Lecture # 14: Diagonalization

Date: Thu 11/8/18

Can we convert a square matrix to a convenient form $\begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \end{bmatrix}$ without changing important properties?

- Value of Det
- Invertibility
- Rank & Nullity
- Char. polynomial
- Eigenvalues
- Eigenspace dimension

$\Big\}$ (*)

<u>Thm</u> If $A \in \mathbb{R}^{n \times n}$ & $P \in \mathbb{R}^{n \times n}$ is invertible, then, the transformation

$$A \longrightarrow P^{-1}AP \qquad (**)$$

preserves all properties (*)

Call (**) a <u>similarity transformation</u>

A is <u>similar</u> to $P^{-1}AP$

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<u>Pf</u> (for the Char. poly)

$$\det(\lambda I - P^{-1}AP) = \det(\lambda P^{-1} I P - P^{-1}AP)$$

$$= \det[P^{-1}(\lambda I - A)P]$$

$$= \det(P^{-1})[\det(\lambda I - A)]\det(P)$$

$$= \frac{1}{\det(P)}[\det(\lambda I - A)]\det(P)$$

$$= \det(\lambda I - A) \qquad \blacksquare$$

<u>Ex 1</u> $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$, $P = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$, $P^{-1} = \frac{1}{5}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$\Rightarrow$ e-vals $\lambda = 2, -3$

<u>Def</u> If there's a similarity transf. $A \mapsto P^{-1}AP$ s.t. $P^{-1}AP$ is diagonal, then A is <u>diagonalizable</u> (diag'ble)

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Thm Let $A \in \mathbb{R}^{n \times n}$, TFAE:

(a) A is diagonalizable

(b) A has n lin. indep. e-vecs

PF (a) $\Rightarrow$ (b) Assume A is diag'ble with

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad (\star)$$

Call the col.s of P: $\vec{p}_1, \vec{p}_2, \dots \vec{p}_n$.
They are L.I. since P is invertible
We are done if we show $\vec{p}_1, \dots, \vec{p}_n$ are e-vecs.

Left-multiply $(\star)$ by P:

$$P(P^{-1}AP) = P(D) \quad (= AP)$$

$$= [\vec{p}_1 \ \dots \ \vec{p}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 \vec{p}_1 \ \dots \ \lambda_n \vec{p}_n]$$

But also

$$AP = A[\vec{p}_1 \ \dots \ \vec{p}_n] = [A\vec{p}_1 \dots Ap_n]$$

So we have

$$A\vec{p}_1 = \lambda_1 \vec{p}_1, \ \dots, \ A\vec{p}_n = \lambda_n \vec{p}_n$$

i.e. $\vec{p}_1, \dots \vec{p}_n$ are lin. indep. e-vecs $\blacksquare$

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Pf. (cont'd)

(b) $\Rightarrow$ (a). Suppose A has lin. indep. e-vecs $P = [\vec{p}_1 \ldots \vec{p}_n]$ has lin. indep. cols, so P is invertible.

Claim: $AP = PD$ where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Reason:

$$AP = [A\vec{p}_1 \ldots A\vec{p}_n] = [\lambda_1 \vec{p}_1 \ldots \lambda_n \vec{p}_n]$$

$$= [\vec{p}_1 \ldots \vec{p}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = PD$$

By right multiplying by P^{-1} we get $P^{-1}AP = D$ ∎

There is no easy way to determine if A has lin. indep. e-vecs. The following thm gives us @ least one shortcut.

Thm If $\lambda_1, \ldots, \lambda_k$ is a set of distinct e-vals of A (so $\lambda_i \neq \lambda_j$ whenever $i \neq j$) then any set $\{\vec{p}_1, \ldots \vec{p}_k\}$ of assoc. e-vecs is lin. indep.

Consequence: If the n e-vals of $A \in \mathbb{R}^{n \times n}$ are distinct then the assoc. e-vecs $\vec{p}_1, \ldots, \vec{p}_n$ are lin. indep. & A is diag'ble by $P = [p_1, \ldots, p_n]$

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Summary of Diagonalization Procedure

1. Find all e-vals $\lambda_1, \ldots, \lambda_n$ & assoc. e-vecs.
2. Decide whether A is diag'ble.

   Determine whether there are n lin. indep. e-vecs (there are if there are n distinct e-vals)
3. If A is diag'ble, form $P = [\vec{p}_1, \ldots, \vec{p}_n]$

   Then $P^{-1}AP$ is diagonal.

Ex. 1 Diagonalize $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Find e-vals:

• Need char. poly:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda) - 12$$

$$= 2 - 2\lambda - \lambda + \lambda^2 - 12$$

$$= -10 - 3\lambda + \lambda^2$$

$$= (\lambda - 5)(\lambda + 2)$$

evals from $\det(A - \lambda I) = 0$

$\Rightarrow (\lambda - 5)(\lambda + 2) = 0$

evals: $\lambda = 5$, $\lambda = -2$

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Ex. 1 (cont'd)

Find e-vecs:

For $\lambda = 5$

$$0 = A - 5I = \begin{bmatrix} 2-5 & 3 \\ 4 & 1-5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$$

$\Rightarrow -3x_1 + 3x_2 = 0$ $\quad\Rightarrow x_1 = x_2$

$4x_1 - 4x_2 = 0$ $\quad$ evec: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{\xi}_1$

For $\lambda = -2$

$$\Rightarrow 0 = A + 2I = \begin{bmatrix} 2+2 & 3 \\ 4 & 1+2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 4x_1 + 3x_2 = 0$ $\quad\Rightarrow 4x_1 = -3x_2$

$\Rightarrow x_1 = -\frac{3}{4}x_2$

$\Rightarrow \vec{\xi}_2 = \begin{bmatrix} -\frac{3}{4} \\ 1 \end{bmatrix}$ $\quad$ Let $x_2 = t$

$x_1 = -\frac{3}{4}t$

or $\vec{\xi}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

So evecs are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

So $P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

Then

$$P^{-1}AP = \frac{1}{7}\begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

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We can also rewrite $P^{-1}AP = D$ to find a representation for A:

$$P^{-1}AP = D$$

Multiplying (on the left) by P

$$P(P^{-1}AP) = PD$$
$$AP = PD$$

Multiplying on the right by P^{-1}

$$(AP)P^{-1} = PDP^{-1}$$

$$\Rightarrow \quad A = PDP^{-1}$$

Ex. 2 From the previous example confirm that $A = PDP^{-1}$

$$\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 5 & -8 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 5 & 6 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 14 & 21 \\ 28 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

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Ex. 3) What is A^4 ?

We know $A = PDP^{-1}$

$$\Rightarrow \quad A^4 = (PDP^{-1})^4$$

$$= (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1}$$

$$= PD^4P^{-1}$$

To find D^4 : (from prev. example)

$$D^4 = \begin{bmatrix} (5)^2 & 0 \\ 0 & (-2)^4 \end{bmatrix} = \begin{bmatrix} 125 & 0 \\ 0 & 16 \end{bmatrix}$$