Lecture # 07: More Inverses & Intro to Determinants

Date: Tue. 10/2/18

Characterizations of Invertible Matrices

Thm **The Invertible Matrix Theorem** (IMT) (TFAE)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Ex. 1)

$$\begin{bmatrix} -7 & 0 & 4 \\ 3 & 0 & -1 \\ 2 & 0 & 9 \end{bmatrix}$$

This matrix is not invertible b/c the columns don't span $\mathbb{R}^3$ (why?) b/c span{cols A} contains the zero vector.

So fails part (h) of IMT

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Ex. 2

$$\underbrace{\begin{bmatrix} 5 & 7 \\ -3 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$

A is invertible b/c System $A\vec{x}=\vec{0}$ has only the trivial solution.

i.e. Since Cols of A are linearly indep.

((e) $\Longleftrightarrow$ (d))

Determinants

Idea: For any square matrix A, calculate a number, det(A), that indicates whether A is invertible.

Recall: The determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ is $\det(A) = ad - cb$

A^{-1} of a 2x2 matrix is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A^{-1} will not exist if $\det(A)=0$

i.e. we determine if A is invertible from det(A)

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For 1×1 matrix:

$$\det([a]) = a$$

For 2×2 matrix:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

For $n\times n$ $(n>2)$ matrices:

things get a bit more complicated.

we define $\det(A)$ in terms of $(n-1)\times(n-1)$ matrices.

Some Definitions

Def The (i,j) th <u>minor</u> of $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{nn} & \cdots & a_{nn} \end{bmatrix}$

is the determinant of the matrix we get by eliminating row i & col. j. Denoted M_{ij}

Ex.3 $A = \begin{bmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$A = \begin{array}{c} \begin{array}{ccc} & & \text{Col 3} \end{array} \\ \begin{bmatrix} 0 & 3 & \cancel{0} \\ \cancel{-1} & \cancel{0} & \cancel{1} \\ 1 & 1 & \cancel{0} \end{bmatrix} \end{array} \text{row 2} \Rightarrow \quad M_{23} = \det\begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} = -3$$

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Def The $(i\,j)$th Cofactor of A is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Ex. 4 For the previous $A = \begin{bmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$C_{23} = (-1)^{2+3} M_{23} = (-1)(-3) = 3$$

The determinant of A is

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{n}$$

Ex. 5 Using same A

$$\det(A) = \begin{vmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

expansion by cofactors using the 1st row.

$$= \underbrace{0 \cdot (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{(1,1)\text{ cofactor}} + \underbrace{3 \cdot (-1)^{1+2} \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}}_{(1,2)\text{ cofactor}} + 0 \cdot (-1)^{1+3} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -3\big(\overset{a}{(-1)}\overset{d}{(0)} - \overset{b}{(1)}\overset{c}{(1)}\big) = 3$$

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Remark: we can expand determinants along any row or column

Ex. 6

$$\det(A) = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cccc} & & & * \\ \end{array}$$

$$\det(A) = \begin{vmatrix} 1 & -1 & 3 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & -1 & 0 & 5 \\ 0 & 1 & 1 & 1 \end{vmatrix}$$

expanding about Col. 4 *

$$= 0\cdot(-1)^{1+4}\begin{vmatrix} -1 & 2 & 1 \\ 3 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 0(-1)^{2+4}\begin{vmatrix} 1 & -1 & 3 \\ 3 & -1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 5(-1)^{3+4}\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 1(-1)^{4+4}\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix}$$

$$= -5\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix}$$

Note: $\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix} = -17$

$$\begin{vmatrix} 1 & -1 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0(-1)^{3+1}\begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 1\cdot(-1)^{3+2}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + 1\cdot(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -[(1)(1) - (3)(-1)] + [(1)(2) - (-1)(-1)]$$

$$= -4 + 1 = -3$$

$$\det(A) = -5(-3) + 1(-17) = 15 - 17 = -2$$

Clearly, calculating determinants can get pretty tedious for larger dimensions!

Save work by choosing to expand by cofactors using a row or column with lots of zeros!

