Gaussian Kernel Fuzzy C-Means with Width Parameter Computation and Regularization

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Abstract

The conventional Gaussian kernel fuzzy c-means clustering algorithms require selecting the width hyper-parameter, which is data-dependent and fixed for the entire execution. Not only that, but these parameters are the same for every dataset variable. Therefore, the variables have the same importance in the clustering task, including irrelevant variables. This paper proposes a Gaussian kernel fuzzy c-means with kernelization of the metric and automated computation of width parameters. These width parameters change at each iteration of the algorithm and vary from each variable and from each cluster. Thus, this algorithm can re-scale the variables differently, thus highlighting those that are relevant to the clustering task. Fuzzy clustering algorithms with regularization have become popular due to their high performance in large-scale data clustering, robustness for initialization, and low computational complexity. Because the width parameters of the variables can also be controlled by entropy, this paper also proposes Gaussian kernel fuzzy c-means algorithms with kernelization of the metric and automated computation of width parameters through entropy regularization. To demonstrate their usefulness, the proposed algorithms are compared with the conventional KFCM-K algorithm and previous algorithms that automatically compute the width parameter of the Gaussian kernel.

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1. Proof of the convergence of the proposed algorithms

The algorithm KFCM-K-W.1 provides a fuzzy partition $\mathbf{U}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_P^*)$, a vector of prototypes $\mathbf{G}^* = (\mathbf{g}_1^*, \ldots, \mathbf{g}_C^*)$ and a width parameters vector $\mathbf{s}^*$ such that:

- $J_{KFCM-K-W.1}(\mathbf{s}^*, \mathbf{G}^*, \mathbf{U}^*) = \min\{J_{KFCM-K-W.1}(\mathbf{s}, \mathbf{G}, \mathbf{U}), \mathbf{s} \in \Xi, \mathbf{G} \in \mathbb{L}^c,$ $\mathbf{U} \in \mathbb{U}^n\}$

where

- Ξ is the space of vectors of width parameters such that $\mathbf{s} \in \Xi$. In this paper, $\Xi = \{\mathbf{s} = (s_1, \ldots, s_p) \in \mathbb{R}^p : \frac{1}{s_j} > 0$ and $\prod_{j=1}^{p} \frac{1}{s_j} = 1\}$.
- $\mathbb{L}$ is the representation space of the prototypes such that $\mathbf{g}_i \in \mathbb{L}$ $(i = 1, \ldots, c)$ and $\mathbf{G} \in \mathbb{L}^c = \mathbb{L} \times \cdots \times \mathbb{L}$. In this paper $\mathbb{L} = \mathbb{R}^p$.
- $\mathbb{U}$ is the space of the fuzzy partition membership degrees such that $\mathbf{u}_k \in \mathbb{U}$ $(k = 1, \ldots, n)$. In this paper $\mathbb{U} = \{\mathbf{u} = (u_1, \ldots, u_c) \in [0,1] \times \cdots \times [0,1] = [0,1]^C : \sum_{i=1}^{c} u_i = 1$ and $u_i \geq 0\}$ and $\mathbf{U} \in \mathbb{U}^n = \mathbb{U} \times \cdots \times \mathbb{U}$.

Moreover, the algorithm KFCM-K-W.2 provides a fuzzy partition $\mathbf{U}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_P^*)$, a vector of prototypes $\mathbf{G}^* = (\mathbf{g}_1^*, \ldots, \mathbf{g}_C^*)$ and a width parameters matrix $\mathbf{S}^*$ such that:

- $J_{KFCM-K-W.2}(\mathbf{S}^*, \mathbf{G}^*, \mathbf{U}^*) = \min\{J_{KFCM-K-W.2}(\mathbf{S}, \mathbf{G}, \mathbf{U}), \mathbf{S} \in \Xi^c,$ $\mathbf{G} \in \mathbb{L}^c, \mathbf{U} \in \mathbb{U}^n\}$

where $\mathbb{L}$ and $\mathbb{U}$ are as before, and

- Ξ is the space of vectors of width parameters such that $\mathbf{s}_i \in \Xi$. In this paper, $\Xi = \{\mathbf{s} = (s_1, \ldots, s_p) \in \mathbb{R}^p : \frac{1}{s_j} > 0$ and $\prod_{j=1}^{p} \frac{1}{s_j} = 1\}$ and $\mathbf{S} \in \Xi^c = \Xi \times \cdots \times \Xi$.

Besides, the algorithm KFCM-K-E$_{W.1}$ provides a fuzzy partition $\mathbf{U}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_P^*)$, a vector of prototypes $\mathbf{G}^* = (\mathbf{g}_1^*, \ldots, \mathbf{g}_C^*)$ and a width parameters vector $\mathbf{s}^*$ such that:

- $J_{KFCM-K-E_{W.1}}(\mathbf{s}^*, \mathbf{G}^*, \mathbf{U}^*) = \min\{J_{KFCM-K-E_{W.1}}(\mathbf{s}, \mathbf{G}, \mathbf{U}), \mathbf{s} \in \Xi, \mathbf{G} \in \mathbb{L}^c,$ $\mathbf{U} \in \mathbb{U}^n\}$

where $\mathbb{L}$ and $\mathbb{U}$ are as before, and

– Ξ is the space of vectors of width parameters such that $\mathbf{s} \in \Xi$. In this paper, $\Xi = \{\mathbf{s} = (s_1, \ldots, s_p) \in \mathbb{R}^p : \frac{1}{s_j} \in [0,1] \text{ and } \sum_{j=1}^{p} \frac{1}{s_j} = 1\}$.

Finally, the algorithm KFCM-K-E$_{W.2}$ provides a fuzzy partition $\mathbf{U}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_P^*)$, a vector of prototypes $\mathbf{G}^* = (\mathbf{g}_1^*, \ldots, \mathbf{g}_C^*)$ and a width parameters matrix $\mathbf{S}^*$ such that:

- $J_{KFCM-K-E_{W.2}}(\mathbf{S}^*, \mathbf{G}^*, \mathbf{U}^*) = \min\{J_{KFCM-K-E_{W.2}}(\mathbf{S}, \mathbf{G}, \mathbf{U}), \mathbf{S} \in \Xi^c,$ $\mathbf{G} \in \mathbb{L}^c, \mathbf{U} \in \mathbb{U}^n\}$

where $\mathbb{L}$ and $\mathbb{U}$ are as before, and

– Ξ is the space of vectors of width parameters such that $\mathbf{s}_i \in \Xi$. In this paper, $\Xi = \{\mathbf{s} = (s_1, \ldots, s_p) \in \mathbb{R}^p : \frac{1}{s_j} \in [0,1] \text{ and } \sum_{j=1}^{p} \frac{1}{s_j} = 1\}$ and $\mathbf{S} \in \Xi^c = \Xi \times \cdots \times \Xi$.

Similarly to Ref. [1], the convergence properties of the proposed algorithms can be studied from the series:

- $v_{KFCM-K-W.1}^{(t)}(\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}) \in \Xi \times \mathbb{L}^c \times \mathbb{U}^n$ and $u_{KFCM-K-W.1}^{(t)} =$ $J_{KFCM-K-W.1}(v_{KFCM-K-W.1}^{(t)}) = J_{KFCM-K-W.1}(\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)})$, where $t = 0, 1, \ldots$ is the iteration number;
- $v_{KFCM-K-W.2}^{(t)}(\mathbf{S}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}) \in \Xi^c \times \mathbb{L}^c \times \mathbb{U}^n$ and $u_{KFCM-K-W.2}^{(t)} =$ $J_{KFCM-K-W.2}(v_{KFCM-K-W.2}^{(t)}) = J_{KFCM-K-W.2}(\mathbf{S}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)})$, where $t = 0, 1, \ldots$ is the iteration number;

- $v^{(t)}_{KFCM-K-E_{W.1}}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)}) \in \Xi \times \mathbb{L}^c \times \mathbb{U}^n$ and $u^{(t)}_{KFCM-K-E_{W.1}} = J_{KFCM-K-E_{W.1}}(v^{(t)}_{KFCM-K-E_{W.1}}) = J_{KFCM-K-E_{W.1}}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, where $t = 0, 1, \ldots$ is the iteration number;

- $v^{(t)}_{KFCM-K-E_{W.2}}(\mathbf{S}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)}) \in \Xi^c \times \mathbb{L}^c \times \mathbb{U}^n$ and $u^{(t)}_{KFCM-K-E_{W.2}} = J_{KFCM-K-E_{W.2}}(v^{(t)}_{KFCM-K-E_{W.2}}) = J_{KFCM-K-E_{W.2}}(\mathbf{S}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, where $t = 0, 1, \ldots$ is the iteration number;

From the initial terms: $v^{(0)}_{KFCM-K-W.1}(\mathbf{s}^{(0)},\mathbf{G}^{(0)},\mathbf{U}^{(0)})$, $v^{(0)}_{KFCM-K-W.2}(\mathbf{S}^{(0)},\mathbf{G}^{(0)},\mathbf{U}^{(0)})$, $v^{(0)}_{KFCM-K-E_{W.1}}(\mathbf{s}^{(0)},\mathbf{G}^{(0)},\mathbf{U}^{(0)})$ and $v^{(0)}_{KFCM-K-E_{W.2}}(\mathbf{S}^{(0)},\mathbf{G}^{(0)},\mathbf{U}^{(0)})$, the algorithms KFCM-K-W.1, KFCM-K-W.2, KFCM-K-E$_{W.1}$ and KFCM-K-E$_{W.2}$ compute the different terms of the series, $v^{(t)}_{KFCM-K-W.1}$, $v^{(t)}_{KFCM-K-W.2}$, $v^{(t)}_{KFCM-K-E_{W.1}}$, and $v^{(t)}_{KFCM-K-E_{W.2}}$, until the respective convergence (to be demonstrated) when the objective functions $J_{KFCM-K-W.1}$, $J_{KFCM-K-W.1}$, $J_{KFCM-K-E_{W.1}}$, and $J_{KFCM-K-E_{W.2}}$ reach stationary values.

Proposition 1.1.

i) *The series* $u^{(t)}_{KFCM-K-W.1} = J_{KFCM-K-W.1}(v^{(t)}_{KFCM-K-W.1}) = J_{KFCM-K-W.1}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, $t = 0, 1, \ldots$, *decreases at each iteration and converges;*

ii) *The series* $u^{(t)}_{KFCM-K-W.2} = J_{KFCM-K-W.2}(v^{(t)}_{KFCM-K-W.2}) = J_{KFCM-K-W.2}(\mathbf{S}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, $t = 0, 1, \ldots$, *decreases at each iteration and converges;*

iii) *The series* $u^{(t)}_{KFCM-K-E_{W.1}} = J_{KFCM-K-E_{W.1}}(v^{(t)}_{KFCM-K-E_{W.1}}) = J_{KFCM-K-E_{W.1}}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, $t = 0, 1, \ldots$, *decreases at each iteration and converges;*

iv) *The series* $u^{(t)}_{KFCM-K-E_{W.2}} = J_{KFCM-K-E_{W.2}}(v^{(t)}_{KFCM-K-E_{W.2}}) = J_{KFCM-K-E_{W.2}}(\mathbf{S}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, $t = 0, 1, \ldots$, *decreases at each iteration and converges;*

Proof.

i) The series $u^{(t)}_{KFCM-K-W.1} = J_{KFCM-K-W.1}(v^{(t)}_{KFCM-K-W.1}) = J_{KFCM-K-W.1}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$, $t = 0, 1, \ldots$, decreases at each iteration and converges;

The objective function $J_{KFCM-K-W.1}$ measures the total heterogeneity of the fuzzy partition as the sum of the heterogeneity in each cluster. We will first show that the inequalities (I), (II) and (III) below hold (i.e., the series decreases at each iteration).

$$\underbrace{J_{KFCM-K-W.1}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})}_{u^{(t)}_{KFCM-K-W.1}} \overbrace{\geq}^{(I)} J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t)},\mathbf{U}^{(t)})$$

$$\overbrace{\geq}^{(II)} J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t+1)},\mathbf{U}^{(t)}) \overbrace{\geq}^{(III)} \underbrace{J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t+1)},\mathbf{U}^{(t+1)})}_{u^{(t+1)}_{KFCM-K-W.1}}$$

The inequality (I) holds because $J_{KFCM-K-W.1}(\mathbf{s}^{(t)},\mathbf{G}^{(t)},\mathbf{U}^{(t)}) = \sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t)})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t)}})}(\mathbf{x}_k,\mathbf{g}_i^{(t)}))$ and $J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t)},\mathbf{U}^{(t)}) = \sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t)})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t+1)}})}(\mathbf{x}_k,\mathbf{g}_i^{(t)}))$, and according to Step 1 of Section 3.1.1 of the paper,

$$\mathbf{s}^{(t+1)}(s_1^{(t+1)},\ldots,s_p^{(t+1)}) = \underbrace{\text{arg min}}_{\mathbf{s}\in\Xi}\sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t)})^m(2-2\,\mathcal{K}^{(\mathbf{s})}(\mathbf{x}_k,\mathbf{g}_i^{(t)}))$$

Moreover, inequality (II) holds because $J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t+1)},\mathbf{U}^{(t)}) = \sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t)})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t+1)}})}(\mathbf{x}_k,\mathbf{g}_i^{(t+1)}))$ and according to Step 2 of Section 3.1.1 of the paper,

$$\mathbf{G}^{(t+1)=(\mathbf{g}_1^{(t+1)},\ldots,\mathbf{g}_c^{(t+1)}}) = \underbrace{\text{arg min}}_{\mathbf{G}=(\mathbf{g}_1,\ldots,\mathbf{g}_c)\in\mathbb{L}^c}\sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t)})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t+1)}})}(\mathbf{x}_k,\mathbf{g}_i))$$

The inequality (III) also holds because $J_{KFCM-K-W.1}(\mathbf{s}^{(t+1)},\mathbf{G}^{(t+1)},\mathbf{U}^{(t+1)}) = \sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki}^{(t+1)})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t+1)}})}(\mathbf{x}_k,\mathbf{g}_i^{(t+1)}))$ and according to Step 3 of Section 3.1.1 of the paper,

$$\mathbf{U}^{(t+1)} = (\mathbf{u}_1^{(t+1)},\ldots,\mathbf{u}_n^{(t+1)}) = \underbrace{\text{arg min}}_{\mathbf{U}=(\mathbf{u}_1,\ldots,\mathbf{u}_n)\in\mathbb{U}^n}\sum_{i=1}^{c}\sum_{k=1}^{n}(u_{ki})^m(2-2\,\mathcal{K}^{(\mathbf{s^{(t+1)}})}(\mathbf{x}_k,\mathbf{g}_i^{(t+1)}))$$

Finally, since the series $u^{(t)}_{KFCM-K-W.1}$ decreases and it is bounded $(J(v^{(t)}_{KFCM-K-W.1}) \geq 0)$, it converges.

The proof of the convergence of the series $u^{(t)}_{KFCM-K-W.2}, t = 0, 1, \ldots, u^{(t)}_{KFCM-K-E_{W.1}}$, $t = 0, 1, \ldots$, and $u^{(t)}_{KFCM-K-E_{W.2}}, t = 0, 1, \ldots$ proceeds similarly to the proof of the convergence of the series $u^{(t)}_{KFCM-K-W.1}, t = 0, 1, \ldots$ presented above.

□

Proposition 1.2.

i) *The series* $v^{(t)}_{KFCM-K-W.1} = (\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}), t = 0, 1, \ldots,$ *converges;*

ii) *The series* $v^{(t)}_{KFCM-K-W.2} = (\mathbf{S}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}), t = 0, 1, \ldots,$ *converges;*

iii) *The series* $v^{(t)}_{KFCM-K-E_{W.1}} = (\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}), t = 0, 1, \ldots,$ *converges;*

iv) *The series* $v^{(t)}_{KFCM-K-E_{W.2}} = (\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}), t = 0, 1, \ldots,$ *converges;*

Proof.

i) The series $v^{(t)}_{KFCM-K-W.1} = (\mathbf{s}^{(t)}, \mathbf{G}^{(t)}, \mathbf{U}^{(t)}), t = 0, 1, \ldots,$ converges;

Assuming that the stationarity of the series $u^{(t)}_{KFCM-K-W.1}$ is achieved in the iteration $t = T$, then, we have $u^{(T)}_{KFCM-K-W.1} = u^{(T+1)}_{KFCM-K-W.1}$ and then $J_{KFCM-K-W.1}(v^{(T)}_{KFCM-K-W.1}) = J_{KFCM-K-W.1}(v^{(T+1)}_{EFCM-M})$.

