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Computing Topologies

Two roads diverged in a mathematical wood, and I—
I took the one less traveled by.

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How far is it from computer science to topology? Surprisingly, it is not far at all—if you take the right path. For a mathematician, the traditional path to topology begins at calculus, goes through real analysis, and continues to the study of abstract spaces. Throughout the journey, the mathematician studies infinite sets of points, and these seem far indeed from the finite world of the computer scientist.

But there is another path to topology. This path begins at computer programming, goes through discrete mathematics, and continues to the study of abstract spaces. The point sets studied are finite, and this is to be expected since only a finite topology can have a computer representation. However, many of the properties of finite spaces are shared by a certain interesting class of infinite spaces. The second path leads eventually to the study of infinite topologies.

The milestones on the mathematician's road to topology are definitions and theorems. The milestones on the computer scientist's road are definitions, theorems, and the computational methods which result from those theorems. For instance, what is meant by the closure of a set? What properties does the closure have? And how (asks the computer scientist) can one write a computer program to find the closure of any given set? Finite topologies can be computed in a way that infinite topologies cannot, and it is reasonable to ask that theorems lead to programs.

As we set out on the computer scientist's path, our first step will be to find suitable data structures for finite topologies. The key to these data structures is the fact that any finite topology can be specified in terms of the dominance relation which generates it. Our aim is not only to *define* topologies in terms of these relations, but also to *represent* the topologies clearly and efficiently both for computer calculation and for our own visualization. We assume knowledge of the mathematics customarily taught in a first course in discrete mathematics for computer science students, and a level of programming skill which can reasonably be expected after a first course in some high-level language such as Pascal, PL/1, or FORTRAN.

As we continue our journey, we look at finite topologies in two ways. First, we ask standard topologists' questions and find computer scientists' answers. Suppose that you are given a finite space. How can you determine which subsets of the space are open? What separation properties does the space have? Is it connected? Which functions from it to other spaces are continuous? As answers we seek both theorems and computational methods.

Next, we take the computer scientist's answers and find the corresponding questions from topology. That is, we see what facts from discrete mathematics can tell us about the topological structure of finite spaces.

Data structures

We begin by defining a dominance relation on a set and giving two standard data structures for the relation: its table and its directed graph. We then show how a dominance relation generates a topology and how the two data structures for the relation function as data structures for the topology. We also show that any finite topology can be generated by a dominance relation. Thus, our data structures can be used to represent any finite topology.

A relation R on a set A is called a **dominance relation** (or a **preorder**) whenever R is both reflexive and transitive. If y is related to x, we write $y \to x$ and say that y **dominates** x. In particular, any equivalence relation or partial order on A is a dominance relation. For instance, let $A = \{1,2,3,4,5,6\}$ and say that y dominates x whenever y divides x. The relation is a dominance relation, in fact a partial order.

If A is finite, a relation on A may easily be represented in matrix form or graphical form. If $A = \{x_1, x_2, \ldots, x_n\}$, the **table** (or **adjacency matrix**) of the relation R is a square matrix (b_{ij}) in which $b_{ij} = 1$ if x_i is related to x_j, and $b_{ij} = 0$ otherwise. The table is a convenient computer representation for the relation and is easy to construct and manipulate in almost any programming language. A relation R may also be represented by a **directed graph** (or **digraph**) in which there is a directed edge from x_i to x_j if and only if x_i is related to x_j.

Since a dominance relation is reflexive, its digraph will have a loop at each vertex. Since the relation is transitive, if there are directed edges from x_i to x_j and from x_j to x_k there will be a directed edge from x_i to x_k. Even for small sets A, the digraph of a dominance relation can become visually confusing. In order to simplify the picture, it is customary to omit the loop at each vertex as well as any arrows implied by the transitivity of the relation. The matrix and the digraph of our example $A = \{1,2,3,4,5,6\}$ with R the division relation are in FIGURE 1. For more about the matrices and digraphs of relations, see [1], [6], [10], [11], or [12].

Some high-level computer languages (for instance, Pascal and PL/1) allow digraphs to be represented as complex linked lists and to be investigated with relative ease. For details about list handling in Pascal and PL/1, see [1] and [2], respectively. Since list handling is not usually taught in an introductory programming course, and since the process is inconvenient in certain common languages (for instance, FORTRAN) we will use the adjacency matrix for machine representation of a relation. The digraph remains a valuable intuitive aid in visualizing the structure of a dominance relation and the topology it generates.

