Worksheet 13 Solutions

Definition of the Laplace Transform

MATH 2310, Spring 2019

Find the Laplace Transform of each of the following functions. Use only the definition and show all the details of your integration process. Results using a table of transforms will receive no credit.

1. (20 pts) $f(t) = t^2$

Solution: Using the definition we have

$$F(s) = \mathcal{L}\left\{t^2\right\} = \int_0^\infty e^{-st} \cdot t^2 \, dt = \lim_{A\to\infty} \int_0^A e^{-st} \cdot t^2 \, dt$$

Now we integrate by parts to obtain

$$= \lim_{A\to\infty} \left[t^2 \left(-\frac{1}{s} e^{-st} \right) \Bigg|_0^A + \frac{2}{s} \underbrace{\int_0^A e^{-st} \cdot t \, dt}_{=\frac{1}{s^2} \text{ by problem 1}} \right] = \lim_{A\to\infty} \left[-\frac{A^2}{s} e^{-sA} - \frac{2}{s}\left(\frac{1}{s^2}\right) \right]$$

$$= \underbrace{\lim_{A\to\infty} \left[-\frac{A^2}{s} e^{-sA} \right]}_{=0} - \frac{2}{s^3} \lim_{A\to\infty} 1 = \frac{2}{s^3}$$

So we have

$$F(s) = \mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$$

2. (20 pts) $f(t) = \cos(at)$, where a is a real constant.

Solution: Using the definition we have

$$F(s) = \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st} \cdot \cos(at)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} \cdot \cos(at)\, dt$$

Now we integrate by parts to obtain

$$\begin{aligned} &= \lim_{A\to\infty} \left[\frac{e^{-st}\sin(at)}{a}\bigg|_0^A + \frac{s}{a}\int_0^A e^{-st} \cdot \sin(at)\, dt \right] \\ &= \underbrace{\lim_{A\to\infty} \left[-\frac{e^{-sA}\sin(aA)}{a} \right]}_{=0} + \frac{s}{a}\int_0^\infty e^{-st} \cdot \cos(at)\, dt \\ &= \frac{s}{a}\int_0^\infty e^{-st} \cdot \sin(at)\, dt \end{aligned}$$

From problem 3 we know that $\displaystyle\int_0^\infty e^{-st} \cdot \sin(at)\, dt = \frac{a}{s^2 + a^2}$ so we have

$$F(s) = \mathcal{L}\{\cos(at)\} = \frac{s}{a}\left[\frac{a}{s^2+a^2}\right] = \frac{s}{s^2+a^2}$$