Atiyah-Singer Index Theorem Seminar.

Clifford algebras & Dirac operators (Jon Mainiero).

Motivation:

Consider E equipped with orthogonal coords x',..., x'. Then we have the Laplace operator:

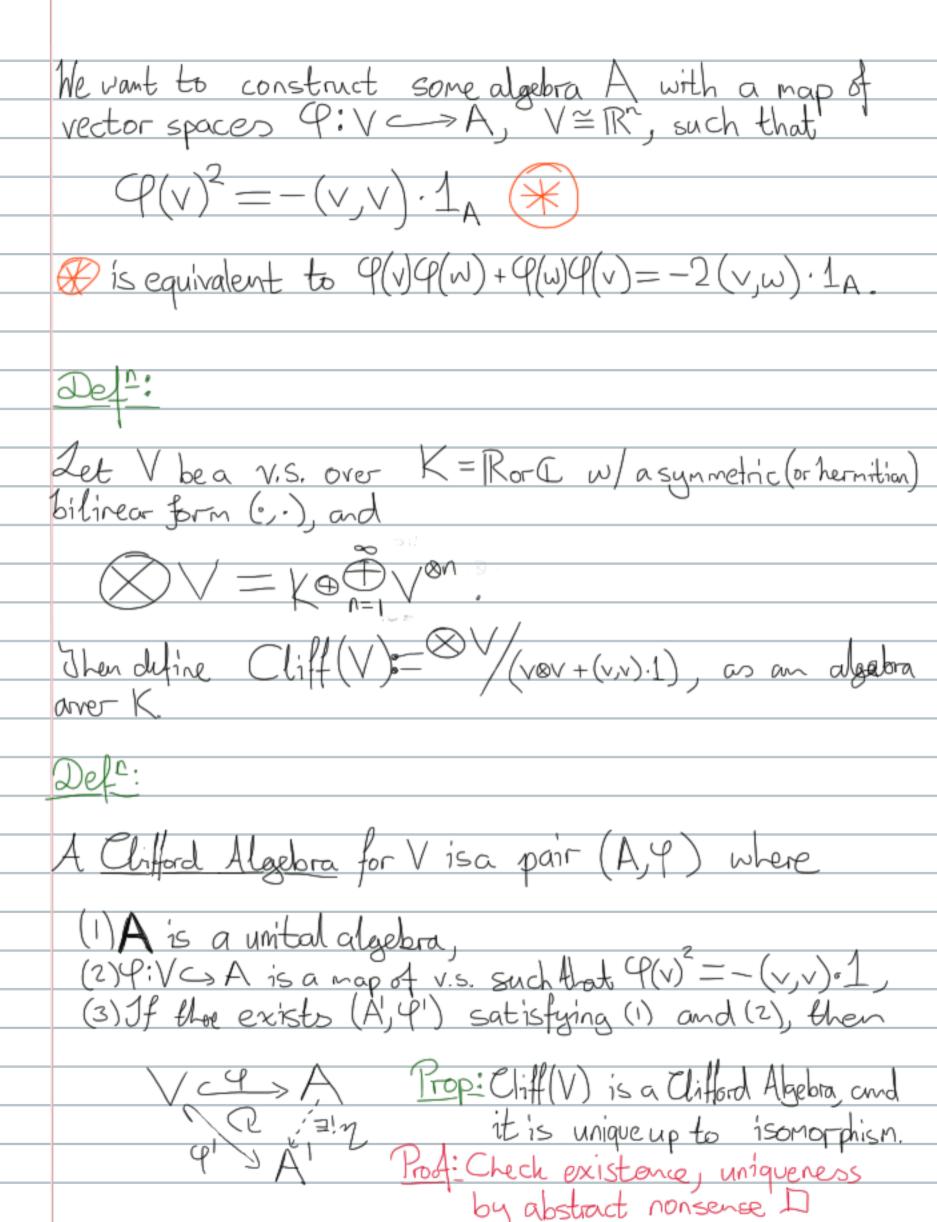
Q: Can we find a 1st order op. D sit. D= 1?

