Worksheet 16 Solutions

Systems & Stability MATH 2250, Fall 2018

Carry out your work on separate pages and attach it to this sheet. If technology is used, clearly indicate what you used it to calculate.

1. Consider the system

$$\frac{dx}{dt} = x - x^2 - xy$$
$$\frac{dy}{dt} = 3y - xy - 2y^2$$

The **critical points** of the system are found by determining all points where $\vec{x}' = 0$. This system has 4 critical points. They are

$$(0,0), \quad \left(0, \frac{3}{2}\right), \quad (1,0), \quad (-1,2)$$

Note also that the **Jacobian** of this system is

$$\vec{J} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1 - 2x - y & -x \\ -y & 3 - x - 4y \end{bmatrix}$$

(a) Find the corresponding linear system near each critical point. Recall that the linear system that approximates this nonlinear system is given by

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \vec{J}(x_0, y_0)\begin{bmatrix} u \\ v \end{bmatrix}$$

(b) Find the eigenvalues of each linear system.

1. (cont'd)

@ $(0,0)$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix}}_{=\begin{bmatrix} u' \\ v' \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{aligned} u_1 &= u \\ v' &= 3v \end{aligned}$$

The linear system that approx's nonlinear system near $(0,0)$

e-vals:

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 3$$

evals are both real & positive & $\lambda_1 \neq \lambda_2$

$\Rightarrow$ unstable Node

@ $(0, 3/2)$

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{aligned} u' &= \tfrac{1}{2}u \\ v' &= -\tfrac{3}{2}u - 3v \end{aligned}$$

Find evals:

$$\begin{vmatrix} -\frac{1}{2}-\lambda & 0 \\ -\frac{3}{2} & -3-\lambda \end{vmatrix} = (-1/2 - r)(-3-r) = 0$$

$$\Rightarrow \lambda = -3, -1/2$$

evals are both real & negative & $\lambda_1 \neq \lambda_2$

$\Rightarrow$ Asymptotically stable Node

1. (cont'd)

@ (1,0)

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{array}{l} u' = -u - v \\ v' = 2v \end{array}$$

Find evals:

$$\begin{vmatrix} -1-\lambda & -1 \\ 0 & 2-\lambda \end{vmatrix} = (-1-\lambda)(2-\lambda) - 0 = 0$$

$$\Rightarrow \lambda = 2, -1$$

evals are real & differ in sign $\Rightarrow$ <u>saddle point</u>

@ (-1,2)

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow \begin{array}{l} u' = u + v \\ v' = -2u - 4v \end{array}$$

Find e-vals:

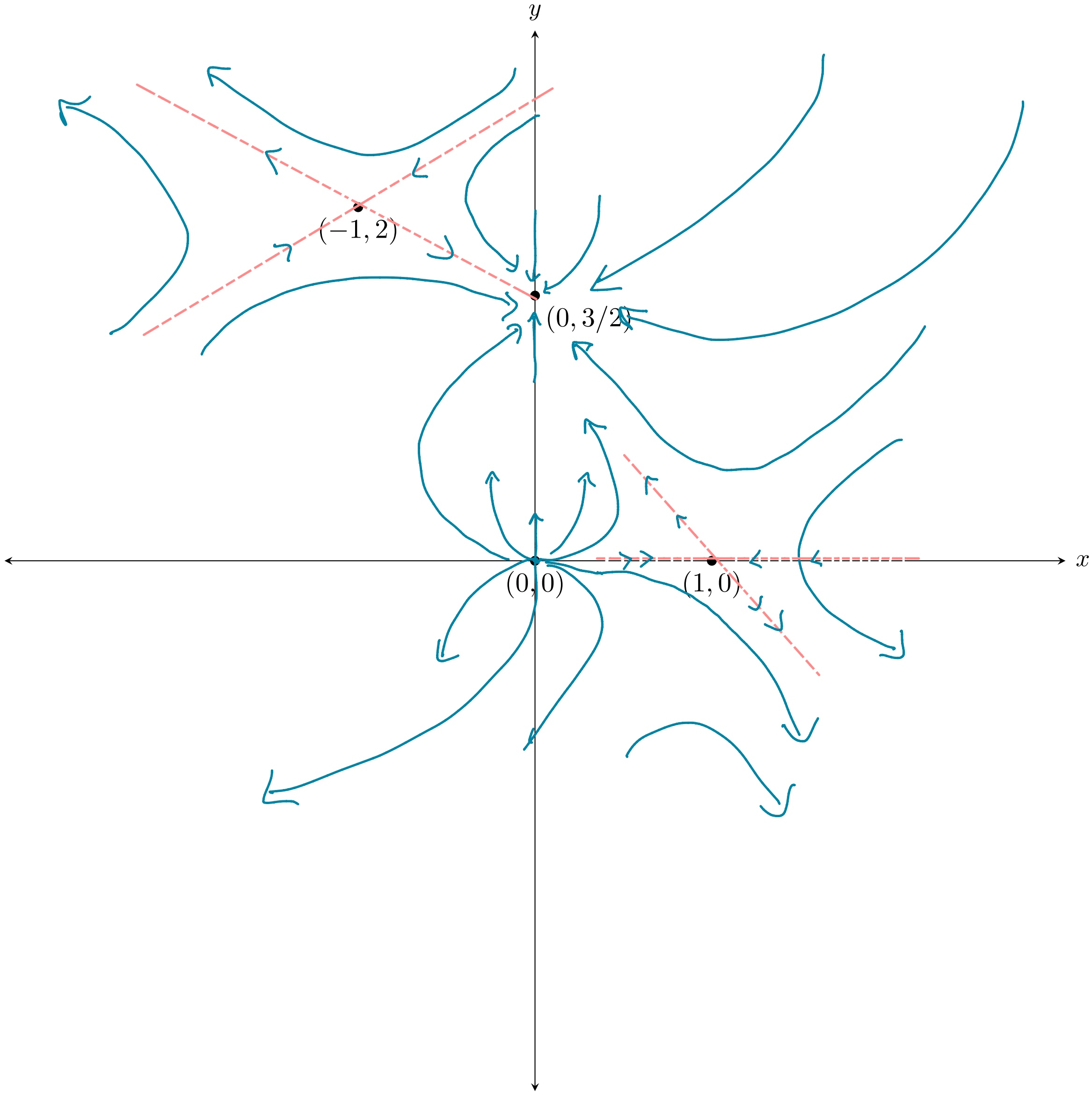
$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) + 2 = 0$$

$$\lambda^2 + 3r - 2 = 0$$

$$\Rightarrow \lambda_1 = \frac{-3-\sqrt{17}}{2}, \quad \lambda_2 = \frac{-3+\sqrt{17}}{2}$$

evals are real & differ in sign $\Rightarrow$ <u>saddle pt.</u>

(c) Draw a phase portrait of the nonlinear system on the axes provided below.



Solution: When drawing the phase portrait without finding the eigenvectors it helps to first sketch any nodes before any saddle points. It is easier to determine behavior from eigenvalues alone for any nodes you have. This way you can see which way solns will approach the saddle points and then draw your arrows from there.

Note also that we have 2 equilibrium solutions on the line $y = 0$. This means that one of the lines for the saddle point will also fall along this line.