Math 2250

Lecture #11: Dimension & Rank

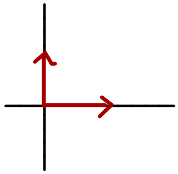
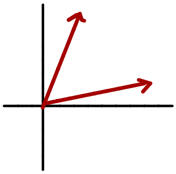
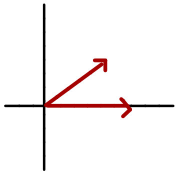
Date: Thu. 10/23/18

Dimension

Objective: If V has a finite basis, all bases for V have the same number of elements. This number is the dimension of V, $\dim(V)$.

Ex. 1) $\dim(\mathbb{R}^2) = 2$. The following are bases:

$$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

Thm. If V has a basis $\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$ then any set $\{\vec{w}_1, \dots, \vec{w}_m\}$ in V having $> n$ elements is linearly dependent.

Pf. Want to find $c_1, \dots, c_m$ not all 0, s.t.

$$c_1 \vec{w}_1 + \dots + c_m \vec{w}_m = \vec{0} \qquad (1)$$

Since $\{v_1, \dots, v_m\}$ is a basis,

$$\vec{w}_1 = a_{11}\vec{v}_1 + \dots + a_{n1}\vec{v}_n$$
$$\vdots$$
$$\vec{w}_m = a_{1m}\vec{v}_1 + \dots + a_{nm}\vec{v}_n$$

So (1) requires

$$c_1(a_{11}v_1 + \dots + a_{n1}v_n) + \dots + c_m(a_{1m}v_1 + \dots + a_{nm}v_m) = \vec{0}$$

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Pf. (cont'd)

Rearranging

$$[a_{11}c_1 + \dots + a_{1m}c_m]\vec{v}_1 + \dots [a_{n1}c_1 + \dots + a_{nm}c_m]\vec{v}_n = \vec{0}$$

Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis, it is lin. indep.
So each expression in [ ] must be 0. i.e.

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

n eqns
m unknowns

Since $m > n$, this system has free variables
Hence, a nontrivial soln i.e.

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

So a set w/ $> n$ elements must be lin. dep. $\square$

Ex. 2 | $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ is the standard basis for $\mathbb{R}^4$. Any set of 5 or more vectors in $\mathbb{R}^4$ is lin. dependent.

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Thm. If V has a basis of n vectors, every basis of V has exactly n vectors.

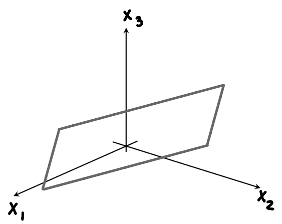
Pf. Let B_1 be a basis with n vectors & let B_2 be another basis. B_2 is lin. indep., so it has $\leq n$ elements by prev. thm.

Since B_2 is a basis & B_1 is lin. indep., B_2 has $\geq n$ elements. i.e. B_2 must have exactly n elements. $\square$

Def. The number of elements in any basis for V is the dimension of V, $\dim(V)$.

Ex. 3 $\dim(\mathbb{R}^n) = n$

Ex. 4 $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + x_2 = x_3 \right\}$ (a plane in $\mathbb{R}^3$)



Typical vector in V:

$$\begin{bmatrix} s \\ t \\ s+t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(V) = 2$

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Ex. 5) If $A \in \mathbb{R}^{m\times n}$,

$\dim(\text{Null}(A))$ = number of free variables in $A\vec{x} = \vec{0}$

$\dim(\text{Col}(A))$ = number of pivot col.s in A

Ex. 6) $A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix}$ Find free variables in $A\vec{x} = \vec{0}$:

Pivot columns

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 6 & 1 & 1 & 0 \end{array}\right] \xRightarrow{\text{REF}} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable $x_3 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ is a basis; $\dim(\text{Null}(A)) = 1$

$\dim(\text{Col}(A))$ = # pivot cols

= # nonzero rows in REF

= 2

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Row space

Given $A \in \mathbb{R}^{m \times n}$, write it as $A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}$ row vector $[a_{11} \dots a_{1n}]$

<u>Def</u> The <u>row space</u> of $A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}$ is

$$\text{Row}(A) = \text{Span}\{\vec{a}_1, \dots, \vec{a}_m\}$$

Facts about Row(A):

1. If $A \to U$ (REF), then $\text{Row}(A) = \text{Row}(U)$

   Example:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix} \xRightarrow{\text{REF}} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$\text{Row}(A) = \text{Row}(U) = \text{Span}\{(1, \tfrac{1}{2}, 0), (0, 1, -\tfrac{1}{2})\}$$

nonzero rows of REF

$$\Rightarrow \dim(\text{Row}(A)) = 2$$

Note: if U is in REF, its nonzero rows from a basis for row(U).