From $J_{KFCM-K-W.1}(v^{(T)}_{KFCM-K-W.1}) = J_{KFCM-K-W.1}(v^{(T+1)}_{KFCM-K-W.1})$ we arrive at $J_{KFCM-K-W.1}(\mathbf{s}^{(T)}, \mathbf{G}^{(T)}, \mathbf{U}^{(T)}) = J_{KFCM-K-W.1}(\mathbf{s}^{(T+1)}, \mathbf{G}^{(T+1)}, \mathbf{U}^{(T+1)})$. This equality, according to Proposition 1.1 , can be rewritten as the equalities (I)-(III):

$$\underbrace{J_{KFCM-K-W.1}(\mathbf{s}^{(T)}, \mathbf{G}^{(T)}, \mathbf{U}^{(T)})}_{u^{(T)}_{KFCM-K-W.1}} \overbrace{=}^{(I)} J_{KFCM-K-W.1}(\mathbf{s}^{(T+1)}, \mathbf{G}^{(T)}, \mathbf{U}^{(T)}) \overbrace{=}^{(II)}$$

$$J_{KFCM-K-W.1}(\mathbf{s}^{(T+1)}, \mathbf{G}^{(T+1)}, \mathbf{U}^{(T)}) \overbrace{=}^{(III)} J_{KFCM-K-W.1}(\mathbf{s}^{(T+1)}, \mathbf{G}^{(T+1)}, \mathbf{U}^{(T+1)})$$

From the first equality (I), the result is $\mathbf{s}^{(T)} = \mathbf{s}^{(T+1)}$ because $\mathbf{s}$ is unique, minimizing $J_{KFCM-K-W.1}$, when the fuzzy partition represented by $\mathbf{U}^{(T)}$ and and the matrix of prototypes $\mathbf{G}^{(T)}$ are maintained fixed. From the second equality (II), the result is $\mathbf{G}^{(T)} = \mathbf{G}^{(T+1)}$, since $\mathbf{G}$ is unique, minimizing $J_{KFCM-K-W.1}$ when the fuzzy partition represented by $\mathbf{U}^{(T)}$ and the width parameter vector $\mathbf{s}^{(T+1)}$ are maintained fixed. Furthermore, from the third equality (III), the result is $\mathbf{U}^{(T)} = \mathbf{U}^{(T+1)}$ since $\mathbf{U}$ is unique minimizing $J_{KFCM-K-W.1}$ when the prototypes $\mathbf{G}^{(T+1)}$ and the width parameter vector $\mathbf{s}^{(T+1)}$ are maintained fixed.

Therefore, it can be concluded that $v^{(T)}_{KFCM-K-W.1} = v^{(T+1)}_{KFCM-K-W.1}$, which stands for all $t \geq T$ and $v^{(t)}_{KFCM-K-W.1} = v^{(T)}_{KFCM-K-W.1}, \forall t \geq T$ and follows that the series $v^{(t)}_{KFCM-K-W.1}$ converges.

The proof of the convergence of the series $v^{(t)}_{KFCM-K-W.2}, t = 0, 1, \ldots,$ $v^{(t)}_{KFCM-K-E_{W.1}}, t = 0, 1, \ldots,$ and $v^{(t)}_{KFCM-K-E_{W.2}}, t = 0, 1, \ldots$ proceeds similarly to the proof of the convergence of the series $v^{(t)}_{KFCM-K-W.1}$ presented above. $\square$

2. Empirical results

This section provides supplementary results of the Section 4 of the main paper.

2.1. Parameters selected by grid search

The parameter T_W was found using a grid search (with the tested values defined in Table 1), selecting the value that maximizes the average of the minimum distance between the elements distinct hard clusters of 30 executions, the idea is to select the parameters that better separate the clusters.

Table 1: Parameters selected by the grid search.

Data set	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$	Data set	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	$T_W = 100.0$	$T_W = 1.5$	Pendigits	$T_W = 0.1$	$T_W = 0.1$
Banknote Authentication	$T_W = 80.0$	$T_W = 0.1$	Pima Indians Diabetes	$T_W = 0.4$	$T_W = 0.2$
Breast Cancer Wdbc	$T_W = 0.1$	$T_W = 0.1$	QSAR Biodegradation	$T_W = 0.1$	$T_W = 10.0$
Breast Cancer Wpbc	$T_W = 0.1$	$T_W = 0.5$	Seeds	$T_W = 0.1$	$T_W = 0.1$
Brest Tissue	$T_W = 10.0$	$T_W = 0.3$	Spambase	$T_W = 0.1$	$T_W = 0.3$
Connectionist Bench Sonar	$T_W = 70.0$	$T_W = 4.5$	Thyroid Disease	$T_W = 25.0$	$T_W = 0.2$
Ecoli	$T_W = 60.0$	$T_W = 3.5$	Two Circles	$T_W = 1.0$	$T_W = 0.1$
German Credit	$T_W = 0.1$	$T_W = 1.5$	Urban	$T_W = 0.4$	$T_W = 40.0$
Glass	$T_W = 2.0$	$T_W = 0.1$	Vehicle Silhouettes	$T_W = 0.1$	$T_W = 7.0$
Heart Disease	$T_W = 30.0$	$T_W = 50.0$	Vertebral Column 2C	$T_W = 0.1$	$T_W = 0.1$
Image Segmentation	$T_W = 0.2$	$T_W = 1.5$	Vertebral Column 3C	$T_W = 100.0$	$T_W = 0.1$
Ionosphere	$T_W = 40.0$	$T_W = 0.5$	Voting Records	$T_W = 0.1$	$T_W = 0.1$
Iris	$T_W = 20.0$	$T_W = 35.0$	Wall Following Readings 2	$T_W = 80.0$	$T_W = 0.1$
Landsat	$T_W = 100.0$	$T_W = 0.2$	Wall Following Readings 4	$T_W = 100.0$	$T_W = 35.0$
Leaf	$T_W = 7.0$	$T_W = 0.2$	Waveform	$T_W = 80.0$	$T_W = 40.0$
Letters	$T_W = 1.0$	$T_W = 0.6$	Wilt	$T_W = 70.0$	$T_W = 10.0$
Liver Disorders	$T_W = 50.0$	$T_W = 0.1$	Wine	$T_W = 6.0$	$T_W = 0.2$
Musk V1	$T_W = 1.5$	$T_W = 1.0$	Wine Quality Red	$T_W = 0.4$	$T_W = 3.5$
Musk V2	$T_W = 15.0$	$T_W = 1.0$	Wine Quality White	$T_W = 0.1$	$T_W = 0.5$
Page Blocks	$T_W = 100.0$	$T_W = 100.0$	Zoo	$T_W = 0.8$	$T_W = 0.1$

2.2. Metrics for hard partitions

To compute the metrics for hard partitions, first we obtain the hard partitions, selecting, for each element, the cluster with the highest membership degree. The metrics for hard partitions computed are the Accuracy, the F-measure [2], external adjusted Rand index (ARI) [3], the normalized mutual information

(NMI) [2], and the Entropy[4]. These metrics are external indexes [5] that compares the hard partition provided by the clustering algorithms with an a priori partition provided by specific expert knowledge.

The ARI index assesses the degree of similarity between an a priori partition and a partition provided by the clustering algorithm. The ARI index takes its values from the interval $[-1, 1]$, in which the value 1 indicates perfect agreement between partitions, whereas values near 0 (or negatives) correspond to cluster agreement found by chance [6]. The F-measure index takes its values on the range $[0, 1]$, in which the value 1 indicates perfect agreement between partitions. The NMI index takes its values on the range $[0, 1]$, in which the value 1 indicates perfect correlation between partitions. The ER index aims to measure the ability of a clustering algorithm to find out a priori classes present in a data set and takes its values on the range $[0, 1]$ in which lower ER values indicate better clustering results.

The Accuracy, F-Measure and Entropy are defined as:

$$Accuracy = \sum_{i=1}^{K} \frac{n_i}{n} P_i \tag{1}$$

$$F - Measure = \frac{1}{n} \sum_{j=1}^{C} n_c j \left[max_{1 \leq i \leq K} \left(\frac{2nij}{n_c j + ni} \right) \right] \tag{2}$$

$$Entropy = \sum_{i=1}^{K} \frac{n_i}{n} \left(- \sum_{j=1}^{C} P_{ij} log_2(P_{ij}) \right) \tag{3}$$

Being P_{ij} the probability that members of the cluster i belongs to the class j, defined by:

$$P_{ij} = \frac{n_{ij}}{n_i} \tag{4}$$

And:

$$P_i = \max_{1 \leq j \leq C} (P_{ij}) \tag{5}$$

Where n_{ij} is the number of elements of the cluster i that belongs to the class j, n_i is the number of elements of the cluster i, n_cj is the number of elements of the original class j, and n the total number of elements.

The Normalized Mutual information (NMI) is defined as:

$$NMI = \frac{2MI}{\left(-\sum_{i=1}^{K} P_{Ki} log\left(P_{Ki}\right)\right) + \left(-\sum_{j=1}^{C} P_{Cj} log\left(P_{Cj}\right)\right)} \tag{6}$$

Where the Mutual information (MI) is defined as:

$$MI = \sum_{i=1}^{K}\sum_{j=1}^{C} P_{ij} log\left(\frac{P_{ij}}{P_{Ki}P_{Cj}}\right) \tag{7}$$

and P_{Ci} is the probability of an element being a members of the class j, and P_{Ki} is the probability of an element being a members of the cluster i, in other words:

$$P_{Cj} = \sum_{i=1}^{K} P_{ij} \qquad P_{Ki} = \sum_{j=1}^{C} P_{ij} \tag{8}$$

The original rand index is defined as:

$$Rand = \frac{a+b}{a+b+c+d} \tag{9}$$

Being:

- a = number of elements from the same class and the same cluster
- b = number of elements from different classes and different clusters
- c = number of elements from the same class, but different clusters
- d = number of elements from different classes, but the same cluster

The adjusted rand index is defined as:

$$Rand = \frac{a+b-f}{a+b+c+d-f} \tag{10}$$

Where:

$$f = \frac{(a+c)(a+d) + (b+c)(b+d)}{a+b+c+d}; \tag{11}$$

2.3. Metrics for fuzzy partitions

The metrics for fuzzy partitions computed are the modified partition coefficient[7], the fuzzy variations of the rand index, proposed both by Frigui[8] and Hullermeier[9], the Jaccard index[10], and the Folkes-Mallows index[11].

The partition coefficient [12] indicates the separation between the fuzzy partitions, being $\frac{1}{c}$ if the membership degrees are the same, and 1 if the membership degrees are similar to a hard partition.

$$PC = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{c} u_{ki}^2 \tag{12}$$

The modified version adjusts the coefficient to the interval (0, 1).

$$MPC = 1 - \frac{c}{c-1}(1 - PC) \tag{13}$$

The Frigui's Rand variation, the Jaccard index and the Folkes-Mallows index are defined by:

$$Rand_{Frigui}(\psi^{(1)}, \psi^{(2)}) = \frac{N_{SS} + N_{DD}}{N_{SS} + N_{SD} + N_{DS} + N_{DD}} \tag{14}$$

$$Jaccard(\psi^{(1)}, \psi^{(2)}) = \frac{N_{SS}}{N_{SS} + N_{SD} + N_{DS}} \tag{15}$$

$$Folkes - Mallows(\psi^{(1)}, \psi^{(2)}) = \frac{N_{SS}}{\sqrt{(N_{SS} + N_{SD})(N_{SS} + N_{DS})}} \tag{16}$$

Being:

$$N_{SS}(\psi^{(1)}, \psi^{(2)}) = \sum_{k=2}^{n}\sum_{j=1}^{k-1} \psi_{kj}^{(1)}\psi_{kj}^{(2)}, \quad N_{DS}(\psi^{(1)}, \psi^{(2)}) = \sum_{k=2}^{n}\sum_{j=1}^{k-1}\left(1 - \psi_{kj}^{(1)}\right)\psi_{kj}^{(2)}, \tag{17}$$

$$N_{SD}(\psi^{(1)}, \psi^{(2)}) = \sum_{k=2}^{n}\sum_{j=1}^{k-1} \psi_{kj}^{(1)} \left(1 - \psi_{kj}^{(2)}\right), \quad N_{DD}(\psi^{(1)}, \psi^{(2)}) = \sum_{k=2}^{n}\sum_{j=1}^{k-1} \left(1 - \psi_{kj}^{(1)}\right) \left(1 - \psi_{kj}^{(2)}\right)$$

Where:

$$\psi_{kj} = \sum_{i=1}^{c} u_{ki} u_{ji} \tag{18}$$

The Hullermeier's Rand variation is defined by:

$$R_E(P, Q) = 1 - \frac{\sum_{k=2}^{n} \sum_{j=1}^{k-1} \left| E_P(k, j) - E_Q(k, j) \right|}{n(n-1)/2} \tag{19}$$

Being:

$$E_P(k, j) = 1 - \frac{\sum_{i=1}^{c} |u_{ki} - u_{ji}|}{2} \tag{20}$$

and

$$E_Q(k, j) = \{[l]0, \text{ if } x_k \mathit{\ and\ } x_j \mathit{\ have\ the\ same\ a\ priori\ class} 1, \mathit{\ otherwise} \tag{21}$$

2.4. Comparison with KFCM-K and KFCM-K-W.1 algorithms

This section shows the average and standard deviation of the execution time (Table 2), Accuracy (Table 3), F-measure (Table 4), Adjusted Rand (Table 5), NMI (Table 6), Entropy (Table 7), Rand Frigui (Table 8), Rand Hullermeier (Table 9), Modified Partition Coefficient (Table 10), Jaccard (Table 11) and Folkes-Mallows (Table 12) indexes were computed for the KFCM-K, KFCM-K-W.1, KFCM-K-W.2, KFCM-K-E$_{W.1}$, and KFCM-K-E$_{W.2}$ algorithms on the data sets.

2.4.1. Execution time

Table 2: Average execution time of the algorithms.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.084310 (0.015685)	0.121770 (0.030829)	**0.077720 (0.011368)**	0.283810 (0.118782)	0.116510 (0.027044)
Banknote Authentication	0.020850 (0.005806)	0.025910 (0.013261)	0.016710 (0.003667)	**0.009010 (0.001723)**	0.014060 (0.002756)
Breast Cancer Wdbc	0.010410 (0.007041)	0.015100 (0.005543)	0.018100 (0.008465)	**0.005110 (0.000488)**	0.005420 (0.001002)
Breast Cancer Wpbc	0.003230 (0.002058)	**0.001650 (0.001590)**	0.002190 (0.002292)	0.004010 (0.001196)	0.003580 (0.001478)
Brest Tissue	0.004770 (0.002611)	0.005610 (0.003295)	0.004410 (0.001940)	**0.003590 (0.001744)**	0.004170 (0.002055)
Connectionist Bench Sonar	**0.000550 (0.000876)**	0.006000 (0.008350)	0.006500 (0.005168)	0.004370 (0.001730)	0.008360 (0.006434)
Ecoli	0.032710 (0.013086)	0.028120 (0.013872)	0.029080 (0.012851)	**0.021910 (0.008868)**	0.034170 (0.017278)
German Credit	0.012230 (0.011445)	0.008290 (0.007417)	**0.006720 (0.003496)**	0.028970 (0.005989)	0.031930 (0.014266)
Glass	0.009190 (0.004004)	0.013730 (0.006945)	0.010790 (0.004680)	**0.008660 (0.003054)**	0.010520 (0.005507)
Heart Disease	0.003140 (0.001882)	0.002690 (0.001611)	0.002990 (0.001752)	**0.002600 (0.000762)**	0.002690 (0.000977)
Image Segmentation	0.177430 (0.082944)	**0.164980 (0.077070)**	0.280910 (0.187560)	0.195280 (0.090172)	0.228940 (0.126954)
Ionosphere	0.003460 (0.002137)	0.009740 (0.007181)	0.011160 (0.008150)	**0.002920 (0.000658)**	0.003270 (0.000705)
Iris	0.002130 (0.000956)	0.003560 (0.004143)	0.003200 (0.002498)	0.002390 (0.000747)	**0.001940 (0.001130)**
Landsat	**0.208210 (0.086456)**	0.335870 (0.194033)	0.406170 (0.244287)	0.251200 (0.058122)	0.481480 (0.105915)
Leaf	**0.107940 (0.052385)**	0.146400 (0.063478)	0.138140 (0.055931)	0.110440 (0.049658)	0.144790 (0.056660)
Letters	**0.938680 (0.374279)**	1.686600 (0.617527)	1.734070 (0.643721)	2.021430 (0.487881)	2.469080 (0.595323)
Liver Disorders	0.008010 (0.003442)	**0.004180 (0.001658)**	0.004640 (0.001764)	0.006350 (0.002571)	0.014880 (0.005315)
Musk V1	**0.000700 (0.000480)**	0.014210 (0.014816)	0.013390 (0.010982)	0.013700 (0.000933)	0.016420 (0.004375)
Musk V2	**0.109320 (0.198760)**	0.684690 (0.637322)	0.889940 (0.612400)	0.359560 (0.033468)	0.673720 (0.428582)
Page Blocks	**0.367450 (0.167159)**	0.400360 (0.152185)	0.719720 (0.481292)	0.616780 (0.178413)	0.385820 (0.125131)
Pendigits	0.122890 (0.066458)	0.196810 (0.093217)	0.226990 (0.112181)	**0.089230 (0.052197)**	0.154300 (0.094274)
Pima Indians Diabetes	0.013030 (0.007417)	0.019550 (0.008585)	0.012470 (0.006051)	**0.006890 (0.001174)**	0.010440 (0.004944)
QSAR Biodegradation	**0.014050 (0.006399)**	0.019780 (0.007417)	0.024550 (0.014890)	0.023910 (0.004848)	0.031510 (0.011291)
Seeds	0.003410 (0.002538)	0.004160 (0.001419)	0.004360 (0.001404)	**0.002150 (0.000654)**	0.002810 (0.000796)
Spambase	**0.055310 (0.041228)**	0.247090 (0.078023)	0.422310 (0.213308)	0.111260 (0.017458)	0.142360 (0.036356)
Thyroid Disease	0.005930 (0.004426)	0.007570 (0.004942)	0.005200 (0.003053)	**0.003300 (0.000831)**	0.004610 (0.002634)
Two Circles	0.008590 (0.002429)	0.005740 (0.001222)	0.006180 (0.002278)	**0.005030 (0.001269)**	0.007760 (0.001839)
Urban	**0.004650 (0.001486)**	0.115760 (0.177958)	0.210760 (0.195267)	0.663670 (0.325977)	0.841430 (0.332560)
Vehicle Silhouettes	**0.044390 (0.017903)**	0.052330 (0.033668)	0.061050 (0.032497)	0.061200 (0.014480)	0.064390 (0.038499)
Vertebral Column 2C	0.002790 (0.000668)	0.006400 (0.001200)	0.004630 (0.001036)	**0.002150 (0.000517)**	0.002490 (0.000728)
Vertebral Column 3C	0.008790 (0.003961)	0.010400 (0.001960)	0.009690 (0.004235)	**0.003830 (0.001010)**	0.004750 (0.001276)
Voting Records	0.005600 (0.003803)	0.003220 (0.001566)	0.003580 (0.002164)	**0.002250 (0.000517)**	0.002740 (0.000658)
Wall Following Readings 2	0.085890 (0.044831)	0.110780 (0.120547)	0.158350 (0.056638)	0.088200 (0.028097)	**0.070200 (0.022668)**
Wall Following Readings 4	0.178680 (0.088104)	0.161110 (0.065606)	0.231820 (0.128518)	**0.084330 (0.010837)**	0.116500 (0.056386)
Waveform	0.236380 (0.145856)	0.379440 (0.176451)	0.563420 (0.298804)	**0.054430 (0.006523)**	0.083910 (0.014595)
Wilt	0.058350 (0.008253)	0.064310 (0.013486)	0.064630 (0.013090)	**0.033450 (0.004809)**	0.048050 (0.009007)
Wine	0.005650 (0.003182)	0.007700 (0.004095)	0.006380 (0.001810)	**0.001180 (0.000433)**	0.001560 (0.000697)
Wine Quality Red	0.142090 (0.058145)	0.207560 (0.083288)	0.295410 (0.103454)	**0.082040 (0.019893)**	0.151660 (0.086309)
Wine Quality White	0.742960 (0.214248)	0.683040 (0.272944)	1.020810 (0.335359)	**0.388310 (0.060073)**	1.123650 (0.366500)
Zoo	0.003530 (0.003705)	**0.001600 (0.000735)**	0.001790 (0.000791)	0.003840 (0.001690)	0.004920 (0.002645)
Average Deviation	0.043963	0.075261	0.095777	**0.040006**	0.065339

Either the KFCM-K or the KFCM-K-$E_{W.1}$ obtained the lowest execution time for most of the data sets. In some data sets, like Abalone and Page Blocks, the extra steps increased the execution time. However, in other data sets, like Banknote Authentication and Waveform, the variants of KFCM-K-E_W converged in less iterations, obtaining a smaller execution time.

