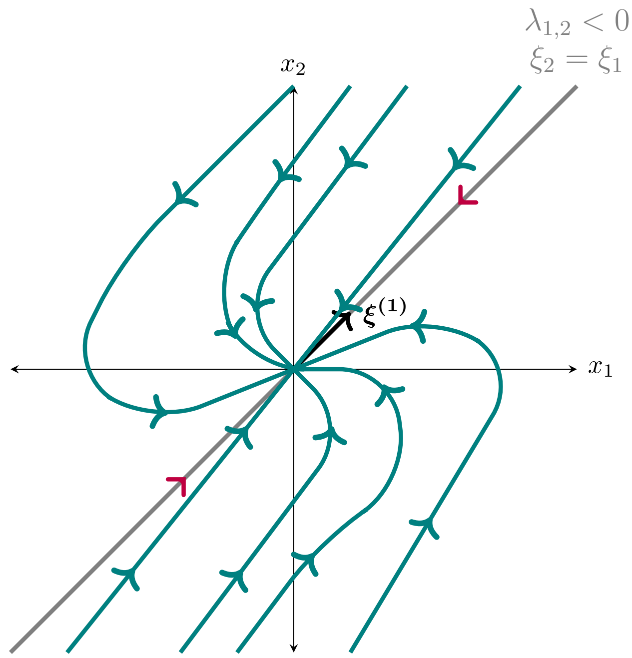
1. (20 pts) Solve the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and draw a phase portrait for the system on the axes provided to the right.

Solution: Since $\lambda_{1,2} = -3 < 0$ we will have an stable improper node.



Solution: First we note the coefficient matrix is $A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$.

Find Eigenvalues

To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (1-\lambda)(-7-\lambda) - (-4)(4) = (\lambda+3)^2 = 0$$

We have a repeated eigenvalue

$$\lambda_1 = \lambda_2 = -3$$

Now we find the eigenvectors corresponding to the eigenvalue.

Find Eigenvectors

For $\lambda_{1,2} = -3$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We note that

$$A - (-3)I = \begin{bmatrix} 1+3 & -4 \\ 4 & 7+3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 4 & -4 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1 - \xi_2 = 0 \qquad \Longrightarrow \xi_1 = \xi_2$$

So the only eigenvector corresponding to $\lambda_{1,2}$ is

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Find Generalized Eigenvector η:

To find the generalized eigenvector corresponding to $\lambda_{1,2} = -3$ we need to solve the system

$$(A - \lambda_1 I)\boldsymbol{\eta} = \boldsymbol{\xi}^{(1)} \tag{1}$$

Solving this system in matrix form we have

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{4} \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\eta_1 - \eta_2 = \tfrac{1}{4} \qquad \Longrightarrow \eta_1 = \eta_2 + \tfrac{1}{4}$$

Let $\eta_2 = k$ so our solution to (1) is

$$\boldsymbol{\eta} = \begin{bmatrix} \frac{1}{4} + k \\ k \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + \underbrace{k \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{Multiple of } \boldsymbol{\xi} \Longrightarrow k=0}$$

The last term is just a multiple of $\boldsymbol{\xi}^{(1)}$ and so we ignore this piece by choosing $k = 0$. The second solution $\mathbf{x}^{(1)}(t)$ will have the general form

$$\begin{aligned} \mathbf{x}^{(2)}(t) &= \boldsymbol{\xi}^{(1)} t e^{\lambda_1 t} + \boldsymbol{\eta} e^{\lambda_1 t} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} t e^{\lambda_1 t} + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} e^{\lambda_1 t} \end{aligned}$$

