Lecture 14: Inverse Laplace Transforms & Solving IVPs

When we take the Laplace transform (LT) of a function we are taking a function $y(t)$ (a function of t), and transforming it into a function $Y(s)$ (a function of s). We have seen that finding Laplace transforms using the definition can be tedious and requires use of all of our integration skills. This is where our table will come in handy.

Inverse Laplace Transforms

To get from $Y(s)$ to a function $y(t)$ we must then take the inverse Laplace transform (ILT) of the function $Y(s)$. i.e.

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

Unfortunately, there is no analytic way to go from any transformed function $Y(s)$ back to a function $y(t)$. Instead, we use the patterns we recognize from finding the Laplace transform of certain functions and manipulate our results to mimic them.

This then allows us to use the second column of our table to get back to a function $y(t)$. These examples cover the common types of manipulations that you may need to perform. The following are two important procedures that you will need to recall for this process.

Completing the Square

For a monic polynomial (the leading coefficient is 1) we have

$$x^2 + bx + c = \left(x + \frac{1}{2}b\right)^2 + \left(c - \frac{b^2}{4}\right)$$

For a non-monic polynomial we have

$$ax^2 + bx + c = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

You may not remember the precise process for completing the square. Luckily, in most situations we will encounter, it will be easy since we will know the form we want it to have rather than needing to find it from scratch.

Partial Fraction Decomposition

For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ we can do partial fraction decomposition if

$$\text{degree of } P(x) < \text{ degree of } Q(x)$$

Our goal is to decompose a rational function into parts that are easier to deal with. For example can obtain

$$f(x) = \frac{1}{(ax+b)(cx+d)\ldots} = \frac{A}{ax+b} + \frac{B}{cx+d} + \ldots$$

where we determine our constants $A, B, \ldots,$ etc. by equating the coefficients in the numerator of our original function.

The following table can be helpful in determining the form you will need for your decomposition.

Factor	**Term in decomposition**
$\dfrac{1}{ax+b}$	$\dfrac{A}{ax+b}$
$\dfrac{1}{(ax+b)^k}$	$\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$
$\dfrac{1}{ax^2+bx+c}$	$\dfrac{Ax+B}{ax^2+bx+c}$
$\dfrac{1}{(ax^2+bx+c)^k}$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 1: Common partial fraction decompositions

Example 1

Find the inverse Laplace transform of $F(s) = \dfrac{19}{s+2} - \dfrac{1}{3s-5} + \dfrac{7}{s^5}$

Solution. We are going to treat each term separately where our goal is to rewrite each of the terms so they look like options in our table:

$$G_1(s) = \frac{19}{s+2} = \frac{19}{s-(-2)} = 19\left(\frac{1}{s-(-2)}\right)$$
$$G_2(s) = \frac{1}{3s-5} = \frac{1}{3\left(s-\frac{5}{3}\right)} = \frac{1}{3}\left(\frac{1}{s-\frac{5}{3}}\right)$$
$$G_3(s) = \frac{7}{s^5} = \frac{7}{s^{4+1}} = 7\left(\frac{1}{s^{4+1}}\right) = 7\left(\frac{4!}{4!}\cdot\frac{1}{s^{4+1}}\right) = \frac{7}{24}\left(\frac{4!}{s^{4+1}}\right)$$

Now we have $F(s) = G_1(s) - G_2(s) + G_3(s)$ and we are ready to take the inverse Laplace transform to get back to a function $f(t)$.

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{G_1(s)\right\} - \mathcal{L}^{-1}\left\{G_2(s)\right\} + \mathcal{L}^{-1}\left\{G_3(s)\right\}$$

Again, we will do each piece separately.

$$\mathcal{L}^{-1}\left\{G_1(s)\right\} = \mathcal{L}^{-1}\left\{19\left(\frac{1}{s-(-2)}\right)\right\} = 19\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\} = 19e^{-2t} \quad \text{by \#2 in the table}$$
$$\mathcal{L}^{-1}\left\{G_2(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-\frac{5}{3}}\right)\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-\frac{5}{3}}\right\} = \frac{1}{3}e^{-5t/3} \quad \text{by \#2 in the table}$$
$$\mathcal{L}^{-1}\left\{G_3(s)\right\} = \mathcal{L}^{-1}\left\{\frac{7}{24}\left(\frac{4!}{s^{4+1}}\right)\right\} = \frac{7}{24}\mathcal{L}^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = \frac{7}{24}t^4 \quad \text{by \#3 in the table}$$

So we have the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = 19e^{-2t} - \frac{1}{3}e^{\frac{5}{3}t} + \frac{7}{24}t^4$$

