Lecture # 10: Linear Independence & Basis   Date: Thu. 10/18/18

Linear Independence

<u>Def</u> A set $\{v_1, \ldots, v_p\}$ of vectors in V is <u>linearly independent</u> if the equation

$$c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution $c_1 = c_2 = \ldots = c_p = 0$

It's <u>linearly dependent</u> otherwise. i.e. if

$\exists\ c_1, c_2, \ldots c_p$ not all zero, s.t.

$$c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = 0$$

<u>Ex. 1</u> $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$ is linearly dependent:

$$-2\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}2\\0\\0\end{bmatrix} = 0$$

<u>Observation:</u>

$\{v_1, \ldots, v_p\}$ (with $\vec{v}_1 \neq 0$) is linearly dependent.

$\Rightarrow$ Some $\vec{v}_i$ is a linear comb. of the others

Suppose

$$c_1\vec{v}_1 + \ldots + c_p\vec{v}_p = 0 \quad \text{with} \quad c_i \neq 0$$

then

$$\vec{v}_i = \underbrace{-\frac{c_1}{c_i}\vec{v}_1 + \ldots + \frac{c_p}{c_i}\vec{v}_p}_{\text{no term involving } \vec{v}_i}$$

Ex. 2) Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Soln.

Try to find c_1, c_2, c_3 s.t.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$

the matrix is invertible so the system has exactly 1 soln $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so vectors are linearly independent. $\blacksquare$

Basis

Def A <u>basis</u> for a subspace $H \subset V$ is a set of vectors $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_p\}$ in V s.t.

H is a subset of V

1. $H = \text{Span}\{\vec{b}_1, \dots, \vec{b}_p\}$

2. $\{\vec{b}_1, \dots, \vec{b}_p\}$ is linearly indep.

Note: this includes the case $H = V$

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Ex. 3 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$

is a basis for $\mathbb{R}^3$ called the <u>standard basis</u>

Check:

Spanning: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Coordinates w.r.t. $\{ \vec{e}_1, \vec{e}_2, \vec{e}_3 \}$

Lin. indep: $\det [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3] = \det I = 1 \neq 0$ ∎

Thm vectors $v_1, \dots v_n$ are a basis for $\mathbb{R}^n$ exactly when they are the col.s of an $n \times n$ invertible matrix.

Ex. 4 $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$

Spanning:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Since every row has a pivot position, this system has a soln $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ for any vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ i.e. the columns span $\mathbb{R}^3$,

Lin. indep.: $\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = 1$, so col.s are lin. ind. ∎

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Thm. The pivot Col.s of A are a basis for its column space. Pivot rows of A are a basis for its row space.

Def The row space of A is the Span of the rows of A. i.e. the row space of A is $Col(A^T)$.

To find the coordinates of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ w.r.t. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

$$y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Known coord.s

unknown coord.s

i.e. Solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow$ $y_3 = x_3$, $y_2 = x_2 - 2x_3$, $y_1 = x_1 - x_3$