Section Summary

Theorem 3.2.1 (Existence and Uniqueness): Consider the initial value problem

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{1}$$

$$y'' + p(t)y' + q(t) = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q, g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution, $y = \Phi(t)$, of this problem and the solution exists throughout the interval I.

Complex Roots

Recall that the general form of a 2nd order homogeneous equation with constant coefficients is

$$ay'' + by' + cy = 0 \tag{2}$$

To find a solution we examine the characteristic equation

$$ar^2 + br + c = 0$$

For the case of complex roots we obtain roots of the form $r = \lambda \pm i\mu$ where λ and μ are real numbers. Our general solution has the form

$$y(t) = e^{\lambda t}\left[c_1 \cos(\mu t) + c_2 \sin(\mu t)\right] \tag{3}$$

1. Determine the longest interval in which the following initial value problems are certain to have a unique twice differentiable solution. Do not solve.

   (a) (6 pts) $y'' + \cos(t)y' + 2\ln|t|y = 0, \quad y(2) = 3, \quad y'(2) = 1.$

   Solution: Our equation is in the general form of (2) so we examine the domains our functions p, q, g:

Function	Domain
$p(t) = \cos(t)$	$(-\infty, \infty)$
$q(t) = 3\ln(t)$	$(0, \infty)$
$g(t) = 0$	$(-\infty, \infty)$

   All of these functions are continuous on $(0, \infty)$. Our initial condition is given at $t = 2$, so our solution will exist on the interval $(0, \infty)$ or $0 < t < \infty$

   (b) (6 pts) $(t-1)y'' - 3ty' + 4y = \sin(t), \quad y(-2) = 2, \quad y'(-2) = 1.$

   Solution: First we must have our equation in the general form of (1). We obtain:

   $$y'' - \frac{3t}{(t-1)}y' + \frac{4}{(t-1)}y = \frac{\sin(t)}{(t-1)}$$

   We can see that our functions p, q, g are all discontinuous at the point $t = 1$. Our options for intervals are either $(-\infty, 1)$ or $(1, \infty)$. Our initial condition is at $t = -2$, so a unique solution will exist on the interval $(-\infty, 1)$ or $-\infty < t < 1$

2. (8 pts) If the Wronskian W of f and g is t^2e^t and if $f(t) = t$, find $g(t)$.

Solution: We know that $W = t^2e^t$ and that $f(t) = t$. From this we also have that $f'(t) = 1$. Now the Wronskian is

$$W = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t)$$

Keeping it general here will be the best strategy. From here we plug in what we know and we have

$$W = f(t)g'(t) - f'(t)g(t) \implies t^2e^t = (t)g'(t) - (1)g(t)$$

Dividing each side by t we obtain

$$g'(t) - \frac{1}{t}g(t) = te^t$$

This has the form of an equation for using an integrating factor. By solving this ODE we find that

$$g(t) = te^t + ct$$

3. (20 pts) Find the solution of the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$$

Solution:

Characteristic Equation:
The characteristic equation is

$$r^2 - 2r + 5 = 0$$

Using the quadratic formula we find our roots $r = 1 \pm 2i$.

General Solution:
We have that $\lambda = 1$ and $\mu = 2$ so our general solution is

$$y(t) = e^t\left[c_1\cos(2t) + c_2\sin(2t)\right]$$

Particular Solution:
Now we find our particular solution by evaluating at our initial conditions. For $y(\pi/2) = 0$ we have

$$\begin{aligned} e^{\pi/2}\left[c_1\cos\left(2\frac{\pi}{2}\right) + c_2\sin\left(2\frac{\pi}{2}\right)\right] &= 0 \\ e^{\pi/2}\left[c_1\cos(\pi) + c_2\sin(\pi)\right] &= 0 \\ e^{\pi/2}[c_1(-1)] &= 0 \qquad \Longrightarrow \; c_1 = 0 \end{aligned}$$

We plug this in to our general solution to obtain

$$y(t) = c_2e^t\sin(2t)$$

we differentiate this to obtain

$$y'(t) = 2c_2e^t[\cos(2t) + \sin(2t)] = 0$$

now we use our initial condition $y'(\pi/2) = 2$

$$\begin{aligned} 2c_2e^{\pi/2}[\cos(\pi) + \sin(\pi)] &= 2 \\ 2c_2e^{\pi/2}(-1) &= 2 \\ -c_2e^{\pi/2} &= 1 \qquad \Longrightarrow \; c_2 = -e^{-\pi/2} \end{aligned}$$

So our particular solution is

$$y(t) = e^t(-e^{-\pi/2}\sin(2t)) = -e^{t-\pi/2}\sin(2t)$$