Chapter 4.2.5 and 4.4.1

The purpose of this lesson is to:

Solve application problems in which a definite integral represents the accumulation of a rate of change.

Set up integrals for calculating the area between two curves.

WARM UP

4-92. A car is traveling along a straight road with a velocity given by the function *v* such that *v*(*t*) represents miles per hour. Examine the equation $\int_3^7 v(t)\,dt = 120$.

a. According to the context of the problem, what does the value 120 represent? What are its units? Explain how you know.

b. Anahit claimed that 120 represents the "area under *v*(*t*)" and the units should be "square units." Discuss her answer with your team. Is she correct?

4-93. While *area under the curve* offers a convenient way to *think* about integrals, integrals can be used in a variety of contexts.

c. For each situation below, write a complete description of what the integral is computing. Use correct units and be sure to mention the meaning of the bounds in your description.

i. $\int_0^{75} P(c)\,dc = 125.93$ where *P* represents the rate that a coffee shop makes a profit in dollars per ounces of coffee sold, *c*.

ii. $\int_{150}^{2000} H(r)\,dr = 13$ where *H* represents the rate that a bee produces honey in teaspoons per round-trips, *r*, that a bee takes to a flower.

iii. $\int_3^7 |v(t)|\,dt = 360$ where $v(t)$ represents the velocity of a car traveling on a north-south highway in miles per hour and $t(0)$ = 8 a.m.

d. What do all of the definite integrals in part (a) have in common? What does the *integration* operation do to the integrand? Discuss these questions with your team and be prepared to share your answers with the class.

e. Antiderivatives are often called *accumulation functions* because they compute net change of the integrand over a given interval. For example, an antiderivative of v, where $v(t)$ represents velocity as function of time (a.k.a. distance/time), will compute displacement (net change in change in distance) over a given interval. Therefore, as an operation, definite integrals can be interpreted as *accumulators*. With this interpretation in mind, describe what accumulates in each of the examples in part (a).

4-94. Luis's hair started going gray when he was a young child. On the morning of his 17th birthday (which is the day of a big Calculus test), he counts 70 strands of white hair. His mother, who keeps data on her precocious child, notices that the rate $R(t)$ that white hair appears on Luis's head can be modeled by the function $R(t) = e^{0.14t}$, where t is time in years.

a. Examine the graph of $y = R(t)$ on your calculator. What does the area under the curve on the interval [17, 25] represent? In other words what is *accumulated*?

b. Does the amount of gray hair on Luis's head increase or decrease as he gets older? Use the graph of $y = R(t)$ to justify your answer.

c. Write and evaluate an integral expression to represent the number of white hairs that Luis had on the day he was born. Then, write and evaluate an integral to represent the number of white hairs that will have accumulated on his 21st birthday.

d. Luis did some research and discovered that an average human head has 10,000 hairs on their head. Assuming that Luis will not go bald, use your calculator to determine what decade of his life he should expect to have all white hair?

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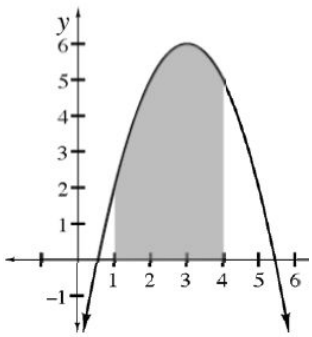
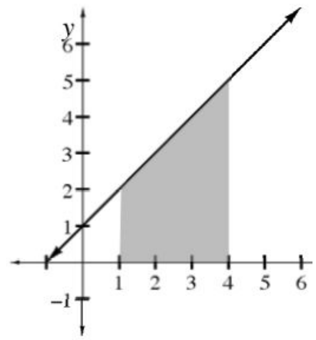
NOTES

4-128. Set up integrals and calculate the exact areas for each of the shaded regions shown.

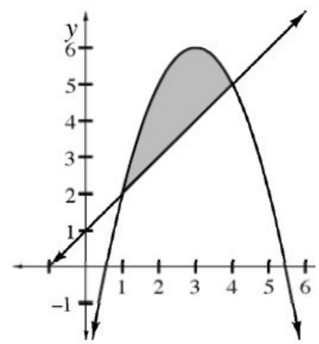
a. $f(x) = -x^2 + 6x - 3$



b. $g(x) = x + 1$

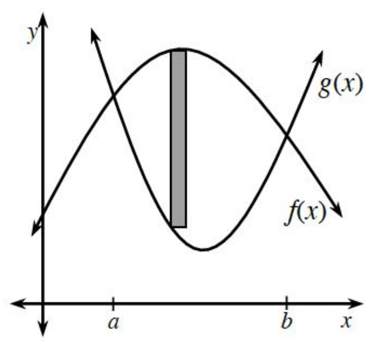


c. Between $f(x)$ and $g(x)$



4-129. AREA BETWEEN TWO CURVES

To calculate the area of the region between two curves, a limit of a sum of rectangle areas is taken. A typical rectangle is shown in the diagram at right.



a. Copy the diagram onto your paper. Label the rectangle with its length, width, and area.

b. Set up an integral that will calculate the area between the curves for $a \le x \le b$.

c. What do a and b represent?

