Worksheet 15 Solutions

Laplace Transform of Step Functions

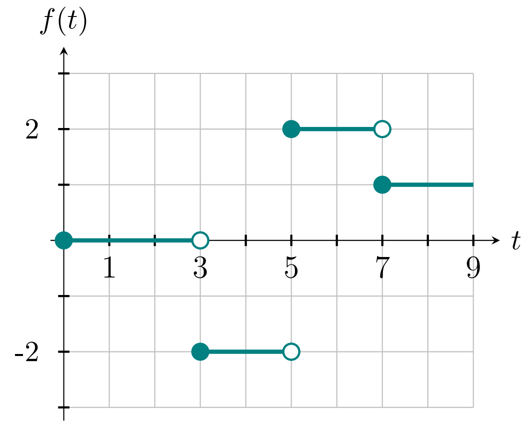
MATH 2310, Spring 2019

You are permitted to use technology to assist you with factoring and partial fraction decompositions. All other work should be done by hand. Solutions that do not have an appropriate amount of detail will not receive credit.

1. (a) (6 pts) Express the function

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

in terms of the unit step function $u_c(t)$. Then, sketch the graph on the axes to the right.



Solution: The notation $u_c(t)$ tells us that c is the point on the t-axis at which the jump occurs in the function. Any constant in front of each piece is the distance that is jumped.

Based on the intervals on which each piece of $f(t)$ is defined we re-write each of these pieces with the $u_c(t)$ notation:

$$\begin{aligned} 0 \le t < 3: &\quad -2u_3(t) \\ 3 \le t < 5: &\quad 4u_5(t) \\ 5 \le t < 7: &\quad -u_7(t) \end{aligned}$$

So our function can now be written as

$$f(t) = -2u_3(t) + 4u_5(t) - u_7(t)$$

(b) (4 pts) Find the Laplace transform of $f(t)$

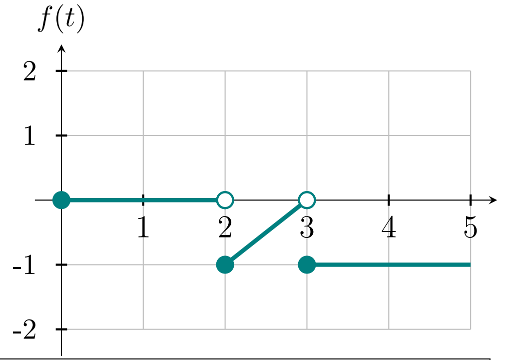
Solution:

$$\begin{aligned} F(s) &= \mathcal{L}\left\{-2u_3(t) + 4u_5(t) - u_7(t)\right\} \\ &= -2\mathcal{L}\left\{u_3(t)\right\} + 4\mathcal{L}\left\{u_4(t)\right\} - \mathcal{L}\left\{u_7(t)\right\} \\ &= -2\left(\frac{e^{-3s}}{s}\right) + 4\left(\frac{e^{-4s}}{s}\right) - \frac{e^{-7s}}{s} \end{aligned}$$

2. (a) (6 pts) Convert the function

$$g(t) = (t-3)u_2(t) - (t-2)u_3(t)$$

to piecewise function notation and sketch the graph on the interval $t \geq 0$.



> **Solution:** We can re-write $g(t)$ as the following:
>
> $$g(t) = (t-3)\begin{cases} 0, & t \leq 2 \\ 1, & t \geq 2 \end{cases} + (t-2)\begin{cases} 0, & t \leq 3 \\ 1, & t \geq 3 \end{cases}$$
>
> Examining the pieces given by $u_2(t)$ and $u_3(t)$ we see that we have some intervals that overlap, which will give us 3 different intervals on which different functions are defined. The 3 intervals are
>
> $$0 \leq t < 2, \quad 2 \leq t < 3, \quad t \geq 3$$
>
> By multiplying the pieces of our functions on the appropriate intervals we obtain
>
> $$g(t) = \begin{cases} (t-3)(0) - (t-2)(0), & 0 \leq t < 2 \\ (t-3)(1) - (t-2)(0), & 2 \leq t < 3 \\ (t-3)(1) - (t-2)(1), & t \geq 3 \end{cases} \implies g(t) = \begin{cases} 0, & 0 \leq t < 2 \\ t-3, & 2 \leq t < 3 \\ -1, & t \geq 3 \end{cases}$$

(b) (4 pts) Find the Laplace transform of $g(t)$

> **Solution:** First, we rewrite $g(t)$ so we can see the appropriate shifts
>
> $$g(t) = \big[(t-2) - 1\big]u_2(t) - \big[(t-3) + 1\big]u_3(t)$$

Then,

$$\begin{aligned}
G(s) &= \mathcal{L}\left\{\big[(t-2)-1\big]u_2(t) - \big[(t-3)+1\big]u_3(t)\right\} \\
&= \mathcal{L}\left\{\big[(t-2)-1\big]u_2(t)\right\} - \mathcal{L}\left\{\big[(t-3)+1\big]u_3(t)\right\} \\
&= \mathcal{L}\left\{u_2(t)f(t-2)\right\} - \mathcal{L}\left\{u_3(t)f(t-3)\right\} \\
&= e^{-2s}F_1(s) - e^{-2s}F_2(s)
\end{aligned}$$

