

Assignment 2 - CS 4071 - Spring 2018

Due: 2018-02-19 Group #13: Robert DiMartino (dimartrt), Hayden Schiff (schiffha), Jeremiah Leak (leakjz)

1. Exercise 2.24

Problem: Give pseudocode for interpolation search, and analyze its worst-case complexity.

2. Exercise 3.6

Problem: Using the Ratio Limit Theorem, prove the following:

$$O(108) \subset O(\ln n) \subset O(n) \subset O(n \ln n) \subset O(n^2) \subset O(n^3) \subset O(2^n) \subset O(3^n)$$

i.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{108}{\ln n} &= 0 \\ \therefore O(108) &\subset O(\ln n) \end{aligned}$$

ii.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\ln n}{n} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} \\ &= \frac{0}{1} = 0 \\ \therefore O(\ln n) &\subset O(n) \end{aligned}$$

iii.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n \ln n} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \ln n} \\ &= 0 \\ \therefore O(n) &\subset O(n \ln n) \end{aligned}$$

iv.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{n \ln n}{n^2} \\
&= \lim_{n \rightarrow \infty} \frac{1 + \ln n}{2n} \\
&= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{2} \\
&= \frac{0}{2} = 0 \\
&\therefore O(n \ln n) \subset O(n^2)
\end{aligned}$$

v.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{n^2}{n^3} \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \\
&= 0 \\
&\therefore O(n^2) \subset O(n^3)
\end{aligned}$$

vi.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{n^3}{2^n} \\
&= \lim_{n \rightarrow \infty} \frac{3n^2}{2^n \ln 2} \\
&= \lim_{n \rightarrow \infty} \frac{6n}{2^n \ln^2 2} \\
&= \lim_{n \rightarrow \infty} \frac{6}{2^n \ln^3 2} \\
&= 0 \\
&\therefore O(n^3) \subset O(2^n)
\end{aligned}$$

vii.

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{2^n}{3^n} \\
&= \lim_{n \rightarrow \infty} \left(\frac{2}{3} \right)^n \\
&= 0 \\
&\therefore O(2^n) \subset O(3^n)
\end{aligned}$$

3. Exercise 3.26

Problem: Obtain a formula for the order of $S(n) = \sum_{i=1}^n (\ln i)^2$.

We start by showing that $S(n) \in O(n \ln^2 n)$.

$$\begin{aligned} S(n) &= \sum_{i=1}^n (\ln i)^2 = \sum_{i=1}^n \ln^2 i \\ &= \ln^2 1 + \ln^2 2 + \cdots + \ln^2 n \end{aligned}$$

It can be shown that $f(x) = \ln^2 x$ has a global minimum as $x = 1$ and is increasing for $x \geq 1$. So it holds that $\ln^2 a \leq \ln^2 b$ when $1 \leq a \leq b$. It follows that,

$$\begin{aligned} \ln^2 1 + \ln^2 2 + \cdots + \ln^2 n &\leq \ln^2 n + \ln^2 n + \cdots + \ln^2 n \\ &= n \ln^2 n \\ \therefore S(n) &\in O(n \ln^2 n) \end{aligned}$$

Now, we intend to show that $S(n) \in \Omega(n \ln^2 n)$. Let $m = \lfloor n/2 \rfloor$. Then,

$$\begin{aligned} S(n) &= \sum_{i=1}^n \ln^2 i = \sum_{i=1}^m \ln^2 i + \sum_{i=m+1}^n \ln^2 i \\ &\geq \sum_{i=m+1}^n \ln^2 i = \ln^2(m+1) + \ln^2(m+2) + \cdots + \ln^2 n \\ &\geq \ln^2(m+1) + \ln^2(m+1) + \cdots + \ln^2(m+1) \\ &= (n-m) \ln^2(m+1) \\ &\geq \frac{n}{2} \ln^2\left(\frac{n}{2}\right) \\ &= \frac{n}{2} (\ln n - \ln 2)^2 \end{aligned}$$

For sufficiently large n ,

$$\begin{aligned} \frac{n}{2} (\ln n - \ln 2)^2 &\geq \frac{n}{2} \left(\ln n - \frac{\ln n}{2} \right)^2 \\ &= \frac{n}{2} \left(\frac{\ln n}{2} \right)^2 \\ &= \frac{n}{8} (\ln n)^2 \\ &= \frac{1}{8} (n \ln^2 n) \\ \therefore S(n) &\in \Omega(n \ln^2 n) \end{aligned}$$

Since $S(n) \in O(n \ln^2 n)$ and $S(n) \in \Omega(n \ln^2 n)$, then $S(n) \in \Theta(n \ln^2 n)$.

4. Exercise 3.37

a.

Problem: Give a recurrence relation for the worst-case complexity $W(n)$ of `TriMergeSort` for an input list of size n .

b.

