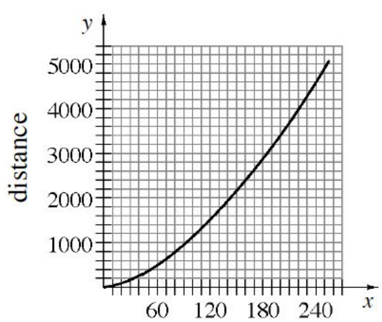
Chapter 4.1.1 and 4.1.2

The purpose of this lesson is to:
Understand how Riemann Sums connect to integrals
Understand the relationship between area under the curve and integrals

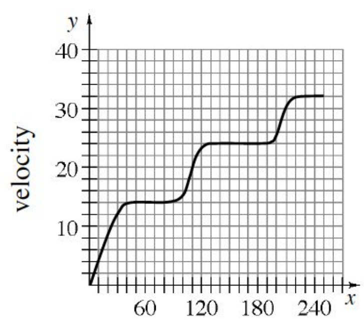
WARM UP

4-1. THE RETURN OF FREDO AND FRIEDA
Examine these new velocity and distance graphs from Fredo and Frieda. Summarize how Fredo's data is reflected in Frieda's graph and how Frieda's data is reflected in Fredo's graph. You may want to review your results from Lesson 1.5.1.

Fredo's Graph



Frieda's Graph



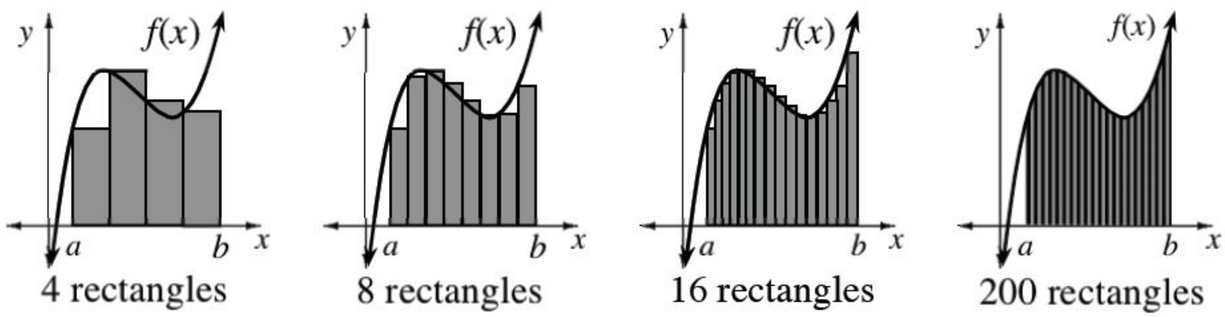
a. Since each student's data is confirmed by the other student's data, using a derivative (to determine the slope of Fredo's graph) must be linked to using an integral (to calculate the area under Frieda's graph). Explain the forward and backward nature of the connection between slope and area.

b. Recall that the slope of a secant line can be used to determine the average velocity (approximate slope) at any point on Fredo's curve. How can the exact slope of a curve at a point be found?

c. Rectangles can be used to approximate the area under Frieda's curve in order to calculate the distance. Make a conjecture about how the exact area under a curve can be determined.

Chapter 4.1.1 and 4.1.2

4-2. Regardless of what value of *n* is chosen, a Riemann sum can only *approximate* the area under *f* on [*a*, *b*] because the rectangles either add extra area or miss some area. Some values of *n* give better approximations than others.



a. Examine the graphs above and write down your observations.

b. How can we calculate an *exact* area? Using complete sentences, describe your ideas thoroughly.

c. The width of each rectangle is $\frac{b-a}{n}$ and represented by Δx. What happens to Δx as more rectangles are used? /

4-3. CALCULATING EXACT AREA

d. Use a Riemann sum to write an expression to represent the *exact* area under *f* on [*a*, *b*].

e. Will Δx ever equal 0? Why or why not?

f. What happens to the area of each individual rectangle as $n \to \infty$?

g. If the area is composed of rectangles with areas that are approaching zero, why does the overall area not approach zero?

Chapter 4.1.1 and 4.1.2

NOTES

Definite Integrals

A Riemann sum is a convenient way to approximate the area under a curve using rectangles. However, in order to get an exact area, an infinite number of rectangles are needed! Since we cannot algebraically evaluate the sum of infinitely many rectangles, we use a limit, *the limit as the number of rectangles approaches infinity*. Another way to express a limit such as this is with a _______________ ___________, shown below:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\tfrac{b-a}{n}\right)f\left(a+\left(\tfrac{b-a}{n}\right)i\right)=\int_a^b f(x)\,dx$$

This "_______ of a Riemann sum," shown above represents the exact area under the graph of $y = f(x)$ on a closed interval $[a, b]$. Each rectangle will have an infinitesimally small _______ expressed as $\lim_{n\to\infty}\frac{b-a}{n}$ or dx. The symbol $\int$ represents the "S" of "summe," the German word for sum.

