Robotics 2
Data Association

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Data Association

> "Data association is the process of associating uncertain measurements to known tracks."

- **Problem types**
  - Track creation, maintenance, and deletion
  - Single or multiple sensors
  - Target detection
  - False alarm model and rates
  - Single or multiple targets

- **Approaches**
  - **Bayesian:** compute a full (or approx.) distribution in DA space from priors, posterior beliefs, and observations
  - **Non-Bayesian:** compute a maximum likelihood estimate from the possible set of DA solutions

Data Association

Overall procedure:

- **Make observations** (= measurements).

  Measurements can be raw data (e.g. processed radar signals) or the output of some target detector (e.g. people detector)

- **Predict the measurements** from the predicted tracks.

  This yields an area in sensor space where to expect an observation. The area is called **validation gate** and is used to narrow the search

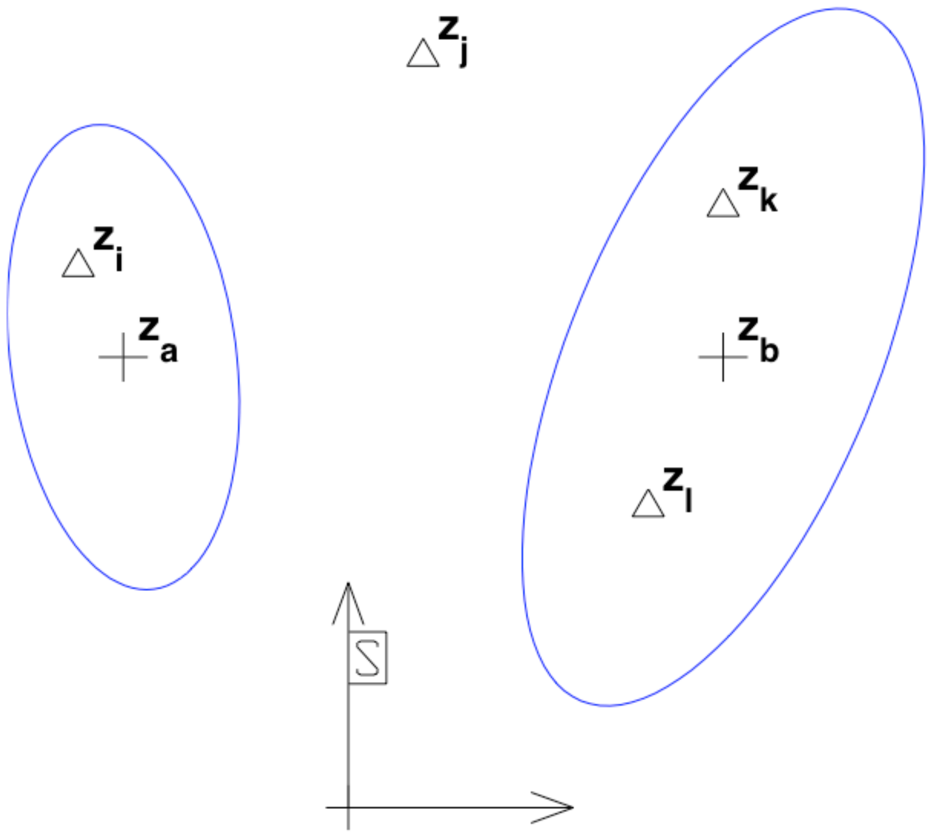
- **Check if a measurement lies in the gate**.

  If yes, then it is a valid candidate for a pairing/match

Data Association

What makes this a difficult problem

- **Multiple targets**
- **False alarms**
- **Detection uncertainty** (occlusions, sensor failures, …)
- **Ambiguities** (several measure-ments in the gate)



Measurement Prediction

- Measurement and measurement cov. prediction
  - This is typically a frame transformation into sensor space

$$\hat{z}(k) = H(k)\hat{x}(k|k-1)$$
$$\hat{R}(k) = H(k)\hat{P}(k|k-1)H^T(k)$$

- If only the **position** of the target is observed (typical case), the measurement matrix is

$$\mathbf{z} = \begin{bmatrix} x & y \end{bmatrix}^T \qquad H = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \end{bmatrix}$$

- Note: One can also observe
  - Velocity (Doppler radar)
  - Acceleration (accelerometers)

Validation Gate

- Assume that measurements are distributed according to a Gaussian, centered at the measurement prediction $\widehat{z}(k)$ with covariance $\widehat{S}(k)$

$$p(z(k)) = \mathcal{N}(z(k); \widehat{z}(k), \widehat{S}(k))$$

  This is the **measurement likelihood model**

- Let further

$$d = \sqrt{(\mathbf{x} - \mu)^{\mathbf{T}} \mathbf{C}^{-1} (\mathbf{x} - \mu)}$$

  be the **Mahalanobis distance** between $\mathbf{x}$ and μ

Validation Gate

- Then, the measurements will be in the area

$$\begin{aligned} \mathcal{V}(k,\gamma) &= \{z : (z-\widehat{z})^T \widehat{S}^{-1}(z-\widehat{z}) \leq \gamma\} \\ &= \{z : d^2 \leq \gamma\} \end{aligned}$$

  with a probability defined by the gate threshold γ (omitting indices k)

- This area is called **validation gate**
- The threshold is obtained from the inverse χ^2 cumulative distribution at a **significance level** α
- χ^2 = "chi square"

Validation Gate

- The **shape** of the validation gate is a hyper-ellipsoid (an ellipse in 2d)
- This follows from setting

$$c = \frac{1}{(2\pi)^{k/2}|S|^{1/2}} \exp\left(-\frac{1}{2}(z-\hat{z})^T S^{-1}(z-\hat{z}))\right)$$

which gives

$$c' = (z-\hat{z})^T S^{-1}(z-\hat{z})$$

- The gate is defined by an **iso-probability contour** obtained when intersecting a Gaussian with a hyper-plane.

Validation Gate

Why a χ^2 distribution?

- Let X_i be a set of k i.i.d. standard normally distributed random variables, $X_i \sim \mathcal{N}(x; 0, 1)$.

  Then, the variable Q

  $$Q = \sum_{i=1}^{k} X_i^2$$

  follows a χ^2 distribution with k "degrees of freedom"

- We will now show that the Mahalanobis distance is a sum of squared standard normally distributed RVs.

Validation Gate in 1D

- Assume 1D measurements and $\mu = \hat{z}(k),\, \sigma^2 = \hat{S}(k)$
- The Mahalanobis distance is then

$$d^2 = (z-\mu)^T(\sigma^2)^{-1}(z-\mu) = \frac{(z-\mu)^2}{\sigma^2}$$

- By changing variables, $y = (z-\mu)/\sigma$, we have

$$y \sim \mathcal{N}(0,1)$$

- Thus, $d^2 = y^2$ and is χ^2 distributed with 1 degree of freedom

Validation Gate in ND

- Assume ND measurements and $\mu = \hat{z}(k),\, \Sigma = \hat{S}(k)$
- The Mahalanobis distance is then

$$d^2 = (z-\mu)^T \Sigma^{-1} (z-\mu)$$

- By changing variables, $y = C^{-1}(z-\mu),\, \Sigma = CC^T$ we have $y \sim \mathcal{N}(0, I)$ and therefore

$$d^2 = y^T I^{-1} y \quad \Rightarrow \quad d^2 = \sum_{i=1}^{k} y_i^2$$

which is χ^2 distributed with k degrees of freedom.
(C is obtained from a Cholesky decomposition)

Validation Gate

Where does the threshold γ come from?

- γ, often denoted $\chi^2_{k,\alpha}$, is taken from the inverse χ^2 cumulative distribution at a level α and k d.o.f.s
- The values are typically given in tables, e.g. in most statistics books (or by the Matlab function `chi2inv`)
- Given the level α, we can now understand the interpretation of the validation gate:

> The validation gate is a **region of acceptance** such that $100(1-\alpha)\%$ of **true measurements** are **rejected**

- Typical values for α are 0.95 or 0.99

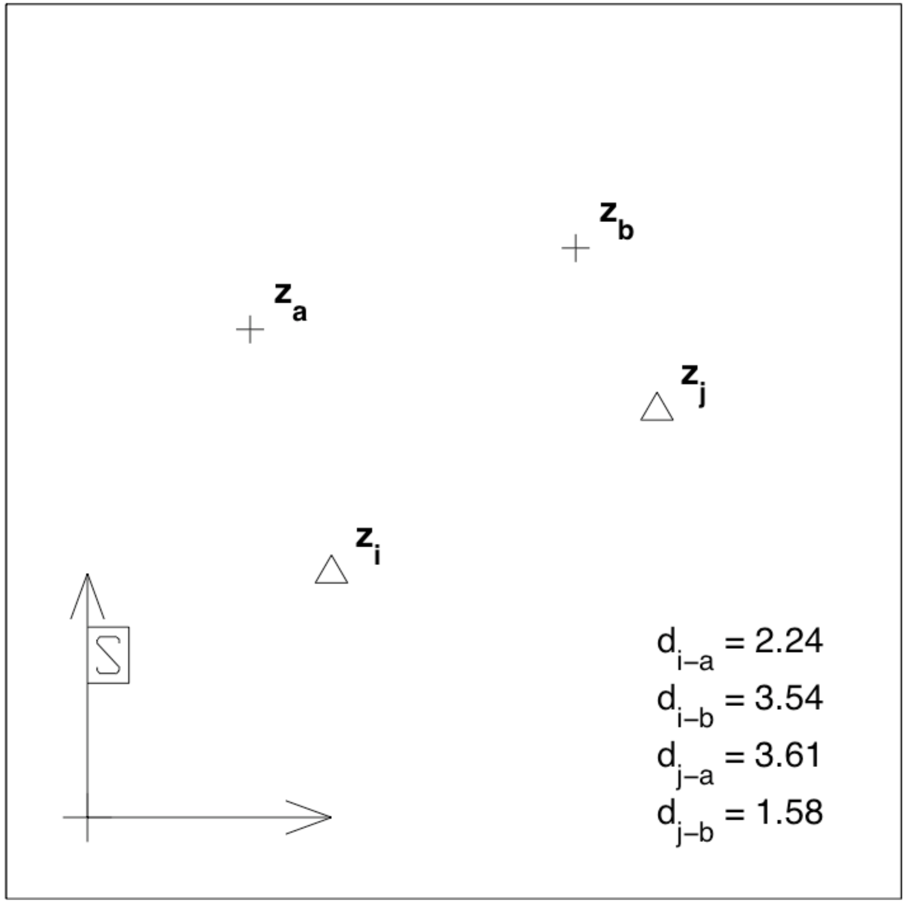
Validation Gate

Euclidian distance

Takes into account:

- ✓ Position
- ✗ Uncertainty
- ✗ Correlations

→ It seems that i-a and j-b belong together



△ Observations + Predictions

Validation Gate

Mahalanobis distance with **diagonal** covariance matrices

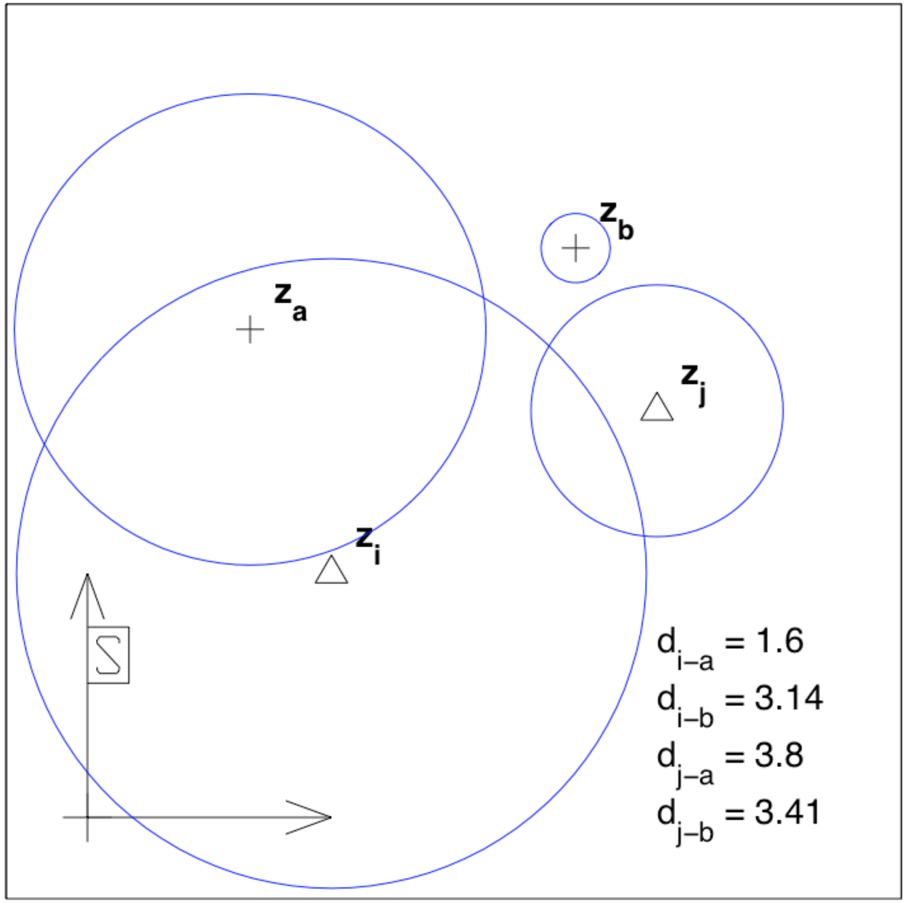
Takes into account:

✓ Position

✓ Uncertainty

✗ Correlations

→ Now, i-b is "closer" than j-b



△ Observations + Predictions

Validation Gate

Mahalanobis distance

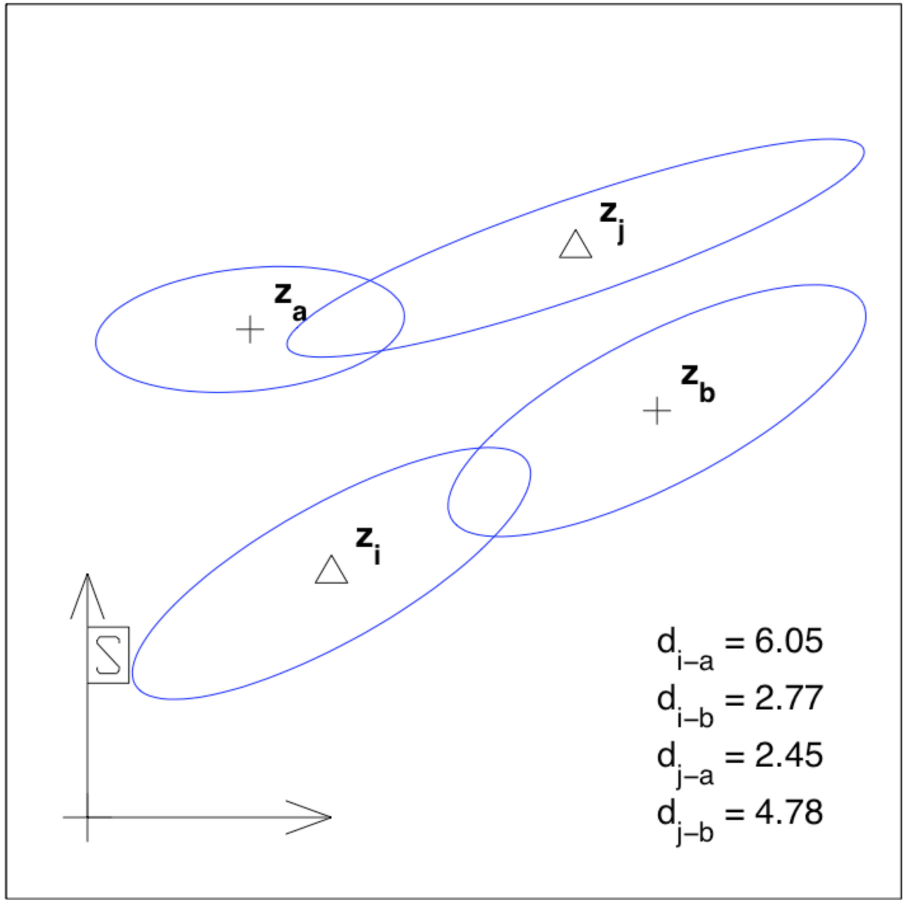
Takes into account:

✓ Position

✓ Uncertainty

✓ Correlations

→ It's actually i-b and j-a that belong together!



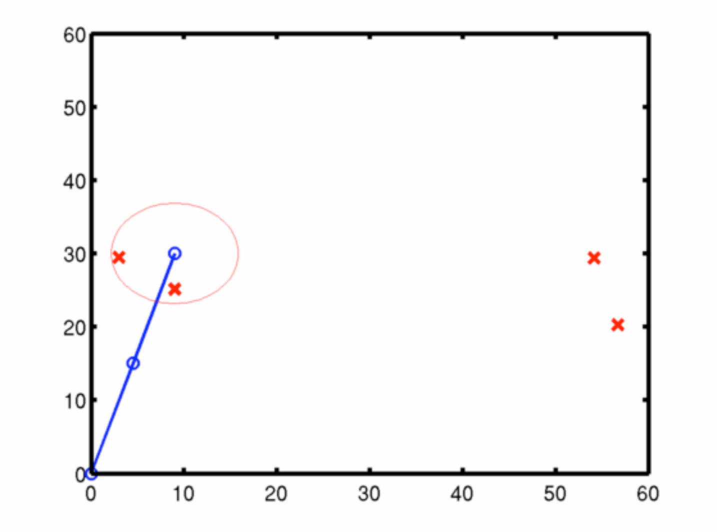
△ Observations + Predictions

False Alarms

- False alarms are **false positives**
- They can come from sensor imperfections or detector failures
- They raise the two questions:

  What is actually inside my **validation gate**?
  - The real measurement or
  - a false alarm?

  How to **model false alarms**?
  - Uniform over sensor space
  - Independent across time

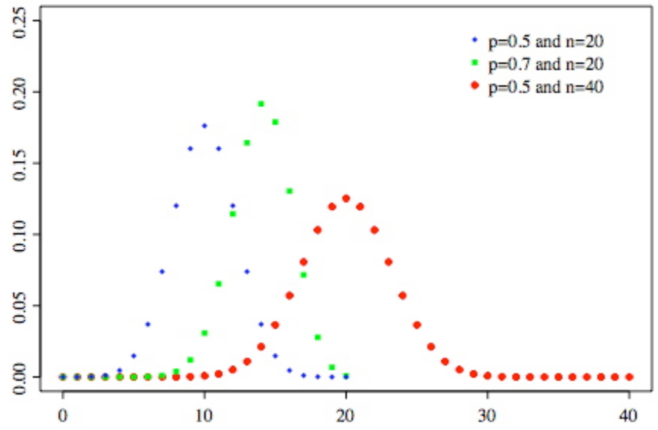


False Alarm Model

- Assume (temporarily) that the sensor field of view V is discretized into N discrete cells, $c_i,\ i = 1, \ldots, N$
- In each cell, false alarms occur with probability P_F
- Assume independence across cells
- The occurrence of false alarms is a Bernoulli process (flipping an unfair coin) with probability $p = P_F$
- Then, the number of false alarms m_F follows a **Binomial distribution**

$$P(K = m_F) = \binom{N}{m_F} p^{m_F} (1-p)^{N-m_F}$$

with expected value Np



False Alarm Model

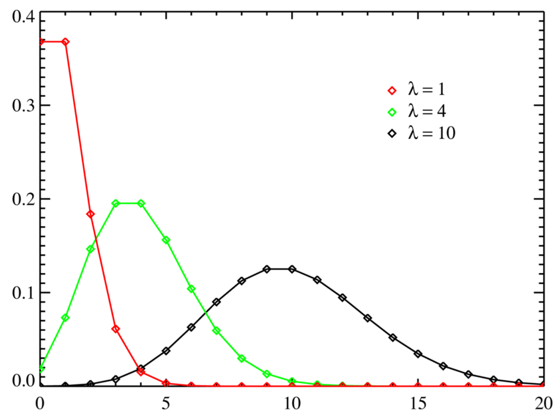
- Let the spatial density λ be the number of false alarms over space

$$\lambda = \frac{Np}{V}$$

[occurrences per m²]