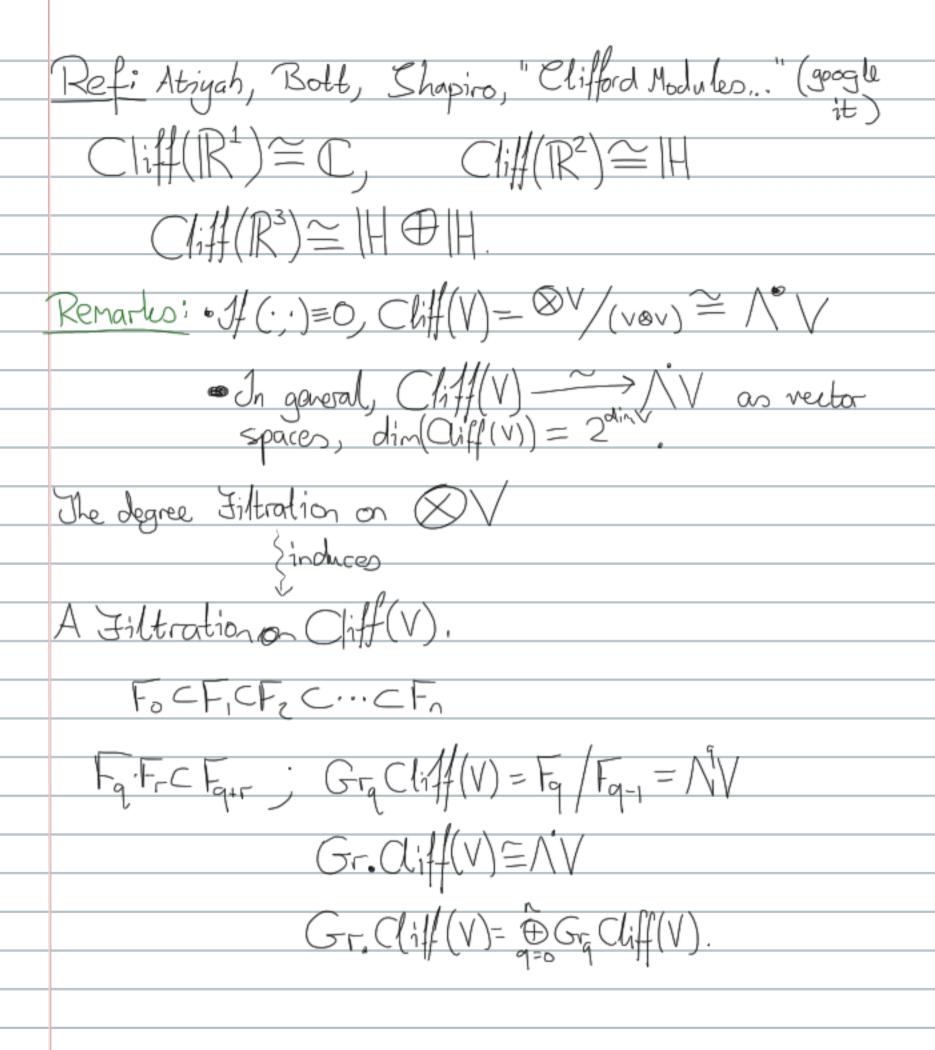
Take D= 213+823+...+83.

Take D = y' = y' = y'Then $A = A \Leftrightarrow \{(y')^2 = -1 \}$ $A = A \Leftrightarrow \{(y'$

