Lecture # 13: Eigenvalues & Eigenvectors

Date: Tue. 11/6/18

Def: An eigenvector of an $n \times n$ Matrix A is a nonzero vector $\vec{x}$ s.t.

$$A\vec{x} = \lambda\vec{x} \qquad (1)$$

for some scalar λ.

The scalar λ is called an eigenvalue of A.

The nontrivial $\vec{x}$ that satisfies (*) is referred to as the eigenvector corresponding to λ.

We can rewrite the eqn (1) as follows

$$A\vec{x} - \lambda\vec{x} = 0$$

Since A is an $n \times n$ matrix

$$\underbrace{(A - \lambda I)}_{\text{a matrix!}}\vec{x} = 0 \qquad (3)$$

Note: $\lambda I = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_3 \end{bmatrix}$

The set of all solns to (3) can be interpreted as the nullspace of the matrix $(A - \lambda I)$.

This is known as the eigenspace of A corr. to λ.

The eigenspace contains all non-zero eigenvectors corr. to λ plus the zero vector.

Lecture # 13: Eigenvalues & Eigenvectors

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Ex 1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \lambda \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}$$

i.e. Find evals (λ) & evecs ($\vec{x}$)

Rewriting:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Recall that to have nonzero vecs in the nullspace our matrix must be <u>non</u> <u>invertible</u>

i.e. we need

$$\boxed{\det(A - \lambda I) = 0}$$

This is the characteristic eqn (or polynomial)

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$$0 = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\right)$$

$\text{Det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$

$$= (1-\lambda)(2-\lambda) - (0)(1)$$

Need to find solns to

$$(1-\lambda)(2-\lambda) = 0$$

$\Rightarrow \lambda = 1$ & $\lambda = 2$ (evals of A)

These are the eigenvalues of A.

To find corr. E-vecs we consider each λ separately to find vecs that satisfy

$$A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = 0$$

For $\lambda = 1$:

$$\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1-(1) & 1 \\ 0 & 2-(1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 \cdot x_1 + x_2 \\ 0\, x_1 + x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x_2 = 0$

$x_1 = t$ (free)

$\Rightarrow \vec{x} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ evec corr. to $\lambda = 1$

For $\lambda = 2$:

$$0 = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow -x_1 + x_2 = 0$$

Let $x_1 = x_2 = t$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: There are ∞ evecs assoc. w/ $\lambda = 1$ & w/ $\lambda = 2$.

Thm The evals of a triangular matrix are the entries on the main diagonal.

Ex $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ Evals are $\lambda = 1, 3, 2$

Thm Zero is an Eval of A iff A is not invertible.

Why? Ex If $\det(A) = 0$ in triangular matrix then an elt of diag. $= 0 \Rightarrow$ eval of A is zero

Lecture # 13: Eigenvalues & Eigenvectors

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The characteristic Eqn.

Given $A \in \mathbb{R}^{n \times n}$ find evals by solving

$$Ax = \lambda x \quad \text{for } \lambda \quad (\vec{x} \neq 0)$$

we want nonzero solns, so we require that

$$\det(A - \lambda I) = 0$$

this is the characteristic eqn.