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Using Degradation Measures to Estimate a Time-to-Failure Distribution

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Some life tests result in few or no failures. In such cases, it is difficult to assess reliability with traditional life tests that record only time to failure. For some devices, it is possible to obtain degradation measurements over time, and these measurements may contain useful information about product reliability. Even with little or no censoring, there may be important practical advantages to analyzing degradation data. If failure is defined in terms of a specified level of degradation, a degradation model defines a particular time-to-failure distribution. Generally it is not possible to obtain a closed-form expression for this distribution. The purpose of this work is to develop statistical methods for using degradation measures to estimate a time-to-failure distribution for a broad class of degradation models. We use a nonlinear mixed-effects model and develop methods based on Monte Carlo simulation to obtain point estimates and confidence intervals for reliability assessment.

KEY WORDS: First crossing time; Nonlinear estimation; Random effect; Reliability.

1. INTRODUCTION

1.1 Problem

Electronic systems, like computers, have many components. Maintaining high reliability for the entire system generally requires that the individual system components have extremely high reliability, even after long periods of time. With short product-development times, reliability tests must be conducted with severe time constraints. Frequently, no failures occur during such tests. Thus it is difficult to assess reliability with traditional life tests that record only time to failure. For some components, degradation measures, taken over time, contain information about product reliability. Then one can define component failure in terms of a specified level of degradation and estimate the time-to-failure distribution from the degradation measures.

Previous works in this area have been restricted to a single observation per unit, relatively simple degradation models, or the estimation of degradation-model parameters. The purpose of this article is to develop more general statistical models and data-analysis methods for using degradation measures to estimate a time-to-failure distribution. The model used here may involve nonlinear relationships for the degradation sample path over time. Among the unknown parameters, there may be some that are fixed and some that are random. Fixed effects describe population (process) characteristics, and random effects describe an individual unit's characteristics.

We review methods for fitting nonlinear regression models to observed degradation measurements and use a *two-stage* method to estimate the mixed-effect path model parameters. When the degradation model is not simple enough to have a closed-form expression for $F_T(t)$, we use Monte Carlo simulation to compute an estimate of the distribution function of the time to failure. We suggest bootstrap methods for setting confidence intervals.

1.2 Motivational Example

We use fatigue-crack-growth data from Hudak, Saxena, Bucci, and Malcolm (1978) to motivate our work. We obtained the data in Table 1 visually from figure 4.5.2 on page 242 of Bogdanoff and Kozin (1985). There are 21 sample paths, one for each of 21 test units. We define a critical crack length of 1.6 inches to be a "failure." We also assume that testing stopped at .12 million cycles. Figure 1 is a plot of the crack-length measurements versus time (in million cycles), connected by straight lines. From this plot, we notice that about half of the units do not fail by the end of the test. Because of their regularity, it is clear that the sample paths contain useful information, beyond the crossing times and running times, to estimate the time-to-failure distribution.

Table 1. Fatigue-Crack-Growth Data From Bogdanoff and Kozin (1985)

	Million cycles												
Path	*.00*	*.01*	*.02*	*.03*	*.04*	*.05*	*.06*	*.07*	*.08*	*.09*	*.10*	*.11*	*.12*
1	.90	.95	1.00	1.05	1.12	1.19	1.27	1.35	1.48	1.64			
2	.90	.94	.98	1.03	1.08	1.14	1.21	1.28	1.37	1.47	1.60		
3	.90	.94	.98	1.03	1.08	1.13	1.19	1.26	1.35	1.46	1.58	1.77	
4	.90	.94	.98	1.03	1.07	1.12	1.19	1.25	1.34	1.43	1.55	1.73	
5	.90	.94	.98	1.03	1.07	1.12	1.19	1.24	1.34	1.43	1.55	1.71	
6	.90	.94	.98	1.03	1.07	1.12	1.18	1.23	1.33	1.41	1.51	1.68	
7	.90	.94	.98	1.02	1.07	1.11	1.17	1.23	1.32	1.41	1.52	1.66	
8	.90	.93	.97	1.00	1.06	1.11	1.17	1.23	1.30	1.39	1.49	1.62	
9	.90	.92	.97	1.01	1.05	1.09	1.15	1.21	1.28	1.36	1.44	1.55	1.72
10	.90	.92	.96	1.00	1.04	1.08	1.13	1.19	1.26	1.34	1.42	1.52	1.67
11	.90	.93	.96	1.00	1.04	1.08	1.13	1.18	1.24	1.31	1.39	1.49	1.65
12	.90	.93	.97	1.00	1.03	1.07	1.10	1.16	1.22	1.29	1.37	1.48	1.64
13	.90	.92	.97	.99	1.03	1.06	1.10	1.14	1.20	1.26	1.31	1.40	1.52
14	.90	.93	.96	1.00	1.03	1.07	1.12	1.16	1.20	1.26	1.30	1.37	1.45
15	.90	.92	.96	.99	1.03	1.06	1.10	1.16	1.21	1.27	1.33	1.40	1.49
16	.90	.92	.95	.97	1.00	1.03	1.07	1.11	1.16	1.22	1.26	1.33	1.40
17	.90	.93	.96	.97	1.00	1.05	1.08	1.11	1.16	1.20	1.24	1.32	1.38
18	.90	.92	.94	.97	1.01	1.04	1.07	1.09	1.14	1.19	1.23	1.28	1.35
19	.90	.92	.94	.97	.99	1.02	1.05	1.08	1.12	1.16	1.20	1.25	1.31
20	.90	.92	.94	.97	.99	1.02	1.05	1.08	1.12	1.16	1.19	1.24	1.29
21	.90	.92	.94	.97	.99	1.02	1.04	1.07	1.11	1.14	1.18	1.22	1.27

1.3 Literature

1.3.1. Degradation Models. Although the literature is not abundant (relative to literature available for the analysis of time-to-failure data), there are important references that have used degradation data to assess reliability. Gertsbackh and Kordonskiy (1969) discussed the degradation problem from an engineering point of view. They pointed out the value of analyzing degradation measures in terms of sample paths to assess product reliability. They presented the Bernstein distribution, which describes the time-to-failure distribution for a simple linear model with random intercept and random slope. Nelson (1981) discussed a special situation in which the degradation measurement is destructive (only one measurement could be made on each item). Tomsky (1982) used a multivariate normal regression model to evaluate component degradation. Amster and Hooper (1983) proposed a simple degradation model for single-, multiple-, and step-stress life tests. They showed how to use this model to estimate the central tendency of the time-to-failure distribution.

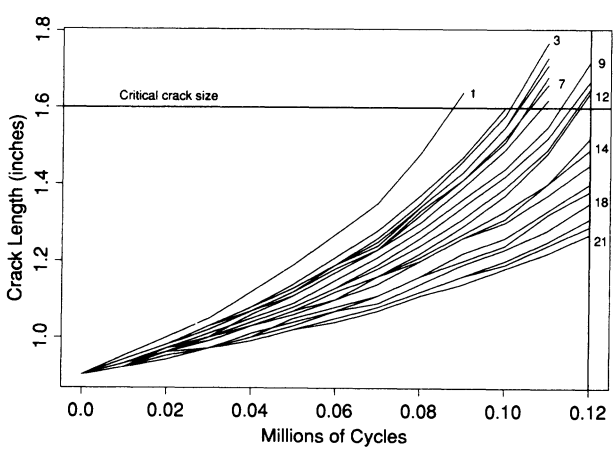

Figure 1. Fatigue-Crack-Growth Data from Bogdanoff and Kozin (1985).

Bogdanoff and Kozin (1985) used a probabilistic approach to model degradation (crack length) for metal fatigue. They included many interesting graphs of sample paths to describe different types of cumulative damage (another term for degradation). They also gave several examples of real data. Carey and Tortorella (1988) described a Markov-process model for degradation data and gave methods of estimating parameters and testing goodness of fit. Similar results were given by Carey (1989). Carey and Koenig (1991) described a data-analysis strategy and model-fitting methods to extract reliability information from observations on the degradation of integrated logic devices that are components in a new generation of submarine cables. Lu and Pantula (1989) used a repeated-measures model to analyze accelerated-test degradation data from silicon devices. Nelson (1990, chap. 11) reviewed the degradation literature, sur-

veyed applications, described basic ideas on accelerated-test degradation models, and, using a specific example, showed how to analyze a type of degradation data.

