Lecture # 17: Orthogonality & Inner Products   Date: Tue. 11/20/18

Def If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

their inner product or dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n \qquad \text{(a scalar)}$$

This can also be written as

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Ex. 1

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = (1)(-1) + (3)(2) + (0)(3) = 5$$

Properties of inner products

(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(ii) $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$

(iii) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

(iv) $\vec{u} \cdot \vec{u} \geq 0$; $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = \vec{0}$

Consequence:

$$(c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_p \vec{u}_p) \cdot \vec{v} = c_1(\vec{u}_1 \cdot \vec{v}_1) + \dots + c_p(\vec{u}_p \cdot \vec{v}_p)$$

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<u>Def</u> The <u>norm</u> (or length) of $\vec{u} \in \mathbb{R}^n$ (from the origin) is

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + \ldots + u_n^2}$$

When $n=2$

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{u_1^2 + u_2^2}$$

$=$ <u>Euclidean</u> <u>distance</u> btwn $\vec{u}$ & $\vec{0}$
(a.k.a. Pythagorean distance)

Properties of Euclidean norm

(i) $\|\vec{u}\| \geq 0$; $\|u\| = 0$ iff $\vec{u} = \vec{0}$

(ii) $\|c\vec{u}\| = |c|\,\|u\|$, any $c \in \mathbb{R}$

(iii) $\|u+v\| \leq \|u\| + \|v\|$ (Triangle ineq.)

Remark: given nonzero $\vec{v}$ in $\mathbb{R}^n$, can construct a unit length vector in same direction

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} \qquad \text{i.e. } \|\vec{u}\| = 1$$

<u>Ex. 2</u> Find a unit vector in direction of $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$

Find $\|\vec{v}\|$

$$\|\vec{v}\| = \left(v_1^2 + v_2^2 + v_3^2 + v_4^2\right)^{1/2}$$

$$= \left((1)^2 + (0)^2 + (2)^2 + (-1)^2 \right)^{1/2} = \sqrt{6}$$

Unit vec in direction of $\vec{v}$:

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{6}}[1, 0, 2, -1]^T = \left[\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]^T$$

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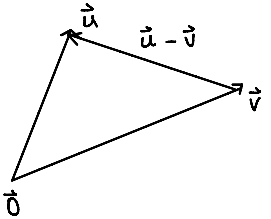
use length to calculate distance btwn 2 points

distance btwn $\vec{u}$ & $\vec{v}$ = $\|\vec{u}-\vec{v}\|$



We can express $\vec{u}\cdot\vec{v}$ in terms of lengths

Thm $\vec{u}\cdot\vec{v} = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$

Pf.

$$\|u+v\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \|u\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

$$\|u-v\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|u\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

subtracting these & solving for $\vec{u}\cdot\vec{v}$ gives the desired result. ∎

Def $\vec{u},\vec{v}\in\mathbb{R}^n$ are orthogonal if $\vec{u}\cdot\vec{v}=\vec{0}$
(i.e. perpendicular)

Ex. 3) $\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

are mutually orthogonal:

$$\vec{e}_i\cdot\vec{e}_j = 0 \quad \text{if} \quad i\neq j$$

Ex. 4 $\vec{0}$ is the only vector orthogonal to every vector in $\mathbb{R}^n$

Orthogonal Complements

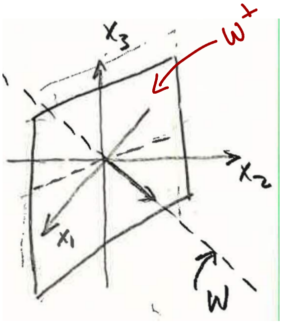
Def Let W be a subset of $\mathbb{R}^n$. The orthogonal complement of W is

$$W^\perp = \{ \vec{z} \in \mathbb{R}^n : \underbrace{\vec{z}\cdot\vec{w} = 0 \ \forall\, \vec{w} \in W}_{\text{orthog. to everything in } W} \}$$

"for all"

"in" "such that"

Ex. 5 Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$,



This is a line in $\mathbb{R}^3$ thru $\vec{0}$

$$W^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

this is a plane thru $\vec{0}$

Thm. Let $W = \text{span}\{\vec{w}_1, \dots \vec{w}_p\}$, subspace of $\mathbb{R}^n$

(a) $\vec{x} \in W^\perp$ iff $\vec{x}\cdot\vec{w}_1 = \dots = \vec{x}\cdot\vec{w}_p = \vec{0}$

(b) $W^\perp$ is a subspace of $\mathbb{R}^n$.