An open set U in a topology is characterized by the property that each point of U is an interior point. Intuitively, each point is "insulated" from the complement of U. We can use the dominance relation to give an appropriate meaning to the term "insulated." We will say that a point x in a subset U of A is **insulated** from $A - U$ if and only if there is no point y in $A - U$ such that y dominates x. We define the set U to be **open** whenever all of its points are insulated from $A - U$. With this definition, both the empty set and A itself are open. The definition implies (as should be the case) that the union of open sets is open, and the intersection of open sets is open. (Here unions and intersections may be arbitrary.)

Note that the process of defining a dominance relation on a set A and using the relation to generate a topology does not require A to be finite. If A is infinite, there will, of course, be no

R	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	0	1	0	1
3	0	0	1	0	0	1
4	0	0	0	1	0	0
5	0	0	0	0	1	0
6	0	0	0	0	0	1

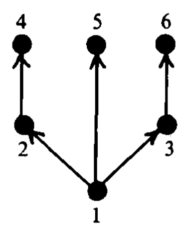


FIGURE 1. **The adjacency matrix and digraph of the dominance relation R on the set $A = \{1,2,3,4,5,6\}$ defined by: y dominates x whenever y divides x.**

computer representation for the relation or for the topology. The topology will have the property that the arbitrary intersection of open sets is open. Thus, not all infinite topologies can be generated by dominance relations. In particular, the ordinary topology on E^n cannot be generated by a dominance relation, nor can any infinite metric topology (except, of course, the discrete topology). The computer scientist's road to topology can lead directly to the study of infinite spaces, but these are not the spaces of classical analysis.

In fact, any topology in which the arbitrary intersection of open sets is open can be generated by a dominance relation. Given such a topology, define the relation by saying that y dominates x whenever y is an element of every open set about x. It is straightforward to verify that this relation is indeed a dominance relation and generates the original topology. In particular, every finite topology can be generated by a dominance relation. Thus, every finite topology has the simple tabular and graphical representations discussed below.

As an example in which a finite topology is used to define a relation, let $B = \{a, b, c, d, e\}$ and let $U = \{a, b, c,\}$, $V = \{d\}$, and $W = \{e\}$. Let the topology $\mathcal{T}$ on B be the collection of open sets $\mathcal{T} = \{\varnothing, U, V, W, U \cup V, U \cup W, V \cup W, B\}$. This is indeed a topology on B and is generated by a certain dominance relation, in fact by an equivalence relation. What is the relation?

We can use the table of a dominance relation to actually construct the open sets of the corresponding topology, and we can also visualize these open sets in the corresponding digraph. For each point x of A, we expect to find a unique smallest open set U_x about x; the set U_x is simply the intersection of all open sets containing x. But, if U is any open set containing x, and if $y \to x$, then $y \in U$ (otherwise, x would not be insulated from $A - U$). Thus, $U_x = \{y | y \to x\}$. This provides a simple criterion for deciding whether or not a given set U is open. The set U is open if and only if

$$U = \bigcup_{x \in U} U_x.$$

In the case of our example in FIGURE 1, $U_1 = \{1\}$, $U_3 = \{1,3\}$, and $U_4 = \{1,2,4\}$. We see immediately that the set $U = \{1,3,4\}$ is not open since $U \neq U_1 \cup U_3 \cup U_4$.

Since every open set can be written as a union of sets U_x, the sets U_x form a basis for the topology. If A has n elements, there are n of these basis sets (not necessarily all distinct). That is, any topology on an n-element set A has a basis of at most n sets.

Moreover, the sets U_x form a computationally useful basis for the topology. If $A = \{x_1, x_2, \ldots, x_n\}$ and if S is a subset of A, the **Boolean** or **bit string representation** of S is an array $(b_1, b_2, \ldots, b_n)$ in which $b_i = 1$ if x_i is an element of S, and $b_i = 0$ otherwise. For instance, in the case of our example in FIGURE 1, the representation of $S = \{1,2,4\}$ is $(1,1,0,1,0,0)$ and the representation of $T = \{4,6\}$ is $(0,0,0,1,0,1)$. The representation of a union of sets is the bit by bit disjunction of the representations of those sets, and the representation of the intersection is the bit by bit conjunction. Thus, for example, the representation of $S \cup T$ is

$$(1 \vee 0, 1 \vee 0, 0 \vee 0, 1 \vee 1, 0 \vee 0, 0 \vee 1) = (1,1,0,1,0,1)$$

and the representation of $S \cap T$ is

$$(1 \wedge 0, 1 \wedge 0, 0 \wedge 0, 1 \wedge 1, 0 \wedge 0, 0 \wedge 1) = (0,0,0,1,0,0).$$

The representation of the complement of a set is the bit by bit complement of its representation. Thus, the representation of $A - S$ is

$$(\bar{1}, \bar{1}, \bar{0}, \bar{1}, \bar{0}, \bar{0}) = (0,0,1,0,1,1).$$

In the matrix of a dominance relation R the jth column provides the Boolean representation of the basis set U_{x_j}. Recall that $b_{ij} = 1$ if and only if $x_i \to x_j$. That is, $b_{ij} = 1$ if and only if x_i is an element of U_{x_j}. In the case of our example in FIGURE 1, we see immediately that the representation of U_3 is simply the third column of the matrix of R, or $(1,0,1,0,0,0)$. Thus, the matrix of R is a data structure not only for the relation itself, but also for the topology it generates. The matrix "stores" the topology in the sense that its columns are the Boolean representations of sets which form a basis.