1.3.2. Relationship to Pharmacokinetics. Pharmacokinetics is the study of the time course of drug concentrations in biological systems. Pharmacokinetic experimental data consist of measured tissue concentrations of a drug for an individual unit over time and can be described in terms of a model that depends on dosage and time. Because pharmacokinetics deals with measurements over time on individual units, there are important similarities between statistical methods that have been developed for pharmacokinetics in biological systems and measured degradation in physical systems. Sheiner and Beal (1980, 1981, 1983) used simulation to compare methods for estimating population pharmacokinetic parameters of specific models. Beal and Sheiner (1985) described the "first order" and "extended least squares estimation" methods of estimating pharmacokinetics model parameters. Steimer, Mallet, Golmard, and Boisvieux (1984) discussed the global two-stage, the iterated two-stage, and the nonlinear filtering estimation methods. Mallet (1986) introduced an optimizing-design-oriented method for estimating the distribution parameters of a random-coefficient regression model. Feldman (1988) gave comparison of several two-stage methods for a linear model. Racine-Poon and Smith (1989) described a Bayesian EM algorithm for estimating the pharmacokinetic parameters.

1.3.3. Nonlinear Regression. Degradation path models are often nonlinear in the parameters. Seber and Wild (1989) provided theory and practice of nonlinear regression, including nonlinear models with dependent errors and useful classes of growth, compartment, and multiphase models. Other important references in nonlinear regression analysis are, for example, Gallant (1987), Bates and Watts (1988), and Ross (1990).

1.4 Overview

In this article, we use a parametric model to describe the degradation measurements. Section 2 introduces definitions and assumptions about the model, including examples and discussion of autocorrelated errors. Section 3 describes the two-stage method of estimating the path-model parameters. Section 4 shows how to estimate the distribution function of the time to failure. Section 5 presents numerical examples, and Section 6 suggests directions for future research.

2. PARAMETRIC MODEL

2.1 Definitions, Notation, and Assumptions

Experimental results provide, for each sample unit, an observed sequence or *path* of degradation readings y over time t. The observed degradation path y is a unit's actual degradation path η, a nondecreasing function of time, which cannot be observed directly, plus measurement error ε. "Time" t could be real time or some other measure like miles for automobile tires or cycles in fatigue tests.

We will use D to denote the critical level for the degradation path above which failure is assumed to have occurred. The failure time T is defined as the time when the actual path η crosses the critical degradation level D. Because we only observe y, however, this implies that we never observe the actual "failure." We use t_s to denote the planned stopping time in the experiment. Inferences are desired on the time-to-failure distribution of a particular product or material.

We make the following general assumptions about the manner in which the test is conducted:

1. Sample units are randomly selected from a population or production process and random measurement errors are independent across time and units.
2. Sample units are tested in a particular homogeneous environment (e.g., the same constant temperature).
3. The measurement (or inspection) times are prespecified, the same across all the test units, and may or may not be equally spaced in time. We use this assumption for constructing confidence intervals for the time-to-failure distribution via the bootstrap simulation method (in Sec. 4.3).

2.2 General Path Model

For each unit in a random sample of size n units, we assume that degradation measurements are available for prespecified times—$t_1, t_2, \ldots, t_s$, generally, until y crosses the prespecified critical level D or until time t_s, whichever comes first. Sometimes measurements are available after the sample path crosses the critical level D. The sample path of ith unit at time t_j is given by

$$y_{ij} = \eta_{ij} + \varepsilon_{ij} = \eta(t_j; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) + \varepsilon_{ij}, \quad i = 1, 2, \ldots, n,$$

$$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2), \qquad j = 1, 2, \ldots, m_{\Theta_i} \le m,$$

where t_j = time of the jth measurement or inspection; ε_{ij} = measurement error with constant variance σ_ε^2; η_{ij} = actual path of the ith unit at time t_j with unknown parameters as listed later; $\boldsymbol{\phi}$ = vector of fixed-effect parameters, common for all units; $\boldsymbol{\Theta}_i$ = vector of the ith unit random-effect parameters, rep-

resenting individual unit characteristics; $\boldsymbol{\Theta}_i$ and ε_{ij} are independent of each other ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m_{\Theta_i}$); m = total number of possible inspections in the experiment; and m_{Θ_i} = total number of inspections on the ith unit, a function of $\boldsymbol{\Theta}_i$. We assume that the $\boldsymbol{\Theta}_i$ ($i = 1, 2, \ldots, n$) follow a multivariate distribution function $G_{\Theta}(\cdot)$, which may depend on some unknown parameters that must be estimated from the data. We also assume that y and t are in appropriately transformed scales, if needed. For example, y might be log-degradation and t log-time.

The distribution function of T, the failure time, can be written as $\Pr\{T \le t\} = F_T(t) = F_T(t; \boldsymbol{\phi}, G_{\Theta}(\cdot), D, \eta)$.

For some simple path models, $F_T(t)$ can be expressed in a closed form. For many path models, however, this is not possible. With more than one random parameter, the problem is especially complicated. In some cases, one can use integral transformations or other methods for finding the distribution of functions of random variables to derive the distribution of T. Springer (1979) provided a comprehensive treatment of these methods. Usually, one will have to evaluate the resulting forms numerically. More generally, however, one can obtain, numerically, the distribution of T for any specified $\boldsymbol{\phi}$, $G_{\Theta}(\cdot)$, D, and η (i.e., the model parameters, the critical degradation level, and the degradation path model), and to any desired degree of precision, by using Monte Carlo simulation.

2.3 Examples

The following examples provide some illustrations of degradation path models that lead to closed-form expressions for the cdf of the time-to-failure distribution. For some models that have such closed-form expressions, the computations for the statistical methods described in this article can be simplified somewhat.

Example 1. Suppose that the actual degradation path of a particular unit is given by $\eta(t) = \phi + \Theta t$, where ϕ is fixed and Θ varies from unit to unit according to a Weibull(α, β) distribution; that is,

$$G_{\Theta}(\vartheta) = \Pr\{\Theta \le \vartheta\} = 1 - \exp\left[-\left(\frac{\vartheta}{\alpha}\right)^{\beta}\right].$$

The parameter ϕ represents the common initial amount of degradation of all the test units at the beginning of the test, $\eta(0) = \phi$, and Θ represents degradation rate. We assume that the component degrades monotonically in time and η is an increasing function, so $\Pr(\Theta > 0) = 1$.

For the critical level D, we can write $D = \phi + \Theta T$, and then $T = \tau(\Theta; \phi, D, \eta) = (D - \phi)/\Theta$, where τ is a transformation from random variable Θ to random variable T.

The distribution function of T is

$$\begin{aligned} F_T(t) = \Pr\{T \le t\} &= \Pr\{\tau(\Theta; \phi, D, \eta) \le t\} \\ &= \Pr\left\{\frac{D - \phi}{\Theta} \le t\right\} = \Pr\left\{\Theta \ge \frac{D - \phi}{t}\right\} \\ &= 1 - G\left(\frac{D - \phi}{t}\right) = \exp\left[-\left(\frac{D - \phi}{\alpha t}\right)^{\beta}\right], \\ & \qquad t > 0. \end{aligned}$$

So the distribution function $F_T(t)$ depends on ϕ, D, η, and distribution parameters α, β. The distribution of T is known as the *reciprocal Weibull* because $1/T$ follows a Weibull distribution.

Similarly, if Θ follows a lognormal distribution (μ, σ^2), then

$$F_T(t) = \Phi\left[\frac{\log t - [\log(D - \phi) - \mu]}{\sigma}\right],$$

where $\Phi(\cdot)$ is the standard normal distribution function. This shows that T follows a lognormal distribution.

