1. Solve the system

$$\begin{array}{rcrcrcl} x_1 & & & - & 3x_3 & = & 8 \\ 2x_1 & + & 2x_2 & + & 9x_3 & = & 7 \\ & & x_2 & + & 5x_3 & = & -2 \end{array} \tag{1}$$

Solution:

Write in matrix form $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$

Perform Row operations:

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 2 & 2 & 9 & 7 \end{array}\right]$

$-2R_1 + R_3 \to R_3$

$\begin{array}{cccc} -2 & 0 & 6 & -16 \\ 2 & 2 & 9 & 7 \\ \hline 0 & 2 & 15 & -9 \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{array}\right]$

$-2R_2 + R_3 \to R_3$

$\begin{array}{cccc} 0 & -2 & -10 & 4 \\ 0 & 2 & 15 & -9 \\ \hline 0 & 0 & 5 & -5 \end{array}$ $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{array}\right]$

Write as a system again

$$\begin{array}{lll} x_1 - 3x_3 = 8 & \Rightarrow & x_1 - (-3) = 8 \Rightarrow x_1 - 5 \\ x_2 + 5x_3 = -2 & \Rightarrow & x_2 + 5(-1) = -2 \Rightarrow x_2 = -2 \\ 5x_3 = -5 & \Rightarrow & x_3 = -1 \qquad\qquad x_3 = 3 \end{array}$$

So solution to system is $x_1 = 5$, $x_2 = -2$, $x_3 = 3$

or $\vec{x} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$

2. Is the given system consistent or inconsistent?

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & -5 \\ x_1 & + & 3x_2 & + & 5x_3 & = & -2 \\ 3x_1 & + & 7x_2 & + & 7x_3 & = & 6 \end{array} \tag{2}$$

Solution:

Write system in matrix Form:

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{array}\right]$$

Perform Row operations:

$R_1 \leftrightarrow R_3$
$$\left[\begin{array}{cccc} 3 & 7 & 7 & 6 \\ 1 & 3 & 5 & -2 \\ 0 & 1 & -4 & -5 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{cccc} 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \\ 0 & 1 & -4 & -5 \end{array}\right]$$

$-3R_1 + R_2 \to R_2$
$$\begin{array}{cccc} -3 & -9 & -15 & 30 \\ 3 & 7 & 7 & 6 \\ \hline 0 & -2 & -8 & 36 \end{array}$$
$$\left[\begin{array}{cccc} 1 & 3 & 5 & -2 \\ 0 & -2 & -8 & 36 \\ 0 & 1 & -4 & -5 \end{array}\right]$$

$\frac{1}{2}R_2 + R_3 \to R_3$
$$\begin{array}{cccc} 0 & -1 & -4 & 16 \\ 0 & 1 & -4 & -5 \\ \hline 0 & 0 & -8 & 11 \end{array}$$
$$\left[\begin{array}{cccc} 1 & 3 & 5 & -2 \\ 0 & -2 & -8 & 36 \\ 0 & 0 & -8 & 11 \end{array}\right]$$

Writing back in system form

$x_1 + 3x_2 + 5x_3 = -2 \implies x_1 + 3\left(-\frac{11}{8}\right) + 5\left(-\frac{11}{8}\right) = -2 \implies x_1 = \frac{121}{8}$

$-2x_2 - 8x_3 = 36 \implies -2x_2 - 8\left(-\frac{11}{8}\right) = 36 \implies -2x_2 = 11$

$-8x_3 = 11 \implies x_3 = -\frac{11}{8} \implies x_2 = -\frac{11}{2}$

Since this system has a solution it is consistent.