Math 2250

Lecture # 09: The Fundamental Subspaces

Date: Tue. 10/16/18

Null space

<u>Def</u> The <u>null space</u> of $A \in \mathbb{R}^{m\times n}$ is

$$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

<u>Ex. 1</u> Determine the null space of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 6 & 1 & 1 \end{bmatrix}$$

Find Null(A) by Solving $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 6 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Back sub. then gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{t}{4} \\ \frac{t}{2} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \implies \text{Null}(A) = \text{span}\left\{\begin{bmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right\}.$$

This is a line in $\mathbb{R}^3$

Thm For any matrix $A \in \mathbb{R}^{m \times n}$, $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^m$.

Pf. Check closure under add'n & scalar mult.

(i.e. $Au = 0$ & $Av = 0$) then by linearity

$$A(u+v) = Au + Av = \vec{0} + \vec{0} = \vec{0}$$

$$A(cu) = cAu = c \cdot \vec{0} = \vec{0}$$

So $u+v,\ cu \in \mathrm{Null}(A)$ $\square$

Column Space

Def The <u>column space</u> of $A \in \mathbb{R}^{m \times n}$ is the span of the columns of A.

Notation:

if $A = [\vec{a}_1, \dots \vec{a}_n]$, $\mathrm{col}(A) = \mathrm{Span}\{\vec{a}_1, \dots \vec{a}_n\}$

Since the span of any set of vectors is a subspace, $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$

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Observation:

Since
$$\begin{bmatrix} a_{n1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathrm{Col}(A)$$

In terms of linear transformations:

For $T: \mathbb{R}^n \to \mathbb{R}^m$:

- Kernel of T = $\mathrm{Null}(A)$, subspace of $\mathbb{R}^n$
- range of T = $\mathrm{Col}(A)$, subspace of $\mathbb{R}^n$

Row Space

Def The row space of A is the span of the rows of A. i.e. the row space of A is $\mathrm{Col}(A^T)$.