Supplementary Table 1: Probabilities of different history classes for gene tree 1, conditional on species tree 1.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	((((WE)X)Y)Z)	{W,X,Y,Z}={A,B,C,D}	24	(W,E)	((WE),X)	(((WE)X),Y)	((((WE)X)Y),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
2	((((XD)E)Y)Z)	{X,Y,Z}={A,B,C}	6	(X,D)	((XD),E)	(((XD)E),Y)	((((XD)E)Y),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
3	((((XD)Y)E)Z)	{X,Y,Z}={A,B,C}	6	(X,D)	((XD),Y)	(((XD)Y),E)	((((XD)Y)E),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
4	((((XD)Y)Z)E)	{X,Y,Z}={A,B,C}	6	(X,D)	((XD),Y)	(((XD)Y),Z)	((((XD)Y)Z),E)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
5	((((XC)E)Y)Z)	{X,Y,Z}={A,B,D}, X≠D	4	(X,C)	((XC),E)	(((XC)E),Y)	((((XC)E)Y),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
6	((((XC)D)E)Y)	{X,Y}={A,B}	2	(X,C)	((XC),D)	(((XC)D),E)	((((XC)D)E),Y)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
7	((((XC)Y)E)D)	{X,Y}={A,B}	2	(X,C)	((XC),Y)	(((XC)Y),E)	((((XC)Y)E),D)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
8	((((XC)D)Y)E)	{X,Y}={A,B}	2	(X,C)	((XC),D)	(((XC)D),Y)	((((XC)D)Y),E)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$

Supplementary Table 2: Probabilities of different history classes for gene tree 1, conditional on species tree 1 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
9	$((((XC)Y)D)E)$	$\{X,Y\}=\{A,B\}$	2	(X,C)	$((XC),Y)$	$(((XC)Y),D)$	$((((XC)Y)D),E)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				3	3	2	1	$\frac{1}{3}g_{22}(T_4)g_{31}(T_3)g_{21}(T_2)$
10	$((((AB)E)X)Y)$	$\{X,Y\}=\{C,D\}$	2	(A,B)	$((AB),E)$	$(((AB)E),X)$	$((((AB)E)X),Y)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
11	$((((AB)D)E)C)$		1	(A,B)	$((AB),D)$	$(((AB)D),E)$	$((((AB)D)E),C)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
12	$((((AB)C)E)D)$		1	(A,B)	$((AB),C)$	$(((AB)C),E)$	$((((AB)C)E),D)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	3	1	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$
13	$((((AB)D)C)E)$		1	(A,B)	$((AB),D)$	$(((AB)D),C)$	$((((AB)D)C),E)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	2	2	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{31}(T_2)$

Supplementary Table 3: Probabilities of different history classes for gene tree 1, conditional on species tree 1 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
14	$((((AB)C)D)E)$		1	(A,B)	$((AB),C)$	$(((AB)C),D)$	$((((AB)C)D),E)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				3	3	2	1	$\frac{1}{3}g_{22}(T_4)g_{31}(T_3)g_{21}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	2	2	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{31}(T_2)$
				4	3	1	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$
				4	3	2	1	$g_{21}(T_4)g_{21}(T_3)g_{21}(T_2)$

Supplementary Table 4: Probabilities of different history classes for gene tree 2, conditional on species tree 1.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((XE)Y)(ZD))	{X,Y,Z}={A,B,C}	6	(X,E)	((XE),Y)	(Z,D)	(((XE)Y),(ZD))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
2	(((XE)Y)(ZC))	{X,Y,Z}={A,B,D}, Z≠D	4	(X,E)	((XE),Y)	(Z,C)	(((XE)Y),(ZC))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
3	(((XE)Y)(AB))	{X,Y}={C,D}	2	(X,E)	((XE),Y)	(A,B)	(((XE)Y),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
4	(((XD)E)(YC))	{X,Y}={A,B}	2	(X,D)	((XD),E)	(Y,C)	(((XD)E),(YC))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	1	3	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
5	(((CD)E)(AB))		1	(C,D)	((CD),E)	(A,B)	(((CD)E),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	1	3	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				2	1	4	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
6	(((XC)E)(YD))	{X,Y}={A,B}	2	(X,C)	((XC),E)	(Y,D)	(((XC)E),(YD))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
7	(((AB)E)(CD))		1	(A,B)	((AB),E)	(C,D)	(((AB)E),(CD))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	1	2	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$

Supplementary Table 5: Probabilities of different history classes for gene tree 2, conditional on species tree 1 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
8	(((XD)Y)(ZE))	{X,Y,Z}={A,B,C}	6	(X,D)	((XD),Y)	(Z,E)	(((XD)Y),(ZE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
9	(((XC)D)(YE))	{X,Y}={A,B}	2	(X,C)	((XC),D)	(Y,E)	(((XC)D),(YE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
10	(((AB)D)(CE))		1	(A,B)	((AB),D)	(C,E)	(((AB)D),(CE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
11	(((XC)Y)(DE))	{X,Y}={A,B}	2	(X,C)	((XC),Y)	(D,E)	(((XC)Y),(DE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
12	(((AB)C)(DE))		1	(A,B)	((AB),C)	(D,E)	(((AB)C),(DE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	3	1	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$