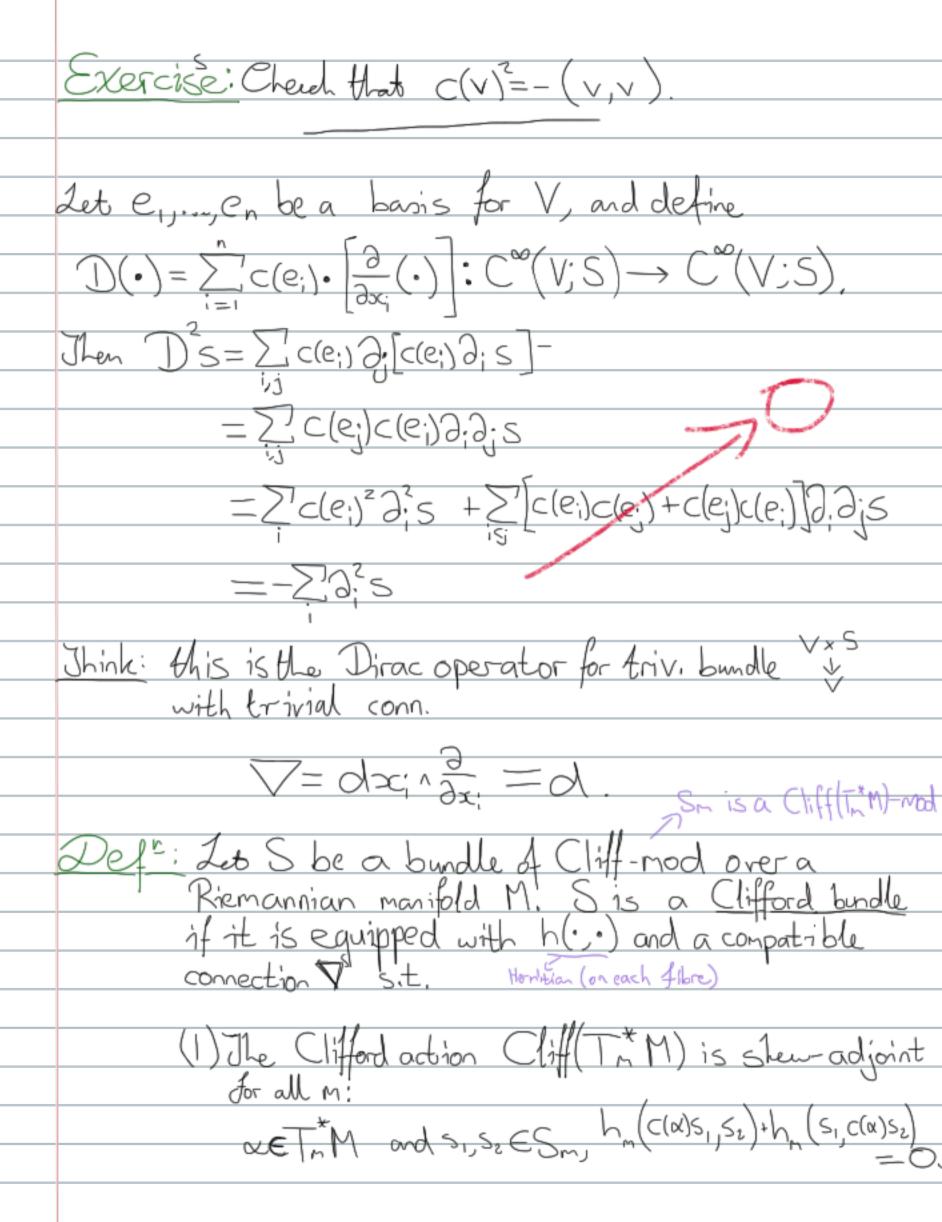
E.g. for n=2 could take the nation rep

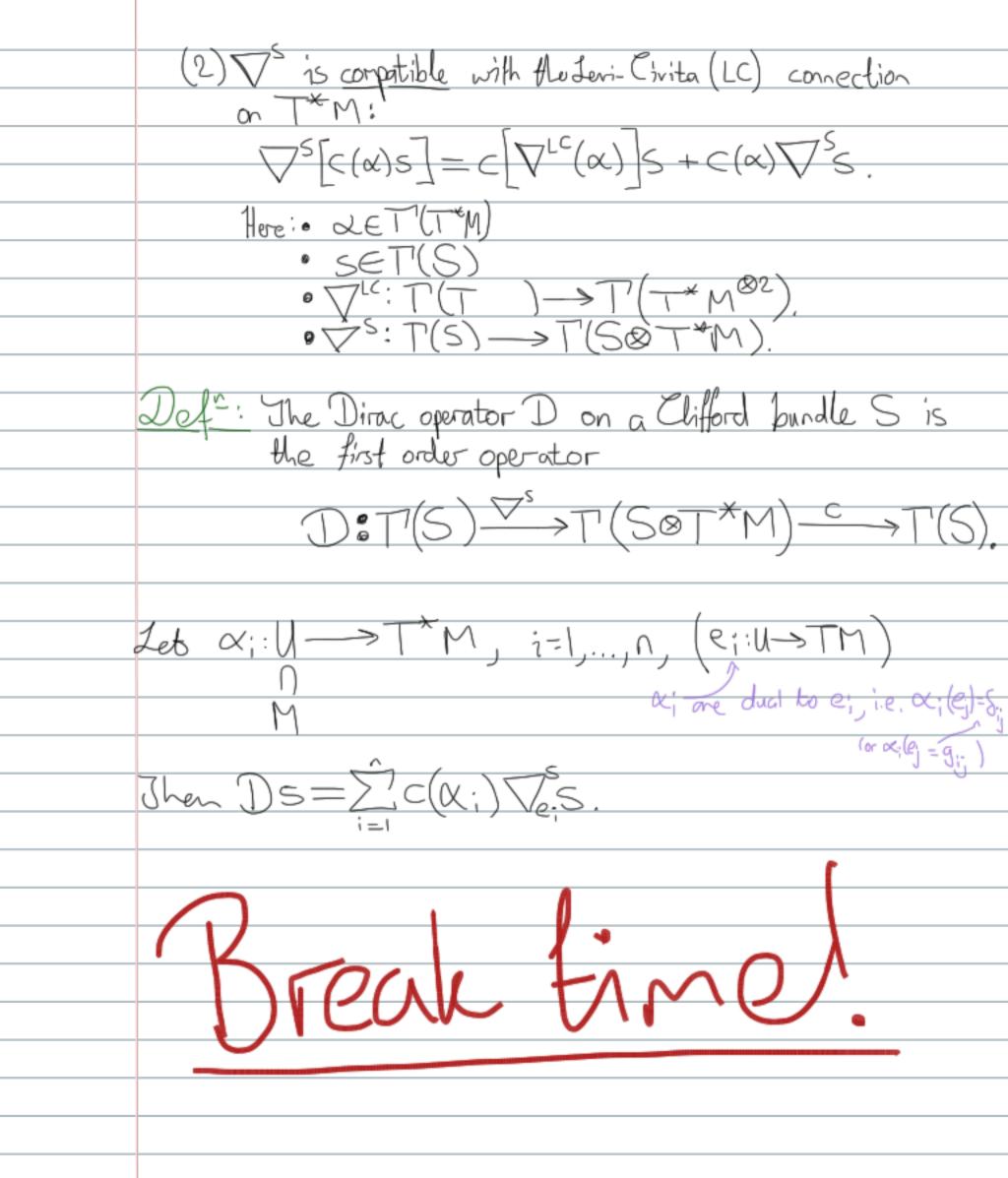
$$\gamma' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
 and $\gamma'' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.