- Let now $N \to \infty$, that is, we reduce the cell size until the continuous case. Then the Binomial becomes a Poisson distribution with

$$\mu_F(m_F) = e^{-\lambda V} \frac{(\lambda V)^{m_F}}{m_F!}$$



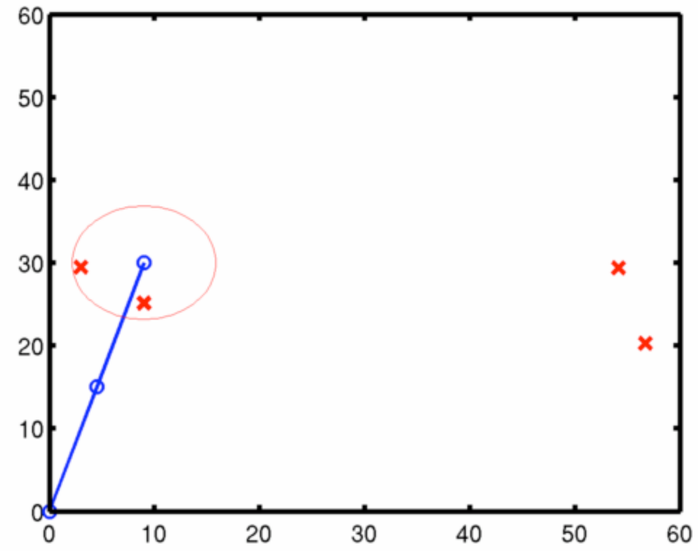
- The **measurement likelihood** of false alarms is assumed to be uni-form,

$$p(z|z \text{ is a false alarm}) = \frac{1}{V}$$

Single Target Data Association

Let us consider

- A **single** target to track
- Track already initialized
- Detection probability < 1
- False alarm probability > 0



Data association approaches

Non-Bayesian:

- Nearest neighbor (NN)
- Track splitting filter

Bayesian:

- Probabilistic Data Association Filter (PDAF)

Single Target DA: NN

Nearest Neighbor filter (NN)

1. Compute Mahalanobis distance to all measurements
2. Accept the **closest measurement**
3. Update the track as if it were the correct measurement

- Problem: with some probability the selected measurement is not the correct one. This can lead to filter divergence (covariances collapse regardless)

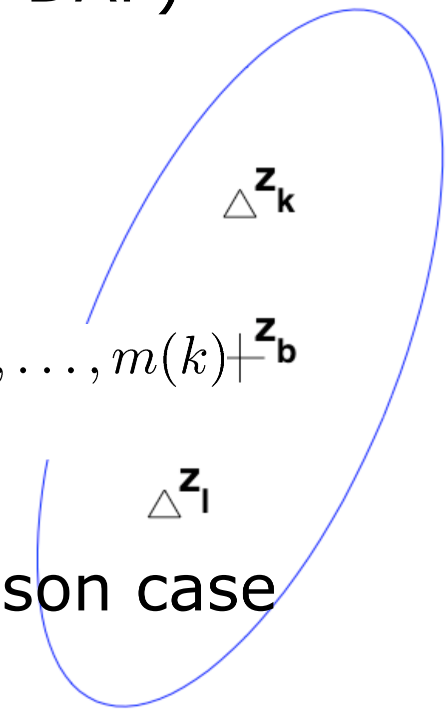
- **Conservative NN variant:**
  Do not associate in case of ambiguities

Single Target DA: PDAF

Probabilistic Data Association filter (PDAF)

- Integrates **all** measurements in the validation gate
  - Conditioning the update on

$$\theta_i(k) = \begin{cases} z_i(k) \text{ is the correct measurement} & i = 1, \ldots, m(k) \\ \text{no correct measurement is present} & i = 0 \end{cases}$$



- With probability $\beta_i \triangleq P(\theta_i | Z^k)$ for the Poisson case

$$\beta_i(k) = \begin{cases} \frac{e_i}{b+\sum_{j=1}^{\mu_F}} & i = 1, \ldots, m(k) \\ \frac{b}{b+\sum_{j=1}^{\mu_F}} & i = 0 \end{cases}$$

$$e_i = \mu_F(m(k) - 1) \cdot P_D P_G \cdot P_G^{-1} \mathcal{N}(\nu_i(k); 0, \hat{S}(k))$$

$$b = \mu_F(m(k))(1 - P_D P_G)$$

Single Target DA: PDAF

- Uses all the measurements in the validation area
  - Conditioning the update on

$$\theta_i(k) = \begin{cases} z_i(k) \text{ is the correct measurement} & i = 1, \ldots, m(k) \\ \text{no correct measurement is present} & i = 0 \end{cases}$$

- With probability $\beta_i \triangleq P(\theta_i | Z^k)$ for the Poisson case

$$\begin{aligned}
\beta_i(k) &= \begin{cases} \frac{e_i}{b + \sum_{j=1}^{\mu_F}} & i = 1, \ldots, m(k) \\ \frac{b}{b + \sum_{j=1}^{\mu_F}} & i = 0 \end{cases} \\
e_i &= \mu_F(m(k) - 1) \cdot P_D P_G \cdot P_G^{-1} \mathcal{N}(\nu_i(k); 0, \hat{S}(k)) \\
b &= \mu_F(m(k))(1 - P_D P_G)
\end{aligned}$$

Single Target DA: PDAF

- State update

$$\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)\nu(k)$$

  - With the combined innovation

$$\nu(k) = \sum_{i=1}^{N} \beta_i(k)\nu_i(k)$$

- Covariance update

$$P(k|k) = \beta_0(k)P(k|k-1) + (1-\beta_0(k))P(k|k) + \tilde{P}(k)$$

  - With the spread of innovations

$$\tilde{P}(k) = K(k)\left[\sum_{i=1}^{N} \beta_i(k)\nu_i(k)\nu_i(k)^T - \nu(k)\nu(k)^T\right]K(k)^T$$

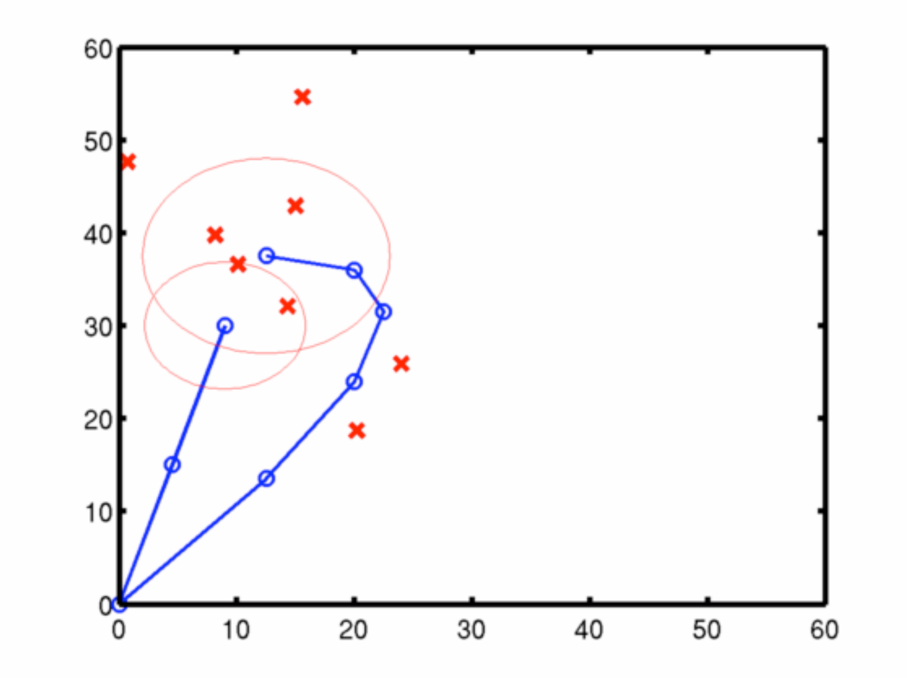
Single Target DA: Summary

- **Nearest Neighbor filter** (NN)
  - Simple to implement
  - Can integrate wrong measurements (false alarms), and thus, produce overconfident estimates
  - Good if prediction and measurement models are accurate

- **Probabilistic Data Association filter** (PDAF)
  - A bit more involved to implement
  - Provide conservative estimates
  - Good in presence of high clutter and noisy models

Multi-Target Data Association

- Multiple targets to track
  - Tracks already initialized
  - Detection probability < 1
  - False alarm probability > 0
- Non Bayesian approaches
  - Nearest neighbor
  - Interpretation tree
  - Joint compatibility (JCBB)
- Bayesian approaches
  - JPDAF
  - MHT
  - MCMC



Multi-Target DA: NN

- Build the assignment matrix $A = \left[d_{ij}^2\right]$

$$d_{ij}^2 = \nu_{ij}(k)^T S_j^{-1}(k) \nu_{ij}(k)$$

- Iterate
  - Find the minimum cost assignment in A
  - Remove the row and column of that assignment

- Check if assignment are in the validation regions
  - Unassociated tracks can be used for track deletion
  - Unassociated measurements can be used for track creation

- Problem: It's not a global minimum
- Conservative NN variant: no association in case of ambiguities

Multi-Target DA: Global NN

- Build the assignment matrix $A = [d_{ij}^2]$

$$d_{ij}^2 = \nu_{ij}(k)^T S_j^{-1}(k) \nu_{ij}(k)$$

- Solve the linear assignment problem

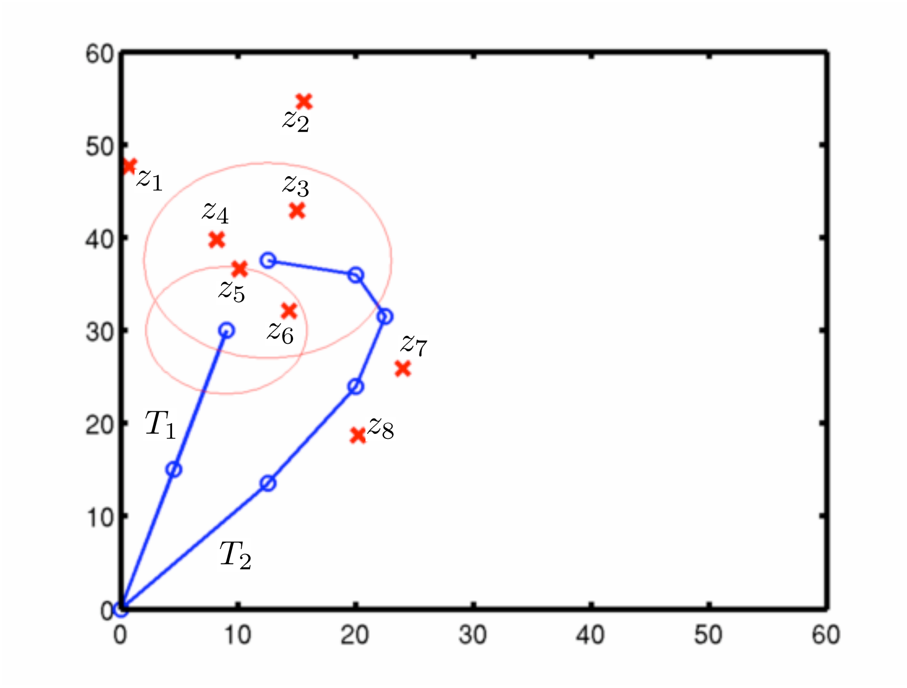
$$min \sum d_{ij}^2 \cdot x_{ij} \quad x_{ij} \in \{0, 1\}$$

$$\sum_i x_{ij} = 1 \quad \sum_j x_{ij} = 1$$

  - **Hungarian** method (blow up to square matrix)
  - **Munkres** algorithm for rectangular matrices
  - Finds **global** cost minimum!