Supplementary Table 6: Probabilities of different history classes for gene tree 3, conditional on species tree 1.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((XD)(YE))Z)	{X,Y,Z}={A,B,C}	6	(X,D)	(Y,E)	((XD),(YE))	(((XD)(YE)),Z)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
2	(((XC)(YE))Z)	{X,Y,Z}={A,B,C}, X≠D	4	(X,C)	(Y,E)	((XC),(YE))	(((XC)(YE)),Z)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
3	(((AB)(XE))Y)	{X,Y}={C,D}	2	(A,B)	(X,E)	((AB),(XE))	(((AB)(XE)),Y)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
4	(((XC)(YD))E)	{X,Y}={A,B}	2	(X,C)	(Y,D)	((XC),(YD))	(((XC)(YD)),E)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	2	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$
5	(((AB)(CD))E)		1	(A,B)	(C,D)	((AB),(CD))	(((AB)(CD)),E)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{44}(T_2)$
				1	2	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{33}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{33}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{33}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{31}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	2	2	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{31}(T_2)$

Supplementary Table 7: Probabilities of different history classes for gene tree 1, conditional on species tree 2.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	$((((VW)X)Y)Z)$	$V \in \{A,B,C\}$, $W \in \{D,E\}$ $\{X,Y,Z\}=\{A,B,C,D,E\} \setminus \{V,W\}$	36	(V,W)	$((VW),X)$	$(((VW)X),Y)$	$((((VW)X)Y),Z)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
2	$((((DE)X)Y)Z)$	$\{X,Y,Z\}=\{A,B,C\}$	6	(D,E)	$((DE),X)$	$(((DE)X),Y)$	$((((DE)X)Y),Z)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				2	1	1	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
3	$((((WC)X)Y)Z)$	$W \in \{A,B\}$, $X \in \{D,E\}$ $\{Y,Z\}=\{A,B,D,E\} \setminus \{W,X\}$	8	(W,C)	$((WC),X)$	$(((WC)X),Y)$	$((((WC)X)Y),Z)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
4	$((((WC)X)Y)Z)$	$\{W,X\}=\{A,B\}$, $\{Y,Z\}=\{D,E\}$	4	(W,C)	$((WC),X)$	$(((WC)X),Y)$	$((((WC)X)Y),Z)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
5	$((((AB)X)Y)Z)$	$\{X,Y,Z\}=\{C,D,E\}$, $X \neq C$	4	(A,B)	$((AB),X)$	$(((AB)X),Y)$	$((((AB)X)Y),Z)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
6	$((((AB)C)X)Y)$	$\{X,Y\}=\{D,E\}$	2	(A,B)	$((AB),C)$	$(((AB)C),X)$	$((((AB)C)X),Y)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
				4	3	1	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$

Supplementary Table 8: Probabilities of different history classes for gene tree 2, conditional on species tree 2.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((VW)X)(YZ))	{V,X,Y}={A,B,C}, {W,Z}={D,E}	12	(V,W)	((VW),X)	(Y,Z)	(((VW)X),(YZ))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
2	(((WX)Y)(ZC))	{W,Z}={A,B}, {X,Y}={D,E}	4	(W,X)	((WX),Y)	(Z,C)	(((WX)Y),(ZC))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
3	(((CX)Y)(AB))	{X,Y}={D,E}	2	(C,X)	((CX),Y)	(A,B)	(((CX)Y),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
4	(((DE)X)(YC))	{X,Y}={A,B}	2	(D,E)	((DE),X)	(Y,C)	(((DE)X),(YC))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				2	1	1	1	$\frac{1}{9}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
				2	1	3	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{21}(T_2)$
5	(((DE)C)(AB))		1	(D,E)	((DE),C)	(A,B)	(((DE)C),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	1	3	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
				2	1	1	1	$\frac{1}{9}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
				2	1	3	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{21}(T_2)$
				2	1	4	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{21}(T_2)$
6	(((WC)X)(YZ))	{W,Y}={A,B}, {X,Z}={D,E}	4	(W,C)	((WC),X)	(Y,Z)	(((WC)X),(YZ))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
7	(((AB)X)(CY))	{X,Y}={D,E}	2	(A,B)	((AB),X)	(C,Y)	(((AB)X),(CY))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
8	(((XC)Y)(DE))	{X,Y}={A,B}	2	(X,C)	((XC),Y)	(D,E)	(((XC)Y),(DE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	1	2	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
				3	1	1	1	$\frac{1}{27}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				3	1	2	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{21}(T_2)$
				3	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{31}(T_3)g_{22}(T_2)$
				3	3	2	1	$\frac{1}{3}g_{22}(T_4)g_{31}(T_3)g_{21}(T_2)$