Problem: Solve the recurrence formula you have given in (a) to obtain an explicit formula for the worst-case complexity $W(n)$ of `TriMergeSort`.

c.

Problem: Which is more efficient in the worst case, `MergeSort` or `TriMergeSort`? Discuss.

5.

Problem: Consider the sorting algorithm Insertion Sort for sorting a list `L[0:n - 1]`. **Derive** a recurrence relation for the worst-case complexity $W(n)$ and **solve**.

6. Exercise 3.35

Problem: Solve the following recurrence relations

a. $t(n) = 3t(n - 1) + n, n \geq 1$, init. cond. $t(0) = 0$

$$\begin{aligned} t(n) &= 3t(n - 1) + n \\ t(n) &= 3(3t(n - 2) + n - 1) + n \\ &= 3^2t(n - 2) + 3(n - 1) + n \\ t(n) &= 3^2(3t(n - 3) + n - 2) + 3(n - 1) + n \\ &= 3^3t(n - 3) + 3^2(n - 2) + 3(n - 1) + n \\ &\vdots \\ t(n) &= 3^k t(n - k) + 3^{k-1}(n - k + 1) + \cdots + 3^2(n - 2) + 3(n - 1) + n \end{aligned}$$

When $k = n$,

$$t(n) = 3^n t(0) + 3^{n-1}(1) + 3^{n-2}(2) + \cdots + 3^2(n - 2) + 3(n - 1) + n$$

Applying the initial condition, $t(0) = 0$,

$$\begin{aligned} t(n) &= 3^{n-1} + 3^{n-2}(2) + \cdots + 3^2(n - 2) + 3(n - 1) + n \\ t(n) &= 3^{n-1} \left(1 + \frac{2}{3} + \cdots + \frac{n-2}{3^{n-3}} + \frac{n-1}{3^{n-2}} + \frac{n}{3^{n-1}} \right) \end{aligned}$$

Note that for the sum of the first $n + 1$ terms of a geometric series (where $r \neq 1$)

$$1 + r + r^2 + r^3 + \dots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r}$$

And differentiation both sides with respect to r yields:

$$0 + 1 + 2r + 3r^2 + \dots + (n-1)r^{n-2} + nr^{n-1} = \frac{nr^{n+1} - (n+1)r^n + 1}{(1-r)^2}$$

And when $r = \frac{1}{3}$,

$$\begin{aligned} 1 + \frac{2}{3} + \dots + \frac{n-2}{3^{n-3}} + \frac{n-1}{3^{n-2}} + \frac{n}{3^{n-1}} &= \frac{n(\frac{1}{3})^{n+1} - (n+1)(\frac{1}{3})^n + 1}{(1 - \frac{1}{3})^2} \\ &= \frac{1}{4} \times \frac{1}{3^{n-1}} \times (-2n + 3^{n+1} - 3) \end{aligned}$$

Plugging that result back into the recurrence relation gives:

$$t(n) = 3^{n-1} \left(\frac{1}{4} \times \frac{1}{3^{n-1}} \times (-2n + 3^{n+1} - 3) \right)$$

Which reduces to:

$$t(n) = \frac{1}{4} \times (-2n + 3^{n+1} - 3)$$

b. $t(n) = 4t(n-1) + 5, n \geq 1$, init. cond. $t(0) = 2$

$$\begin{aligned} t(n) &= 4t(n-1) + 5 \\ t(n) &= 4(4t(n-2) + 5) + 5 \\ &= 4^2 t(n-2) + 4 \times 5 + 5 \\ t(n) &= 4^2 (4t(n-3) + 5) + 4 \times 5 + 5 \\ &= 4^3 t(n-3) + 4^2 \times 5 + 4 \times 5 + 5 \\ &\vdots \\ t(n) &= 4^k t(n-k) + 4^{k-1} \times 5 + \dots + 4^2 \times 5 + 4 \times 5 + 5 \end{aligned}$$

When $k = n$,

$$t(n) = 4^k t(0) + 4^{n-1} \times 5 + \dots + 4^2 \times 5 + 4 \times 5 + 5$$

Applying the initial conditon, $t(0) = 2$,

$$\begin{aligned}
t(n) &= 4^n \times 2 + 4^{n-1} \times 5 + \dots + 4^2 \times 5 + 4 \times 5 + 5 \\
t(n) &= 4^n \times 5 - 4^n \times 3 + 4^{n-1} \times 5 + \dots + 4^2 \times 5 + 4 \times 5 + 5 \\
t(n) &= 4^n \times 5 \left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^{n-2}} + \frac{1}{4^{n-1}} \right) - 4^n \times 3 \\
t(n) &= 4^n \times 5 \left(\frac{1 - \left(\frac{1}{4}\right)^n}{1 - \frac{1}{4}} \right) - 4^n \times 3
\end{aligned}$$

Which reduces to:

$$t(n) = \frac{1}{3}(4^n \times 11 - 5)$$