Bridging Informal and Formal Mathematical Reasoning with AI

Sean Welleck

April 14, 2025

Carnegie Mellon University

AI agents in expert domains

AI agents in expert domains

- Finance
- Medicine
- Mathematics

AI agents in expert domains

AI agents in expert domains

- Finance
- Medicine
- Mathematics
  - Open-ended dialogue
  - Come up with counterexamples
  - Help write proofs
  - ...

“Informal” mathematics

Math as raw data (text, images, ...)

- Flexible
- Widely used
- Difficult to check

Input:
Let r be a root of $x^2 + 5x + 7 = 0$. Compute $(r-1)(r+2)(r+6)(r+3)$.

LLEMMA 34B solution:
We have

$$\begin{aligned}(r-1)(r+2)(r+6)(r+3) &= (r^2+r-2)(r^2+9r+18)\\ &= r^4+10r^3+25r^2+19r-36\\ &= r^4+5r^3+7r^2+5r+7+5r^3+20r^2+14r-36\\ &= r^4+5r^3+25r^2+19r-29\\ &= (r^2+5r+7)(r^2+4r-5)\\ &= (0)(r^2+4r-5)\\ &= \boxed{0}.\end{aligned}$$

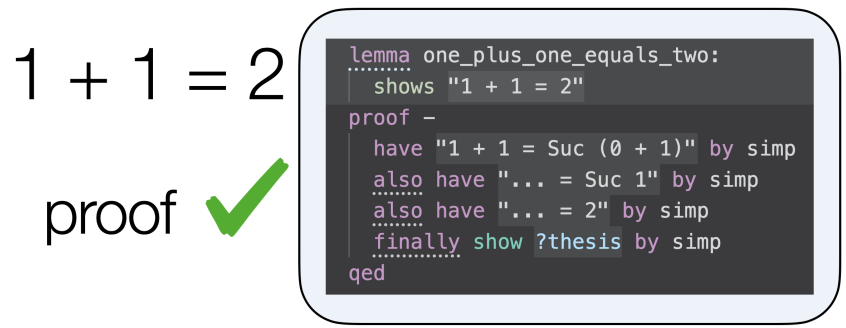
Final Answer: The final answer is 0.

Language model solution.

Formal mathematics

Math as **source code**

- Write a specification (e.g., 1+1=2)
- Write a proof
- **Automatically checked**
  - Code compiles ≡ correct proof



Math as source code.

Formal mathematics

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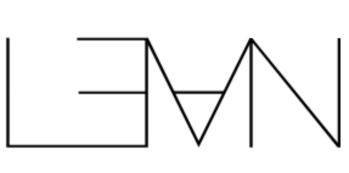


Lean

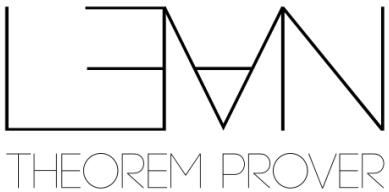
Isabelle

Coq

Theorem proving languages

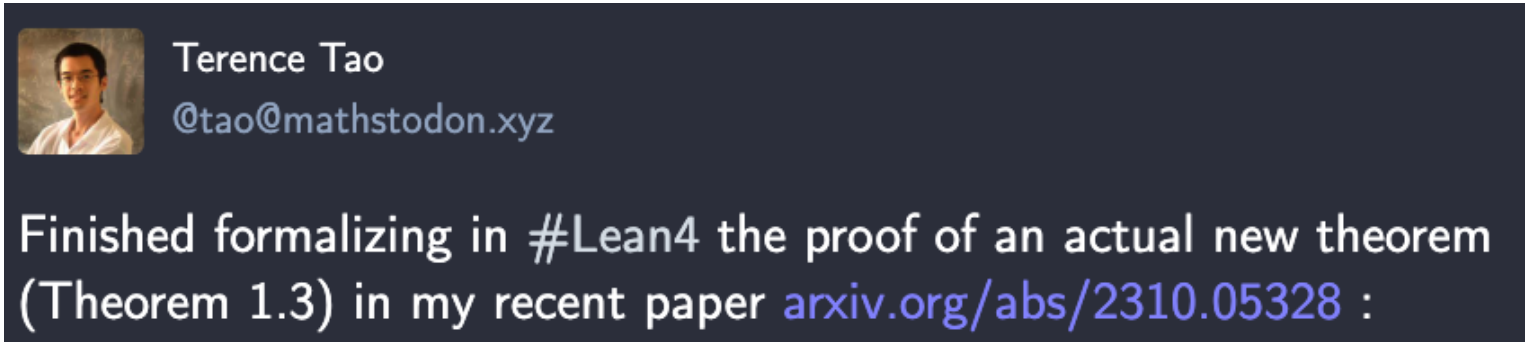
Formal mathematics (Demo)

If $R \subseteq S$ and $S \subseteq T$ then $R \subseteq T$



How is formal math used in practice?

Growing use in mathematics:



Terence Tao's Lean formalization project (October 2023)

How is formal math used in practice?

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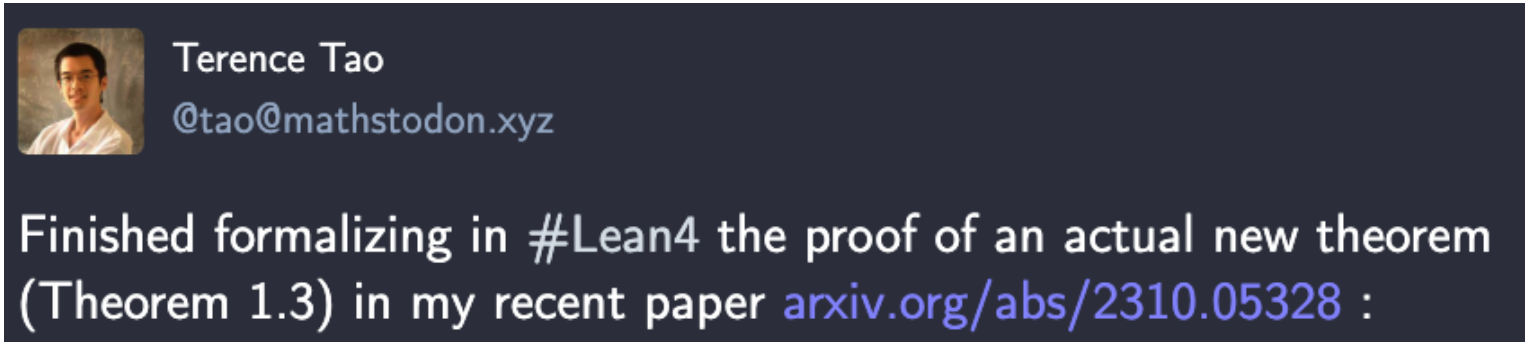

Terence Tao's Lean formalization project (October 2023)

- **Lean Mathlib** project: 1+ million lines of code, 300+ contributors
- **Courses** at CMU, Imperial College London, Fordham, JHU, ...
- **New journal (2024)**: *Annals of Formalized Mathematics*

How is formal math used in practice?

Why?[1]

- **Collaboration**
  - Break down a big problem into multiple pieces
  - Anyone can submit code to solve a piece
  - We know we can trust the code since it is automatically checked!

[1] See e.g., *Mathematics and the formal turn*, AFM Aims and Scope

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- ...

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Why is AI ∩ formal math important?

Formal math for AI

- **Verifiable**
  - Prevent incorrect math and code generation
  - Feedback signal for learning

Why is AI ∩ formal math important?

Formal math for AI

- **Verifiable**
  - Prevent incorrect math and code generation
  - Feedback signal for learning
- Tests **reasoning**
  - From easy: $1+1 = 2$
  - To hard: Fermat's Last Theorem

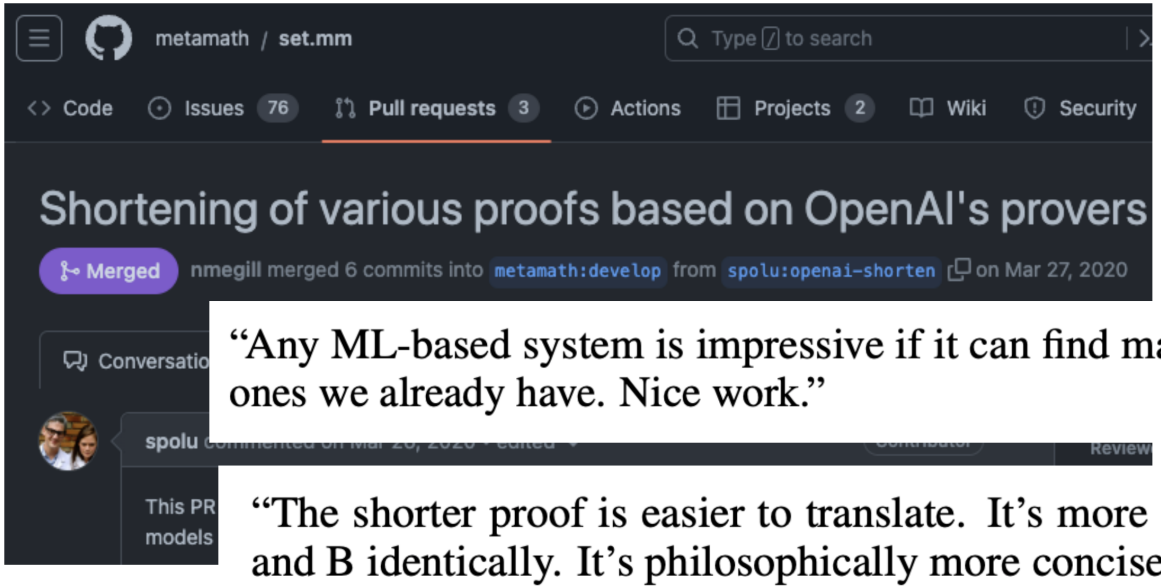
Generative Language Modeling for Automated Theorem Proving

Stanislas Polu
OpenAI
spolu@openai.com

Ilya Sutskever
OpenAI
ilyasu@openai.com

gpt-f (2020)

LLMs ∩ formal math



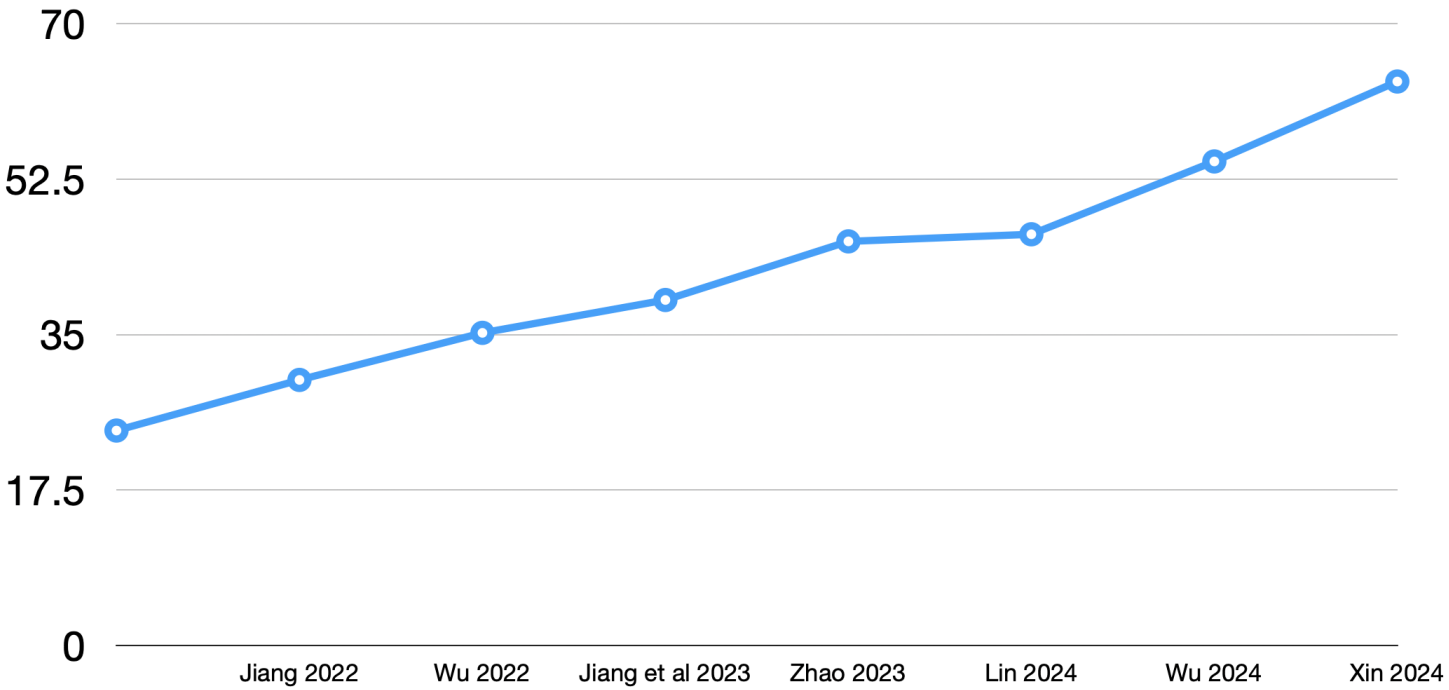
"Any ML-based system is impressive if it can find many shorter proofs than the ones we already have. Nice work."

"The shorter proof is easier to translate. It's more symmetric in that it treats A and B identically. It's philosophically more concise in that it doesn't rely on the existence of a universal class of all sets."

gpt-f (2020)

LLMs ∩ formal math

Rapid progress in methods based on language models:



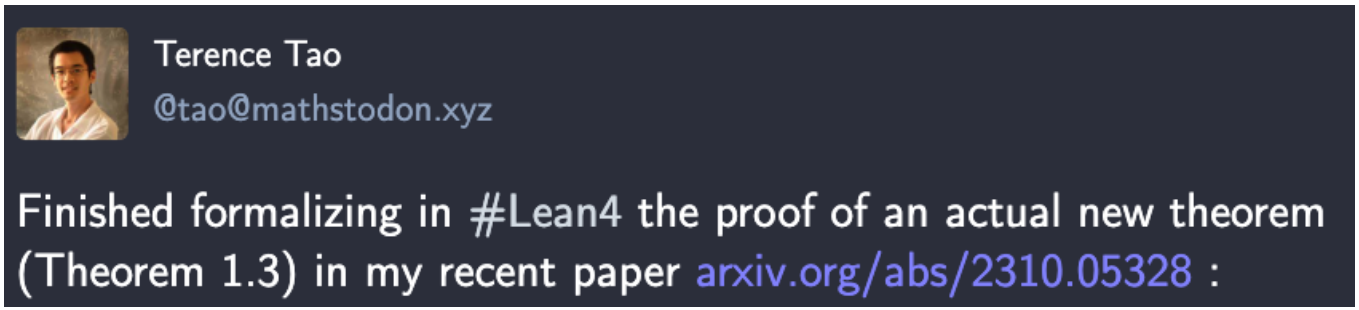
miniF2F benchmark performance, 2022-2024

LLMs ∩ formal math

```
theorem imo_1960_p2 (x : ℝ) (h₀ : 0 ⩽ 1 + 2 * x) (h₁ : (1 - Real.sqrt (1 + 2 *
    x)) ^ 2 ≠ 0)
    (h₂ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 < 2 * x + 9) : -(1 / 2)
   ⩽ x ∧ x < 45 / 8 := by
  norm_num at h₀ h₁ h₂
  have h₃ : 0 ⩽ 1 + 2 * x := by linarith
  have h₄ : 0 < 1 + Real.sqrt (1 + 2 * x) := by
    nlinarith [Real.sqrt_nonneg (1 + 2 * x)]
  have h₅ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 < 2 * x + 9 := by
    linarith
  have h₆ : 1 - Real.sqrt (1 + 2 * x) ≠ 0 := by
    intro h
    apply h₁
    nlinarith
  have h₇ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 = (1 + Real.sqrt (1 +
    2 * x)) ^ 2 := by
    field_simp [h₆]
    nlinarith [sq_sqrt (show 0 ⩽ 1 + 2 * x by linarith)]
  rw [h₇] at h₅
  constructor <;> nlinarith [sq_sqrt (show 0 ⩽ 1 + 2 * x by linarith)]
```

