Lecture # 15: Systems of ODEs

Date: Tue. 11/13/18

A system of n order linear ordinary differential Eqns (ODEs) has the Form

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned} \qquad (1)$$

This can be written as the matrix eqn:

$$\vec{x}' = \vec{P}(t)\vec{x} + \vec{g}(t) \qquad (2)$$

A vector $\vec{x} = \vec{\Phi}(t)$ is a soln of this system if its components satisfy the system.

We assume that the fcns $\vec{P}$ & $\vec{g}$ are cont. on some interval $\alpha < t < \beta$.

This condition is enough to guarantee existence of a solns on $\alpha < t < \beta$

The eqn in (2) is a nonhomogeneous eqn. Just as in dealing w/ a single ODE we first examine the homogeneous eqn & deal w/ the nonhomogeneous part separately.

The homog. eqn is far easier to deal with.

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A homogeneous system of ODEs has the form

$$\vec{x}' = \frac{d\vec{x}}{dt} = \vec{p}(t)\,\vec{x}$$

Found by setting $\vec{g}(t) = 0$.

Let $\vec{P}(t) = A$, a matrix w/ cst entries

$$\frac{d\vec{x}}{dt} = \underset{\substack{\uparrow \\ \text{coeff} \\ \text{matrix}}}{A}\,\vec{x}$$

This is a homog. system of ODEs w/ cst. coeff.s

For $n=1$:

$$\frac{dx}{dt} = ax \quad , \quad a \neq 0$$

This soln has gen. form

$$x(t) = x_0 e^{rt} \quad , \quad x(t_0) = x_0$$

The only cst soln will be given by $x = 0$

This is known as the equilibrium soln.

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So for system we seek solns of form

$$\vec{x} = \vec{\xi} e^{rt} = \begin{bmatrix} \xi_1 e^{rt} \\ \xi_2 e^{rt} \end{bmatrix}$$

want this to be a soln to $\frac{d\vec{x}}{dx} = A\vec{x}$

$$\Rightarrow \underbrace{r \vec{\xi} e^{rt}}_{\frac{d\vec{x}}{dt}} = A \underbrace{\vec{\xi} e^{rt}}_{\vec{x}}$$

Canceling e^{rt} terms ($e^{rt} \neq 0$)

$$\Rightarrow A\vec{\xi} = r\vec{\xi} \quad \Rightarrow \quad \underbrace{(A - rI)\vec{\xi} = \vec{0}}_{\substack{\text{only true for} \\ \det(A - rI) = 0}}$$

$$(A\vec{x} = \lambda \vec{x})$$

Characteristic Eqn

i.e.

r must be an eigenvalue of A which means $\vec{\xi}$ is an eigenvector (of A corresponding to r)

<u>Thm</u> (Principle of Superposition)

If the vector Fcns $\vec{x}^{(1)} \dots \vec{x}^{(k)}$ are solns of the system $\vec{x}' = \vec{P}(t)\vec{x}$ then the linear combination

$$\phi(t) = c_1 \vec{x}^{(1)}(t) + \dots + c_k \vec{x}^{(k)}(t) \quad (3)$$

is also a soln for any csts $c_1, \dots, c_k$

If $\vec{x}^{(1)}, \dots, \vec{x}^{(k)}$ are linearly independent then this linear combo is unique & is called the <u>fundamental set of solutions</u>

The soln (3) is the general soln to the system & contains all solns of the system

In other words, the fund. set of solns forms a basis for the set of solns.

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Ex. 1 Solve $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

i.e. we want soln to system

$$\frac{dx_1}{dt} = x_1 + x_2$$

$$\frac{dx_2}{dt} = 4x_1 + x_2$$

Soln. Find e-vals of A:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)^2 - 4 = 0$$

$$(1-\lambda)^2 = 4$$

$$1-\lambda = \pm 2$$

$$\Rightarrow \quad \lambda = 1 \pm 2$$

$$\lambda = -1, 3$$

Find e-vecs:

For $\lambda = 3$

$$A - 3I = \begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2\xi_1 + \xi_2 = 0$

$4\xi_1 - 2\xi_2 = 0$

$\Rightarrow \quad \xi_2 = 2\xi_1$

So e-vec is $\vec{\xi}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda = -1$

$$A + 1I = \begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2\xi_1 + \xi_2 = 0$

$4\xi_1 + 2\xi_2 = 0$

$\Rightarrow \quad \xi_2 = -2\xi_1$

So e-vec is $\vec{\xi}^{(2)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

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Ex. 1) (cont'd)

So we have 2 solns to the system

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} \quad \& \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

The general soln to system is a lin. combo of $\vec{x}^{(1)}$ & $\vec{x}^{(2)}$

$$\Rightarrow \Phi(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}$$