Wirac Operators. Fix V to be a R v.s. (w/inner product), let S be a V.s. over K= R or C that is also a left module for Cliff (V). Cliff(V) C: V > End_K(S) R-linear and $(v)^2 = -(v,v) \cdot 1_s$ Renark: In the case K=C, can extend the action to a Cliff (V):= Cliff(V) & C action. Example: MV is a Cliff(V)-mod when equipped with C(V) = C(V) + I(V),where for WEV E(v)w=vnw (exterior product) 2(v) is "adjoint" to ∈(v), (1(v)q,r)=(q,E(v)r)Can take: $L(V)w = \langle V, W \rangle$ for $w \in \bigwedge V = V$, and extend by $L(V)(w_1, w_2) = (L(V)w_1) \wedge w_2 + (-1)^{\deg(w_1)} w_1 \wedge L(V)w_2$.





	Weitzentöch/Lichrevouicz Formala.
	We wish to compute D2 and see how it is related to the Hodge Laplacion.
	Laplacian.
	Let {e; }, {\alpha; } be dual orthonormal local coordinate systems.
	Assume Mese are "synchronous" coords at a point mEM:
	1) V; «; = 0, V; e; = 0 [ei,ei]=0 (at a point m).
	Compute D's at MEM in these coordinates:
	$\mathbb{D}_{S}^{2} = \sum_{c} c(\alpha_{j}) \nabla_{j} \left[c(\alpha_{i}) \nabla_{j} \right]$
	$\frac{(1)}{-\sum C(\alpha_j)C(\alpha_i)} \sqrt{j} \sqrt{j} \sqrt{j} \sqrt{j} \sqrt{j} \sqrt{j} \sqrt{j} j$
	$= -\sum_{i} \nabla_{i}^{2} + \sum_{j \leq i} C(\alpha_{j}) c(\alpha_{j}) \left[\nabla_{j} \nabla_{i} - \nabla_{j} \nabla_{j}^{2} \right]$
	=SZZ(ej,ei), curvature of VS
	End(S) (no read towory about $\nabla_{E,j}$ term due to synchronous courds)
-	Remarki - ZV; is the coordinate expression for the Laplacian VIV, where
	$\nabla^*: \Gamma(S \otimes T^*M) \rightarrow \Gamma(S)$
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is the "formal" adjoint of Vs.

When M is Spin, then F = 0. So, $D^2 = \nabla^* \nabla + \frac{1}{4} R \cdot 1_s$ If R>O and M is compact, there are no non-trivial solutions to Dis=0.

Example: Let S=/TMOC The Cliffe (TM) structure is given fibrarise by $C(\infty) = E(\infty) + 2(\infty)$, $\alpha \in T_{\infty}^{*}M$. Calculations (exercises) show this is a Clifford bundle. What is the Dirac greater for this Clifford budle? Dw=Zc(a)V,w, wET(NTMOC) $=\sum_{i}^{\prime}\alpha_{i}^{\prime}\sqrt{V_{i}}\omega+\sum_{i}^{\prime}\iota(\alpha_{i})\nabla_{i}^{\prime}\omega$. If ∇ is a forsion-free correction, $\nabla_{\mu} \omega_{\nu} = \partial_{\mu} \omega_{\nu}$. So, d=EV d:/TM->/"TM. VWET (NTMOTM)-T (NTM) Then D=d+d* where d*=*d*.