Let $\Theta \sim N(\mu, \sigma^2)$ with $\sigma \ll \mu$ so that $\Pr(\Theta \le 0)$ is negligible. Then we can derive the distribution $F_T(t)$ from another direction: The proportion failing by time t is equal to the proportion of degradation measure exceeding the critical level D at that time. With the path function $\eta(t) = \phi + \Theta t$, we have $\eta(t) \sim N(\phi + \mu t, \sigma^2 t^2)$. Then, for the critical level D, $T = (D - \phi)/\Theta$ and

$$F_T(t) = \Pr\{T \le t\} \approx \Phi\left(\frac{t - [D - \phi]/\mu}{\sigma t/\mu}\right), \qquad t > 0.$$

The approximation in the preceding equation is due to the small $\Pr(\Theta \le 0)$. Note that, because the variance of $\eta(t)$ depends on t, T does not follow a normal distribution. This is a special case of the Bernstein distribution (see Gertsbakh and Kordonskiy 1969, p. 88, and the following example).

Example 2. When both the intercept and the slope in the simple linear path model are assumed to be random, normally distributed, and independent of each other, T follows a Bernstein distribution. The path model is $\eta(t) = \Theta_1 + \Theta_2 t$, where $\Theta_1 \sim N(\mu_1, \sigma_1^2)$, $\Theta_2 \sim N(\mu_2, \sigma_2^2)$, and Θ_1 is independent of Θ_2. In practice, both $\Pr\{\Theta_1 < 0\}$ and $\Pr\{\Theta_2 \le 0\}$ would be negligible. Then, $\eta(t) \sim N(\mu_1 + \mu_2 t, \sigma_1^2 + \sigma_2^2 t^2)$ and

$$F_T(t) \approx \Phi\left(\frac{t - [D - \mu_1]/\mu_2}{\sqrt{[\sigma_1^2 + \sigma_2^2 t^2]/\mu_2^2}}\right).$$

The approximation is due to the small probability of a negative Θ_2. Again, T does not follow a normal distribution because the variance of $\eta(t)$ depends on t. See Ahmad and Sheikh (1984) and Sheikh, Younas, and Ahmad (1986) for further discussion about this distribution. It is possible to generalize the Bernstein distribution if Θ_1 and Θ_2 are correlated (as might be expected in some applications).

Example 3. Suppose that a unit path is given by $\eta(t) = \phi_1 + \Theta \exp(\phi_2 t)$, $\phi_2 > 0$, where $\boldsymbol{\phi} = (\phi_1, \phi_2)'$ are fixed and $\Theta \sim \text{lognormal}(\mu, \sigma^2)$, so

$$G_\Theta(\vartheta) = \Phi\left(\frac{\log \vartheta - \mu}{\sigma}\right).$$

Then T can be expressed as follows:

$$T = \tau(\Theta; \boldsymbol{\phi}, D, \eta) = \frac{\log(D - \phi_1) - \log \Theta}{\phi_2}.$$

The distribution function of T for the critical level D is

$$F_T(t) = \Phi\left(\frac{t - [\log(D - \phi_1) - \mu]/\phi_2}{\sigma/\phi_2}\right), \qquad t > 0.$$

Therefore, we have

$$T \sim N\left(\frac{\log(D - \phi_1) - \mu}{\phi_2}, \frac{\sigma^2}{\phi_2^2}\right).$$

The possibility of negative T arises because, if $\Theta > D - \phi_1$, then $\eta(0) > D$ and $\eta(t)$ crosses D before time 0.

2.4 Multivariate Normal Model

To effectively broaden the range of application, we use a somewhat more general model that assumes that the vector of random effects $\boldsymbol{\Theta}$, or some appropriate reparameterization $\boldsymbol{\theta} = \mathbf{H}(\boldsymbol{\Theta})$, follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}_\theta$ and variance–covariance matrix $\boldsymbol{\Sigma}_\theta$. The assumption of multivariate normal random effects, after reparameterization if needed, allows us to summarize the information in the sample paths, without loss of substantial information, with only a mean vector and variance–covariance matrix. This also allows the use of standard methods for adjusting our two-stage estimation (described in Sec. 3) for measurement error. Extension to other assumed joint distributions is possible but would require estimation methods like maximum likelihood (ML) which, for estimating random parameters, are computationally intensive.

The reparameterization $\boldsymbol{\theta}$ of the random effects $\boldsymbol{\Theta}$ may depend on physical knowledge of the process or on assumed ranges for some components of $\boldsymbol{\Theta}$ (e.g., a more adequate model can often be expected, instead, from using the logarithm of positive components of $\boldsymbol{\Theta}$). If there is a known reparameterization available to transform the random-effect parameters $\boldsymbol{\Theta}$ to normal, then we will use that reparameterization. When the reparameterization is unknown, an appropriate reparameterization may be suggested by the data. In any case, possible transformations should be investigated using standard graphical techniques and appropriate tests. For many problems, the Box–Cox family of transformations (Box and Cox 1964) will be suitable. Andrews, Gnanadesikan, and Warner (1971, 1973) described the multivariate generalization of the Box–Cox transformation.

We assume that $\mathbf{H}(\boldsymbol{\Theta}_i) = \boldsymbol{\theta}_i = (\theta_{1i}, \theta_{2i}, \ldots, \theta_{p\theta i})' \sim MVN(\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)$ $(i = 1, 2, \ldots, n)$. With this model, the distribution of T, time to failure, can be rewritten as $\Pr\{T \le t\} = F_T(t) = F_T(t; \boldsymbol{\phi}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta, D, \eta)$. In general, however, there is no closed-form expression for this function.

2.5 Autocorrelated Errors

The observed degradation on a specimen over time is a time series. Time series data can exhibit autocorrelation caused by modeling error or by cyclic changes in ambient conditions (e.g., temperature), in the measurement errors, or in the degradation process itself. Generally, autocorrelation becomes stronger when the times between measurements are relatively short and becomes less noticeable when they are longer. Gallant (1987) and Seber and Wild (1989) described methods for estimation of nonlinear regression models with autocorrelated errors.

In this article, we assume that autocorrelation is negligible. To extend the parametric model introduced in Section 2 to include autocorrelated errors will involve modifications to our model, the two-stage estimation method in Section 3, and parametric bootstrap simulation in Section 4.3.

3. THE TWO-STAGE METHOD OF ESTIMATION

When parameters appear nonlinearly in the path model, full ML estimation of random-effect parameters $\boldsymbol{\mu}_\theta$ and $\boldsymbol{\Sigma}_\theta$ is, in general, algebraically intractable and computationally intensive, as pointed out by Brillinger (1987). Lindstrom and Bates (1990) proposed a general nonlinear mixed-effects model for repeated-measures data and defined estimators for its parameters. Palmer, Phillips, and Smith (1991) described the use of the EM algorithm to the ML estimation for some nonlinear random-coefficient models in animal studies. In this section, we discuss an alternative two-stage estimation method for estimating these random-effect parameters. We use this two-stage method because it is computationally sim-

ple and intuitively appealing and because our simulation studies (details not given here) showed that the method compares well with the more computationally intensive methods. Applications of similar two-stage methods can be found in the work of Mowers, Fuller, and Shrader (1981), Beal and Sheiner (1985), and Carey and Tortorella (1988). Steimer et al. (1984) proposed similar methods that use iterative refinement. To carry out the two-stage method, we do the following:

1. In the first stage, for each sampled unit, fit the degradation model to the sample path and obtain the Stage 1 estimates of the model parameters.
2. Transform, if necessary, the Stage 1 estimates (in effect reparameterizing the random-effect model parameters) so that the random-effect parameters can be modeled with a (multivariate) normal distribution.
3. In the second stage, combine the (reparameterized) Stage 1 estimates of the model parameters to produce estimates of $\boldsymbol{\phi}$, $\boldsymbol{\mu}_\theta$, and $\boldsymbol{\Sigma}_\theta$.

