Lecture # 7: Intro to 2nd order Eqns

Date: Wed. 2/27/19

2nd Order Eqns

<u>Def</u> A 2nd order linear ODE has gen form

$$y'' = f(t, y, y')$$

or

$$= g(t) - q(t)y - p(t)y'$$

$$y'' + p(t)y' + q(t)y = g(t) \qquad (*)$$

<u>or</u>

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

- Eqns not in this form are <u>nonlinear</u>
- when $G(t) = 0$ the ODE is <u>homogeneous</u>
- when $G(t) \neq 0$ the ODE is <u>nonhomogeneous</u>

For 2nd order ODEs we now need 2 initial conditions

$$y(t_0) = y_0 \quad \& \quad y'(t_0) = y_0'$$

<u>Why?</u> Essentially we are doing 2 integrations to arrive @ our soln. i.e. we have 2 "constants" of integration to find.

Lecture # 7: Intro to 2^{nd} order Eqns

Date: Wed. 2/27/19

We will focus on cases with constant coefficients i.e. when $p(t)$, $Q(t)$, $R(t)$ are all constants

$$ay'' + by' + cy = 0$$

Homogeneous Eqns

First we discuss the homogeneous case.

We will assume solns have form

$$y = e^{rt}$$

where r is a constant.

Since $y' = re^{rt}$
$y'' = r^2 e^{rt}$

Sub this into $ay'' + by' + cy = 0$

$$\Rightarrow ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\underbrace{e^{rt}}_{\substack{\text{Never}\\ =0}} [\underbrace{ar^2 + br + c}_{\substack{\text{want when this}\\ =0}}] = 0$$

So we need to solve

$$ar^2 + br + c = 0 \quad \underline{\text{Characteristic Eqn}}$$

Lecture #7: Intro to 2^{nd} order Eqns

Date: Wed. 2/27/19

Use the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 cases for r_1 & r_2

Look @ discriminant $b^2 - 4ac$

i) $b^2 - 4ac > 0 \Rightarrow r_1 \neq r_2, \quad r_1, r_2 \in \mathbb{R}$ "real & different"

ii) $b^2 - 4ac = 0 \Rightarrow r_1 = r_2, \quad r_1, r_2 \in \mathbb{R}$ "real & repeated"

iii) $b^2 - 4ac < 0 \Rightarrow r_1, r_2 = \lambda \pm i\mu, \quad \lambda, \mu \in \mathbb{R}$ "complex roots"

Real & Different Roots

When $r_1 \neq r_2$ both real we will have 2 solns to the ODE (**)

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

Def A linear combination is an expression that results from mult. each term by a cst & summing the result.

Lecture # 7: Intro to 2^{nd} order Eqns Date: Wed. 2/27/19

Thm (3.2.2) Principle of Superposition

Suppose y_1, y_2 are solns to

$$y'' + p(t)y' + q(t)y = 0$$

then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a soln for any values c_1 & c_2

Def y_1 & y_2 are called a fundamental set of solns & the gen. soln of (*) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

So if $y_1 = e^{r_1 t}$ & $y_2 = e^{r_2 t}$ are solns then

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

must also be a soln by Thm 3.2.2.

Lecture # 7: Intro to 2^{nd} order Eqns

Date: Wed. 2/27/19

Ex. 1) $y'' + 4y' + 3y = 0$, $y(0) = 2$, $y'(0) = -1$

This is a 2^{nd} order linear ODE w/ gen. form

$$ay'' + by' + c = 0$$

Char. eqn is

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$\Rightarrow r_1 = -3, \quad r_2 = -1$$

Gen. soln given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$= c_1 e^{-3t} + c_2 e^{-t}$$

Apply Initial Conditions (I.C.s)

@ $y(0) = 2$ $\quad (y(t=0) = 2)$

$$2 = c_1 e^{-3(0)} + c_2 e^{-(0)}$$

$$\Rightarrow 2 = c_1 + c_2 \quad \Rightarrow \quad c_1 = 2 - c_2$$

@ $y'(0) = -1$

$$y'(t) = -3c_1 e^{-3t} - c_2 e^{-t}$$

$$\Rightarrow -1 = -3c_1 e^{-3(0)} - c_2 e^{-(0)}$$

$$\Rightarrow -1 = -3c_1 - c_2 \quad \Rightarrow c_2 = 1 - 3c_1$$

Lecture #7: Intro to 2^{nd} order Eqns

Date: Wed. 2/27/19

Ex. 1 (cont'd)

$$c_1 = 2 - c_2 \qquad \Rightarrow \quad c_1 = -\frac{1}{2}$$

$$c_2 = 1 - 3c_1 \qquad \quad c_2 = \frac{5}{2}$$

So particular soln is

$$y(t) = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}$$