Assignment 2 - CS 4071 - Spring 2018

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1. Exercise 2.24

Problem: Give pseudocode for interpolation search, and analyze its worst-cast complexity.

2. Exercise 3.6

Problem: Using the Ratio Limit Theorem, prove the following:

$$O(108) \subset O(\ln n) \subset O(n) \subset O(n \ln n) \subset O(n^2) \subset O(n^3) \subset O(2^n) \subset O(3^n)$$

i.

$$\lim_{n\to\infty} \frac{108}{\ln n} = 0$$
$$\therefore O(108) \subset O(\ln n)$$

ii.

$$\lim_{n\to\infty} \frac{\ln n}{n}$$
$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$
$$= \frac{0}{1} = 0$$
$$\therefore O(\ln n) \subset O(n)$$

iii.

$$\lim_{n\to\infty} \frac{n}{n \ln n}$$
$$= \lim_{n\to\infty} \frac{1}{1 + \ln n}$$
$$= 0$$
$$\therefore O(n) \subset O(n \ln n)$$

iv.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{n \ln n}{n^2} \\
&= \lim_{n\to\infty} \frac{1+\ln n}{2n} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{2} \\
&= \frac{0}{2} = 0 \\
&\therefore O(n \ln n) \subset O(n^2)
\end{aligned}
$$

v.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{n^2}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0 \\
&\therefore O(n^2) \subset O(n^3)
\end{aligned}
$$

vi.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{n^3}{2^n} \\
&= \lim_{n\to\infty} \frac{3n^2}{2^n \ln 2} \\
&= \lim_{n\to\infty} \frac{6n}{2^n \ln^2 2} \\
&= \lim_{n\to\infty} \frac{6}{2^n \ln^3 2} \\
&= 0 \\
&\therefore O(n^3) \subset O(2^n)
\end{aligned}
$$

vii.

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{2^n}{3^n} \\
&= \lim_{n\to\infty} \left(\frac{2}{3}\right)^n \\
&= 0 \\
&\therefore O(2^n) \subset O(3^n)
\end{aligned}
$$

3. Exercise 3.26

Problem: Obtain a formula for the order of $S(n) = \sum_{i=1}^{n} (\ln i)^2$.

We start by showing that $S(n) \in O(n \ln^2 n)$.

$$
\begin{aligned}
S(n) &= \sum_{i=1}^{n} (\ln i)^2 = \sum_{i=1}^{n} \ln^2 i \\
&= \ln^2 1 + \ln^2 2 + \cdots + \ln^2 n
\end{aligned}
$$

It can be shown that $f(x) = \ln^2 x$ has a global minimum as $x = 1$ and is increasing for $x \geq 1$. So it holds that $\ln^2 a \leq \ln^2 b$ when $1 \leq a \leq b$. It follows that,

$$
\begin{aligned}
\ln^2 1 + \ln^2 2 + \cdots + \ln^2 n &\leq \ln^2 n + \ln^2 n + \cdots + \ln^2 n \\
&= n \ln^2 n \\
\therefore S(n) &\in O(n \ln^2 n)
\end{aligned}
$$

Now, we intend to show that $S(n) \in \Omega(n \ln^2 n)$. Let $m = \lfloor n/2 \rfloor$. Then,

$$
\begin{aligned}
S(n) = \sum_{i=1}^{n} \ln^2 i &= \sum_{i=1}^{m} \ln^2 i + \sum_{i=m+1}^{n} \ln^2 i \\
\geq \sum_{i=m+1}^{n} \ln^2 i &= \ln^2 (m+1) + \ln^2 (m+2) + \cdots + \ln^2 n \\
&\geq \ln^2 (m+1) + \ln^2 (m+1) + \cdots + \ln^2 (m+1) \\
&= (n-m) \ln^2 (m+1) \\
&\geq \frac{n}{2} \ln^2 \left(\frac{n}{2}\right) \\
&= \frac{n}{2} (\ln n - \ln 2)^2
\end{aligned}
$$

For sufficiently large n,

$$
\begin{aligned}
\frac{n}{2} (\ln n - \ln 2)^2 &\geq \frac{n}{2} \left(\ln n - \frac{\ln n}{2}\right)^2 \\
&= \frac{n}{2} \left(\frac{\ln n}{2}\right)^2 \\
&= \frac{n}{8} (\ln n)^2 \\
&= \frac{1}{8} (n \ln^2 n) \\
\therefore S(n) &\in \Omega(n \ln^2 n)
\end{aligned}
$$

Since $S(n) \in O(n \ln^2 n)$ and $S(n) \in \Omega(n \ln^2 n)$, then $S(n) \in \Theta(n \ln^2 n)$.