3.1 Stage 1

In the first stage, for each unit i ($i = 1, 2, \ldots, n$) the least squares estimates $(\hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\Theta}}_i)$ of $(\boldsymbol{\phi}, \boldsymbol{\Theta}_i)$ are the values of the path parameters that minimize

$$\mathcal{Q}(\boldsymbol{\varphi}, \boldsymbol{\vartheta}) = \sum_{j=1}^{m_i} \{y_{ij} - \eta(t_j; \boldsymbol{\varphi}, \boldsymbol{\vartheta})\}^2.$$

An estimator of the error variance σ_ε^2 from the ith unit is

$$\hat{\sigma}_{\varepsilon i}^2 = \frac{1}{m_{\Theta_i} - p} \mathcal{Q}(\hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\Theta}}_i),$$

where $p = p_\phi + p_\theta$ and p_ϕ and p_θ are the dimensions of $\boldsymbol{\phi}$ and $\boldsymbol{\Theta}$, respectively. Under appropriate regularity conditions (e.g., Seber and Wild 1989, sec. 12.2.2), $\hat{\boldsymbol{\phi}}_i$, $\hat{\boldsymbol{\Theta}}_i$, and $\hat{\sigma}_{\varepsilon i}^2$ are consistent and, for large m_{Θ_i}, we have the following asymptotic distributions, conditional on $\boldsymbol{\Theta}_i = \boldsymbol{\Theta}_i^*$,

$$\begin{pmatrix} \hat{\boldsymbol{\phi}}_i \\ \hat{\boldsymbol{\Theta}}_i \end{pmatrix} \sim MVN\left(\begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\Theta}_i^* \end{pmatrix}, \sigma_\varepsilon^2 [\dot{\boldsymbol{\eta}}'(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i^*) \dot{\boldsymbol{\eta}}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i^*)]^{-1} \right)$$

$$\frac{(m_{\Theta_i} - p)\hat{\sigma}_{\varepsilon i}^2}{\sigma_\varepsilon^2} \sim \chi^2_{m_{\Theta_i} - p},$$

where $\boldsymbol{\Theta}_i^*$ and m_{Θ_i} are realizations of $\boldsymbol{\Theta}_i$ and m_{Θ_i}, $\mathbf{t}_i = (t_1, t_2, \ldots, t_{m_i})'$, $\boldsymbol{\eta}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) = (\eta(t_1; \boldsymbol{\phi}, \boldsymbol{\Theta}_i), \eta(t_2; \boldsymbol{\phi}, \boldsymbol{\Theta}_i), \ldots, \eta(t_{m_i}; \boldsymbol{\phi}, \boldsymbol{\Theta}_i))'$, and $\dot{\boldsymbol{\eta}}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) = \partial \boldsymbol{\eta}(\mathbf{t}_i; \boldsymbol{\varphi}, \boldsymbol{\vartheta}) / \partial(\boldsymbol{\varphi}'\boldsymbol{\vartheta}')|_{(\boldsymbol{\varphi}, \boldsymbol{\vartheta}) = (\boldsymbol{\phi}, \boldsymbol{\Theta}_i)}$. Moreover, $(\hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\Theta}}_i)$ and $\hat{\sigma}_{\varepsilon i}^2$ are asymptotically independent. Note that the asymptotic results in this and the following sections depend on large m_{Θ_i}.

3.2 Preliminaries for Stage 2

We assume that, by some appropriate reparameterization (e.g., using a Box–Cox tranformation), $\hat{\boldsymbol{\theta}}_i = \mathbf{H}(\hat{\boldsymbol{\Theta}}_i)$ ($i = 1, 2, \ldots, n$) is approximately multivariate normally distributed with the asymptotic mean $E_\varepsilon(\hat{\boldsymbol{\theta}}_i) = \boldsymbol{\theta}_i$ and the asymptotic variance–covariance, obtained by Taylor series approximation, as $\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i) = [\mathbf{J}(\boldsymbol{\Theta}_i)]' \text{var}_\varepsilon(\hat{\boldsymbol{\Theta}}_i) \mathbf{J}(\boldsymbol{\Theta}_i)$ ($i = 1, 2, \ldots, n$), where $\mathbf{J}(\boldsymbol{\Theta}_i) = (\partial / \partial \boldsymbol{\Theta}') \mathbf{H}(\boldsymbol{\Theta})|_{\boldsymbol{\Theta} = \boldsymbol{\Theta}_i}$ is the $p_\theta \times p_\theta$ Jacobian matrix of $\boldsymbol{\theta}_i = \mathbf{H}(\boldsymbol{\Theta}_i)$ and $\text{var}_\varepsilon(\hat{\boldsymbol{\Theta}}_i) = \sigma_\varepsilon^2 [\dot{\boldsymbol{\eta}}'(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) \dot{\boldsymbol{\eta}}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i)]^{p_\theta p_\theta}$ is the asymptotic variance–covariance matrix of $\hat{\boldsymbol{\Theta}}_i$ based on the sample path from the ith unit, where $[\dot{\boldsymbol{\eta}}'(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) \cdot \dot{\boldsymbol{\eta}}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i)]^{p_\theta p_\theta}$ is the lower right $p_\theta \times p_\theta$ submatrix of $[\dot{\boldsymbol{\eta}}'(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i) \dot{\boldsymbol{\eta}}(\mathbf{t}_i; \boldsymbol{\phi}, \boldsymbol{\Theta}_i)]^{-1}$.

Then taking the variability of the random effects into account, the unconditional asymptotic distribution of $\hat{\boldsymbol{\theta}}_i$, based on data from the ith sample path, has mean vector and variance–covariance matrix as follows: $E_\Theta(\hat{\boldsymbol{\theta}}_i) = E_\Theta[E_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)] = E_\Theta(\boldsymbol{\theta}_i) = \boldsymbol{\mu}_\theta$ and

$$\begin{aligned} \text{var}_\Theta(\hat{\boldsymbol{\theta}}_i) &= \text{var}_\Theta[E_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)] + E_\Theta[\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)] \\ &= \text{var}_\Theta(\boldsymbol{\theta}_i) + E_\Theta[\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)] = \boldsymbol{\Sigma}_\theta + \boldsymbol{\Sigma}_{\hat{\theta}}, \end{aligned}$$

where $\boldsymbol{\Sigma}_{\hat{\theta}} = E_\Theta[\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)]$ is the component of asymptotic variability due to measurement errors in the response values from the ith sample path.

Note that $E_\Theta(\hat{\boldsymbol{\phi}}_i) = E_\Theta[E_\varepsilon(\hat{\boldsymbol{\phi}}_i | \boldsymbol{\Theta}_i)] = E_\Theta(\boldsymbol{\phi}) = \boldsymbol{\phi}$.

3.3 Stage 2

In the second stage, we combine the unconditional estimators, from the preceding discussion, $(\hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\theta}}_i)$ ($i = 1, 2, \ldots, n$), to construct the two-stage estimators of the path-model parameters. The two-stage estimators of the path-model parameters $\boldsymbol{\phi}$ and $\boldsymbol{\mu}_\theta$ are, respectively,

$$\hat{\boldsymbol{\phi}} = \frac{1}{n} \sum_{i=1}^n \hat{\boldsymbol{\phi}}_i \quad \text{and} \quad \hat{\boldsymbol{\mu}}_\theta = \frac{1}{n} \sum_{i=1}^n \hat{\boldsymbol{\theta}}_i.$$

Because $E_\Theta(\hat{\boldsymbol{\phi}}_i) = \boldsymbol{\phi}$, $E_\Theta(\hat{\boldsymbol{\theta}}_i) = \boldsymbol{\mu}_\theta$, and $\text{var}_\Theta(\hat{\boldsymbol{\theta}}_i) = \boldsymbol{\Sigma}_\theta$ and $\boldsymbol{\Sigma}_{\hat{\theta}}$, we have, for large m_{Θ_i} ($i = 1, 2, \ldots, n$) the following asymptotic result:

$$E_\Theta\left[\frac{1}{n-1} \sum_{i=1}^n (\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)' \right] = \boldsymbol{\Sigma}_\theta + \boldsymbol{\Sigma}_{\hat{\theta}}.$$