Our simple criterion for determining whether or not a set U is open can be developed immediately into a computer program. In the matrix of the relation R, form the disjunction of the columns corresponding to the elements of U. If the result is the Boolean representation of U, then U is open; otherwise U is not open. In the case of our example in FIGURE 1, if $U = \{1,3,4\}$, then the disjunction of columns 1, 3, and 4 is $(1,1,1,1,0,0)$ which does not equal U, so that U is not open. (What does the bit string $(1,1,1,1,0,0)$ represent? It does *not* represent the interior of U.)

Not only do the basis sets U_x appear in a natural way in the table of the relation R, but they are also easy to see in the digraph of R. The basis set U_x consists of all those points y such that x is "reachable" from y, i.e., there is a path from y to x. Intuitively, the set U_x begins at x and "spreads" backwards along the directed edges of the graph. The digraph can help one to answer, by inspection, simple questions about the corresponding topology. In our example in FIGURE 1, is there an open set which does not contain the point 1? Since the digraph shows every point is reachable from 1, there is no such open set. Consequently, no two open sets are disjoint.

Before we leave the matrix and the digraph of the relation R, let us look at them from another point of view. We have seen that the columns of the matrix have a topological significance, but do the rows? In the digraph, the points y such that x is reachable from y have a significance. What about the points y which are reachable from x? We expect a certain duality, and we find it by means of the dual topology.

The Dual topology

Topologies generated by dominance relations can also be approached by means of their closed sets. If F is a subset of A, recall that F is closed whenever $A - F$ is open. If F is closed, x is an element of F, and $x \to y$, then y cannot be an element of $A - F$ (since $A - F$ is open). Thus, y is also an element of F. If we let $F_x = \{y | x \to y\}$, then F_x is the smallest closed set about x. That is, $F_x = \overline{\{x\}}$, where $\overline{\{x\}}$ denotes the closure of $\{x\}$. This is clear since F_x is closed (as the relation R is transitive), contains x (as the relation R is reflexive), and is a subset of every closed set about x. A set F is closed if and only if

$$F = \bigcup_{x \in F} F_x.$$

This is the dual of our representation of open sets. Note that, in a general topology, a closed set F has the representation given above. However, a union of closures of single point sets is not necessarily closed.

If the set A is finite, the Boolean representation of the set F_{x_i} is simply the ith row of the matrix of the relation R. A set F is closed if and only if its Boolean representation is the disjunction of the rows corresponding to its elements. Closed sets, like open sets, can be easily constructed using Boolean operations on the matrix of R. In the digraph, the points y contained in F_x are those points y which are reachable from x. Intuitively, F_x "spreads" forwards from x along the edges of the digraph. The key to the duality between open and closed sets is the dual of the relation R.

If R is a dominance relation on a set A, then its **dual** or **converse** R^d is defined by the requirement that $y \to^d x$ if and only if $x \to y$. The dual is also a dominance relation, and its table is the transpose of the original table. Its digraph is generated from the digraph of R by reversing the orientation of each edge. The topology it generates is called the dual of that generated by the original relation R.

If we let $U^d{}_x$ and $F^d{}_x$ be the smallest open and smallest closed sets about x in the dual topology, then $U^d{}_x = F_x$ and $F^d{}_x = U_x$. This is immediate from the definitions and is what we expect from the representations of these sets in the table of R and its transpose. Thus, the closed sets of the original topology are exactly the open sets of its dual, and conversely.

The ability of the topology to distinguish between two points x and y by open sets is equivalent to its ability to distinguish between x and y by closed sets, and both are equivalent to the ability of the relation R to distinguish between x and y. More precisely, the following are equivalent:

$$x \leftrightarrow y$$
$$U_x = U_y$$
$$F_x = F_y.$$

The second condition says that any open set contains both points x and y or else neither point. The third condition says the same thing about closed sets. An easy proof is to notice that the first condition is equivalent to the second and also to the condition that $x \leftrightarrow {}^d y$.