Supplementary Table 9: Probabilities of different history classes for gene tree 2, conditional on species tree 2 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
				(A,B)	((AB),C)	(D,E)	(((AB)C),(DE))	
9	(((AB)C)(DE))		1					
				1	1	1	1	$\frac{1}{60} g_{22}(T_4) g_{33}(T_3) g_{22}(T_2)$
				1	1	2	1	$\frac{1}{18} g_{22}(T_4) g_{33}(T_3) g_{21}(T_2)$
				3	1	1	1	$\frac{1}{27} g_{22}(T_4) g_{32}(T_3) g_{22}(T_2)$
				3	1	2	1	$\frac{1}{9} g_{22}(T_4) g_{32}(T_3) g_{21}(T_2)$
				3	3	1	1	$\frac{1}{9} g_{22}(T_4) g_{31}(T_3) g_{22}(T_2)$
				3	3	2	1	$\frac{1}{3} g_{22}(T_4) g_{31}(T_3) g_{21}(T_2)$
				4	1	1	1	$\frac{1}{9} g_{21}(T_4) g_{22}(T_3) g_{22}(T_2)$
				4	1	2	1	$\frac{1}{3} g_{21}(T_4) g_{22}(T_3) g_{21}(T_2)$
				4	3	1	1	$\frac{1}{3} g_{21}(T_4) g_{21}(T_3) g_{22}(T_2)$
				4	3	2	1	$g_{21}(T_4) g_{21}(T_3) g_{21}(T_2)$

Supplementary Table 10: Probabilities of different history classes for gene tree 3, conditional on species tree 2.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((XD)(YE))Z)	{X,Y,Z}={A,B,C}	6	(X,D)	(Y,E)	((XD),(YE))	(((XD)(YE)),Z)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
2	(((VW)(XY))Z)	{V,W,X}={A,B,C}, {V,W} ≠ {A,B} {Y,Z}={D,E}	4	(V,W)	(X,Y)	((VW),(XY))	(((VW)(XY)),Z)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
3	(((AB)(CX))Y)	{X,Y}={D,E}	2	(A,B)	(C,X)	((AB),(CX))	(((AB)(CX)),Y)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
4	(((XC)(DE))Y)	{X,Y}={A,B}	2	(X,C)	(D,E)	((XC),(DE))	(((XC)(DE)),Y)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	2	1	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				3	2	1	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{21}(T_2)$
5	(((AB)(DE))C)		1	(A,B)	(D,E)	((AB),(DE))	(((AB)(DE)),C)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{33}(T_3)g_{22}(T_2)$
				1	2	1	1	$\frac{1}{18}g_{22}(T_4)g_{33}(T_3)g_{21}(T_2)$
				3	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{32}(T_3)g_{22}(T_2)$
				3	2	1	1	$\frac{1}{9}g_{22}(T_4)g_{32}(T_3)g_{21}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{22}(T_2)$
				4	2	1	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{21}(T_2)$

Supplementary Table 11: Probabilities of different history classes for gene tree 1, conditional on species tree 3.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	((((WE)X)Y)Z)	{W,X,Y,Z}={A,B,C,D}	24	(W,E)	((WE),X)	(((WE)X),Y)	((((WE)X)Y),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
2	((((WX)E)Y)Z)	W $\in$ {A,B}, X $\in$ {C,D}, {Y,Z}={A,B,C,D} \ {W,X}	8	(W,X)	((WX),E)	(((WX)E),Y)	((((WX)E)Y),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
3	((((WX)Y)E)Z)	W $\in$ {A,B}, X $\in$ {C,D}, {Y,Z}={A,B,C,D} \ {W,X}	8	(W,X)	((WX),Y)	(((WX)Y),E)	((((WX)Y)E),Z)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
4	((((WX)Y)Z)E)	W $\in$ {A,B}, X $\in$ {C,D}, {Y,Z}={A,B,C,D} \ {W,X}	8	(W,X)	((WX),Y)	(((WX)Y),Z)	((((WX)Y)Z),E)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{22}(T_3)g_{41}(T_2)$
5	((((CD)E)X)Y)	{X,Y}={A,B}	2	(C,D)	((CD),E)	(((CD)E),X)	((((CD)E)X),Y)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
6	((((CD)X)E)Y)	{X,Y}={A,B}	2	(C,D)	((CD),X)	(((CD)X),E)	((((CD)X)E),Y)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$
7	((((CD)X)Y)E)	{X,Y}={A,B}	2	(C,D)	((CD),X)	(((CD)X),Y)	((((CD)X)Y),E)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{22}(T_3)g_{41}(T_2)$
				3	1	1	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$
				3	2	2	1	$\frac{1}{3}g_{22}(T_4)g_{21}(T_3)g_{31}(T_2)$
8	((((AB)E)X)Y)	{X,Y}={C,D}	2	(A,B)	((AB),E)	(((AB)E),X)	((((AB)E)X),Y)	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$

