Higher order equations have the general form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Similar to 2nd order homogeneous equations we examine the roots of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

We will have 3 cases for the general form of our solutions.

Real, Distinct Roots

For distinct roots r_1, r_2, $\ldots$, r_k will have terms

$$e^{r_1 t}, \quad e^{r_2 t}, \ \ldots \ , \quad e^{r_k t}$$

Real, Repeated Roots

If r_1 is a repeated root (with multiplicity m) solutions corresponding to r_1 will have the form

$$e^{r_1 t} + te^{r_1 t} + \cdots + t^{m-1} e^{r_1 t}$$

Distinct, Complex Roots

If the characteristic polynomial has complex roots, they must occur in conjugate pairs, i.e. $\lambda + i\mu$ and $\lambda - i\mu$. If none of the complex roots are repeated, then solutions for these roots will have the form

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

Repeated, Complex Roots

The case for repeated complex roots is a bit more tricky as it involves some more knowledge of complex numbers. All roots will need to be considered and can often be difficult to find. If we have a complex root $\lambda \pm i\mu$ repeated k times we will have solutions with general form

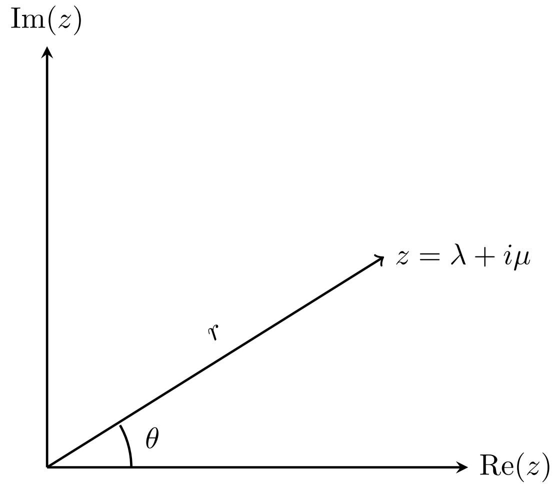
$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t, \quad te^{\lambda t} \cos(\mu t), \quad te^{\lambda t} \sin \mu t, \ \ldots \ , \quad t^{k-1} e^{\lambda t} \cos(\mu t), \quad t^{k-1} e^{\lambda t} \sin \mu t$$

Review of Complex Numbers

Complex numbers have the general form $z = \lambda \pm i\mu$ where λ is the real part of z and μ is the imaginary part of z i. e.

$$\mathrm{Re}(z) = \lambda \qquad \text{and} \qquad \mathrm{Im}(z) = \mu$$

Geometrically, complex numbers can be represented as a vector in the complex plane.



Note that here

$$r = |z| \qquad \text{and} \qquad \theta = \text{ Arg } z$$

The **modulus** is denoted by $|z|$ and is defined as

$$|z| = |\lambda + i\mu| = \sqrt{\lambda^2 + \mu^2}$$

Remember that the absolute value bars indicate finding distance from zero. For complex numbers this is done by the formula given above. Since a complex number is represented as a vector in the complex plane, this is the same operation as finding the magnitude of a vector.

Exponential Form

We have already seen **Euler's formula**, which tells us that

$$e^{it} = \cos t + i \sin t$$

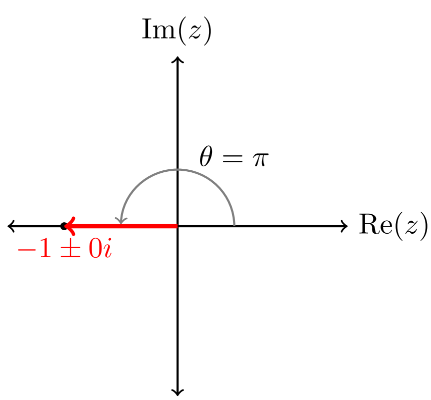
This can also be used to represent a complex number in **exponential form**. First, we'll need to understand the geometry of complex numbers.

Recall that we can write a complex number $z = \lambda + i\mu$ in exponential form where

$$z = \underline{re^{(i(\theta + 2\pi n))}} = r \exp\left(i[\theta + 2\pi n]\right)$$

which just means the angle between z and the positive part of the real line (see figure).

Example 1



$z = -1 = -1 \pm 0i$

$r = |z| = |-1| = \sqrt{(1)^2 + (0)^2} = \sqrt{1} = 1$

$\theta = \pi$

$z = re^{i\theta} = 1 \cdot e^{i\pi}$

or $z = 1e^{i(\pi + 2\pi n)}$

Lecture # 12: Higher Order Equations Date: Wed 3/13/19

Ex. Find gen. soln of

$$y^{(4)} - y = 0$$

The characteristic eqn is

$$r^4 - 1 = 0$$

$$(r^2-1)(r^2+1) = 0$$

$r^2 - 1 = 0$ $r^2 + 1 = 0$

$r^2 = 1$ $r^2 = -1$

$r = \pm 1$ $r = \pm i$

so $r = -1, 1, -i, i = -1, 1, \pm i$

Gen. soln is

$$y(t) = c_1 e^{-1t} + c_2 e^{1t} + e^{0t}(c_3\cos(t) + c_4\sin(t))$$

$$= c_1 e^{-t} + c_2 e^{t} + c_3\cos(t) + c_4\sin(t)$$

Note: to find a particular soln you need four initial conditions

Lecture # 12: Higher Order Equations

Date: Wed 3/13/19

Ex. Find gen. soln to

$$y^{(4)} + y = 0$$

char. eqn

$$r^4 + 1 = 0$$

$$r^4 = -1$$

$$r = (-1)^{1/4}$$

Rep. $r = (-1)^{1/4}$ as a complex number in exponential form. Let $z = -1 + 0i$

From prev. example

$$z = -1 = e^{i(\pi + 2\pi n)}$$

so $r = (z)^{1/4} = \left[e^{i(\pi + 2\pi n)}\right]^{1/4}$

$$= e^{i\frac{1}{4}(\pi + 2\pi n)}$$

$$= e^{i\left(\frac{\pi}{4} + \frac{\pi}{2}n\right)}$$

$$= \exp\left[i\left(\frac{\pi}{4} + \frac{\pi}{2}n\right)\right], \text{ for } n = 0, 1, 2, \ldots$$

$$= \cos\left(\frac{\pi}{4} + \frac{\pi}{2}n\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{2}n\right)$$

Need 4 roots: use $n = 0, 1, 2, 3$

$n=0$: $r_1 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1+i}{\sqrt{2}}$

$n=1$: $r_2 = \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{-1+i}{\sqrt{2}}$

$n=2$: $r_3 = \cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) = \frac{-1-i}{\sqrt{2}}$

$n=3$: $r_4 = \cos\left(\frac{\pi}{4} + \frac{4\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{2}\right) = \frac{1-i}{\sqrt{2}}$

Lecture # 12: Higher Order Equations

$$r_{1,2} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i \quad \& \quad r_{3,4} = -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$$

$$\lambda_{1,2} = \frac{1}{\sqrt{2}}, \ \mu_{1,2} = \frac{1}{\sqrt{2}} \qquad \lambda_{3,4} = -\frac{1}{\sqrt{2}}, \ \mu_{3,4} = \frac{1}{\sqrt{2}}$$

so gen. soln is

$$y(t) = e^{\frac{t}{\sqrt{2}}}\left[c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right)\right] + e^{-\frac{t}{\sqrt{2}}}\left[c_3 \cos\left(\frac{t}{\sqrt{2}}\right) + c_4 \sin\left(\frac{t}{\sqrt{2}}\right)\right]$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$