Lecture # 6: Exact Equations

Date: mon. 2/25/19

Most 1^{st} order ODEs cannot be solved as a separable equation or by integrating factors.

Another type of 1^{st} order eqn has the general form

$$M(x,y) + N(x,y)\,y'(x) = 0$$

or

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

<u>Note:</u> this form differs from the general form of a sep. eqn. Here M & N are fcns of both x & y.

Suppose there is a fcn $\psi(x,y) = 0$ where

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \& \quad \frac{\partial \psi}{\partial y} = N(x,y)$$

By Chain Rule we have

$$0 = \frac{d}{dx}\psi(x,y)$$

$$= \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\,y'(x)$$

$$= M(x,y) + N(x,y)\,y'(x) \quad (*)$$

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So our eqn in form of $(*)$ can be reduced to

$$\frac{d}{dx}\psi(x,y)=0$$

By a simple integration we obtain the soln

$$\int \frac{d}{dx}\psi(x,y)\,dx = \int 0\,dx \quad \Rightarrow \quad \psi(x,y)=c$$

where $\psi(x,y)=c$ is an implicit fcn

Def An eqn of form $M(x,y)+N(x,y)\,y'(x)=0$ is exact if $\exists$ a fcn $\psi(x,y)$ which satisfies

"there exists"

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$$

i.e. $M_y = N_x$

We'll apply this definition to develop the process to solve these types of ODEs

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Suppose $\exists$ a fcn $\psi(x,y)$ s.t.

$$\frac{\partial \psi}{\partial x} = M \quad \& \quad \frac{\partial \psi}{\partial y} = N$$

taking partial derivatives we find that

$$M_y = \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = \psi_{xy} \quad \& \quad N_x = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} = \psi_{yx}$$

Note: These are the mixed partials of ψ

If $\psi(x,y)$ is continuous & 1^{st} order derivatives are continuous then we must have that

$$\psi_{yx} = \psi_{xy}$$

So we must have that $M_y = N_x$.

How do we find $\psi(x,y)$?

Since $\psi_x = M$ & $\psi_y = N$ we first integrate ψ_x w.r.t. x

$$\psi(x,y) = \int \psi_x \, dx = \int M \, dx = Q(x,y) + h(y)$$

"fcn of integration"

Where $Q(x,y)$ is some diff'ble fcn of x & y. Our "constant" of integration is the fcn $h(y)$.

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Now we need to find $h(y)$. We need to use our N to do this. Since $N = \varphi_y$

Take $\varphi(x,y)$ that we just found & diff w.r.t. y

$$\varphi(x,y) = \overbrace{Q(x,y)}^{\text{Known}} + \overbrace{h(y)}^{\text{Unknown}}$$

$$\Rightarrow \varphi_y = \frac{\partial}{\partial y} Q(x,y) + h'(y)$$

$$= Q_y + h'(y)$$

Since $N = \varphi_y$ then

$$N = Q_y + h'(y)$$

Solving for $h'(y)$ we have

$$h'(y) = N - Q_y$$

Integrate w.r.t. y

$$h(y) = \int h'(y)\,dy = \int (N - Q_y)\,dy$$

So gen. form of soln will be

$$\varphi(x,y) = Q(x,y) + h'(y)$$

$$= Q(x,y) + \int (N - Q_y)\,dy$$

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Ex. 1 Solve

$$(y\cos(x) + 2xe^y) + (\sin(x) + x^2e^y - 1)y' = 0$$

Has general form $M(x,y) + N(x,y)y' = 0$

Let

$$M(x,y) = y\cos(x) + 2xe^y$$
$$N(x,y) = \sin(x) + x^2e^y - 1$$
$$\Rightarrow \quad M_y = \cos(x) + 2xe^y$$
$$N_x = \cos(x) + 2xe^y$$

Since $N_x = M_y$ eqn is exact

Then $\exists\ \psi(x,y) = c$ s.t.

$$M = \psi_x = y\cos(x) + 2xe^y$$
$$N = \psi_y = \sin(x) + x^2e^y - 1$$

Take $M = \psi_x$ & Integrate w.r.t. y

$$\Rightarrow \psi(x,y) = \int \psi_y\, dx = \int M\, dx$$
$$= \int (y\cos(x) + 2xe^y)\, dx$$
$$= y\sin(x) + x^2e^y + h(y)$$

So

$$\psi(x,y) = y\sin(x) + x^2e^y + h(y)$$

Find ψ_y

$$\psi_y = \sin(x) + x^2e^y\ h'(y)$$

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Ex. 1 (cont'd)

Since $N = \psi_y$ as well, set equal & solve for $h'(y)$.

$$\underbrace{\sin(x) + x^2e^y - 1}_{N} = \underbrace{\sin(x) + x^2e^y + h'(y)}_{\psi_y}$$

Solving for $h'(y)$

$$h'(y) = -1$$

Integrate wrt y to find $h(y)$

$$h(y) = \int h'(y)\,dy = \int -1\,dy = -y$$

So gen. (implicit) soln is

$$C = \psi(x,y) = y\sin(x) + x^2e^y + h(y)$$

$$= y\sin(x) + x^2e^y - y$$

$$\Rightarrow \quad y\sin(x) + x^2e^y - y = C$$