- Check if assignments are in the validation gate
- Performs DA jointly!

Assignment Matrix Example



- Rectangular

$$A = \begin{bmatrix} d_{11}^2 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 & d_{16}^2 & d_{17}^2 & d_{18}^2 \\ d_{21}^2 & d_{22}^2 & d_{23}^2 & d_{24}^2 & d_{25}^2 & d_{26}^2 & d_{27}^2 & d_{28}^2 \end{bmatrix}$$

- Square

$$A = \begin{bmatrix} d_{11}^2 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 & d_{16}^2 & d_{17}^2 & d_{18}^2 \\ d_{21}^2 & d_{22}^2 & d_{23}^2 & d_{24}^2 & d_{25}^2 & d_{26}^2 & d_{27}^2 & d_{28}^2 \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \\ p_F & p_F & p_F & p_F & p_F & p_F & p_F & p_F \end{bmatrix}$$

- Entries
  - $d_{ij}^2 = \nu_{ij}(k)^T S_j^{-1}(k) \nu_{ij}(k)$
  - False alarm probability

GNN vs. Interpretation Tree

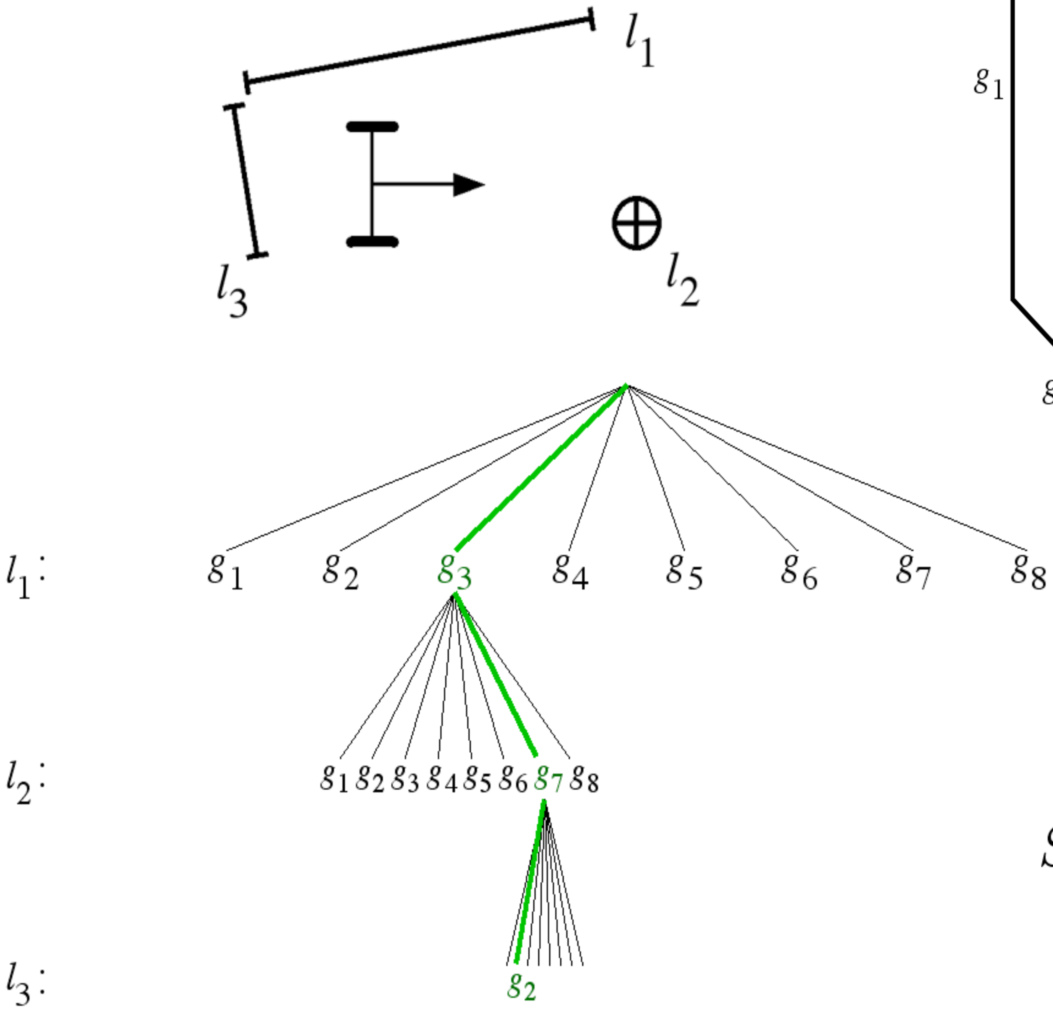
- A solution to GNN in presence of constraints
  - Introduced in [Grimson 87], used in feature-based SLAM

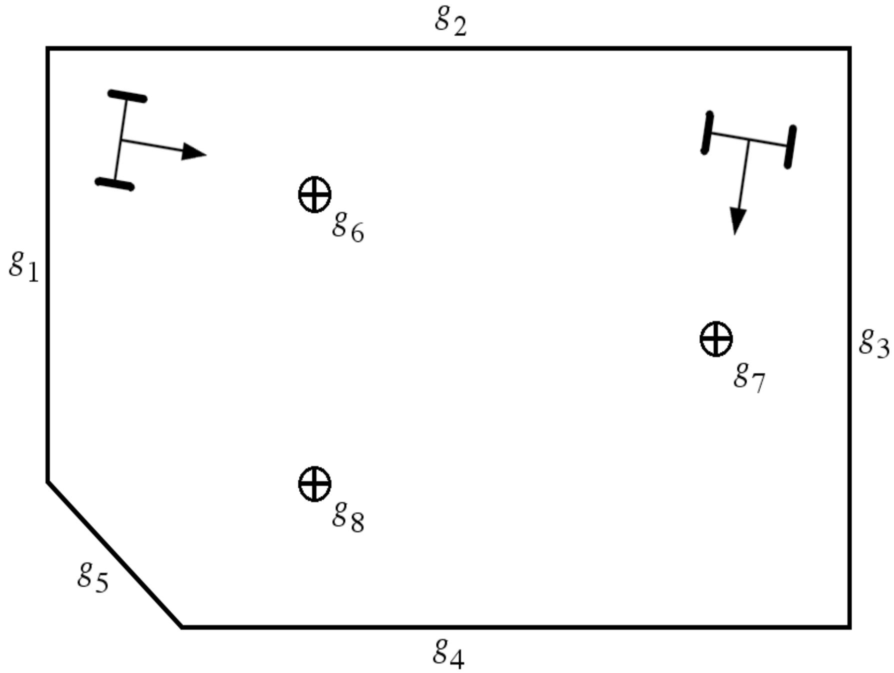
- Main idea: consider all possible association among measurements and tracks
  - The association are built over a tree representation
  - A Depth-first visit on the tree is performed
  - Constraints are used to prune the tree

- Worst case: exponential complexity
  - The whole number of associations is $(\#T + 1)^{\#m}$