Thus the asymptotic variance–covariance matrix $\boldsymbol{\Sigma}_\theta$ can be expressed as

$$\boldsymbol{\Sigma}_\theta = E_\Theta\left[\frac{1}{n-1} \sum_{i=1}^n (\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)' \right] - \boldsymbol{\Sigma}_{\hat{\theta}}.$$

We can estimate $\boldsymbol{\Sigma}_{\hat{\theta}} = E_\Theta[\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i | \boldsymbol{\Theta}_i)]$ by the sample average $n^{-1} \sum_{i=1}^n \hat{\text{var}}_\varepsilon(\hat{\boldsymbol{\theta}}_i)$, where $\hat{\text{var}}_\varepsilon(\hat{\boldsymbol{\theta}}_i)$ is obtained by evaluating $\text{var}_\varepsilon(\hat{\boldsymbol{\theta}}_i) = [\mathbf{J}(\boldsymbol{\Theta}_i)]' \text{var}_\varepsilon(\hat{\boldsymbol{\Theta}}_i) \mathbf{J}(\boldsymbol{\Theta}_i)$ at

$\hat{\boldsymbol{\phi}}_i$, $\hat{\boldsymbol{\Theta}}_i$, and $\hat{\sigma}^2_{\varepsilon i}$ (see Sec. 3.2). Therefore, we could estimate $\boldsymbol{\Sigma}_\theta$ by using

$$\frac{1}{n-1}\sum_{i=1}^{n}(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)' - \frac{1}{n}\sum_{i=1}^{n}\widehat{\mathrm{var}}_\varepsilon(\hat{\boldsymbol{\theta}}_i) = \mathbf{M}_a - \mathbf{M}_b.$$

The matrix $\mathbf{M}_a - \mathbf{M}_b$, however, may not always be nonnegative definite (a nonnegative definite matrix is one that is either positive definite or positive semidefinite), which would put it outside the parameter space of a variance–covariance matrix. Thus a modified estimator is required. This is a common problem in the estimation of variance components.

Following the procedure of Amemiya (1985), a modified (by a constraint) estimator of $\boldsymbol{\Sigma}_\theta$, which will always be nonnegative definite, is

$$\begin{aligned}\hat{\boldsymbol{\Sigma}}_\theta &= \mathbf{M}_a - \mathbf{M}_b && \text{if } \mathbf{M}_a - \mathbf{M}_b \\ &&& \text{is nonnegative definite} \\ &= \mathbf{0} && \text{if } \mathbf{M}_a - \mathbf{M}_b \text{ is negative definite} \\ &= \boldsymbol{\Gamma}_+(\boldsymbol{\Lambda}_+ - \mathbf{I})\boldsymbol{\Gamma}'_+ && \text{otherwise,}\end{aligned}$$

where definitions and computational formulas for $\boldsymbol{\Lambda}_+$ and $\boldsymbol{\Gamma}_+$ and the derivation are as given in the Appendix. When $\mathbf{M}_a - \mathbf{M}_b$ is not nonnegative definite, this estimator $\hat{\boldsymbol{\Sigma}}_\theta$ is the nonnegative definite matrix that is, in a sense, "closest" to $\mathbf{M}_a - \mathbf{M}_b$ (see Amemiya 1985). Other approaches for constructing nonnegative definite estimators of a variance–covariance matrix can be found in, for example, the work of Carter and Yang (1986).

3.4 Discussion of Asymptotic Results

The maximum number of inspections m affects the estimation accuracy of the model parameters for the individual paths. In the derivation of the two-stage method, we assume large m_{Θ_i} ($i = 1, 2, \ldots, n$). If m is too small, the asymptotic properties of our estimators might not provide adequate approximations. It is impossible to determine the necessary sample size in general because it depends on the form of the path model and the distribution of random effects. For the model used in our example in Section 5, simulation results (details not given here) indicate that the asymptotic normal approximation is very good with m as small as 7 (our example has $m = 13$). The number of sample units n affects the estimation accuracy of the distribution of random effects. The size of n is also important in determining the accuracy of approximate asymptotic confidence intervals for the path-model parameters and for functions of these parameters.

4. ESTIMATION OF $F_T(t)$

4.1 Point Estimation

One can estimate the time-to-failure distribution $F_T(t)$ by substituting the estimates $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\mu}}_\theta$, and $\hat{\boldsymbol{\Sigma}}_\theta$ into $F_T(t; \boldsymbol{\phi}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta, D, \eta)$; that is, $\hat{F}_T(t) = F_T(t; \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\mu}}_\theta, \hat{\boldsymbol{\Sigma}}_\theta, D, \eta)$. This is straightforward for the case in which $F_T(t)$ can be expressed in a closed form.

When there is no closed-form expression for $F_T(t)$ and when numerical transformation methods are too complicated, one can evaluate the estimate $\hat{F}_T(t)$ to any desired degree of precision by using Monte Carlo simulation. This is done by generating a sufficiently large number of random sample paths from the assumed path model with the estimated parameters and using the proportion failing as a function of time as an estimate of $F_T(t)$.

We use the following algorithm:

1. Estimate the path-model parameters $\boldsymbol{\phi}$, $\boldsymbol{\mu}_\theta$, and $\boldsymbol{\Sigma}_\theta$ from the n sample paths by using the two-stage method of the previous section, giving $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\mu}}_\theta$, and $\hat{\boldsymbol{\Sigma}}_\theta$.
2. Generate N simulated realizations $\tilde{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ from $N(\hat{\boldsymbol{\mu}}_\theta, \hat{\boldsymbol{\Sigma}}_\theta)$ and obtain the corresponding N simulated realizations $\tilde{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$ from $\mathbf{H}^{-1}(\tilde{\boldsymbol{\theta}})$, where N is a large number (e.g., $N = 100{,}000$) and $\mathbf{H}^{-1}$ is the inverse transformation of $\mathbf{H}$.
3. Compute the corresponding N simulated failure times $\tilde{t}$ by substituting $\tilde{\boldsymbol{\Theta}}$ into $T = \tau(\boldsymbol{\Theta}; \hat{\boldsymbol{\phi}}, D, \eta)$.
4. Estimate $F_T(t)$ from the simulated empirical distribution

$$\hat{F}_T(t) = \frac{\text{number of } \tilde{t} \le t}{N}$$

for any desired values of t.

The Monte Carlo approximation error is easy to evaluate by using the binomial distribution. This error can be made arbitrarily small by choosing the Monte Carlo sample size N to be large enough. We choose N to be large enough so that the Monte Carlo approximation error is dominated by the sampling error in the path parameters.

4.2 Pointwise Confidence Intervals for $F_T(t)$

There are many methods to construct confidence intervals for a point on a distribution function (e.g., see Lawless 1982; Nelson 1982). For a parametric model, assuming that $\hat{F}_T(t)$ or some normalizing transformation of $\hat{F}_T(t)$ follows a normal distribution is the simplest and most commonly used method. Thomas and Grunkemeier (1975) showed how to obtain better intervals by inverting the likelihood ratio test. These methods cannot, however, be applied directly in the present situation, unless there is a

closed-form expression for $F_T(t)$. One could use the jackknife method (e.g., Efron 1982) to estimate the standard error of $\hat{F}_T(t)$, choose an appropriate normalizing transformation, and use the asymptotic normal approximation for the distribution of the transformed $\hat{F}_T(t)$ to set pointwise confidence intervals. Instead, because it essentially finds a good normalizing transformation automatically, we apply a parametric bootstrap (or simulation) method to obtain pointwise confidence intervals for $F_T(t)$.