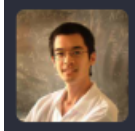
Generated International Math Olympiad solution in Lean
(DeepSeek Prover-1.5B, Xin et al 2024)

LLMs ∩ formal math



Terence Tao's Lean formalization project (October 2023)

LLMs ∩ formal math



Terence Tao
@tao@mathstodon.xyz

Finished formalizing in #Lean4 the proof of an actual new theorem (Theorem 1.3) in my recent paper arxiv.org/abs/2310.05328 :

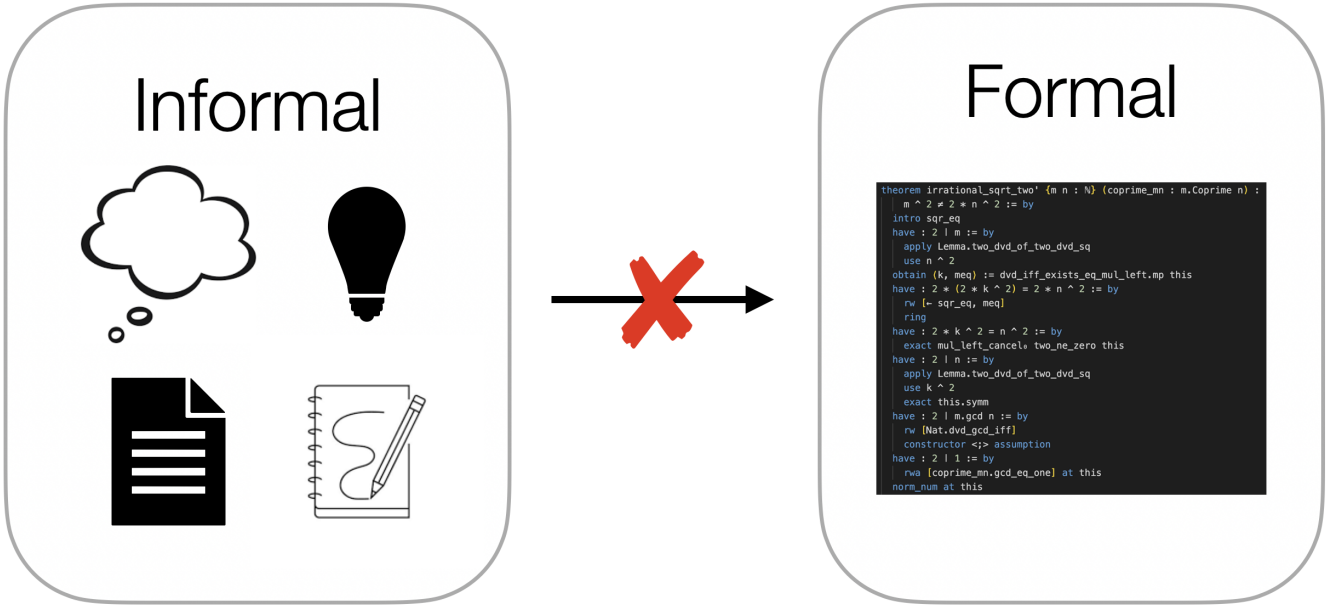
The ability of Github copilot to correctly anticipate multiple lines of code for various routine verifications, and inferring the direction I want to go in from clues such as the names I am giving the theorems, continues to be uncanny.

Terence Tao's Lean formalization project (October 2023)

So...why don't people and AI always use formal math?

Key challenge: the informal-formal gap

Informal ideas, intuitions, and even proofs are difficult to express formally:



- Each step of reasoning needs to be specified in detail
- Requires a deep knowledge of the formal system

Bridging Informal and Formal Mathematical Reasoning

Informal

Formal

```
theorem irrational_sqrt_two' {m n : ℕ} (coprime_mn : m.Coprime n) :
  m ^ 2 ≠ 2 * n ^ 2 := by
intro sqr_eq
have : 2 ∣ m := by
  apply Lemma.two_dvd_of_two_dvd_sq
  use n ^ 2
obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
  rw [← sqr_eq, meq]
  ring
have : 2 * k ^ 2 = n ^ 2 := by
  exact mul_left_cancel₀ two_ne_zero this
have : 2 ∣ n := by
  apply Lemma.two_dvd_of_two_dvd_sq
  use k ^ 2
  exact this.symm
have : 2 ∣ m.gcd n := by
  rw [Nat.dvd_gcd_iff]
  constructor <;> assumption
have : 2 ∣ 1 := by
  rwa [coprime_mn.gcd_eq_one] at this
norm_num at this
```

Flexible

Easy to check

This talk: Bridging Informal and Formal Mathematical Reasoning with AI

This talk: Bridging Informal and Formal

1. Informal thoughts

2. Informal provers

3. Research-level mathematics

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1. Informal thoughts
    - Training models to think informally
        - Lean-STaR

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1. Informal thoughts
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    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer

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This talk: Bridging Informal and Formal

1. Informal thoughts
    - Training models to think informally
        - Lean-STaR

2. Informal provers
    - Sketching proofs and filling in the gaps
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3. Research-level mathematics
    - Assisting in research-level projects
        - Practical tools
        - MiniCTX

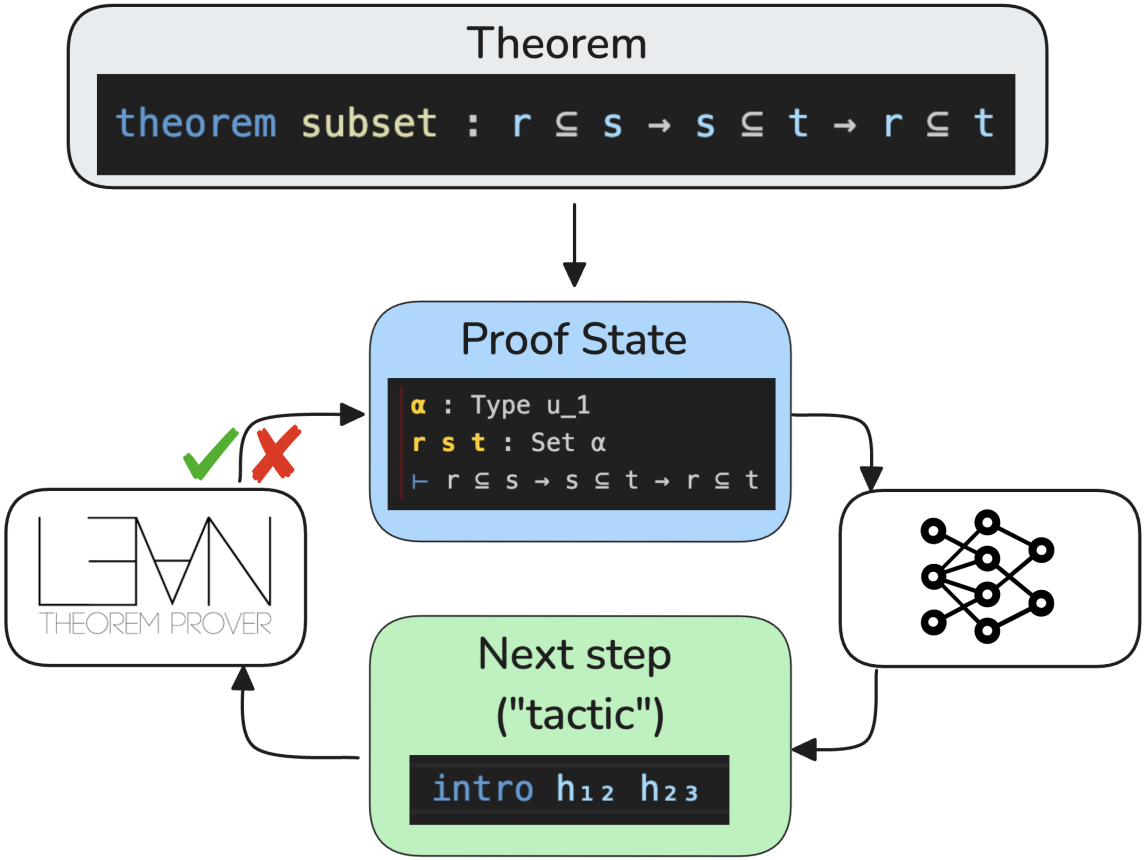
I: Informal thoughts

1. Training models to "think" — Lean-STaR

Lean-STaR: Learning to Interleave Thinking and Proving
Haohan Lin, Zhiqing Sun, Yiming Yang, Sean Welleck
ICLR 2025 (**Spotlight**)

1. Training models to “think” — Neural theorem proving

Neural theorem proving



- Math as checkable code
- Proof: sequence of (state, step)

1. Training models to "think" — Neural theorem proving

Language model-based proving:

- **Train** a model $p_\theta(y|x)$ on a dataset $\mathcal{D} = \{(x, y)\}$, e.g.,
  - x: proof state
  - y: next tactic (next "step")
  - $\mathcal{D}$: extracted from theorems and proofs

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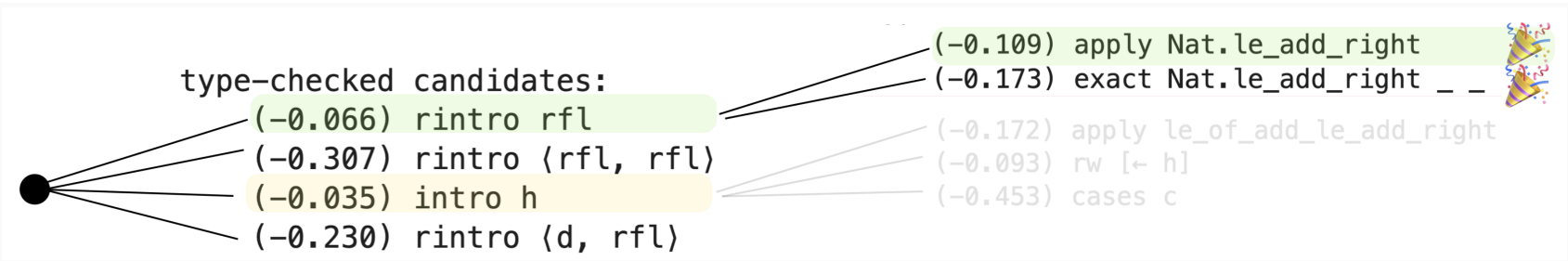
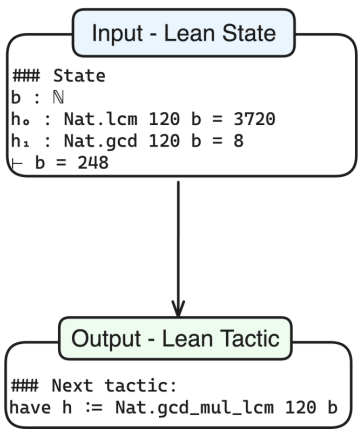
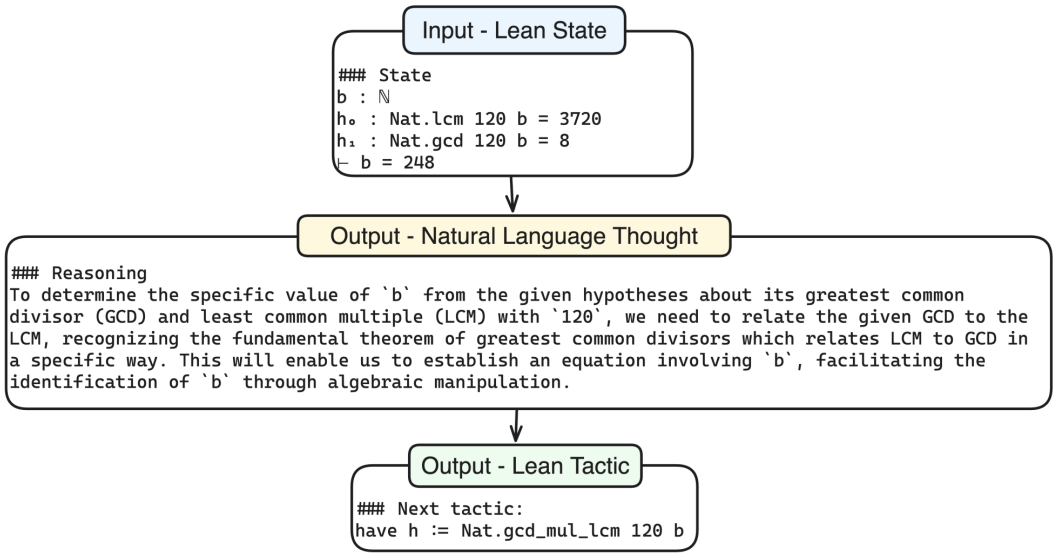
  - x: proof state
  - y: next tactic (next "step")
  - $\mathcal{D}$: extracted from theorems and proofs
- **Generate** proofs:



Best-first search

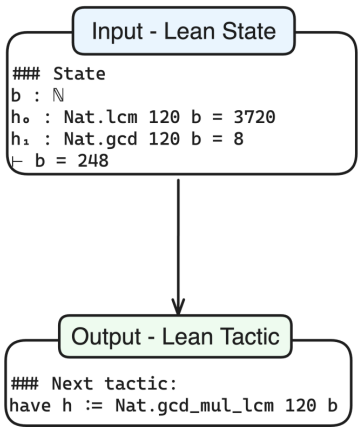
1. Training models to “think” — Lean-STaR

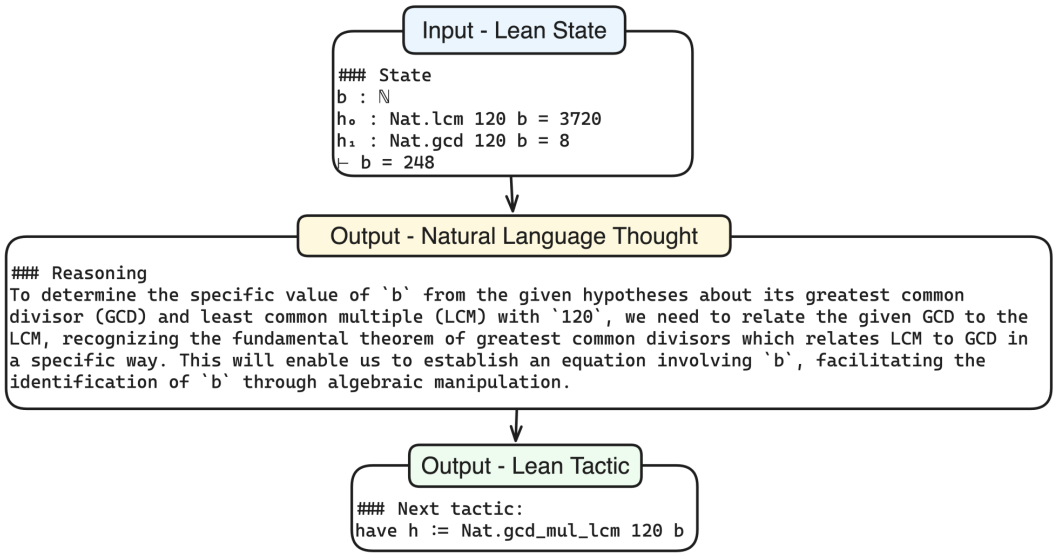




Can we train a model to “think” before each step of formal reasoning?

1. Training models to “think” — Lean-STaR





Why?

