Section Summary

Theorem 2.6.1: A differential equation of the form

$$M(x, y) + N(x, y)y' = 0$$

is *exact* if $M_y = N_x$. Then there exists a function $\psi(x, y)$ for which

$$\psi_x(x, y) = M(x, y) \quad \text{and} \quad \psi_y(x, y) = N(x, y)$$

Procedure

1. First verify if your function is exact by checking if $M_y = N_x$.
2. Let $\psi_x = M$ and $\psi_y = N$.
3. Integrate ψ_x to get ψ.
4. Use the function $h(y)$ to represent your constant of integration in ψ.
5. Differentiate ψ with respect to y noting that your $h(y)$ is now $h'(y)$.
6. Set this ψ_y equal to N and solve for $h'(y)$.
7. Now find $h(y)$ by integrating $h'(y)$ with respect to y.
8. Plug $h(y)$ into $\psi(x, y)$.
9. General solution is then $\psi(x, y) = c$.

Integrating Factors and Exact Equations

If your equation is not exact, it may be possible to find an integrating factor which will make your equation exact. We need to check if either

$$\frac{M_y - N_x}{N} = g(x) \quad \text{or} \quad \frac{N_x - M_y}{M} = f(y)$$

Where $g(x)$ is some function of only x and $f(y)$ is some function of only y. So if

$$\begin{aligned} \frac{M_y - N_x}{N} &= \text{a function of only } x \quad \Longrightarrow \text{ find } \mu(x) \text{ by solving } \quad \frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu \\ \frac{N_x - M_y}{M} &= \text{a function of only } y \quad \Longrightarrow \text{ find } \mu(y) \text{ by solving } \quad \frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu \end{aligned}$$

Multiplying your equation by this μ will then yield an exact equation. Note that it may not be possible to find an integrating factor which will allow you to solve your equation in this way.

1. (10 pts) Determine whether the equation

$$(2x + 4y) + (2x - 2y)y' = 0$$

is exact. If it is exact, find the solution.

Solution: Since

$$M_y = 4 \quad \text{and } N_x = 2$$

we see that $M_y \neq N_x$. Therefore, our equation is not exact and we're done!

2. (10 pts) Determine whether the equation

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0$$

is exact. If it is exact, find the solution.

Solution: This equation has the form form $M(x, y) + N(x, y)y' = 0$. We must first check if it's exact:

$$\begin{matrix} M = 2xy^2 + 2y \\ \\ N = 2x^2y + 2x \end{matrix} \implies \begin{matrix} M_y = 4xy + 2 \\ \\ N_x = 4xy + 2 \end{matrix} \implies M_y = N_x$$

$M_y = N_x$ so our equation is exact. Now by Theorem 2.6.1 we know that there exists an equation $\psi(x, y)$ such that

$$\begin{aligned} \psi_x &= M = 2xy^2 + 2y \\ \psi_y &= N = 2x^2y + 2x \end{aligned}$$

To find $\psi(x, y)$ we must integrate either ψ_x or ψ_y. (It doesn't matter which you choose as long as you keep it straight what you are using. Changing which equation to integrate usually depends on which function will be easier to integrate.)

In this case it doesn't matter so we'll integrate ψ_x with respect to x. In other words,

$$\psi = \int \psi_x \, dx = \int M \, dx$$

and so we have

$$\int M \, dx = \int 2xy^2 + 2y \, dx = x^2y^2 + 2xy + h(y)$$

Where did $h(y)$ come from? Essentially, $h(y)$ is representing our "constant" of integration. This could be an actual constant or some function of y only. Notice that if we differentiate a function $f(x, y)$ with respect to x any part of f which was either constant or only containing the variable y, will be zero. We need to find out what this missing piece of our equation is so we need to find this $h(y)$.

