Math 2250

Lecture # 18: Orthogonal sets & projections

Date: Tue. 11/27/18

Def A set $\{\vec{u}_1, \dots, \vec{u}_p\}$ of vectors in $\mathbb{R}^n$ is an orthogonal set if

$$\vec{u}_i \cdot \vec{u}_j = 0$$

whenever $i \neq j$. The set is an orthogonal basis, if the set also forms a basis.

Thm If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is orthogonal, it's linearly independent.

Pf Suppose

$$c_1 u_1 + \dots + c_p \vec{u}_p = \vec{0}$$

inner products w/ $\vec{u}_1$:

$$c_1 \vec{u}_1 \cdot \vec{u}_1 + c_2 \vec{u}_2 \cdot \vec{u}_1 + \dots + c_p \vec{u}_p \cdot \vec{u}_1 = \vec{0} \cdot \vec{u}_1 = 0$$

and so $c_1 = 0$. Similarly, taking inner products w/ $\vec{u}_2, \dots, \vec{u}_p$ shows $c_2 = \dots = c_p = 0$

■

Orthogonal bases make it easy to calculate the coord's in a lin. combo

$$\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_p \vec{u}_p$$

$$\vec{v} \cdot \vec{u}_j = c_1 \vec{u}_1 \cdot \vec{u}_j + \dots + c_p \vec{u}_p \cdot \vec{u}_j = c_j \vec{u}_j \cdot \vec{u}_j$$

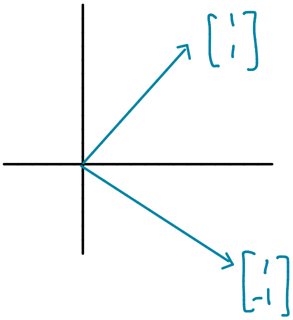
$$c_j = \frac{\vec{v} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} = \frac{\vec{v} \cdot \vec{u}_j}{\|\vec{u}_j\|^2} = \begin{array}{c} \text{Coord. of } \vec{v} \\ \text{w.r.t. } \vec{u}_j \end{array}$$

Lecture #18: Orthogonal sets & projections

Date: Tue. 11/27/18

Ex. 1) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is an orthog. basis for $\mathbb{R}^2$

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Find c_i:

$$c_1 = \frac{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{4}{2} = 2$$

$$c_2 = \frac{\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} = 1 \qquad \Rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Def. The orthogonal projection of $\vec{v}$ onto $\vec{u}$ is

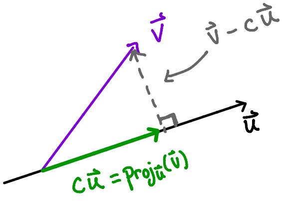
$$\text{Proj}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2} \vec{u}$$

Geometric idea:

Find c that makes $\vec{v} - c\vec{u} \perp \vec{u}$:

$$0 = (\vec{v} - c\vec{u}) \cdot \vec{u} = \vec{v} \cdot \vec{u} - c \underbrace{\vec{u} \cdot \vec{u}}_{\|\vec{u}\|^2}$$

$$\Rightarrow c = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}$$



Math 2250

Lecture # 18: Orthogonal sets & projections

Date: Tue. 11/27/18

Coord.s are even simpler if all orthogonal basis vectors have length 1:

$$\vec{v} = \frac{\vec{v}\cdot\vec{u}_1}{\|\vec{u}_1\|^2} u_1 + \dots + \frac{\vec{v}\cdot\vec{u}_p}{\|\vec{u}_p\|} \vec{u}_p$$

$$= \underbrace{(\vec{v}\cdot\vec{u}_1)}_{c_1}\vec{u}_1 + \dots + \underbrace{(\vec{v}\cdot\vec{u}_p)}_{c_p}\vec{u}_p$$

Def $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an <u>orthonormal set</u> if it's an orthogonal set & each $\|\vec{u}_j\| = 1$. If it's also a basis, it's an orthonormal basis.

Ex.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$$

is an orthonormal basis for $\mathbb{R}^3$

so is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\left(\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 \right)^{1/2}$$
$$= \left(\frac{1}{2} + \frac{1}{2} + 0\right)^{1/2} = 1^{1/2} = 1$$

$$\sqrt{1^2 + 1^2 + 0^2}$$
$$= \sqrt{2}$$

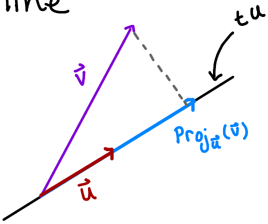
Minimum Distance Property

Given a fixed $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$, the set

$$\{tu : t \in \mathbb{R}\}$$

is a line thru $\vec{0}$ in the direction of $\vec{u}$

$\text{Proj}_{\vec{u}}(\vec{v})$ is the vector on this line that is closest to $\vec{v}$, i.e. the best approx. to $\vec{v}$ in Span$\{\vec{u}\}$.



Justification:

$$0 = \frac{d}{dt}\|\vec{v} - t\vec{u}\|^2 = \frac{d}{dt}[(\vec{v}-t\vec{u})\cdot(\vec{v}-t\vec{u})]$$

$$(\text{distance})^2 = \frac{d}{dt}[\|\vec{v}\|^2 - 2t\,\vec{v}\cdot\vec{u} + t^2\|\vec{u}\|^2]$$

$$= -2\vec{v}\cdot\vec{u} + 2t\|\vec{u}\|^2$$

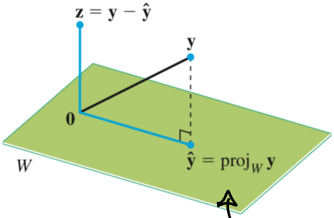
$$\Rightarrow t = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}$$

Generalization of Orthogonal Projections

Given an orthog. set $\{\vec{u}_1, \dots \vec{u}_p\}$ of vectors in $\mathbb{R}^n$, the orthog. projection of $\vec{y}$ onto $\text{Span}\{\vec{u}_1, \dots, \vec{u}_p\} = W$ is

$$\text{Proj}_W \vec{y} = \hat{y} = \underbrace{\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}_{}\vec{u}_1 + \dots + \underbrace{\frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}}_{}\vec{u}_p$$

Coeff.s that gaurantee that $\vec{y} - \hat{y} \perp$ every $\vec{u}_i$



$W = \text{span}\{\vec{u}_1, \vec{u}_2\}$ (a plane in $\mathbb{R}^3$)

Ex) Find $\text{proj}_W \vec{y}$ for $W = \text{Span}\left\{\begin{bmatrix}-4\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ & $\vec{y} = \begin{bmatrix}6\\4\\1\end{bmatrix}$

$$\begin{array}{cc} u_1 & u_2 \end{array}$$

$$\hat{y} = \text{Proj}_W \vec{y} = \underbrace{\frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}}_{}\vec{u}_1 + \underbrace{\frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}}_{}\vec{u}_2 = \frac{\vec{y}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$= \frac{-27}{18}\vec{u}_1 + \frac{5}{2}\vec{u}_2 = -\frac{27}{18}\begin{bmatrix}-4\\1\\1\end{bmatrix} + \frac{5}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}6\\1\\1\end{bmatrix}$$