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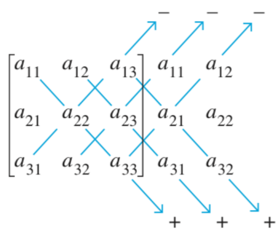
There is a shortcut for calculating the determinant of a 3x3 matrix.

multiplying diagonals:

1) re-copy the 1st two cols of the matrix to the right of the matrix.



2) multiply entries in diagonals w/ 3 entries.

3) add or subtract the products of these entries according the pattern in the figure

Note: This method only applies to 3x3 matrices!

Ex. 7) $A = \begin{bmatrix} 0 & 3 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$\begin{array}{ccc|cc} 0 & 3 & 0 & 0 & 3 \\ -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{array} = 0 + 3 + 0 - 0 - 0 - 0$$

$$= 3$$

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Properties of Determinants

__Thm__ Let B be the result of

(1) mult. a single row of A by r. Then

$$\det(B) = r\cdot\det(A)$$

(2) interchanging 2 rows of A. Then

$$\det(B) = -\det(A)$$

(3) adding $c\cdot(\text{row } i)$ to row_j of A. Then,

$$\det(B) = \det(A)$$

__Ex. 8__

$$\det\begin{bmatrix}1&0&0\\0&5&0\\0&0&1\end{bmatrix} = 5\cdot\det\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = 5 \qquad \text{by (1)}$$

$$\det\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \det\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = -1 \qquad \text{by (2)}$$

$$\det\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = \det\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} = 1 \qquad \text{by (3)}$$

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Determinants of Products (of $n \times n$ matrices)

Property 1: If E is an elem. Matrix,

$$\det(EB) = \det(E)\det(B)$$

For Example:

If E multiplies a row by c, then

$$\det(EB) = c \cdot \det(B) = \det(E)\det(B)$$

If E interchanges 2 rows, then

$$\det(EB) = -\det(B) = \det(E)\det(B)$$

Property 2: A is invertible iff $\det(A) \neq 0$

Pf A is invertible if there are elem. matrices $E_1, E_2, \dots E_m$ s.t.

$$E_1 E_2 \cdots E_m A = I$$

Take det of both sides

$$\det(E_1)\det(E_2)\dots\det(E_m)\det(A) = 1$$

This is possible iff $\det(A) \neq 0$ $\square$

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Ex. 9 $A = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 0 & 2 & -2 \\ 3 & 0 & 8 & -1 \\ 1 & 0 & 5 & 0 \end{bmatrix}$ isn't invertible, since $\det(A) \neq 0$.

Hence, $A\vec{x} = \vec{0}$ doesn't have a unique soln.

Check that the soln is $\vec{x} = \begin{bmatrix} 0 \\ t \\ 0 \\ 0 \end{bmatrix}$ for $t \in \mathbb{R}$.

Property 3: $\det(AB) = (\det A)(\det B)$

Pf

Case 1: If A isn't invertible, neither is AB

Otherwise, $(AB)^{-1} = B^{-1}A^{-1}$ & A^{-1} DNE)

Hence,

$$\begin{aligned} \det(AB) &= 0 \\ &= 0 \cdot \det(B) \\ &= \det(A) \cdot \det(B) \end{aligned}$$

Case 2: If A is invertible, then it's the product of elem. matrices:

$$\begin{aligned} \det(AB) &= \det(E_1 E_2 E_3 \dots E_m B) \\ &= \det(E_1)\det(E_2)\dots\det(E_m)\det(B) \\ &= \det(\underbrace{E_1 E_2 \dots E_m}_{A})\det(B) \end{aligned}$$

(by Prop. 1)

$\square$

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Thm If A is triangular, then $\det(A)$ is the product of entries on the main diagonal.

Pf

Case 1: upper triangular, Expand $\det(A)$ along Col. 1:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & \cdots & a_{2n} \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix} = a_{11} a_{22} \det \begin{bmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$$

Case 2: lower triangular. Expand $\det(A)$ along row 1 $\square$

Ex. 10) $\det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 2 & 7 \end{bmatrix} = 1 \cdot 3 \cdot 7 = 21$