Lecture #12: Change of Basis

Date: Thu. 10/25/18

Coordinate Systems

Thm (Unique Representation Theorem)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

Def

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Notation: If c_i are $\mathcal{B}$ coordinates of $\vec{x}$ then in $\mathbb{R}^n$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the coordinate vector of $\vec{x}$ (relative to $\mathcal{B}$)

and $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is the coordinate mapping

Ex. 1) Graphically, we could think of a coord. system like using different types of graph paper.

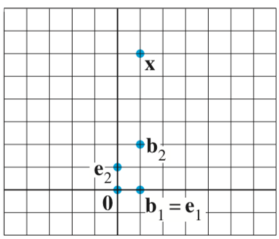

FIGURE 1 Standard graph paper.

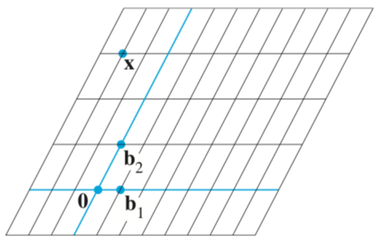

FIGURE 2 $\mathcal{B}$-graph paper.

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Ex. 2) Basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Find Coord vec of $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Solve system $[\vec{b}_1 \ \ \vec{b}_2]\,\vec{c} = \vec{b}$

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \Longrightarrow \vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

then coord. vec of $\vec{x}$ is

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

If $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$ a basis then <u>change of coord's matrix</u> wrt $\mathcal{B}$ is

$$P_{\mathcal{B}} = [\vec{b}_1 \ \dots \ \vec{b}_n]$$

So that $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ which is equivalent to $\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

Since $P_{\mathcal{B}}$ Consists of basis vecs $P_{\mathcal{B}}$ is invertible. This implies that

$$P_{\mathcal{B}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$$

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Coordinate mapping

By choosing a basis for a space V we are introducing a coord. system in V.

In other words, the coord. mapping $\vec{x} = [\vec{x}]_{\mathcal{B}}$ connects V (an unfamiliar space) to $\mathbb{R}^n$ (a familiar space.

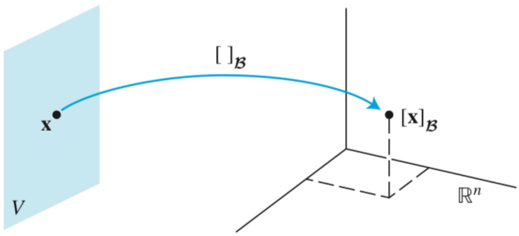

FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Thm

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Def A one-to-one linear transf. from V to W is an isomorphism.

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Change of Basis

Sometimes we need to change basis in order to make a process/calculation "easier"

Thm

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. Multiplication by $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.[2] Figure 2 illustrates the change-of-coordinates equation (4).

The matrix $\underset{C \leftarrow B}{P}$ is the <u>change of coord.s matrix</u> (from B to C)

Multiplying by $\underset{C \leftarrow B}{P}$ converts B coord.s to C coord.s

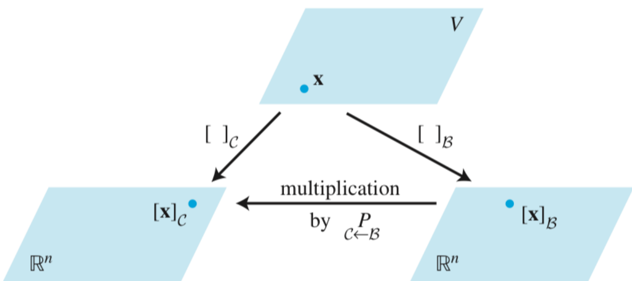

FIGURE 2 Two coordinate systems for V.

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To find the change of coord.s matrix from $\mathcal{C}$ to $\mathcal{B}$:

$$[\vec{b}_1\ \vec{b}_2\ \dots\ \vec{b}_n \mid \vec{c}_1\ \vec{c}_2\ \dots\ \vec{c}_n] \sim [I \mid \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}]$$

i.e. row reduce matrix on LHS until it has form on LHS.

Ex. 3 Find change of coord.s matrix from $\mathcal{C}$ to $\mathcal{B}$ given

Basis for $\mathcal{B}$ $= \left\{ \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$ & Basis for $\mathcal{C}$ $= \left\{ \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$.

$$\Rightarrow [\vec{b}_1\ \vec{b}_2 \mid \vec{c}_1\ \vec{c}_2] = \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & 1 & -5 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & 1 & -5 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{7}{11} & \frac{2}{11} \\ 0 & 1 & -\frac{20}{11} & \frac{12}{11} \end{array}\right]$$

So $$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} -\frac{7}{11} & \frac{2}{11} \\ -\frac{20}{11} & \frac{12}{11} \end{bmatrix}$$