Lecture 15: Laplace Transforms of Step Functions

Math 2310-360: Differential Equations Spring 2019

Step Functions

In some applications you will encounter discontinuous forcing functions. These types of functions appear in models involving electrical circuits or a force with an impulse. This is where the **unit step function** (or **Heaviside function**) comes in to play.

Definition: Unit Step Function, $u_c(t)$

The **unit step function** is defined as

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

and is shown in the Fig. 1.

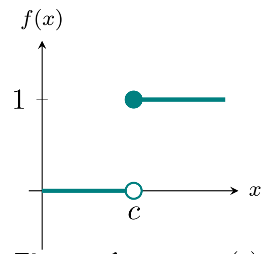


Figure 1: $y = u_c(t)$

Conceptually, think of this as a switch that is "off" until it switches "on" and jumps "up" a (positive) distance of 1 at $t = c$.

There are several alternative notations for the Unit Step function

$$u_c(t) = u(t - c) = H(t - c)$$

There is also the alternative case where the unit step function is "on" until $t = c$ when it jumps "down" a (negative) distance of 1 and turns "off".

Definition: Unit Step Function, $1 - u_c(t)$

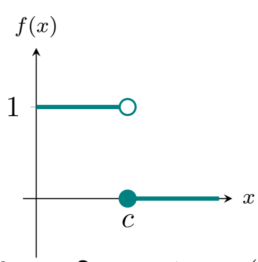


Figure 2: $y = 1 - u_c(t)$

This function is defined as

$$1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$

and is shown in Figure 2.

In general, try to think of the constant (or function) in front of the $u_c(t)$ as the distance "jumped" either in a positive or negative direction (i.e. "up" vs "down"). The time $t = c$ is when the function "jumps".

There are two key Laplace tranforms for these types of functions given in your table.

$$\textbf{\# 12} \quad \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \qquad\qquad \textbf{\# 13} \quad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

When using # 13 we are taking the Laplace transform of a *shifted* function. We may not see this directly in $f(t)$ so we need to account for the shift in our function (if it isn't there already) i.e. we need to rewrite it so it has the form of $f(t-c)$ before we can use the transform. Notice that we don't have a formula for $u_c(t)f(t)$!

The result is the Laplace transform of the *shifted* function. When we take the inverse Laplace transform of $\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\}$ we will get $\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$ which is the inverse Laplace transform of the *unshifted* function. This means we will need to add the shift back in to get back to our $f(t-c)$.

For shifting trig functions you may find the following identities useful.

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \qquad \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\cos(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \qquad \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\cos(\beta)$$

The Paul's Online Math Notes table has the following two identities which allow you to skip the heavy trig work.

 # 15 $\mathcal{L}\left\{\sin(at+b)\right\} = \dfrac{s\sin(b) + a\cos(b)}{s^2+a^2}$ **# 16** $\mathcal{L}\left\{\cos(at+b)\right\} = \dfrac{s\cos(b) - a\sin(b)}{s^2+a^2}$

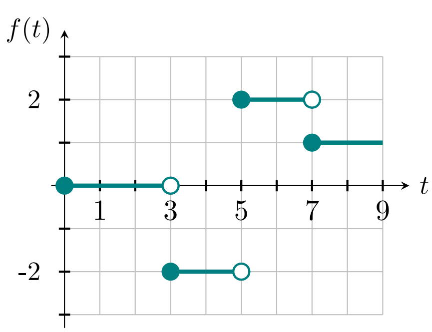
Example 1

Express the function

$$f(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 5 \\ 2, & 5 \le t < 7 \\ 1, & t \ge 7 \end{cases}$$

in terms of the unit step function $u_c(t)$ and sketch the graph.

Solution. The notation $u_c(t)$ tells us that c is the point on the t-axis at which the jump occurs in the function. Any constant in front of each piece is the distance that is jumped.



