Lecture # 06: matrix Operations & Inverses

Date: Thu. 9/27/18

Matrix Operations

Notation:

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{mn} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = [\vec{a}_1 \ \vec{a}_2 \ \cdots\cdots \ \vec{a}_n] \in \mathbb{R}^{m \times n}$$

row i

column $j = \vec{a}_j$

The entries $a_{11}, a_{22}, \ldots a_{mm}$ are <u>diagonal entries</u>

$$0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note: Must be a square matrix

Adding 2 $m \times n$ matrices:

add corresponding matrix entries

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2+1 & 3+0 & 5+(-1) \\ 0+0 & 1+0 & 1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 4 \\ 0 & 1 & 4 \end{bmatrix}$$

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Properties of matrix addition:

If $A, B, C \in \mathbb{R}^{m\times n}$, then

$$A+B = B+A \qquad \text{(commutivity)}$$
$$(A+B)+C = A+(B+C) \qquad \text{(associativity)}$$
$$A+0 = A \qquad \text{(identity)}$$

Scalar multiplication:

If $r \in \mathbb{R}$,

$$r\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{mn} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ra_{11} & \cdots & ra_{1j} & \cdots & ra_{1n} \\ \vdots & & \vdots & \ddots & \vdots \\ ra_{mn} & \cdots & ra_{mj} & \cdots & ra_{mn} \end{bmatrix}$$

Def If $A \in \mathbb{R}^{m\times n}$ & $B = [\vec{b}_1 \ \dots \ \vec{b}_p] \in \mathbb{R}^{n\times p}$, then

$$AB = A[\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_p] = [A\vec{b}_1 \ \ A\vec{b}_2 \ \dots \ A\vec{b}_p] \in \mathbb{R}^{m\times p}$$

each is $m\times 1$

Note: AB is defined only when

cols of A = # rows of B

Ex. 1

$$\underset{2\times 3}{\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}} \underset{\substack{3\times 3 \\ (I_3)}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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Look @ how AB acts as a linear transf.

$$(AB)\vec{x} = A\left(B\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}\right) = A[x_1\vec{b}_1 + x_2\vec{b}_2 + \dots x_p\vec{b}_p)$$

Composition of A w/ B

$$= x_1 A\vec{b}_1 + \dots + x_p A\vec{b}_p$$

$$= [Ab_1 \; \dots \; Ab_p]\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}$$

Properties of matrix multiplication

1.) $(AB)C = A(BC)$
2.) $A(B+C) = AB + BC$
3.) $(B+C)A = BA + CA$
4.) $r(AB) = (rA)B = A(rB)$

Provided products are actually defined

5.) If $A \in \mathbb{R}^{m \times n}$, $I_m A = A = A I_n$

Matrix mult. is not generally commutative

- Even if AB defined, BA may not be
- Even if BA is defined, possible to have $AB \neq BA$

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Transpose

<u>Def</u> If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$,

then the <u>Transpose</u> of A is

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

i.e. we interchange rows w/cols

Properties of Transposes

1.) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$ & $(rA)^T = r(A^T)$
3) $(AB)^T = B^T A^T$

Inverse of a Matrix

<u>Def</u> $A \in \mathbb{R}^{n\times n}$ (must be square!) is <u>invertible</u> if $\exists$ $C \in \mathbb{R}^{n\times n}$ s.t.

$$AC = I_n = CA$$

We denote this matrix $C = A^{-1}$.
Invertible matrices are also called nonsingular.

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Ex. 2 $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & -3/2 \\ 0 & 1/2 \end{bmatrix}$

Check if $AA^{-1} = I$:

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} (1)(1) + (0)(-3/2) \\ (0)(2) + (2)(\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

and similarly for $A^{-1}A$

Note that not all square matrices are invertible.

Ex. 3 $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ Try to find $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Need $A^{-1}A = I$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6a + 2b & 3a + b \\ 6c + 2d & 3c + 2d \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Re-writing in system form:

$$\begin{aligned} 6a + 2b &= 1 \\ 3a + b &= 0 \\ \hline 6c + 2d &= 0 \\ 3c + 2d &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 6a + 2b &= 1 \\ 3a + b &= \tfrac{1}{2} \end{aligned}$$

(divide by 2)

but we also have $3a + b = 0$

$\rightarrow$ a contradiction!

Def Matrices that are not invertible are <u>singular</u> matrices.

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Special Case:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then,

$$A^{-1} = \begin{cases} \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \text{DNE}, \quad \text{otherwise} \end{cases}$$

We call $ad-bc = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ the <u>determinant</u>

If $ad-bc=0$ then A^{-1} DNE

<u>Question</u> How does the inverse relate to linear systems?

<u>Thm</u> If A is invertible then $A\vec{x}=b$ has exactly 1 solution.

<u>Pf</u>

To solve the matrix eqn

$$A\vec{x} = b$$

$$\Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}b$$

$$\Rightarrow \vec{x} = A^{-1}b$$

$\square$

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Some properties of Inverses:

1.) If A is invertible then A^{-1} unique

2.) If $A, B \in \mathbb{R}^{n \times n}$ are both invertible then so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= A I_n A^{-1} \\ &= AA^{-1} \\ &= I_n\end{aligned}$$

similarly,

$$(B^{-1}A^{-1})AB = I_n$$

Extension: $(A_1 A_2 A_3 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1} A_1^{-1}$

Question: How do we find A^{-1} if A is not a 2×2 matrix?

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Elementary Matrices

Idea: Write elem. row ops as matrices & use them to compute A^{-1}

Def A matrix $E \in \mathbb{R}^{n \times n}$ is an elementary matrix if it's the result of applying an elem. row operation to I.

Operation	Inverse operation
Mult. row i by $c \neq 0$	Mult. row i by $1/c$
Interchange rows i & j	Interchange rows i & j
Add $c \cdot$ (row i) to row j	Add $-c \cdot$ (row i) to row j

Ex. 4) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ interchange rows 1 & 2

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Mult. row 2 by 18

$18R_2 \rightarrow R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ Add $-3 \cdot$ (row 1) to row 3

$-3R_1 + R_3 \rightarrow R_3$

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Observation: Applying an ERO to A is equivalent to left multiplying A by the corresponding elem matrix E.

Ex. 5 | Interchange rows 1 & 2 of $A = \begin{bmatrix} 1 & 7 & 0 \\ 3 & -1 & 5 \\ 0 & 7 & 2 \end{bmatrix}$

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 0 \\ 3 & -1 & 5 \\ 0 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 5 \\ 1 & 7 & 0 \\ 0 & 7 & 2 \end{bmatrix}$$

Thm Every elem. matrix E is invertible, & it's inverse is an elem. matrix.

Pf. If E applies an elem. op., E^{-1} applies the inverse operation.

Ex |

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \Rightarrow E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

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Use elem. matrices to find A^{-1} as follows:

1. Find a sequence of elem. Row ops that reduce A to I:

$$\underbrace{E_k \cdots E_2 E_1}_{A^{-1} = E_k \cdots E_1 I} A = I$$

2. So apply the same sequence to I to get A^{-1}!

Ex.) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Augment A w/ I

$$\left[\underbrace{\begin{matrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right]$$

$-2R_3 + R_1 \to R_3$ $\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & -2 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$\frac{1}{2} R_2 \to R_2$ $\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$-2R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$-R_1 \to R_1$ $$\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} -1 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{matrix}}_{A^{-1}}\right]$$

$\Rightarrow$ $A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$