4.3 Parametric Bootstrap Simulation

The bootstrap procedure, introduced by Efron (1982), is a sample-reuse method that is used frequently to assess sampling error or construct confidence intervals when there is no known alternative approach that is both tractable and sufficiently accurate. A useful introduction to the bootstrap methods was given by Efron and Tibshirani (1986). DiCiccio and Romano (1988) gave a survey of bootstrap procedures for constructing confidence regions.

We use the following bias-corrected percentile *parametric* bootstrap (simulation) procedure for constructing pointwise confidence intervals for $F_T(t)$. This procedure is similar to that described by Efron (1985), which is a refinement of the percentile method given by Efron (1982). We use the following steps:

1. Estimate the model parameters $\boldsymbol{\phi}$, $\boldsymbol{\mu}_\theta$, and $\boldsymbol{\Sigma}_\Theta$ and the error variance σ_ε^2 from the n sample paths by using the two-stage method, giving $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\mu}}_\theta$, $\hat{\boldsymbol{\Sigma}}_\theta$, and $\hat{\sigma}_\varepsilon^2$, where $\hat{\sigma}_\varepsilon^2$ is the pooled estimate of σ_ε^2:

$$\hat{\sigma}_\varepsilon^2 = \frac{\sum_{i=1}^n (m_{\Theta_i} - p)\hat{\sigma}_{\varepsilon i}^2}{\sum_{i=1}^n (m_{\Theta_i} - p)}.$$

2. Generate n simulated realizations $\tilde{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ from $N(\hat{\boldsymbol{\mu}}_\theta, \hat{\boldsymbol{\Sigma}}_\theta)$ and obtain the corresponding n simulated realizations $\tilde{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$ from $\mathbf{H}^{-1}(\tilde{\boldsymbol{\theta}})$, where $\mathbf{H}^{-1}$ is the inverse transformation of $\mathbf{H}$.

3. Compute n simulated paths from $\tilde{y}_{ij} = \eta(t_j; \hat{\boldsymbol{\phi}}, \tilde{\boldsymbol{\Theta}}_i) + \tilde{\varepsilon}_{ij}$, where $\tilde{\varepsilon}_{ij}$ are pseudo errors generated from $N(0, \hat{\sigma}_\varepsilon^2)$ and t_j are the same measurement times used in the original experiment, up to the planned stopping time t_s. The simulated paths that cross D before time t_s are truncated at the time point after the crossing.

4. Use the n simulated paths to estimate parameters of the path model, giving the bootstrap estimates $\hat{\tilde{\boldsymbol{\phi}}}$, $\hat{\tilde{\boldsymbol{\mu}}}_\theta$, and $\hat{\tilde{\boldsymbol{\Sigma}}}_\theta$.

5. Generate N_B simulated realizations $\check{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ from $N(\hat{\tilde{\boldsymbol{\mu}}}_\theta, \hat{\tilde{\boldsymbol{\Sigma}}}_\theta)$ and obtain the corresponding N_B simulated realizations $\check{\boldsymbol{\Theta}}$ of $\boldsymbol{\Theta}$ from $\mathbf{H}^{-1}(\check{\boldsymbol{\theta}})$, where N_B is a large number (e.g., $N_B = 100{,}000$) and $\mathbf{H}^{-1}$ is the inverse transformation of $\mathbf{H}$.

6. Compute the corresponding N_B simulated failure times $\check{t}$ by substituting $\check{\boldsymbol{\Theta}}$ into $T = \tau(\boldsymbol{\Theta}, \hat{\tilde{\boldsymbol{\phi}}}, D, \eta)$.

7. Compute the bootstrap estimate $\hat{\tilde{F}}_T(t)$ from the simulated empirical distribution

$$\hat{\tilde{F}}_T(t) = \frac{\text{number of } \check{t} \le t}{N_B}$$

for any desired values of t.

8. Do Steps 2–7 B times (e.g., $B = 4{,}000$) to obtain the bootstrap estimates $\hat{\tilde{F}}_T(t)_1, \hat{\tilde{F}}_T(t)_2, \ldots, \hat{\tilde{F}}_T(t)_B$.

9. Sort the $\hat{\tilde{F}}_T(t)_1, \hat{\tilde{F}}_T(t)_2, \ldots, \hat{\tilde{F}}_T(t)_B$ in increasing order for each desired time t to give $\hat{\tilde{F}}_T(t)_{[1]}, \hat{\tilde{F}}_T(t)_{[2]}, \ldots, \hat{\tilde{F}}_T(t)_{[B]}$.

10. Determine the lower and upper bounds of pointwise $100(1 - \alpha)\%$ confidence intervals for the distribution function $F_T(t)$: $[\hat{\tilde{F}}_T(t)_{[l]}, \hat{\tilde{F}}_T(t)_{[u]}]$, where $l = \Psi(2\Psi^{-1}(q) + \Psi^{-1}(\alpha/2))B$, $u = \Psi(2\Psi^{-1}(q) + \Psi^{-1}(1 - \alpha/2))B$,

$$q = \frac{\text{number of } \hat{\tilde{F}}_T(t)_b \le \hat{F}_T(t)}{B}, \quad b = 1, 2, \ldots, B,$$

and Ψ is a distribution function symmetric about zero. Following Efron (1985), we use the standard normal distribution function Φ for Ψ. Then $\hat{\tilde{F}}_T(t)_{[l]}$ and $\hat{\tilde{F}}_T(t)_{[u]}$ are approximate pointwise lower and upper one-sided $100(1 - \alpha/2)\%$ bias-corrected confidence bounds for $F_T(t)$.

If $F_T(t)$ can be expressed in a closed form, we can skip Steps 5–7 and directly substitute the bootstrap estimates $\hat{\tilde{\boldsymbol{\phi}}}$, $\hat{\tilde{\boldsymbol{\mu}}}_\theta$, and $\hat{\tilde{\boldsymbol{\Sigma}}}_\theta$, obtained in Step 4, into the closed-form expression to get the bootstrap estimate $\hat{\tilde{F}}_T(t) = F_T(t; \hat{\tilde{\boldsymbol{\phi}}}, \hat{\tilde{\boldsymbol{\mu}}}_\theta, \hat{\tilde{\boldsymbol{\Sigma}}}_\theta, D, \eta)$. Most of the computation time will be spent on Step 4 when we have nonlinear model. Note that Steps 4–7 are similar to Steps 1–4 in Section 4.1.

5. NUMERICAL EXAMPLE

In this section, we return to the example in Section 1.2, using the degradation data in Table 1 and Figure 1. Because the fatigue experiment was conducted on notched specimens, each with the same initial crack length of .90 inches, we are interested in the time to grow a crack from .90 inches to the critical crack length 1.60 inches. Table 2 gives the failure times and indicates which ones would have been censored if the test had ended at $t_s = .12$. First, we give the degradation data analysis. Then, we take the traditional approach of fitting parametric models to the censored time-to-failure data. We also compare the results with the nonparametric estimate based on the actual failure times of all of the 21 units.

Table 2. Time-to-Failure Data From Bogdanoff and Kozin (1985), Defined as the Time at Which Fatigue Cracks First Crossed the D = 1.60-Inch Threshold

Path	Failure time (million cycles)
1	.088
2	.100
3	.101
4	.103
5	.103
6	.106
7	.106
8	.109
9	.113
10	.115
11	.118
12	.118
13	.129*
14	.133*
15	.138*
16	.144*
17	.146*
18	.151*
19	.160*
20	.167*
21	.170*

NOTE: Observations marked with an asterisk would have been censored for a test that ended at .12 million cycles.