2. If U is in REF then

$$\dim(\text{Row}(U)) = \#\text{ of nonzero rows} = \#\text{ of basic variables}$$

Ex. 7 | Find a basis for

$$\text{Span}\{(2,1,0),(4,0,1),(6,1,1)\}$$

Use the vectors as rows of a matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix} \xRightarrow{\text{REF}} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = U$$

2 nonzero rows $\Rightarrow \dim(\text{Row}(U)) = 2$

$$\Rightarrow \text{Row}(A) = \text{Row}(U) = \text{Span}\{(1,\tfrac{1}{2},0),(0,1,-\tfrac{1}{2})\}$$

Thm. For any matrix $A \in \mathbb{R}^{m\times n}$,

$$\dim(\text{Row}(A)) = \dim(\text{Col}(A))$$

Pf Reduce A to REF U. Then

$$\begin{aligned} \dim(\text{Col}(A)) &= \dim(\text{Col}(U)) \\ &= \#\text{ of basic variables} \\ &= \#\text{ nonzerow rows } U \\ &= \dim(\text{Row}(U)) \\ &= \dim(\text{Row}(A)) \end{aligned}$$

$\square$

Rank

Def. The rank of A is

$$\boxed{\text{rank}(A) = \dim(\text{Col}(A))}$$

$(= \dim(\text{Row}(A))$ by Prev. thm$)$

The nullity of A is $\dim(\text{Null}(A))$

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Ex. 8

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 10 & 2 & 6 \\ 0 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

So $\{(1,3,-1,2), (0,1,1,\frac{1}{2})\}$ is a basis for $\text{Row}(U) = \text{row}(A)$

$$\Rightarrow \text{rank}(A) = \dim(\text{Row}(A)) = 2.$$

Remark:

to find a basis for Col(A) we use the cols of A in which U has leading 1's

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ 2 \end{bmatrix} \right\}.$$

To find a basis for Null(A) use REF:

Since $A\vec{x} = \vec{0}$ iff $U\vec{x} = \vec{0}$, $\text{Null}(A) = \text{Null}(U)$

$$U\vec{x} = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Free variables: $x_3 = t$, $x_4 = s$

$x_1 = 4t - \frac{1}{2}s$

$x_2 = -t - \frac{1}{2}s$

General Soln:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4t - \frac{1}{2}s \\ -t - \frac{1}{2}s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

So, $\left\{ \begin{bmatrix} 4 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\text{Null}(U) = \text{Null}(A)$

$$\begin{aligned} \Rightarrow \text{nullity}(A) &= \dim(\text{Null}(A)) \\ &= \#\text{ of Free variables} \\ &= 2 \end{aligned}$$

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Observation: for this matrix $A \in \mathbb{R}^{3\times 4}$,

$$\text{Rank}(A) + \dim(\text{Null}(A)) = 4$$
$$\text{Rank}(A) + \text{nullity}(A) = 4$$

<u>Thm.</u> For any $A \in \mathbb{R}^{m\times n}$

$$\boxed{\text{rank}(A) + \dim(\text{Null}(A)) = n} \quad (\# \text{ of cols})$$

<u>Pf.</u> Reduce A to REF. Then

$$\underbrace{\#\text{ of basic variables}}_{\substack{\text{rank}(A) = \dim(\text{Col}(A)) \\ = \dim(\text{Row}(A))}} + \underbrace{\#\text{ of free variables}}_{\dim(\text{Null}(A))} = n$$

Rank and dimension allow us to say even more regarding invertible matrices:

<u>Thm.</u> If $A \in \mathbb{R}^{n\times n}$ (square!), TFAE:

(1) A is invertible

(2) Cols of A form a basis for $\mathbb{R}^n$

(3) $\text{Col}(A) = \mathbb{R}^n$

(4) $\text{rank}(A) = n$

(5) $\text{Null}(A) = \{\vec{0}\}$

(6) $\dim(\text{Null}(A)) = 0$

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In Summary:

For $A \in \mathbb{R}^{m \times n}$:

Null(A)

$\{\underset{\sim}{x} \in \mathbb{R}^n \mid A\underset{\sim}{x} = \underset{\sim}{0}\}$

$\subset \mathbb{R}^n$

$\dim(\text{Null}(A))$ = number of free variables in REF

basis: write general solution to $A\underset{\sim}{x} = \underset{\sim}{0}$

Separate the parameters

Col(A)

span{columns of A}

$\subset \mathbb{R}^m$

$\dim(\text{Col}(A)) = \text{rank}(A)$

= number of basic variables in REF

= number of pivot columns

basis: columns of A corresp. to pivot columns in REF

Row(A)

span{rows of A}

$\subset \mathbb{R}^n$

$\dim(\text{Row}(A)) = \dim(\text{Col}(A))$

basis: nonzero rows of the REF