Christie, in [3], uses a slightly different approach to generating finite topologies from preorders. Christie uses the preorder to define an interior operator on the subsets of A and notices that, if A is finite, the topology is completely determined by the interiors of the complements of single point sets. A topology on A can also be defined in terms of a closure operator on the subsets of A or in terms of a boundary operator. For more about such operators, see [4], [7], and [9]. Wilansky, in [13], p. 50, shows how a topology on a finite set A generates a preorder on A. The fact that every finite topology is generated by a preorder has also been used to count the number of topologies on a finite set. The problem is reduced to one of counting the number of matrices of a certain form. For the details, see [5] and [8].

As an exercise, suppose that S is a subset of A. How would one write a computer program to find the interior of S? Recall that $\bigcup_{x \in S} U_x$ is *not* the interior of S. However, if $T = A - S$, then $\bigcup_{x \in T} F_x$ is the closure of T. Also, it is well known that the interior of S is simply $A - \overline{T}$. How would one write a program to find the boundary of S? How does one visualize the interior, the closure, or the boundary of S in the digraph of the relation R?

If R is any relation on the set A, we cannot expect that R will generate a topology. There is, however, a smallest preorder R' on A which contains R. R' is simply the intersection of all preorders on A which contain R (here considered as subsets of $A \times A$). The matrix of R' can be found efficiently by first placing 1's along the main diagonal of the matrix of R (extending R to a reflexive relation) and then finding the transitive closure of the reflexive extension. The transitive closure can be found efficiently by means of Warshall's algorithm (see [1] or [2]). Thus, for each point x of a finite set A, one can specify arbitrarily those points y which must lie in every open set about x (in a reasonable specification, x would be one such point). One can then construct a topology on A in which each basic open set U_x (and thus each open set) contains the required points y and as few others as possible.

Friends and relations

Suppose that the dominance relation is required to be antisymmetric and so is a *partial order*, denoted by $\leqslant$. In this case, for each point x of A, $U_x = \{y \mid y \to x\} = \{y \mid y \leqslant x\}$, and $F_x = \{y \mid x \to y\} = \{y \mid x \leqslant y\}$. (Technically, U_x is the principal ideal generated by the element x.) The single-element open (closed) sets are those consisting of a minimal (maximal) element alone. Two points can be separated by disjoint open (closed) sets if and only if they have no common lower (upper) bound. Thus, in particular, in a lattice no two points can be separated by either disjoint open sets or by disjoint closed sets. In our example in FIGURE 1, there is one single element open set, $\{1\}$, and three single element closed sets. No two points can be separated by disjoint open sets, but some points can be separated by disjoint closed sets (for instance, 6 and 4 can be so separated).

Since a partial order is antisymmetric, for each element x of A, $U_x \cap F_x = \{y \mid y \leftrightarrow x\} = \{x\}$. Conversely, if for each element x of A $U_x \cap F_x = \{x\}$, then the relation is a partial order. A **total** or **linear order** is characterized by the fact that, for each element x of A, $U_x \cup F_x = A$. Here note that, if x is not a minimal element, then U_x is the smallest open neighborhood about one of its points (namely x) but not about any of its other points (for if $y < x$, then $U_y \subsetneq U_x$).

Suppose that the dominance relation R is required to be symmetric and so is an **equivalence relation**. Then, for each point x of A, $U_x = F_x = C_x$, where C_x is the equivalence class containing

x. This follows directly from the definitions of U_x and F_x. If x and y are equivalent, these points lie in exactly the same open sets (since $U_x = U_y$) and exactly the same closed sets (since $F_x = F_y$). If x and y are not equivalent, they are separated by disjoint open sets and by disjoint closed sets (since $U_x \cap U_y = \varnothing$ and $F_x \cap F_y = \varnothing$). Since an equivalence relation is its own dual, the dual topology is simply the original. Any subset S of A is either both open and closed (in case it is the union of equivalence classes) or neither open nor closed. Here each basic neighborhood U_x is the smallest open neighborhood about not just x but about any of its points. The topology $\mathscr{T}$ of our earlier example space $B = \{a, b, c, d, e\}$ is generated by the equivalence relation for which the corresponding equivalence classes are $U = \{a, b, c\}$, $V = \{d\}$, $W = \{e\}$. The points a and b lie in exactly the same open sets and exactly the same closed sets. However, the points a and d can be separated by disjoint open sets and by disjoint closed sets. The topology was originally given in terms of its open sets, and these are precisely the sets which are unions of equivalence classes.