5.1 Degradation Analysis

The Paris Law, in the notation of materials science

$$\frac{da}{dN} = C(\Delta K(a))^m,$$

is widely used to describe the growth of fatigue cracks, where da/dN is the crack growth rate (a: crack length, N: number of loading cycles), $\Delta K(a)$ is the stress-intensity factor, and C, m are empirically derived constants (e.g., see Ostergaard and Hillberry 1983; Trantina and Johnson 1983). We use a special case of this model in which $\Delta K(a) = a$, for which a closed-form solution

$$a(N) = \frac{a(0)}{[1 - [a(0)]^{m-1}C(m - 1)N]^{1/(m-1)}}$$

is available. Here $a(0)$ is initial crack length at $N = 0$. This suggests, by taking logs and having $a(0) = .90$, the following nonlinear path model for the 21 test units:

$$\begin{aligned} y_{ij} &= \eta(t_j; \boldsymbol{\theta}_i) + \varepsilon_{ij} \\ &= -\frac{1}{\Theta_{2i}} \log(1 - .90^{\Theta_{2i}}\Theta_{1i}\Theta_{2i}t_j) + \varepsilon_{ij}, \\ &\qquad i = 1, 2, \ldots, 21; j = 1, 2, \ldots, m_i, \end{aligned}$$

where $y = \log(a(N)/a(0)) = \log(\text{crack length}/.90)$, $t_j = N =$ millions of cycles, and $\varepsilon_{ij} \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$. The random effects $\boldsymbol{\Theta} = (\Theta_1, \Theta_2)'$, corresponding to $(C, m - 1)'$ in the Paris Law, are from an unknown multivariate distribution $G_\Theta(\cdot)$.

Table 3 gives, for each sample path, the Stage 1 estimates of the path parameters $\hat{\Theta}_1$ and $\hat{\Theta}_2$, the corresponding standard errors $s_{\hat{\Theta}1}$ and $s_{\hat{\Theta}2}$, an estimate of the measurement error standard deviation $\hat{\sigma}_{\varepsilon i}$, and the first-order autocorrelation r_1 of the residuals. After having examined the fitted model for each sample path (through residual analysis and the results in Table

Table 3. Stage 1 Estimates for the Fatigue-Crack-Growth Data

Path	m_i	$\hat{\Theta}_1$	$\hat{\Theta}_2$	$s_{\hat{\Theta}1}$	$s_{\hat{\Theta}2}$	$\hat{\sigma}_\varepsilon$	r_1
1	10	5.32	1.229	.06948	.1087	.00679	.2066
2	11	4.66	1.257	.01618	.0309	.00193	−.3937
3	12	4.47	1.533	.04461	.0734	.00624	.3854
4	12	4.39	1.515	.04936	.0873	.00690	.1772
5	12	4.39	1.470	.04765	.0852	.00663	.0258
6	12	4.32	1.416	.06410	.1228	.00877	.0437
7	12	4.27	1.481	.03864	.0761	.00549	.4114
8	12	4.17	1.480	.03071	.0657	.00447	−.0536
9	13	3.96	1.574	.04046	.0813	.00663	−.0764
10	13	3.80	1.711	.02722	.0616	.00476	.0968
11	13	3.69	1.780	.03334	.0832	.00586	.2072
12	13	3.51	2.129	.04084	.1142	.00792	.4742
13	13	3.38	1.784	.04456	.1570	.00833	.0131
14	13	3.53	.851	.03059	.1064	.00482	−.1507
15	13	3.48	1.426	.02528	.0842	.00447	−.0833
16	13	3.04	1.991	.02240	.1175	.00505	.1740
17	13	3.05	1.569	.03502	.1918	.00726	−.1026
18	13	2.92	1.623	.02658	.1738	.00595	.1431
19	13	2.72	1.957	.00764	.0667	.00201	−.4661
20	13	2.70	1.621	.01121	.1023	.00287	−.2595
21	13	2.60	1.601	.01084	.1161	.00292	−.2820

3), we can say that the proposed path model fits the data with no major discrepancies and without serious autocorrelation in residuals.

Figure 2 is a scatterplot of the Stage 1 estimates. The multivariate transformation and test suggested by Andrews et al. (1971, 1973) can be used to determine if and how the random-effect model parameters $\boldsymbol{\Theta}_i$ should be reparameterized such that the $\mathbf{H}(\boldsymbol{\Theta})$ is approximately multivariate normal. Gnanadesikan (1977, sec. 5.4.2) and Seber (1984, sec. 4.3.2) discussed other approaches to assessing multivariate normality. We applied these methods to the example data (see Lu 1992 for details) and, after finding that reasonable transformations had little effect on the final answers, we concluded that no transformation was needed in this case. This was not unexpected given the small ranges in the values of the Stage 1 estimates.

Following Sections 3.2 and 3.3, we obtain the two-stage estimates of the basic model parameters:

$$\hat{\boldsymbol{\mu}}_\theta = \begin{pmatrix} 3.732 \\ 1.571 \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\Sigma}}_\theta = \begin{pmatrix} .5456 & -.09554 \\ -.09554 & .06654 \end{pmatrix}.$$

Because $\mathbf{M}_a - \mathbf{M}_b$ is positive definite, there is no need, in this case, for the adjustment given by the Amemiya (1985) procedure. Figure 3 shows the point estimate $\hat{F}_T(t)$ and pointwise two-sided 90% and 80% bias-corrected percentile bootstrap confidence intervals for $F_T(t)$. The confidence intervals were obtained by using the bootstrap simulation with B = 4,000 and N_B = 10,000. The points on the graph are the nonparametric estimate

$$\hat{F}_T^{NP}(t) = \frac{\{\text{number of units fail up to time } t\} - .5}{21},$$

based on the failure times for all of the 21 paths in Table 2 (even those that would have been censored at t_s = .12).

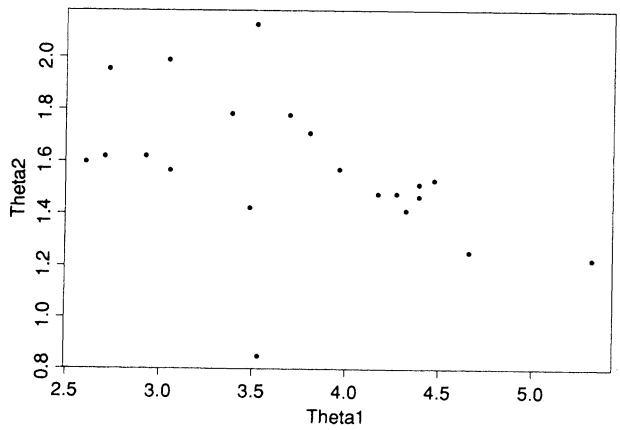

Figure 2. Scatterplot of the Stage 1 Estimates.

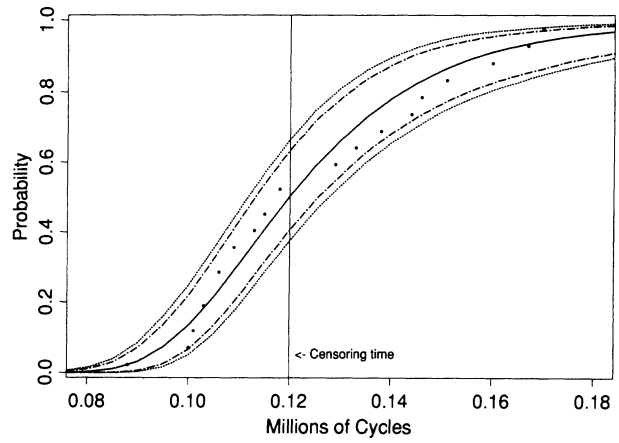

Figure 3. Degradation-Model Estimate of $F_T(t)$ With Pointwise Two-Sided 90% and 80% Bootstrap Bias-Corrected Percentile Confidence Intervals, Based on the Time-to-Failure Data Censored at t_s = .12. The nonparametric estimate is also shown with dots: cdf estimate, ———; 80% confidence intervals, —·—·—; 90% confidence intervals, ·····.

To run the bootstrap with B = 4,000 took approximately 144 hours using a program written in S-Plus (Statistical Sciences, Inc. 1990) and run on a DECstation 5000/200 workstation. Most of the time was spent doing the required 84,000 nonlinear least squares estimations. The analysis could be done more rapidly if programmed 100% in a language like C or FORTRAN or if the bootstrap simulations were done in parallel.

