Lecture #03: Matrix Eqns & Solution Sets

Date: Tue. 9/18/18

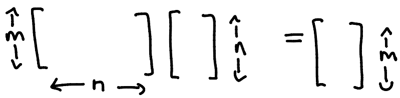
The matrix Eqn $A\vec{x} = \vec{b}$

Def If A is an $m \times n$ matrix $[\vec{a}_1\ \vec{a}_2, \dots \vec{a}_n]$ & $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, then

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n$$

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$$

Notice:



Ex.

$$\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 0x_1 - x_2 \\ 4x_1 + 3x_2 \\ x_1 + 0x_2 \end{bmatrix}$$

Def A <u>matrix equation</u> has the form

$$\underbrace{[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]}_{\text{Known columns}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{Unknown}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\text{Known vector}}$$

Shorthand: $A\vec{x} = \vec{b}$

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The Following have the same soln. set:

(i) $A\vec{x} = b$

(ii) $x_1\vec{a}_1 + \ldots + x_n\vec{a}_n = \vec{b}$

(iii) $[\vec{a}_1 \ \ldots \ \vec{a}_n \mid \vec{b}\,]$

We find soln using REF or RREF & back sub.

Consequence: $A\vec{x} = \vec{b}$ has a soln if & only if (iff) $\vec{b}$ is a linear combination of the cols of A, i.e.

$\vec{b} \in$ Span{ columns of A }

<u>Thm</u> Let A be an $m \times n$ matrix.
The Following are equivalent (TFAE)

a) For any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a soln.

b) Every $\vec{b} \in \mathbb{R}^m$ is a linear combo of cols. of A

c) Span{ Cols of A } $= \mathbb{R}^m$

d) A has a pivot position in every row

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Ex) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a soln $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

No matter what we pick for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Check:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pivot Position in every Row

Think of matrices as <u>transformations</u> that transform vectors into other vectors: $\vec{v} \rightarrow A\vec{v}$

The transformations are <u>linear</u>:

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{v}) = c(A\vec{v}), \quad \text{any } c \in \mathbb{R}$$

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Solution sets of Linear Systems

Def: A linear system is homogeneous if it can be written as

$$A\vec{x} = \vec{0}$$

Ex 1

$$3x_1 + 2x_2 = 0$$
$$-x_1 - 3x_2 = 0$$

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thm Every homogeneous linear system $A\vec{x} = \vec{0}$ has @ least 1 soln.

Pf $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$ is a soln $\square$

We call $\vec{0}$ the trivial solution.

Question: When do nontrivial solns exist?

Answer: $A\vec{x} = \vec{0}$ has a nontrivial soln (= nonzero) solns iff the linear system has a free variable

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Ex

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 6 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \begin{bmatrix} 3 & -1 & 2 & 0 \\ 0 & 3 & -5 & 0 \end{bmatrix} \Rightarrow \begin{array}{r} 3x_1 - x_2 + 2x_3 = 0 \\ 3x_2 - 5x_3 = 0 \end{array}$$

$\Rightarrow$

$$x_3 = t$$

$$3x_2 - 5t = 0 \Rightarrow x_2 = \frac{5}{3}t$$

$$3x_1 - \frac{5}{3}t + 2t = 0 \Rightarrow x_1 = \frac{1}{9}t$$

includes $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (choose $t=0$) <u>and</u> ∞ many other solns

Conclusion: soln set consists of vectors of the form $t \begin{bmatrix} \frac{1}{9} \\ \frac{5}{3} \\ 1 \end{bmatrix}$, any t.

i.e. Soln set $= \text{Span}\left\{ \begin{bmatrix} \frac{1}{9} \\ \frac{5}{3} \\ 1 \end{bmatrix} \right\}$ which contains non trivial solns

More generally:

The soln set of $A\vec{x} = \vec{0}$ is always $\text{span}\{\vec{v}_1 \dots \vec{v}_k\}$ for some set of vectors $\vec{v}_1, \dots \vec{v}_k$.

(If $\vec{0}$ is the only soln then soln set $= \text{span}\{\vec{0}\} = \{\vec{0}\}$)

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Nonhomogeneous Systems

Thm Suppose $A\vec{x}=\vec{b}$ is consistent (i.e. has ≥ 1 soln). If $\vec{x}_p$ is the only soln, then the soln set has the form $\vec{x}_p+\vec{x}_h$, where $\vec{x}_h$ ranges over all solns to $A\vec{x}=\vec{0}$.

Check: $A(x_p+x_h) = \underbrace{Ax_p}_{\vec{b}} + \underbrace{Ax_h}_{\vec{0}} = \vec{b}$

so $\vec{x}_p+\vec{x}_h$ is a soln.

Remarks:

1. If $A\vec{x}=0$ has exactly 1 soln, so does $A\vec{x}=\vec{b}$
2. If $A\vec{x}=\vec{0}$ has ∞ many solns so does $A\vec{x}=\vec{b}$
3. If $A\vec{x}=\vec{b}$ is inconsistent, theorm doesn't apply

Ex

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 2 \\ 6 & 1 & -1 & 7 \end{array}\right] \xrightarrow{REF} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 2 \\ 0 & 3 & -5 & 3 \end{array}\right]$$

free

$x_3 = t$

$3x_2 - 5t = 3 \Rightarrow x_2 = \frac{5}{3}t+1$

$3x_1 - (\frac{5}{3}t+1) + 2t = 2 \Rightarrow x_1 = -\frac{1}{9}t+1$

Parametric Form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{t\begin{bmatrix} -\frac{1}{9} \\ \frac{5}{3} \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_p}$$

$\Rightarrow \infty$ many solns