- Plan proof steps
- Diversify search space
- More tokens can give more computational capacity

1. Training models to "think" — Lean-STaR

Lean-STaR (Self-taught reasoner[2])

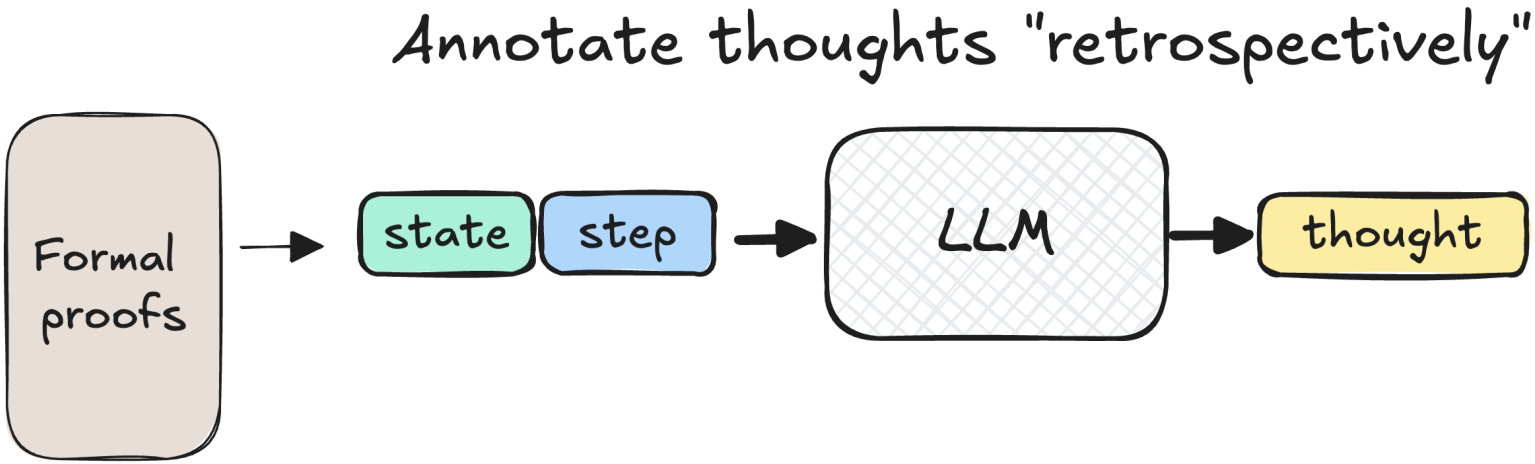
Learn to generate thoughts via reinforcement learning

1. Initialization
2. Reinforcement learning

[2]Inspired by *STaR: Bootstrapping Reasoning with Reasoning*, Zelikman et al 2022

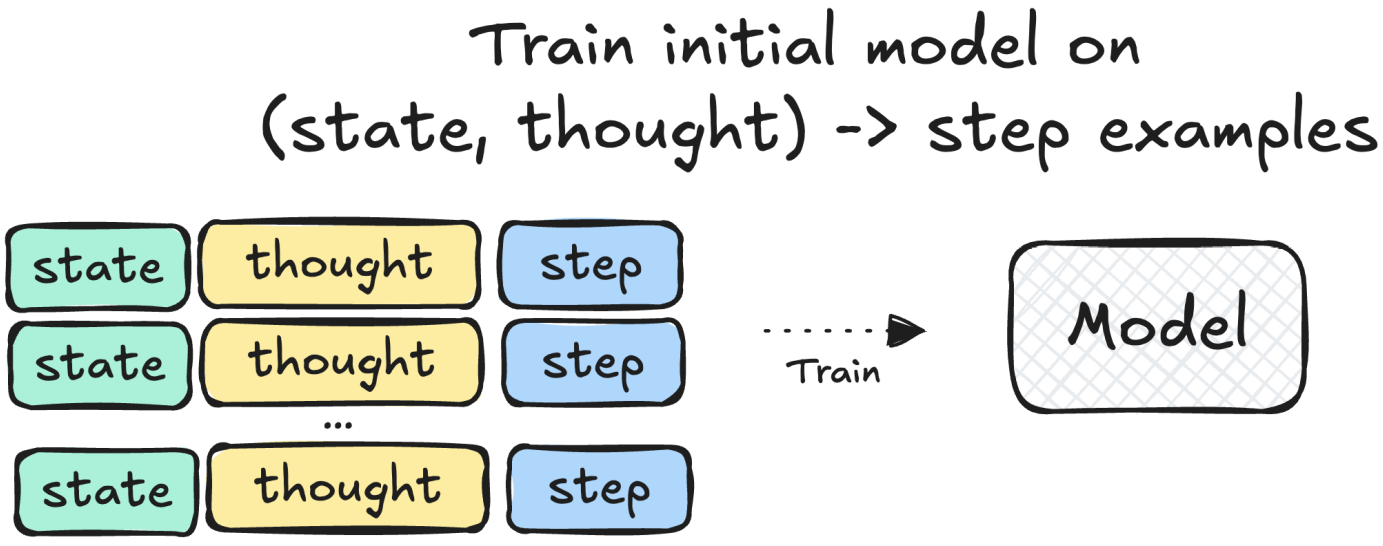
1. Training models to “think” — Lean-STaR

1. **Initialization**



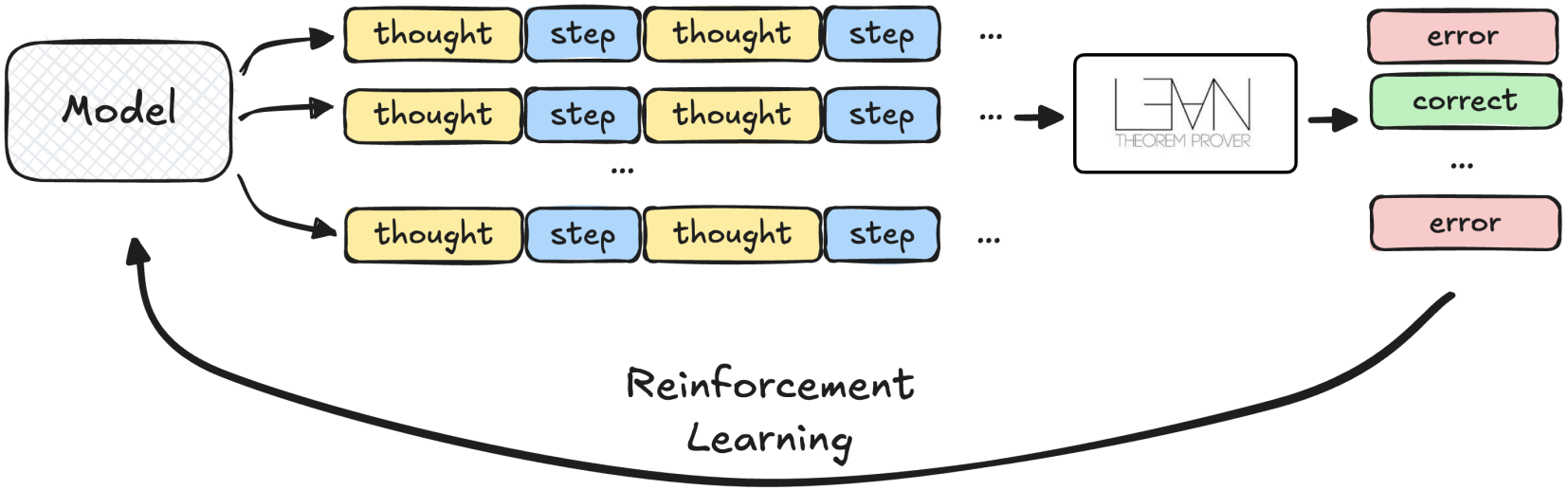
1. Training models to “think” — Lean-STaR

1. **Initialization**



1. Training models to “think” — Lean-STaR

2: **Reinforcement learning**



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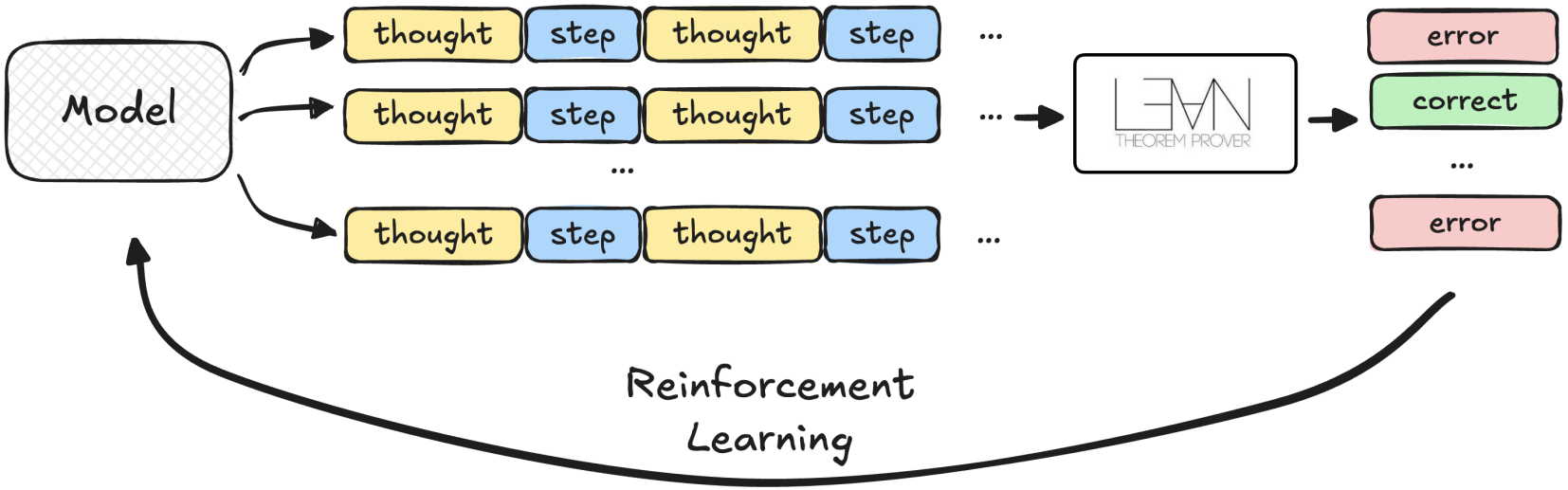
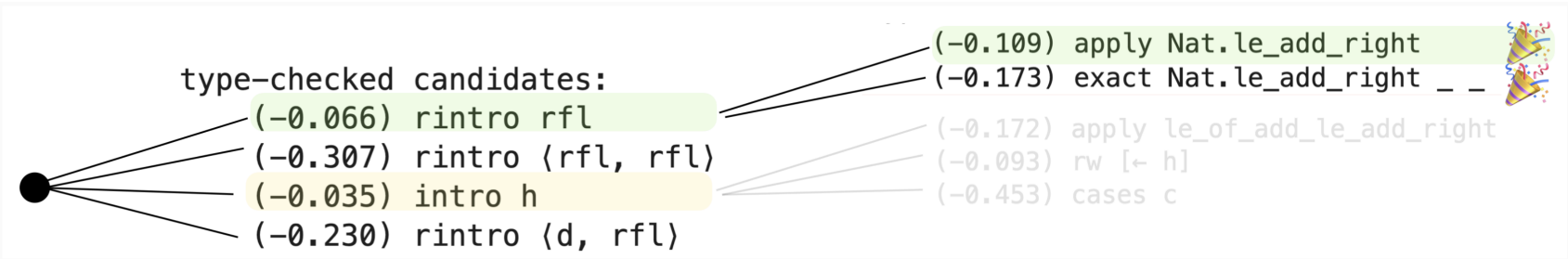


Need:

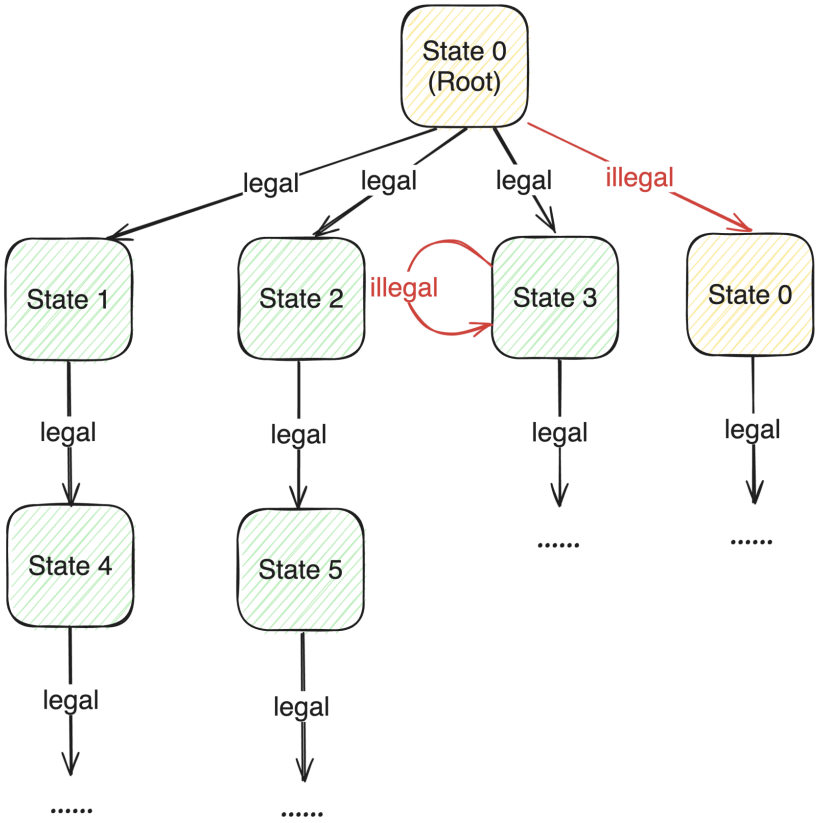
- Method to generate proofs
- Learning algorithm

1. Training models to “think” — Lean-STaR



Best-first search: difficult to score (thought, tactic) candidates

1. Training models to "think" — Lean-STaR



New sampling method

1. Training models to "think" — Lean-STaR

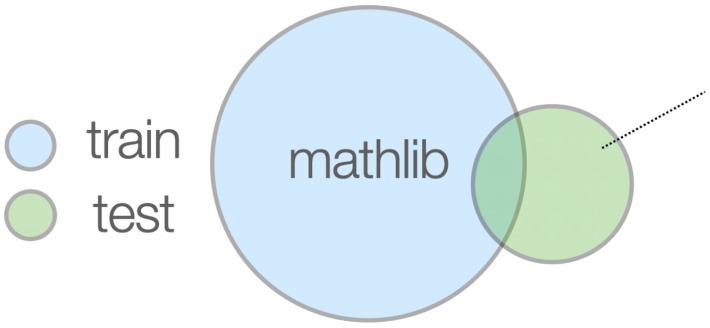
Algorithm: train on the successful proofs, and repeat:[3]

- Collect (state, thought, tactic) from successful proofs
- Train a new model $p_\theta^1(\mathit{thought}, \mathit{tactic} | \mathit{state})$
- Generate proofs
- ...