Now for our $\psi(x, y)$ we have

$$\psi(x, y) = x^2y^2 + 2xy + h(y)$$

To find our $h(y)$ we now take the partial derivative of $\psi(x, y)$ that we just found with respect to y and we have

$$\psi_y = 2x^2y + 2x + h'(y)$$

But Theorem 2.6.1 tells us that $\psi_y = N$. So we equate these two equations to solve for $h'(y)$

$$\underbrace{2x^2y + 2x + h'(y)}_{\psi_y} = \underbrace{2x^2y + 2x}_{N}$$

We see by canceling terms that $h'(y) = 0$. Now we integrate with respect to y to get $h(y)$

$$h(y) = \int h'(y)\, dy = \int 0\, dy = c$$

This constant will just get smushed (a technical math term) with any other constants in our solution so we can assume the simplest case and say that $h(y) = 0$. This gives us

$$\psi(x, y) = x^2y^2 + 2xy$$

So our general solution is the implicit function $\psi(x, y) = c$ which is

$$x^2y^2 + 2xy = c$$

If we had an initial condition we could plug it in here to find our c, but we don't in this case so we're done!

3. (20 pts) Determine if the equation

$$y\, dx + (2xy - e^{-2y})\, dy = 0$$

is exact. If it is not exact, find an integrating factor and solve it.

Solution: First we check if our equation is exact.

$$\begin{matrix} M = y \\ \\ N = 2xy - e^{-2y} \end{matrix} \implies \begin{matrix} M_y = 1 \\ \\ N_x = 2y \end{matrix} \implies M_y \neq N_x$$

so our equation is not exact. We evaluate $\dfrac{M_y - N_x}{N}$ and $\dfrac{N_x - M_y}{M}$ to see if we get a function solely of x or y respectively. We see that

$$\frac{M_y - N_x}{N} = \frac{1-2y}{2xy - e^{-2y}} \quad \text{and} \quad \frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}$$

We see that we get a function solely of y in the second case. So we now solve the differential equation

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu \implies \frac{d\mu}{dy} = \left(2 - \frac{1}{y}\right)\mu$$

This is just a separable equation. We solve this to obtain

$$\mu(y) = y^{-1}e^{2y}$$

We now multiply each side by $\mu(y)$ to obtain an exact equation.

$$\begin{aligned} y^{-1}e^{2y}[y\,dx + (2xy - e^{-2y})\,dy] &= 0 \\ e^{2y}\,dx + (2xe^{2y} - y^{-1})\,dy &= 0 \end{aligned}$$

Let's check that this equation is now exact

$$\begin{matrix} M = e^{2y} & & M_y = 2e^{2y} & & \\ & \implies & & \implies & M_y = N_x \\ N = 2xe^{-2y} - y^{-1} & & N_x = 2e^{2y} & & \end{matrix}$$

so our equation is now exact. Now by Theorem 2.6.1 we know that there exists an equation $\psi(x, y) = c$ such that

$$\begin{aligned} \psi_x &= M = e^{2y} \\ \psi_y &= N = 2xe^{-2y} - y^{-1} \end{aligned}$$

To find $\psi(x, y)$ we must integrate either ψ_x or ψ_y. In this case it looks like $M = \psi_x$ will be easier to integrate with respect to x In other words,

$$\psi = \int \psi_x\,dx = \int M\,dx = \int e^{2y}\,dx = xe^{2y} + h(y)$$

Our goal is to find this function $h(y)$ and so we now differentiate this with respect to y and obtain

$$\psi_y = 2xe^{2y} + h'(y)$$

But we said earlier that $N = \phi_y$ so we equate these two equations and solve for $h'(y)$

$$2xe^{2y} + h'(y) = 2xe^{2y} - y^{-1} \implies h'(y) = -y^{-1}$$

We integrate $h'(y)$ with respect to y to obtain $h(y)$

$$\int h'(y)\,dy = \int -y^{-1}\,dy = -\ln(y)$$

So $\psi(x, y) = xe^{2y} - \ln(y)$ and so our general solution is

$$xe^{2y} - \ln(y) = c$$