1. Assume that A is row equivalent to B where

$$A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find bases for Null(A) and Col(A).

Solution: Null(A) is found by solving $A\mathbf{x} = \mathbf{0}$. Since B is the row reduced form of A we can find the solution to $B\mathbf{x} = \mathbf{0}$. This gives us

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & & & + & 4x_4 & + & 5x_5 & = & 0 \\ & & & + & 5x_3 & - & 7x_4 & + & 8x_5 & = & 0 \\ & & & & & & & - & 9x_5 & = & 0 \end{array}$$

performing back substitution we have

$$\mathbf{x} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ \frac{7}{5}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix}$$

so the basis for Null(A) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Col(A) is given by the pivot columns of A. We can see from the matrix B that the pivot columns of A are the 1st, 3rd, and 5th columns. So a basis for Col(A) is then

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$$

2. Find a basis for the space spanned by the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Solution: By constructing a matrix A where the vectors are the column vectors. The basis for the set of vectors will then be given by the basis of the column space of A. This basis will be given by finding the pivot columns of A.

$$\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \underset{\substack{\text{Row} \\ \text{Reduce}}}{=} \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Referencing back to our original matrix A we take the first 3 columns (i.e. the pivot columns). So a basis for $\text{Col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

and this is the same basis for the given set of vectors.