Supplementary Table 12: Probabilities of different history classes for gene tree 1, conditional on species tree 3 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
9	$((((AB)X)E)Y)$	$\{X,Y\}=\{C,D\}$	2	(A,B)	$((AB),X)$	$(((AB)X),E)$	$((((AB)X)E),Y)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
10	$((((AB)X)Y)E)$	$\{X,Y\}=\{C,D\}$	2	(A,B)	$((AB),X)$	$(((AB)X),Y)$	$((((AB)X)Y),E)$	
				1	1	1	1	$\frac{1}{180}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{18}g_{22}(T_4)g_{22}(T_3)g_{41}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	2	2	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{31}(T_2)$

Supplementary Table 13: Probabilities of different history classes for gene tree 2, conditional on species tree 3.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((WE)X)(YZ))	Y ∈ {A,B}, Z ∈ {C,D} {W,X}={A,B,C,D} \ {Y,Z}	8	(W,E)	((WE),X)	(Y,Z)	(((WE)X),(YZ))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
2	(((XE)Y)(CD))	{X,Y}={A,B}	2	(X,E)	((XE),Y)	(C,D)	(((XE)Y),(CD))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
3	(((XE)Y)(AB))	{X,Y}={C,D}	2	(X,E)	((XE),Y)	(A,B)	(((XE)Y),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
4	(((WX)E)(YZ))	W ∈ {A,B}, X ∈ {C,D}, {Y,Z}={A,B,C,D} \ {W,X}	4	(W,X)	((WX),E)	(Y,Z)	(((WX)E),(YZ))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
5	(((CD)E)(AB))		1	(C,D)	((CD),E)	(A,B)	(((CD)E),(AB))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				1	1	4	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	1	4	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				3	1	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				3	1	2	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$
				3	1	4	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$
6	(((AB)E)(CD))		1	(A,B)	((AB),E)	(C,D)	(((AB)E),(CD))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	1	2	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				1	1	3	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	1	2	1	$\frac{1}{27}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	1	3	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	1	2	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	1	3	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$
7	(((WX)Y)(ZE))	W ∈ {A,B}, X ∈ {C,D} {Y,Z}={A,B,C,D} \ {W,X}	8	(W,X)	((WX),Y)	(Z,E)	(((WX)Y),(ZE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$

Supplementary Table 14: Probabilities of different history classes for gene tree 2, conditional on species tree 3 (continued).

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
8	(((AB)X)(YE))	{X,Y}={C,D}	2	(A,B)	((AB),X)	(Y,E)	(((AB)X),(YE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				4	1	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
9	(((CD)X)(YE))	{X,Y}={A,B}	2	(C,D)	((CD),X)	(Y,E)	(((CD)X),(YE))	
				1	1	1	1	$\frac{1}{60}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{54}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				3	1	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				3	2	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$

Supplementary Table 15: Probabilities of different history classes for gene tree 3, conditional on species tree 3.

Number for history class	Description of class of labeled topologies	Definitions of V, W, X, Y, Z	Number of labeled topologies in class	Description of coalescences and the branches on which they occur				Probability of coalescent history
1	(((WX)(YE))Z)	W ∈ {A,B}, X ∈ {C,D} {Y,Z}={A,B,C,D} \ {W,X}	8	(W,X)	(Y,E)	((WX),(YE))	(((WX)(YE)),Z)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
2	(((CD)(XE))Y)	{X,Y}={A,B}	2	(C,D)	(X,E)	((CD),(XE))	(((CD)(XE)),Y)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				3	1	1	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
3	(((AB)(XE))Y)	{X,Y}={C,D}	2	(A,B)	(X,E)	((AB),(XE))	(((AB)(XE)),Y)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
4	(((XC)(YD))E)	{X,Y}={A,B}	2	(X,C)	(Y,D)	((XC),(YD))	(((XC)(YD)),E)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	2	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{22}(T_3)g_{41}(T_2)$
5	(((AB)(CD))E)		1	(A,B)	(C,D)	((AB),(CD))	(((AB)(CD)),E)	
				1	1	1	1	$\frac{1}{90}g_{22}(T_4)g_{22}(T_3)g_{44}(T_2)$
				1	2	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				1	3	1	1	$\frac{1}{18}g_{22}(T_4)g_{21}(T_3)g_{33}(T_2)$
				2	1	1	1	$\frac{1}{108}g_{22}(T_4)g_{22}(T_3)g_{43}(T_2)$
				2	2	1	1	$\frac{1}{27}g_{22}(T_4)g_{22}(T_3)g_{42}(T_2)$
				2	2	2	1	$\frac{1}{9}g_{22}(T_4)g_{22}(T_3)g_{41}(T_2)$
				2	3	1	1	$\frac{1}{9}g_{22}(T_4)g_{21}(T_3)g_{32}(T_2)$
				2	3	2	1	$\frac{1}{3}g_{22}(T_4)g_{21}(T_3)g_{31}(T_2)$
				4	1	1	1	$\frac{1}{18}g_{21}(T_4)g_{22}(T_3)g_{33}(T_2)$
				4	2	1	1	$\frac{1}{9}g_{21}(T_4)g_{22}(T_3)g_{32}(T_2)$
				4	2	2	1	$\frac{1}{3}g_{21}(T_4)g_{22}(T_3)g_{31}(T_2)$
				4	3	1	1	$\frac{1}{3}g_{21}(T_4)g_{21}(T_3)g_{22}(T_2)$
				4	3	2	1	$g_{21}(T_4)g_{21}(T_3)g_{21}(T_2)$