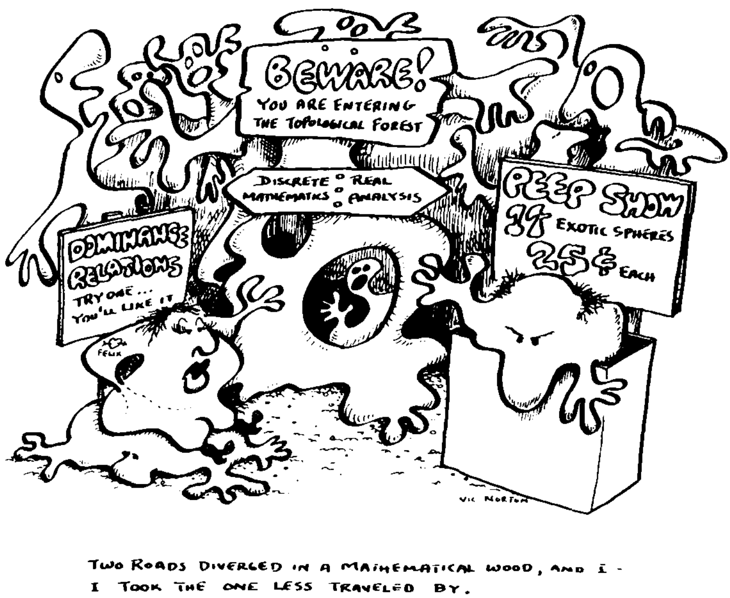
In what other ways do partial orders and equivalence relations generate different types of topologies?

Topological properties

What do relations reveal about the topologies that they generate? One can approach this question from the point of view of the topologist or from that of the computer scientist. That is, one can ask questions about basic topological properties, or one can explore the topological interpretations of familiar theorems from discrete mathematics.

Let us begin with the first point of view. What do preorders reveal about the separation properties of the topologies they induce? What do they reveal about connectivity, compactness, or the continuity of functions?

The familiar spaces of analysis are at the very least Hausdorff. Our dominance relation spaces are hardly so tidy. If the induced topology is Hausdorff, then the relation is the trivial relation in which each element is related only to itself. Recall that a **Hausdorff space** is one in which every two distinct points can be separated by disjoint open sets and that, as a direct consequence of the definition, single point sets are closed. If the space is Hausdorff, for each element x of A, $\{x\} = \overline{\{x\}} = F_x = \{y \mid x \to y\}$. But the trivial relation induces the discrete topology (since $U_x = \{y \mid y \to x\} = \{x\}$). Thus, if the space is Hausdorff, it is a discrete space. Since any finite Hausdorff space is a discrete space, the result is not unexpected.



TWO ROADS DIVERGED IN A MATHEMATICAL WOOD, AND I -
I TOOK THE ONE LESS TRAVELED BY.

The separation properties to explore are clearly those below Hausdorff or T_2. Recall that a **T_0 space** is one in which, whenever x and y are distinct, at least one point has a neighborhood not containing the other. In our case, what is required is that whenever $x \neq y$, then either $x \notin U_y$ or $y \notin U_x$. That is, if $x \neq y$, then either $y \nrightarrow x$ or $x \nrightarrow y$. If the relation is a nontrivial equivalence relation, the resulting space is not even T_0. The T_0 spaces are precisely those induced by partial orders (see [13], p. 50). Recall that a **T_1 space** is one in which, whenever x and y are distinct, each point has a neighborhood not containing the other. In our case, the requirement is that $x \notin U_y$ and $y \notin U_x$ whenever $x \neq y$. Again, the relation must be the trivial one and the topology discrete. In particular, our example in FIGURE 1 is a T_0 space, but no more, and our space B is not even T_0.

Using preorders, we may easily construct nonhomogeneous topologies in which the neighborhood structures at distinct points x and y are quite different. Separation properties might better be studied locally. If we define $h(x, y)$ to be the number of elements in $U_x \cap U_y$, then any neighborhoods of x and y will have in their intersection at least this many points. Thus, $h(x, y)$ is a measure of how separated x and y are. What properties must the function h have? For a finite space, the function is easily computed. In the matrix of the relation $R, h(x_i, x_j)$ is the number of 1's in the conjunction of columns i and j.

Connectivity

Whether or not the space A is connected is revealed in a straightforward way by the digraph of the corresponding relation. The space A is connected exactly when the digraph of the relation R is (weakly) connected, that is, whenever the underlying undirected graph is connected. For a discussion of connectivity in graphs, see [6] or [11]. What is required for connectivity is that, for any points x and y in A, there is a semipath in the digraph from x to y. A semipath differs from a path in that it is permissible to travel along an edge from its terminal point to its initial point. If A is finite, then the connectivity of the digraph can be checked using Warshall's algorithm. (The algorithm is to be applied, not to the original digraph, but to the corresponding symmetric digraph in which each edge is given both orientations. Again, see [1], or [2].) Thus, our example space of FIGURE 1 is connected, but our space B is not.