[3] I.e. Expert Iteration [Polu et al 2022], Rest-EM [Singh et al 2024]

1. Training models to "think" — Lean-STaR

- **miniF2F** [1]: competition problems (AMC, AIME, IMO)

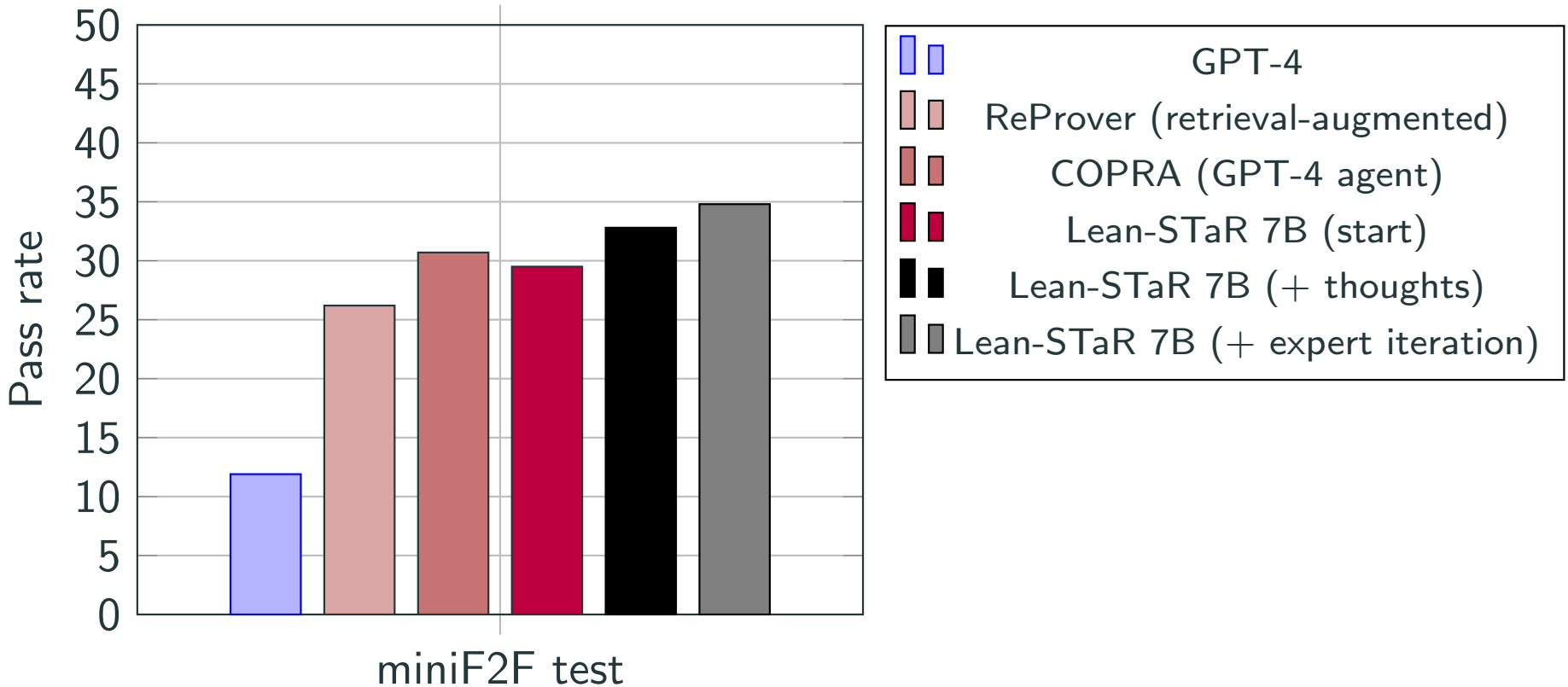


Problem 1959 IMO Problems/Problem 1

Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

```
theorem imo_1959_p1
  (n : ℕ)
  (h₀ : 0 < n) :
  Nat.gcd (21*n + 4) (14*n + 3) = 1 := by sorry
```

1. Training models to “think” — Lean-STaR



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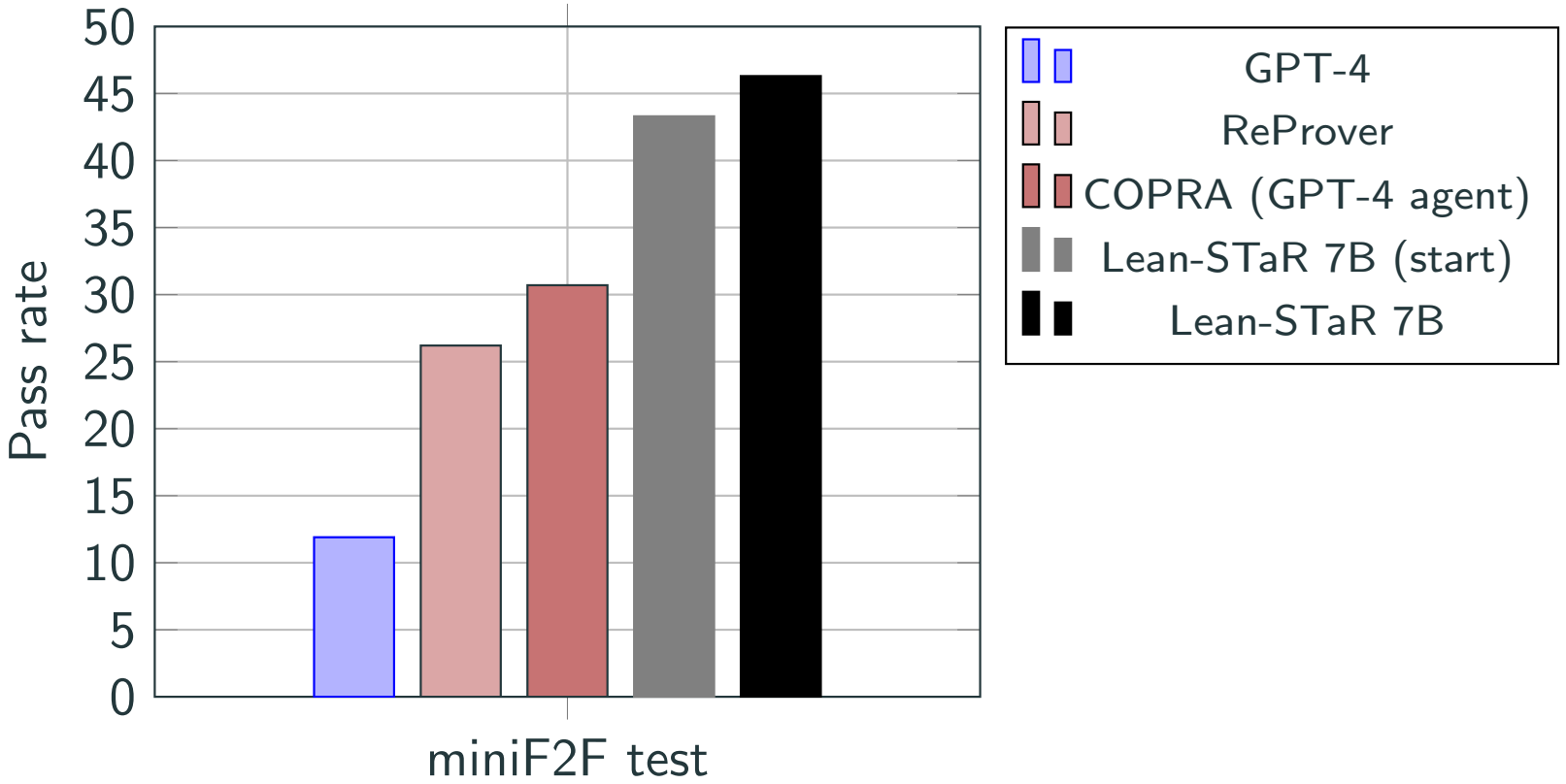

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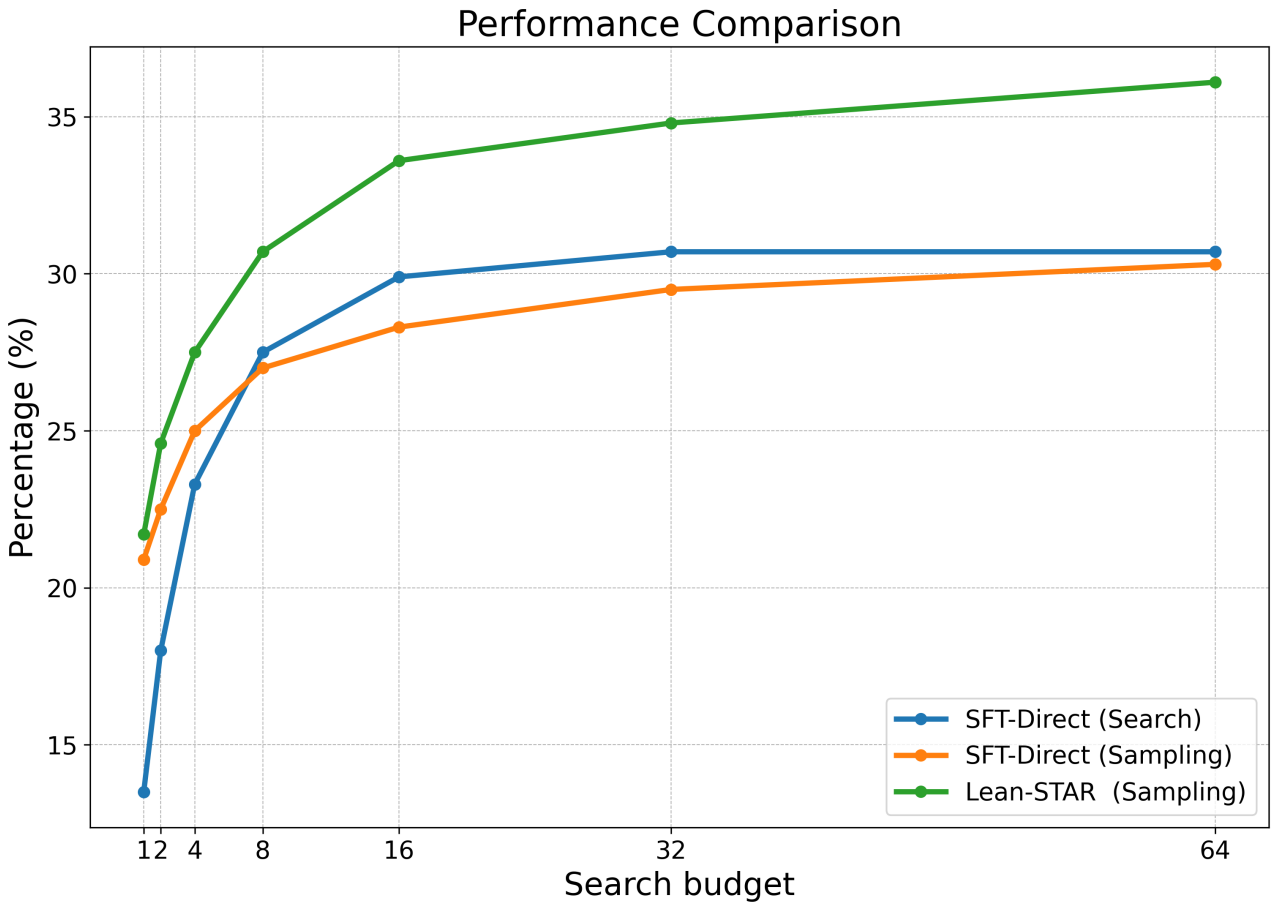


Increasing the search budget is more effective with thoughts

1. Training models to “think” — Lean-STaR

```
theorem amc12a_2002_p6 (n : ℕ) (h₀ : 0 < n)
    : ∃ m, m > n ∧ ∃ p, m * p ≤ m + p :=
    use n + 1, by linarith only [h₀] -- Reasoning: To demonstrate the existence of
    such an `m` and `p` that satisfy the given inequalities, we leverage the
    property of natural numbers that ensures an increase in `n` from `0` leads to
    the possibility of finding such `m` and `p`. This approach is supported by the
    fundamental property of natural numbers and their arithmetic operations,
    suggesting that for a given `n` greater than `0`, there exists a successor `m`
    and a number `p` that satisfy the conditions outlined.
    exact ⟨0, by simp ⟩ -- Reasoning: To demonstrate that there exists an \(n \in ℕ
    \) such that \((n + 1) * p \leq n + 1 + p\) holds for some \(p \in ℕ\), we
    recognize that setting \(p = 0\) satisfies the inequality by the properties of
    zero in arithmetic. This is because \(n + 1 > 0\) by assumption, ensuring the
    left-hand side is non-negative and the right-hand side simplifies to \(n +
    1\), making the inequality trivially true.
```

Example generated thoughts and proof from Lean-STaR

1. Training models to “think” — Lean-STaR

```
theorem aime_1983_p2 (x p : ℝ) (f : ℝ → ℝ) (h₀ : 0 < p ∧ p < 15) (h₁ : p ≤ x ∧ x
    ≤ 15)
    (h₂ : f x = abs (x - p) + abs (x - 15) + abs (x - p - 15))
    : 15 ≤ f x :=
    cases' h₁ with h₁ h₂ -- Reasoning: Given a function \(f\) defined over the real
    numbers, where \(f(x)\) equals the sum of the absolute values of the
    differences \(x - p\), \(x - 15\), and \(x - p - 15\), along with conditions
    on \(p\) and \(x\) that ensure \(x\) lies between \(p\) and \(15\) and \(p\)
    is positive, we aim to show that \(f(x) \geq 15\). To demonstrate this, we
    consider all possible cases for the position of \(x\) relative to \(p\) and
    \(15\), specifically at \(p\) itself, between \(p\) and \(15\), and at \(15\).
    By evaluating \(f(x)\) for each case, we can establish a lower bound of
    \(f(x)\) that is greater than or equal to \(15\).
    simp [h₀, abs_of_nonneg (sub_nonneg.2 h₁), abs_of_nonneg (sub_nonneg.2 h₂), *]
    -- Reasoning: The objective is to show that \(f(x)\) is at least 15 for \(x\)
    within the bounds \(p \le x \le 15\), where \(p\) is positive but less than
    15. The provided expression for \(f(x)\) suggests using properties of absolute
    values and inequalities to simplify and analyze \(f(x)\). By applying
    simplification rules for absolute values and inequalities, along with the
    given bounds, we aim to demonstrate the desired inequality \(f(x) \ge 15\).
    simp only [abs_of_nonneg] at * -- Reasoning: The objective is to establish an
    inequality involving an expression with absolute values and a sum, where all
    terms are functions of \(x\). By recognizing that the absolute value functions
    are always non-negative, we can simplify the given expression by removing the
    absolute value symbols, thus focusing on the arithmetic manipulation of the
    terms to demonstrate the inequality.
    simp only [abs_of_nonneg, sub_add_eq_add_sub] at * -- Reasoning: The objective
    is to demonstrate that the expression `15 ≤ x - p + |x - 15| + |x - p - 15|`
    holds given the conditions on `p`, `x`, and the function `f`. Since the
    expression in question involves absolute values which inherently involve
    non-negativity, manipulating these expressions to highlight their
    non-negativity or equality is a logical initial step. This involves
    recognizing that the absolute value of a difference can be rewritten as the
    difference of absolute values, which simplifies the manipulation of the
    expression to prove the inequality.
    congr 1 -- Reasoning: To establish the inequality `15 ≤ x + |x - 15| + |x - p
    - 15| - p`, we recognize that the left-hand side of the inequality involves a
    mix of additions and subtractions of real numbers. By introducing a congruence
    transformation that simplifies operations involving absolute values, we aim to
    align both sides of the inequality in a form that directly reflects the
    properties of real numbers under these operations.
    simp only [abs_of_nonpos (sub_nonpos_of_le h₂), abs_of_nonneg (sub_nonneg_of_le
    h₁)] -- Reasoning: The objective is to prove an inequality involving absolute
    values and subtractions that aligns the given conditions and constraints.
    Simplification using the properties of absolute values and the provided bounds
    can directly lead to the desired conclusion by expressing the absolute values
    in terms of their underlying expressions and applying the given constraints.
    simp [abs_of_pos, *] -- Reasoning: To prove the given inequality involving
    absolute values, simplification using algebraic properties and given
    hypotheses will directly lead us to the desired inequality by recognizing the
    algebraic manipulation and application of the known bounds `h₁` and `h₂`, as
    well as the non-negativity of `p`.
    rw [abs_of_nonpos] -- Reasoning: The objective is to demonstrate that `15 ≤ 15
    + |x - p - 15| - p` under the given constraints on `p` and `x`. Given `p < 15`
    and `p ≤ x ≤ 15`, it's apparent that `p - 15` is non-positive. This
    observation allows us to apply the property that the absolute value of a
    non-positive number is equal to its negative value, thereby simplifying the
    expression on the right-hand side of the inequality.
    all_goals linarith -- Reasoning: The goal involves demonstrating a relationship
    between expressions involving inequalities and subtraction. The simplification
    of these expressions into a form that directly compares their numerical values
    can lead to a straightforward application of known inequalities and algebraic
    properties, demonstrating the inequality's validity under the given conditions.
```

Example generated thoughts and proof from Lean-STaR

1. Training models to "think" — Lean-STaR

Recap: **Lean-STaR**

- Learn to generate "thoughts" before each step
- Benefits from scaling up the inference budget

This talk: Bridging Informal and Formal

1. Informal thoughts
2. **Informal provers**
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer
3. Research-level mathematics

II: Informal and formal provers

Combining informal and formal provers

Overall goal: combine the high-level reasoning of neural provers with the low-level reasoning of classical symbolic provers.

