1. Find a unit vector in the direction of the given vectors

(a) $\mathbf{x} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

Solution: The unit vector in the direction of $\mathbf{x}$ is given by $\dfrac{\mathbf{x}}{\|\mathbf{x}\|}$. We have

$$\|\mathbf{x}\| = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} = \sqrt{(-6)^2 + (4)^2 + (-3)^2} = \sqrt{61}$$

then the unit vector is

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{1}{\|\mathbf{x}\|}\mathbf{x} = \frac{1}{\sqrt{61}} \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\sqrt{61}} \\ \frac{4}{\sqrt{61}} \\ \frac{-3}{\sqrt{61}} \end{bmatrix}$$

(b) $\mathbf{w} = \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix}$

Solution: The unit vector in the direction of $\mathbf{x}$ is given by $\dfrac{\mathbf{w}}{\|\mathbf{w}\|}$. We have

$$\|\mathbf{w}\| = \sqrt{(w_1)^2 + (w_2)^2} = \sqrt{(\tfrac{8}{3})^2 + (2)^2} = \frac{10}{3}$$

then the unit vector is

$$\frac{\mathbf{w}}{\|\mathbf{w}\|} = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{1}{\frac{10}{3}} \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix} = \frac{3}{10} \begin{bmatrix} \frac{8}{3} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

2. Determine which set of vectors are orthogonal.

(a) $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

Solution: Two vectors are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$. Since

$$\mathbf{u} \cdot \mathbf{v} = (12)(2) + (3)(-3) + (-5)(3) = 0$$

So $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(b) $\mathbf{z} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

Solution: Two vectors are orthogonal if $\mathbf{z} \cdot \mathbf{w} = 0$. Since

$$\mathbf{z} \cdot \mathbf{w} = (-3)(1) + (7)(-8) + (4)(15) + (0)(-7) = 1$$

So $\mathbf{z}$ and $\mathbf{w}$ are NOT orthogonal.

3. Verify that $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$. (Hint: Use the definition of the inner product that involves the transpose)

Solution: Recall that $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^\top\mathbf{v}$. Then

$$\begin{aligned}(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} &= (\mathbf{u}+\mathbf{v})^\top\mathbf{w}\\ &= (\mathbf{u}^\top+\mathbf{v}^\top)\mathbf{w}\\ &= \mathbf{u}^\top\mathbf{w}+\mathbf{v}^\top\mathbf{w}\\ &= \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}\end{aligned}$$

4. Verify that $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$. Use the same hint as in problem 3.

Solution: Recall that $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^\top\mathbf{v}$. Then

$$\begin{aligned}(c\mathbf{u})\cdot\mathbf{v} &= (c\mathbf{u})^\top\mathbf{v}\\ &= c\mathbf{u}^\top\mathbf{v}\\ &= c(\mathbf{u}\cdot\mathbf{v})\\ &= \mathbf{u}^\top(c\mathbf{v})\\ &= \mathbf{u}\cdot(c\mathbf{v})\end{aligned}$$