In order to see that the connectivity of the space is equivalent to the connectivity of the digraph, note first that if K is any component of the digraph, then K is both open and closed in the space. If x is an element of K and $y \rightarrow x$ or $x \rightarrow y$, then y is also an element of K since there is an edge in the undirected graph between x and y. Thus, K is open and closed. Conversely, if a set K is both open and closed in the space, there can be no edge in the undirected graph between K and $A - K$, and K is a component of the digraph. The space A is connected if and only if A itself is the only nonempty set both open and closed in A. The graph is connected if and only if its only component is A.

Suppose that we construct "at random" a relation R on an n-element set A. That is, we write a computer program to assign each entry of an $n \times n$ matrix the value 0 or 1 with specified probabilities $p(0) = p$ and $p(1) = 1 - p$. If we then construct the smallest preorder R' containing R, what is the probability that the space A is connected under the topology generated by R'? We can find an approximate answer, simply and by elementary methods. The process of constructing the relation R, generating R', and checking whether the associated digraph is connected can be programmed efficiently, as outlined above, using Warshall's algorithm. The process can be considered a Bernoulli trial. Our program can make m trials, and we can (by the methods of elementary undergraduate statistics) find confidence intervals for the probability that the space is connected. For an easier project, what is the probability that the space is T_0? It is trivial to check that a matrix is antisymmetric.

If the set A has at least two elements, then an equivalence relation on A produces either the indiscrete topology or a topology in which A is not connected. Here the components of the graph are exactly the equivalence classes. Partial orders may clearly produce disconnected spaces, but lattice structures never do.

However, any preorder produces a locally connected space. The sets U_x form a basis of connected open sets. If we had $x \in V \subsetneq U_x$, where V was both open and closed in U_x, then V would be open in A, and U_x would not be the smallest open set about x. In particular, then, all finite spaces are locally connected. There are, of course, infinite locally connected spaces (such as E^n) whose topology is not generated by a preorder.

By considering the dual topology we know that, for each element x of A, $F_x = \overline{\{x\}}$ is connected. This is not peculiar to topologies generated by dominance relations. Closures of single point sets are always connected (see [4], p. 109). Sometimes the dual topology leads us to new information, and sometimes it leads us back to well known facts.

Compactness

What can be said about the compactness of our dominance relation spaces? If A is finite, it is always a compact space. If A is infinite, a preorder may generate either a compact or a noncompact topology for A. For example, both the discrete and indiscrete topology for A are generated by preorders (in fact, by equivalence relations).

If A is a finite space, let us call A ***m*-compact** if and only if every open cover of A has a subcover of at most m sets. For a given space A, what is the least value of m, and how may the sets of a subcover be generated? One approach to this question is in terms of maximal elements. An element x of A is called a **maximal element** if and only if, whenever y is an element of A and $x \to y$, then $y \to x$. In our terminology, x is maximal if and only if F_x is a subset of U_x, and this condition can be checked easily using Boolean operations on the matrix of the relation R. If the relation R is restricted to maximal elements, then it is an equivalence relation. If the number of equivalence classes is k, then it is straightforward to verify that every open cover of A contains a subcover of at most k sets. That is, A is k-compact.

Given an open cover of A, a subcover can be found by choosing, for each equivalence class C_i $(1 \leqslant i \leqslant k)$ a maximal element x_i of C_i and a set U_i from the open cover with x_i an element of U_i. Note that if x_i is an element of U_i, then C_i is a subset of U_i since U_i is open. If x is an element of A, then either x is maximal and $x \in C_i \subseteq U_i$ for some i, or else there is a maximal element x_i with $x \to x_i$. In this case, x is an element of U_i since U_i is open. The number k is best possible since the cover $\mathscr{U} = \{U_x | x \in A\}$ of basic open sets has no subcover of fewer than k sets. In particular, our space in FIGURE 1 is 3-compact, and our example space B is also 3-compact. (What does the dual topology reveal about closed covers of A?)

The process for generating the subcover is straightforward. We inspect the elements of A in order, and we produce a collection of nonequivalent maximal elements as follows: given an element x, if $x \to y$, where y is already part of our collection, we discard x. Otherwise, we add x to our collection. If the open sets of the cover $\mathscr{U}$ are given as a list of Boolean representations, we search the list until we have found enough open sets to cover our collection of maximal elements. This will require us to find, at most, one open set per maximal element. The number of sets we find may, of course, depend on the order of the list of open sets. For instance, try the above process with our example space B and the cover $\mathscr{U} = \{U, V, W, B\}$. If the cover sets are listed in the order given, the subcover will consist of U, V, and W. If the cover sets are listed in the opposite order, the subcover will consist of B alone.

