Worksheet 17 Solutions

Convolution Integrals MATH 2310, Spring 2019

Use the convolution integral to find the inverse Laplace transform of the following. Evaluate the integral if possible.

1. (10 pts) $F(s) = \dfrac{1}{s(s-3)}$

Solution:

Let $H(s) = \dfrac{1}{s}$ & $G(s) = \dfrac{1}{s-3}$

Take the ILT to Find $h(t)$ & $g(t)$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

By Convolution:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{H(s)G(s)\} \\ &= (f * g)(t) \\ &= \int_0^t h(t-\tau)\,g(\tau)\,d\tau \\ &= \int_0^t (1)\,e^{3\tau}\,d\tau \\ &= \frac{1}{3}e^{3\tau}\Big|_0^t \\ &= \frac{1}{3}(e^{3t}-1) \end{aligned}$$

So $f(t) = \dfrac{1}{3}(e^{3t}-1)$

2. (10 pts) $F(s) = \dfrac{s}{(s-3)(s^2+1)}$

Solution:

$$F(s) = \frac{s}{(s-3)(s^2+1)} = \left(\frac{1}{s-3}\right)\left(\frac{s}{s^2+1}\right)$$

Let $H(s) = \dfrac{1}{s-3}$ & $G(s) = \dfrac{s}{s^2+1}$

Take the ILT to find $h(t)$ & $g(t)$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

By Convolution:

$$f(t) = \mathcal{L}^{-1}\{H(s)\,G(s)\}$$

$$= (h * g)(t)$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^t e^{3(t-\tau)}\cos(\tau)\,dt$$

$$= e^{3t}\int_0^t e^{-3\tau}\cos(\tau)\,d\tau$$

$$= e^{3t}\left[\frac{e^{-3\tau}}{10}(\sin(\tau) - 3\cos(\tau))\right]_0^t$$

$$= e^{3t}\left[\frac{e^{-3t}}{10}(\sin(t) - 3\cos(t)) - \frac{e^0}{10}(\sin(0) - 3\cos(0))\right]_0^t$$

$$= e^{3t}\left[\frac{e^{-3t}}{10}(\sin(t) - 3\cos(t)) + \frac{3}{10}\right]$$

$$= \frac{1}{10}\left[\sin(t) - 3\cos(t) + 3e^{3t}\right]$$

3. (10 pts) $F(s) = \dfrac{G(s)}{s^2+1}$

Solution:

$$F(s) = \frac{G(s)}{s^2+1} = G(s)\left(\frac{1}{s^2+1}\right)$$

Let $H(s) = \dfrac{1}{s^2+1}$

Take the ILT to find $h(t)$ & $g(t)$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\}$$

By Convolution:

$$f(t) = \mathcal{L}^{-1}\{H(s)\cdot G(s)\}$$

$$= (h*g)(t)$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^t \sin(t-\tau)\,g(\tau)\,d\tau$$

or

$$= \int_0^t g(t-\tau)\,h(\tau)\,d\tau$$

$$= \int_0^t g(t-\tau)\sin(\tau)\,d\tau$$

Since we don't know $g(t)$ explicitly we cannot evaluate the integral.

4. (10 pts) $F(s) = \dfrac{1}{(s^2+9)^2}$

Solution: $F(s) = \dfrac{1}{(s^2+9)^2} = \dfrac{1}{(s^2+3^2)} \cdot \dfrac{1}{(s^2+3^2)}$

Let

$$H(s) = \frac{1}{s^2+3^2} = \frac{1}{3}\left[\frac{3}{s^2+3^2}\right] \quad \& \quad G(s) = \frac{1}{s^2+3^2} = \frac{1}{3}\left[\frac{3}{s^2+3^2}\right]$$

Take the ILT to Find $h(t)$ & $g(t)$

$$\Rightarrow h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{3}{s^2+3^2}\right)\right\} = \frac{1}{3}\sin(3t)$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{3}{s^2+3^2}\right)\right\} = \frac{1}{3}\sin(3t)$$

By Convolution:

$$f(t) = \mathcal{L}^{-1}\{H(s)\cdot G(s)\}$$

$$= (h*g)(t)$$

$$= \int_0^t h(t-\tau)\,g(\tau)\,d\tau$$

$$= \frac{1}{9}\int_0^t \sin(3(t-\tau))\sin(\tau)\,d\tau$$

Note: $\sin(3t-3\tau) = \sin(3t)\cos(3\tau) - \cos(3t)\sin(3\tau)$

$$= \frac{1}{9}\int_0^t \left[\sin(3t)\cos(3\tau) - \cos(3t)\sin(3\tau)\right]\sin(3\tau)\,d\tau$$

$$= \frac{1}{9}\left[\sin(3t)\left[\int_0^t \sin(3\tau)\cos(3\tau)\,d\tau\right] - \cos(t)\left[\int_0^t \sin^2(3\tau)\,d\tau\right]\right]$$

4. (cont'd)

$$F(t) = \frac{1}{9}\left(\sin(3t)\left[-\frac{1}{12}\cos(6\tau)\right]_0^t - \cos(3t)\left[\frac{\tau}{2} - \frac{1}{12}\sin(6\tau)\right]_0^t\right)$$

$$= \frac{1}{9}\left(\sin(3t)\left[-\frac{1}{12}\cos(6t) + \frac{1}{12}\right] - \cos(3t)\left[\frac{t}{2} - \frac{1}{12}\sin(6t)\right]\right)$$

$$= \frac{1}{9}\left[-\frac{1}{12}\sin(3t)\cos(6t) + \frac{1}{12}\sin(3t) - \frac{t}{2}\cos(3t) + \frac{1}{12}\cos(3t)\sin(6t)\right]$$

By double angle ID's

$$\sin(2\theta) = 2\sin\theta\cos\theta \quad \& \quad \cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\Rightarrow \sin(3t)\cos(6t) = \sin(3t)(2\cos^2(3t) - 1)$$

$$= 2\cos^2(3t)\sin(3t) - \sin(3t)$$

and

$$\cos(3t)\sin(6t) = \cos(3t)(2\sin(3t)\cos(3t))$$

$$= 2\cos^2(3t)\sin(3t)$$

$$\Rightarrow F(t) = \frac{1}{9}\left[-\frac{1}{12}\left[2\cos^2(3t)\sin(3t) - \sin(3t)\right] + \frac{1}{12}\sin(3t) - \frac{t}{2}\cos(3t) + \frac{1}{12}\left[2\cos^2(3t)\sin(3t)\right]\right]$$

$$= \frac{1}{9}\left[\cancel{-\frac{1}{6}\cos^2(3t)\sin(3t)} + \frac{1}{12}\sin(3t) + \frac{1}{12}\sin(3t) - \frac{t}{2}\cos(3t) + \cancel{\frac{1}{6}\cos^2(3t)\sin(3t)}\right]$$

$$= \frac{1}{9}\left[\frac{1}{6}\sin(3t) - \frac{t}{2}\cos(3t)\right]$$

$$= \frac{1}{54}\left[\sin(3t) - 3t\cos(3t)\right]$$