Lecture #20: Intro to Homog. Systems Date: Mon 4/29/19

We will consider first order homogeneous linear systems given by

$$\vec{x}\,' = \vec{P}(t)\,\vec{x}$$

Let $p(t) = A$ be a matrix w/ entries that are csts

$$\Rightarrow \frac{d\vec{x}}{dt} = \underset{\substack{\uparrow \\ \text{Coeff Matrix}}}{A}\,\vec{x}$$

We are dealing w/ systems of n eqns w/ n unknowns
We can extend our theory for one eqns to systems.

For $n=1$: $\dfrac{dx}{dt} = ax$, $a \neq 0$

Which has soln of the form $x(t) = x_0 e^{rt}$
Where the only constant soln is $x = 0$.

So for a system of ODEs we seek solns of the form

$$\vec{x} = \vec{\xi} e^{rt} = \begin{bmatrix} \xi_1 e^{rt} \\ \xi_2 e^{rt} \end{bmatrix}$$

We want this to be a soln to $\dfrac{d\vec{x}}{dt} = A\vec{x}$

$$\Rightarrow \underbrace{r\vec{\xi} e^{rt}}_{\frac{d\vec{x}}{dt}} = A \underbrace{\vec{\xi} e^{rt}}_{\vec{x}}$$

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Canceling e^{rt} terms (since $e^{rt} \neq 0$)

$$\Rightarrow A\vec{\xi} = r\vec{\xi}$$

$$\Rightarrow (A - rI)\vec{\xi} = 0$$

This last statement is only true if

$$\det(A - rI) = 0$$

In other words, r must be an eigenvalue and $\vec{\xi}$ the eigenvector corresponding to r

Thm (7.4.1) Principle of Superposition

If the vector fcns $\vec{x}^{(1)}(t)$ & $\vec{x}^{(2)}(t)$ are solns of the system $\vec{x}' = \vec{P}(t)\vec{x}$ then the linear combination

$$c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t)$$

is also a soln for any constants c_1, c_2

For a system of n eqns then for n properly chosen solns if $\vec{x}^{(1)}(t), \ldots \vec{x}^{(n)}(t)$ are solns to the system then

$$\vec{x} = c_1\vec{x}^{(1)}(t) + \ldots + c_n\vec{x}^{(n)}(t)$$

is also a soln.

Then the matrix

$$\mathbf{X}(t) = [\vec{x}^{(1)}(t) \quad \vec{x}^{(2)}(t) \quad \ldots \quad \vec{x}^{(n)}(t)]$$

$$= \begin{bmatrix} x_{11}(t) & x_{12}(t) & \ldots & x_{1n}(t) \\ \vdots & & & \\ x_{n1}(t) & \ldots & \ldots & x_{nn}(t) \end{bmatrix}$$

will have linearly independent columns iff $\det(\mathbf{X}) \neq 0$. The determinant of this matrix is the <u>wronskian.</u> So $\mathbf{X}(t)$ will have linearly indep. columns for every value for which

$$W[\vec{x}^{(1)}(t) \ \ldots \ \vec{x}^{(2)}(t)] = \det(\mathbf{X}) \neq 0$$

<u>Thm</u> (7.4.2)

If vector fcns $\vec{x}^{(1)}, \ldots \vec{x}^{(n)}$ are linearly independent solns of the system $\vec{x}' = A\vec{x}$ for each pt in interval $\alpha < t < \beta$ then each soln $\vec{x} = \vec{\phi}(t)$ of the system can be expressed as a linear combination of $\vec{x}^{(1)}, \ldots \vec{x}^{(n)}$

$$\phi(t) = c_1 \vec{x}^{(1)}(t) + \ldots c_n \vec{x}^{(n)}(t)$$

in exactly one way.