Continuity

In order to investigate a function f from some finite set A to a finite set B, we need a representation of the function itself as well as representations of both topologies. If the function has a simple closed form, for instance $f(x) = x^2$, this presents no problem. For a more arbitrary function f there is a standard tabular representation of f as a relation from A to B which is convenient in that both the image of each point x of A and the preimage of each point y of B are easily accessible. If $A = \{x_1, x_2, \dots, x_n\}$ and $B = \{y_1, y_2, \dots, y_m\}$, the **table** of f is a matrix (f_{ij}) in which $f_{ij} = 1$ if $f(x_i) = y_j$, and otherwise $f_{ij} = 0$. Thus, the ith row is the Boolean representation of the (one element) image set of the point x_i, and the jth column is the Boolean representation of the preimage of the point y_j. The disadvantage of this representation is the storage requirement. A more compact representation of f is simply a one-dimensional array (a_i) in which $a_i = j$ when $f(x_i) = y_j$. Here the preimage of the point y_j is not so easily accessible.

Since the sets A and B are finite, their topologies are given by dominance relations. This fact helps us to determine the continuity of the function f without explicitly recovering preimages. Let $\rightarrow$ be used to denote each dominance relation. Then the function f is continuous if and only if $f(y) \rightarrow f(x)$ whenever $y \rightarrow x$. That is, f is continuous if and only if f preserves dominance. The proof follows directly from the fact that open sets V are those for which z is an element of V whenever w is an element of V and $z \rightarrow w$ (see [**13**], p. 60).

The continuity of a function f can be checked easily with the array representation. Suppose that a function f from our space A in FIGURE 1 to our space B is given by the array $F = (1,1,2,4,4,3)$. That is, $f(1) = a$, $f(2) = a$, $f(3) = b$, $f(4) = d$, $f(5) = d$, $f(6) = c$. Let the matrices of the dominance relations on A and B be denoted by T^A and T^B, respectively. We check each entry of T^A. If for some i and j $(1 \leqslant i \leqslant 6, 1 \leqslant j \leqslant 6)$ we have $T^A{}_{ij} = 1$ and $T^B{}_{F_iF_j} = 0$, then the function f is not continuous. The check can be programmed as a simple double do-loop.

In particular, a homeomorphism f from A to B is a one-to-one correspondence for which $y \rightarrow x$ if and only if $f(y) \rightarrow f(x)$. In other words, f is an isomorphism between the digraphs of the corresponding dominance relations. This characterization allows the use of graph-theoretical methods in determining whether two spaces are homeomorphic. Although a number of isomorphism invariants are known for graphs, there is as yet no simple complete set of such invariants.

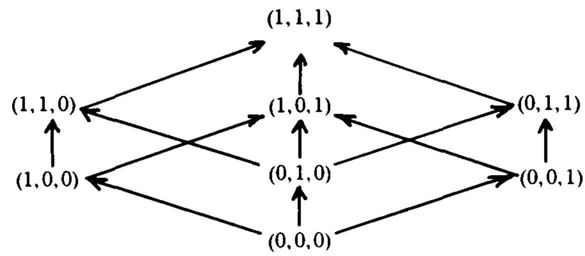
If the dominance relations on both A and B are equivalence relations, then any continuous function f from A to B must preserve equivalence classes. If x is an element of A, there is some element w of B such that $f(C_x)$ is a subset of C_w, where C_x and C_w denote the equivalence classes containing x and w, respectively. This follows directly from the fact that $f(x) \equiv f(y)$ whenever $x \equiv y$. Since any partitions of A and B are induced by corresponding equivalence relations, it is easy to find topologies on finite sets of real numbers (or even on the reals themselves) under which such functions as $f(x) = 2x$ or $f(x) = x^2$ are not continuous. The derivation in terms of equivalence relations even makes these topologies seem (in some sense) natural.

If the dominance relations on A and B are partial orders, the function f is continuous if and only if it is order-preserving. That is, f is continuous if and only if $f(x) \leqslant f(y)$ whenever $x \leqslant y$, where $\leqslant$ is the generic symbol for a partial order.

If the relation on A is an equivalence relation while that on B is a partial order, then under any continuous function f each equivalence class must map to a single point in B. If $x \equiv y$, then $f(x) = f(y)$ since a partial order is antisymmetric. What if the relation on A is a partial order while that on B is an equivalence relation? What if A is a lattice? Why must A then map entirely into a single equivalence class?

The above discussion illustrates how the properties of dominance relations illuminate the topological structures of all finite (and some infinite) spaces, and how the properties of continuous functions illuminate the interplay between two different structures on different spaces or on the same space. A point set topology text (such as [**3**], [**4**], [**7**], [**9**], or [**13**]) will suggest a number of other topological properties that can be explored using dominance relations.