Example 2

Find the inverse Laplace transform of $F(s) = \dfrac{6s-5}{s^2+7}$

Solution. First we rewrite each term so it fits a form we see in the table.

$$\begin{aligned} F(s) = \frac{6s}{s^2+7} - \frac{5}{s^2+7} &= 6\left(\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right) - 5\left(\frac{1}{s^2+\left(\sqrt{7}\right)^2}\right) \\ &= 6\left(\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right) - 5\left(\frac{\frac{\sqrt{7}}{\sqrt{7}}}{s^2+\left(\sqrt{7}\right)^2}\right) \\ &= 6\left(\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right) - \frac{5}{\sqrt{7}}\left(\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right) \end{aligned}$$

Now taking the inverse Laplace transform

$$\begin{aligned} \mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{6\left(\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{5}{\sqrt{7}}\left(\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right)\right\} \\ &= 6\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\sqrt{7}\right)^2}\right\}}_{\text{use \# 6}} - \frac{5}{\sqrt{7}}\underbrace{\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+\left(\sqrt{7}\right)^2}\right\}}_{\text{use \#5}} \\ &= 6\cos(\sqrt{7}t) - \frac{5}{\sqrt{7}}\sin(\sqrt{7}t) \end{aligned}$$

So we have the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = 6\cos(\sqrt{7}t) - \frac{5}{\sqrt{7}}\sin(\sqrt{7}t)$$

Example 3

Find the inverse Laplace transform of $F(s) = \dfrac{2s+2}{s^2+2s+5}$.

Solution. In this case we need to complete the square in the denominator:

$$s^2+2s+5 = \left(s^2+2s+1\right) - 1 + 5 = (s+1)^2 + 4$$

We can then rewrite $F(s)$ in the following way

$$F(s) = \frac{2s+2}{(s+1)^2+4} = \frac{2\,(s+1)}{(s+1)^2+2^2} = 2\left[\frac{s-(-1)}{(s-(-1))^2+2^2}\right]$$

From #10 in the table we know

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$$

where in this case we have $a = -1$ and $b = 2$ and so

$$f(t) = \mathcal{L}\left\{F(s)\right\} = 2\left[e^{-t}\cos(2t)\right]$$

Example 4

Find the inverse Laplace transform of $F(s) = \dfrac{s+7}{s^2-3s-10}$

Solution. In this case we need to do a partial fraction decomposition

$$F(s) = \frac{s+7}{(s+2)(s-5)} = \frac{A}{s+2} + \frac{B}{s-5}$$

To find our unknown constants we have

$$\begin{aligned}\frac{s+7}{(s+2)(s-5)} &= \frac{A(s-5)+B(s+2)}{(s+2)(s-5)} \\ \implies s+7 &= A(s-5)+B(s+2) \\ &= As - 5A + Bs + 2B \\ &= (A+B)s + (-5A+2B)\end{aligned}$$

Equating these coefficients to what we have on the LHS we have the system of equations

$$\begin{aligned}A + B &= 1 \\ -5A + 2B &= 7\end{aligned}$$

Solving this we obtain $A = -\frac{5}{7}$ and $B = \frac{12}{7}$. Now our $F(s)$ becomes

$$F(s) = \frac{\frac{-5}{7}}{s+2} + \frac{\frac{12}{7}}{s-5} = -\frac{5}{7}\left(\frac{1}{s-(-2)}\right) + \frac{12}{7}\left(\frac{1}{s-5}\right)$$

Now we can take the inverse Laplace transform

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\} &= -\frac{5}{7}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}}_{\text{use \# 2}} + \frac{12}{7}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}}\text{use \# 2} \\ &= -\frac{5}{7}e^{-2t} + \frac{12}{7}e^{5t}\end{aligned}$$

So we have the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = -\frac{5}{7}e^{-2t} + \frac{12}{7}e^{5t}$$

Solving IVPs with the Laplace Transform

The Laplace transform allows us to find the solution to many differential equations far more easily than methods learned thus far. The following corollary is stated in your text but has been simplified here. This gives us a simple method for finding the Laplace transform of a differential equation.

Recall that the notation $f^{(n)}(t)$ refers to the nth derivative of the function $f(t)$.