The execution time of proposed algorithms has an average deviation lower than 9.6%, being a consistent time, specially for the KFCM-K-$E_{W.1}$, which obtained a execution time more stable than the reference algorithms, being close to 4.0%.

2.4.2. Accuracy

Table 3: Average Accuracy of the algorithms.

Data set	KFCM-KM	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	0.512808 (0.000000)	0.509548 (0.000116)	0.519512 (0.000000)	**0.527192 (0.002135)**	0.515035 (0.003846)
Banknote Authentication	0.561953 (0.000000)	**0.590357 (0.090449)**	0.563411 (0.000000)	0.578243 (0.021092)	0.559650 (0.000491)
Breast Cancer Wdbc	0.729244 (0.110987)	0.756450 (0.103724)	0.751810 (0.106425)	**0.920914 (0.000000)**	**0.920914 (0.000000)**
Breast Cancer Wpbc	0.594596 (0.076337)	0.595000 (0.084682)	**0.613737 (0.092568)**	0.550505 (0.000000)	0.609899 (0.094162)
Brest Tissue	0.563679 (0.030288)	0.582264 (0.044929)	**0.582453 (0.027330)**	0.553868 (0.016626)	0.571887 (0.023368)
Connectionist Bench Sonar	0.553413 (0.029275)	0.540000 (0.014941)	0.540481 (0.013686)	**0.560721 (0.013744)**	0.559279 (0.009807)
Ecoli	0.783125 (0.030542)	**0.808095 (0.021509)**	0.781310 (0.027582)	0.794673 (0.020038)	0.806369 (0.025330)
German Credit	**0.700360 (0.002865)**	0.700000 (0.000000)	0.700030 (0.000222)	0.700000 (0.000000)	0.700000 (0.000000)
Glass	0.519766 (0.058274)	0.528411 (0.061992)	0.487710 (0.033688)	0.533645 (0.013017)	**0.537804 (0.016443)**
Heart Disease	0.537327 (0.017229)	0.532508 (0.008982)	0.533960 (0.010799)	**0.557294 (0.010510)**	0.542013 (0.014460)
Image Segmentation	0.586062 (0.072408)	0.521400 (0.042767)	0.527943 (0.038350)	**0.658995 (0.019865)**	0.625319 (0.039836)
Ionosphere	0.659630 (0.026912)	0.657350 (0.033419)	0.654929 (0.024592)	0.700313 (0.006702)	**0.702422 (0.002221)**
Iris	0.749867 (0.098140)	0.827733 (0.133379)	0.827133 (0.118086)	0.821800 (0.008530)	**0.893867 (0.120737)**
Landsat	0.613520 (0.073498)	0.598430 (0.068913)	0.591200 (0.072584)	**0.734345 (0.000879)**	0.726140 (0.020917)
Leaf	0.524559 (0.026511)	0.527382 (0.020291)	0.550088 (0.029972)	**0.553647 (0.017044)**	0.550588 (0.019002)
Letters	0.267566 (0.018655)	0.287160 (0.032202)	0.189037 (0.014249)	0.310614 (0.007463)	**0.312890 (0.008976)**
Liver Disorders	**0.579710 (0.000000)**	**0.579710 (0.000000)**	**0.579710 (0.000000)**	**0.579710 (0.000000)**	**0.579710 (0.000000)**
Musk V1	**0.572416 (0.011695)**	0.567395 (0.004188)	0.567332 (0.004840)	0.565126 (0.000000)	0.565126 (0.000000)
Musk V2	**0.846469 (0.002208)**	0.846096 (0.000805)	0.846061 (0.000783)	0.845862 (0.000000)	0.845862 (0.000000)
Page Blocks	0.901122 (0.004877)	0.899496 (0.004879)	**0.905547 (0.015207)**	0.902584 (0.001608)	0.904804 (0.003738)
Pendigits	0.538791 (0.055079)	0.520926 (0.091541)	0.497993 (0.042680)	**0.705429 (0.025375)**	0.680566 (0.034331)
Pima Indians Diabetes	0.651185 (0.001007)	0.651120 (0.000481)	0.651042 (0.000000)	**0.673177 (0.000000)**	0.671628 (0.005648)
QSAR Biodegradation	0.667517 (0.017861)	0.692720 (0.043873)	0.694370 (0.043667)	**0.736493 (0.000000)**	0.730512 (0.019105)
Seeds	0.831667 (0.120453)	0.846286 (0.084155)	0.841905 (0.087619)	**0.923810 (0.000000)**	0.923571 (0.002553)
Spambase	0.613517 (0.018035)	0.689993 (0.074341)	0.675536 (0.075531)	0.605955 (0.000000)	**0.850552 (0.024591)**
Thyroid Disease	0.763953 (0.023929)	0.793023 (0.047544)	**0.873674 (0.050707)**	0.795814 (0.040710)	0.804837 (0.042445)
Two Circles	0.563280 (0.071545)	0.581940 (0.080589)	**0.595040 (0.085893)**	0.519920 (0.000627)	0.519460 (0.001830)
Urban	0.281185 (0.027901)	0.193363 (0.008509)	0.211600 (0.011205)	**0.691837 (0.015737)**	0.683659 (0.025346)
Vehicle Silhouettes	0.396868 (0.020480)	0.412541 (0.037419)	**0.448345 (0.039291)**	0.388889 (0.000000)	0.389504 (0.010972)
Vertebral Column 2C	0.694710 (0.003430)	0.698581 (0.001601)	**0.720613 (0.004628)**	0.693516 (0.000321)	0.693419 (0.001284)
Vertebral Column 3C	0.666194 (0.001675)	0.673194 (0.007194)	0.677871 (0.009124)	**0.728516 (0.010269)**	0.672129 (0.007090)
Voting Records	0.791770 (0.098504)	0.712483 (0.099424)	0.732897 (0.100579)	**0.880460 (0.000000)**	**0.880460 (0.000000)**
Wall Following Readings 2	0.673545 (0.030786)	0.619342 (0.058614)	**0.682872 (0.084232)**	0.637529 (0.073021)	0.674474 (0.030473)
Wall Following Readings 4	0.517914 (0.053527)	0.506246 (0.061679)	**0.585983 (0.066405)**	0.487028 (0.017512)	0.511163 (0.050701)
Waveform	0.581286 (0.062607)	**0.582050 (0.064048)**	0.567346 (0.059437)	0.528468 (0.000616)	0.529072 (0.002067)
Wilt	**0.946063 (0.000000)**	**0.946063 (0.000000)**	**0.946063 (0.000000)**	**0.946063 (0.000000)**	**0.946063 (0.000000)**
Wine	0.706011 (0.133953)	0.730112 (0.110593)	0.708876 (0.125296)	**0.966404 (0.000787)**	0.966124 (0.001245)
Wine Quality Red	0.536954 (0.014920)	0.526254 (0.014534)	0.539925 (0.011078)	**0.557849 (0.003513)**	0.554647 (0.009877)
Wine Quality White	0.459065 (0.003271)	0.456307 (0.006969)	0.477613 (0.007869)	**0.479967 (0.001365)**	0.475282 (0.009061)
Zoo	0.765545 (0.074500)	0.724158 (0.075417)	0.787624 (0.054471)	**0.869307 (0.024851)**	0.851881 (0.034637)
Average Deviation	0.038104	0.043517	0.039767	**0.009349**	0.017915

The algorithms obtained a similar accuracy for most of the data sets, but we want to emphasis cases like:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E$_W$

obtained an accuracy of 92% (compared to the 73 76% of the other algorithms),

- the Pendigits data set, in which the KFCM-K-E$_{W.1}$ obtained an accuracy of 70% (compared to the 52 54% of the reference algorithms),
- the Urban data set, in which the variants of KFCM-K-E$_W$ obtained an accuracy of 68-69% (compared to the 19 28% of the other algorithms),
- the Wine data set, in which the variants of KFCM-K-E$_W$ obtained an accuracy of 97% (compared to the 71 73% of the other algorithms)

The proposed algorithms obtained an accuracy with an average deviation lower than 4.0%, demonstrating it robustness, specially for the variants of KFCM-K-E$_W$, which obtained an accuracy more stable than the reference algorithms, being lower than 1.8%.

2.4.3. *F-measure*

Table 4: Average performance of the algorithms according to the F-measure index.

Data set	KFCM-KM	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.524313 (0.000062)	0.523263 (0.000080)	**0.528138 (0.000000)**	0.526937 (0.005951)	0.521693 (0.002956)
Banknote Authentication	0.561988 (0.000000)	**0.590848 (0.090388)**	0.563830 (0.000000)	0.574732 (0.026322)	0.559936 (0.002046)
Breast Cancer Wdbc	0.722007 (0.112804)	0.750307 (0.108142)	0.748870 (0.111387)	**0.920277 (0.000000)**	**0.920277 (0.000000)**
Breast Cancer Wpbc	0.577987 (0.085250)	0.579389 (0.091714)	**0.595397 (0.100484)**	0.519139 (0.002121)	0.588999 (0.106424)
Brest Tissue	0.561738 (0.032517)	0.570679 (0.044295)	**0.581018 (0.034947)**	0.528712 (0.021236)	0.555522 (0.028582)
Connectionist Bench Sonar	0.614465 (0.047219)	**0.663458 (0.007960)**	0.660315 (0.011550)	0.562049 (0.013963)	0.560305 (0.009959)
Ecoli	0.584155 (0.043384)	0.630633 (0.042293)	**0.645723 (0.060239)**	0.606562 (0.014408)	0.632144 (0.036967)
German Credit	**0.646662 (0.047233)**	0.643551 (0.054690)	0.612954 (0.064100)	0.564152 (0.021169)	0.601484 (0.044348)
Glass	0.430123 (0.057929)	0.459742 (0.052110)	0.446550 (0.043824)	**0.478753 (0.009419)**	0.478396 (0.017901)
Heart Disease	0.505307 (0.061683)	0.460508 (0.043199)	0.464818 (0.046338)	**0.583458 (0.012469)**	0.562880 (0.028380)
Image Segmentation	0.586518 (0.065688)	0.555173 (0.044469)	0.552282 (0.044071)	**0.658047 (0.017686)**	0.635002 (0.038115)
Ionosphere	0.652016 (0.039141)	0.656170 (0.052952)	0.634743 (0.058843)	0.705679 (0.006726)	**0.707759 (0.002245)**
Iris	0.755611 (0.093815)	0.828693 (0.130814)	0.827265 (0.115873)	0.821638 (0.008528)	**0.898900 (0.111786)**
Landsat	0.592428 (0.070581)	0.581514 (0.065910)	0.575212 (0.067002)	**0.713449 (0.000720)**	0.707820 (0.014141)
Leaf	0.516160 (0.025427)	0.516834 (0.020888)	0.545536 (0.029343)	**0.548200 (0.015512)**	0.545129 (0.019747)
Letters	0.281723 (0.021145)	0.301341 (0.034155)	0.211654 (0.014597)	**0.337466 (0.009712)**	0.336782 (0.011565)
Liver Disorders	0.552907 (0.034357)	0.565363 (0.021683)	0.573871 (0.021442)	**0.597577 (0.002635)**	0.592327 (0.015452)
Musk V1	0.668350 (0.002419)	**0.670336 (0.001289)**	0.670234 (0.001182)	0.565216 (0.000000)	0.565229 (0.000091)
Musk V2	0.812363 (0.003845)	**0.814550 (0.002767)**	0.814355 (0.002231)	0.632218 (0.000053)	0.630396 (0.014464)
Page Blocks	0.447748 (0.039970)	**0.622696 (0.045776)**	0.530091 (0.024185)	0.617841 (0.039482)	0.590074 (0.049550)
Pendigits	0.525636 (0.054819)	0.522404 (0.073963)	0.494756 (0.041628)	**0.696358 (0.022858)**	0.673170 (0.032520)
Pima Indians Diabetes	0.615735 (0.028996)	0.624556 (0.030694)	0.548565 (0.034368)	**0.675137 (0.000000)**	0.674758 (0.008836)
QSAR Biodegradation	0.644003 (0.045324)	0.670067 (0.071505)	0.674370 (0.068447)	**0.741439 (0.000000)**	0.736005 (0.016370)
Seeds	0.831009 (0.122824)	0.844868 (0.086461)	0.840643 (0.089468)	**0.924133 (0.000000)**	0.923890 (0.002554)
Spambase	0.639502 (0.040369)	0.686903 (0.070502)	0.672754 (0.080010)	0.678213 (0.000000)	**0.850660 (0.017186)**
Thyroid Disease	0.583473 (0.018265)	0.641958 (0.092465)	**0.734240 (0.134631)**	0.686832 (0.086057)	0.712333 (0.096515)
Two Circles	0.555155 (0.057700)	0.571137 (0.061461)	**0.585982 (0.062450)**	0.519886 (0.000658)	0.520645 (0.005469)
Urban	0.281414 (0.026922)	0.238334 (0.003886)	0.240852 (0.007856)	**0.648949 (0.028982)**	0.641757 (0.034858)
Vehicle Silhouettes	0.413402 (0.025743)	0.433883 (0.036911)	**0.463050 (0.038628)**	0.411953 (0.000085)	0.418729 (0.009796)
Vertebral Column 2C	0.704450 (0.003462)	0.708087 (0.001520)	**0.729080 (0.004307)**	0.702874 (0.000414)	0.702792 (0.001228)
Vertebral Column 3C	0.642357 (0.044968)	0.659389 (0.042760)	**0.667245 (0.030352)**	0.630737 (0.011184)	0.588084 (0.005103)
Voting Records	0.787160 (0.111175)	0.720527 (0.087250)	0.741227 (0.088343)	**0.881847 (0.000000)**	**0.881847 (0.000000)**
Wall Following Readings 2	0.583262 (0.053543)	0.532271 (0.053255)	0.588610 (0.111065)	0.580629 (0.076279)	**0.594550 (0.060100)**
Wall Following Readings 4	0.456740 (0.044962)	0.447236 (0.061125)	**0.521012 (0.061359)**	0.433944 (0.019272)	0.457521 (0.050471)
Waveform	**0.587196 (0.060669)**	0.579777 (0.060054)	0.559915 (0.052447)	0.530983 (0.000291)	0.531549 (0.001393)
Wilt	**0.687496 (0.023399)**	0.660442 (0.037249)	0.655966 (0.028521)	0.670751 (0.000372)	0.674412 (0.005493)
Wine	0.715509 (0.129501)	0.735119 (0.101654)	0.725599 (0.113300)	**0.966128 (0.000803)**	0.965844 (0.001258)
Wine Quality Red	**0.374571 (0.020737)**	0.361178 (0.019433)	0.360287 (0.019403)	0.359992 (0.008188)	0.362941 (0.012460)
Wine Quality White	0.301076 (0.016478)	0.309088 (0.030899)	**0.311595 (0.016422)**	0.302845 (0.004617)	0.308764 (0.011356)
Zoo	0.678364 (0.097988)	0.660337 (0.099005)	0.730605 (0.076268)	**0.812686 (0.050551)**	0.794360 (0.071019)
Average Deviation	0.047808	0.051941	0.050273	**0.013468**	0.024968

The algorithms obtained a similar F-measure for most of the data sets, but we want to emphasis cases like:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E_W

obtained a F-measure of 92% (compared to the 72 75% of the other algorithms),

- the Landsat data set, in which the KFCM-K-E$_{W.1}$ obtained a F-measure of 71% (compared to the 58 59% of the other algorithms),
- the Thyroid Disease data set, in which the KFCM-K-W.2 obtained a F-measure of 73% (compared to the 58 64% of the reference algorithms),
- the Wine data set, in which the variants of KFCM-K-E$_W$ obtained a F-measure of 98% (compared to the 72 74% of the other algorithms),
- the Zoo data set, in which the variants of KFCM-K-E$_W$ obtained a F-measure of 79-81% (compared to the 66 68% of the reference algorithms)

The proposed algorithms obtained a F-measure with an average deviation lower than 5.1%, demonstrating it robustness, specially for the variants of KFCM-K-E$_W$, which obtained a F-measure more stable than the reference algorithms, being lower than 2.5%.