To find $F_1(s)$ and $F_2(s)$ we need to take the Laplace transform of the *unshifted* functions $f_1(t)$ and $f_2(t)$. We have

$$\begin{array}{ccc}
\text{The shifted function} & & \text{the unshifted function} \\
f_1(t-2) = (t-2)-1 & \Longrightarrow & f_1(t) = t-1 \\
f_2(t-3) = (t-3)+1 & \Longrightarrow & f_2(t) = t+1
\end{array}$$

so we have

$$F_1(s) = \mathcal{L}\left\{f_1(t)\right\} = \mathcal{L}\left\{t-1\right\} = \mathcal{L}\left\{t\right\} - \mathcal{L}\left\{1\right\} = \frac{1}{s^2} - \frac{1}{s}$$

$$F_2(s) = \mathcal{L}\left\{f_2(t)\right\} = \mathcal{L}\left\{t+1\right\} = \mathcal{L}\left\{t\right\} + \mathcal{L}\left\{1\right\} = \frac{1}{s^2} + \frac{1}{s}$$

So the Laplace transform of $g(t)$ is then

$$\begin{aligned}
G(s) &= e^{-2s}F_1(s) - e^{-2s}F_2(s) \\
&= e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) - e^{-2s}F_2(s)\left(\frac{1}{s^2} + \frac{1}{s}\right)
\end{aligned}$$

3. (10 pts) Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$

Solution: Here we will need to complete the square $t^2 - 2t + 2 = (t-1)^2 + 1$ and so

$$f(t) = \begin{cases} 0, & t < 1 \\ (t-1)^2 + 1, & t \geq 1 \end{cases} \quad \Longrightarrow \quad f(t) = [(t-1)^2 + 1]u_1(t)$$

We see that $f(t)$ has the form $f(t) = g(t-1)u_1(t)$. This implies that the unshifted function $g(t) = t^2 + 1$. From #13 in the table we know that

$$\mathcal{L}^{-1}\{f(t)\} = F(s) = e^{-1s} \cdot G(s) \tag{1}$$

$$G(s) = \mathcal{L}\left\{t^2 + 1\right\} = \mathcal{L}\left\{t^2\right\} + \mathcal{L}\{1\} = \frac{2}{s^3} + \frac{1}{s}$$

We plug this back in to (1) and obtain

$$F(s) = e^{-1s} \cdot G(s) = e^{-1s} \cdot \left[\frac{2}{s^3} + \frac{1}{s}\right]$$

4. (10 pts) Find the Laplace transform of $g(t) = \begin{cases} 0 & t < 4 \\ 3\sin(\pi t), & 4 \le t < 5 \\ 0, & t \ge 5 \end{cases}$

Solution: In $u_c(t)$ notation we have

$$g(t) = \sin(t) - \sin(t)u_\pi(t)$$

In order to use the Laplace transform given by # 13 in the table we need to represent the functions with a shift that "matches" the value in the unit step function. Here we must re-write the $\sin(t)$ so that it has a shift of π. To do this we write

$$\begin{aligned} \sin(t) &= \sin(t - \pi + \pi) \\ &= \sin((t - \pi) - \pi) \end{aligned}$$

Note that we technically added nothing here! Now we can use the sum to product formula. We let $\alpha = t - \pi$ and $\beta = \pi$ to obtain

$$\begin{aligned} \sin((t-\pi)+\pi) &= \sin(t-\pi)\cos(\pi) + \cos(t-\pi)\sin(\pi) \\ &= \sin(t-\pi)(-1) + \cos(t-\pi)(0) \\ &= -\sin(t-\pi) \end{aligned}$$

So,

$$g(t) = \sin(t) + \sin(t-\pi)u_\pi(t)$$

Now we take the Laplace transform to obtain

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{\sin(t-\pi)u_\pi(t)\}$$

Where

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1^2} \tag{2}$$

$$\mathcal{L}\{\sin(t-\pi)u_\pi(t)\} = e^{-\pi s}F(s) \tag{3}$$

Now $F(s)$ is the Laplace transform of the *unshifted* function. In other words, $\mathcal{L}\{f(t)\} = F(s)$. Since

$$f(t-\pi) = \sin(t-\pi) \quad \Longrightarrow \quad f(t) = \sin(t)$$

then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1^2}$$

and so

$$\mathcal{L}\{\sin(t-\pi)u_\pi(t)\} = e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)$$

and so the Laplace tranform of $g(t)$ is

$$\mathcal{L}\left\{g(t)\right\} = \frac{1}{s^2+1^2} + e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)$$

Alternatively, we can skip using sum formula by using #15 from the Paul's Online Math Notes table. Starting from (3) we recognize that for

$$f(t-\pi) = \sin(t-\pi)$$

we have the unshifted function

$$f(t) = \sin(t+\pi)$$

So now using #15 we let $a = 1$ and $b = \pi$ we have

$$\mathcal{L}\left\{\sin(t+\pi)\right\} = \frac{s\sin(\pi) + (1)\cos(\pi)}{s^2+1^2} = \frac{-1}{s^2+1^2}$$

and so the Laplace tranform of $g(t)$ is

$$\begin{aligned}\mathcal{L}\left\{g(t)\right\} &= \frac{1}{s^2+1^2} - e^{-\pi s}\left(\frac{-1}{s^2+1^2}\right)\\ &= \frac{1}{s^2+1^2} + e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)\end{aligned}$$

which is the same thing as we got above.

Warning: Do not just assume signs change when you are adjusting your shifts in trig functions. You will not always have a shift involving π!