1. Determine if the vectors are linearly independent

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}; \quad \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix} \tag{1}$$

Solution: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. So to determine if the given set of vectors we must find the solution to the system

$$\begin{array}{rcrcrcr} 5c_1 & + & 7c_2 & + & 9c_3 & = & 0 \\ & & 2c_2 & + & 4c_3 & = & 0 \\ & - & 6c_2 & - & 8c_3 & = & 0 \end{array}$$

We construct the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{0}]$ and then perform elementary row operations to get the system in row echelon form.

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \underset{\substack{\text{Row} \\ \text{Reduce}}}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Writing this in system form we have

$$\begin{array}{ccccc} c_1 & & & = & 0 \\ & c_2 & & = & 0 \\ & & c_3 & = & 0 \end{array}$$

and so we have only the trivial solution $c_1 = c_2 = c_3 = 0$. Therefore, the set of vectors is linearly independent.

2. Determine if the vectors are linearly independent

$$\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}; \quad \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}; \quad \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \tag{2}$$

Solution: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. So to determine if the given set of vectors we must find the solution to the system

$$\begin{array}{ccccccc} & & & - & 3c_3 & = & 0 \\ & & 5c_2 & + & 4c_3 & = & 0 \\ 2 & - & 8c_2 & + & c_3 & = & 0 \end{array}$$

In this system form, we can see that $c_3 = 0$. Using forward substitution we also have $c_1 = c_2 = 0$ and so we have only the trivial solution $c_1 = c_2 = c_3 = 0$. Therefore, the set of vectors is linearly independent.

An Extra Problem (not on the original worksheet)

3. Determine if the vectors are linearly independent

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \tag{3}$$

Solution: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. So to determine if the given set of vectors we must find the solution to the system

$$\begin{array}{ccccccccc} c_1 & + & c_2 & + & c_3 & + & 2c_4 & = & 0 \\ & & c_2 & + & c_3 & + & 3c_4 & = & 0 \\ & & & & c_3 & + & 4c_4 & = & 0 \end{array}$$

We construct the augmented matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{0}]$ and then perform elementary row operations to get the system in row echelon form.

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix} \underset{\substack{\text{Row} \\ \text{Reduce}}}{=} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{bmatrix}$$

Writing this in system form we have

$$\begin{array}{ccccccc} c_1 & & & + & c_4 & = & 0 \\ & c_2 & & - & 3c_4 & = & 0 \\ & & c_3 & + & 4c_4 & = & 0 \end{array}$$

and so we we see that

$$\begin{aligned} c_1 &= -c_4 \\ c_2 &= 3c_4 \\ c_3 &= -4c_4 \end{aligned}$$

Which means we have a free variable. This means there exists c_1, c_2, c_3, c_4 not all zero for which

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Therefore, the set of vectors is NOT linearly indepdent. In other words, this set of vectors is linearly dependent.