Lecture #2: Vector Equations

Date: Thu. 9/13/18

Vectors

Def A matrix w/ only one column or row is called a <u>vector</u>.

Column vector:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\Rightarrow$ 2 x 1 Matrix

"2 by 1"

"2 rows by 1 column"

Row vector:

$$\vec{v} = [v_1, v_2]$$

component of a vector

$\Rightarrow$ 1 x 2 matrix

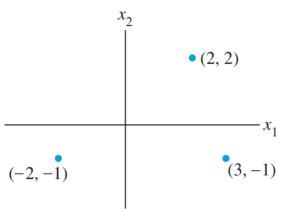
"1 by 2"

Notation: $m \times n$ "m rows by n columns"

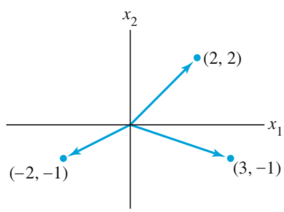
Note: 2D space of real #'s: $\mathbb{R}^2$

3D space : $\mathbb{R}^3$

n-D space : $\mathbb{R}^n$



Ordered pairs in the plane



Vectors in the plane

Note: you may also see < , > used for vectors instead of ( )

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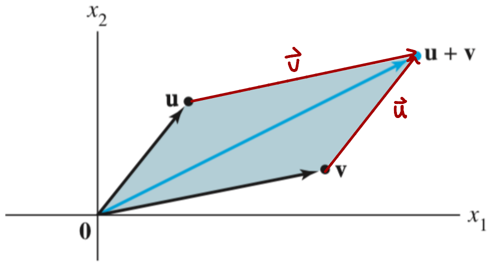
Vector Operations

Vector Addition

To add 2 vectors by adding the components of each vector

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

Geometrically:



Parallelogram Rule for Addition

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$. See Figure 3.

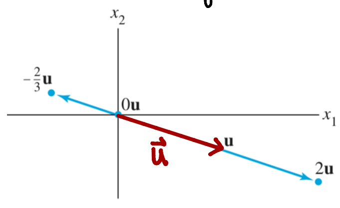
Scalar Multiplication

<u>Def</u> A constant c is also called a <u>scalar</u>.

We can multiply a vector by a scalar by multiplying the scalar w/ each component

$$c\vec{u} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$$

Geometrically:



Typical multiples of $\mathbf{u}$

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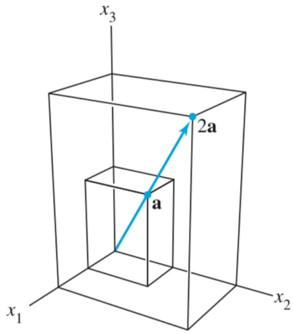
Vectors in $\mathbb{R}^3$

a vector in $\mathbb{R}^3$ has the form

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Geometrically:



Vectors in $\mathbb{R}^n$

a vector in $\mathbb{R}^n$ has the form $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

Properties of Vectors

Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(viii) $1\mathbf{u} = \mathbf{u}$

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Linear Combinations

Def Given vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_p$ & scalars $c_1, c_2, \dots, c_p$ the vector $\vec{y}$ defined by

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

is a <u>linear</u> <u>Combination</u> of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$

Ex Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Determine whether $\vec{b}$ can be written as a linear combination of $\vec{v}_1$ & $\vec{v}_2$

"such that"

Want to find constants c_1, c_2 s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$$

$$\Rightarrow \quad c_1\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} c_1 + 2c_2 \\ -2c_1 + 5c_2 \\ 5c_1 + 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

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Ex (cont'd)

This is the system of equations

$$c_1 + 2c_2 = 7$$
$$-2c_1 + 5c_2 = 4$$
$$-5c_1 + 6c_2 = -3$$

which can be written as the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \underset{\substack{\text{Reduced} \\ \text{Row} \\ \text{Echelon} \\ \text{form}}}{\overset{\text{RREF}}{\Longrightarrow}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So the solution to the system is $c_1 = 3$
So we have $c_2 = 2$

$$3\vec{v}_1 + 2\vec{v}_2 = \vec{b}$$

$$\Longrightarrow \quad 3\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

So $\vec{b}$ can be written as a linear combination of $\vec{v}_1$ & $\vec{v}_2$. $\square$

Note that this vector eqn can also be written as $[\,\vec{v}_1 \;\; \vec{v}_2 \,\vdots\, \vec{b}\,]$ $\leftarrow$ this is the augmented matrix.

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Def

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{5}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

$\boxed{\text{Span}}$

"in"

Def If $\vec{v}_1, \ldots \vec{v}_p \in \mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \ldots \vec{v}_p$ is denoted by

$$\text{Span}\{\vec{v}_1, \ldots \vec{v}_p\}$$

We say that this is the <u>subset of $\mathbb{R}^n$</u> <u>spanned</u> (or generated) by $\vec{v}_1, \ldots \vec{v}_p$.

In other words,

$$\text{Span}\{\vec{v}_1, \ldots \vec{v}_p\}$$

Is the collection of all vectors that can be written in the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_p v_p$$

Where $c_1, \ldots c_p$ are scalars

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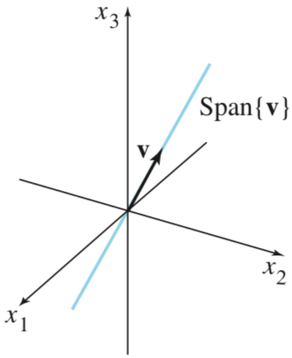
Geometrically (in $\mathbb{R}^3$)

Span$\{\vec{v}\}$: the set of all scalar multiples of $\vec{v}$ which pass thru the origin.



Span$\{\vec{u}, \vec{v}\}$: the plane that contains $\vec{u}$, $\vec{v}$, $\vec{0}$