Lecture # 1: Intro to Linear Systems

Date: Tue. 9/11/18

Linear Systems of Equations

Def a linear equation with n unknowns has the general form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$$

where a_i are the coefficients
& x_i are the unknowns.

Notation:
The subscript i is generally a positive integer. and is referred to as the index or index variable

Def If an eqn is not linear it is nonlinear.

Ex.1 Linear & Nonlinear Equations

Linear

$$2x + 3y = 7$$

$$x_1 + x_2 + x_3 = 1$$

Nonlinear

$$2x^2 + 3x + 2 = 0$$

$$\sin(x) + \cos(y) = 1$$

$$xy + 2y = 4$$

$$\sqrt{x} + y = 2$$

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Def a system of m linear equations with n unknowns has the general form

$$a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n} = b_1$$
$$a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2n}x_{2n} = b_2$$
$$a_{31}x_{31} + a_{32}x_{32} + \cdots + a_{3n}x_{3n} = b_3$$
$$\vdots \quad \leftarrow \text{"and so on"}$$
$$a_{m1}x_{m1} + a_{m2}x_{m2} + \cdots + a_{mn}x_{mn} = b_m$$

where a_{ij} are the coefficients

& x_{ij} are the unknowns

Notation:

$$a_{ij} \iff a_{i,j}$$

"i^{th} row" → a_{ij} ← "j^{th} column"

the comma is usually dropped unless double digit indices are needed

Double digit indices will rarely be needed in this course (if at all).

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Solving Systems of Equations

In algebra, we learn how to solve basic systems of eqns, usually w/ only 2 unknowns.

For example,

$$\begin{aligned} 4x + y &= 4 \quad (1) \\ 2x - 2y &= 4 \quad (2) \end{aligned}$$

is a system of 2 eqns w/ 2 unknowns.

We know 2 ways to solve this system

→ Substitution

Solve for a single variable in one eqn & sub into the other eqn.

→ Elimination

We multiply one equation by a constant & add it to the other eqn.

The goal is to eliminate one of the unknowns.

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Ex. 2 Solve the system of equations

$$\begin{aligned} 4x + y &= 4 \quad ① \\ 2x - 2y &= 4 \quad ② \end{aligned}$$

We'll solve this system using both substitution and elimination to compare the two processes.

Using Substitution

Solving ② for y we have

$$y = x - 2 \quad ③$$

sub ③ into ① & we have

$$\begin{aligned} 4x + (x-2) &= 4 \\ 5x - 2 &= 4 \\ 5x &= 6 \Rightarrow x = \frac{6}{5} \end{aligned}$$

subbing $x = \frac{6}{5}$ in ③ we have

$$y = \frac{6}{5} - 2 = \frac{6}{5} - \frac{10}{5} = \frac{-4}{5}$$

So the solution to our system of eqns is $\left(\frac{6}{5}, \frac{-4}{5}\right)$

Ex. 2 (cont'd)

Using Elimination

Multiplying ① by 2 & add it to ②:

$$2(4x + y = 4) \quad ①$$
$$2x - 2y = 4 \quad ②$$

$$\Rightarrow \begin{array}{rcrcl} & 8x & + 2y & = & 8 \\ + & 2x & - 2y & = & 4 \\ \hline & 10x & & = & 12 \end{array}$$

$$x = \frac{12}{10} = \frac{6}{5}$$

Then, using this in eqn ② we find

$$2\left(\frac{6}{5}\right) - 2y = 4$$
$$\Rightarrow \frac{12}{5} - 2y = 4$$
$$\frac{12}{5} - 4 = 2y$$
$$\frac{12}{5} - \frac{20}{5} = 2y \quad \Rightarrow y = \frac{1}{2}\left(-\frac{8}{5}\right) = -\frac{4}{5}$$

So the solution to our system of eqns is $\left(\frac{6}{5}, \frac{-4}{5}\right)$ □

as we can see, either method yields the same soln. The choice btwn methods is usually in regards to whichever involves the least amt of effort.