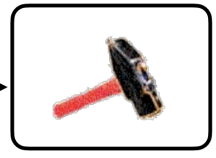
Motivation: high-level sketch and low-level steps

```
have c1: "10*280 = n*40"
using assms
    by (smt (z3) prod_gcd_lcm_nat)
then have c2: "n = 10*280/40"
    by auto
then show ?thesis
    by auto
```

A proof with a high-level sketch and low-level proof steps.

Symbolic provers: Sledgehammer

```
have c1: “10*280 = n*40”
using assms
```



```
have c1: “10*280 = n*40”
using assms
by (smt (z3) prod_gcd_lcm_nat)
```

Sledgehammer [Paulson 2010] calls out to external automated provers.

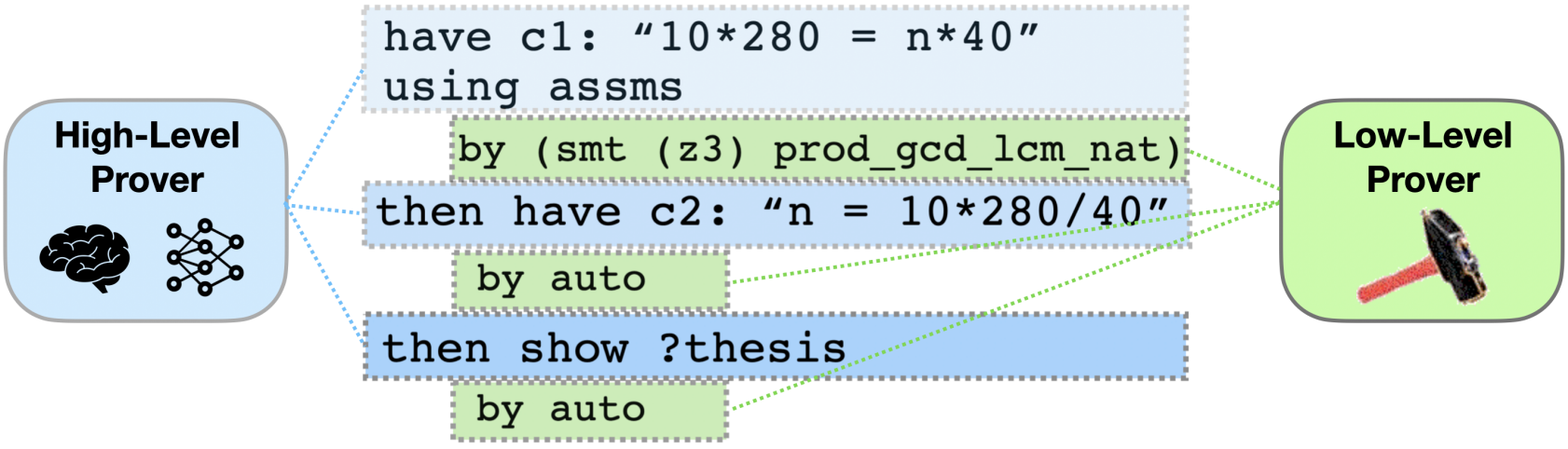
- First-order logic, higher-order logic, SMT

Symbolic provers: Sledgehammer

Sledgehammer [Paulson 2010]

Struggles due to the large search space of possible proofs

Idea: combine high-level and low-level reasoning



Example: Draft-Sketch-Prove

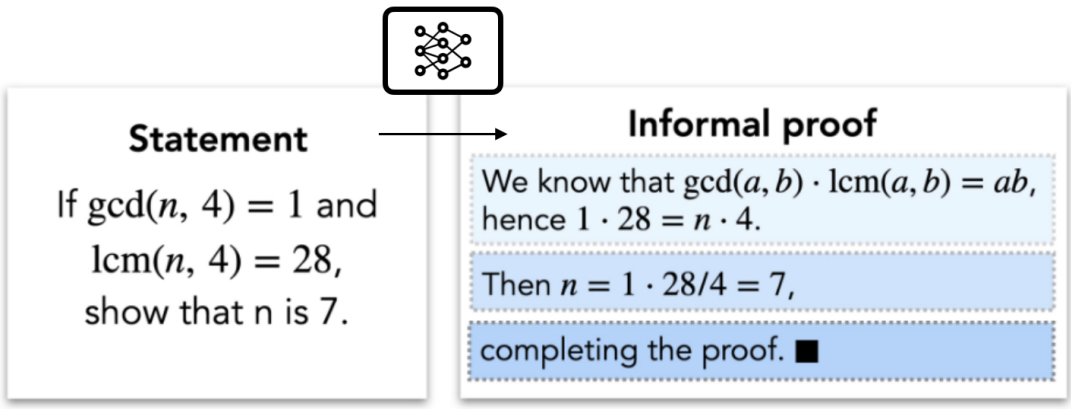
Draft, Sketch, Prove: Guiding Formal Theorem Provers with Informal Proofs

Albert Q. Jiang, Sean Welleck, Jin Peng Zhou
Jiacheng Liu, Wenda Li, Mateja Jamnik
Guillaume Lample, Yuhuai Wu
*ICLR 2023 (**Oral**)*

Draft, Sketch, Prove

Given informal theorem x_I,
formal theorem x_F

1. **Draft** $y_I \sim p(\cdot|x_I)$

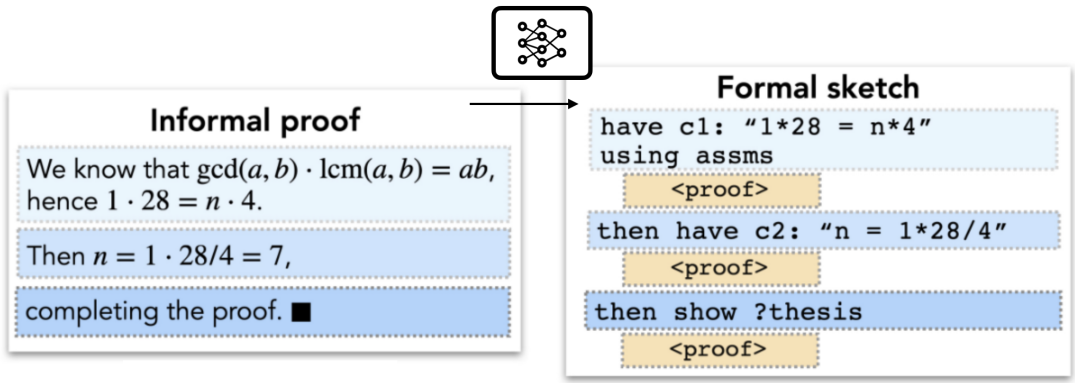


Human-written or LLM-generated draft

Draft, Sketch, Prove

Given informal theorem x_I,
formal theorem x_F

1. Draft $y_I \sim p(\cdot|x_I)$
2. **Sketch** $z_F \sim p(\cdot|x_F, x_I, y_I)$

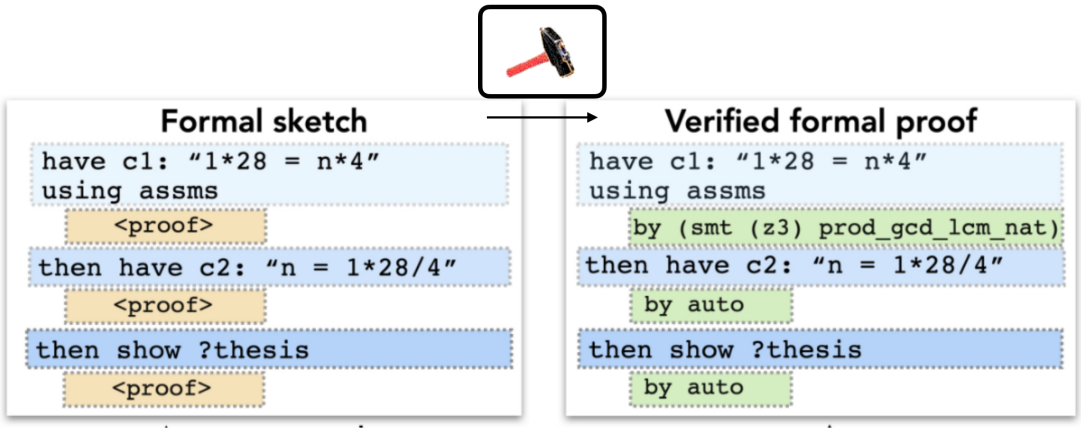


LLM-generated sketch

Draft, Sketch, Prove

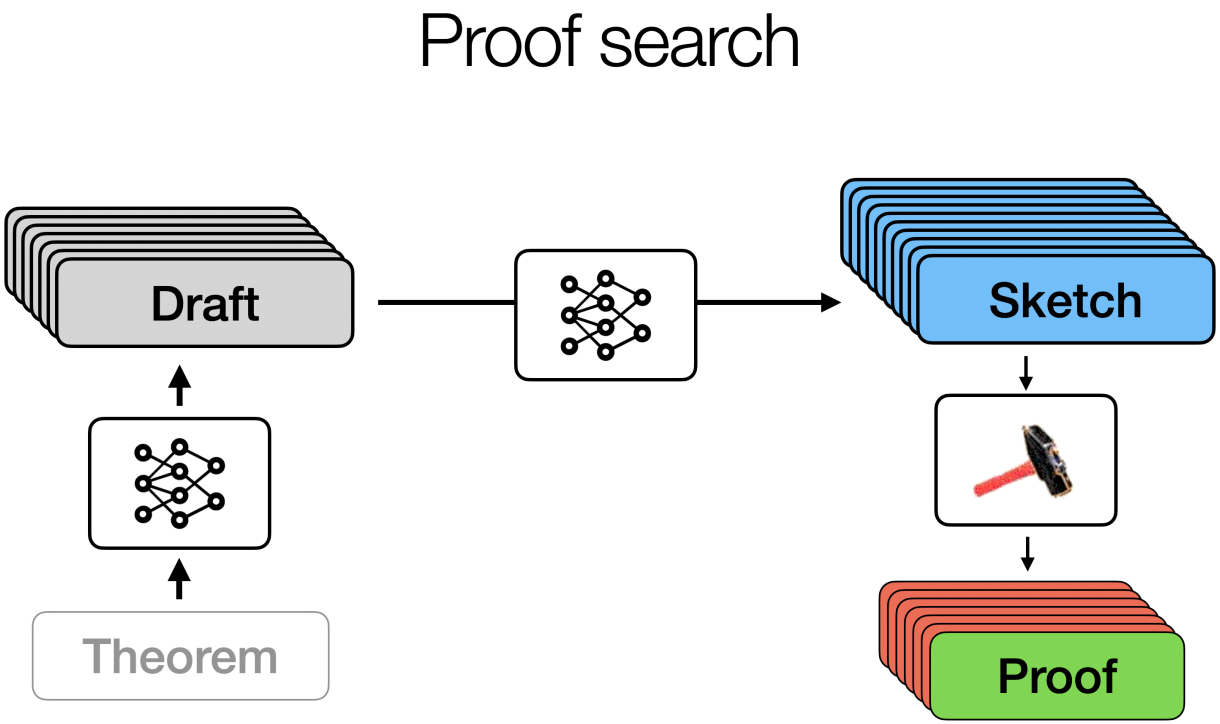
Given informal theorem x_I,
formal theorem x_F

1. Draft $y_I \sim p(\cdot|x_I)$
2. Sketch $z_F \sim p(\cdot|x_F, x_I, y_I)$
3. **Prove** $y_F = f(x_F, z_F)$



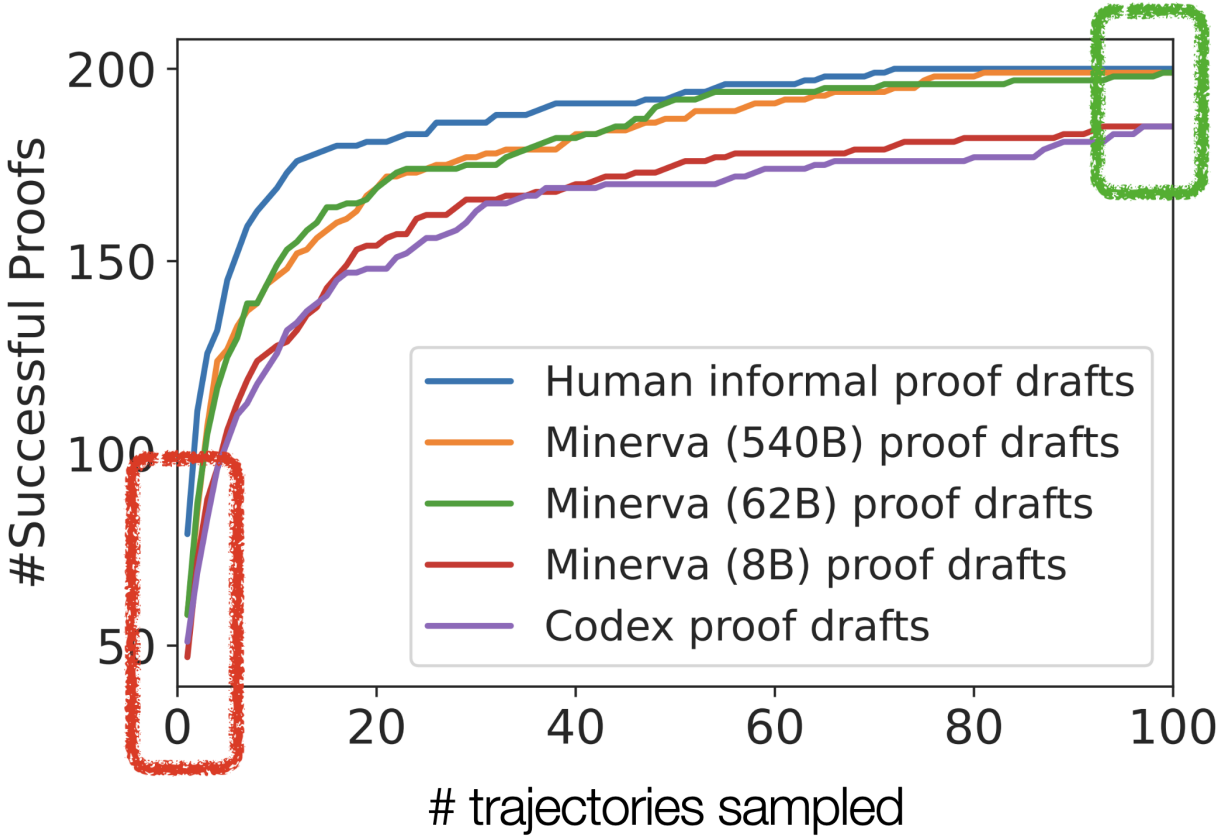
Symbolic prover (*Sledgehammer*) fills in the gaps

Draft-sketch-prove



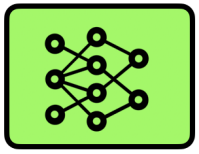
Proof search

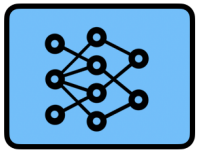
Draft-sketch-prove



Inference-time proof search scaling

Draft-sketch-prove





Statement: Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

Informal Proof (Minerva 62B):

We must show that $\gcd(21n+4, 14n+3) = 1$.
The Euclidean algorithm gives

$$21n+4 = 1 \cdot (14n+3) + 7n + 1$$
$$14n+3 = 2 \cdot (7n+1) + 1.$$

Since $\gcd(7n+1, 1) = 1$, we have $\gcd(21n+4, 14n+3) = 1$.

