We will now consider how to find solutions to the *nonhomogeneous* equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where

$$L[y] = y'' + p(t)y' + q(t)y = 0 \tag{2}$$

is the homogeneous equation corresponding to (1).

Theorem 1: (3.5.1)

If Y_1 and Y_2 are two solutions of the nonhomogeneous equation, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation. If, in addition, y_1 and y_2 are a fundamental set of solutions of (1) then

$$Y_1(t) - Y_2(t) = c_1y_1 + c_2y_2$$

where c_1 and c_2 are constants.

Theorem 2: (3.5.2)

The general solution of the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

can be written in the form

$$y = \phi(t) = c_1y_1 + c_2y_2 + Y(t)$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and c_1, c_2 are arbitrary constants and $Y(t)$ is some particular solution to the nonhomogeneous equation (1).

By Theorem 3.5.1 we can select some arbitrary solution where $Y_1(t) = \phi(t)$, so that

$$\begin{aligned} \phi(t) - Y(t) &= c_1y_1 + c_2y_2 \\ \implies \phi(t) &= c_1y_1 + c_2y_2 + Y(t) \end{aligned}$$

This $\phi(t)$ will include all solutions of (1) and so is referred to as the general solution.

Essentially, to find the solution to a nonhomogeneous equation we must

- Find the general solution to the homogeneous equation. This solution is referred to as the **complimentary solution** and is denoted by $y_c(t)$.
- Find a solution of the nonhomogeneous equation $Y(t)$. This solution is referred to as the **particular solution**.

- Write the solution as the sum of $y_c(t)$ and $Y(t)$:

$$y(t) = y_c(t) + Y(t) = c_1 y_1 + c_2 y_2 + Y(t) \tag{3}$$

We already know how to find solutions to the homogeneous equation, $y_c(t)$. There are two main methods that can be used to find the solution to the nonhomogeneous equation.

- Method of Undetermined Coefficients
- Variation of Parameters

Method of Undetermined Coefficients

This method assumes that $Y(t)$ should have a form similar to $g(t)$. Initially using generic coefficients we can then determine a more specific form for our particular solution. The general process entails the following steps:

- Determine a good guess for a solution to $Y(t)$ using Table 1 based on the form of $g(t)$.
- Let $y_p(t) = Y(t)$ and find $Y''(t)$ and $Y'(t)$.
- Plug $Y(t)$, $Y''(t)$, and $Y'(t)$ into the differential equation (1) and simplify the right hand side of the equation.
- Equate coefficients on the right hand side with those on the left hand side to obtain a system of equations, then solve to get your coefficients.
- Plug coefficients into $y_p(t)$ to obtain the $Y(t)$ for your solution.

Recall that to determine if a solution satisfies a differential equation we must plug in the relevant derivatives and see that it results in a true statement. In other words, our coefficients at the beginning of this process are *undetermined coefficients.*

We can use the table below to determine a guess for $y_p(t)$ based on the form of $g(t)$. This table is more clear than the one in your text (on pg. 182).

$g(t)$	$Y_p(t)$ guess
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n-th degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$

Table 1: General Form for Particular Solutions

One thing is important to note. Just as with homogeneous equations with repeated roots, we can not have repeated solutions in our general solution. This means that a solution that appears in the complimentary solution cannot appear in the particular solution of a nonhomogeneous equation. If a solution is repeated in the particular solution, we must multiply by a t^n, where n is the smallest integer that will differentiate it from all solutions in the complimentary solution $y_c(t)$.

Example 1: Find a general solution to

$$y'' - 2y' - 3y = 3e^{2t}$$

Nonhomogeneous eqn:

Find Homog. soln for

$$y'' - 2y' - 3y = 0$$

Char. eqn:

$$r^2 - 2r - 3 = 0$$

$$\Rightarrow (r+1)(r-3) = 0 \quad \Rightarrow r_1 = -1, \ r_2 = 3$$

Complimentary soln is:

$$y_c(t) = c_1 e^{-t} + c_2 e^{3t}$$

Find particular soln.

Choose $y_p(t) = Ae^{2t}$

Let $y_p(t) = Y(t)$

$$Y(t) = Ae^{2t}$$

$$Y'(t) = 2Ae^{2t}$$

$$Y''(t) = 4Ae^{2t}$$

Plug into ODE

$$4Ae^{2t} - 2(2Ae^{2t}) - 3(Ae^{2t}) = 3e^{2t}$$

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Ex. 1 (cont'd)

$$\Rightarrow -3Ae^{2t} = 3e^{2t}$$

Equate coeff.s

$$\Rightarrow -3A = 3 \quad \Rightarrow A = -1$$

so $y_p(t) = -e^{2t}$

Gen. soln to Nonhomog. ODE is

$$y(t) = y_c(t) + y_p(t)$$
$$= c_1 e^{-t} + c_2 e^{3t} - e^{2t}$$

Example 2: Find a general solution to

$$y'' + 2y' + 5y = 3\sin(2t)$$

Find homog. soln for

$$y'' + 2y' + 5y = 0$$

Char. eqn: $r^2 + 2r + 5 = 0$

$$r_{1,2} = -1 \pm 2i$$

$$\lambda = -1, \quad \mu = 2$$

Complimentary soln.

$$y_c(t) = e^{-t}(C_1\cos(2t) + C_2\sin(2t))$$

$$= C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$$

Particular soln.

Choose. $y_p(t) = A\cos(2t) + B\sin(2t)$

Let $y_p(t) = Y(t)$

$$Y(t) = A\cos(2t) + B\sin(2t)$$

$$Y'(t) = -2A\sin(2t) + 2B\cos(2t)$$

$$Y''(t) = -4A\cos(2t) - 4B\sin(2t)$$

Plug into ODE

$$[-4A\cos(2t) - 4B\sin(2t)] + 2[-2A\sin(2t) + 2B\cos(2t)]$$

$$+5[A\cos(2t) + B\sin(2t)]$$

$$= 3\sin(2t)$$

Ex. 2 (cont'd)

$$-4A\cos(2t) - 4B\sin(2t) - 4A\sin(2t) + 4B\cos(2t)$$
$$+5A\cos(2t) + 5B\sin(2t) = 3\sin(2t)$$

$$A\cos(2t) + B\sin(2t) - 4A\sin(2t) + 4B\cos(2t)$$
$$= 3\sin(2t)$$

Combine like terms

$$\Rightarrow (A+4B)\cos(2t) + (-4A+B)\sin(2t) = 3\sin(2t)$$

Equate coeffs

$$A + 4B = 0$$
$$-4A + B = 3$$

Solving system (w/ tech)

$$A = -\frac{12}{17} \quad \& \quad B = \frac{3}{17}$$

$$y_p(t) = -\frac{12}{17}\cos(2t) + \frac{3}{17}\sin(2t)$$

So general soln to non-homog. ODE is

$$y(t) = y_c(t) + y_p(t)$$
$$= e^{-t}(c_1\cos(2t) + c_2\sin(2t))$$
$$-\frac{12}{17}\cos(2t) + \frac{3}{17}\sin(2t)$$