Interpretation Tree: SLAM

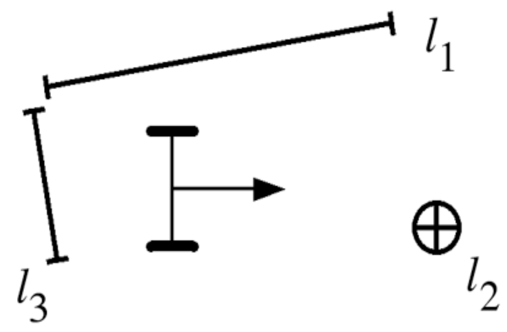
Interpretation tree

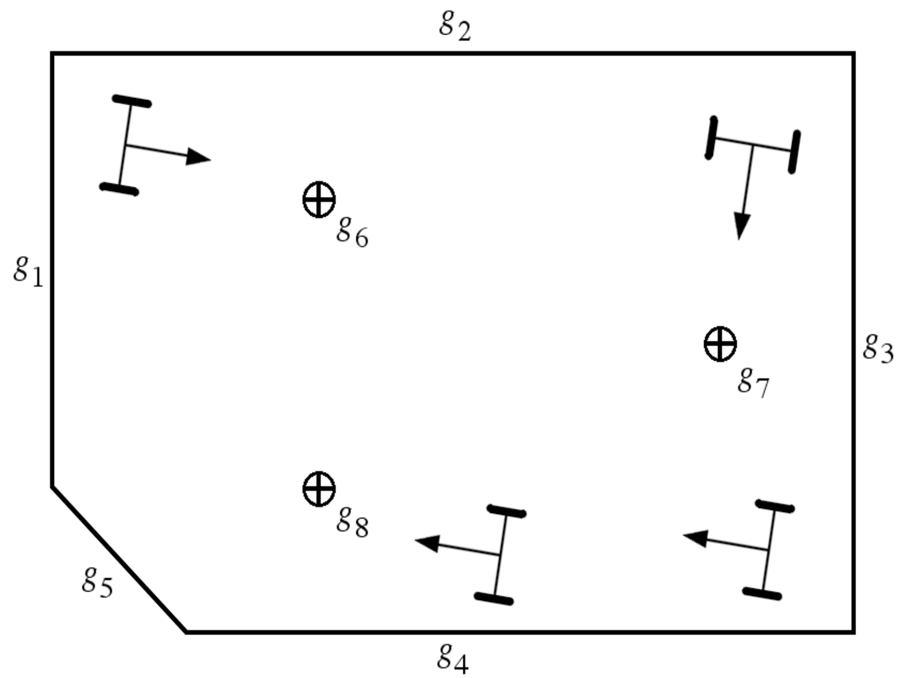


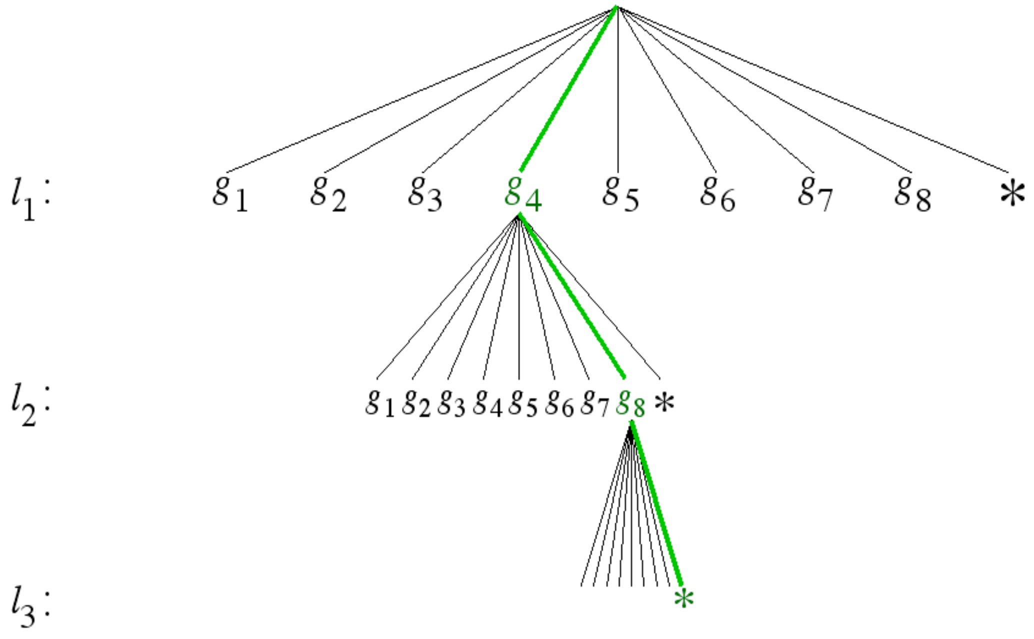


$$S_{h_2} = \{\{l_1, g_3\}, \{l_2, g_7\}, \{l_3, g_2\}\}$$

Interpretation Tree: SLAM







$$S_h = \{\{l_1, g_4\}, \{l_2, g_8\}, \{l_3, *\}\}$$

GNN Comparison

Assignment solver

- Pros
  - Fast: polynomial
  - Libraries available
  - Extension to k-best
- Cons
  - Only linear constraints
  - Blow-up of cost matrix

Interpretation tree

- Pros
  - General constraints
  - Extensive search
  - Provide k-best
- Cons
  - Slow: exponential

Interpretation trees are good when several constrains are available, in other cases assignment solver are preferable

Joint Compatibility

- Individual compatibility (e.g. independent tracks)
  - One measurement integration influences only one track
  - One measurement per track assumption
  - Typical of target tracking

- Joint compatibility (e.g. correlated tracks)
  - One measurement integration influences several track
  - Multiple measurement per track association
  - Typical of localization and SLAM

Joint Compatibility

- Given the joint hypothesis $\mathcal{H} = \{j_1, \ldots, j_m\}$
  - the k-th measurement is associated with track $j_k = t$

- And the joint measurement function

$$
\begin{aligned}
\mathbf{z}_{\mathcal{H}} &= \mathbf{h}_{\mathcal{H}}(\mathbf{x}) + \mathbf{w} \\
\mathbf{h}_{\mathcal{H}} &= \begin{bmatrix} \mathbf{h}_{ij_1} \\ \vdots \\ \mathbf{h}_{ij_m} \end{bmatrix}
\end{aligned}
$$

- The hypothesis is jointly compatible if

$$
\begin{aligned}
D^2_{\mathcal{H}} &= (\mathbf{z}_{\mathcal{H}} - \mathbf{h}_{\mathcal{H}}(\hat{\mathbf{x}}))^T C^{-1}_{\mathcal{H}} (\mathbf{z}_{\mathcal{H}} - \mathbf{h}_{\mathcal{H}}(\hat{\mathbf{x}})) \leq \chi^2_{d,\alpha} \\
C_{\mathcal{H}} &= H_{\mathcal{H}} \hat{P} H^T_{\mathcal{H}} + R_{\mathcal{H}}
\end{aligned}
$$

Joint Compatibility – JCBB

- Joint compatibility branch and bound
- Initialize with empty hypothesis and first obs.

- For all tracks
  - If is individually and jointly compatible assign and recursively call JCBB
  - Otherwise consider it a false alarm

```
procedure JCBB (H, i):
-- H : current hypothesis
-- i : observation to be matched
if i > m
   if pairings(H) > pairings(Best)
      Best = H
   fi
else
   for j in {1...n}
      if unary(i, j)
            ∧ joint_compatibility([H j])
         JCBB([H j], i + 1)
      fi
   rof
   if pairings(H) + m - i > pairings(Best)
      JCBB([H 0], i + 1)
   fi
fi
```

[Neira et al.'03]

Multi-Target DA: MHT

- Reason about the associations of sequences measurements with tracks and false alarm
- Evaluate the probability of association hypotheses
- Optimal Bayesian solution

- Algorithm
  - State and measurement prediction
  - Hypotheses generation
  - Hypotheses probability evaluation
  - State update
  - Hypotheses management (i.e. pruning, elimination, creation)

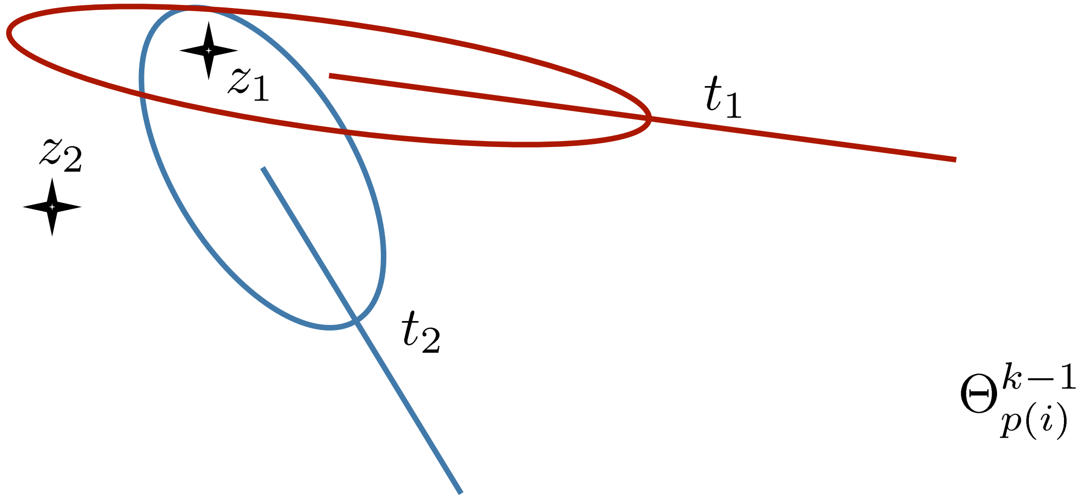
- Exponential complexity of the full solution
  - Pruning strategies
  - K-best hypotheses

MHT: Hypothesis Generation

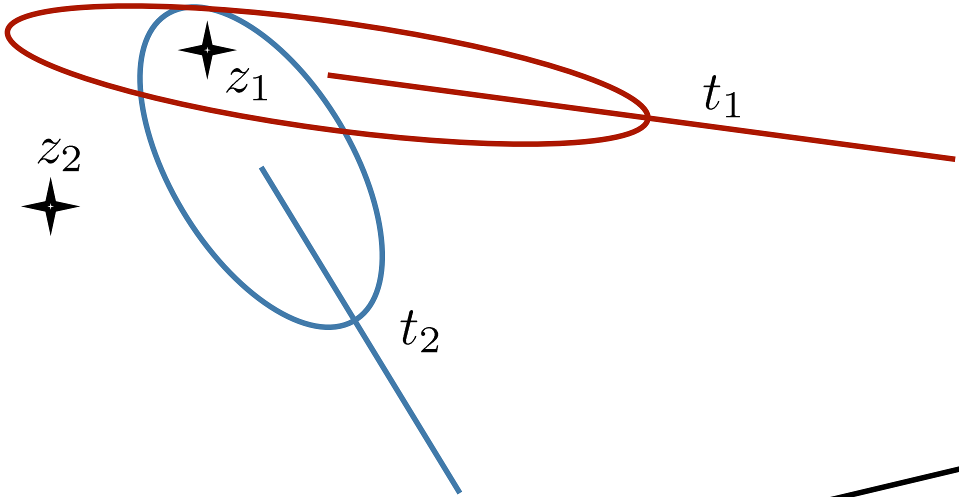
- A hypothesis $\Theta_i^k = \{\Theta_{p(i)}^{k-1}, \theta_{c(i)}(k)\}$ at time k is a history of assignment sets $\theta = \{\theta_{obs}, \theta_{track}\}$ to time k
  - $\theta_{obs} = \{z_1, \ldots, z_{m(k)}\}$ is a set of measurement associations, where a measurement is either associated to track $z_i = t$, treated as a new track $z_i = n$ or as a false alarm $z_i = f$
  - $\theta_{track} = \{l_1, \ldots, l_{t(k)}\}$ is a set of track label, where a track can be matched $l_i = m$, occluded $l_i = o$ or deleted $l_i = d$

- Hypotheses are generated recursively in a tree-based structure
  - Unlikely branch are avoided by validation gating
  - Exponential growth of the trees
  - Only a subset of hypotheses are generated in practice