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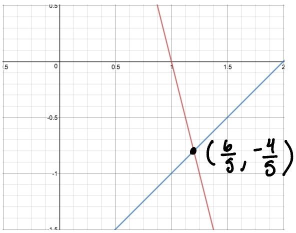
Graphical Solutions

Graphically, the solution to these types of systems is the point where the two lines intersect.

The solution to the system in Ex. 2 is shown below



So the solution to the system is <u>unique</u> since two lines can only intersect @ a single point.

There are 3 possibilities for the solution of a system of equations

A linear system can have
- No solution
- a unique solutions
- infinitely many solutions

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Ex. 3) System w/ infinitely many solns

$$x_1 - 2x_2 = -1 \quad ①$$
$$-x_1 + 2x_2 = 1 \quad ②$$

using elimination:

adding both eqns:

$$\begin{array}{r} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \\ \hline 0 + 0 = 0 \end{array}$$

Let's try substitution:

Solving for ① for x_1:

$$x_1 = -1 + 2x_2 \quad ③$$

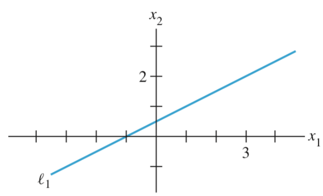
Sub ③ into ②:

$$-(-1 + 2x_2) + 2x_2 = 1$$
$$1 - 2x_2 + 2x_2 = 1$$
$$\Rightarrow \quad 1 = 1$$

While neither of these results are false, it is not possible to solve for either variable.

This means there are infinitely many soln.s

Graphically: both eqns give the same line



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Ex. 4 a system w/no soln

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$

using elimination:

adding both eqns:

$$\begin{array}{r} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \\ \hline 0 + 0 = 2 \end{array}$$

$$\Rightarrow \quad 0 = 1$$

a false statement!

Let's try substitution:

Solving for ① for x_1: $\quad x_1 = -1 + 2x_2$ ③

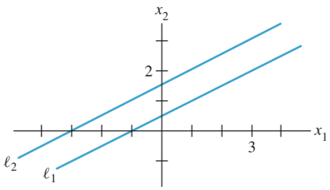
Sub ③ into ②: $-(-1 + 2x_2) + 2x_2 = 3$

$$1 - 2x_2 + 2x_2 = 3$$

$$\Rightarrow \quad 1 = 3$$

a false statement!

Graphically, this means the 2 lines are parallel



To Summarize:

A system of linear equations has

1. no solution, or (i.e. system is inconsistent)
2. exactly one solution, or
3. infinitely many solutions.

} (i.e. system is consistent)

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Question: What do we do when the number of eqns & unknowns is greater than 2?

We will need a more concise way of peforming our algebraic manipulations.

Matrix Notation

For systems w/ more than eqns & unknowns it is convenient to represent systems of eqns in a new way.

Consider the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

We can represent the coefficients of the unknowns in a Coefficient Matrix.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

Note that for any unknown that does not appear in our system is represented by a zero in our coeff. matrix.

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we also need to represent the information from the right hand side (RHS)

To do this we augment our coeff. matrix by adding an add'l column to our coeff. matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

a dotted line is usually drawn so we know that the last column is our RHS

This is known as an augmented matrix.

Note that will will not always be dealing with augmented matrices in this course. This type of matrix is only used when our goal is to solve a particular system.

This matrix notation makes it easier to perform steps necessary to solve a system by removing the need to rewrite unknowns & equals signs, etc.

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Solving systems using a Matrix

We will solve systems using a sequence of <u>elementary</u> <u>row</u> <u>operations</u>.

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

We perform these operations until our matrix is in a special form.

Row Echelon Form

<u>Def</u> Row echelon form

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Ex. 5 The matrix

"Leading" entry in this row

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix}$$

is in row echelon form

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Def Reduced Row echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Ex. 6 The matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

is in Reduced row echelon form.

Translating this back into a system we can see why this is a nice form

$$\begin{aligned} x_1 \qquad\qquad &= 29 \\ x_2 \qquad &= 16 \\ x_3 &= 3 \end{aligned}$$

Thm Each matrix is row equivalent to one & only one reduced echelon matrix i.e. the reduced row echelon form is unique.