Based on the intervals on which each piece of $f(t)$ is defined we re-write each of these pieces with the $u_c(t)$ notation:

$$\begin{aligned} 0 \le t < 3: &\quad -2u_3(t) \\ 3 \le t < 5: &\quad 4u_5(t) \\ 5 \le t < 7: &\quad -u_7(t) \end{aligned}$$

So our function can now be written as

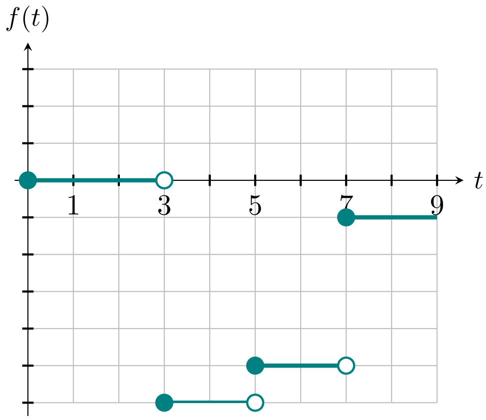
$$f(t) = -2u_3(t) + 4u_5(t) - u_7(t)$$

Example 2

Convert the function

$$g(t) = -6u_3(t) + u_5(t) + 4u_7(t)$$

to piecewise function notation and sketch the graph on the interval $t \geq 0$.



Solution. Remember that each constant in front of the $u_c(t)$ as the distance "jumped" either in a positive or negative direction (i.e. "up" vs "down"). This let's us re-write the function $g(t)$ as

$$g(t) = \begin{cases} 0, & t \leq 3 \\ -6, & 3 \leq t < 5 \\ -5, & 5 \leq t < 7 \\ -1, & t \geq 7 \end{cases}$$

Example 3

Find the Laplace transform of $f(t) = \begin{cases} 0, & t < 4 \\ t^2 - 8t + 19, & t \geq 4 \end{cases}$

Solution. Here we will need to complete the square $t^2 - 8t + 19 = (t-4)^2 + 3$ and so

$$f(t) = \begin{cases} 0, & t < 4 \\ (t-4)^2 + 3, & t \geq 4 \end{cases} \quad \Longrightarrow \quad f(t) = [(t-4)^2 + 3] \cdot u_4(t)$$

We see that $f(t)$ has the form $f(t) = g(t-4)u_1(t)$. This implies that the unshifted function is $g(t) = t^2 + 3$. From #13 in the table we know that

$$\mathcal{L}^{-1}\{f(t)\} = F(s) = e^{-4s} \cdot G(s) \tag{1}$$

where we have

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\left\{t^2 + 3\right\} = \mathcal{L}\left\{t^2\right\} + 3\mathcal{L}\{1\} = \frac{2}{s^3} + \frac{3}{s}$$

We plug this back in to (1) and obtain

$$F(s) = e^{-4s} \cdot G(s) = e^{-4s} \cdot \left[\frac{2}{s^3} + \frac{3}{s}\right]$$

Example 4

Find the Laplace transform of $g(t) = \begin{cases} \sin(t), & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Solution. In $u_c(t)$ notation we have

$$g(t) = \sin(t) - \sin(t)u_\pi(t)$$

In order to use the Laplace transform given by # 13 in the table we need to represent the functions with a shift that "matches" the value in the unit step function. Here we must re-write the $\sin(t)$ so that it has a shift of π. To do this we write

$$\begin{aligned} \sin(t) &= \sin(t - \pi + \pi) \\ &= \sin((t - \pi) - \pi) \end{aligned}$$

Note that we technically added nothing here! Now we can use the sum to product formula. We let $\alpha = t - \pi$ and $\beta = \pi$ to obtain

$$\begin{aligned} \sin((t-\pi) + \pi) &= \sin(t-\pi)\cos(\pi) + \cos(t-\pi)\sin(\pi) \\ &= \sin(t-\pi)(-1) + \cos(t-\pi)(0) \\ &= -\sin(t-\pi) \end{aligned}$$

So,

$$g(t) = \sin(t) + \sin(t-\pi)u_\pi(t)$$

Now we take the Laplace transform to obtain

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{\sin(t-\pi)u_\pi(t)\}$$

Where

$$\mathcal{L}\{\sin(t)\} = \frac{1}{s^2 + 1^2} \tag{2}$$

$$\mathcal{L}\{\sin(t-\pi)u_\pi(t)\} = e^{-\pi s}F(s) \tag{3}$$

Now $F(s)$ is the Laplace transform of the *unshifted* function. In other words, $\mathcal{L}\{f(t)\} = F(s)$. Since

$$f(t-\pi) = \sin(t-\pi) \quad \Longrightarrow \quad f(t) = \sin(t)$$

then

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1^2}$$

and so

$$\mathcal{L}\{\sin(t-\pi)u_\pi(t)\} = e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)$$

and so the Laplace tranform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2+1^2} + e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)$$

