Section Summary

Higher order equations have the general form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Similar to 2nd order homogeneous equations we examine the roots of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

We will have 3 cases for the general form of our solutions.

- **Real, distinct roots** for distinct roots r_1, r_2, ..., r_k will have terms

$$e^{r_1 t}, \quad e^{r_2 t}, \ \ldots\ , \quad e^{r_k t}$$

- **Repeated roots** if r_1 is a repeated root (with multiplicity m) solutions corresponding to r_1 will have the form

$$e^{r_1 t} + t e^{r_1 t} + \cdots + t^{m-1} e^{r_1 t}$$

- **Complex roots** this is the tricky case since all roots will need to be considered and can often be difficult to find. If we have a complex root $\lambda \pm i\mu$ repeated k times we will have solutions with general form

$$e^{\lambda t}\cos\mu t, \quad e^{\lambda t}\sin\mu t, \quad t e^{\lambda t}\cos(\mu t), \quad t e^{\lambda t}\sin\mu t, \ \ldots\ , \quad t^{k-1}e^{\lambda t}\cos(\mu t), \quad t^{k-1}e^{\lambda t}\sin\mu t$$

The case for multiple complex roots is demonstrated in Problem (3)

Find the general solution to the given differential equations.

1. (10 pts) $y^{(4)} - 5y'' + 4y = 0$

Solution: The characteristic equation is

$$\begin{aligned} r^4 - 5r^2 + 4 &= 0 \\ (r^2-1)(r^2-4) &= 0 \\ (r-1)(r+1)(r-2)(r+2) &= 0 \\ r &= 1, -1, 2, -2 \end{aligned}$$

Here we have real, distinct roots, so the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}$$

2. (10 pts) $y^{(4)} - 4y''' + 4y'' = 0$

Solution: The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 4r^2 &= 0 \\ r^2(r^2 - 4r + r) &= 0 \\ r^2(r-2)(r-2) &= 0 \\ r &= 0, 0, 2, 2 \end{aligned}$$

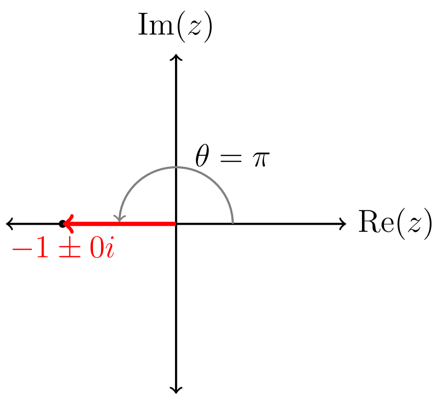
Here we have repeated roots so the general solution is

$$y(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$$

3. (20 pts) $y^{(6)} + y = 0$

Solution: The characteristic equation is

$$
\begin{aligned}
r^6 + 1 = 0 &\implies r^6 = -1 \\
&\implies r = (-1)^{1/6} = \sqrt[6]{-1} = i^{1/6}
\end{aligned}
$$



So we need to compute the sixth roots of -1. Recall that we can write a complex number $z = \lambda + i\mu$ in exponential form where

$$z = re^{(i(\theta + 2\pi n))} = r \exp\big(i[\theta + 2\pi n]\big)$$

where $r = |z|$ and $\theta =$ Arg z which just means the angle between z and the positive part of the real line (see figure). Note that $|z|$ is the modulus and not the absolute value! The modulus is defined as

$$|z| = |\lambda + i\mu| = \sqrt{\lambda^2 + \mu^2}$$

So for $z = -1 + 0i$ we have that $r = |-1| = 1$ and $\theta = \pi$ and so

$$(-1)^{1/6} = \exp(i(\theta + 2n\pi))^{1/6} = e^{i(\frac{\pi}{6} + \frac{2n\pi}{6})} = e^{i(\frac{\pi}{6} + \frac{n\pi}{3})}$$

Now we use Euler's Formula to re-write the exponentials in terms of sines and cosines $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and so we have

$$e^{i\left(\frac{\pi}{6} + \frac{n\pi}{3}\right)} = \cos\left(\frac{\pi}{6} + \frac{n\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{n\pi}{3}\right) \qquad n = 0, 1, 2, 3, 4, 5$$

So we have the following roots found by plugging in each value of n.

$$
\begin{aligned}
\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\
\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) &= i \\
\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \\
\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right) &= -\frac{\sqrt{3}}{2} - \frac{1}{2}i \\
\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right) &= -i \\
\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) &= \frac{\sqrt{3}}{2} - \frac{1}{2}i
\end{aligned}
$$

and so are roots are $\pm i$, $\dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i$, $-\dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i$ and our general solution is

$$y(t) = c_1 \cos t + c_2 \sin t + e^{\frac{\sqrt{3}t}{2}} \left[c_3 \cos\left(\frac{t}{2}\right) + c_4 \sin\left(\frac{t}{2}\right)\right] + e^{-\frac{\sqrt{3}t}{2}} \left[c_5 \cos\left(\frac{t}{2}\right) + c_6 \sin\left(\frac{t}{2}\right)\right]$$

Note that we find the first 6 roots since we know that the characteristic equation has at most 6 roots. Recall that complex numbers are defined on a circle and so higher values of n would simply give us a multiple of one the above values.