Worksheet 18 Solutions

Linear Algebra Tutorial I MATH 2310, Spring 2019

You may use your calculator to assist in your calculations. You must cite specifically where you do so and which calculation was performed. The emphasis here is to explain what you are doing and why, not necessarily being able to do all of the matrix operations by hand.

1. (a) (4 pts) Write a matrix equation that is equivalent to the system of equations.

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ 5x_1 & & & - & 5x_3 & = & 10 \end{array} \tag{1}$$

Solution: Writing the system of equations as a matrix equation we have

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

(b) (6 pts) Solve the system from part (a).

Solution: The solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

2. (10 pts) Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$ by hand. Verify the result by calculating AA^{-1} and $A^{-1}A$ (by hand).

Solution: Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and so

$$A^{-1} = \frac{1}{(1)(7) - (2)(4)} \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

3. (10 pts) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

Solution: To find the inverse of A by hand we first augment the matrix with the identity matrix I_3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

We then perform elementary row operations until the LHS of the augmented matrix is the identity matrix I_3.

$$3R_1 + R_2 \to R_2$$

$$\begin{array}{ccc|ccc} 3 & 0 & 6 & 3 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ \hline 0 & 1 & 10 & 3 & 1 & 0 \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$-2R_1 + R_3 \to R_3$$

$$\begin{array}{ccc|ccc} -2 & 0 & 4 & -2 & 0 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \\ \hline 0 & -3 & 0 & -2 & 0 & 1 \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$R_3 \leftrightarrow R_2$$

$$\implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \\ 0 & 1 & 10 & 3 & 1 & 0 \end{array}\right]$$

$$\tfrac{1}{3}R_2 \leftrightarrow R_2$$

$$\implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 10 & 3 & 1 & 0 \end{array}\right]$$

$$-R_2 + R_3 \to R_3$$

$$\begin{array}{ccc|ccc} 0 & -1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 10 & \frac{9}{3} & 1 & 0 \\ \hline 0 & 0 & 10 & \frac{7}{3} & 1 & \frac{1}{3} \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 10 & \frac{7}{3} & 1 & \frac{1}{3} \end{array}\right]$$

$$-\frac{1}{10}R_3 \to R_3 \qquad \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{array}\right]$$

$$-2R_3 + R_1 \to R_1$$

$$\begin{array}{ccc|ccc} 0 & 0 & -2 & -\frac{14}{30} & -\frac{2}{10} & -\frac{2}{30} \\ 1 & 0 & 2 & \frac{30}{30} & 0 & 0 \\ \hline 1 & 0 & 0 & \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \end{array} \qquad \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{array}\right]$$

Since the LHS is now the identity matrix I_3 then the RHS of this augmented matrix is the inverse of A and so

$$A^{-1} = \begin{bmatrix} \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{bmatrix} = \begin{bmatrix} \frac{8}{15} & -\frac{1}{5} & -\frac{1}{15} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{bmatrix}$$

4. (10 pts) Find the determinant of $B = \begin{bmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{bmatrix}$

Solution: Since there is a zero in the first row, we will use a cofactor expansion with this row. The determinant of B is then

$$\begin{aligned} \det(B) = \begin{bmatrix} 4 & 3 & 0 \\ 6 & 5 & 2 \\ 9 & 7 & 3 \end{bmatrix} &= 4\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} - 3\begin{vmatrix} 6 & 2 \\ 9 & 3 \end{vmatrix} + 0\begin{vmatrix} 6 & 5 \\ 9 & 7 \end{vmatrix} \\ &= 4\Big[(5)(3) - (7)(2)\Big] - 3\Big[(6)(3) - (2)(9)\Big] + 0 \\ &= 4(1) - 3(0) \\ &= 4 \end{aligned}$$