1. Find the characteristic polynomial and then the eigenvalues and corresponding eigenvectors of the matrix
$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Solution:

Characteristic Polynomial:

The characteristic polynomial is given by $\det(A - \lambda I)$.

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} \\ &= (5-\lambda)^2 - 9 \\ &= (8-\lambda)(2-\lambda)\end{aligned}$$

So the characteristic polynomial is

$$P(\lambda) = (8-\lambda)(2-\lambda)$$

Eigenvalues:

Eigenvalues are the solutions to the characteristic equation $\det(A - \lambda I) = 0$ so we solve
$$(8-\lambda)(2-\lambda) = 0 \implies \lambda_1 = 2,\ \lambda_2 = 8$$
So eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 8$.

Eigenvector(s) for $\lambda_1 = 2$:

Eigenvectors are solutions to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda_1 I = A - 2I = \begin{bmatrix} 5-2 & 3 \\ 3 & 5-2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

Solving the $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we have

$$\left[\begin{array}{cc:c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\longrightarrow} \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies x_1 = -x_2$$

So the eigenvector corresponding to $\lambda_1 = 2$ is $\xi^{(1)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenvector(s) for $\lambda_2 = 8$:

Eigenvectors are solutions to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

$$A - \lambda_2 I = A - 8I = \begin{bmatrix} 5-8 & 3 \\ 3 & 5-8 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

Solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ we have

$$\left[\begin{array}{rr|r} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \underset{\substack{\text{Row} \\ \text{Reduce}}}{\rightarrow} \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies x_1 = x_2$$

So the eigenvector corresponding to $\lambda_2 = 8$ is $\xi^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. Find the characteristic polynomial and then the eigenvalues and corresponding eigenvectors of the matrix

$$B = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & 2 \end{bmatrix}$$

Solution:

Characteristic Polynomial:

The characteristic polynomial is given by $\det(B - \lambda I)$.

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)\left(\lambda^2 - 7\lambda - 8\right) \\ &= -(\lambda-8)(\lambda-1)(\lambda+1) \end{aligned}$$

So the characteristic polynomial is

$$P(\lambda) = -(\lambda-8)(\lambda-1)(\lambda+1)$$

Eigenvalues:

Eigenvalues are the solutions to the characteristic equation $\det(A - \lambda I) = 0$ so we solve

$$-(\lambda-8)(\lambda-1)(\lambda+1) = 0 \implies \lambda_1 = 8,\ \lambda_2 = 1,\ \lambda_3 = -1$$

So eigenvalues of A are $\lambda_1 = 8$, $\lambda_2 = 1$ and $\lambda_3 = -1$.

Eigenvector(s) for $\lambda_1 = 8$:

Eigenvectors are solutions to the equation $(B - \lambda_1 I)\mathbf{x} = \mathbf{0}$.

$$B - \lambda_1 I = A - 8I = \begin{bmatrix} 5-8 & -2 & 3 \\ 0 & 1-8 & 0 \\ 6 & 7 & 2-8 \end{bmatrix} = \begin{bmatrix} -3 & -2 & 3 \\ 0 & -7 & 0 \\ 6 & 7 & -6 \end{bmatrix}$$

Solving the $(B - \lambda_1 I)\mathbf{x} = \mathbf{0}$ we have

$$\left[\begin{array}{ccc|c} -3 & -2 & 3 & 0 \\ 0 & -7 & 0 & 0 \\ 6 & 7 & -6 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \implies x_1 = -x_3,\ x_2 = 0$$

So the eigenvector corresponding to $\lambda_1 = 8$ is $\xi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigenvector(s) for $\lambda_2 = 1$:

Eigenvectors are solutions to the equation $(B - \lambda_2 I)\mathbf{x} = \mathbf{0}$.

$$B - \lambda_2 I = A - 1I = \begin{bmatrix} 5-1 & -2 & 3 \\ 0 & 1-1 & 0 \\ 6 & 7 & 2-1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 0 & 0 \\ 6 & 7 & 1 \end{bmatrix}$$

Solving the $(B - \lambda_2 I)\mathbf{x} = \mathbf{0}$ we have

$$\left[\begin{array}{ccc:c} 4 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 7 & 1 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{ccc:c} 1 & 0 & \frac{23}{40} & 0 \\ 0 & 1 & -\frac{7}{20} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies x_1 = -\tfrac{23}{40}x_3,\ x_2 = \tfrac{7}{20}x_3$$

So the eigenvector corresponding to $\lambda_2 = 1$ is $\xi^{(2)} = \begin{bmatrix} -\frac{23}{40} \\ \frac{7}{20} \\ 1 \end{bmatrix}$ or $\xi^{(2)} = \begin{bmatrix} -23 \\ 14 \\ 40 \end{bmatrix}$

Eigenvector(s) for $\lambda_3 = -1$:

Eigenvectors are solutions to the equation $(B - \lambda_3 I)\mathbf{x} = \mathbf{0}$.

$$B - \lambda_3 I = A + 1I = \begin{bmatrix} 5+1 & -2 & 3 \\ 0 & 1+1 & 0 \\ 6 & 7 & 2+1 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 3 \\ 0 & 2 & 0 \\ 6 & 7 & 3 \end{bmatrix}$$

Solving the $(B - \lambda_3 I)\mathbf{x} = \mathbf{0}$ we have

$$\left[\begin{array}{ccc:c} 6 & -2 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 7 & 3 & 0 \end{array}\right] \underset{\substack{\text{Row}\\\text{Reduce}}}{\rightarrow} \left[\begin{array}{ccc:c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies x_1 = -\frac{1}{2}x_3,\ x_2 = 0$$

So the eigenvector corresponding to $\lambda_3 = -1$ is $\xi^{(3)} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ or $\xi^{(3)} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$