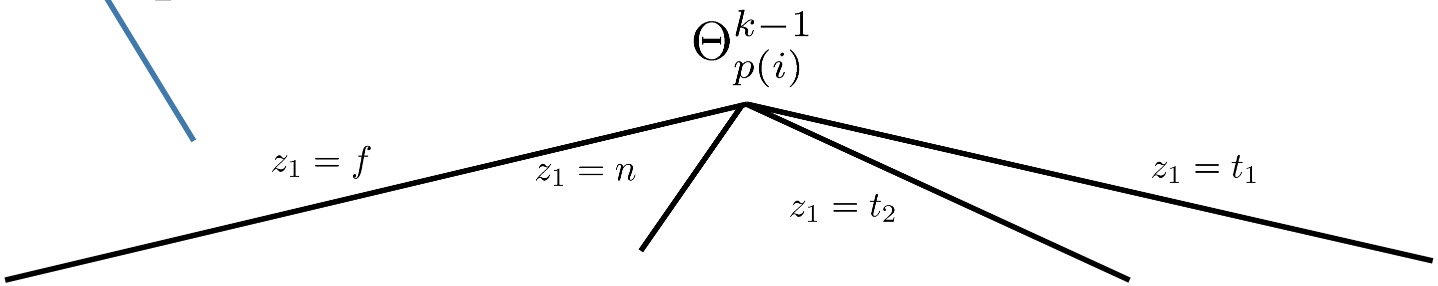
MHT: Hypothesis Generation



MHT: Hypothesis Generation

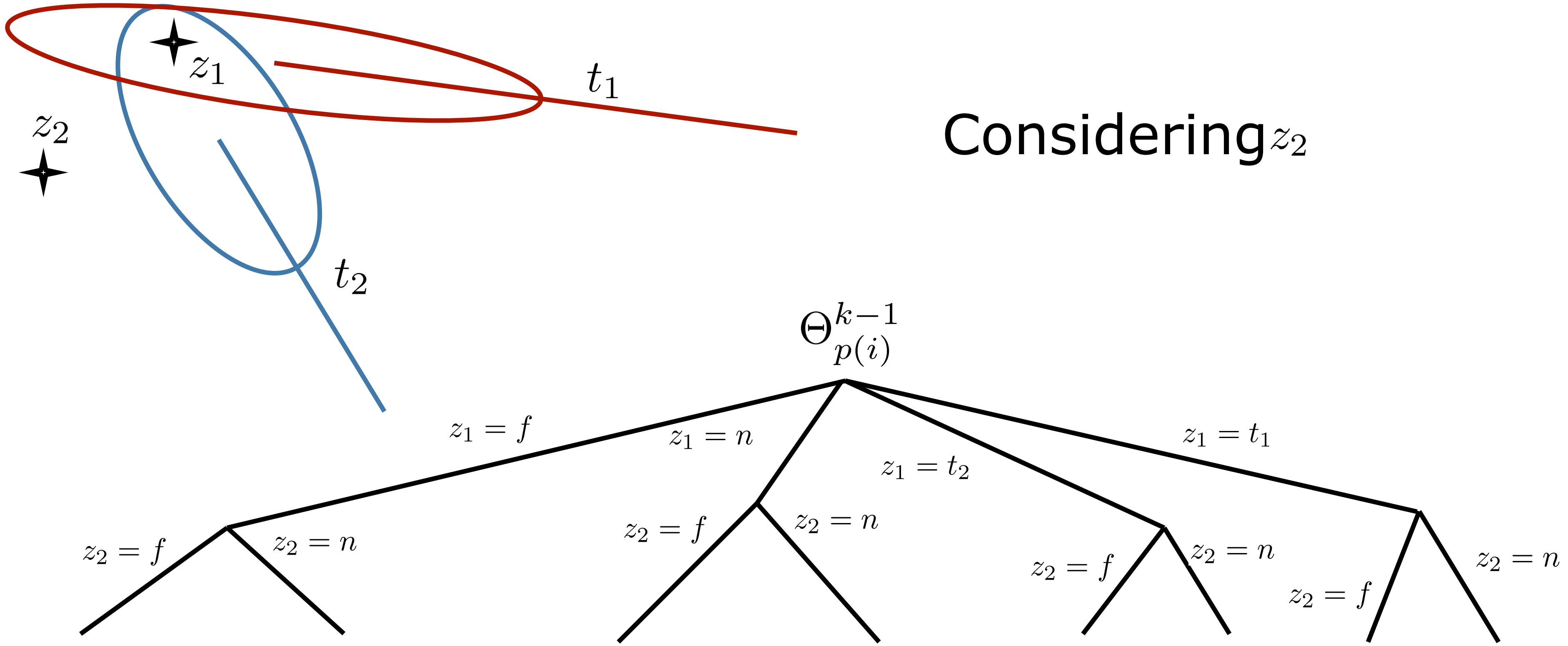


Considering z_1

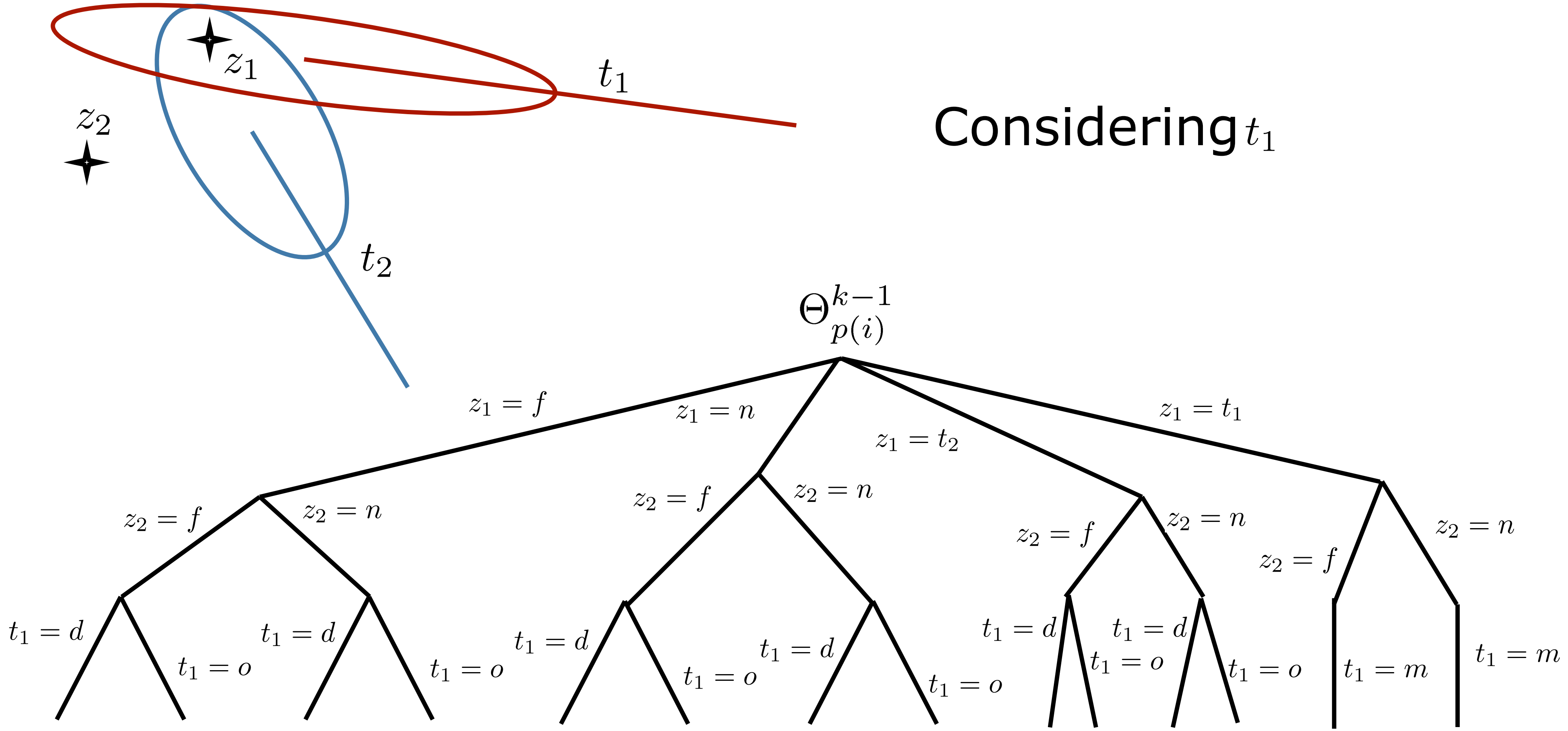


MHT: Hypothesis Generation

Considering z_2

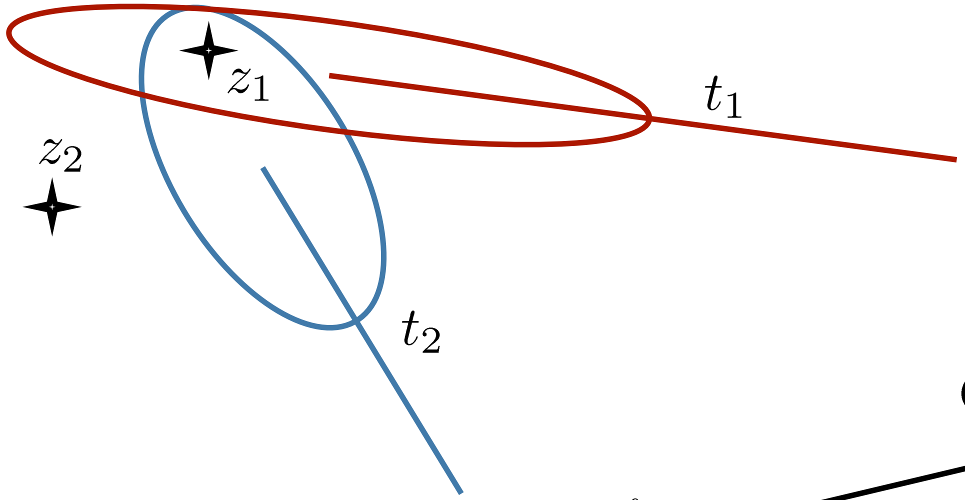


MHT: Hypothesis Generation

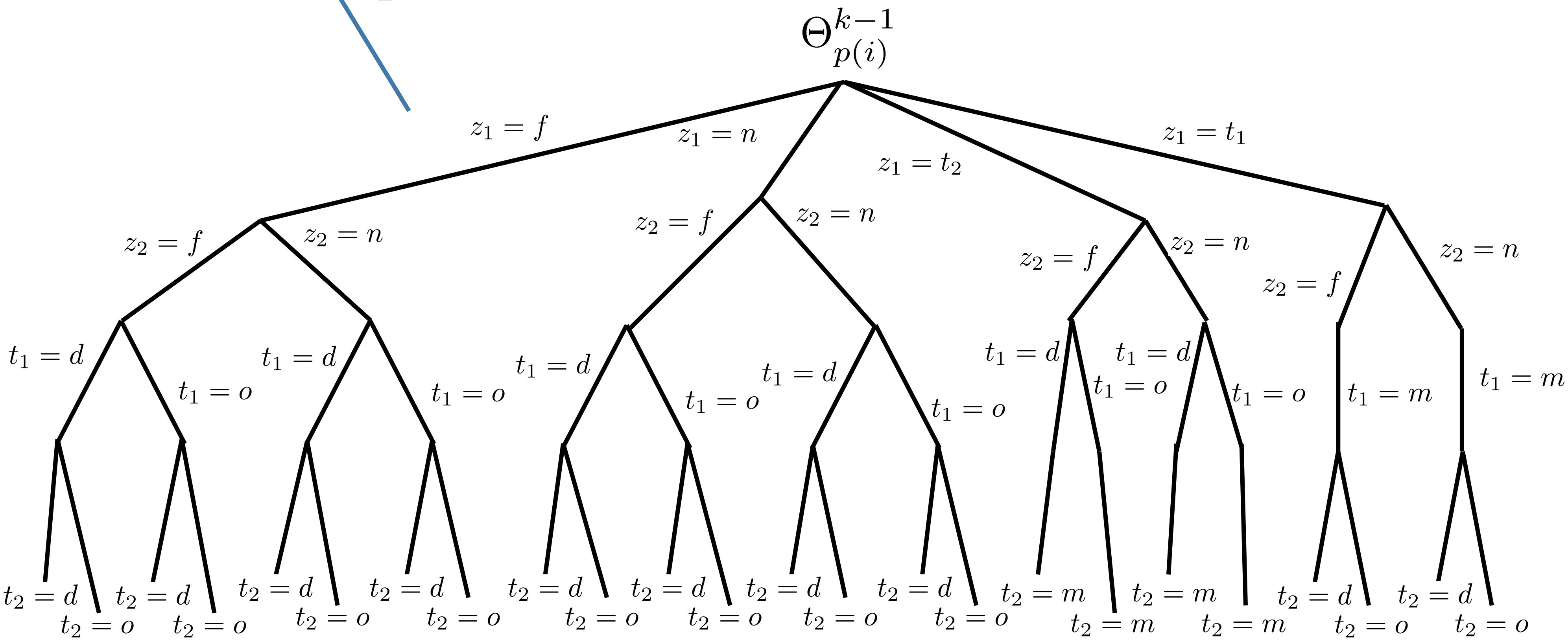


Considering t_1

MHT: Hypothesis Generation



Considering t_2



MHT: Hypothesis Evaluation

- The probability of an hypothesis $\Theta_i^k = \{\Theta_{p(i)}^{k-1}, \theta_{c(i)}(k)\}$:an be calculated using Bayes rules