Supplementary Table 16: Equations that characterize the anomaly zone for gene tree 1 and species tree 1.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_1	γ_1	1	No	$-\frac{1}{540x^6y^3z}(-4 - 5x^3 - 6x^5 + 30x^6 - 15x^3y^2 - 90x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z$ $-360x^5y^3z + 540x^6y^3z) > 0$
		2	No	$-\frac{1}{180x^6y^3z}(2 - 5x^3 - 2x^5 + 10x^6 - 5x^3y^2 - 30x^5y^2 + 60x^6y^2 + 80x^5y^3$ $-120x^6y^3 + 10x^3y^2z + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z$ $+180x^6y^3z) > 0$
		3	No	$-\frac{1}{180x^6y^3z}(-2 + 5x^3 - 8x^5 + 10x^6 - 5x^3y^2 - 30x^5y^2 + 60x^6y^2$ $+80x^5y^3 - 120x^6y^3 + 10x^3y^2z + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z$ $+180x^6y^3z) > 0$
		4	No	$-\frac{1}{36x^3y^3z}(-1 + 2x^2 - y - 6x^2y^2 + 12x^3y^2 + 16x^2y^3 - 24x^3y^3$ $+2y^2z + 12x^2y^2z - 24x^3y^2z - 24x^2y^3z + 36x^3y^3z) > 0$
		5	No	$-\frac{1}{90x^6y^3z}(1 - x^5 + 5x^6 - 5x^3y^2 - 15x^5y^2 + 30x^6y^2 + 40x^5y^3$ $-60x^6y^3 + 5x^3y^2z + 30x^5y^2z - 60x^6y^2z - 60x^5y^3z$ $+90x^6y^3z) > 0$
		6	No	$-\frac{1}{180x^6y^3z}(-2 - 5x^3 + 7x^5 + 10x^6 + 5x^3y^2 - 45x^5y^2 + 60x^6y^2$ $+80x^5y^3 - 120x^6y^3 + 10x^3y^2z + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z$ $+180x^6y^3z) > 0$
		7	No	$-\frac{1}{180x^6y^3z}(-2 - 5x^3 - 3x^5 + 10x^6 + 5x^3y^2 - 15x^5y^2 60x^6y^2$ $+60x^5y^3 - 120x^6y^3 + 10x^3y^2z + 60x^5y^2z + 60x^5y^2z - 120x^6y^2z$ $-120x^5y^3z + 180x^6y^3z) > 0$
		8	No	$-\frac{1}{18x^3y^3z}(-2x^2 + 3x^3 - y^2 + 3x^3y^2 + 8x^2y^3 - 12x^3y^3 + y^2z$ $+6x^2y^2z - 12x^3y^2z - 12x^2y^3z + 18x^3y^3z) > 0$
		9	No	$-\frac{1}{18x^3yz}(1 + 6x^2 - 12x^3 - 12x^2y + 18x^3y)(-1 + z) > 0$
		10	No	$-\frac{1}{90x^6y^3z}(1 - x^5 + 5x^6 - 15x^5y^2 + 30x^6y^2 + 40x^5y^3 - 60x^6y^3$ $+30x^5y^2z - 60x^6y^2z - 60x^5y^3z + 90x^6y^3z) > 0$
		11	No	$-\frac{1}{180x^6y^3z}(-2 - 5x^3 + 7x^5 + 10x^6 - 15x^3y^2 - 15x^5y^2 + 60x^6y^2$ $+80x^5y^3 - 120x^6y^3 + 30x^3y^2z + 30x^5y^2z - 120x^6y^2z - 120x^5y^3z$ $+180x^6y^3z) > 0$
		12	No	$-\frac{1}{180x^6y^3z}(-1 + x)(2 + 2x + 2x^2 + 7x^3 + 7x^4 + 10x^5 + 15x^3y^2$ $+15x^4y^2 + 60x^5y^2 - 120x^5y^3 - 30x^3y^2z - 30x^4y^2z - 120x^5y^2z$ $+180x^5y^3z) > 0$
		13	No	$-\frac{1}{180xy^3z}(-2 + 3x)(-1 + y)(-1 - y - 4y^2 + 6y^2z) > 0$
		14	-	This history class matches the species tree topology