So the general solution to the system is

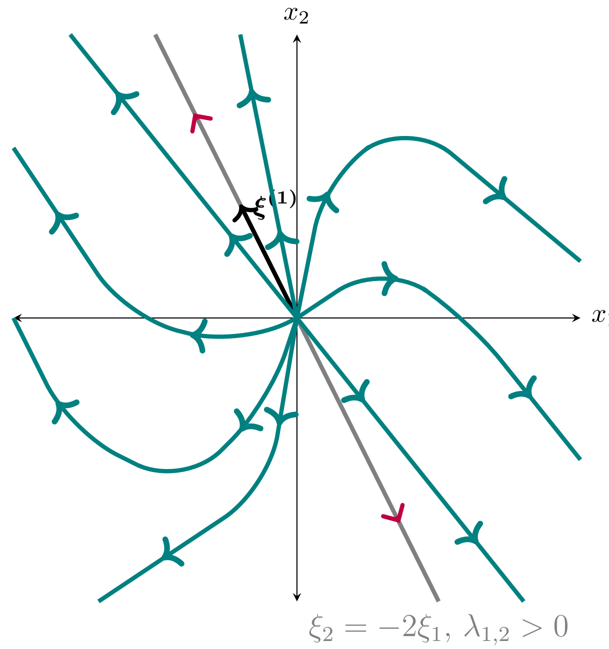
$$
\begin{aligned}
\mathbf{x}(t) &= c_1\boldsymbol{\xi}^{(1)}e^{\lambda_1 t} + c_2\boldsymbol{\xi}^{(1)}te^{\lambda_1 t} + \boldsymbol{\eta}e^{\lambda_1 t} \\
&= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} te^{-3t} + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} e^{-3t}\right) \\
&= e^{-3t}\left[c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}\right)\right] \\
&= e^{-3t}\left[c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} t+\frac{1}{4} \\ t \end{bmatrix}\right]
\end{aligned}
$$

2. (20 pts) Find the general solution to the system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \mathbf{x}$$

and draw a phase portrait for the system on the axes provided to the right.

Solution: Since $\lambda_{1,2} = 5 > 0$ we will have an unstable improper node.



Solution: First we note the coefficient matrix is $A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$.

Find Eigenvalues

To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) - (-4)(1) = (\lambda - 5)^2 = 0$$

We have a repeated eigenvalue

$$\lambda_1 = \lambda_2 = 5$$

Now we find the eigenvectors corresponding to the eigenvalue.

Find Eigenvectors

For $\lambda_{1,2} = 5$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We note that

$$A-(-3)I=\begin{bmatrix} 7-5 & 1 \\ -4 & 3-5 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \left[\begin{array}{cc:c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{cc:c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1+\frac{1}{2}\xi_2=0 \qquad \Longrightarrow \xi_1=-\frac{1}{2}\xi_2 \quad \text{or} \quad \xi_2=-2\xi_1$$

So the only eigenvector corresponding to $\lambda_{1,2}$ is

$$\boldsymbol{\xi}^{(1)}=\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\xi}^{(1)}=\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Find Generalized Eigenvector $\boldsymbol{\eta}$:

To find the generalized eigenvector corresponding to $\lambda_{1,2}=5$ we need to solve the system

$$(A-\lambda_1 I)\boldsymbol{\eta}=\boldsymbol{\xi}^{(1)} \tag{2}$$

Solving this system in matrix form we have

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=\begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \Longrightarrow \quad \left[\begin{array}{cc:c} 2 & 1 & -1 \\ -4 & -2 & 2 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{cc:c} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\eta_1+\tfrac{1}{2}\eta_2=-\tfrac{1}{2} \qquad \Longrightarrow \eta_1=-\tfrac{1}{2}-\tfrac{1}{2}\eta_2$$

Let $\eta_2=k$ so our solution to (2) is

$$\boldsymbol{\eta}=\begin{bmatrix} -\frac{1}{2}-\frac{1}{2}k \\ k \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}+\underbrace{k\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}}_{\text{Multiple of } \boldsymbol{\xi}}$$

The last term is just a multiple of $\boldsymbol{\xi}^{(1)}$ and so we ignore this piece by choosing $k=0$. The second solution $\mathbf{x}^{(1)}(t)$ will have the general form

$$\begin{aligned} \mathbf{x}^{(2)}(t) &= \boldsymbol{\xi}^{(1)}te^{\lambda_1 t}+\boldsymbol{\eta}e^{\lambda_1 t} \\ &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}te^{\lambda_1 t}+\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}e^{\lambda_1 t} \end{aligned}$$

So the general solution to the system is

$$\begin{aligned}\mathbf{x}(t) &= c_1\boldsymbol{\xi}^{(1)}e^{\lambda_1 t} + c_2\boldsymbol{\xi}^{(1)}te^{\lambda_1 t} + \boldsymbol{\eta}e^{\lambda_1 t}\\ &= c_1\begin{bmatrix}-1\\2\end{bmatrix}e^{5t} + c_2\left(\begin{bmatrix}-1\\2\end{bmatrix}te^{5t} + \begin{bmatrix}-\frac{1}{2}\\0\end{bmatrix}e^{5t}\right)\end{aligned}$$