Lecture # 04: Linear Independence

Date: Thu. 9/20/18

Def

An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (2)$$

Ex

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$ is linearly dependent

$$-2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

not all coeffs are zero.

Observation:

If $\{v_1, \dots v_p\}$ is lin dep, then some $\vec{v}_i$ is a linear comb. of the others.

Suppose

$$c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$$

with $c_i \neq 0$ then

$$\vec{v}_i = \underbrace{-\frac{c_1}{c_i}\vec{v}_1 - \dots - \frac{c_p}{c_i}\vec{v}_p}_{\text{no term involving } v_i}$$

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Ex) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ lin. indep.?

Goal: Try to find $c_1, c_2\ c_3$ s.t.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

REF $\Rightarrow$ no free variable

$\Rightarrow$ exactly one soln $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then set is linealy indep.

Ex) If $\{ v_1, v_2, \ldots, v_p \}$ contains $\vec{0}$ then its linearly dependent.

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<u>Thm</u> If $v_1, \ldots, v_p \in \mathbb{R}^n$ & $p > n$ then $\{v_1, \ldots v_p\}$ is linearly dependent

<u>Pf</u> Let $A = [v_1, \ldots v_p]$ which is $n \times p$. Since $p > n$, there must be @ least 1 free variable: Hence, there are ∞ many solns to

$$A\vec{c} = c_1 v_1 + \ldots + c_p v_p = \vec{0}$$

so $c_1 = c_2 = \ldots = c_p = 0$ isn't the only soln. $\square$

<u>Summary</u>: $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is

- linearly dependent if $\exists\ c_1, \ldots c_p \in \mathbb{R}$ not all 0 s.t.

$$c_1 \vec{v}_1 + \ldots + c_p \vec{v}_p = 0$$

Equivalently, the matrix eqn $[\vec{v}_1, \ldots, \vec{v}_p]\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \vec{0}$ has nontrivial

Solutions $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

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- Linearly indep. if the only constants $c_1, \dots c_p$ for which $c_1\vec{v}_1 + \dots + c_p\vec{v}_p = 0$ are $c_1 = c_2 = \dots = c_p = 0$

  Equivalently, the matrix eqn

  $$[\vec{v}_1, \dots \vec{v}_p]\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \vec{0}$$

  has only the trivial soln. $\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \vec{0}$