1. (20 pts) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

Solution: To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)-(-1)(3) = \lambda^2-6\lambda+8 = (\lambda-4)(\lambda-2)$$

So solution to $\det(A - \lambda I) = 0$ is

$$(\lambda - 4)(\lambda - 2) = 0 \qquad \Longrightarrow \qquad \lambda_1 = 2, \quad \lambda_2 = 4$$

Now e find the eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = 2$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We have

$$A - (2)I = \begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

then

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we have

$$\left[\begin{array}{cc:c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \underset{\substack{\text{Row} \\ \text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc:c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1 - \xi_2 = 0 \qquad \Longrightarrow \xi_1 = \xi_2$$

So the eigenvector corresponding to λ_2 is

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 4$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_2 I)\boldsymbol{\xi}^{(2)} = 0$$

We have

$$A - (4)I = \begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

then

$$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system we have

$$\left[\begin{array}{cc:c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc:c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1 - \tfrac{1}{3}\xi_2 = 0 \qquad \Longrightarrow \xi_1 = \tfrac{1}{3}\xi_2$$

So the eigenvector corresponding to λ_2 is

$$\boldsymbol{\xi}^{(2)} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

2. (20 pts) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

Solution: To find eigenvalues we need to find solutions to the characteristic equation. In other words, we need to find values of λ for which

$$\det(A - \lambda I) = 0$$

We have

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3+3\lambda^2-7\lambda+5 = -(\lambda-1)\left(\lambda^2 - 2\lambda + 5\right)$$

So solution to $\det(A - \lambda I) = 0$ is

$$-(\lambda - 1)\left(\lambda^2 - 2\lambda + 5\right) = 0 \qquad \Longrightarrow \qquad \lambda_1 = 1, \quad \lambda_{2,3} = 1 \pm 2i$$

Now to find the eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = 1$:

To find eigenvalues we need to find the solution to

$$(A - \lambda_1 I)\boldsymbol{\xi}^{(1)} = 0$$

We have

$$A - (1)I = \begin{bmatrix} 1-1 & 0 & 0 \\ 2 & 1-1 & -2 \\ 3 & 2 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system we have

$$\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 2 & 0 & 0 \end{array}\right] \underset{\substack{\text{Row} \\ \text{Reduce}}}{\rightarrow} \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have

$$\xi_1 - \xi_3 = 0 \qquad \Longrightarrow \xi_1 = \xi_3$$

$$\xi_2 + \frac{3}{2}\xi_3 = 0 \qquad \Longrightarrow \xi_2 = \frac{3}{2}\xi_2$$

So the eigenvector corresponding to λ_2 is

$$\boldsymbol{\xi}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$