$$P(\Theta_i^k | Z^k) = P(\Theta_{p(i)}^{k-1}, \theta_{c(i)}(k) | Z^k) =$$

$$\frac{1}{\eta} \cdot \underbrace{p(Z(k) | \Theta_{p(i)}^{k-1}, \theta_{c(i)}(k), Z^{k-1})}_{\text{Likelihood}} \cdot \underbrace{P(\theta_{c(i)}(k) | \Theta_{p(i)}^{k-1}, Z^k)}_{\text{Assignment probability}} \cdot \underbrace{P(\Theta_{p(i)}^{k-1} | Z^{k-1})}_{\text{Prior}}$$

MHT: Hypothesis Evaluation

- Likelihood

$$p(Z(k)|\Theta^{k-1}, \theta(k), Z^{k-1}) = \prod_{l=1}^{m(k)} p(z_l(k)|\Theta^{k-1}, \theta(k), Z^{k-1})$$

  - Case 1: associated with track t

$$p(z_l(k)|\Theta^{k-1}, \theta(k), Z^{k-1}) = \mathcal{N}(z_l(k); \hat{z}_t(k|k-1), S_t(k))$$

  - Case 2: false alarm

$$p(z_l(k)|\Theta^{k-1}, \theta(k), Z^{k-1}) = V^{-1}$$

  - Case 3: new track

$$p(z_l(k)|\Theta^{k-1}, \theta(k), Z^{k-1}) = V^{-1}$$

MHT: Hypothesis Evaluation

- Assignment probability

$$P(\theta(k)|\Theta^{k-1}, Z^k) \quad = \quad P(\theta(k)|N_M, N_O, N_D, N_N, N_F) \cdot \\ \cdot P(N_M, N_O, N_D, N_N, N_F|\theta(k), \Theta^{k-1})$$

  - $P(N_M, N_O, N_D, N_N, N_F|\theta(k), \Theta^{k-1})$ is the probability of having N_M matched tracks, N_O occluded tracks, N_D deleted tracks, N_F false alarm and N_N new tracks.

  - $P(\theta(k)|N_M, N_O, N_D, N_N, N_F)$ is the probability of a possible configuration $\theta(k)$ given the number of events defined before

MHT: Hypothesis Evaluation

- Assignment probability 1: $P(N_M, N_O, N_D, N_N, N_F | \theta(k), \Theta^{k-1})$
  - Assuming a multinomial distribution for track labels

$$P(N_M, N_O, N_D | \theta(k), \Theta^{k-1}) = \frac{N_T!}{N_M! N_O! N_D!} p_M^{N_M} p_O^{N_O} p_D^{N_D}$$

  - Assuming a Poisson distribution for new tracks

$$P(N_N | \theta(k), \Theta^{k-1}) = \frac{(V\lambda_N)^{N_N} e^{-V\lambda_N}}{N_N!}$$

  - Assuming a Poisson distribution for false alarm

$$P(N_F | \theta(k), \Theta^{k-1}) = \frac{(V\lambda_F)^{N_F} e^{-V\lambda_F}}{N_F!}$$

  - We obtain

$$\boxed{P(\cdot) = \frac{N_T!(e^{-V\lambda_N})(e^{-V\lambda_F})}{N_N! N_F! N_M! N_O! N_D!} (V\lambda_N)^{N_N} (V\lambda_F)^{N_F} p_M^{N_M} p_O^{N_O} p_D^{N_D}}$$

MHT: Hypothesis Evaluation

- Assignment probability 2: $P(\theta(k)|N_M, N_O, N_D, N_N, N_F)$
  - The possible choices of taken as matched tracks

$$\binom{m(k)}{N_M} Perm(N_M, N_T) = \frac{m(k)!}{N_M!(m(k)-N_M)!}\frac{N_T!}{(N_T-N_M)!}$$

  - The combinations of taken as new tracks or false alarms

$$\binom{m(k)-N_M}{N_N}\underbrace{\cancel{\binom{m(k)-N_M-N_N}{N_F}}}_{=1} = \frac{(m(k)-N_M)!}{N_N!(m(k)-N_M-N_N)!}$$

  - The combinations of taken as occluded or deleted

$$\binom{N_T-N_M}{N_O}\underbrace{\cancel{\binom{N_T-N_M-N_O}{N_D}}}_{=1} = \frac{(N_T-N_M)!}{N_O!(N_T-N_M-N_O)!}$$

  - The probability is 1 over all the possible choices

$$\left[\frac{m(k)!}{N_M!(m(k)-N_M)!}\frac{N_T!}{(N_T-N_M)!}\frac{(m(k)-N_M)!}{N_N!(m(k)-N_M-N_N)!}\frac{(N_T-N_M)!}{N_O!(N_T-N_M-N_O)!}\right]^{-1}$$

MHT: Hypothesis Evaluation

- Assignment probability 2: $P(\theta(k)|N_M, N_O, N_D, N_N, N_F)$
  - The possible choices of taken as matched tracks

$$\binom{m(k)}{N_M} Perm(N_M, N_T) = \frac{m(k)!}{N_M!(m(k)-N_M)!}\frac{N_T!}{(N_T-N_M)!}$$

  - The combinations of taken as new tracks or false alarms

$$\binom{m(k)-N_M}{N_N}\underbrace{\cancel{\binom{m(k)-N_M-N_N}{N_F}}}_{=1} = \frac{(m(k)-N_M)!}{N_N!(m(k)-N_M-N_N)!}$$

  - The combinations of taken as occluded or deleted

$$\binom{N_T-N_M}{N_O}\underbrace{\cancel{\binom{N_T-N_M-N_O}{N_D}}}_{=1} = \frac{(N_T-N_M)!}{N_O!(N_T-N_M-N_O)!}$$

  - The probability is 1 over all the possible choices

$$\left[\frac{m(k)!}{N_M!\cancel{(m(k)-N_M)!}}\frac{N_T!}{\cancel{(N_T-N_M)!}}\frac{\cancel{(m(k)-N_M)!}}{N_N!(\underbrace{m(k)-N_M-N_N}_{N_D})!}\frac{\cancel{(N_T-N_M)!}}{N_O!(\underbrace{N_T-N_M-N_O}_{N_F})!}\right]^{-1}$$

MHT: Hypothesis Evaluation

- Assignment probability 2: $P(\theta(k) | N_M, N_O, N_D, N_N, N_F)$
  - The possible choices of taken as matched tracks

$$\binom{m(k)}{N_M} Perm(N_M, N_T) = \frac{m(k)!}{N_M!(m(k)-N_M)!} \frac{N_T!}{(N_T - N_M)!}$$

  - The combinations of taken as new tracks or false alarms

$$\binom{m(k)-N_M}{N_N} \underbrace{\cancel{\binom{m(k)-N_M-N_N}{N_F}}}_{=1} = \frac{(m(k)-N_M)!}{N_N!(m(k)-N_M-N_N)!}$$

  - The combinations of taken as occluded or deleted

$$\binom{N_T-N_M}{N_O} \underbrace{\cancel{\binom{N_T-N_M-N_O}{N_D}}}_{=1} = \frac{(N_T-N_M)!}{N_O!(N_T-N_M-N_O)!}$$

  - The probability is 1 over all the possible choices

$$\boxed{P(\theta(k) | N_M, N_O, N_D, N_N, N_F) = \frac{N_M! N_N! N_F! N_O! N_D!}{m(k)! N_T!}}$$

MHT: Hypothesis Evaluation

- Assignment probability

$$\begin{aligned} P(\theta(k)|\Theta^{k-1}, Z^k) &= P(\theta(k)|N_M, N_O, N_D, N_N, N_F) \cdot \\ &\cdot P(N_M, N_O, N_D, N_N, N_F|\theta(k), \Theta^{k-1}) \end{aligned}$$

- Combining everything together we have

$$P(\theta(k)|\Theta^{k-1}, Z^k) =$$

$$= \quad \frac{N_T!(e^{-V\lambda_N})(e^{-V\lambda_F})}{N_N!N_F!N_M!N_O!N_D!}(V\lambda_N)^{N_N}(V\lambda_F)^{N_F} p_M^{N_M} p_O^{N_O} p_D^{N_D} \frac{N_M!N_N!N_F!N_O!N_D!}{m(k)!N_T!}$$

MHT: Hypothesis Evaluation

- Assignment probability

$$
\begin{aligned}
P(\theta(k)|\Theta^{k-1}, Z^k) \quad &= \quad P(\theta(k)|N_M, N_O, N_D, N_N, N_F) \cdot \\
&\quad \cdot P(N_M, N_O, N_D, N_N, N_F|\theta(k), \Theta^{k-1})
\end{aligned}
$$

- Combining everything together we have

$$
P(\theta(k)|\Theta^{k-1}, Z^k) =
$$

$$
= \quad \frac{\cancel{N_F!}(e^{-V\lambda_N})(e^{-V\lambda_F})}{\cancel{N_N! N_F! N_M! N_O! N_D!}} (V\lambda_N)^{N_N} (V\lambda_F)^{N_F} p_M^{N_M} p_O^{N_O} p_D^{N_D} \frac{\cancel{N_M! N_N! N_F! N_O! N_D!}}{m(k)! \cancel{N_T!}}
$$

