1. Find the inverse Laplace transforms of the following functions

   (a) $F(s) = \dfrac{2s-3}{s^2-4}$

Solution:

$$\begin{aligned} F(s) &= \frac{2s}{s^2-2^2} - \frac{3}{s^2-2^2} \\ &= 2\left(\frac{s}{s^2-2^2}\right) - 3\left(\frac{2/2}{s^2-2^2}\right) \\ &= 2\left(\frac{s}{s^2-2^2}\right) - \frac{3}{2}\left(\frac{2}{s^2-2^2}\right) \end{aligned}$$

Now we take the inverse Laplace transform to get back a function $f(t)$

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= 2\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2-2^2}\right\}}_{\text{by \#8}} + \frac{3}{2}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\}}_{\text{by \#7}} \\ &= 2\cosh(t) - \frac{3}{2}\sinh(t) \end{aligned}$$

You can also do this by partial fraction decomposition

$$F(s) = \frac{2s-3}{(s-2)(s+2)} = \frac{7}{4}\left(\frac{1}{s+2}\right) + \frac{1}{4}\left(\frac{1}{s-2}\right)$$

Taking the inverse Laplace transform to get back a function $f(t)$

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{7}{4}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}}_{\text{by \#2}} + \frac{1}{4}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}}_{\text{by \#2}} \\ &= \frac{7}{4}e^{-2t} + \frac{1}{4}e^{2t} \end{aligned}$$

Note that this is the same as what we obtain from using the table directly. Using the following identities for hyperbolic cosine and sine

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

we can see that

$$\begin{aligned}
f(t) &= 2\cosh(t) - \frac{3}{2}\sinh(t) \\
&= 2\left(\frac{e^{2t}+e^{-2t}}{2}\right) - \frac{3}{2}\left(\frac{e^{2t}-e^{-2t}}{2}\right) \\
&= e^{2t}+e^{-2t} - \frac{3}{4}\left(e^{2t}-e^{-2t}\right) = e^{2t}-e^{-2t}+\frac{3}{4}e^{2t}-\frac{3}{4}e^{-2t} \\
&= \frac{7}{4}e^{-2t}+\frac{1}{4}e^{2t}
\end{aligned}$$

(b) $F(s) = \dfrac{1}{s+1} + \dfrac{8-s}{s^2-2s+2}$

Solution:

$$\begin{aligned}
F(s) &= \frac{1}{s-(-1)} + \frac{8-s}{(s-1)^2+1} && \text{Completing the square} \\
&= \frac{1}{s-(-1)} + \frac{7-s+1}{(s-1)^2+1} && \text{Since } 8 = 7+1 \\
&= \frac{1}{s-(-1)} + \frac{7}{(s-1)^2+1} - \frac{s-1}{(s-1)^2+1} && \text{Pull out } -1 \text{ from last term}
\end{aligned}$$

Now we take the inverse Laplace transform to get back a function $f(t)$

$$\begin{aligned}
f(t) = \mathcal{L}^{-1}\{F(s)\} &= \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}}_{\text{by \#2}} + 7\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}}_{\text{by \# 9}} - \underbrace{\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}}_{\text{by\#10}} \\
&= e^{-t} + 7e^{t}\sin(t) - e^{t}\cos(t)
\end{aligned}$$

You will need at least a full page to do the work for each of these problems so do your work for the next two questions on separate paper and staple to your worksheet. There is a lot of tedious algebra and arithmetic that needs to be done for these problems and it can be very easy to make little errors along the way. So as always, it is highly recommended that you DON'T SKIP STEPS!

