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MAT 169
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Lecture 21 Notes

These notes correspond to Section 10.3 in the text.

The Dot Product

One of the most fundamental problems concerning vectors is that of computing the angle between two given vectors. It has numerous applications in mathematics and other sciences. In physics, it plays a role in the decomposition of forces into component forces that act in various directions. In computer science, it is useful for creating two-dimensional visualizations of three-dimensional objects. In computational mathematics, it is a vital ingredient in algorithms for data fitting, approximation of functions, and other essential problems.

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be two vectors with a common initial point. Then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ form a triangle, as shown in Figure 1. By the Law of Cosines,

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Using the formula for the magnitude of a vector, we obtain

$$(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2 = (u_1^2+u_2^2+u_3^2) + (v_1^2+v_2^2+v_3^2) - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Simplifying yields

$$u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

We therefore define the *dot product*, also known as the *inner product*, of $\mathbf{u}$ and $\mathbf{v}$ to be the number $\mathbf{u}\cdot\mathbf{v}$ given by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

An equivalent definition, typically used in physics, is

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Example Let $\mathbf{u} = \langle 1, -1, 2\rangle$ and $\mathbf{v} = \langle -2, 1, 3\rangle$. Then

$$\mathbf{u}\cdot\mathbf{v} = 1(-2) + (-1)(1) + 2(3) = -2 - 1 + 6 = 3.$$

To obtain the angle θ between $\mathbf{u}$ and $\mathbf{v}$, we compute

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{3}{2\sqrt{21}},$$

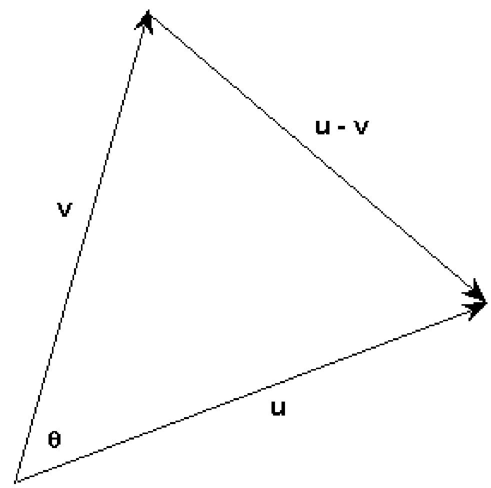

Figure 1: By the Triangle Law, the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ form a triangle. The angle between $\mathbf{u}$ and $\mathbf{v}$ is θ.

which yields $\theta \approx 1.237$ radians, or 70.893 degrees. □

Example Let $\mathbf{u}$ and $\mathbf{v}$ be vectors such that $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, and the angle between them is $\pi/3$ radians, or 60 degrees. Then

$$\mathbf{u} \cdot \mathbf{v} = 3(4) \cos \frac{\pi}{3} = 12 \frac{1}{2} = 6.$$

□

The dot product has the following properties, which can be proved from the definition.

1. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
2. Commutativity: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
3. Distributive property: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$, for any scalar c

5. $\mathbf{0} \cdot \mathbf{u} = 0$

Example By the first, second and third properties, the length of a sum of vectors $\mathbf{u} + \mathbf{v}$ can be expressed in terms of inner products as follows:

$$\begin{aligned} |\mathbf{u} + \mathbf{v}|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= |\mathbf{u}|^2 + 2\mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2. \end{aligned}$$

□

Suppose that two *nonzero* vectors $\mathbf{u}$ and $\mathbf{v}$ have an angle between them that is $\theta = \pi/2$. That is, $\mathbf{u}$ and $\mathbf{v}$ are *perpendicular*, or *orthogonal*. Then, we have

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \frac{\pi}{2} = 0.$$

On the other hand, if $\mathbf{u} \cdot \mathbf{v} = 0$, then we must have $\cos\theta = 0$, where θ is the angle between them, which implies that $\theta = \pi/2$, and therefore $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. In summary, $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Example Let $\mathbf{u} = \langle \alpha, \beta \rangle$ be any nonzero vector in V_2. Then a vector that has the same length as $\mathbf{u}$, and is orthogonal to $\mathbf{u}$ is $\mathbf{v} = \langle \beta, -\alpha \rangle$. To verify this, we compute

$$\mathbf{u} \cdot \mathbf{v} = \langle \alpha, \beta \rangle \cdot \langle \beta, -\alpha \rangle = \alpha\beta + \beta(-\alpha) = 0.$$

By the fourth property of the dot product, $\mathbf{w} = \langle -\beta, \alpha \rangle$ also satisfies $|\mathbf{w}| = |\mathbf{u}|$, and is orthogonal to $\mathbf{u}$. □

Summary

- The dot product, or inner product, of two vectors, is the sum of the products of corresponding components. Equivalently, it is the product of their magnitudes, times the cosine of the angle between them.
- The dot product of a vector with itself is the square of its magnitude.
- The dot product of two vectors is commutative; that is, the order of the vectors in the product does not matter.
- Multiplying a vector by a constant multiplies its dot product with any other vector by the same constant.
- The dot product of a vector with the zero vector is zero.
- Two nonzero vectors are perpendicular, or orthogonal, if and only if their dot product is equal to zero.