2.4.4. *Adjusted Rand*

Table 5: Average performance of the algorithms according to the Adjusted Rand index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	0.134525 (0.000048)	0.132457 (0.000103)	0.142638 (0.000000)	**0.146162 (0.004382)**	0.135416 (0.003093)
Banknote Authentication	0.014507 (0.000000)	**0.064628 (0.137505)**	0.015279 (0.000000)	0.024266 (0.015376)	0.013373 (0.000763)
Breast Cancer Wdbc	0.225287 (0.254243)	0.281760 (0.231675)	0.276413 (0.240505)	**0.706433 (0.000000)**	**0.706433 (0.000000)**
Breast Cancer Wpbc	0.110025 (0.157933)	0.134332 (0.180324)	0.165985 (0.199800)	0.083362 (0.000631)	**0.183322 (0.181676)**
Brest Tissue	0.308422 (0.053299)	0.332151 (0.063171)	**0.359701 (0.037136)**	0.266940 (0.031298)	0.296722 (0.021701)
Connectionist Bench Sonar	0.008853 (0.019791)	0.001606 (0.009263)	0.001443 (0.007582)	**0.010666 (0.006960)**	0.009579 (0.004900)
Ecoli	0.353779 (0.061378)	0.398884 (0.051388)	**0.455799 (0.091936)**	0.355093 (0.025585)	0.387260 (0.050771)
German Credit	**0.013793 (0.031125)**	0.007076 (0.018688)	0.000425 (0.014619)	0.006329 (0.009784)	0.013150 (0.022584)
Glass	0.118594 (0.054536)	0.136541 (0.054095)	0.104062 (0.055079)	0.156951 (0.013149)	**0.162772 (0.021612)**
Heart Disease	0.122910 (0.116527)	0.032049 (0.085015)	0.039521 (0.092629)	**0.297560 (0.027724)**	0.238219 (0.066055)
Image Segmentation	0.396485 (0.086178)	0.329851 (0.045615)	0.334458 (0.051193)	**0.515440 (0.014915)**	0.469156 (0.043198)
Ionosphere	0.054291 (0.067640)	0.040920 (0.083358)	0.034948 (0.066836)	0.157783 (0.010680)	**0.161036 (0.003622)**
Iris	0.487999 (0.167482)	0.652009 (0.232043)	0.646800 (0.194047)	0.603551 (0.013640)	**0.779993 (0.189102)**
Landsat	0.362835 (0.088669)	0.343987 (0.077518)	0.338602 (0.075745)	**0.524578 (0.000668)**	0.514089 (0.027397)
Leaf	0.340337 (0.031334)	0.341382 (0.023013)	0.378163 (0.034398)	**0.381794 (0.015832)**	0.366307 (0.022029)
Letters	0.105051 (0.013939)	0.123237 (0.028782)	0.040476 (0.009429)	**0.154365 (0.006435)**	0.151361 (0.007844)
Liver Disorders	0.002631 (0.002645)	0.011628 (0.006958)	**0.011635 (0.005070)**	-0.007261 (0.002268)	-0.006786 (0.001897)
Musk V1	**0.003721 (0.009854)**	0.000217 (0.003710)	0.000162 (0.004170)	0.001675 (0.000000)	0.001687 (0.000081)
Musk V2	**0.004393 (0.028432)**	-0.001546 (0.013181)	-0.002391 (0.011500)	-0.032366 (0.000020)	-0.022004 (0.020247)
Page Blocks	0.023865 (0.023358)	0.056314 (0.043145)	0.035992 (0.013310)	**0.095381 (0.005540)**	0.088003 (0.016770)
Pendigits	0.347785 (0.058965)	0.311906 (0.120351)	0.257431 (0.057885)	**0.546005 (0.024088)**	0.512109 (0.041583)
Pima Indians Diabetes	0.029956 (0.010723)	0.044254 (0.021760)	0.002203 (0.005026)	**0.113386 (0.000000)**	0.106331 (0.025745)
QSAR Biodegradation	0.013406 (0.060987)	0.083248 (0.118620)	0.093624 (0.114876)	**0.220144 (0.000000)**	0.201974 (0.061324)
Seeds	0.637723 (0.179469)	0.640007 (0.118328)	0.633854 (0.123184)	**0.785036 (0.000000)**	0.784479 (0.006452)
Spambase	0.017711 (0.032001)	0.150269 (0.127646)	0.128689 (0.136302)	-0.003967 (0.000000)	**0.492049 (0.049485)**
Thyroid Disease	0.157082 (0.031987)	0.245381 (0.155464)	**0.447578 (0.248637)**	0.257740 (0.165220)	0.308832 (0.170907)
Two Circles	0.034709 (0.060350)	0.051193 (0.064166)	**0.064032 (0.068305)**	-0.000415 (0.000086)	-0.000473 (0.000226)
Urban	0.005495 (0.008238)	-0.000275 (0.001396)	-0.000143 (0.003614)	**0.421729 (0.027418)**	0.412212 (0.032239)
Vehicle Silhouettes	0.071354 (0.018470)	0.081875 (0.034333)	**0.115547 (0.040629)**	0.072537 (0.000036)	0.079589 (0.011476)
Vertebral Column 2C	0.144368 (0.028955)	0.154777 (0.002501)	**0.189758 (0.022918)**	0.146738 (0.000625)	0.146609 (0.001909)
Vertebral Column 3C	0.300882 (0.072707)	0.334479 (0.076250)	**0.339386 (0.047985)**	0.304695 (0.015674)	0.220546 (0.010972)
Voting Records	0.370751 (0.214629)	0.188864 (0.215160)	0.234175 (0.218805)	**0.577941 (0.000000)**	**0.577941 (0.000000)**
Wall Following Readings 2	0.275818 (0.074739)	0.201856 (0.080968)	**0.299806 (0.143806)**	0.266750 (0.116841)	0.291276 (0.083600)
Wall Following Readings 4	0.103194 (0.054931)	0.100584 (0.068331)	**0.185745 (0.073316)**	0.077587 (0.023858)	0.096449 (0.062656)
Waveform	0.252006 (0.088409)	**0.256831 (0.088444)**	0.235788 (0.085860)	0.252940 (0.000166)	0.253205 (0.000742)
Wilt	**0.013210 (0.003717)**	0.000889 (0.005758)	-0.001703 (0.001353)	0.009839 (0.000097)	0.011024 (0.001646)
Wine	0.439507 (0.220091)	0.446210 (0.174001)	0.425909 (0.184431)	**0.897843 (0.002434)**	0.897018 (0.003668)
Wine Quality Red	0.054701 (0.011126)	0.047776 (0.011974)	0.048616 (0.008488)	0.066252 (0.004438)	**0.067004 (0.007413)**
Wine Quality White	0.027400 (0.002654)	0.033899 (0.008142)	0.034928 (0.004704)	**0.037365 (0.000369)**	0.034947 (0.003251)
Zoo	0.531158 (0.164841)	0.407226 (0.171738)	0.567205 (0.137619)	**0.692971 (0.087820)**	0.684113 (0.117317)
Average Deviation	0.066660	0.076347	0.073318	**0.016851**	0.034949

At least one of the proposed algorithm obtained the best Adjusted Rand index for 34 out of the 40 data sets.

The proposed algorithms obtained an accuracy with an Adjusted Rand deviation lower than 7.4%, demonstrating it robustness, specially for the variants of KFCM-K-E_W, which obtained an Adjusted Rand more stable than the reference algorithms, being lower than 3.5%.

2.4.5. NMI

Table 6: Average performance of the algorithms according to the NMI index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.160978 (0.000035)	0.161069 (0.000124)	**0.162451 (0.000000)**	0.162264 (0.001937)	0.161146 (0.001699)
Banknote Authentication	0.013876 (0.000000)	**0.055477 (0.115564)**	0.014003 (0.000000)	0.021891 (0.013513)	0.012442 (0.000464)
Breast Cancer Wdbc	0.195809 (0.209345)	0.244071 (0.196465)	0.245548 (0.202262)	**0.588047 (0.000000)**	**0.588047 (0.000000)**
Breast Cancer Wpbc	0.184219 (0.164883)	0.206401 (0.201022)	0.240791 (0.219698)	0.159265 (0.000533)	**0.254642 (0.196866)**
Brest Tissue	0.490323 (0.041594)	0.519029 (0.045457)	**0.535479 (0.030584)**	0.484241 (0.012271)	0.509792 (0.024399)
Connectionist Bench Sonar	**0.027971 (0.033046)**	0.017832 (0.018320)	0.018038 (0.016141)	0.012032 (0.004786)	0.011080 (0.003457)
Ecoli	0.539205 (0.028293)	**0.574594 (0.029776)**	0.552426 (0.035579)	0.561926 (0.009581)	0.568658 (0.026994)
German Credit	**0.009174 (0.012287)**	0.003136 (0.003936)	0.002425 (0.003426)	0.006610 (0.002327)	0.007057 (0.005821)
Glass	0.229900 (0.065026)	0.263354 (0.047926)	0.243675 (0.051693)	0.292019 (0.023041)	**0.304983 (0.032623)**
Heart Disease	0.132475 (0.086155)	0.059919 (0.042399)	0.075819 (0.041100)	**0.253181 (0.027039)**	0.208340 (0.041918)
Image Segmentation	0.542495 (0.066712)	0.477881 (0.045006)	0.485799 (0.038100)	**0.625105 (0.016465)**	0.602100 (0.030252)
Ionosphere	0.044435 (0.043241)	0.050021 (0.061443)	0.044665 (0.044104)	0.115729 (0.009666)	**0.118198 (0.003492)**
Iris	0.558258 (0.164047)	0.706957 (0.161893)	0.714926 (0.120144)	0.648645 (0.009237)	**0.796046 (0.118200)**
Landsat	0.481836 (0.069957)	0.469246 (0.065495)	0.462252 (0.066266)	**0.617959 (0.000552)**	0.608931 (0.024160)
Leaf	0.682496 (0.016966)	0.686754 (0.012515)	**0.710941 (0.017705)**	0.707689 (0.009311)	0.700507 (0.009784)
Letters	0.336078 (0.018927)	0.360523 (0.036985)	0.222283 (0.017275)	**0.391925 (0.006711)**	0.391026 (0.008580)
Liver Disorders	0.004876 (0.001773)	**0.010564 (0.004262)**	0.007330 (0.002629)	0.005115 (0.002123)	0.003838 (0.001308)
Musk V1	**0.017004 (0.019183)**	0.009133 (0.008596)	0.008964 (0.008163)	0.010436 (0.000000)	0.010456 (0.000135)
Musk V2	0.008135 (0.016922)	0.004086 (0.006344)	0.003822 (0.006038)	**0.028417 (0.000020)**	0.019069 (0.012898)
Page Blocks	0.083393 (0.025691)	0.088858 (0.037424)	0.121548 (0.031217)	0.129585 (0.001286)	**0.131134 (0.015499)**
Pendigits	0.537546 (0.047697)	0.527035 (0.085956)	0.471960 (0.039386)	**0.695025 (0.013088)**	0.681444 (0.019867)
Pima Indians Diabetes	0.025948 (0.014446)	0.041936 (0.024526)	0.001631 (0.001824)	**0.063837 (0.000000)**	0.059757 (0.015020)
QSAR Biodegradation	0.046779 (0.045239)	0.070311 (0.067694)	0.075661 (0.069053)	**0.151263 (0.000000)**	0.141438 (0.029633)
Seeds	0.648775 (0.099833)	0.640949 (0.065608)	0.638376 (0.066624)	**0.738407 (0.000000)**	0.737967 (0.005580)
Spambase	0.026420 (0.032102)	0.134219 (0.103949)	0.131021 (0.105006)	0.008353 (0.000000)	**0.378163 (0.038043)**
Thyroid Disease	0.253485 (0.047030)	0.338876 (0.135886)	**0.523457 (0.171555)**	0.322800 (0.121698)	0.363631 (0.139440)
Two Circles	0.065328 (0.110879)	0.098589 (0.120028)	**0.123574 (0.128134)**	0.001161 (0.000061)	0.001122 (0.000163)
Urban	0.163178 (0.032840)	0.030036 (0.014912)	0.064208 (0.018637)	**0.560025 (0.021316)**	0.553089 (0.020008)
Vehicle Silhouettes	0.111035 (0.029621)	0.129727 (0.034342)	**0.151232 (0.041132)**	0.091072 (0.000024)	0.107596 (0.024532)
Vertebral Column 2C	0.114209 (0.017747)	0.123343 (0.001423)	0.146820 (0.010680)	**0.163691 (0.002026)**	0.163229 (0.002571)
Vertebral Column 3C	0.319648 (0.021779)	0.339168 (0.024121)	0.371031 (0.013291)	**0.398513 (0.020390)**	0.294656 (0.014074)
Voting Records	0.321992 (0.180343)	0.245725 (0.165065)	0.274178 (0.172117)	**0.494724 (0.000000)**	**0.494724 (0.000000)**
Wall Following Readings 2	0.360727 (0.051296)	0.310083 (0.052201)	0.370705 (0.100728)	0.359184 (0.073367)	**0.374892 (0.059519)**
Wall Following Readings 4	0.196294 (0.049377)	0.198429 (0.061465)	**0.265829 (0.066189)**	0.199905 (0.024515)	0.194854 (0.055465)
Waveform	0.294135 (0.074455)	0.307263 (0.082269)	0.288562 (0.083490)	0.364015 (0.000686)	**0.364261 (0.001454)**
Wilt	0.017464 (0.001299)	0.001012 (0.001412)	0.000890 (0.000463)	0.018593 (0.000122)	**0.019306 (0.001978)**
Wine	0.502710 (0.200188)	0.503145 (0.156352)	0.461618 (0.180446)	**0.876227 (0.002337)**	0.875390 (0.003887)
Wine Quality Red	0.101599 (0.012322)	0.093597 (0.013708)	0.085169 (0.006362)	**0.102951 (0.003834)**	0.099247 (0.006325)
Wine Quality White	0.056590 (0.002944)	0.066893 (0.005839)	0.073271 (0.003280)	**0.076336 (0.000821)**	0.071915 (0.005847)
Zoo	0.650752 (0.101445)	0.612584 (0.098826)	0.691254 (0.068574)	**0.816138 (0.035923)**	0.791756 (0.054758)
Average Deviation	0.056424	0.061413	0.057477	**0.011765**	0.026418

At least one of the proposed algorithm obtained the best NMI for 33 out of the 40 data sets. With emphasis on:

- the Thyroid Disease data set, in which the KFCM-K-W.2 obtained a NMI of 52% (compared to the 25 33% of the reference algorithms),
- the Voting Records data set, in which the variants of KFCM-K-E$_W$ obtained a NMI of 49% (compared to the 25 32% of the other algorithms),
- the Wine data set, in which the variants of KFCM-K-E$_W$ obtained a an NMI of 88% (compared to the 50% of the reference algorithms).

The proposed algorithms obtained a NMI with an average deviation lower than 5.8%, demonstrating it robustness, specially for the variants of KFCM-K-E$_W$, which obtained a NMI more stable than the reference algorithms, being lower than 2.7%.

