# Worksheet 14 Solutions

## Diagonalization

MATH 2250, Fall 2018

1. Let $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Compute $A^4$.

**Solution:** Since $A$ is diagonalizable then

$$A^4 = PD^4P^{-1}$$

The inverse of $P$ is

$$P^{-1} = \begin{bmatrix} -\frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{2}{5} \end{bmatrix}$$

and since $D$ is a diagonal matrix then

$$D^4 = \begin{bmatrix} (1)^4 & 0 \\ 0 & \left(\frac{1}{2}\right)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$$

then we have

$$\begin{aligned} A^4 &= PD^4P^{-1} \\ &= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} -\frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11}{16} & -\frac{9}{8} \\ \frac{9}{8} & \frac{7}{4} \end{bmatrix} \end{aligned}$$

2. Diagonalize the matrix
$$A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$$
If it is not possible, explain why.

**Solution:** Since $A$ is lower triangular, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$. The corresponding eigenvectors for each eigenvalue are as follows

$$\lambda_1 = -1 : \quad \xi^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\lambda_2 = 1 : \quad \xi^{(2)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Our matrix $D$ is then
$$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
and so our matrix $P$ has columns which are the corresponding eigenvectors to the eigenvalues in $D$:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \qquad \text{and} \qquad P^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$$

As a check we can compute $P^{-1}DP$ to ensure it is equal to the matrix $A$.

3. Diagonalize the matrix

$$B = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

If it is not possible, explain why.

**Solution:** Since $A$ is lower triangular, the eigenvalues are $\lambda_{1,2} = 5$ and $\lambda_3 = 4$. The corresponding eigenvectors for each eigenvalue are as follows

$$\lambda_{1,2} = 4: \quad \xi^{(1)} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \xi^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 5: \quad \xi^{(3)} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Our matrix $D$ is then

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and so our matrix $P$ has columns which are the corresponding eigenvectors to the eigenvalues in $D$:

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

As a check we can compute $P^{-1}DP$ to ensure it is equal to the matrix $A$.

4. Diagonalize the matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

If it is not possible, explain why.

**Solution:** Since $A$ is lower triangular, the eigenvalues are $\lambda_{1,2} = 4$ and $\lambda_3 = 5$. The corresponding eigenvectors for each eigenvalue are as follows

$$\lambda_{1,2} = 4 : \quad \xi^{(1,2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 5 : \quad \xi^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

While $\lambda_{1,2} = 4$, we only have one eigenvector. Thus, the basis for that eigenspace has only one eigenvector. This means we only have two eigenvectors for this $3 \times 3$ system. Therefore, it is impossible to construct a matrix $P$ with linearly independent columns comprised of the eigenvectors of $A$ (and so $P$ will not be invertible).

This means that $A$ is NOT diagonalizable.