2. Use the Laplace transform to solve

$$y'' - y' - 6y = 0, \qquad y(0) = 1, \quad y'(0) = -1$$

Solution: Take the Laplace Transform of the ODE. Technically, we are applying it to both sides of the equation but the Laplace transform of 0 is 0 and so the right hand side of the ODE stays the same.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

Applying Corollary 6.2.2 we obtain

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - s + 1 \\ \mathcal{L}\{y'\} &= s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} - 1 \end{aligned}$$

Let $Y(s) = \mathcal{L}\{y\}$ and plug into $\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$

$$\left(s^2Y(s) - s + 1\right) - (sY(s) - 1) - 6Y(s) = 0$$

Now we combine like terms and solve for $Y(s)$

$$\begin{aligned} \left(s^2 - s - 6\right)Y(s) - s + 2 &= 0 \\ \left(s^2 - s - 6\right)Y(s) &= s - 2 \\ Y(s) &= \frac{s-2}{s^2 - s - 6} = \frac{s-2}{(s-3)(s+2)} \end{aligned}$$

The goal is for the right hand side to be in the form of something seen in our table. To get $Y(s)$ in an appropriate form we need to do a partial fraction decomposition. We will skip the details here.

$$\frac{s-2}{(s+2)(s-3)} = \frac{A}{s+2} + \frac{B}{s-3} = \frac{\frac{4}{5}}{s+2} + \frac{\frac{1}{5}}{s-3}$$

So we now have

$$Y(s) = \frac{4}{5}\left(\frac{1}{s+2}\right) + \frac{1}{5}\left(\frac{1}{s-3}\right)$$

We can now take the Inverse Laplace Transform of $Y(s)$ to get back to a function $y(t)$

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} \\ &= \frac{4}{5}e^{-2t} + \frac{1}{5}e^{3t} \end{aligned}$$

3. Use the Laplace transform to solve

$$y'' - 2y' + 2y = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 1$$

Solution: Now we have a non-homogeneous equation. When we take the Laplace Transform of the ODE we must apply it to both sides.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\left\{e^{-t}\right\} \tag{1}$$

Applying Corollary 6.2.2 we have the following

$$\begin{aligned}
\mathcal{L}\{y''\} &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2\mathcal{L}\{y\} - 1 \\
\mathcal{L}\{y'\} &= s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} - 0 \\
\mathcal{L}\left\{e^{-t}\right\} &= \frac{1}{s+1}
\end{aligned}$$

Now we let $Y(s) = \mathcal{L}\{\{\}\, y\}$ and plug into (1) to obtain

$$\left(s^2Y(s) - 1\right) - 2\left(sY(s)\right) + 2Y(s) = \frac{1}{s+1}$$

Now we need to solve for $Y(s)$

$$\begin{aligned}
\left(s^2 - 2s + 2\right) Y(s) - 1 &= \frac{1}{s+1} \\
Y(s) &= \left[\frac{1}{s+1} + 1\right]\left[\frac{1}{s^2 - 2s + 2}\right] \\
&= \frac{s+2}{(s+1)(s^2 - 2s + 2)}
\end{aligned}$$

By Partial Fractions:

$$\begin{aligned}
Y(s) &= \frac{s+2}{(s+1)(s^2 - 2s + 2)} = \frac{As + B}{s+1} + \frac{Cs + D}{s^2 - 2s + 2} \\
&= \frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{8 - s}{s^2 - 2s + 2}\right)
\end{aligned}$$

By completing the square in the denominator of the second term we obtain

$$\begin{aligned}
Y(s) &= \frac{1}{5}\left(\frac{1}{s-(-1)}\right) + \frac{1}{5}\left(\frac{8 - s}{(s-1)^2 + 1}\right) \\
&= \frac{1}{5}\left(\frac{1}{s-(-1)}\right) + \frac{1}{5}\left(\frac{7 - s + 1}{(s-1)^2 + 1}\right) \\
&= \frac{1}{5}\left(\frac{1}{s-(-1)}\right) + \frac{1}{5}\left(\frac{7}{(s-1)^2 + 1}\right) - \frac{1}{5}\left(\frac{s-1}{(s-1)^2 + 1}\right)
\end{aligned}$$

We now take the inverse Laplace transform to get back to a function $y(t)$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} \\ &= \frac{1}{5}e^{-t} + \frac{7}{5}e^{t}\sin(t) - \frac{1}{5}e^{t}\cos(t) \end{aligned}$$