Corollary 1: (6.2.2)

Suppose that $f, f', \dots, f^{(n-1)}$ are all continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$ then

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\left\{f(t)\right\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Laplace Transform of a 2nd Order Non-homogeneous IVP

We have primarily dealt with 2nd order equations so consider the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_0'$$

Since the Laplace transform is a linear operator we have

$$\mathcal{L}\left\{ay'' + by' + cy\right\} = \mathcal{L}\left\{g(t)\right\} \quad \Longrightarrow \quad a\mathcal{L}\left\{y''\right\} + b\mathcal{L}\left\{y'\right\} + c\mathcal{L}\left\{y\right\} = \mathcal{L}\left\{g(t)\right\}$$

By Corollary 6.2.2 we have that

$$\begin{aligned}\mathcal{L}\left\{y''\right\} &= s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0) = s^2Y(s) - sy_0 - y_0' \\ \mathcal{L}\left\{y'\right\} &= s\mathcal{L}\left\{y\right\} - y(0) = sY(s) - y_0'\end{aligned}$$

Here we are simply using $\mathcal{L}\left\{y\right\} = Y(s)$ to simplify writing as you do your work. Sometimes I have found it helpful to maintain the $\mathcal{L}\left\{y\right\}$ notation so that I remember to take the inverse Laplace transform to get back to a function of t (i.e. $y(t)$). Also note that $\mathcal{L}\left\{g(t)\right\} = G(s)$ is simply the Laplace transform of the function $g(t)$.

We use these two equations in our differential equation to obtain

$$a(s^2Y(s) - sy_0 - y_0') + b(sY(s) - y_0') + cY(s) = G(s)$$

Our goal will then be to solve for $Y(s)$ by collecting like terms. To get back to a function $y(t)$ we need to take the inverse Laplace transform of $Y(s)$, i.e. $\mathcal{L}^{-1}\left\{Y(s)\right\} = y(t)$.

This is most intensive part of the process as it requires algebraic manipulation to get $Y(s)$ in a form that we see in our table of Laplace transforms.

After we find the inverse Laplace transform we are done! Since we have already used our initial conditions there will be no need to worry about them at the end of our problem as we have in the past. The corollary above will also allow us to take the Laplace transform of higher order equations. This results in a far easier process for solving higher order equations than those seen in Chapter 4.

Example 5

Use the Laplace transform to solve

$$y'' + 2y' + y = 4e^{-t}, \qquad y(0) = 2, \quad y'(0) = -1$$

Solution.

Take L.T. of each side:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$$

By Thm. 6.2.2

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$$

$$= s^2 Y(s) - 2s - (-1) = s^2 Y(s) - 2s + 1$$

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$$= sY(s) - 2$$

Note

$$\mathcal{L}\{e^{-t}\} = \frac{1}{s-(-1)} = \frac{1}{s+1}$$

Plug into ODE

$$(s^2 Y(s) - 2s + 1) + 2(sY(s) - 2) + Y(s) = 4\left(\frac{1}{s+1}\right)$$

$$s^2 Y(s) - 2s + 1 + 2sY(s) - 4 + Y(s) = \cdots$$

Combine like terms & solve for $Y(s)$

$$(s^2 + 2s + 1)Y(s) - 2s - 3 = 4\left(\frac{1}{s+1}\right)$$

$$\Rightarrow (s^2 + 2s + 1)Y(s) = 4\left(\frac{1}{s+1}\right) + 2s + 3$$

$$Y(s) = \frac{1}{s^2+2s+1}\left(4\left(\frac{1}{s+1}\right) + 2s + 3\right)$$

Goal is to get RHS into "nice" form to take the ILT.

$$Y(s) = \frac{1}{s^2+2s+1}\left[4\left(\frac{1}{s+1}\right)+2s+3\right]$$

$$= \frac{1}{(s+1)^2}\left[\frac{4}{s+1}+(2s+3)\left(\frac{s+1}{s+1}\right)\right]$$

$$= \frac{1}{(s+1)^2}\left[\frac{4+(2s^2+2s+3s+3)}{(s+1)}\right]$$

$$= \frac{1}{(s+1)^2}\left[\frac{2s^2+5s+7}{s+1}\right]$$

$$= \frac{2s^2+5s+7}{(s+1)^3}$$

$$\Rightarrow Y(s) = \frac{2}{s+1}+\frac{1}{(s+1)^2}+\frac{4}{(s+1)^3}$$

Now take the ILT of each side to get back a fcn $y(t)$

Get in "nice" form:

$$Y(s) = 2\left(\frac{1}{s-(-1)}\right)+\frac{1}{(s-(-1))^{1+1}}+4\left(\frac{1}{(s-(-1))^{2+1}}\right)$$

$$= 2\left(\frac{1}{s-(-1)}\right)+\frac{1}{(s-(-1))^{1+1}}+\frac{4}{2!}\left(\frac{2!}{(s-(-1))^{2+1}}\right)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s-(-1))^{1+1}}\right\}$$

$$+\frac{4}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{(s-(-1))^{2+1}}\right\}$$

So, particular soln to ODE is

$$y(t) = 2e^{-t}+te^{-t}+\frac{4}{2!}t^2e^{-t}$$