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 17: Equations that characterize the anomaly zone for gene tree 2 and species tree 1.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_1	γ_2	1	Yes	$-\frac{1}{180x^3y^3z}(-5 - 2x^2 + 10x^3 - 5y^2 - 30x^2y^2 + 60x^3y^2 + 80x^2y^3$ $-120x^3y^3 + 10y^2z + 60x^2y^2z - 120x^3y^2z - 120x^2y^3z$ $+180x^3y^3z) > 0$
		2	Yes	$-\frac{1}{90x^3y^3z}(-x^2 + 5x^3 - 5y^2 - 15x^2y^2 + 30x^3y^2 + 40x^2y^3$ $-60x^3y^3 + 5y^2z + 30x^2y^2z - 60x^3y^2z - 60x^2y^3z + 90x^3y^3z) > 0$
		3	Yes	$-\frac{1}{90xy^3z}(-1 + 5x - 15y^2 + 40y^3 + 30xy^2 - 60xy^3 + 30x^2y$ $-60xy^2z - 60y^3z + 90xy^3z) > 0$
		4	Yes	$-\frac{1}{540x^6y^3z}(-4 - 5x^3 + 3x^5 + 30x^6 + 15x^3y^2 - 135x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z$ $-360x^5y^3z + 540x^6y^3z) > 0$
		5	Yes	$-\frac{1}{540x^6y^3z}(-4 - 5x^3 + 3x^5 + 30x^6 - 45x^3y^2 - 45x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 90x^3y^2z + 90x^5y^2z - 360x^6y^2z$ $-360x^5y^3z + 540x^6y^3z) > 0$
		6	Yes	$-\frac{1}{540x^6y^3z}(-4 + 10x^3 + 3x^5 + 30x^6 - 135x^5y^2 + 180x^6y^2 + 240x^5y^3$ $-360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z - 360x^5y^3z$ $+540x^6y^3z) > 0$
		7	Yes	$-\frac{1}{540x^6y^3z}(-4 + 10x^3 + 3x^5 + 30x^6 - 30x^3y^2 - 45x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 60x^3y^2z + 90x^5y^2z - 360x^6y^2z$ $-360x^5y^3z + 540x^6y^3z) > 0$
		8	Yes	$-\frac{1}{540x^6y^3z}(-2 + 5x^3 - 24x^5 + 30x^6 - 15x^3y^2 - 90x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z$ $-360x^5y^3z + 540x^6y^3z) > 0$
		9	Yes	$-\frac{1}{540x^6y^3z}(-2 - 10x^3 + 21x^5 + 30x^6 - 135x^5y^2 + 180x^6y^2 + 240x^5y^3$ $-360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z - 360x^5y^3z$ $+540x^6y^3z) > 0$
		10	Yes	$-\frac{1}{540x^6y^3z}(-2 - 10x^3 + 21x^5 + 30x^6 - 30x^3y^2 - 45x^5y^2 + 180x^6y^2$ $+240x^5y^3 - 360x^6y^3 + 60x^3y^2z + 90x^5y^2z - 360x^6y^2z - 360x^5y^3z$ $+540x^6y^3z) > 0$
		11	Yes	$-\frac{1}{540x^6y^3z}(-2 - 10x^3 - 9x^5 + 30x^6 - 45x^5y^2 + 180x^6y^2 + 180x^5y^3$ $-360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z - 360x^5y^3z$ $+540x^6y^3z) > 0$
		12	Yes	$-\frac{1}{540x^6y^3z}(-2 - 10x^3 - 9x^5 + 30x^6 - 30x^3y^2 - 135x^5y^2 + 180x^6y^2$ $+360x^5y^3 - 360x^6y^3 + 60x^3y^2z + 270x^5y^2z - 360x^6y^2z - 540x^5y^3z$ $+540x^6y^3z) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 18: Equations that characterize the anomaly zone for gene tree 3 and species tree 1.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_1	γ_3	1	Yes	$-\frac{1}{180x^6y^3z}(1 - 5x^3 - 2x^5 + 10x^6 - 5x^3y^2 - 30x^5y^2 + 60x^6y^2 + 80x^5y^3 - 120x^6y^3 + 10x^3y^2z + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z + 180x^6y^3z) > 0$
		2	Yes	$-\frac{1}{180x^6y^3z}(1 - 2x^5 + 10x^6 - 10x^3y^2 - 30x^5y^2 + 60x^6y^2 + 80x^5y^3 - 120x^6y^3 + 10x^3y^2z + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z + 180x^6y^3z) > 0$
		3	Yes	$-\frac{1}{180x^6y^3z}(1 - 2x^5 + 10x^6 - 30x^5y^2 + 60x^6y^2 + 80x^5y^3 - 120x^6y^3 + 60x^5y^2z - 120x^6y^2z - 120x^5y^3z + 180x^6y^3z) > 0$
		4	Yes	$-\frac{1}{540x^6y^3z}(1 - 10x^3 - 24x^5 + 60x^6 - 30x^3y^2 + 90x^6y^2 + 240x^5y^3 - 360x^6y^3 + 30x^3y^2z + 180x^5y^2z - 360x^6y^2z - 360x^5y^3z + 540x^6y^3z) > 0$
		5	Yes	$-\frac{1}{540x^6y^3z}(1 - 10x^3 - 24x^5 + 60x^6 - 180x^5y^2 + 270x^6y^2 + 240x^5y^3 - 360x^6y^3 + 360x^5y^2z - 540x^6y^2z - 360x^5y^3z + 540x^6y^3z) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 19: Equations that characterize the anomaly zone for gene tree 1 and species tree 2.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_2	γ_1	1	No	$-\frac{1}{90xy^3z}(-4 + 5x - 20y^2 + 40x^3 + 30xy^2 - 60xy^3 + 40y^2z$ $-60xy^2z - 60y^2z + 90xy^3z) > 0$
		2	No	$-\frac{1}{90xy^3z}(1 - 20y^2 + 40y^3 + 30xy^2 - 60xy^3 + 40y^2z - 60xy^2z$ $-60y^3z + 90xy^3z) > 0$
		3	No	$-\frac{1}{180xy^3z}(-3 + 10x - 45y^2 + 80y^3 + 60xy^2 - 120xy^3 + 80y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		4	No	$-\frac{1}{180xy^3z}(-13 + 10x - 15y^2 + 60y^3 + 60xy^2 - 120xy^3 + 80y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		5	No	$-\frac{1}{180xy^3z}(-3 + 10x - 35y^2 + 80y^3 + 60xy^2 - 120xy^3 + 70y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		6	No	$-\frac{1}{180xy^3z}(-13 + 10x - 65y^2 + 120y^3 + 60xy^2 - 120xy^3 + 130y^2z$ $-120xy^2z - 180y^3z + 180xy^3z) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 20: Equations that characterize the anomaly zone for gene tree 2 and species tree 2.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_2	γ_2	1	No	$-\frac{1}{18xy^3z}(-1+x-4y^2+8y^3+6xy^2-12xy^3+8y^2z-12xy^2z-12y^3z+18xy^3z) > 0$
		2	No	$-\frac{1}{36xy^3z}(-1+2x-9y^2+16y^3+12xy^2-24xy^3+16y^2z-24xy^2z-24y^3z+36xy^3z) > 0$
		3	No	$-\frac{1}{36xy^3z}(-1+2x-7y^2+16y^3+12xy^2-24xy^3+14y^2z-24xy^2z-24y^3z+36xy^3z) > 0$
		4	Yes	$-\frac{1}{36xy^3z}(-3+4x-3y^2+16y^3+6xy^2-24xy^3+16y^2z-24xy^2z-24y^3z+36xy^3z) > 0$
		5	Yes	$-\frac{1}{36xy^3z}(-3+4x-13y^2+16y^3+18xy^2-24xy^3+26y^2z-36xy^2z-24y^3z+36xy^3z) > 0$
		6	No	$-\frac{1}{18xy^3z}(x-5y^2+8y^3+6xy^2-12xy^3+8y^2z-12xy^2z-12y^2z+18xy^3z) > 0$
		7	No	$-\frac{1}{18xy^3z}(x-3y^2+8y^3+6xy^2-12xy^3+6y^2z-12xy^2z-12y^2z+18xy^3z) > 0$
		8	No	$-\frac{1}{9xyz}(-2+3x)(-2+3y)(-1+z) > 0$
		9	-	This history class matches the species tree topology

