Worksheet 11 Solutions

Dimension & Rank MATH 2250, Fall 2018

1. Find the dimension of the subspace spanned by the vectors

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}$$

Solution: Let

$$A = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \underset{\text{Row Reduce}}{\rightarrow} \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that A has 3 pivot columns. In other words, the basis of $\text{Col}(A)$ has 3 vectors in it. So

$$\dim(\text{Col}(A)) = 3$$

2. Determine the dimensions of $\text{Null}(A)$ and $\text{Col}(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Be sure to clearly explain how you acheived your result.

Solution:

Find $\dim(\text{Col}(A))$:

Since A is already in reduced echelon form, we see that A has 3 pivot columns. This means that the basis of $\text{Col}(A)$ has 3 vectors in it, and so

$$\dim(\text{Col}(A)) = 3$$

Find $\dim(\text{Null}(A))$:

You can either determine the basis for $\text{Null}(A)$ by solving $A\mathbf{x} = \mathbf{0}$ or you can apply the Rank-Nullity theorem where $\text{Rank}(A) = 3$ and $n = 6$

$$\begin{aligned} \text{Rank}(A) + \text{Nullity}(A) &= n \\ 3 + \text{Nullity}(A) &= 6 \\ \text{Nullity}(A) &= 3 \end{aligned}$$

3. Suppose $A \in \mathbb{R}^{6\times 3}$ has rank 3 find $\dim(\text{Null}(A))$, $\dim(\text{Row}(A))$ and rank $A^\top$.

Solution:

Find $\text{Nullity}(A)$:

Since $\text{Rank}(A) = 3$ and A has 3 columns, then by the Rank-Nullity theorem

$$\begin{aligned}
\text{Rank}(A) + \text{Nullity}(A) &= n \\
3 + \text{Nullity}(A) &= 3 \\
\text{Nullity}(A) &= 0
\end{aligned}$$

Find $\dim(\text{Row}(A))$:

We know that

$$\dim(\text{Row}(A)) = \dim(\text{Col}(A))$$

in other words

$$\dim(\text{Row}(A)) = \text{Rank}(A) = 3$$

Find $\text{Rank}(A^\top)$:

Since $\text{Col}(A^\top) = \text{Row}(A)$ then

$$\begin{aligned}
\text{Rank}(A^\top) &= \dim(\text{Col}(A^\top)) \\
&= \dim(\text{Row}(A)) \\
&= \dim(\text{Col}(A)) \\
&= \text{Rank}(A) \\
&= 3
\end{aligned}$$

4. Given

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}; \qquad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

assume A is row equivalent to B. Find $\text{Rank}(A)$ and $\dim(\text{Null}(A))$ and find bases for $\text{Col}(A)$, $\text{Row}(A)$, and $\text{Null}(A)$. Be sure to clearly explain how you acheived your results.

Solution:

Find $\text{Rank}(A)$:

$\text{Rank}(A)$ is equivalent to the number of pivot columns of A. Examining the matrix B we see that columns 1, 3, and 5 are each pivot columns. Since there are 3 pivot columns so

$$\text{Rank}(A) = 3$$

Find $\dim(\text{Null}(A))$:

Recall that $\dim(\text{Null}(A)) = \text{Nullity}(A)$. By the Rank-Nullity Theorem we know that for an $m \times n$ matrix A

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

Since $\text{Rank}(A) = 3$ and $n = 5$then

$$3 + \text{Nullity}(A) = 5 \implies \text{Nullity}(A) = 2$$

Basis for $\text{Col}(A)$:

The column space of A is found by locating the pivot columns of A. Examining the matrix B we see that columns 1, 3, and 5 are each pivot columns. Therefore, the column space of A is

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix} \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$$

and so a basis for $\text{Col}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix} \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\}$$

Basis for $\text{Null}(A)$:

To find the $\text{Null}(A)$ we must find all solutions to $A\mathbf{x} = \mathbf{0}$. Row reducing A we have

$$\left[\begin{array}{ccccc:c} 1 & -3 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -\dfrac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{rcrcrcrcl} x_1 & - & 3x_2 & & & + & 5x_4 & & = 0 \\ & & & & x_3 & - & \dfrac{3}{2}x_4 & & = 0 \\ & & & & & & & x_5 & = 0 \end{array}$$

We see that we will have two free variables where x_2 and x_4 are free. So the solution is

$$\mathbf{x} = \begin{bmatrix} 3x_2 - 5x_2 \\ x_2 \\ \frac{3}{2}x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

and so a basis for $\text{Null}(A)$ is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Basis for $\text{Row}(A)$:

A basis for $\text{Row}(A)$ is given by the pivot rows of B. Thus, a basis for $\text{Row}(A)$ is

$$\{(1, -3, 0, 5, -7), (0, 0, 0, 2, -3, 8), (0, 0, 0, 0, 5)\}$$