Formal Proof:

```
theorem imo_1959_p1:
  fixes n :: nat
  shows "gcd (21*n + 4) (14*n + 3) = 1"
proof -
  (* The Euclidean algorithm gives
  21n+4=1\cdot(14n+3)+7n+1
  14n+3=2\cdot(7n+1)+1. *)
  have c0: "21*n + 4 = 1*(14*n + 3) + 7*n + 1"
    by auto  [ATP]
  have c1: "14*n + 3 = 2*(7*n + 1) + 1" using c0
    by auto  [ATP]

  (* Since \gcd(7n+1,1)=1, we have \gcd(21n+4,14n+3)=1. *)
  then have "gcd (7*n + 1) 1 = 1"
    using c1
    by auto  [ATP]
  then have "gcd (21*n + 4) (14*n + 3) = 1"
    using c1
    by (smt (z3) BitM_plus_one ab_semigroup_add_class.add_ac(1)
    add.assoc c0 gcd.commute gcd_add2 gcd_add_mult mult_numeral_1
    numeral_One numeral_eq_Suc numerals(1) semiring_norm(3))  [ATP]
  then show ?thesis
    using c1
    by blast  [ATP]
qed
```

International Math Olympiad problem

Draft, Sketch, Prove

Demo notebook: github.com/cmu-l3/ntptutorial-II

Next: combining high-level and low-level reasoning in Lean

Recap:

- Draft-Sketch-Prove: generate high-level sketches and fill in low-level details
- Isabelle's *Sledgehammer* calls out to external provers to fill in the details

Next: combining high-level and low-level reasoning in Lean

Recap:

- Draft-Sketch-Prove: generate high-level sketches and fill in low-level details
- Isabelle's *Sledgehammer* calls out to external provers to fill in the details

Next: can we enable a Sledgehammer for Lean?

Premise Selection for a Lean Hammer

Thomas Zhu, Joshua Clune

Jeremy Avigad, Albert Q. Jiang, Sean Welleck

Under Review 2025

Recap: Hammers

Recap: What is a hammer?

- Integrates an automated theorem prover into a formal proof checker
  - Formal proof checker: Lean, Isabelle, Coq
  - Automated theorem prover: higher-order logic provers, SMT solvers

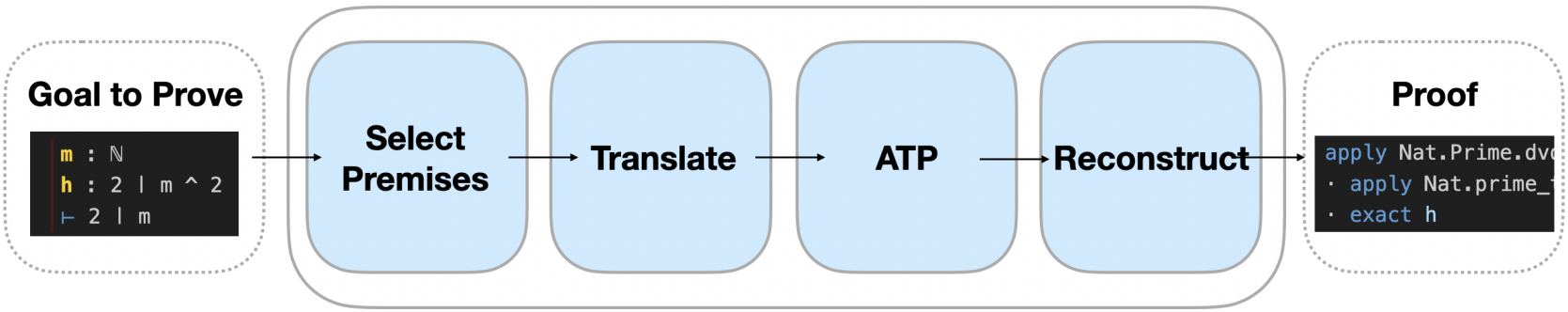
Key challenge: Premise selection

Automated theorem provers (ATPs) struggle with the large search space of possible proofs.

- **Premise selection**: select a small subset of theorems that are likely to be useful for proving a given theorem.
- Cuts down the search space.

A hammer pipeline

A standard hammer pipeline:



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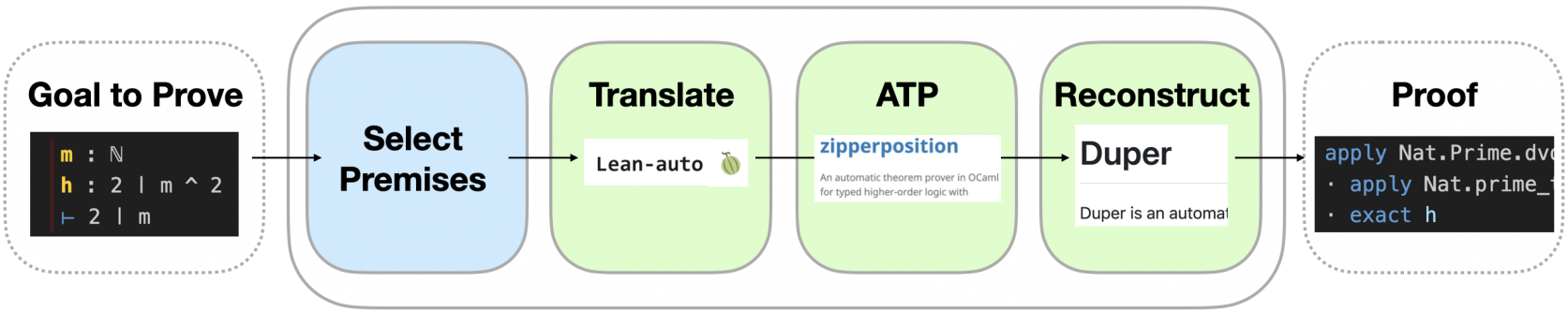
A standard hammer pipeline:

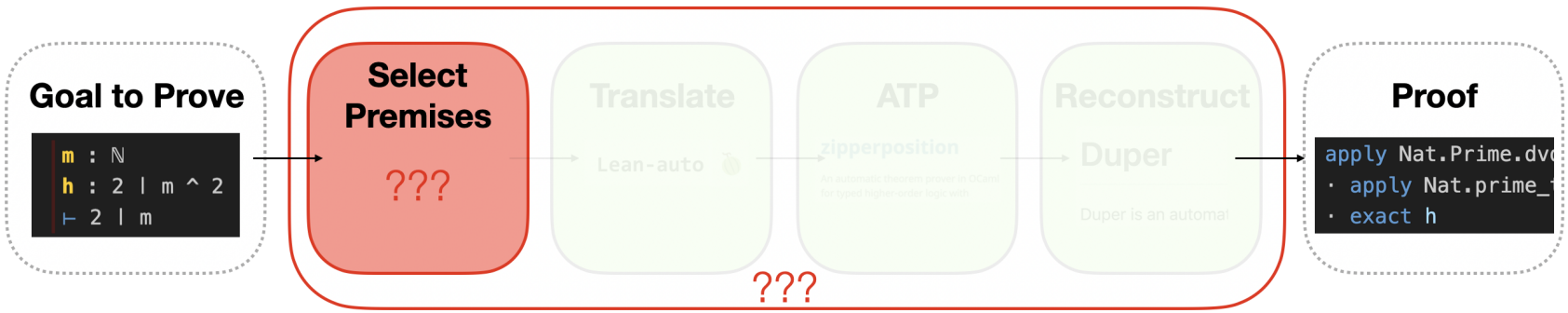


Pre-existing components:

- **Translation**: LeanAuto [Qian et al 2025]
- **ATP**: Zipperposition [Cruanes et al 2015]
- **Reconstruction**: Duper [Clune et al 2024]

A hammer pipeline

A standard hammer pipeline:

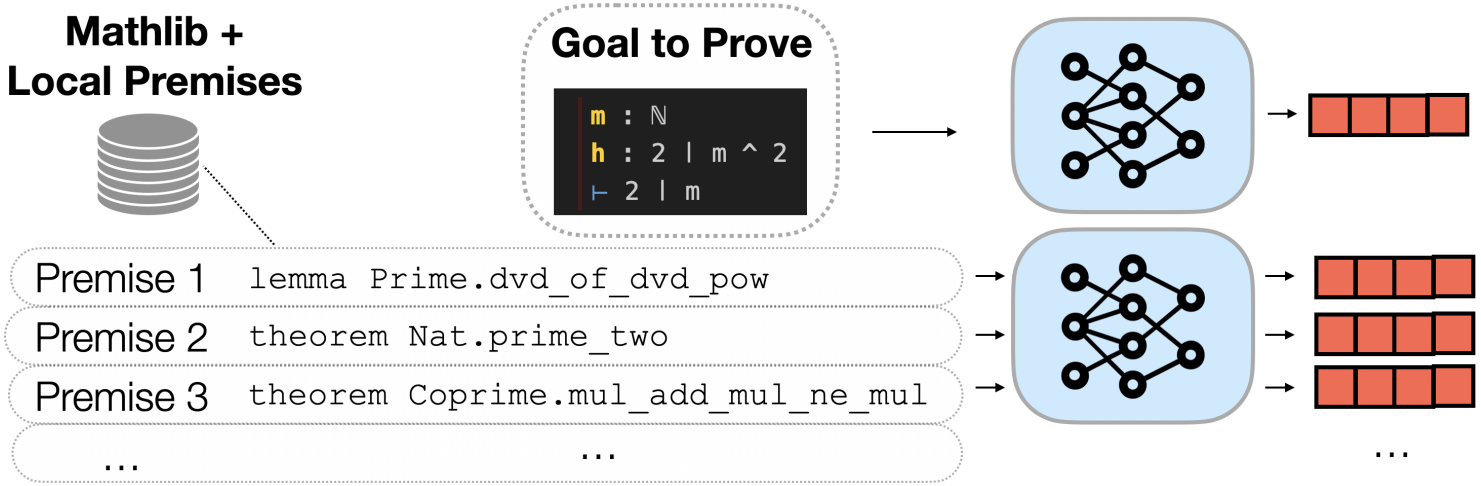


Our challenge:

- **Premise selection**
- **Put it all together** to create LeanHammer

LeanHammer — Neural premise selection

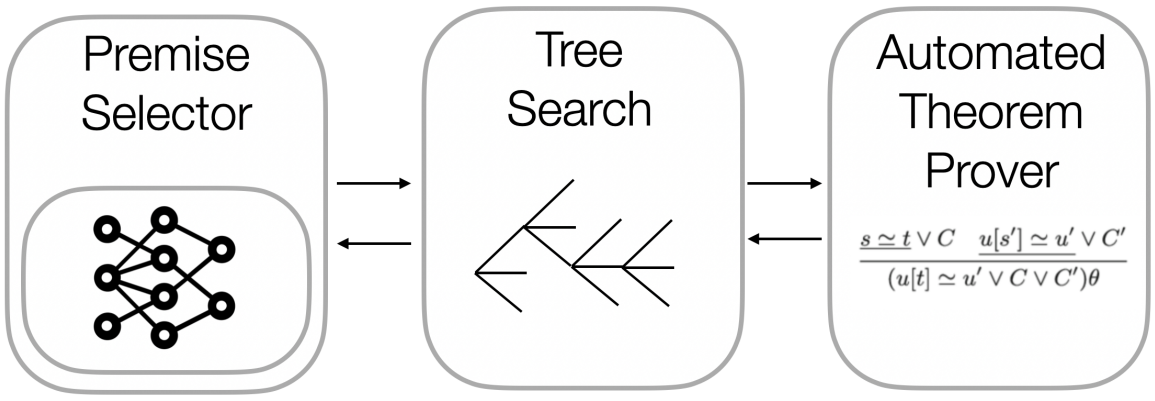
Idea: frame premise selection as retrieval with a neural language model



- Transformer encoder embeds the state and candidate premises
- Contrastive loss on (state, $\{\text{premise}^+\}$, $\{\text{premise}^-\}$) examples
  - Nuance in how to collect and format examples

LeanHammer — Putting it all together

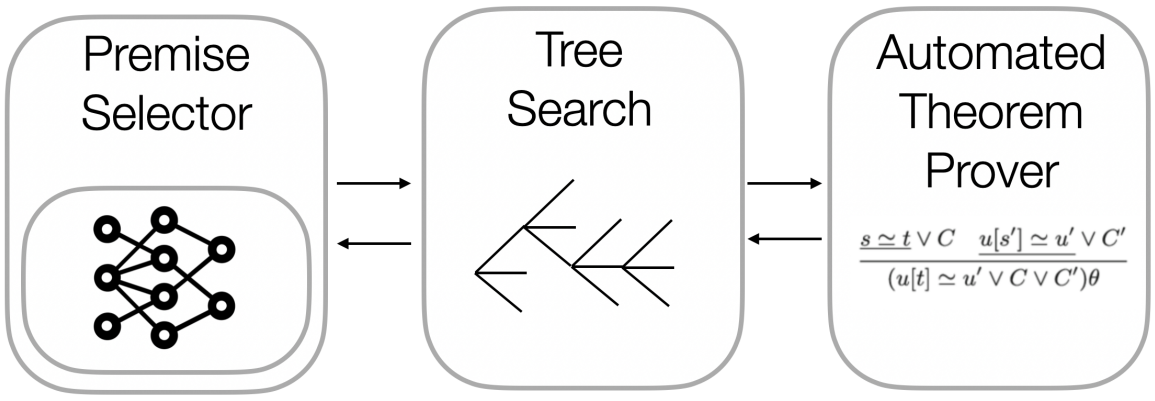
Idea: combine the premise selector and ATP with a tree search



Tree search: *Aesop* [Limperg & From 2023]

LeanHammer — Putting it all together

Idea: combine the premise selector and ATP with a tree search

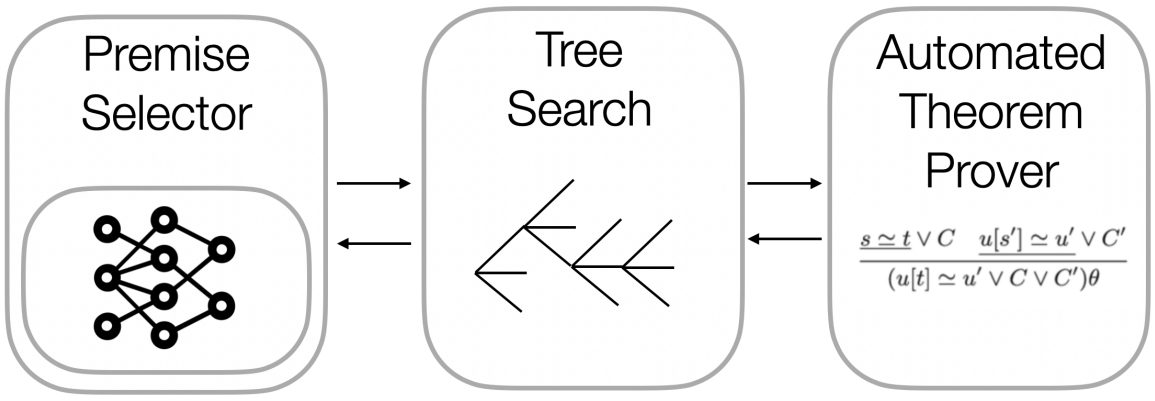


Tree search: *Aesop* [Limperg & From 2023]

1. Queries the automated theorem prover using the premises
2. Applies tactics (e.g. `apply`, `simp_all`) using the premises

LeanHammer — Putting it all together

Idea: combine the premise selector and ATP with a tree search



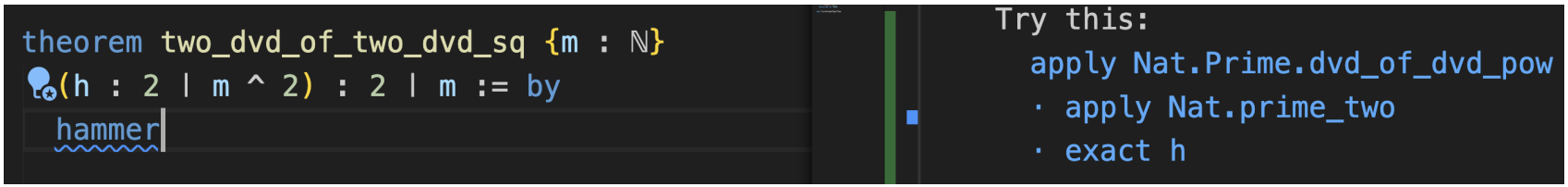
Tree search: *Aesop* [Limperg & From 2023]

1. Queries the automated theorem prover using the premises
2. Applies tactics (e.g. `apply`, `simp_all`) using the premises

Goes beyond the standard hammer pipeline!