2.4.6. *Entropy*

Table 7: Average entropy of the algorithms.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	0.043199 (0.000011)	0.041631 (0.000026)	**0.036924 (0.000004)**	0.042777 (0.002572)	0.043967 (0.000960)
Banknote Authentication	0.222295 (0.000002)	0.211752 (0.022772)	0.174225 (0.000001)	**0.063627 (0.008065)**	0.068118 (0.000247)
Breast Cancer Wdbc	0.650838 (0.016019)	0.601947 (0.028717)	0.593329 (0.028143)	0.069112 (0.000000)	**0.069111 (0.000001)**
Breast Cancer Wpbc	0.298308 (0.088965)	0.226560 (0.135259)	0.192537 (0.115465)	0.201726 (0.000036)	**0.175684 (0.033961)**
Brest Tissue	0.495081 (0.060953)	0.406919 (0.085788)	0.259651 (0.067634)	**0.065804 (0.010816)**	0.067125 (0.011129)
Connectionist Bench Sonar	0.686195 (0.001002)	0.683383 (0.003106)	0.682225 (0.003795)	**0.209780 (0.007575)**	0.210761 (0.001899)
Ecoli	0.484507 (0.054016)	0.118988 (0.026198)	**0.092828 (0.026779)**	0.109255 (0.004777)	0.126952 (0.026302)
German Credit	0.688474 (0.003461)	0.362217 (0.095941)	**0.326368 (0.079905)**	0.414474 (0.023559)	0.360561 (0.085811)
Glass	0.680349 (0.060879)	0.461827 (0.062464)	0.179617 (0.076477)	0.160271 (0.020049)	**0.157916 (0.024267)**
Heart Disease	0.656456 (0.015627)	0.423328 (0.087742)	0.393482 (0.085462)	0.169796 (0.017353)	**0.168930 (0.022174)**
Image Segmentation	1.189610 (0.103975)	0.415459 (0.125566)	0.258779 (0.083591)	0.159916 (0.015191)	**0.145019 (0.027433)**
Ionosphere	0.582102 (0.053554)	0.594461 (0.057789)	0.566842 (0.056952)	**0.129638 (0.004312)**	0.129895 (0.000192)
Iris	0.203972 (0.154055)	0.188155 (0.181058)	0.166090 (0.159335)	0.028124 (0.001330)	**0.014345 (0.006904)**
Landsat	1.323790 (0.081400)	1.283730 (0.082174)	1.260090 (0.087684)	**0.078009 (0.000229)**	0.079404 (0.003506)
Leaf	0.915268 (0.087966)	0.846380 (0.097805)	0.272763 (0.094755)	**0.043462 (0.008832)**	0.046499 (0.010274)
Letters	2.925900 (0.035530)	2.710260 (0.374851)	1.896090 (0.125969)	0.344463 (0.012993)	**0.342699 (0.012016)**
Liver Disorders	0.371215 (0.017209)	0.284755 (0.006982)	0.258003 (0.041759)	**0.113909 (0.009812)**	0.130768 (0.009208)
Musk V1	0.690220 (0.000145)	0.686865 (0.003880)	0.686968 (0.002825)	**0.080057 (0.000001)**	0.080248 (0.000324)
Musk V2	0.692448 (0.000749)	0.688640 (0.004793)	0.687987 (0.003733)	**0.053206 (0.000035)**	0.075933 (0.028841)
Page Blocks	0.370310 (0.027482)	0.227567 (0.012459)	0.140969 (0.007366)	0.126296 (0.006115)	**0.123152 (0.013960)**
Pendigits	1.667460 (0.083839)	1.479350 (0.122453)	0.782795 (0.145771)	**0.113937 (0.005297)**	0.114083 (0.008357)
Pima Indians Diabetes	0.489948 (0.002899)	0.471878 (0.023102)	0.270364 (0.051101)	0.166401 (0.000006)	**0.159587 (0.027866)**
QSAR Biodegradation	0.668113 (0.015631)	0.339168 (0.094599)	0.294714 (0.101492)	0.232981 (0.000021)	**0.221462 (0.039900)**
Seeds	0.357704 (0.096849)	0.233711 (0.077515)	0.225472 (0.082226)	0.055119 (0.000001)	**0.054904 (0.001027)**
Spambase	0.664133 (0.020267)	0.439656 (0.088374)	0.208177 (0.070395)	**0.035290 (0.000000)**	0.359386 (0.032210)
Thyroid Disease	0.290674 (0.006754)	0.232387 (0.025331)	0.144975 (0.044628)	**0.100037 (0.031133)**	0.120470 (0.038783)
Two Circles	0.137313 (0.012511)	0.117830 (0.020654)	0.115946 (0.013232)	**0.075557 (0.001875)**	0.075653 (0.001497)
Urban	2.167860 (0.000465)	2.165660 (0.002067)	2.162740 (0.003793)	0.317910 (0.013958)	**0.311679 (0.018583)**
Vehicle Silhouettes	1.116250 (0.029441)	0.872780 (0.097010)	0.763045 (0.096354)	0.124417 (0.000017)	**0.115575 (0.014241)**
Vertebral Column 2C	0.281042 (0.017825)	0.264633 (0.000008)	0.255629 (0.011942)	**0.075137 (0.001183)**	0.075271 (0.000140)
Vertebral Column 3C	0.400052 (0.030684)	0.350290 (0.029681)	0.313836 (0.020832)	**0.072792 (0.006783)**	0.091442 (0.002584)
Voting Records	0.645081 (0.024371)	0.404276 (0.108718)	0.365919 (0.101494)	**0.064073 (0.000000)**	**0.064073 (0.000000)**
Wall Following Readings 2	0.108843 (0.019588)	0.098082 (0.022511)	0.087982 (0.019792)	**0.059872 (0.026284)**	0.073919 (0.023089)
Wall Following Readings 4	0.267232 (0.056003)	0.270768 (0.034161)	0.210128 (0.024707)	0.092648 (0.007383)	**0.087433 (0.025170)**
Waveform	1.095410 (0.001179)	1.094260 (0.001295)	1.093980 (0.001292)	**0.113436 (0.000311)**	0.114010 (0.001165)
Wilt	0.245393 (0.000693)	0.161057 (0.000393)	0.156169 (0.000246)	0.132004 (0.000052)	**0.131834 (0.001127)**
Wine	0.947818 (0.047262)	0.898299 (0.035981)	0.831584 (0.046547)	**0.098642 (0.000786)**	0.098886 (0.000766)
Wine Quality Red	1.263790 (0.034441)	1.201180 (0.033386)	1.099150 (0.020339)	0.276200 (0.003330)	**0.259160 (0.021798)**
Wine Quality White	1.571580 (0.012732)	1.447590 (0.026473)	1.378870 (0.014974)	0.390208 (0.001365)	**0.382606 (0.020173)**
Zoo	1.435140 (0.084311)	0.689355 (0.222618)	0.346515 (0.203809)	**0.066107 (0.017208)**	0.157085 (0.117883)
Average Deviation	0.036519	0.064043	0.055565	**0.006766**	0.017894

At least one of the proposed algorithm obtained the best Entropy for every one of the tested data sets.

The proposed algorithms obtained an Entropy with an average deviation lower than 5.6%, demonstrating it robustness, specially for the variants of KFCM-K-E_W, which obtained an Entropy more stable than the reference algorithms, being lower than 1.8%.

2.4.7. *Rand Frigui*

Table 8: Average performance of the algorithms according to the Rand Frigui index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.612084 (0.000004)	0.611203 (0.000003)	**0.616882 (0.000000)**	0.614555 (0.001591)	0.611882 (0.001141)
Banknote Authentication	0.501469 (0.000002)	**0.519394 (0.049219)**	0.501963 (0.000001)	0.511583 (0.007786)	0.506002 (0.000403)
Breast Cancer Wdbc	0.503882 (0.003609)	0.510345 (0.008646)	0.508581 (0.009342)	**0.840877 (0.000000)**	**0.840877 (0.000006)**
Breast Cancer Wpbc	0.515873 (0.018901)	0.521628 (0.024966)	0.532538 (0.035358)	0.509329 (0.000015)	**0.533545 (0.047315)**
Brest Tissue	0.746335 (0.018855)	0.758468 (0.019121)	0.772097 (0.017410)	0.775461 (0.012522)	**0.782078 (0.014695)**
Connectionist Bench Sonar	0.500005 (0.000038)	0.500041 (0.000071)	0.500052 (0.000091)	0.503963 (0.001743)	**0.504337 (0.001225)**
Ecoli	0.766015 (0.016432)	0.796871 (0.014223)	**0.810007 (0.025822)**	0.785900 (0.007137)	0.792295 (0.015098)
German Credit	0.500008 (0.000018)	0.509301 (0.006790)	0.511164 (0.007173)	0.503117 (0.004578)	**0.512561 (0.016072)**
Glass	0.672401 (0.016476)	0.673062 (0.022599)	0.623431 (0.035848)	**0.689915 (0.006872)**	0.683584 (0.010191)
Heart Disease	0.501496 (0.001468)	0.509593 (0.021028)	0.518270 (0.023051)	**0.630746 (0.008178)**	0.607155 (0.029468)
Image Segmentation	0.785033 (0.010248)	0.812600 (0.023230)	0.820668 (0.026451)	**0.875718 (0.004298)**	0.854796 (0.017669)
Ionosphere	0.514782 (0.009899)	0.512299 (0.011897)	0.513223 (0.011696)	0.557653 (0.001917)	**0.557699 (0.000217)**
Iris	0.732955 (0.087748)	0.797357 (0.142275)	0.796005 (0.127979)	0.825312 (0.005567)	**0.894684 (0.095027)**
Landsat	0.732345 (0.007381)	0.734952 (0.007776)	0.737041 (0.007807)	**0.856589 (0.000041)**	0.852276 (0.011731)
Leaf	0.949562 (0.002416)	0.948572 (0.002204)	0.955699 (0.003705)	**0.959088 (0.001614)**	0.957351 (0.003054)
Letters	0.926856 (0.000119)	0.926893 (0.000583)	0.921489 (0.003099)	**0.934899 (0.000674)**	0.933617 (0.001111)
Liver Disorders	0.500917 (0.000439)	0.501573 (0.000810)	**0.501651 (0.000188)**	0.499159 (0.001005)	0.499339 (0.000453)
Musk V1	0.500000 (0.000004)	0.500016 (0.000043)	0.500016 (0.000039)	**0.500567 (0.000000)**	0.500551 (0.000026)
Musk V2	0.500001 (0.000001)	0.500014 (0.000038)	0.500009 (0.000014)	0.502509 (0.000004)	**0.503136 (0.010721)**
Page Blocks	0.352023 (0.022333)	0.428243 (0.031375)	0.373437 (0.011392)	**0.434035 (0.012693)**	0.428747 (0.021720)
Pendigits	0.837159 (0.004478)	0.842078 (0.004705)	0.855662 (0.010269)	**0.911932 (0.005563)**	0.902370 (0.011519)
Pima Indians Diabetes	0.510958 (0.011514)	0.513999 (0.010221)	0.503661 (0.007294)	**0.559283 (0.000008)**	0.557670 (0.006641)
QSAR Biodegradation	0.500264 (0.000359)	0.516412 (0.013928)	0.515640 (0.016662)	**0.576807 (0.000008)**	0.573242 (0.014378)
Seeds	0.758131 (0.077777)	0.792441 (0.070754)	0.793380 (0.072956)	**0.897494 (0.000013)**	0.897260 (0.001843)
Spambase	0.500831 (0.000898)	0.537412 (0.019451)	0.557447 (0.051026)	0.517756 (0.000000)	**0.627293 (0.010974)**
Thyroid Disease	0.548049 (0.013789)	0.612080 (0.073511)	**0.718634 (0.111351)**	0.621300 (0.086645)	0.640474 (0.089020)
Two Circles	0.523537 (0.041254)	0.539488 (0.048737)	**0.544998 (0.046954)**	0.499633 (0.000004)	0.499622 (0.000053)
Urban	0.783010 (0.000002)	0.783015 (0.000007)	0.783021 (0.000010)	**0.863838 (0.004155)**	0.860633 (0.007149)
Vehicle Silhouettes	0.633410 (0.002917)	0.641852 (0.007712)	0.646349 (0.008387)	**0.650697 (0.000005)**	0.648675 (0.008075)
Vertebral Column 2C	0.522931 (0.004153)	0.524405 (0.000011)	0.532705 (0.003145)	**0.564074 (0.000381)**	0.563989 (0.000416)
Vertebral Column 3C	0.607116 (0.006884)	0.620016 (0.007789)	0.628969 (0.003978)	**0.668810 (0.007254)**	0.637503 (0.003946)
Voting Records	0.505516 (0.006114)	0.568049 (0.079752)	0.584258 (0.079044)	0.776785 (0.000001)	**0.776786 (0.000002)**
Wall Following Readings 2	0.678947 (0.024291)	0.647585 (0.035354)	**0.696116 (0.060952)**	0.667931 (0.047640)	0.684411 (0.025646)
Wall Following Readings 4	0.598691 (0.029689)	0.601533 (0.031319)	**0.646805 (0.029305)**	0.588355 (0.011471)	0.595465 (0.035629)
Waveform	0.555636 (0.000042)	0.555691 (0.000062)	0.555711 (0.000058)	0.665108 (0.000086)	**0.665126 (0.000205)**
Wilt	**0.506462 (0.021339)**	0.505361 (0.037634)	0.502650 (0.026785)	0.504179 (0.000021)	0.504415 (0.000604)
Wine	0.575146 (0.015007)	0.581302 (0.015076)	0.595086 (0.022140)	**0.927103 (0.000530)**	0.926878 (0.000927)
Wine Quality Red	0.599173 (0.001369)	0.599030 (0.001498)	0.601379 (0.001103)	**0.616853 (0.000485)**	0.615357 (0.003376)
Wine Quality White	0.626724 (0.000437)	0.627058 (0.000672)	0.627841 (0.000594)	**0.636406 (0.000085)**	0.633928 (0.002438)
Zoo	0.713218 (0.007281)	0.810525 (0.046296)	0.843441 (0.045954)	**0.898259 (0.025136)**	0.884340 (0.038816)
Average Deviation	0.012150	0.022285	0.023611	**0.006693**	0.013975

At least one of the proposed algorithm obtained the best Rand Frigui for 38 out of the 40 data sets. With emphasis on:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E_W obtained a Rand Frigui of 84% (compared to the 50 51% the other algorithms),
- the Thyroid Disease data set, in which the KFCM-K-W.2 obtained a Rand Frigui of 72% (compared to the 55 61% of the reference algorithms),
- the Wine data set, in which the variants of KFCM-K-E_W obtained a an Rand Frigui of 93% (compared to the 58% of the reference algorithms).

The proposed algorithms obtained a Rand Frigui with an average deviation lower than 2.4%, demonstrating it robustness, specially for the KFCM-K-$E_{W.1}$, which obtained a Rand Frigui more stable than the reference algorithms, being lower than 0.7%.

2.4.8. Rand Hullermeier

Table 9: Average performance of the algorithms according to the Rand Hullermeier index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.611854 (0.000004)	0.610995 (0.000002)	**0.616713 (0.000000)**	0.614337 (0.001568)	0.611644 (0.001145)
Banknote Authentication	0.505733 (0.000002)	**0.521813 (0.049084)**	0.504053 (0.000001)	0.511764 (0.007741)	0.506179 (0.000407)
Breast Cancer Wdbc	0.520714 (0.007643)	0.520673 (0.012426)	0.517425 (0.013705)	**0.841082 (0.000000)**	**0.841082 (0.000006)**
Breast Cancer Wpbc	0.510555 (0.030799)	0.530743 (0.044381)	**0.544833 (0.046594)**	0.518899 (0.000031)	0.543807 (0.048970)
Brest Tissue	0.734101 (0.021726)	0.761409 (0.019431)	0.779595 (0.018768)	0.775171 (0.012529)	**0.781811 (0.014754)**
Connectionist Bench Sonar	0.499815 (0.000520)	0.499931 (0.000732)	0.499910 (0.000803)	0.505810 (0.001957)	**0.506443 (0.001617)**
Ecoli	0.750798 (0.018403)	0.796132 (0.014548)	**0.809338 (0.025934)**	0.786225 (0.007141)	0.792129 (0.015352)
German Credit	**0.577928 (0.001741)**	0.530740 (0.012186)	0.528399 (0.011783)	0.518065 (0.003361)	0.524255 (0.012999)
Glass	0.620823 (0.019160)	0.656385 (0.022568)	0.622279 (0.034697)	**0.687269 (0.007133)**	0.680948 (0.010247)
Heart Disease	0.354167 (0.010517)	0.446306 (0.040536)	0.462814 (0.042955)	**0.626628 (0.008269)**	0.602540 (0.029678)
Image Segmentation	0.605933 (0.047545)	0.792791 (0.028021)	0.810947 (0.025827)	**0.873484 (0.004155)**	0.852900 (0.017437)
Ionosphere	0.524368 (0.009459)	0.527080 (0.011687)	0.523438 (0.013956)	0.562471 (0.001759)	**0.562601 (0.000220)**
Iris	0.745489 (0.070032)	0.816375 (0.123352)	0.814178 (0.108169)	0.825222 (0.005573)	**0.894668 (0.095023)**
Landsat	0.491136 (0.044232)	0.511179 (0.042243)	0.521051 (0.044333)	**0.855929 (0.000035)**	0.851551 (0.011897)
Leaf	0.900114 (0.011863)	0.906485 (0.012124)	0.951946 (0.005776)	**0.959003 (0.001621)**	0.957250 (0.003067)
Letters	0.253794 (0.019892)	0.343867 (0.149705)	0.646961 (0.039457)	**0.929407 (0.000729)**	0.927978 (0.001253)
Liver Disorders	**0.505194 (0.000079)**	0.504101 (0.000960)	0.504066 (0.000885)	0.500110 (0.000951)	0.500315 (0.000414)
Musk V1	0.507411 (0.000387)	0.507561 (0.000987)	**0.507628 (0.000855)**	0.500744 (0.000000)	0.500727 (0.000029)
Musk V2	**0.738603 (0.000815)**	0.735700 (0.004207)	0.735336 (0.002805)	0.503904 (0.000001)	0.505240 (0.011017)
Page Blocks	0.352189 (0.024228)	0.426713 (0.031296)	0.373047 (0.011537)	**0.434259 (0.012442)**	0.429346 (0.021556)
Pendigits	0.510914 (0.043814)	0.590222 (0.047009)	0.788504 (0.037437)	**0.911247 (0.005535)**	0.901687 (0.011508)
Pima Indians Diabetes	0.514216 (0.012209)	0.516769 (0.010095)	0.504546 (0.008020)	**0.559319 (0.000008)**	0.557714 (0.006564)
QSAR Biodegradation	0.554626 (0.006662)	0.562727 (0.019407)	0.556874 (0.019784)	**0.586842 (0.000006)**	0.582537 (0.016136)
Seeds	0.757468 (0.069944)	0.797956 (0.054547)	0.799777 (0.056263)	**0.897329 (0.000013)**	0.897095 (0.001843)
Spambase	0.516270 (0.004541)	0.547723 (0.022653)	0.563919 (0.051469)	0.517882 (0.000000)	**0.641036 (0.012319)**
Thyroid Disease	0.541021 (0.013865)	0.611123 (0.075018)	**0.722449 (0.109691)**	0.620251 (0.087149)	0.639448 (0.089562)
Two Circles	0.529808 (0.046663)	0.546583 (0.053318)	**0.554500 (0.052309)**	0.499914 (0.000013)	0.499901 (0.000077)
Urban	0.153276 (0.000779)	0.154677 (0.001193)	0.156585 (0.002096)	**0.857490 (0.003941)**	0.854391 (0.006937)
Vehicle Silhouettes	0.434402 (0.016751)	0.517628 (0.028070)	0.554063 (0.029795)	**0.648261 (0.000004)**	0.646546 (0.007758)
Vertebral Column 2C	0.549324 (0.002805)	0.549339 (0.000004)	0.559931 (0.002539)	**0.565011 (0.000417)**	0.564920 (0.000443)
Vertebral Column 3C	0.614756 (0.010312)	0.632055 (0.012687)	0.643615 (0.004759)	**0.668699 (0.007339)**	0.637225 (0.003853)
Voting Records	0.536121 (0.018010)	0.612660 (0.073634)	0.625478 (0.066999)	0.776988 (0.000001)	**0.776990 (0.000002)**
Wall Following Readings 2	0.678942 (0.024326)	0.647603 (0.035370)	**0.696084 (0.061033)**	0.667754 (0.047665)	0.684236 (0.025682)
Wall Following Readings 4	0.589409 (0.028737)	0.592245 (0.030880)	**0.641377 (0.029509)**	0.587199 (0.011496)	0.594483 (0.035456)
Waveform	0.341238 (0.002763)	0.344014 (0.002803)	0.344671 (0.002576)	0.664537 (0.000083)	**0.664543 (0.000210)**
Wilt	**0.542083 (0.021794)**	0.518889 (0.038064)	0.515499 (0.027163)	0.513739 (0.000036)	0.514020 (0.000609)
Wine	0.464123 (0.043001)	0.490298 (0.030338)	0.523103 (0.041960)	**0.929152 (0.000490)**	0.928934 (0.000968)
Wine Quality Red	0.502129 (0.010433)	0.511736 (0.009739)	0.529031 (0.005144)	**0.613613 (0.000238)**	0.612239 (0.003325)
Wine Quality White	0.469198 (0.004626)	0.494656 (0.007402)	0.511807 (0.003678)	**0.629335 (0.000107)**	0.626658 (0.002539)
Zoo	0.468195 (0.035532)	0.762380 (0.077444)	0.843873 (0.050215)	**0.898081 (0.025159)**	0.892162 (0.037227)
Average Deviation	0.018915	0.031254	0.027782	**0.006667**	0.014003