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 21: Equations that characterize the anomaly zone for gene tree 3 and species tree 2.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_2	γ_3	1	No	$-\frac{1}{180xy^3z}(-9 + 10x - 40y^2 + 80y^3 + 60xy^2 - 120xy^3 + 80y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		2	No	$-\frac{1}{180xy^3z}(-4 + 10x - 45y^2 + 80y^3 + 60xy^2 - 120xy^3 + 80y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		3	No	$-\frac{1}{180xy^3z}(-4 + 10x - 35y^2 + 80y^3 + 60xy^2 - 120xy^3 + 70y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		4	No	$-\frac{1}{180xy^3z}(-24 + 30x - 15y^2 + 80y^3 + 30xy^2 - 120xy^3 + 80y^2z$ $-120xy^2z - 120y^3z + 180xy^3z) > 0$
		5	No	$-\frac{1}{180xy^3z}(-24 + 30x - 65y^2 + 80y^3 + 90xy^2 - 120xy^3 + 130y^2z$ $-180xy^2z - 120y^3z + 180xy^3z) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 22: Equations that characterize the anomaly zone for gene tree 1 and species tree 3.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_3	γ_1	1	No	$-\frac{1}{540x^6yz}(-5 - 40x^3 - 72x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		2	No	$-\frac{1}{540x^6yz}(5 - 50x^3 - 72x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		3	No	$-\frac{1}{540x^6yz}(-7 - 20x^3 - 90x^5 + 240x^6 - 30x^3y + 180x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		4	No	$-\frac{1}{540x^6yz}(-1 - 50x^3 - 36x^5 + 210x^6 + 30x^3y + 180x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		5	No	$-\frac{1}{540x^6yz}(5 - 20x^3 - 72x^5 + 240x^6 + 180x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		6	No	$-\frac{1}{540x^6yz}(-7 - 80x^3 + 240x^6 + 90x^3y + 90x^5y - 360x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		7	No	$-\frac{1}{540x^6yz}(-1 - 20x^3 - 216x^5 + 390x^6 + 360x^5y - 540x^6y + 30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		8	No	$-\frac{1}{540x^6yz}(5 - 20x^3 - 72x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		9	No	$-\frac{1}{540x^6yz}(-7 - 80x^3 + 240x^6 + 30x^3y + 180x^5y - 360x^6y + 90x^3z + 90x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		10	No	$-\frac{1}{540x^6yz}(-1 - 20x^3 - 216x^5 + 390x^6 + 30x^3y + 180x^5y - 360x^6y + 360x^5z - 540x^6z - 360x^5yz + 540x^6yz) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 23: Equations that characterize the anomaly zone for gene tree 2 and species tree 3.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_3	γ_2	1	Yes	$-\frac{1}{540x^6yz}(-1 - 50x^3 - 72x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y$ $+30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		2	Yes	$-\frac{1}{540x^6yz}(-1 - 20x^3 - 72x^5 + 240x^6 + 180x^5y - 360x^6y + 30x^3z$ $+180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		3	Yes	$-\frac{1}{540x^6yz}(-1 - 20x^3 - 72x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y$ $+180x^5z - 360x^6y + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		4	Yes	$-\frac{1}{540x^6yz}(-5 - 10x^3 - 108x^5 + 240x^6 + 30x^3y + 180x^5y - 360x^6y$ $+30x^3z + 180x^5z - 360x^6z - 360x^5yz + 540x^6yz) > 0$
		5	Yes	$-\frac{1}{540x^6yz}(-5 - 100x^3 - 108x^5 + 240x^6 + 60x^3y + 270x^5y - 360x^6y$ $+90x^3z + 270x^5z - 360x^6z - 540x^5yz + 540x^6yz) > 0$
		6	Yes	$-\frac{1}{540x^6yz}(-5 - 100x^3 - 108x^5 + 240x^6 + 90x^3y + 270x^5y - 360x^6y$ $+60x^3z + 270x^5z - 360x^6z - 540x^5yz + 540x^6yz) > 0$
		7	Yes	$-\frac{1}{180x^6yz}(-1 - 10x^3 - 30x^5 + 80x^6 + 10x^3y + 60x^5y - 120x^6y$ $+10x^3z + 60x^5z - 120x^6z - 120x^5yz + 180x^6yz) > 0$
		8	Yes	$-\frac{1}{180x^6yz}(-1 - 20x^3 + 80x^6 + 10x^3y + 60x^5y - 120x^6y + 20x^3z$ $+30x^5z - 120x^6z - 120x^5yz + 180x^6yz) > 0$
		9	Yes	$-\frac{1}{180x^6yz}(-1 - 20x^3 + 80x^6 + 20x^3y + 30x^5y - 120x^6y + 10x^3z$ $+60x^5z - 120x^6z - 120x^5yz + 180x^6yz) > 0$