LeanHammer

As a user, simply issue `hammer` at any step of a proof:



LeanHammer in action

LeanHammer — Demo

Demo: start with human-written proof sketch (from *Mathematics in Lean*)

```
/-- Theorem taken from Mathematics in Lean -/
theorem irrational_sqrt_two {m n : ℕ} (coprime_mn : m.Coprime n) :
    m ^ 2 ≠ 2 * n ^ 2 := by
  intro sqr_eq
  have : 2 ∣ m := by
    sorry
  obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
  have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
    sorry
  have : 2 * k ^ 2 = n ^ 2 := by
    sorry
  have : 2 ∣ n := by
    sorry
  have : 2 ∣ m.gcd n := by
    sorry
  have : 2 ∣ 1 := by
    sorry
  sorry
```

LeanHammer — Demo

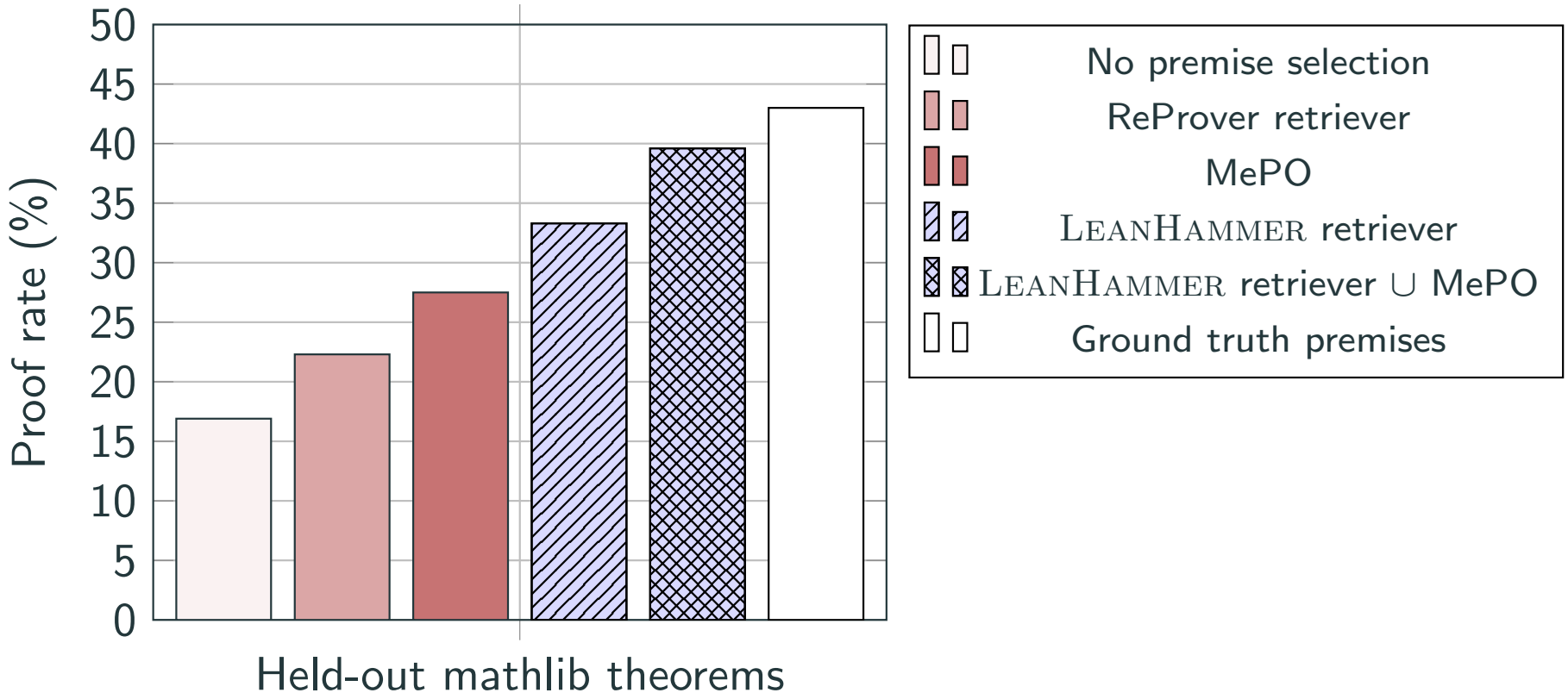
Demo: fill in the gaps (`sorrys`) with LeanHammer

Human Written


```
theorem irrational_sqrt_two {m n : ℕ} (coprime_mn : m.Coprime n) :
    m ^ 2 ≠ 2 * n ^ 2 := by
  intro sqr_eq
  have : 2 ∣ m := by
    apply Lemma.two_dvd_of_two_dvd_sq
    simp_all only [dvd_mul_right]
  obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
  have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
    subst meq
    simp_all only [dvd_mul_left, mul_eq_mul_left_iff, OfNat.ofNat_ne_zero, or_false]
    (linarith)
  have : 2 * k ^ 2 = n ^ 2 := by
    subst meq
    simp_all only [mul_eq_mul_left_iff, OfNat.ofNat_ne_zero, or_false, dvd_mul_left]
  have : 2 ∣ n := by
    subst meq
    simp_all only [dvd_mul_left]
    (hammerCore [] [*, dvd_mul_right, Nat.modEq_zero_iff_dvd, Nat.ModEq.symm, Nat.modEq_iff_dvd',
        Nat.Coprime.mul_add_mul_ne_mul, frobeniusNumber_pair, add_le_mul, FrobeniusNumber, Nat.ModEq.trans,
        Nat.cast_mul, sq, Nat.Coprime.dvd_of_dvd_mul_left, AddSubmonoid.mem_closure_pair, Nat.chineseRemainder_lt_mul,
        Nat.Coprime.eq_one_of_dvd, Lemma.two_dvd_of_two_dvd_sq] {simpTarget := no_target})
  have : 2 ∣ m.gcd n := by
    clear coprime_mn
    subst meq
    simp_all only [dvd_mul_left]
    apply Nat.dvd_gcd
    · simp_all only [dvd_mul_left]
    · simp_all only
  have : 2 ∣ 1 := by
    subst meq
    simp_all only [dvd_mul_left, Nat.dvd_one, OfNat.ofNat_ne_one]
  subst meq
  simp_all only [Nat.dvd_one, OfNat.ofNat_ne_one]
```

LeanHammer — Quantitative results

Varying the premise selector within LeanHammer:



Recap

Two approaches for combining informal and formal provers:

Recap

Two approaches for combining informal and formal provers:

- **Draft-Sketch-Prove (DSP)**
  - LLM drafts informal proof, generates formal sketch
  - Symbolic prover (Sledgehammer) fills in low-level details

Recap

Two approaches for combining informal and formal provers:

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- **LeanHammer**
  - Brings "hammer" functionality to Lean
  - Neural premise selection + tree search + automated theorem proving
  - Enables filling in proof sketches with `hammer` command

Recap

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Key idea: Combine high-level reasoning (LLM or human) with low-level reasoning (symbolic provers)

Recap

Two approaches for combining informal and formal provers:

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Key idea: Combine high-level reasoning (LLM or human) with low-level reasoning (symbolic provers)

Even small neural networks are powerful! (retriever: < 100M params)

This talk: Bridging Informal and Formal

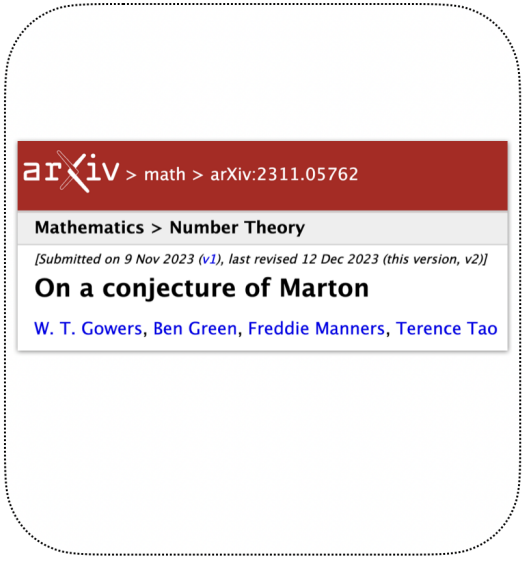
1. Informal thoughts

2. Informal provers

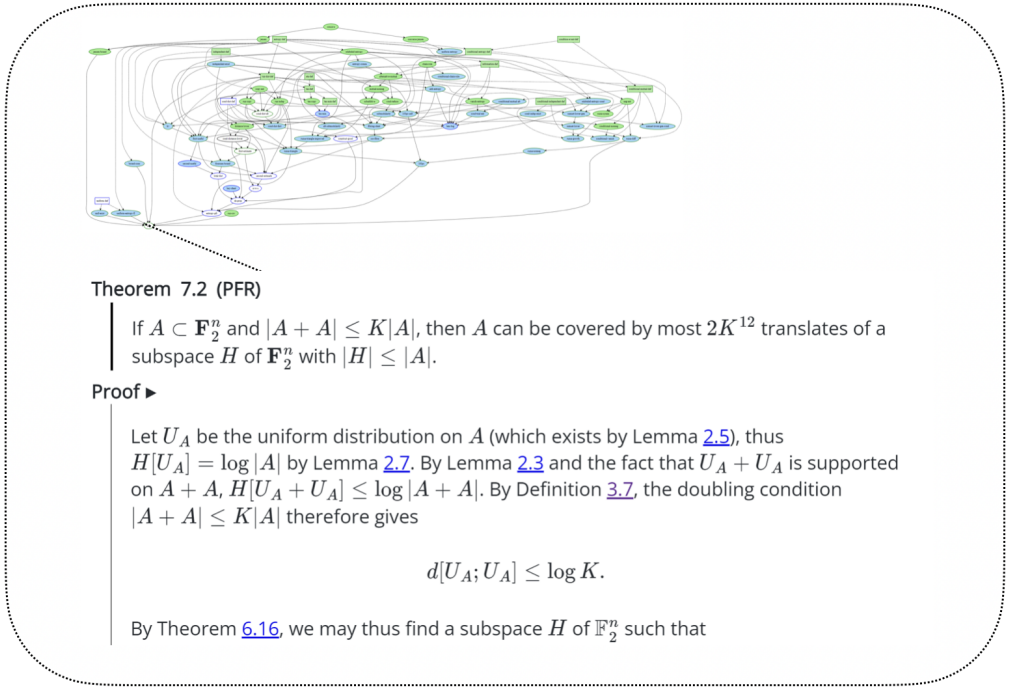
3. **Research-level mathematics**
   - Assisting in research-level projects
   - Practical tools
   - MiniCTX

III: Research-level mathematics

What does it look like to formalize research-level math?[4]



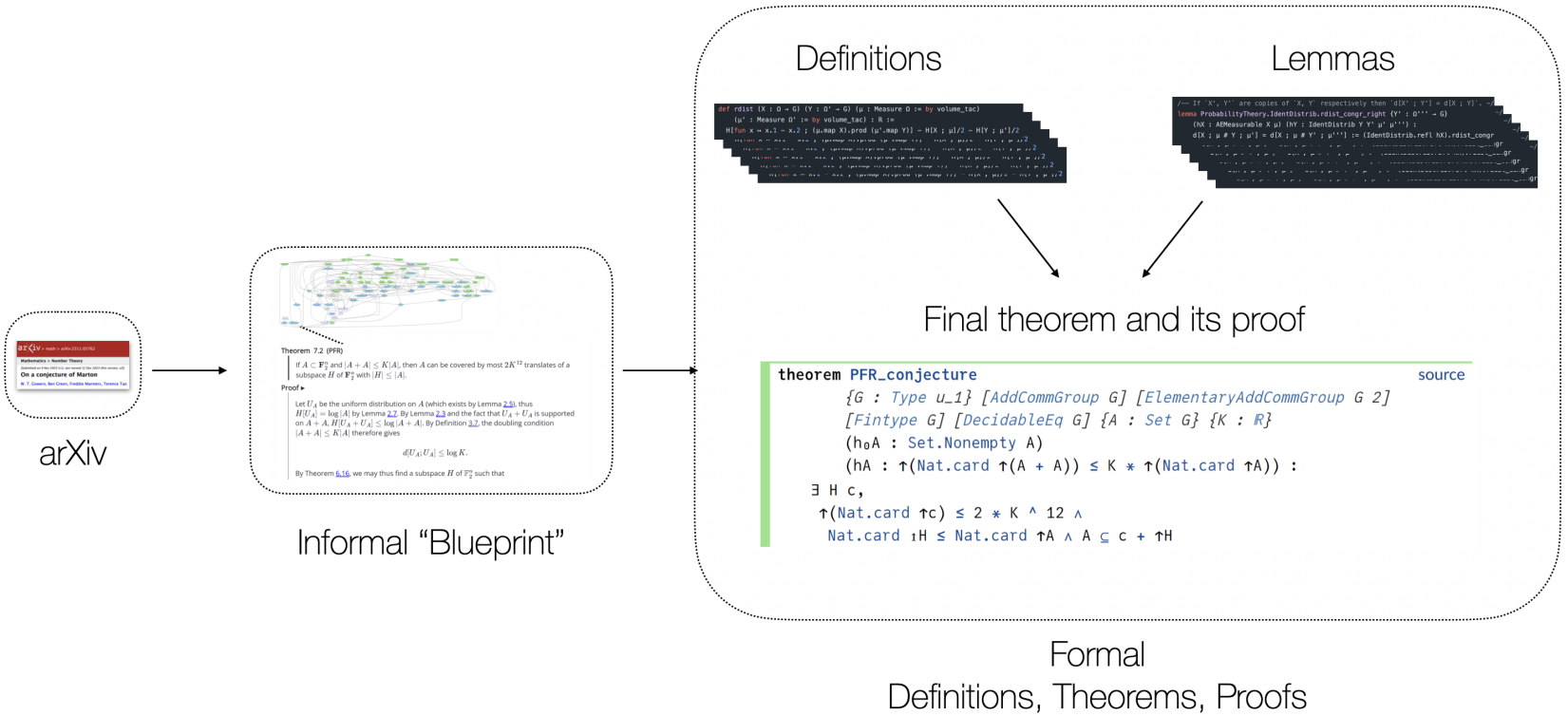
arXiv



Informal "Blueprint"

[4] *Formalizing the proof of PFR in Lean4 using Blueprint: a short tour* by Terence Tao

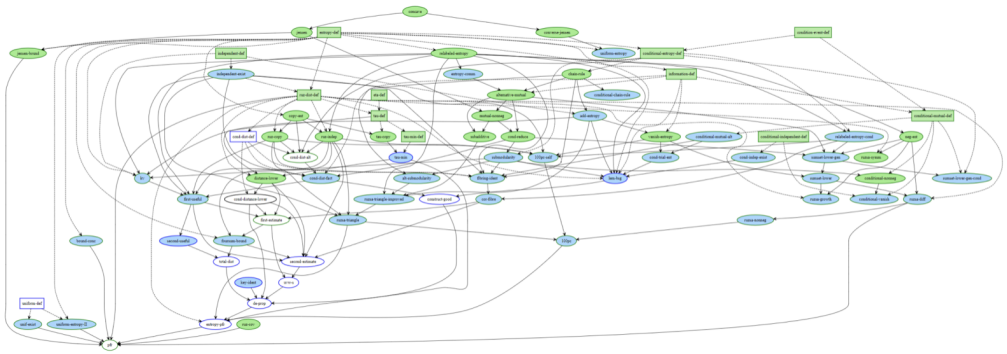
What does it look like to formalize research-level math?[4]



[4] *Formalizing the proof of PFR in Lean4 using Blueprint: a short tour* by Terence Tao

Where can AI help?