At least one of the proposed algorithm obtained the best Rand Hullermeier for 35 out of the 40 data sets. With emphasis on:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E_W

obtained a Rand Hullermeier of 84% (compared to the 52% the other algorithms),

- the Landsat data set, in which the variants of KFCM-K-E_W obtained a Rand Hullermeier of 85-86% (compared to the 49 52% of the other algorithms),

- the Letters data set, in which the variants of KFCM-K-E_W obtained a Rand Hullermeier of 93% (compared to the 25 34% of the reference algorithms),

- the Pendigits data set, in which the variants of KFCM-K-E_W obtained a Rand Hullermeier of 91% (compared to the 51 59% of the reference algorithms),

- the Thyroid Disease data set, in which the KFCM-K-W.2 obtained a Rand Hullermeier of 72% (compared to the 54 61% of the reference algorithms),

- the Urban data set, in which the variants of KFCM-K-E_W obtained a Rand Hullermeier of 85-86% (compared to the 15 16% of the other algorithms),

- the Wine data set, in which the variants of KFCM-K-E_W obtained a an Rand Hullermeier of 93% (compared to the 46-49% of the reference algorithms).

The proposed algorithms obtained a Rand Hullermeier with an average deviation lower than 2.8%, demonstrating it robustness, specially for the variants of KFCM-K-E_W, which obtained a Rand Hullermeier more stable than the reference algorithms, being lower than 1.5%.

2.4.9. *Modified Partition Coefficient*

Table 10: Average performance of the algorithms according to the Modified Partition Coefficient.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	0.960758 (0.000009)	0.962047 (0.000030)	**0.966726 (0.000004)**	0.960697 (0.002463)	0.959954 (0.001008)
Banknote Authentication	0.713360 (0.000003)	0.723006 (0.032655)	0.775763 (0.000002)	**0.925812 (0.010557)**	0.920018 (0.000232)
Breast Cancer Wdbc	0.069784 (0.026618)	0.146294 (0.045584)	0.159169 (0.044395)	**0.915806 (0.000000)**	**0.915806 (0.000002)**
Breast Cancer Wpbc	0.587202 (0.132358)	0.680826 (0.196252)	0.725428 (0.166694)	0.720097 (0.000040)	**0.757060 (0.048516)**
Brest Tissue	0.720347 (0.033160)	0.771635 (0.047624)	0.857307 (0.037793)	**0.960646 (0.006708)**	0.959334 (0.006628)
Connectionist Bench Sonar	0.010111 (0.001690)	0.014164 (0.004517)	0.015864 (0.005542)	**0.736948 (0.008154)**	0.734767 (0.003270)
Ecoli	0.754347 (0.026043)	0.931678 (0.013071)	**0.948748 (0.013787)**	0.932847 (0.002503)	0.924346 (0.013109)
German Credit	0.007330 (0.005616)	0.480423 (0.136367)	0.529880 (0.115202)	0.463521 (0.034849)	**0.540205 (0.115850)**
Glass	0.615502 (0.034691)	0.737143 (0.035604)	0.899479 (0.042933)	0.904736 (0.012244)	**0.905162 (0.014543)**
Heart Disease	0.057585 (0.024703)	0.393568 (0.128701)	0.436291 (0.125755)	0.791840 (0.024783)	**0.792540 (0.030253)**
Image Segmentation	0.388988 (0.053778)	0.782605 (0.064870)	0.864302 (0.043145)	0.903575 (0.009262)	**0.911145 (0.016930)**
Ionosphere	0.166936 (0.080636)	0.148074 (0.087165)	0.189729 (0.086056)	**0.838115 (0.006219)**	0.837585 (0.000283)
Iris	0.822264 (0.141469)	0.835122 (0.168452)	0.854851 (0.147667)	0.974593 (0.001307)	**0.989022 (0.006170)**
Landsat	0.264445 (0.045957)	0.286746 (0.046505)	0.300193 (0.049520)	**0.946533 (0.000120)**	0.945475 (0.002732)
Leaf	0.716103 (0.025533)	0.737699 (0.028076)	0.915831 (0.027504)	**0.978744 (0.003779)**	0.976888 (0.004392)
Letters	0.096383 (0.010385)	0.159200 (0.107471)	0.409749 (0.037629)	**0.831469 (0.005389)**	0.831056 (0.005636)
Liver Disorders	0.498479 (0.023319)	0.619983 (0.007010)	0.653148 (0.051953)	**0.858247 (0.013635)**	0.835527 (0.011583)
Musk V1	0.004224 (0.000209)	0.009109 (0.005723)	0.008922 (0.004090)	**0.899800 (0.000002)**	0.899489 (0.000527)
Musk V2	0.001011 (0.001082)	0.006620 (0.007136)	0.007547 (0.005504)	**0.945368 (0.000033)**	0.913120 (0.041145)
Page Blocks	0.762982 (0.016966)	0.854028 (0.008214)	0.908014 (0.005319)	0.912729 (0.004197)	**0.914330 (0.009598)**
Pendigits	0.274289 (0.036510)	0.354713 (0.051991)	0.659081 (0.063349)	**0.936103 (0.003511)**	0.935688 (0.004885)
Pima Indians Diabetes	0.321517 (0.004938)	0.345314 (0.032564)	0.630846 (0.072283)	0.793146 (0.000009)	**0.802126 (0.036595)**
QSAR Biodegradation	0.040030 (0.025392)	0.522765 (0.138448)	0.579540 (0.147244)	0.703991 (0.000031)	**0.718940 (0.051588)**
Seeds	0.693657 (0.091491)	0.802336 (0.073139)	0.810309 (0.077482)	0.949252 (0.000003)	**0.949489 (0.001114)**
Spambase	0.045258 (0.031721)	0.388310 (0.119739)	0.702605 (0.102581)	**0.955769 (0.000000)**	0.531302 (0.042182)
Thyroid Disease	0.740952 (0.007044)	0.793160 (0.022663)	0.872757 (0.040305)	**0.913794 (0.027072)**	0.894734 (0.034642)
Two Circles	0.834628 (0.016100)	0.862300 (0.032458)	0.865475 (0.020219)	**0.910568 (0.002282)**	0.910332 (0.001777)
Urban	0.013364 (0.000211)	0.014363 (0.000948)	0.015673 (0.001729)	0.816844 (0.008707)	**0.820631 (0.010780)**
Vehicle Silhouettes	0.202689 (0.022291)	0.378266 (0.069056)	0.459778 (0.070472)	0.903207 (0.000012)	**0.911070 (0.011038)**
Vertebral Column 2C	0.625316 (0.026619)	0.647618 (0.000012)	0.656730 (0.018037)	**0.908098 (0.001726)**	0.907892 (0.000280)
Vertebral Column 3C	0.653283 (0.029263)	0.696687 (0.030463)	0.731539 (0.019612)	**0.936739 (0.006183)**	0.919589 (0.002962)
Voting Records	0.075705 (0.038623)	0.417213 (0.156704)	0.472482 (0.146264)	**0.921252 (0.000000)**	**0.921252 (0.000000)**
Wall Following Readings 2	0.919700 (0.013631)	0.927580 (0.016601)	0.935075 (0.014763)	**0.955037 (0.019141)**	0.944639 (0.016989)
Wall Following Readings 4	0.808516 (0.041240)	0.805766 (0.025926)	0.849931 (0.018371)	0.933236 (0.005613)	**0.936041 (0.018725)**
Waveform	0.003310 (0.001300)	0.004503 (0.001443)	0.004776 (0.001438)	**0.902607 (0.000419)**	0.902084 (0.001247)
Wilt	0.678905 (0.001224)	0.790619 (0.000700)	0.797606 (0.000260)	0.832016 (0.000082)	**0.832333 (0.001567)**
Wine	0.149143 (0.048080)	0.196591 (0.036720)	0.259414 (0.047201)	**0.915040 (0.000807)**	0.914785 (0.000722)
Wine Quality Red	0.293875 (0.019509)	0.327483 (0.019517)	0.385551 (0.011389)	0.825745 (0.000963)	**0.836133 (0.013214)**
Wine Quality White	0.186427 (0.006476)	0.252517 (0.014156)	0.287458 (0.008084)	0.765289 (0.001137)	**0.768688 (0.011107)**
Zoo	0.253060 (0.049649)	0.623525 (0.133575)	0.809406 (0.118442)	**0.962881 (0.010214)**	0.896896 (0.089685)
Average Deviation	0.029888	0.053696	0.050250	**0.005854**	0.017088

At least one of the proposed algorithm obtained the best Modified Partition Coefficient for every one of the tested data sets. With an considerable difference for most of the data sets.

The proposed algorithms obtained a Modified Partition Coefficient with an average deviation lower than 5.1%, demonstrating it robustness, specially for the variants of KFCM-K-E$_W$, which obtained a Modified Partition Coefficient more stable than the reference algorithms, being lower than 1.8%.

2.4.10. *Jaccard index*

Table 11: Average performance of the algorithms according to the Jaccard index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-E$_{W.1}$	KFCM-K-E$_{W.2}$
Abalone	0.272444 (0.000002)	0.271881 (0.000002)	0.275135 (0.000001)	**0.281813 (0.003134)**	0.273227 (0.002135)
Banknote Authentication	0.339260 (0.000002)	**0.355673 (0.048117)**	0.338378 (0.000001)	0.348756 (0.009224)	0.342530 (0.000105)
Breast Cancer Wdbc	0.351951 (0.004419)	0.360379 (0.011060)	0.357541 (0.011221)	0.743864 (0.000000)	**0.743865 (0.000009)**
Breast Cancer Wpbc	0.339911 (0.059192)	0.347990 (0.080472)	**0.366582 (0.088226)**	0.292121 (0.000019)	0.354963 (0.086936)
Brest Tissue	0.181786 (0.016645)	0.211676 (0.029212)	0.240187 (0.023394)	0.242026 (0.013174)	**0.260620 (0.012486)**
Connectionist Bench Sonar	0.333266 (0.000034)	0.333302 (0.000070)	0.333309 (0.000084)	**0.338513 (0.004702)**	0.338378 (0.003439)
Ecoli	0.272806 (0.041626)	0.345640 (0.041980)	**0.403222 (0.082228)**	0.306713 (0.019399)	0.333753 (0.041494)
German Credit	0.366951 (0.000041)	0.391998 (0.019851)	**0.398674 (0.020837)**	0.371267 (0.008739)	0.393246 (0.041251)
Glass	0.176096 (0.021576)	0.197455 (0.028832)	0.217602 (0.033482)	0.229869 (0.008989)	**0.234846 (0.018552)**
Heart Disease	0.254202 (0.001700)	0.270920 (0.022514)	0.276928 (0.025231)	**0.393213 (0.009865)**	0.365965 (0.029722)
Image Segmentation	0.153472 (0.022031)	0.256916 (0.044910)	0.285053 (0.030892)	**0.405741 (0.011900)**	0.375724 (0.030765)
Ionosphere	0.368826 (0.012506)	0.365348 (0.014697)	0.366131 (0.013866)	**0.406642 (0.001810)**	0.406629 (0.000176)
Iris	0.451200 (0.124670)	0.585579 (0.248986)	0.576251 (0.220689)	0.580879 (0.010934)	**0.765125 (0.152956)**
Landsat	0.143233 (0.014027)	0.148074 (0.014823)	0.150761 (0.014886)	**0.435046 (0.000126)**	0.428119 (0.018912)
Leaf	0.152659 (0.016736)	0.146507 (0.014135)	0.224532 (0.025773)	**0.251560 (0.011643)**	0.240385 (0.015641)
Letters	0.024119 (0.000700)	0.028425 (0.006416)	0.039679 (0.003069)	**0.099451 (0.003057)**	0.097488 (0.003750)
Liver Disorders	0.347319 (0.021711)	0.346720 (0.013473)	0.344924 (0.013664)	**0.383770 (0.001796)**	0.380204 (0.010144)
Musk V1	0.336625 (0.000004)	0.336643 (0.000044)	0.336641 (0.000040)	**0.355197 (0.000000)**	0.355116 (0.000136)
Musk V2	0.425024 (0.000001)	0.425041 (0.000043)	0.425037 (0.000025)	**0.438558 (0.000007)**	0.436011 (0.012238)
Page Blocks	0.235044 (0.026299)	**0.333403 (0.035167)**	0.260981 (0.013868)	0.328658 (0.017280)	0.321221 (0.027601)
Pendigits	0.108848 (0.014271)	0.134167 (0.017851)	0.193315 (0.031164)	**0.414355 (0.020180)**	0.388259 (0.032077)
Pima Indians Diabetes	0.368115 (0.019326)	0.371773 (0.016742)	0.363124 (0.009111)	0.416928 (0.000006)	**0.422138 (0.023730)**
QSAR Biodegradation	0.356491 (0.000451)	0.390606 (0.021533)	0.391113 (0.020455)	0.429100 (0.000011)	**0.433303 (0.015791)**
Seeds	0.479279 (0.112935)	0.533410 (0.110138)	0.535812 (0.114112)	**0.731377 (0.000031)**	0.730861 (0.004074)
Spambase	0.344599 (0.001637)	0.395075 (0.018597)	0.407259 (0.046888)	**0.508235 (0.000000)**	0.473718 (0.003834)
Thyroid Disease	0.330058 (0.028730)	0.425963 (0.105963)	**0.548092 (0.177651)**	0.451606 (0.127695)	0.474337 (0.131571)
Two Circles	0.368445 (0.062411)	0.391765 (0.071550)	**0.403152 (0.072427)**	0.332243 (0.000080)	0.332558 (0.001808)
Urban	0.065162 (0.000005)	0.065175 (0.000020)	0.065189 (0.000026)	**0.302540 (0.014816)**	0.299712 (0.019444)
Vehicle Silhouettes	0.154915 (0.003362)	0.172442 (0.009464)	0.182503 (0.011904)	0.180089 (0.000013)	**0.186659 (0.010302)**
Vertebral Column 2C	0.382260 (0.005135)	0.382729 (0.000013)	0.389990 (0.001948)	**0.418864 (0.000603)**	0.418757 (0.000439)
Vertebral Column 3C	0.295596 (0.001531)	0.310258 (0.003839)	0.321035 (0.005385)	**0.390540 (0.012759)**	0.339059 (0.004914)
Voting Records	0.349867 (0.005324)	0.417973 (0.081120)	0.434283 (0.078368)	0.642822 (0.000001)	**0.642825 (0.000003)**
Wall Following Readings 2	0.334056 (0.060931)	0.293401 (0.054811)	**0.358828 (0.100104)**	0.351220 (0.079084)	0.353852 (0.071452)
Wall Following Readings 4	0.236710 (0.020164)	0.234411 (0.036154)	**0.261401 (0.038592)**	0.237173 (0.011358)	0.251920 (0.032371)
Waveform	0.200060 (0.000043)	0.200114 (0.000057)	0.200117 (0.000058)	0.332977 (0.000148)	**0.333043 (0.000406)**
Wilt	**0.479161 (0.023624)**	0.478966 (0.041000)	0.475997 (0.029179)	0.476581 (0.000018)	0.476795 (0.000485)
Wine	0.228157 (0.017776)	0.236723 (0.018580)	0.250639 (0.026268)	**0.803898 (0.001294)**	0.803352 (0.002248)
Wine Quality Red	0.137329 (0.003071)	0.139162 (0.004297)	0.138245 (0.003057)	0.167875 (0.004445)	**0.172028 (0.009857)**
Wine Quality White	0.113172 (0.000929)	0.114304 (0.001122)	0.114768 (0.000947)	0.128824 (0.000268)	**0.131201 (0.004605)**
Zoo	0.142751 (0.015546)	0.376249 (0.126195)	0.461280 (0.132236)	**0.603987 (0.092847)**	0.565347 (0.125773)
Average Deviation	0.019528	0.035346	0.038034	**0.012536**	0.025091

At least one of the proposed algorithm obtained the best Jaccard index for 37 out of the 40 data sets. With emphasis on:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E$_W$

obtained a Jaccard index of 74% (compared to the 35 26% the other algorithms),

- the Iris data set, in which the KFCM-K-E$_{W.2}$ obtained a Jaccard index of 72% (compared to the 45 56% of the other algorithms),
- the Seeds data set, in which the variants of KFCM-K-E$_W$ obtained a Jaccard index of 72% (compared to the 48 54% of the other algorithms),
- the Voting Records data set, in which the variants of KFCM-K-E$_W$ obtained a Jaccard index of 72% (compared to the 35 43% of the other algorithms),
- the Wine data set, in which the variants of KFCM-K-E$_W$ obtained a an Jaccard index of 93% (compared to the 23-25% of the other algorithms).

