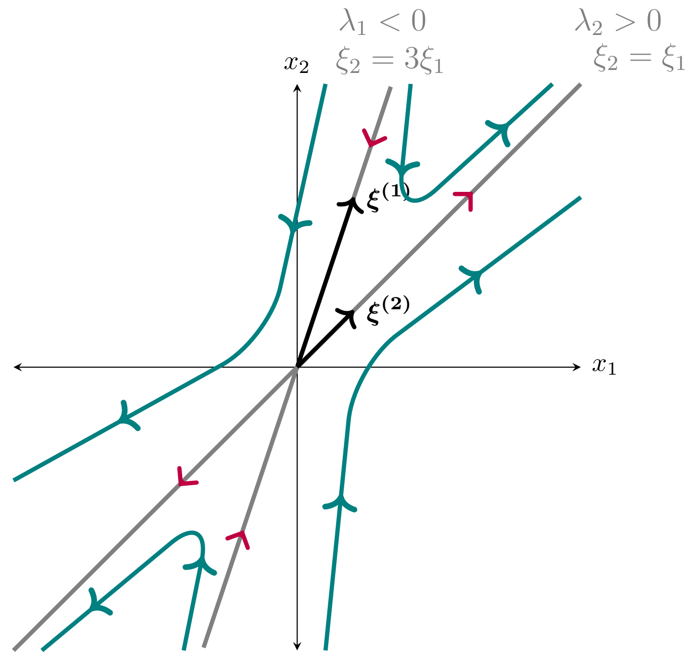
Worksheet 20 Solutions

Intro to Homogeneous Linear Systems MATH 2310, Spring 2019

1. (20 pts) Find the general solution of the system of equations

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$$

and draw a phase portrait for the system on the axes provided to the right.



Solution: Since our eigenvalues are both real, but differ in sign, the origin is a saddle point. We plot the two lines along our eigenvectors:

- $\xi_2 = 3\xi_1$, arrows point towards the origin since $\lambda_1 < 0$
- $\xi_2 = \xi_1$, arrows point towards the origin since $\lambda_2 > 0$.

Solution: First we note the coefficient matrix is $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$.

Find Eigenvalues

To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (-1)(3) = (\lambda - 1)(\lambda + 1) = 0$$

So the eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 1$$

Now we find the eigenvectors corresponding to each eigenvalue.

Find Eigenvectors

For $\lambda_1 = -1$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We note that

$$A-(-1)I=\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix}=\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longrightarrow\quad \left[\begin{array}{cc:c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{cc:c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1-\frac{1}{3}\xi_2=0 \qquad \Longrightarrow \xi_1=\frac{1}{3}\xi_2 \quad \text{or} \quad \xi_2=3\xi_1$$

So the eigenvector corresponding to λ_1 is

$$\boldsymbol{\xi}^{(1)}=\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\xi}^{(1)}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

For $\lambda_2 = 1$:

To find eigenvalues we need to find the solution to

$$(A-\lambda_2 I)\boldsymbol{\xi}^{(1)}=0$$

We note that

$$A-(1)I=\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 3 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longrightarrow\quad \left[\begin{array}{cc:c} 3 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1-\xi_2=0 \qquad \Longrightarrow \xi_1=\xi_2$$

So the eigenvector corresponding to λ_2 is

$$\boldsymbol{\xi}^{(2)}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution to the system will have the form

$$\mathbf{x}(t)=c_1\boldsymbol{\xi}^{(1)}e^{\lambda_1 t}+c_2\boldsymbol{\xi}^{(2)}e^{\lambda_2 t}$$

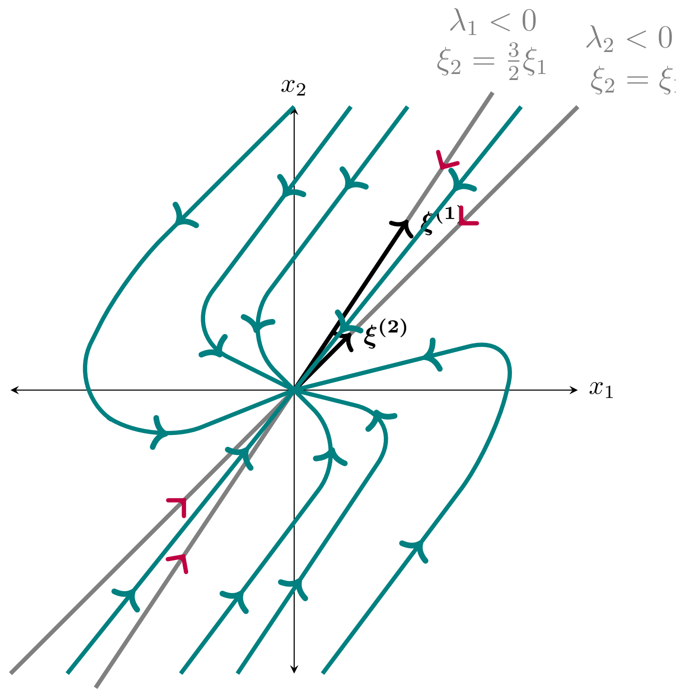
So the general solution to the system is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{-t}+c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t}$$

2. (20 pts) Find the general solution of the system of equations

$$\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \mathbf{x}$$

and draw a phase portrait for the system on the axes provided to the right.



Solution: Since our eigenvalues are both real and negative, the origin is a stable (improper) node. We plot the two lines along our eigenvectors:

- $\xi_2 = \frac{3}{2}\xi_1$, arrows point towards the origin since $\lambda_1 < 0$
- $\xi_2 = \xi_1$, arrows point towards the origin since $\lambda_2 < 0$.

Solution: First we note the coefficient matrix is $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$.

Find Eigenvalues

To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - (-2)(3) = (\lambda+1)(\lambda+2) = 0$$

So the eigenvalues are

$$\lambda_1 = -2, \quad \lambda_2 = -1$$

Now we find the eigenvectors corresponding to each eigenvalue.

Find Eigenvectors

For $\lambda_1 = -2$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We note that

$$A-(-2)I=\begin{bmatrix} 1+2 & -2 \\ 3 & -4+2 \end{bmatrix}=\begin{bmatrix} 3 & -1 \\ 3 & -2 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 3 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longrightarrow\quad \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1-\frac{2}{3}\xi_2=0 \qquad \Longrightarrow \xi_1=\frac{2}{3}\xi_2 \quad \text{or} \quad 2\xi_2=3\xi_1 \quad \text{or} \quad \xi_2=\frac{3}{2}\xi_1$$

So the eigenvector corresponding to λ_1 is

$$\boldsymbol{\xi}^{(1)}=\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\xi}^{(1)}=\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\xi}^{(1)}=\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$

For $\lambda_2=-1$:

To find eigenvalues we need to find the solution to

$$(A-\lambda_2 I)\boldsymbol{\xi}^{(1)}=0$$

We note that

$$A-(-1)I=\begin{bmatrix} 1+1 & -2 \\ 3 & -4+1 \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$$

then solving the system we have

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Longrightarrow\quad \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1-\xi_2=0 \qquad \Longrightarrow \xi_1=\xi_2$$

So the eigenvector corresponding to λ_2 is

$$\boldsymbol{\xi}^{(2)}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general solution to the system will have the form

$$\mathbf{x}(t)=c_1\boldsymbol{\xi}^{(1)}e^{\lambda_1 t}+c_2\boldsymbol{\xi}^{(2)}e^{\lambda_2 t}$$

So the general solution to the system is

$$\mathbf{x}(t)=c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{-2t}+c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{t}$$