As a start, can AI help with filling in small parts of the blueprint?



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/-- `H[X | Y=y] = ∑_s P[X=s | Y=y] log 1/(P[X=s | Y=y])`. -/
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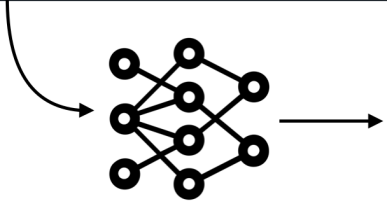
```



???

Where can AI help? — Challenges

Accessibility gap:

- Some methods are hard to integrate into tools
  - Not open-source (AlphaProof, ...)
  - Expensive to run (MCTS, ...)

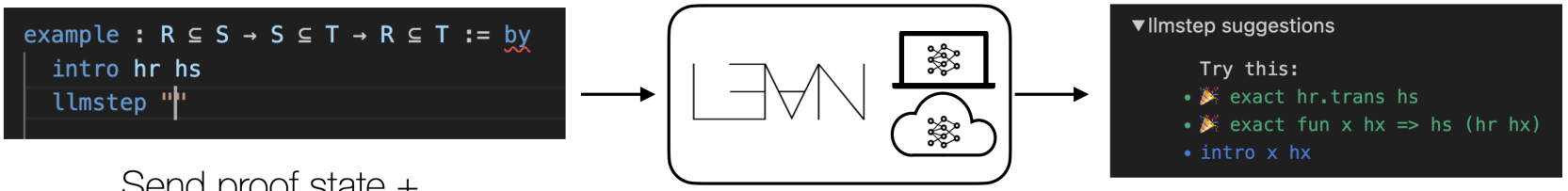
Where can AI help? — Challenges

Accessibility gap:

- Some methods are hard to integrate into tools
  - Not open-source (AlphaProof, ...)
  - Expensive to run (MCTS, ...)

However, there **are** already tools available!

Where can AI help? — Existing tools



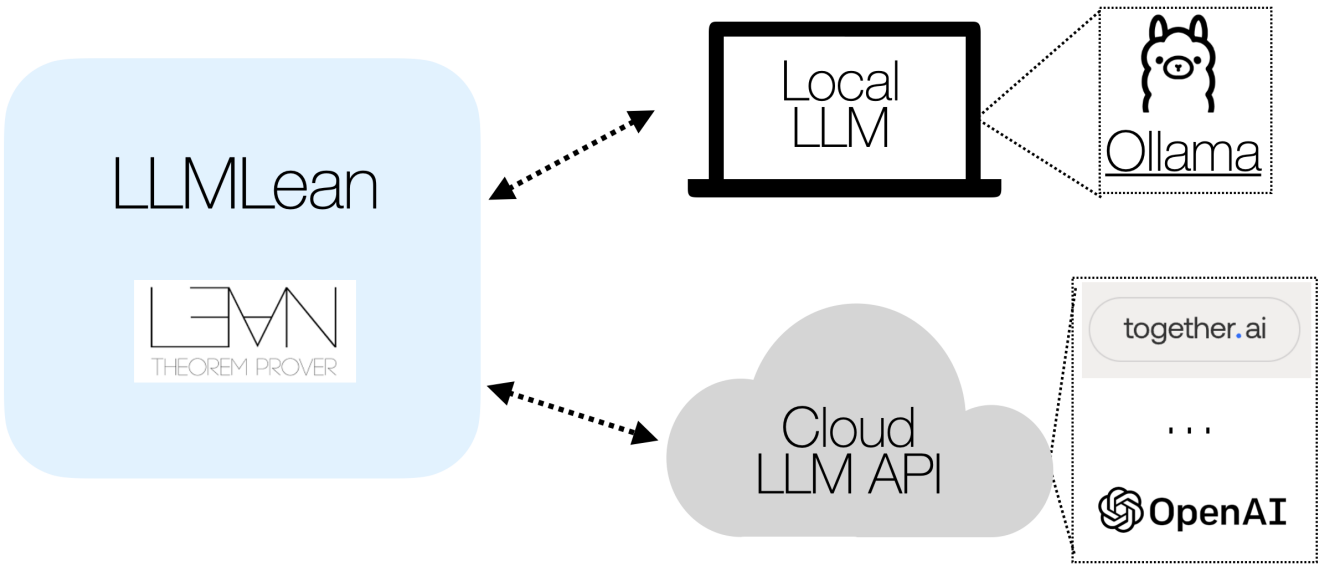
Send proof state + doc context

LLMLean

Verified suggestions

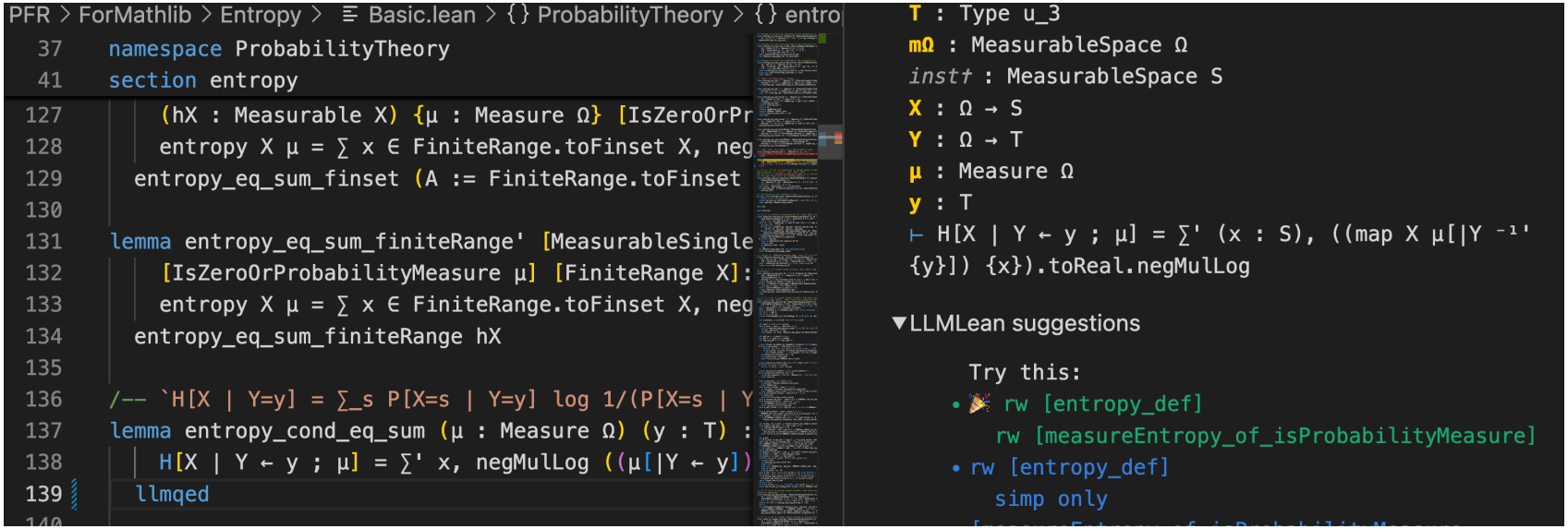
LLMLean: https://github.com/cmu-l3/llmlean

Where can AI help? — Existing tools



LLMLean: https://github.com/cmu-l3/llmlean

Where can AI help? — Existing tools



LLMLean example on Polynomial Freiman Rusza Conjecture project

Where can AI help? — Benchmarking gap

Math competition problems

Theorem to prove

```
theorem imo_1964_p1_2 (n : ℕ) : ¬7 ∣ 2 ^ n + 1 := by
  sorry
```

- Self-contained
- Uses standard results

Where can AI help? — Benchmarking gap

Math competition problems

Theorem to prove

```
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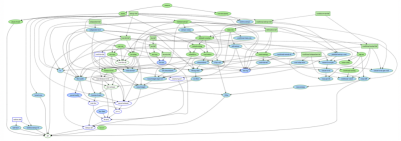
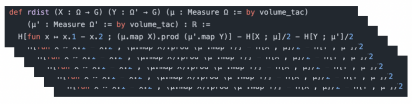
Real projects



Definitions

Lemmas



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```

- Self-contained
- Uses standard results

- Part of a project
- Uses new definitions and lemmas

miniCTX: Neural Theorem Proving with (Long-)Contexts

Jiewen Hu, Thomas Zhu, Sean Welleck

*ICLR 2025 (**Oral**)*

miniCTX

Research-level theorems depend on newly-formalized **context**

- (context, theorem) $\rightarrow$ proof
  - Context: repository of code, new definitions, auxiliary lemmas

miniCTX

miniCTX:

Test models on real Lean projects:[5]

- "Future mathlib": theorems added after a time cutoff
- Recent projects: PFR, PrimeNumberTheorem
- Textbook exercises: How To Prove It, Math 2001

[5] + tools for easily adding new projects: https://github.com/cmu-l3/ntp-toolkit

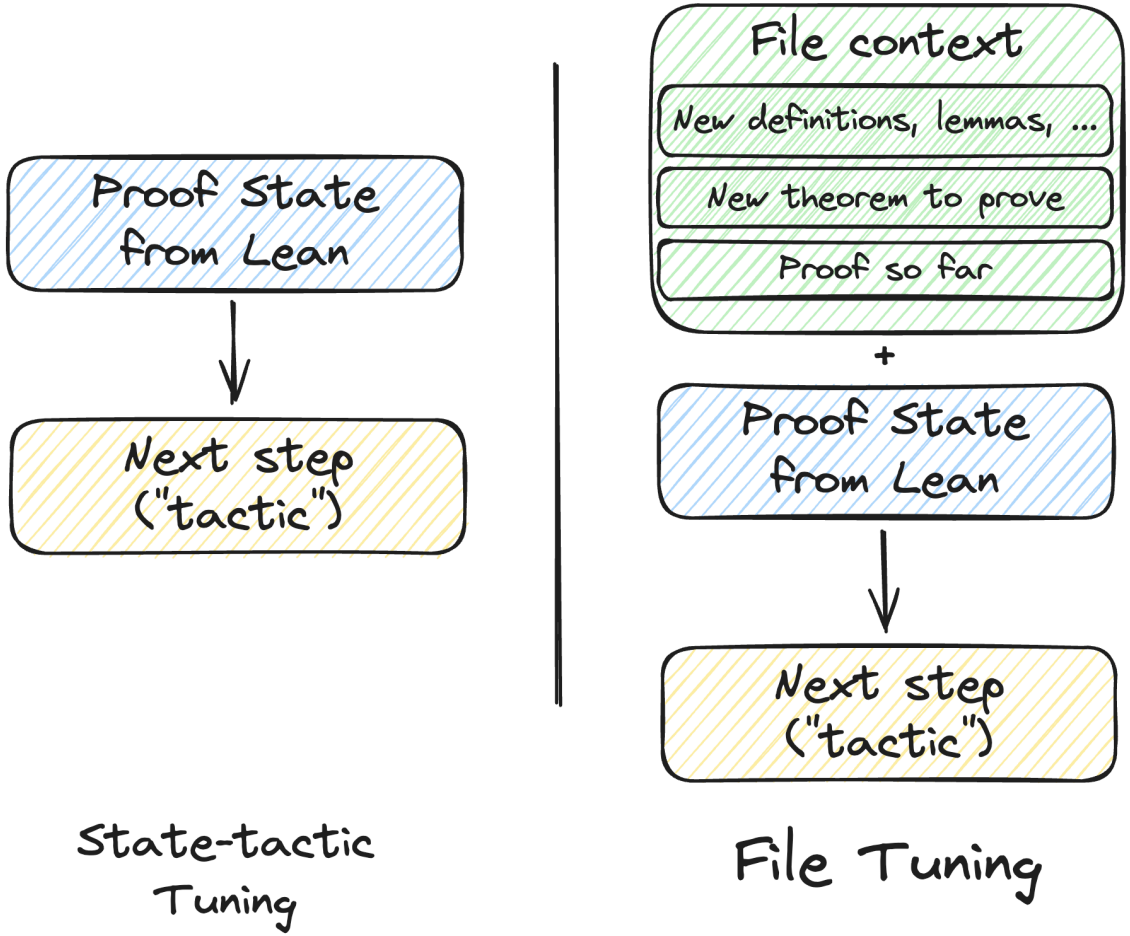
miniCTX

Context:

- Preceding code in the file
- All accessible premises
- All code in the project

miniCTX

Does context actually matter? A simple experiment.



"File tuning": train on (preceding code, state, next-tactic) examples

miniCTX

Two methods can have similar performance on competition problems, but vastly difference performance on actual projects:

	MiniF2F	MiniCTX				
Models	**Test**	**Prime**	**PFR**	**Mathlib**	**HTPI**	**Avg.**
GPT-4o (full proof)	-	1.15%	5.56%	2.00%	9.73%	5.59%
GPT-4o (+ context)	-	13.79%	1.85%	18.00%	31.89%	22.07%
State-tactic prompting	28.28%	19.54%	5.56%	16.00%	19.15%	20.61%
State-tactic tuning	32.79%	11.49%	5.56%	22.00%	5.95%	9.31%
File tuning	**33.61%**	**32.18%**	**5.56%**	**34.00%**	**38.38%**	**31.65%**

miniCTX — LLMLean

File-tuned model is deployed in LLMLean:

https://github.com/cmu-l3/llmlean

miniCTX

Several open-source artifacts:

- Data/models: `https://huggingface.co/l3lab`
- Data extraction: `https://github.com/cmu-l3/ntp-toolkit`
- Evaluation: `https://github.com/cmu-l3/minictx-eval`

Recap: Towards AI for Research-Level Formalization

Formalizing research-level math has unique challenges

- Accessibility gap exists between AI advances and real-world formalization
  - But we have some tools available!
- Benchmarking gap exists between competition problems and real-world formalization
  - miniCTX tests the ability to generalize to new, real-world projects

This talk: Bridging Informal and Formal

1. Informal thoughts
    - Training models to think informally
        - Lean-STaR

2. Informal provers
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer

3. Research-level mathematics
    - Assisting in research-level projects
    - Practical tools
    - MiniCTX

Thank you!

Sean Welleck

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References i


K. Zheng, J. M. Han, and S. Polu.
minif2f: a cross-system benchmark for formal olympiad-level mathematics.
In *International Conference on Learning Representations*, 2022.