Worksheet 11 Solutions

Section Summary

Recall that the general form of a non-homogeneous equation is

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where the general solution is given by

$$y = y_c(t) + Y(t)$$

Variation of Parameters

For this method the particular solution $Y(t)$ is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{2}$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W}\,dt \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W}\,dt$$

and W is the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1y_2' - y_1'y_2$$

This method is a more general method that can be used for a wider variety of equations. There are two main difficulties with this method. The first is finding a solution to the homogeneous equation when we have non-constant coefficients. The second is in the evaluation of the integrals obtained for the particular solution.

1. (20 pts) Find the general solution of the differential equation

$$y'' + y = \tan(t), \quad 0 < t < \frac{\pi}{2}$$

Solution:

Homogeneous Solution $y_c(t)$:

Using the methods for homogeneous equations with constant coefficients we solve

$$y'' + y = 0$$

Characteristic equation:

$$r^2 + 1 = 0 \implies \quad r = \pm i$$

Recall that the general solution for complex roots $r = \lambda \pm i\mu$, has the form $y_c(t) = e^{\lambda t}[c_1 \cos(\mu t) + c_2 \sin(\mu t)]$.So our general solution for $y_c(t)$ is

$$y_c(t) = c_1 \cos(t) + c_2 \sin(t)$$

Particular Solution $Y(t)$:

Our particular solution is given by (2). So we need to find $u_1(t)$, $u_2(t)$ and our Wronskian.

Wronskian: we have $y_1(t) = \cos t$ and $y_2(t) = \sin t$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

Now we find $u_1(t)$ and $u_2(t)$:

$$\begin{aligned} u_1(t) = -\int \frac{y_2(t)g(t)}{W}\,dt = -\int \frac{\sin t \tan t}{1}\,dt &= -\ln\left(\frac{\cos t}{\sin t - 1}\right) + \sin t \\ &= -\ln(\tan t + \sec t) + \sin t \end{aligned}$$

Note that:

$$\frac{\cos t}{\sin t - 1} = \frac{\cos t \sin t + \cos t}{\sin^2 t - 1} = \frac{\sin t}{\cos t} + \frac{1}{\cos t} = \tan t + \sec t$$

$$\begin{aligned} u_2(t) = \int \frac{y_1(t)g(t)}{W}\,dt = \int \cos t \tan t\,dt = \int \frac{\cos t \sin t}{\cos t}\,dt &= \int \sin t\,dt \\ &= -\cos t \end{aligned}$$

So we have

$$\begin{aligned} Y(t) &= [-\ln(\tan t + \sec t) + \sin t]\cos t + [-\cos t]\sin t \\ &= -\ln(\tan t + \sec t)\cos t + \sin t\cos t - \sin t\cos t \\ &= -\ln(\tan t + \sec t)\cos t \end{aligned}$$

General Solution $y(t)$:

Recall that our general solution is $y(t) = y_c(t) + Y(t)$ and so we have

$$y(t) = c_1\cos(t) + c_2\sin(t) - \ln(\tan t + \sec t)\cos t$$

If you were given initial conditions you would proceed similarly to the process used for homogeneous equations with constant coefficients.

2. (20 pts) Find the general solution of the differential equation

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$$

Solution:

Homogeneous Solution $y_c(t)$:

We do this by using the methods for homogeneous equations with constant coefficients to solve

$$y'' + 4y' + 4y = 0$$

Characteristic equation:

$$r^2 + 4r + 4 = (r+2)^2 = 0 \qquad \Longrightarrow r_1 = r_2 = -2$$

So our general solution for $y_c(t) = c_1e^{rt} + c_2te^{rt}$ is

$$y_c(t) = c_1e^{-2t} + c_2te^{-2t}$$

Particular Solution $Y(t)$:

Recall that our particular solution is given by $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$. So we need $u_1(t)$ and $u_2(t)$ and our Wronskian.

Wronskian: we have $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & -2te^{-2t} + e^{-2t} \end{vmatrix} = -2te^{-4t} + e^{-4t} + 2te^{-4t} = e^{-4t}$$

Now we find $u_1(t)$ and $u_2(t)$:

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W}\,dt = -\int \frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}\,dt = -\int \frac{t^{-1}e^{-4t}}{e^{-4t}}\,dt = -\int t^{-1}\,dt$$
$$= -\ln(t)$$

$$u_2(t) = \int \frac{y_1(t)g(t)}{W}\,dt = \int \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}\,dt = \int t^{-2}\,dt$$
$$= -t^{-1}$$

So we have
$$Y(t) = -\ln(t)e^{-2t} + t^{-1}te^{-2t} = -\ln(t)e^{-2t} - e^{-2t}$$

General Solution $y(t)$:

Recall that our general solution is $y(t) = y_c(t) + Y(t)$ and so we have

$$\begin{aligned} y(t) &= c_1e^{-2t} + c_2te^{-2t} - \ln(t)e^{-2t} - e^{-2t} \\ &= (c_1 - 1)e^{-2t} + c_2te^{-2t} - \ln(t)e^{-2t} \\ &= c_1e^{-2t} + c_2te^{-2t} - \ln(t)e^{-2t} \end{aligned}$$

Note that the $(c_1 - 1)$ becomes simply c_1. Since c_1 is just an arbitrary constant then $c_1 - 1$ is just another arbitrary constant in our solution. We could label it something else like C or K to be more precise in showing that the value has changed. Since the value is arbitrary it is not really necessary to do so here. If you were given initial conditions you would proceed similarly to the process used for homogeneous equations with constant coefficients.