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Def Any set of linearly independent set of solns $\vec{x}^{(1)}, \ldots \vec{x}^{(n)}$ is the <u>Fundamental set of solns.</u>

Ex. 1 Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Find eigenvalues:

i.e. soln to $\det(A - \lambda I) = 0$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$\Rightarrow (1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4$$

$$1-\lambda = \pm 2$$

$$\lambda = 1 \pm 2 \Rightarrow \lambda = -1, 3$$

Find eigenvectors for each eigenvalue

For $\lambda = -1$

$$A + I = \begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \Rightarrow \text{Solve system} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \underset{\text{rref}}{\Rightarrow} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \xi_1 + \frac{1}{2}\xi_2 = 0$$

$$\Rightarrow \xi_1 = -\frac{1}{2}\xi_2$$

$$\Rightarrow \xi_2 = -2\xi_1$$

So evec is

$$\vec{\xi}^{(1)} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$$A - 3I = \begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

Solve

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \underset{\text{rref}}{\Longrightarrow} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \xi_1 - \frac{1}{2}\xi_2 = 0$$

$$\Rightarrow \xi_1 = \frac{1}{2}\xi_2$$

or

$$\xi_2 = 2\xi_1$$

evec for $\lambda = 3$ is

$$\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So gen. soln has form

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{\xi}^{(1)} + c_2 e^{\lambda_2 t} \vec{\xi}^{(2)}$$

$$= c_1 e^{-t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-t} + e^{3t} \\ e^{-t} + 2e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

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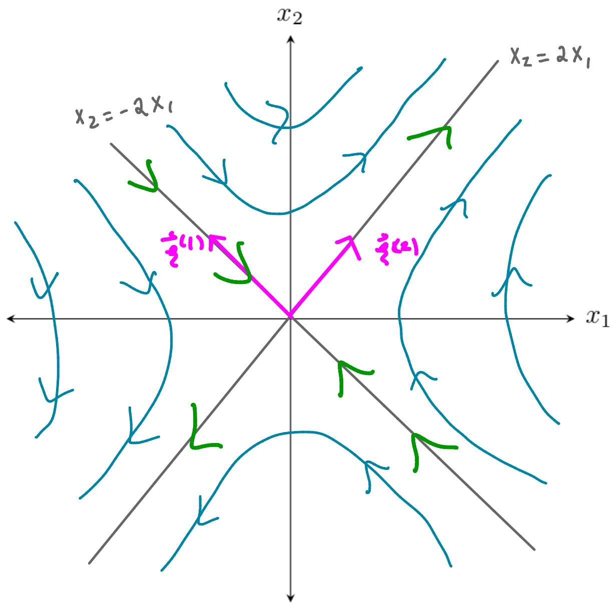
The Phase Plane

For 2x2 systems, we can visualize solns on an x_1x_2 plane known as the phase plane

Sketching representative trajectories of our system is called a phase portrait



In our example we found our evecs via the following eqns:

$\lambda_1 = -1$: $x_2 = -2x_1$

$\lambda_2 = 3$: $x_2 = 2x_1$

These are lines.

Origin is a saddle point

When e-val is negative: solns move toward the origin

When e-val is positive: solns move away from origi

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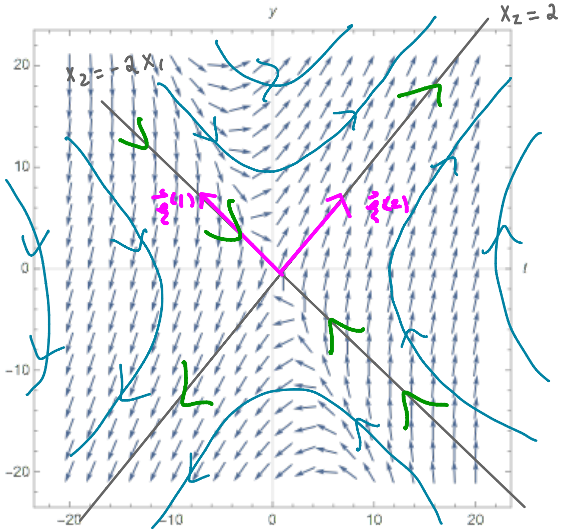
Looking @ the direction field along w/ our phase portrait we can see that our trajectories follow the same behavior.



Math ________

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