The proposed algorithms obtained a Jaccard index with an average deviation lower than 3.9%, demonstrating it robustness, specially for the KFCM-K-E$_{W.1}$, which obtained a Jaccard index more stable than the reference algorithms, being lower than 1.3%.

2.4.11. *Folkes-Mallows index*

Table 12: Average performance of the algorithms according to the Folkes-Mallows index.

Data set	KFCM-K	KFCM-K-W.1	KFCM-K-W.2	KFCM-K-$E_{W.1}$	KFCM-K-$E_{W.2}$
Abalone	0.428259 (0.000002)	0.427569 (0.000002)	0.431548 (0.000001)	**0.439859 (0.003855)**	0.429236 (0.002648)
Banknote Authentication	0.506638 (0.000002)	**0.522996 (0.048628)**	0.505658 (0.000001)	0.517106 (0.010156)	0.510278 (0.000117)
Breast Cancer Wdbc	0.520837 (0.004796)	0.529857 (0.011880)	0.526811 (0.012053)	0.853268 (0.000000)	**0.853269 (0.000006)**
Breast Cancer Wpbc	0.516670 (0.075265)	0.526659 (0.102527)	**0.548259 (0.109904)**	0.455066 (0.000024)	0.532296 (0.105458)
Brest Tissue	0.309602 (0.024859)	0.352282 (0.044141)	0.390610 (0.033861)	0.392112 (0.016384)	**0.416851 (0.017343)**
Connectionist Bench Sonar	0.499925 (0.000039)	0.499965 (0.000079)	0.499973 (0.000094)	**0.505830 (0.005392)**	0.505668 (0.003937)
Ecoli	0.451127 (0.046686)	0.536235 (0.042733)	**0.584393 (0.074957)**	0.498648 (0.020857)	0.523711 (0.041982)
German Credit	0.538352 (0.000041)	0.563540 (0.019960)	**0.570244 (0.020981)**	0.542793 (0.008780)	0.565207 (0.042812)
Glass	0.301508 (0.030502)	0.331297 (0.041058)	0.360171 (0.046678)	0.374351 (0.011717)	**0.380773 (0.024549)**
Heart Disease	0.413372 (0.002265)	0.435815 (0.029062)	0.442881 (0.032149)	**0.576639 (0.010806)**	0.546420 (0.033444)
Image Segmentation	0.265580 (0.033069)	0.409352 (0.058411)	0.446235 (0.036620)	**0.577506 (0.011942)**	0.549535 (0.032082)
Ionosphere	0.538939 (0.013307)	0.535215 (0.015623)	0.536070 (0.014743)	**0.578383 (0.001831)**	0.578373 (0.000179)
Iris	0.612425 (0.128491)	0.707475 (0.207363)	0.706844 (0.185156)	0.734823 (0.008462)	**0.859683 (0.107541)**
Landsat	0.250484 (0.021473)	0.257837 (0.022672)	0.261933 (0.022605)	**0.606394 (0.000122)**	0.599519 (0.018769)
Leaf	0.265705 (0.025410)	0.256667 (0.021998)	0.368232 (0.034253)	**0.403496 (0.014670)**	0.389566 (0.019665)
Letters	0.047105 (0.001336)	0.055226 (0.012040)	0.076801 (0.005857)	**0.181079 (0.005060)**	0.177957 (0.006201)
Liver Disorders	0.515849 (0.024879)	0.515034 (0.015579)	0.513047 (0.015777)	**0.556977 (0.002102)**	0.552900 (0.011505)
Musk V1	0.503708 (0.000004)	0.503728 (0.000049)	0.503726 (0.000045)	**0.524487 (0.000001)**	0.524397 (0.000152)
Musk V2	0.607949 (0.000001)	0.607964 (0.000039)	0.607959 (0.000020)	**0.617657 (0.000006)**	0.616036 (0.010224)
Page Blocks	0.454540 (0.028174)	0.549330 (0.034612)	0.482948 (0.013596)	**0.552720 (0.014950)**	0.546279 (0.024577)
Pendigits	0.196057 (0.023157)	0.236397 (0.028079)	0.323619 (0.043803)	**0.586818 (0.020122)**	0.561418 (0.031625)
Pima Indians Diabetes	0.538211 (0.020120)	0.542147 (0.017412)	0.532938 (0.009511)	0.588589 (0.000006)	**0.594572 (0.026822)**
QSAR Biodegradation	0.526235 (0.000471)	0.562729 (0.023242)	0.562940 (0.021827)	0.601073 (0.000011)	**0.605943 (0.018364)**
Seeds	0.639669 (0.110367)	0.688053 (0.107050)	0.689639 (0.109578)	**0.844851 (0.000020)**	0.844499 (0.002763)
Spambase	0.512670 (0.001791)	0.567263 (0.020888)	0.577524 (0.046215)	**0.705032 (0.000000)**	0.643203 (0.006505)
Thyroid Disease	0.504394 (0.028771)	0.596130 (0.090354)	**0.700861 (0.131202)**	0.616639 (0.107710)	0.637713 (0.110542)
Two Circles	0.536900 (0.065688)	0.561275 (0.074974)	**0.573652 (0.076369)**	0.498773 (0.000090)	0.499129 (0.002047)
Urban	0.122984 (0.000009)	0.123007 (0.000035)	0.123032 (0.000046)	**0.465597 (0.017371)**	0.461765 (0.022942)
Vehicle Silhouettes	0.268259 (0.005050)	0.294119 (0.013841)	0.308632 (0.017246)	0.305222 (0.000019)	**0.314812 (0.015129)**
Vertebral Column 2C	0.553951 (0.005329)	0.554399 (0.000013)	0.562018 (0.001993)	**0.591323 (0.000578)**	0.591219 (0.000434)
Vertebral Column 3C	0.456668 (0.001814)	0.473907 (0.004668)	0.486432 (0.006132)	**0.561648 (0.013215)**	0.506471 (0.005457)
Voting Records	0.518480 (0.005828)	0.585299 (0.076305)	0.601637 (0.074198)	0.782778 (0.000001)	**0.782780 (0.000002)**
Wall Following Readings 2	0.499669 (0.065371)	0.452852 (0.061009)	**0.522126 (0.103932)**	0.515976 (0.086919)	0.520712 (0.077052)
Wall Following Readings 4	0.383568 (0.026744)	0.380013 (0.046609)	**0.414733 (0.048245)**	0.383403 (0.013625)	0.402790 (0.041274)
Waveform	0.333417 (0.000060)	0.333492 (0.000079)	0.333496 (0.000081)	0.499603 (0.000167)	**0.499677 (0.000458)**
Wilt	**0.674969 (0.016087)**	0.673994 (0.027694)	0.671992 (0.019707)	0.673223 (0.000016)	0.673407 (0.000454)
Wine	0.371210 (0.023526)	0.382472 (0.024336)	0.400117 (0.033701)	**0.891318 (0.000791)**	0.890981 (0.001393)
Wine Quality Red	0.257889 (0.004015)	0.260334 (0.005541)	0.259913 (0.004084)	0.304261 (0.005363)	**0.308893 (0.011757)**
Wine Quality White	0.220291 (0.001223)	0.222075 (0.001434)	0.223045 (0.001181)	0.246447 (0.000338)	**0.248778 (0.005754)**
Zoo	0.255730 (0.024046)	0.539529 (0.132562)	0.625126 (0.123235)	**0.756989 (0.066599)**	0.721952 (0.099918)
Average Deviation	0.022252	0.037115	0.038291	**0.012002**	0.024597

At least one of the proposed algorithm obtained the best Folkes-Mallows index for 38 out of the 40 data sets. With emphasis on:

- the Breast Cancer Wdbc data set, in which the variants of KFCM-K-E_W

obtained a Folkes-Mallows index of 85% (compared to the 52 53% the other algorithms),

- the Landsat data set, in which the variants of KFCM-K-E_W obtained a Folkes-Mallows index of 60-61% (compared to the 25 26% of the other algorithms),
- the Pendigits data set, in which the variants of KFCM-K-E_W obtained a Folkes-Mallows index of 56-59% (compared to the 20 24% of the reference algorithms),
- the Urban data set, in which the variants of KFCM-K-E_W obtained a Folkes-Mallows index of 46-47% (compared to the 12% of the other algorithms),
- the Wine data set, in which the variants of KFCM-K-E_W obtained a Folkes-Mallows index of 89% (compared to the 37 40% of the other algorithms),
- the Zoo data set, in which the variants of KFCM-K-E_W obtained a an Folkes-Mallows index of 72-76% (compared to the 26-54% of the reference algorithms).

The proposed algorithms obtained a Folkes-Mallows index with an average deviation lower than 3.9%, demonstrating it robustness, specially for the KFCM-K-$E_{W.1}$, which obtained a Folkes-Mallows index more stable than the reference algorithms, being close to 1.2%.

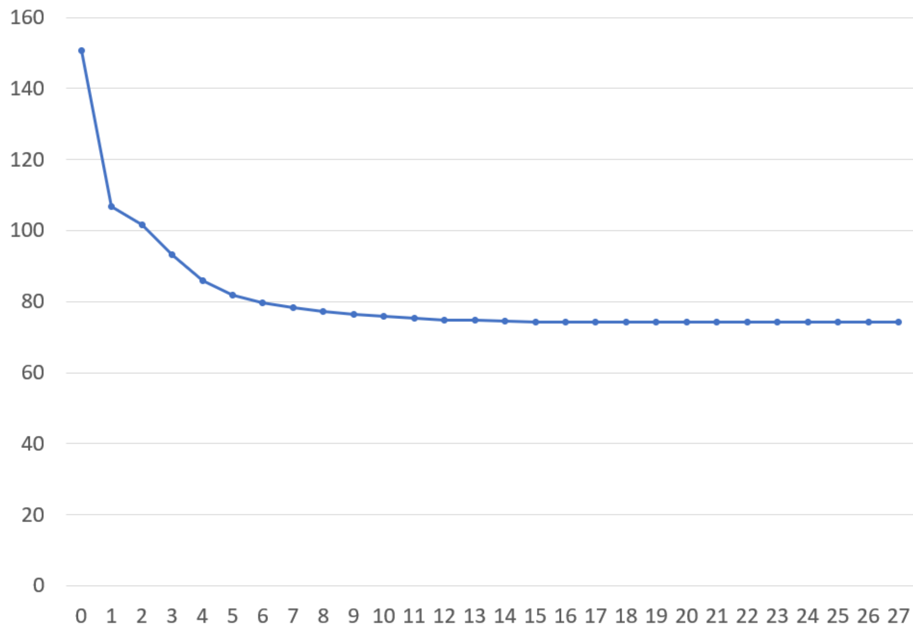
2.4.12. Main Findings

We found that the proposed algorithms obtained the best result for most of the data sets considering every metric. Being the execution time the one that obtained a more even distribution between proposed and reference. With an emphasis for the KFCM-K-E_W variants, which obtained most of the best results. In particular, the algorithms KFCM-K-$E_{W.1}$ and KFCM-K-$E_{W.2}$ were the best in the data sets with many clusters (Leaf, Letters, Pendigits, and Urban) according to the average of all considered indexes.

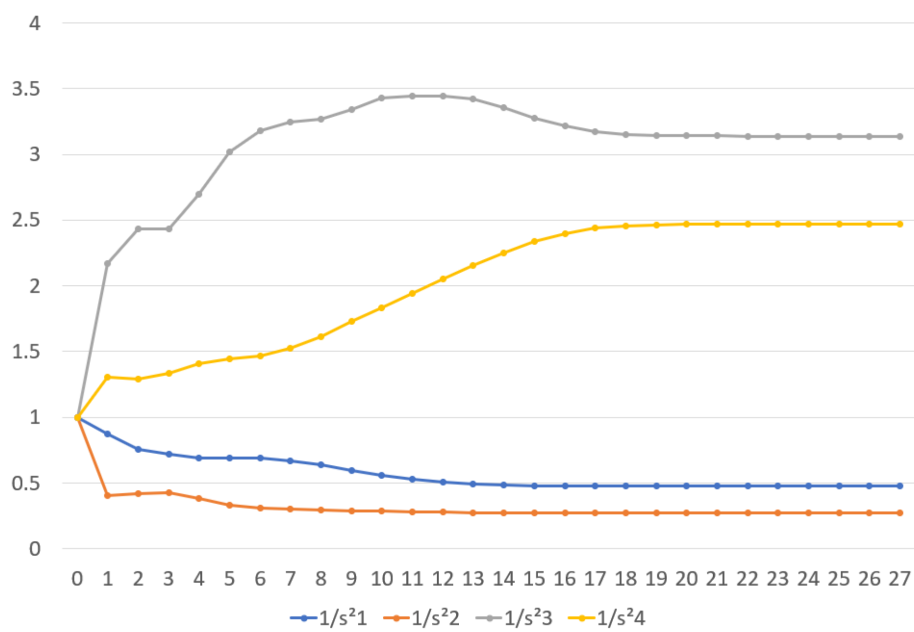
The proposed algorithms shown to be stable, with an average deviations bellow 9.6% for every metric, specially for the KFCM-K-E$_{W.1}$, which obtained an average deviations bellow 4.1% for every metric, demonstrating it's robustness.

2.5. Detailed Results for the Iris Data Set

The following graphs represents the evolution of the objective functions and width parameters versus iterations for the iris data set.