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Task Card

4-95. You have entered a lollipop eating competition. Each contestant is given a large spherical lollipop with a volume of 2400 cm^3. The first contestant to lick their way to the center of the lollipop is the winner. You have been practicing for months and have found that as you lick away towards the center, the rate that the radius of the lollipop changes (in centimeters per minute) can be modeled by the function $R(t) = \left(-\frac{3}{2500}t\right)\ln(9t)$.

a. Use your calculator to examine the graph of $y = R(t)$ during the first hour of the competition. Is $R(t)$ positive or negative? Is $R(t)$ increasing or decreasing? Interpret the graph in the context of the problem.

b. Interpret the expression $\int_0^{10} R(t)\,dt$ in the context of this problem. What accumulates? Then use your calculator to evaluate this expression. Use correct units in your answer.

c. According to the function $R(t)$, how much of the lollipop (in cubic centimeters) will remain after 10 minutes?

d. Last year, the winner took exactly 50 minutes to finish a 2400 cm^3 lollipop. Do you have a good chance of beating last year's record?

4-96. Daniel the Dinosaur sees a meteor heading directly towards Earth and fears doom. He knows that the mass of the meteor (in kilograms) will decrease as it enters Earth's atmosphere, and the rate the mass decreases can be modeled by the function $K(m) = -\frac{1}{20}e^{m/20}$, where m is measured in kilograms. At the moment Daniel spots the meteor, it is 100 kilometers from Earth and has a mass of 199 kg. If the mass is greater than 50 kg on impact, then Daniel and all dinosaurs will be destroyed. Set up and evaluate an integral expression that will determine if Daniel and the dinosaurs will survive.

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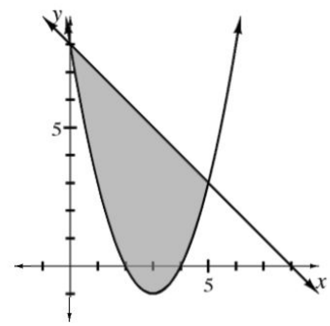
Task Card

4-130. Use the graph shown below and shade the region bounded by $f(x) = (x - 3)^2 - 1$ and $g(x) = -x + 8$.

a. On your diagram, draw a typical rectangle. Label the rectangle with its dimensions and calculate its area.

b. Set up and evaluate an integral to calculate the area of the shaded region. Check your solution with your graphing calculator.



c. Even though $y = f(x)$ dips below the x-axis, explain why we do not subtract off this portion.

d. **4-131.** Given $f(x) = -x$ and $g(x) = x^2 - 6$:
   a. Set up and evaluate an integral to calculate the area bounded by the curves in Quadrant IV.

   b. Explain why the area is positive even though the graphs are below the x-axis.

e. **4-132.** Calculate the area of each of the enclosed regions below. A complete solution includes:
   a. A sketch with the shaded region.

   b. A typical rectangle with width and length labeled.

   c. An integral expression to add up the areas of all rectangles.

   d. An analytical solution, checked with a graphing calculator.

   e. The area between $y = -(x - 3)^2 + 9$ and $y = x + 6$.

   f. The area between $y = \sin(x)$ and $y = x^2 - 1$.

   4-133. Describe how to determine the bounds of integration when calculating the area between two curves.

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Homework

4-97. Brianna is babysitting for her calculus teacher because she broke her phone and needs to buy a new one. She will get paid at a flat rate of $3.50 per hour, per child. Natalie is away at a friend's house for the first two hours while Brianna is babysitting Morgan and Lydia. Natalie returns and Brianna continues to babysit for three additional hours.

e. Sketch a graph and write a piecewise-defined function to represent Brianna's pay rate.

f. How much will Brianna get paid for the five hours of work?

g. Represent Brianna's total pay using definite integrals.

4-134. Examine the following integrals. Consider the multiple tools available for evaluating integrals and use the best strategy for each part. Evaluate each integral and briefly describe your method. Homework Help ✎

a. $\int_{-5}^{-2} \frac{3m^3+2m^2-9m}{m}\, dm$

b. $\int_{-1}^{2} t(2t+3)dt$

c. $\int_{-4}^{-1} \left(1+\frac{1}{x}\right)^2 dx$

d. $\int_{2}^{3} (ax+b)dx$

4-135. The area under the graph of the function *f* from $t = 0$ to $t = x$ can be calculated using the function $F(x) = 3(x - 4)^3 + 6$. What is the equation of $f(t)$? Explain how this is an application of the Fundamental Theorem of Calculus. Homework Help ✎

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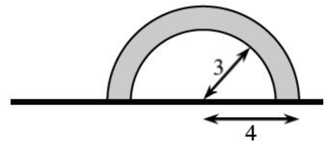
Homework

4-136. A horizontal flag is shown at right. The radius of the outer semicircle is 4. The radius of the inner semicircle is 3. Homework Help ✎



a. Imagine rotating the flag about its pole and describe the resulting three-dimensional figure. Sketch this figure on your paper.

b. Calculate the volume of the rotated flag.

4-137. Graph $y = g(x)$, given below, and determine if *g* is differentiable at $x = 2$. Homework Help ✎

$$g(x) = \begin{cases} (x-1)^2 & \text{for } x < 2 \\ 2\sin(x-2)+1 & \text{for } x \geq 2 \end{cases}$$

4-138. As a log falls in a waterfall, its velocity is given by $v(t) = -32t - 18$ in feet per second. The position of the log at time $t = 0$ was at the top of the waterfall, 500 feet above sea level. 4-138 HW eTool (Desmos) Homework Help ✎.

a. Where is the log after 1 second? 2 seconds? 3 seconds?

b. Where is the log after *t* seconds? This is the position function. What is its relationship to the velocity function?