Lecture # 11: Variation of Parameters

Date: Mon 3/11/19

Recall:

Gen. form of 2^{nd} order nonhomog. eqn is

$$y'' + p(t)y' + q(t)y = g(t)$$

With general soln of form

$$y(t) = y_c(t) + y_p(t)$$

We have a 2^{nd} method for determining $y_p(t)$ Known as variation of parameters

The method (General case)

Since $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ (complimentary soln)

We assume that

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Goal: Determine $u_1(t)$ & $u_2(t)$ so that $y_p(t)$ satisfies the ODE

$$y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{\text{Want} = 0} + u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y''$$

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Put all back into ODE

$$(u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y'') + p(t)(u_1 y_1' + u_2 y_2') + q(t)(u_1 y_1 + u_2 y_2) = g(t)$$

Factor out u_1, u_2:

$$u_1 \underbrace{(y_1'' + p(t) y_1' + q(t) y_1)}_{=0} + u_2 \underbrace{(y_2'' + p(t) y_2' + q(t) y_2)}_{=0} + u_1' y_1' + u_2' y_2' = g(t)$$

(Since y_1, y_2 satisfy the homog. Eqn)

Eqn Reduces to

$$u_1' y_2' + u_2' y_2' = g(t)$$

This coupled w/ need for $u_1' y_1 + u_2' y_2 = 0$ yields the system

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g(t)$$

Solving this system for u_1 & u_1 we obtain

$$u_1'(t) = -\frac{y_2(t)\, g(t)}{y_1 y_2' - y_1' y_2} \quad \& \quad u_2'(t) = \frac{y_1(t)\, g(t)}{y_1 y_2' - y_1' y_2}$$

Note that $W = y_1 y_2' - y_1' y_2$ (the wronskian)

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Integrating each of these we obtain u_1 & u_2

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt \quad \& \quad u_2(t) = \int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

Plugging these into $y_p(t)$ we obtain:

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

$$= -\,y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt \;+\; y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

so the soln to the nonhomog. eqn

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t)$$

w/Variation of Parameters is

$$y(t) = y_c(t) + y_p(t)$$

$$= c_1 y_1(t) + c_2 y_2(t) \;-\; y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)(t)}\,dt \;+\; y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)(t)}\,dt$$

Recall that the general form of a non-homogeneous equation is

$$y'' + p(t)y' + q(t)y = g(t) \tag{1}$$

where the general solution is given by

$$y = y_c(t) + Y(t)$$

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For this method the particular solution $Y(t)$ is given by

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{2}$$

where

$$u_1(t) = -\int \frac{y_2(t)g(t)}{W}\,dt \qquad\qquad u_2(t) = \int \frac{y_1(t)g(t)}{W}\,dt$$

and W is the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1 y_2' - y_1' y_2$$

This method is a more general method that can be used for a wider variety of equations. There are two main difficulties with this method. The first is finding a solution to the homogeneous equation when we have non-constant coefficients. The second is in the evaluation of the integrals obtained for the particular solution.

Example 1: Find a general solution to

$$y'' - 2y' - 3y = 3e^{2t}$$

From Ex. 1 of Lecture 10:

$$y_c(t) = c_1 e^{-t} + c_2 e^{3t}$$

Find particular soln:

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$y_1(t) = e^{-t} \qquad y_1'(t) = -e^{-t}$$

$$\Rightarrow$$

$$y_2(t) = e^{3t} \qquad y_2'(t) = 3e^{3t}$$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$= 3e^{3t} \cdot e^{-t} + e^{3t} \cdot e^{-t}$$

$$= 3e^{2t} + e^{2t} = 4e^{2t}$$

$$u_1 = -\int \frac{y_2 g(t)}{W} dt = -\int \frac{e^{3t}(3e^{2t})}{4e^{2t}} dt = -\int \frac{3e^{5t}}{4e^{2t}} dt$$

$$= -\frac{3}{4}\int e^{3t} dt = -\frac{3}{4}\left[\frac{1}{3}e^{3t}\right]$$

$$= -\frac{3}{12} e^{3t}$$

$$u_2 = \int \frac{y_1 \cdot g(t)}{W} dt = \int \frac{e^{-t}(3e^{2t})}{4e^{2t}} dt$$

$$= \int \frac{3e^t}{4e^{2t}} dt = \frac{3}{4}\int e^{-t} dt$$

$$= \frac{3}{4}\left(-e^{-t}\right)$$

$$= -\frac{3}{4} e^{-t}$$

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Ex. 1) (cont'd)

$$y_p(t) = u_1 y_1 + u_2 y_2$$

$$= -\frac{3}{12} e^{3t} e^{-t} - \frac{3}{4} e^{-t} e^{3t}$$

$$= -\frac{3}{12} e^{2t} - \frac{3}{4} e^{2t}$$

$$= -\frac{3}{12} e^{2t} - \frac{9}{12} e^{2t} = -\frac{12}{12} e^{2t}$$

$$= -e^{2t}$$

So gen. soln to non homog. ODE is

$$y(t) = y_c(t) + y_p(t)$$

$$= C_1 e^{-t} + C_2 e^{3t} - e^{2t}$$