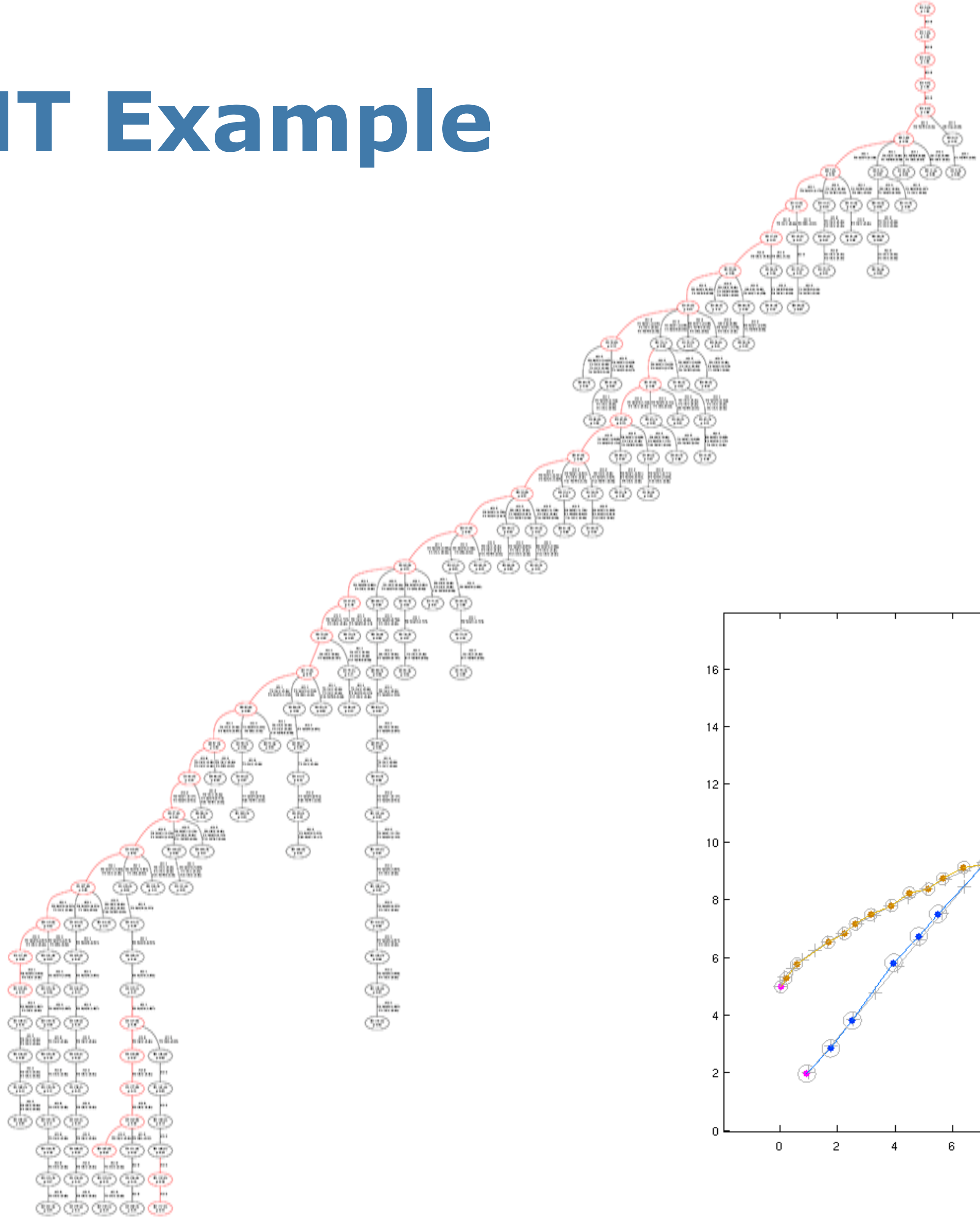
- Simplifying the expression we obtain

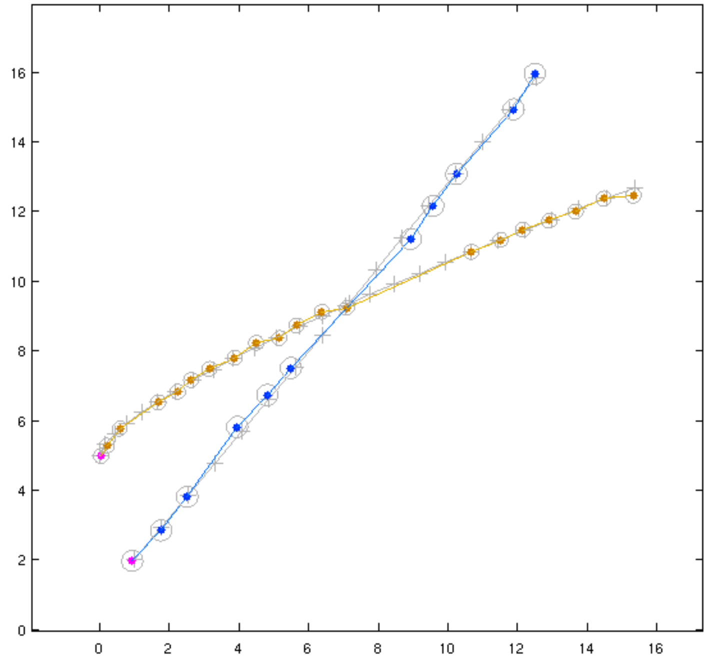
$$
\boxed{P(\theta(k)|\Theta^{k-1}, Z^k) = \frac{(e^{-V\lambda_N})(e^{-V\lambda_F})}{m(k)!} (V\lambda_N)^{N_N} (V\lambda_F)^{N_F} p_M^{N_M} p_O^{N_O} p_D^{N_D}}
$$

MHT Approximations

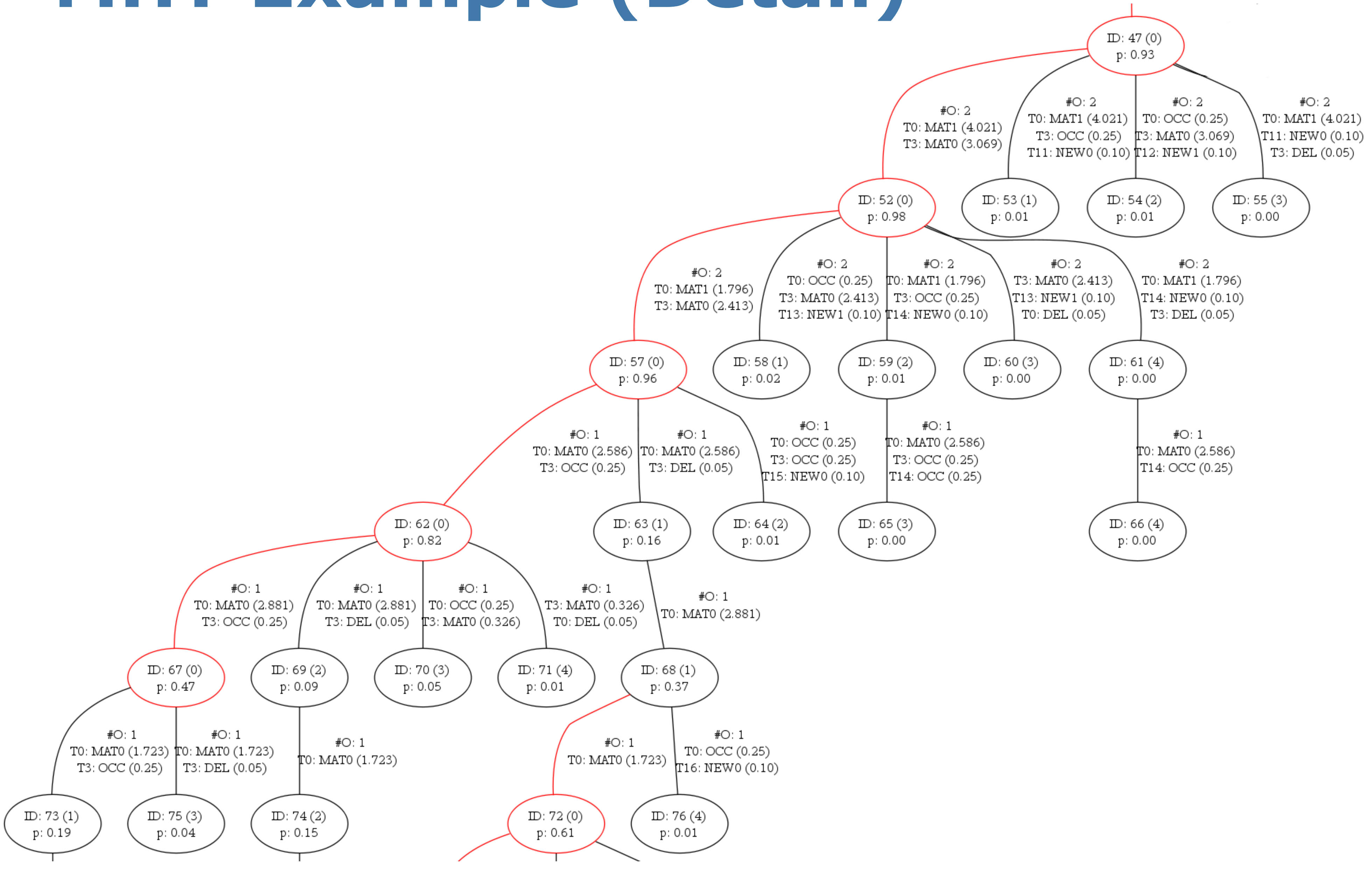
- Spatially disjoint hypothesis trees
  - Tracks can be partitioned in clusters
  - A separate tree is grown for each cluster
- K-Best hypothesis tree
  - Directly generate the k-best hypothesis
  - Generation and evaluation are integrated in a single step
  - Use Murty algorithm and a liner assignment solver
- N-Scan back pruning
  - Ambiguities are supposed to be resolved after N steps
  - Children at step k+N give the prob. of parents at step k
  - Keep only the most probable branch

MHT Example





MHT Example (Detail)



Multi-Target DA: Summary

- **Nearest Neighbor filters** (NN and GNN)
  - Simple to implement
  - NN: Good if tracks are well separated and not noisy
  - NN+GNN: No integration over time

- **Interpretation tree**
  - More involved to implement
  - Good in case of general constraints among associations

- **MHT**
  - Fully Bayesian
  - Most general framework for multiple targets
  - Complex and expensive
  - Only approximations are practically implemented