Lecture #08: Vector Spaces

Date: Thu. 10/11/18

also true for $\mathbb{C}$

Def. A (real) vector space is a set V with 2 operations, addition & scalar multiplication for which the following properties hold:

1.) V is closed under addition: $u, v \in V \Rightarrow u + v \in V$

2.) addition is commutative: $u + v = v + u$

3.) addition is associative: $(u+v) + w = u + (v+w)$

4.) V has an additive identity $\vec{0}$: $u + \vec{0} = u$

5.) V has additive inverses: $u + (-u) = u - u = \vec{0}$

6.) V is closed under scalar mult. $c \in \mathbb{R}, v \in V \Rightarrow cv \in V$

7.) $c(u+v) = cu + cv$

8.) $(c+d)u = cu + du$

9.) $c(du) = (cd)u$

10.) $1 \cdot u = u$

A vector is any element of V.

A scalar is any element of $\mathbb{R}$

Example: $V = \mathbb{R}^n$

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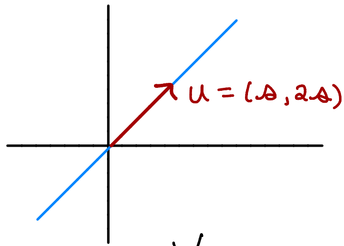
Ex. 1 $V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 2x_1\}$



Check closure:

let $\vec{u} = (s, 2s)$, $v = (t, 2t)$

$$u + v = (s, 2s) + (t, 2t) = (s+t, 2(s+t)) \in V$$

$$cu = c(s, 2s) = (cs, 2cs) \in V$$

Ex. 2 $V = \{x_2 = 2x_1\} \cup \{x_2 = -x_1\}$

Check closure:

$$u = (t, 2t), \quad v = (s, -s)$$

$$u + v = (t + s, 2t - s) \notin V$$

Ex. 3 $\mathbb{Z} = \{\text{integers}\}$

not closed under scalar mult.

$u = 3$
$c = \frac{1}{2}$ $\Rightarrow cu = \frac{3}{2} \notin \mathbb{Z}$

$\in \mathbb{Q} = \{\text{rationals}\}$

Ex. 4 $\mathbb{R}^{m \times n} = \{m \times n \text{ matrices}\}$

By matrix addition & scalar mult.

Ex. 5| $C([a,b]) = \{\text{cont. fcns } f:[a,b]\to\mathbb{R}\}$ is a vector space.

Check closure:

$$\left.\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (cf)(x) &= cf(x)\end{aligned}\right\}\ \text{cont. whenever } f \,\&\, g \text{ are cont.}$$

Ex. 6| $\{\vec{0}\}$ is a vector space

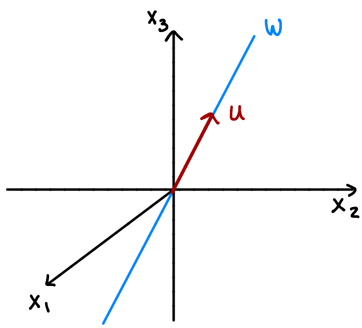
Check closure:

$$\vec{0}+\vec{0} = \vec{0}$$
$$c\vec{0} = \vec{0}$$

$\boxed{\text{Subspaces}}$

Def. A nonempty subset W of a vector space V is a subspace of V if W is also a vector space (under the same op.s)

Ex. 6| Any line thru $\vec{0}$ is a subspace of $\mathbb{R}^3$ (pick any nonzero u on the line. Every vector of the form cu is also on the line.)



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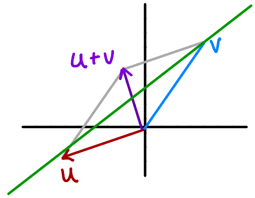
Ex. 7 The line $x_2 = 3x_1 + 1$ is not a subspace of $\mathbb{R}^2$ (not closed under addition)



Observation: any subspace must contain $\vec{0}$

Thm If $\{v_1, \dots, v_m\}$ is a set of vectors in a vector space V, then $\text{span}\{v_1, \dots, v_m\}$ is a subspace of V (possibly all of V)

Pf. Need only to check closure under addition & scalar mult.

Let $u, v \in \text{span}\{v_1, \dots, v_m\}$. We must show $u + v \in \text{span}\{v_1, \dots, v_m\}$

Since $u = k_1 v_1 + \dots + k_m v_m$

& $v = l_1 v_1 + \dots + l_m v_m$,

$u + v = (k_1 + l_1) v_1 + \dots + (k_m + l_m) v_m \in \text{span}\{v_1, \dots, v_m\}$

$cu = (ck_1) v_1 + \dots (ck_m) v_m \in \text{span}\{v_1, \dots, v_m\}$ $\square$

Ex. 8 If $A \in \mathbb{R}^{m \times n}$ & $x \in \mathbb{R}^n$, then

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$$

So $\{A\vec{x} \in \mathbb{R}^m : \vec{x} \in \mathbb{R}^n\} = \{\text{all possible products } A\vec{x}\}$

$= \text{Span}\{\text{cols of } A\}$

which is a subspace of A.