As an exercise, how would we apply what we have learned to a type of finite set which is a major focus of discussion in elementary discrete mathematics, namely, the finite Boolean algebra? Consider the algebra as a subset algebra. If A is any finite set, then the power set of A with the subset relation $\subseteq$ is a partially ordered set, in fact a distributive complemented lattice (for details, see [**10**] or [**11**]). As a space with the topology induced by the relation $\subseteq$, what properties does it have? (See FIGURE 2 for the case of A a three-element set.) If S is any subset of A, U_S is simply the collection of all subsets of S, and F_S is the collection of all supersets of S. If S and T are any subsets of A, then S and T cannot be separated by disjoint open sets. In this case, $h(S,T)$ is the number of subsets of $S \cap T$. Since the subset relation $\subseteq$ is a partial order, the space is T_0. Since the set A itself is a universal upper bound, the space is 1-compact. The space is connected since its digraph is connected.



R	(0,0,0)	(0,0,1)	(0,1,0)	(0,1,1)	(1,0,0)	(1,0,1)	(1,1,0)	(1,1,1)
(0,0,0)	1	1	1	1	1	1	1	1
(0,0,1)	0	1	0	1	0	1	0	1
(0,1,0)	0	0	1	1	0	0	1	1
(0,1,1)	0	0	0	1	0	0	0	1
(1,0,0)	0	0	0	0	1	1	1	1
(1,0,1)	0	0	0	0	0	1	0	1
(1,1,0)	0	0	0	0	0	0	1	1
(1,1,1)	0	0	0	0	0	0	0	1

FIGURE 2. **The digraph and adjacency matrix of the set inclusion relation R on the power set of A, where A is any three element set. The Boolean representations of the subsets are used, and these representations are the key to decomposing the relation as a product relation and the corresponding topological space as a product space. Notice that only the part of the matrix above the main diagonal need be stored.**

If A is large, the digraph is extremely complex, and this suggests that the resulting topology is also complex. To the contrary, the space can be naturally decomposed as the product of n copies of a simple two element space. The key is to represent the subset order as a product order. This is the approach often taken in elementary discrete mathematics (for instance, see [**11**] and [**12**]). As a project, verify that the product of dominance relations is also a dominance relation, and the topology it generates is the product topology, if the product is finite (for information about product topologies, see [**4**] or [**7**]). Use the product representation to investigate the topology of the power set of A.

From discrete mathematics to topology

In the previous discussion, we have asked topological questions about spaces whose topologies are given by preorders. We may also study these spaces by considering the topological interpretations of known facts from discrete mathematics. We will simply illustrate this second approach by an example.

If A is a partially ordered set with order relation denoted by $\leqslant$, then a **chain** in A is a subset of A in which any two elements are related, and an **antichain** is a subset in which no two elements are related. A familiar theorem from discrete mathematics is a dual of Dilworth's theorem: *If A is a partially ordered set in which the length of a longest chain is n, then A can be partitioned into n (but no fewer) disjoint antichains* (see [**11**]). In topological terms, what does this say? The elements $x_1 < x_2 < \cdots < x_n$ form a chain if and only if the corresponding basic open sets $U_{x_1} \subsetneq U_{x_2} \cdots \subsetneq U_{x_n}$ form a chain. The subset S of A forms an antichain if and only if S has, as a subspace, the discrete topology. This follows immediately from the fact that, for each element y of S, $U_y \cap S = \{y\}$. Thus, in topological terms, if the length of a longest chain of basic open sets is n, then A can be partitioned into n (but no fewer) subspaces, each of which has the discrete topology. Can we omit the requirement that the open sets be basic open sets U_x? We expect that our space A in FIGURE 1 can be partitioned into three (but no fewer) discrete subspaces. What are the subspaces? How would you write a computer program to generate them? (The method for finding the antichains described in [**11**] leads immediately to a program.) What are the topological interpretations of other theorems from discrete mathematics?

Postscript

We have seen how any topology on a finite set A (and some topologies on an infinite set A) can be generated by dominance relations or preorders on A. If A is finite, the properties of the topology are reflected in both the matrix and the digraph of the relation. In particular, the matrix is an efficient, easily maintained data structure for the topology. Basis sets appear as columns, and open sets are disjunctions of columns. Closures of single point sets appear as rows, and closed sets are disjunctions of rows.

In asking standard topological questions, we found as answers both theorems and computational methods. We have also briefly illustrated how theorems from discrete mathematics can have topological interpretations.

How far is it from computer science to topology? Not far at all.

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Author's Note

In preparing this manuscript
we were not assisted
by any grant;
in fact,
we were seriously distracted
by teaching duties,
inadequate facilities,
requirements that we attend
to details of administrative fervor
and making do
with inferior pay
and insecure position.

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