the square root of each side.

$$\begin{aligned} y^2 &= 2e^{x}(1-x) - 1 \\ \sqrt{y^2} &= \sqrt{2e^{x}(1-x) - 1} \\ y &= \pm\sqrt{2e^{x}(1-x) - 1} \end{aligned}$$

Our solution needs to satisfy our initial condition which is positive. Thus we choose the positive square root. This gives us our *explicit* particular solution

$$y = \sqrt{2e^{x}(1-x) - 1}$$