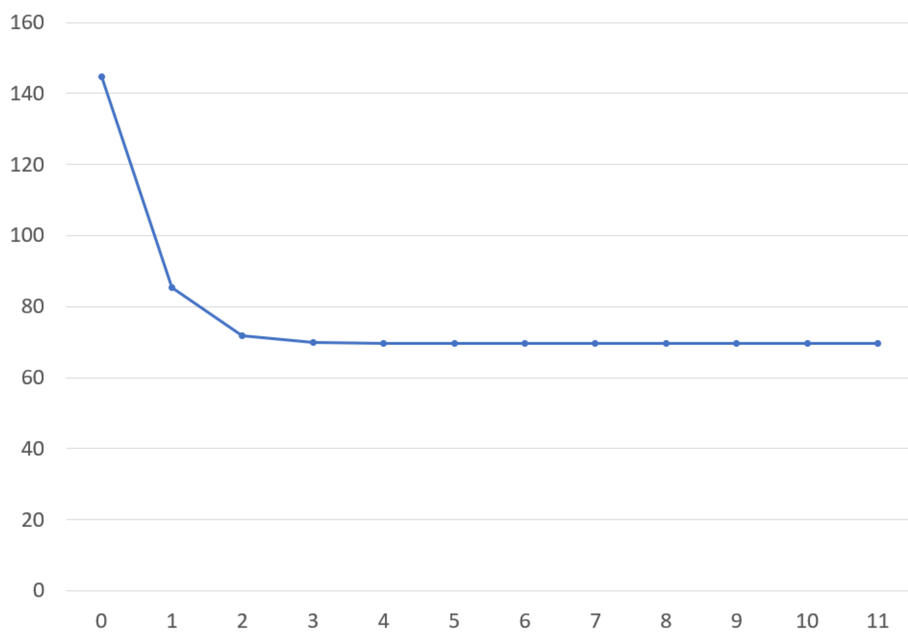


(a) KFCM-K-W.1 Objective Function

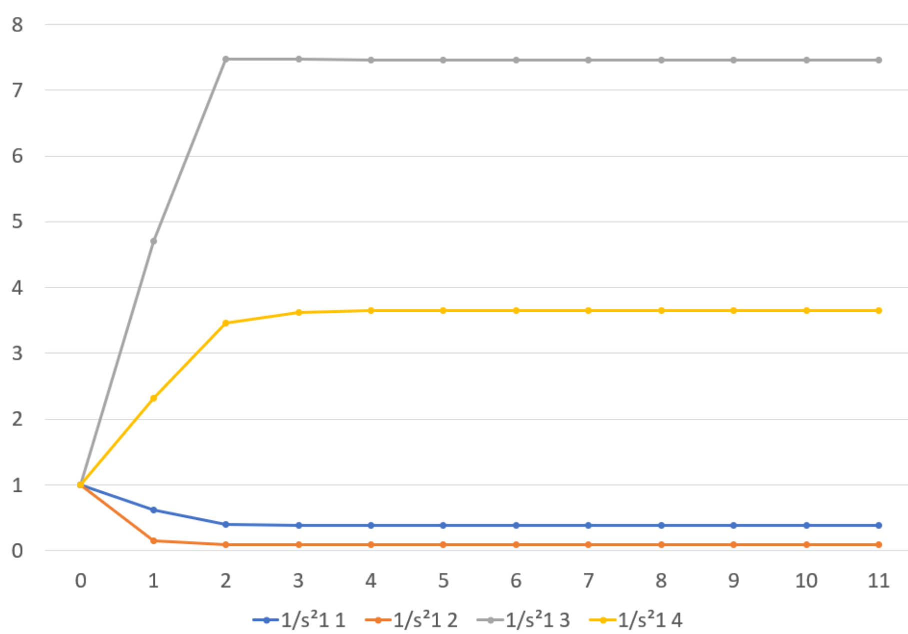


(b) KFCM-K-W.1 Weights

Figure 1: Iteration of the KFCM-K-W.1 algorithm for the iris data set

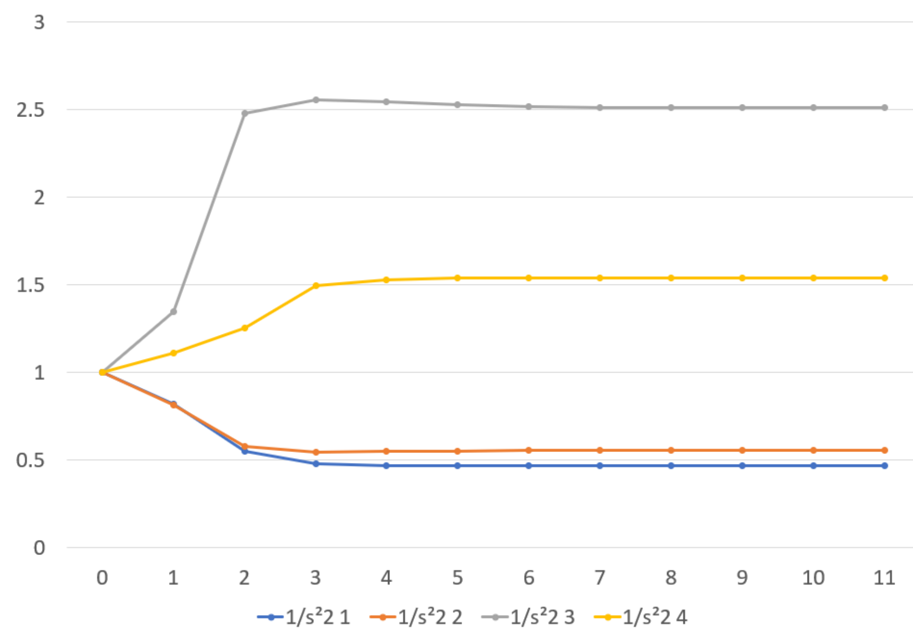


(a) KFCM-K-W.2 Objective Function

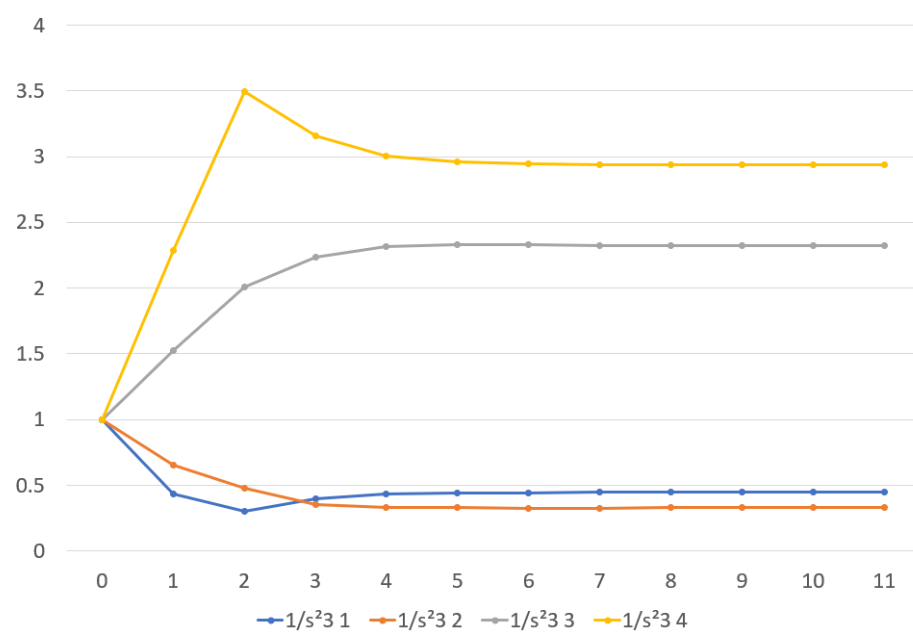


(b) KFCM-K-W.2 Weights (cluster 1)

Figure 2: Iteration of the KFCM-K-W.2 algorithm for the iris data set

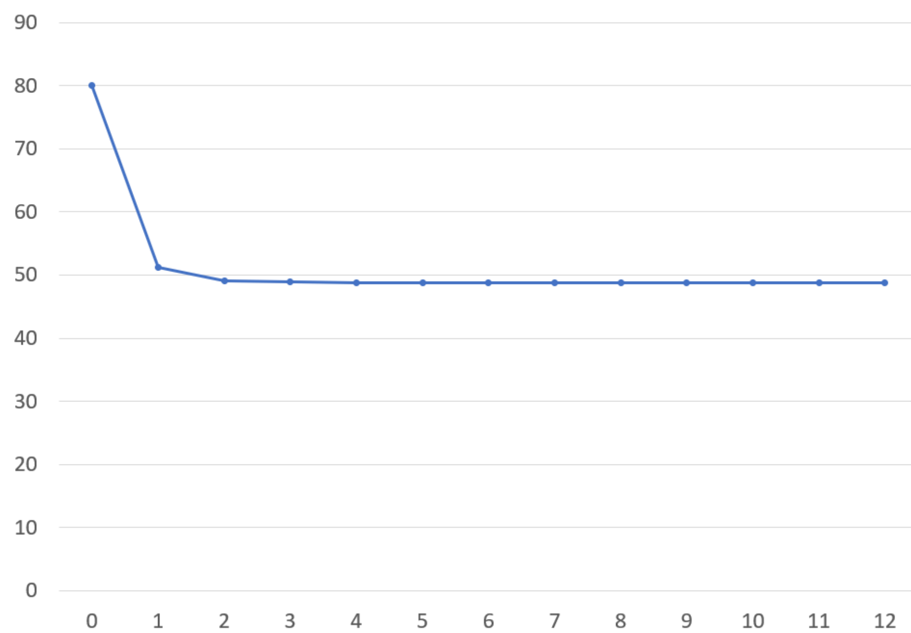


(a) KFCM-K-W.2 Weights (cluster 2)

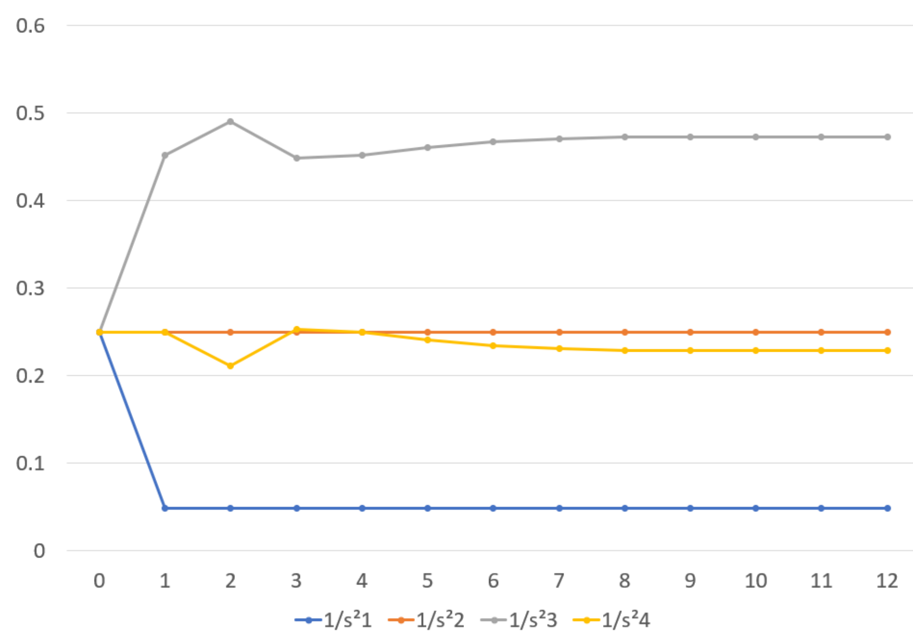


(b) KFCM-K-W.2 Weights (cluster 3)

Figure 3: Iteration of the KFCM-K-W.2 algorithm for the iris data set

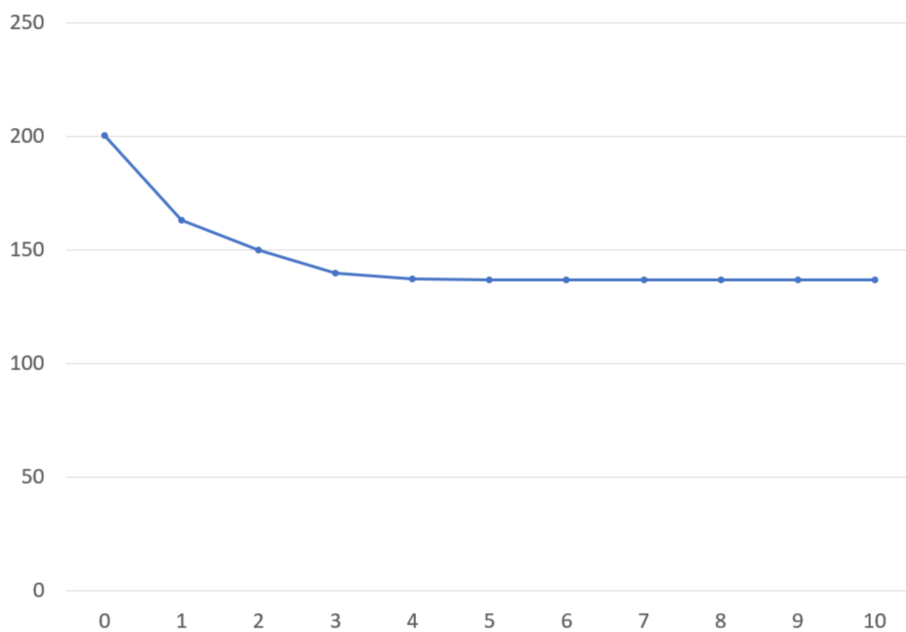


(a) KFCM-K-$E_{W.1}$ Objective Function

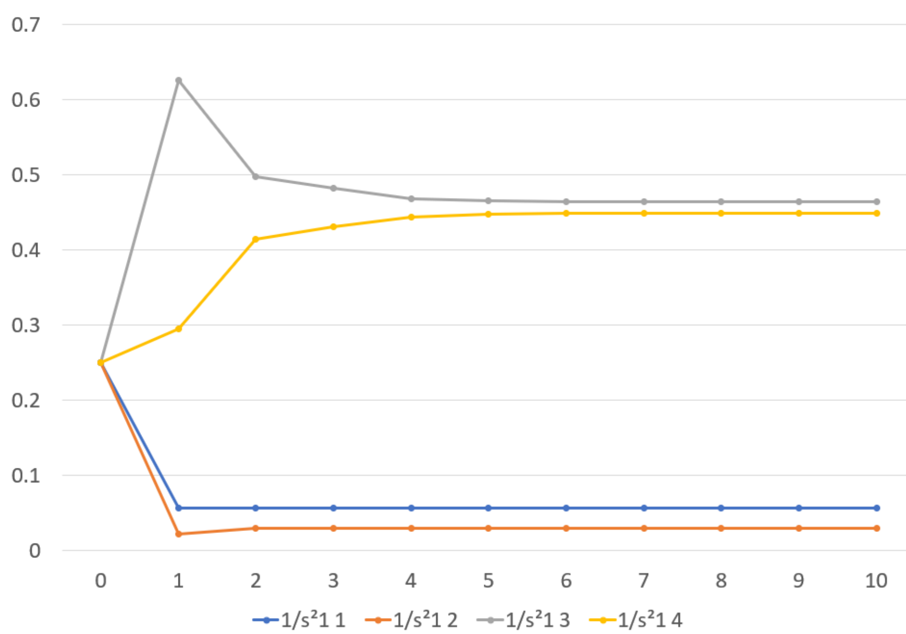


(b) KFCM-K-$E_{W.1}$ Weights

Figure 4: Iteration of the KFCM-K-$E_{W.1}$ algorithm for the iris data set

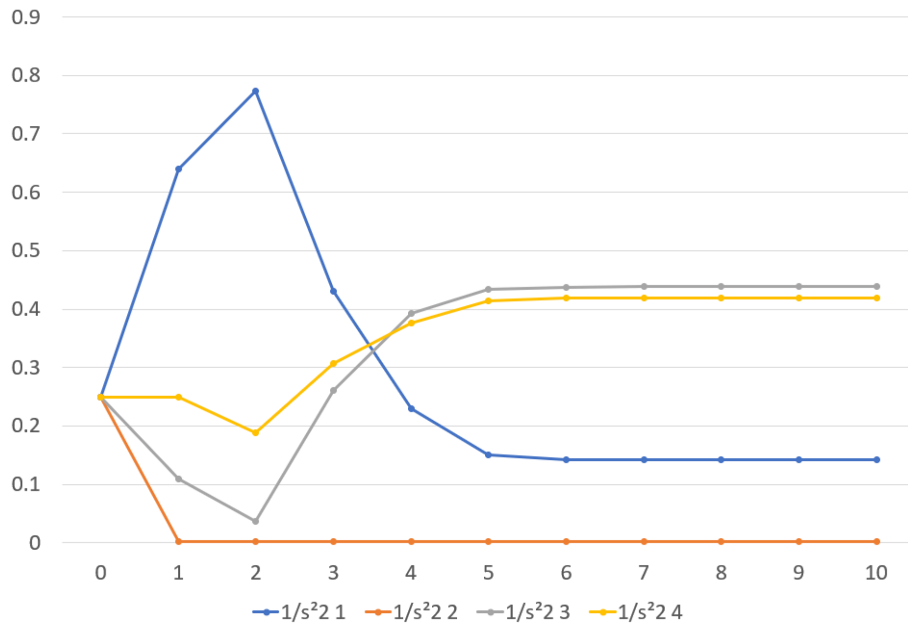


(a) KFCM-K-$E_{W.2}$ Objective Function

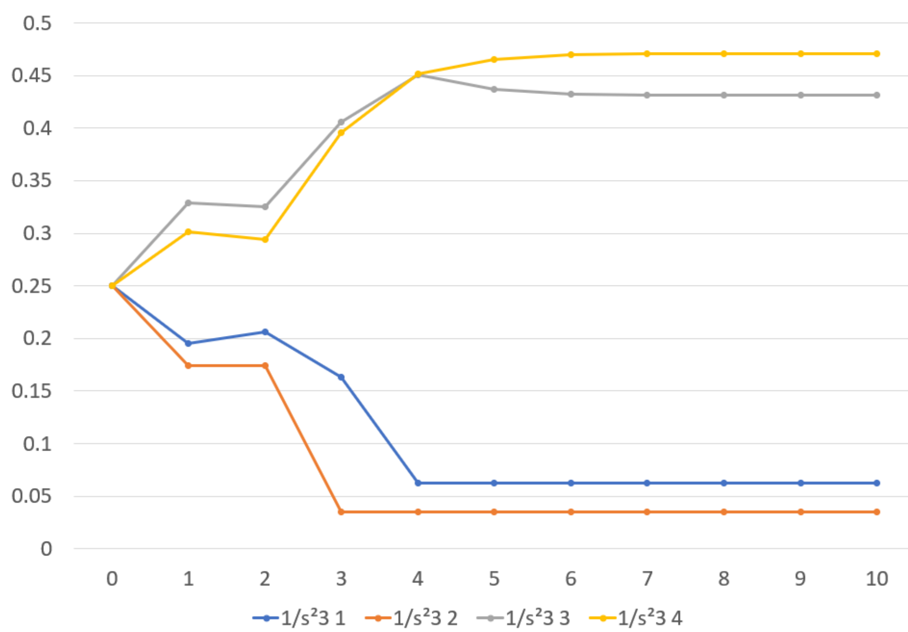


(b) KFCM-K-$E_{W.2}$ Weights (cluster 1)

Figure 5: Iteration of the KFCM-K-$E_{W.2}$ algorithm for the iris data set



(a) KFCM-K-E$_{W.2}$ Weights (cluster 2)



(b) KFCM-K-E$_{W.2}$ Weights (cluster 3)

Figure 6: Iteration of the KFCM-K-E$_{W.2}$ algorithm for the iris data set

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