1. Let H be the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that H is a subspace of $\mathbb{R}^4$.

Solution: Since
$$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$
then by Theorem 1 in the textbook we must have that
$$\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}\right\}$$
is a subspace of $\mathbb{R}^4$.

2. Let
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(a) Is $\mathbf{w} \in \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are there in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Solution: No. By inspection $\mathbf{w} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. There are 3 vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Solution: Since Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ there are infinitely many vectors in this set.

3. Again consider the set of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

(a) Is $\mathbf{w}$ in the subpsace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why or why not?

Solution: If the $\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ then the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$$

has at least one solution. By setting up the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{w}]$ and row reducing we obtain

$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \underset{\substack{\text{Row}\\ \text{Reduce}}}{=} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So this system has the solution

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 1 - 2c_3 \end{aligned}$$

Since the solution involves a free variable, we know this system has at least one solution. Thus $\mathbf{w}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Thus, $\mathbf{v}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Let $\mathbf{u} = \begin{bmatrix} 8 \\ 4 \\ 7 \end{bmatrix}$. Is $\mathbf{u}$ in the subpsace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why or why not?

Solution: If the $\mathbf{u}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ then the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{u}$$

has at least one solution. By setting up the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{u}]$ and row reducing we obtain

$$\left[\begin{array}{rrr|r} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{array}\right] \underset{\substack{\text{Row}\\ \text{Reduce}}}{=} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This system has no solution, so $\mathbf{u}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Thus, $\mathbf{u}$ is not in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.