Math 2250

Lecture # 05: Linear Transformations

Date: Tue. 9/25/18

We don't always need to consider linear Algebra in the context of solving a system of equations

In a more abstract sense, linear algebra is the study of <u>linear</u> <u>maps</u> on finite dimensional vector spaces (we'll learn exactly what this means later).

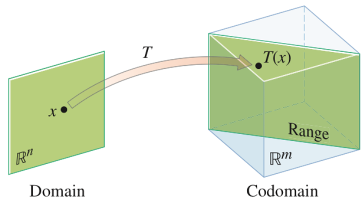
Now we will examine matrices & how they <u>act</u> on (or <u>transform</u>) a vector

Definitions

<u>Def</u> Every $m \times n$ matrix A defines a <u>transformation</u> $T: \underset{\text{Domain}}{\mathbb{R}^n} \to \underset{\text{Codomain}}{\mathbb{R}^m}$

as the <u>mapping</u> (or function)

$$T(\underset{\in \mathbb{R}^n}{\vec{x}}) = \underbrace{A\vec{x}}_{\in \mathbb{R}^m}$$



$T: \mathbb{R}^n \to \mathbb{R}^m$

<u>Def</u> The <u>range</u> of T is

$$\{ \underbrace{T(\vec{u})}_{\text{image (of } u \text{ under action of } T)} \in \mathbb{R}^m : \vec{u} \in \mathbb{R}^n \}$$

<u>Note</u>: the range is not the same as the codomain!

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Recall: a transformation is <u>linear</u> if

$$\begin{aligned} T(\vec{u}+\vec{v}) &= A(\vec{u}+\vec{v}) \\ &= A\vec{u} + A\vec{v} \\ &= T(\vec{u}) + T(\vec{v}) \qquad \text{(additivity)} \end{aligned}$$

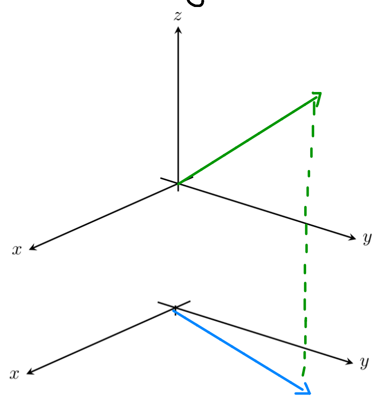
$$\begin{aligned} T(c\vec{u}) &= A(c\vec{v}) \\ &= c\,A\vec{u} \\ &= c\,T(\vec{u}) \qquad \text{(homogeneity)} \end{aligned}$$

Ex. 1 Define $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

T_1 projects $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$ onto $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$

Geometrically:



The range of T_1 is

$$\{ T\vec{u} \in \mathbb{R}^2 : \vec{u} \in \mathbb{R}^3 \} = \mathbb{R}^2$$

This is a <u>projection</u>

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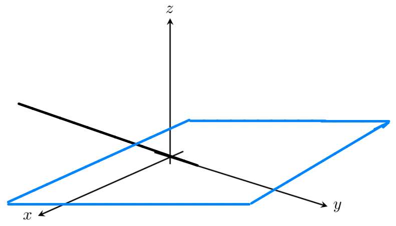
Ex. 2 $T_2 : \mathbb{R}^2 \to \mathbb{R}^3$,

$$T_2\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$



The plane thru x-axis

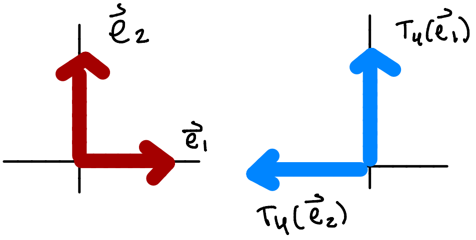
Ex. 3 $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \frac{1}{2}y \\ y \end{bmatrix}$$

Shearing

Ex. 4 $T_4 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_4\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$



Rotation by $\frac{\pi}{2}$

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Def $T: \mathbb{R}^m \to \mathbb{R}^n$ is

(a) <u>one to one</u> (1-1) if

$$T(\vec{u}) = T(\vec{v}) \implies \vec{u} = \vec{v}$$

i.e. different vectors map to different images.

(b) <u>onto</u> if every $\vec{x} \in \mathbb{R}^n$ is $T(\vec{u})$ for some $\vec{u} \in \mathbb{R}^m$

Ex. 5 • T_1 (projection) is onto, not 1-1

$$T_1\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = T_1\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

• T_2 (plane) is 1-1, not onto

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \notin \text{range}(T_2)$$

• T_3 (shearing) & T_4 (rotation) are 1-1 & onto

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Matrix of a Linear Transformation

<u>Shown:</u> Every $m \times n$ matrix defines a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

<u>Next:</u> Every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ has a matrix representation.

Ex. 6 In $\mathbb{R}^3$ let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We can write any $x \in \mathbb{R}^3$ as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3$$

If $T: \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation

$$T\vec{x} = T(x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + x_3 T(\vec{e}_3)$$

$$= [\underbrace{T(\vec{e}_1) \;\; T(\vec{e}_2) \;\; T(\vec{e}_3)}_{\text{a } 4 \times 3 \text{ matrix}}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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In general, if $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transf. the <u>standard</u> <u>matrix</u> for T is

$$A = [T(\vec{e}_1) \;\ldots\; T(\vec{e}_m)] \qquad (n \times m)$$

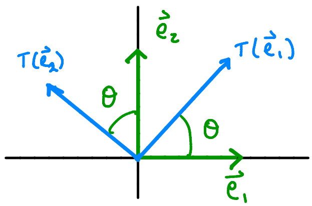
where cols of length n

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots \quad \vec{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

These are the <u>standard</u> <u>basis</u> <u>vectors</u> in $\mathbb{R}^m$

<u>Ex. 7</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$, counterclockwise by θ radians

In $\mathbb{R}^2$: $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

So the standard matrix for T is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transf. then T is 1-1 iff the eqn $T(\vec{x}) = \vec{0}$ has only the trivial soln.

Pf ($\Rightarrow$) Assume T is 1-1. Then $T(\vec{x}) = \vec{0}$ for only one $\vec{x}$. That vector must be $\vec{0}$ (the trivial soln)

Since we must have $T(\vec{0}) = \vec{0}$ for any linear transformation.

($\Leftarrow$) (by contrapositive: i.e. to show $P \Rightarrow Q$ it is equivalent to show that not $P \Rightarrow$ not Q)

Assume T is not 1-1. Then for some $\vec{b} \in \mathbb{R}^m$ we have

$$T(\vec{u}) = T(\vec{v}) = \vec{b} \quad \text{for } \vec{u} \neq \vec{v}$$

We want to show that $T(\vec{x}) = \vec{0}$ for some $\vec{x} \neq 0$

Let

$$\vec{x} = \vec{u} - \vec{v} \neq 0$$

Then,

$$\begin{aligned} T(\vec{x}) &= T(\vec{u} - \vec{v}) \\ &= T(\vec{u}) - T(\vec{v}) \\ &= \vec{b} - \vec{b} \\ &= \vec{0} \end{aligned}$$

This means that T does not have only the trivial soln (since $\vec{x} \neq 0$). ∎