Alternatively, we can skip using sum formula by using #15 from the Paul's Online Math Notes table. Starting from (3) we recognize that for

$$f(t-\pi) = \sin(t-\pi)$$

we have the unshifted function

$$f(t) = \sin(t+\pi)$$

So now using #15 we let $a = 1$ and $b = \pi$ we have

$$\mathcal{L}\{\sin(t+\pi)\} = \frac{s\sin(\pi) + (1)\cos(\pi)}{s^2+1^2} = \frac{-1}{s^2+1^2}$$

and so the Laplace tranform of $g(t)$ is

$$\begin{aligned}\mathcal{L}\{g(t)\} &= \frac{1}{s^2+1^2} - e^{-\pi s}\left(\frac{-1}{s^2+1^2}\right)\\ &= \frac{1}{s^2+1^2} + e^{-\pi s}\left(\frac{1}{s^2+1^2}\right)\end{aligned}$$

which is the same thing as we got above.

Warning: Do not just assume signs change when you are adjusting your shifts in trig functions. You will not always have a shift involving π!

Example 5

Find the Laplace transform of $f(t) = -t^2u_3(t) + \cos(t)u_5(t)$

Solution. Again, we need to re-write our functions so that they have a shift that "matches" the unit step function. We have

$$f(t) = -(t-3+3)^2u_3(t) + \cos(t-5+5)u_5(t)$$

For $-(t-3+3)^2u_3(t)$ recall that $(a+b)^2 = a^2+2ab+b^2$. So,

$$\begin{aligned}((t-3)+3)^2 &= (t-3)^2 + 2(t-3)3 + 3^2\\ &= (t-3)^2 + 6(t-3) + 9\end{aligned}$$

For $\cos(t-5+5)u_5(t)$ we can use the sum to product formula here. So our second function becomes

$$\cos((t-5)+5) = \cos(t-5)\cos(5) - \sin(t-5)\sin(5)$$

So, we have

$$f(t) = -((t-3)^2 + 6(t-3) + 9)u_3(t) + \cos(5)\cos(t-5)u_5(t) - \sin(5)\sin(t-5)u_5(t)$$

taking the Laplace transform we have

$$\begin{aligned}F(s) &= -\mathcal{L}\{u_3(t)g_1(t-3)\} + \mathcal{L}\{u_5(t)g_2(t-5)\} - \mathcal{L}\{u_5(t)g_3(t-5)\}\\ &= -e^{-3s}G_1(s) + e^{-5s}(G_2(s) - G_3(s))\end{aligned}$$

where

$$\begin{aligned} g_1(t-3) &= (t-3)^2 + 6(t-3) + 9 &&\implies g_1(t) = t^2 + 6t + 9 \\ g_2(t-5) &= \cos(5)\cos(t-5) &&\implies g_2(t) = \cos(5)\cos(t) \\ g_3(t-5) &= \sin(5)\sin(t-5) &&\implies g_3(t) = \sin(5)\sin(t) \end{aligned}$$

and

$$\begin{aligned} G_1(s) &= \mathcal{L}\left\{g_1(t)\right\} = \frac{2}{s^5} + \frac{6}{s^2} + \frac{9}{s} \\ G_2(s) &= \mathcal{L}\left\{g_2(t)\right\} = \cos(5)\left[\frac{s}{s^2+1}\right] \\ G_3(s) &= \mathcal{L}\left\{g_3(t)\right\} = \sin(5)\left[\frac{1}{s^2+1}\right] \end{aligned}$$

so the Laplace transform of $f(t)$ is

$$F(s) = -e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right) + e^{-5s}\left(\cos(5)\left(\frac{s}{s^2+1}\right) + \sin(5)\left(\frac{1}{s^2+1}\right)\right)$$

Of course you can also use #16 from the POMN table for the cosine function. For

$$u_5(t)\cos((t-5)+5) = u_5(t)g(t-5) \qquad \implies \qquad g(t) = \cos(t+5)$$

taking the Laplace transform we have

$$\mathcal{L}\left\{u_5(t)g(t-5)\right\} = e^{-5s} \cdot G(s)$$

where

$$G(s) = \mathcal{L}\left\{g(t)\right\} = \mathcal{L}\left\{\cos(t+5)\right\}$$

Let $a = 1$ and $b = 5$ and we obtain

$$\mathcal{L}\left\{u_5(t)\cos((t-5)+5)\right\} = e^{-5s}\left(\frac{s\cos(5) - \sin(5)}{s^2+1}\right)$$

which is the same as what we found using the first method.