Note that all rectangles in this definition are of _______ ______. It is possible to define the definite integral when the widths of the intervals are different, as long as all of the widths approach ______.

For example, the *exact* area under the curve $f(x) = x^2 + 1$ over the interval $-2 \le x \le 3$ is

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(\tfrac{3-(-2)}{n}\right)f\left(-2+\left(\tfrac{3-(-2)}{n}\right)i\right)=\int_a^b f(x)\,dx.$$

In the definite integral above, –2 is the ________ ________, 3 is the _________ _________, and the expression $x^2 + 1$ is called the ____________.

Definite integrals can also be used on functions with removable or jump discontinuities.

Chapter 4.1.1 and 4.1.2

Task Card

4-4. Examine the general form of a definite integral, $\int_a^b f(x)\,dx$,as shown in the preceding Math Notes box.

a. What do the upper "*b*," and lower "*a*," bounds of the definite integral represent?

b. The Math Notes box states that $\int_a^b f(x)\,dx$ is equivalent to $\lim\limits_{n\to\infty} \sum\limits_{i=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a+\left(\frac{b-a}{n}\right)i\right)$. Compare $\frac{b-a}{n}$ in the limit to *dx* in the definite integral.

c. Explain why it is important to remember that we are multiplying (x^2 + 1) by *dx*.

4-5. For the following definite integrals, sketch each function and shade the appropriate region. Describe the region and then calculate the area without using a calculator.

d. $\int_0^{2\pi} \sin(2t)\,dt$

e. $\int_{-2}^{3} \left(\frac{1}{2}x+3\right) dx$

f. $\int_0^6 4\,dx$

g. $\int_4^4 (3k^2)\,dk$

Chapter 4.1.1 and 4.1.2

Task Card

4-13. Use the numerical integration feature of a graphing calculator to evaluate each of the integrals below. Then, for the region associated with each integral, evaluate a Riemann sum to approximate the area under the curve with 20 rectangles. Finally, compare these results.

a. $\int_{-\pi}^{\pi/2} 3\sin(2x)\,dx$

b. $\int_{-5}^{5} (x^2 - 3)\,dx$

4-14. Without a calculator, graph and shade the region represented by the integrals below. Then, rewrite this expression using only *one* integral. Use a calculator to verify that the sum of the three integrals is equal to the single integral.

$$\int_{1}^{3} (9x-2)\,dx + \int_{3}^{8} (9x-2)\,dx + \int_{-2}^{1} (9x-2)\,dx$$

4-15. Draw and shade the region representing $\int_{5}^{0} 4x\,dx$.

a. Evaluate the integral geometrically and then verify your result using the numerical integration feature of a graphing calculator. What happened?

b. Why is $\int_{5}^{0} 4x\,dx$ different than $\int_{0}^{5} 4x\,dx$?

4-16. Does $\int_{a}^{b} f(x)\,dx = \int_{b}^{a} f(x)\,dx$ if $a \neq b$? Why or why not?

Examine the limit of a Riemann sum, $\lim\limits_{n\to\infty} \sum_{i=0}^{n-1} \frac{b-a}{n} \cdot f(a + \frac{b-a}{n} \cdot i)$ as you answer this question.

Chapter 4.1.1 and 4.1.2

Homework

4-17. COMBINING REGIONS

Rewrite each of the following expressions as a single integral.

a. $\int_{1}^{6} f(x)\,dx - \int_{3}^{6} f(x)\,dx$

b. $\int_{3}^{10} f(x)\,dx + \int_{9}^{3} f(x)\,dx$

c. $\int_{c}^{d} f(x)\,dx + \int_{e}^{c} f(x)\,dx$

d. $\int_{a}^{x+h} f(t)\,dt - \int_{a}^{x} f(t)\,dt$

e. When can we combine multiple regions? When can we rewrite them? Use the above examples to justify your answer

4-18. Evaluate the following definite integrals. Homework Help ✎

a. $-\int_{0}^{3} x\,dx$

b. $\int_{0}^{3} (-x)\,dx$

c. $\int_{3}^{0} x\,dx$

d. $-\int_{3}^{0} (-x)\,dx$

4-19. For each function below, write the equation of its general antiderivative, F. Homework Help ✎

e. $f(x) = -2$

f. $f(x) = \frac{3}{2} x^{-1/2}$

g. $f(x) = -3x^2 + 6x$

h. $f(x) = 2(x + 3)$

4-20. Differentiate each function below. That is, write an equation for the slope function, f'.

Homework Help ✎

i. $f(x) = 6(x - 2)^3$

j. $f(x) = 2 \sin(x)$

k. $f(x) = (x + 5)(2x - 1)$

l. $f(x) = \frac{x^3 - 6x^2 + 2x}{x}$