1. Let

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

Verify that $AB = AC$ but $B \neq C$.

Solution:

$$AB = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

and

$$BC = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$$

So $AB = BC$ but clearly $B \neq C$.

2. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$ by hand. Verify the result by calculating AA^{-1} and $A^{-1}A$.

Solution: Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and so

$$A^{-1} = \frac{1}{(1)(7) - (2)(4)} \begin{bmatrix} 7 & -2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

3. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ by hand. Verify the result with your calculator.

Solution: To find the inverse of A by hand we first augment the matrix with the identity matrix I_3

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

We then perform elementary row operations until the LHS of the augmented matrix is the identity matrix I_3.

$3R_1 + R_2 \to R_2$

$$\begin{array}{ccc|ccc} 3 & 0 & 6 & 3 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ \hline 0 & 1 & 10 & 3 & 1 & 0 \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$-2R_1 + R_3 \to R_3$

$$\begin{array}{ccc|ccc} -2 & 0 & 4 & -2 & 0 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \\ \hline 0 & -3 & 0 & -2 & 0 & 1 \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$\implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \\ 0 & 1 & 10 & 3 & 1 & 0 \end{array}\right]$$

$\frac{1}{3}R_2 \leftrightarrow R_2$

$$\implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 1 & 10 & 3 & 1 & 0 \end{array}\right]$$

$-R_2 + R_3 \to R_3$

$$\begin{array}{ccc|ccc} 0 & -1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 10 & \frac{9}{3} & 1 & 0 \\ \hline 0 & 0 & 10 & \frac{7}{3} & 1 & \frac{1}{3} \end{array} \implies \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 10 & \frac{7}{3} & 1 & \frac{1}{3} \end{array}\right]$$

$-\dfrac{1}{10}R_3 \to R_3$

$$\Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{array}\right]$$

$-2R_3 + R_1 \to R_1$

$$\begin{array}{ccc:ccc} 0 & 0 & -2 & -\frac{14}{30} & -\frac{2}{10} & -\frac{2}{30} \\ 1 & 0 & 2 & \frac{30}{30} & 0 & 0 \\ \hline 1 & 0 & 0 & \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \end{array} \quad \Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{array}\right]$$

Since the LHS is now the identity matrix I_3 then the RHS of this augmented matrix is the inverse of A and so

$$A^{-1} = \begin{bmatrix} \frac{16}{30} & -\frac{2}{10} & -\frac{2}{30} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{bmatrix} = \begin{bmatrix} \frac{8}{15} & -\frac{1}{5} & -\frac{1}{15} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{30} & \frac{1}{10} & \frac{1}{30} \end{bmatrix}$$