Introduction to Causal Inference

Solutions to Quiz 4

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Instructions:

- Write your name in the space provided below before you begin.
- You have **20 minutes** to complete the quiz.
- You may answer questions either in English or Japanese.
- The exam is closed book and calculators are not allowed.
- Please turn off your phone before you begin.
- Note that I have allocated more space than we anticipate you will need for each problem. Just answer the questions as best you can, don't try to fill the available space.
- Good luck!

Your Name: ______________________________

Problem 1

Are the following statements true or false? Why? For each question, state whether you think the statement is true or false, and explain why you think so in a few sentences.

a) The sharp regression discontinuity design (RDD) exploits the deterministic treatment assignment governed by the forcing variable around a threshold. The identification of causal effects under RDD therefore rests on the assumption that average potential outcomes discontinuously jump at the threshold as a function of the forcing variable.

False. The continuity assumption requires potential outcomes to be continuous around the threshold. In a RDD setup observed outcomes are generally discontinuous at the threshold.

b) Under the encouragement design, if encouragement is not exogenously assigned then the exclusion restriction is also necessarily violated because potential outcomes are correlated with the encouragement variable.

False. This would be a violation of ignorability of the encouragement, but not necessarily exclusion. The encouragement Z_i can be assigned such that it is not independent of potential outcomes of $Y_i(z)$, without having any *effect* on Y_i other than through D_i, the treatment. It could, for instance, be that Z_i and Y_i are caused by a mutual confounder. Conversely, while a violation of exogeneity implies that $Y_i(1,d) \neq Y_i(0,d)$ for $d = 0, 1$, it does not necessarily imply that $\{Y_i(1), Y_i(0)\} \not\perp\!\!\!\perp Z_i$.

c) Amongst compliers, the intention-to-treat effect and the average treatment effect are always equal.

True. For compliers the treatment status always equals the encouragement status, so the effects of those two causal variables on an outcome must be identical.

Problem 2

Consider a simple random sample of N units from the target population of interest. Let $Z_i \in \{0, 1\}$ be the binary encouragement (or instrumental variable) where $Z_i = 1$ if unit i is encouraged to take the treatment and $Z_i = 0$ if not. Let $D_i(z) \in \{0, 1\}$ be the treatment status of unit i that would be realized if i is assigned to the encouragement status $z \in \{0, 1\}$. Finally, we use $Y_i(d)$ to denote the potential outcome for unit i that would be realized if i actually received the treatment $d \in \{0, 1\}$. For the rest of this problem, assume randomization of the encouragement, the Stable Unit Treatment Value Assumption, non-zero average encouragement effect, and the exclusion restrictions. Suppose that the analyst is interested in estimating the local average treatment effect among the compliers, which we denote by:

$$\tau_C \equiv \mathbb{E}[Y_i(1) - Y_i(0) \mid i \text{ is a complier}]$$

Now, answer the following questions.

a) Suppose that the analyst is also willing to assume that there is no defier in the population. Write down the assumption formally, using the notation introduced above.

The assumption of monotonicity, or no defiers, is $D_i(1) \geq D_i(0)$ for all i.

b) Prove that under the current set of assumptions, τ_C can be identified as follows,

$$\tau_C \;=\; \frac{\mathbb{E}[Y_i \mid Z_i = 1] - \mathbb{E}[Y_i \mid Z_i = 0]}{\Pr(D_i = 1 \mid Z_i = 1) - \Pr(D_i = 1 \mid Z_i = 0)}$$

(make sure to justify each step of your proof, referring to specific assumptions if necessary).

$$\begin{aligned}
\tau_C &= \mathbb{E}[Y_i(1) - Y_i(0)] \mid i \text{ is a complier}] \\
&= \frac{1}{\Pr(D_i(1) = 1, D_i(0) = 0)}\{\mathbb{E}[Y_i(1) - Y_i(0)] - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 1]\Pr(D_i(1) = 1, D_i(0) = 1) - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(0) = 0, D_i(0) = 0]\Pr(D_i(1) = 0, D_i(0) = 0) - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 0, D_i(0) = 1]\Pr(D_i(1) = 0, D_i(0) = 1)\} \\
&= \frac{1}{\Pr(D_i(1) = 1, D_i(0) = 0)}\{\mathbb{E}[Y_i(1) - Y_i(0)] - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 1]\Pr(D_i(1) = 1, D_i(0) = 1) - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(0) = 0, D_i(0) = 0]\Pr(D_i(1) = 0, D_i(0) = 0) - 0\} \quad \because \text{ no defiers} \\
&= \frac{1}{\mathbb{E}[D_i|Z_i = 1] - \mathbb{E}[D_i|Z_i = 0]}\{\mathbb{E}[Y_i(1) - Y_i(0)] - \\
&\quad 0 \cdot \Pr(D_i(1) = 1, D_i(0) = 1) - 0 \cdot \Pr(D_i(1) = 0, D_i(0) = 0)\} \quad \because \text{ exclusion} \\
&= \frac{\mathbb{E}[Y_i|Z_i = 1] - [Y_i|Z_i = 0]}{\Pr[D_i = 1|Z_i = 1] - \Pr[D_i = 0|Z_i = 0]}
\end{aligned}$$

Bonus Question: Solve this question if you have finished the earlier problems and do not know what to do for the rest of your time. Suppose that the assumption in part (a) was actually violated. What is the large sample bias in the expression in part (b)? Derive it formally, and discuss conditions under which the bias is large.

$$
\begin{aligned}
\tilde{\tau}_C - \tau_C &= \frac{\mathbb{E}[Y_i(1)|Z_i = 1] - \mathbb{E}[Y_i(0)|Z_i = 0]}{\Pr[D_i = 1|Z_i = 1] - \Pr[D_i = 0|Z_i = 0]} - \mathbb{E}[Y_i(1) - Y_i(0)] \mid i \text{ is a complier}] \\
&= \frac{\mathbb{E}[Y_i(1)|Z_i = 1] - \mathbb{E}[Y_i(0)|Z_i = 0]}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} - \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0] \\
&= \frac{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0]\Pr(D_i(1) = 1, D_i(0) = 0)}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} + \\
&\quad \frac{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 0, D_i(0) = 1]\Pr(D_i(1) = 0, D_i(0) = 1)}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} - \\
&\quad \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0] \quad \because \text{ exclusion} \\
&= \frac{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0]\Pr(D_i(1) = 1, D_i(0) = 0)}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} + \\
&\quad \frac{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 0, D_i(0) = 1]\Pr(D_i(1) = 0, D_i(0) = 1)}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} - \\
&\quad \frac{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0][\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)]}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0|D_i(0) = 1)} \\
&= \{\mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 0, D_i(0) = 1] + \mathbb{E}[Y_i(1) - Y_i(0)|D_i(1) = 1, D_i(0) = 0]\} \\
&\quad \frac{\Pr(D_i(1) = 0, D_i(0) = 1)}{\Pr(D_i(1) = 1, D_i(0) = 0) - \Pr(D_i(1) = 0, D_i(0) = 1)}
\end{aligned}
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The third line comes from the decomposition of the ITT under the assumption of exclusion but not monotonicity. (Compare this to the second step in the previous question.) The final expression is the sum of the LATE among compliers and the ITT among defiers, multiplied by a ratio of probabilities. Note that the ITT for compliers and defiers will be oppositely signed; we could also write the first term as the difference of LATEs between compliers and defiers. Bias thus increases with both the number of defiers and the difference in LATEs between compliers and defiers.