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.

Supplementary Table 24: Equations that characterize the anomaly zone for gene tree 3 and species tree 3.

Species tree topology	Gene tree topology	History class	Anomaly zone nonempty?	Inequality describing anomaly zone
ψ_3	γ_3	1	No	$-\frac{1}{270x^6yz}(1 - 25x^3 - 36x^5 + 120x^6 + 15x^3y + 90x^5y - 180x^6y$ $+15x^3z + 90x^5z - 180x^6z - 180x^5yz + 270x^6yz) > 0$
		2	No	$-\frac{1}{270x^6yz}(1 - 10x^3 - 36x^5 + 120x^6 + 90x^5y - 180x^6y + 15x^3z$ $+90x^5z - 180x^6z - 180x^6z - 180x^5yz + 270x^6yz) > 0$
		3	No	$-\frac{1}{270x^6yz}(1 - 10x^3 - 36x^5 + 120x^6 + 15x^3y + 90x^5y - 180x^6y$ $+90x^5z - 180x^6z - 180x^5yz + 270x^6yz) > 0$
		4	No	$-\frac{1}{18x^3yz}(-2 + 6x^3 + 6x^2y - 12x^3y + z + 6x^2z - 12x^3z - 12x^2yz$ $+18x^3yz) > 0$
		5	-	This history class matches the species tree topology

The anomaly zone is the set of all points (T_2, T_3, T_4) that satisfy eq. 1. For convenience, in these inequalities, we substitute $x = e^{T_2}$, $y = e^{T_3}$, and $z = e^{T_4}$. The inequalities are polynomials in $1/x$, $1/y$, and $1/z$.