5.2 Comparison With Time-to-Failure Data Analysis

In this section we compare the degradation and time-to-failure data analyses. Based on the time-to-failure data censored at t_s = .12, Figure 4 gives a lognormal probability plot of $\hat{F}_T^{NP}(t)$ with the lognormal distribution ML estimate of $F_T(t)$ superimposed. Figure 5 gives $\hat{F}_T^{NP}(t)$, the lognormal distribution ML estimate of $F_T(t)$, and the corresponding 90% asymptotic pointwise confidence intervals. Figure 6 compares the degradation data/model estimates with the ML estimates of the lognormal, normal, and Weibull time-to-failure distributions, based on the censored time-to-failure data. Figures 5 and 6 also show the nonparametric estimate of the time-to-failure distribution using the actual crossing times that occurred after t_s = .12. The censored data analysis was done with CENSOR (Meeker and Duke 1981). Some observations from these figures are:

1. Figures 4 and 5 show that the lognormal distribution provides a good fit to the time-to-failure data up to t_s = .12, but not beyond.
2. Figure 6 shows that the other commonly used parametric models, which fit almost as well before

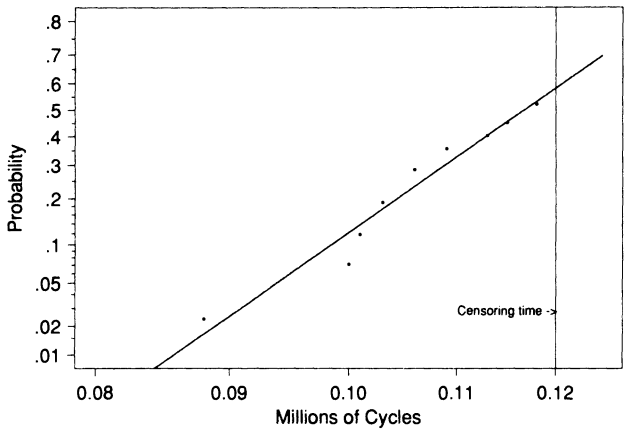

Figure 4. Lognormal Probability Plot and Lognormal Distribution ML Estimate Based on the Time-to-Failure Data Censored at t_s = .12.

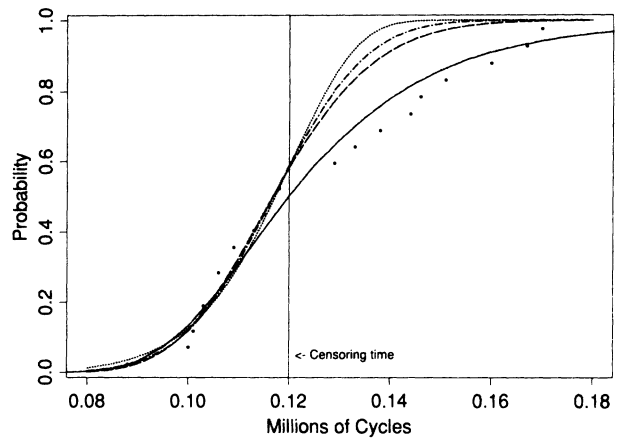

Figure 6. Comparison of Degradation and Time-to-Failure Data Analysis Based on Data Censored at the Nonparametric Estimate (dots): Weibull, ····; Normal, – – –; Lognormal, — ·; Degradation, ———.

t_s = .12, do not do any better beyond t_s = .12. The degradation analysis, however, does provide a reasonable extrapolation beyond t_s = .12. This is because the degradation-analysis method directly models the relationship between degradation and time and takes account of the amount of degradation in the censored observations when estimating $F_T(t)$. See the distribution of crack lengths for the units that had not failed before t_s = .12, shown in Figure 1. The traditional time-to-failure data analysis ignores this important information.

3. Comparing Figures 3 and 5 shows that the confidence intervals based on the degradation and time-to-failure data have similar widths from .10 < t < .12. Outside of this range, however, the confidence intervals are narrower for the degradation method.

4. Figure 6 shows that all of the point estimates are similar in the lower tail of the time-to-failure distribution. Comparing Figures 3 and 5 shows, however, that the degradation method provides a much tighter upper confidence bound on the cdf in this region. This can be explained by the fact that the rates and levels of crack growth observed in the early degradation measurements provide much more information about the lower tail of the time-to-failure distribution than do the times that the cracks reach the critical level.

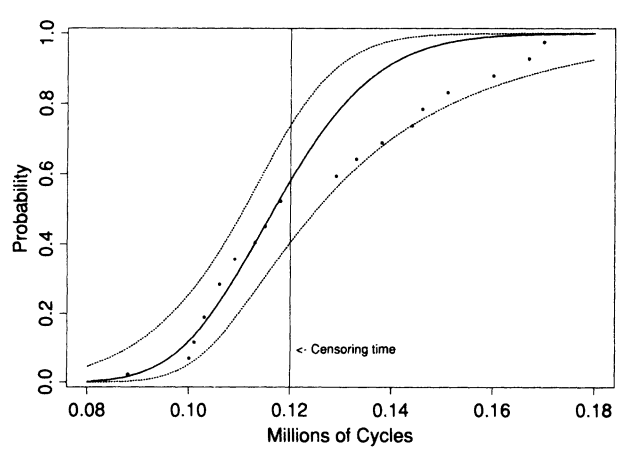

Figure 5. Lognormal Distribution ML Estimate, 90% Pointwise Approximate Confidence Intervals, and Nonparametric Estimate (dots) Based on the Censored Time-to-Failure Data.

6. CONCLUDING REMARKS AND FUTURE WORK

There remain many open questions about how to collect and analyze degradation data. Some of these include the following:

1. We used the bootstrap simulation method to construct *pointwise* confidence intervals for $F_T(t)$. This same method can be used to construct *simultaneous* confidence bands for $F_T(t)$. Such bands are often needed in practical problems in which one is interested in the probability of survival over a range of time.

2. In some high-reliability applications, the rate of degradation is so slow that it is impossible to make useful inferences in a reasonable amount of time. In such cases, as with traditional life tests, an alternative is to use stress acceleration. Often higher stress (e.g., temperature or cycling rate) will lead to faster degradation. One can then use physical models that relate degradation rate to stress to extrapolate and estimate the time-to-failure distribution at a design stress. Some accelerated tests are run at constant stress. Others use progressive or step stress. See Nelson (1990) for models and data-analysis methods. The methods used in this article can be used with such models.

3. For some models, degradation measurements will provide substantially more information than the traditional time-to-failure measurements. The relative benefits depend on the actual model and at least some of its parameters. It would be useful to assess the benefits of taking degradation measurements and to evaluate their cost effectiveness.

4. There are important questions about how to design an experiment to provide the most efficient use of one's resources and to assess the precision that one can expect to achieve with a specified design. This approach can lead to better designs for conducting degradation tests. Important questions include how to choose the interval between inspections (or frequency of recordings in the cases in which continuous readings can be taken), the number of units that should be tested and, for accelerated testing, the levels of the stress(es) and the allocation of the units to the different levels of stress. Carey and Escobar (1991) studied this issue for a special class of degradation models.

5. Sensitivity to possible model departures is of concern to all who analyze data. Sensitivity can be assessed, to some extent, by changing assumptions and reanalyzing. The systematic methods suggested by Cook (1986) and further refined by Escobar and Meeker (1992) could be applied to our model. These methods allow the analyst to assess, systematically, the "general influence" or to assess the impact of specified model departures on particular inferences of interest.

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APPENDIX: NONNEGATIVE DEFINITE ESTIMATOR FOR $\boldsymbol{\Sigma}_\theta$

We will follow the procedure described by Amemiya (1985) to define a modified nonnegative definite estimator for $\boldsymbol{\Sigma}_\theta$ discussed in Section 3.2. Assume that $\mathbf{M}_a = (n-1)^{-1}\sum_{i=1}^n(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)(\hat{\boldsymbol{\theta}}_i - \hat{\boldsymbol{\mu}}_\theta)'$ is nonnegative definite and $\mathbf{M}_b = n^{-1}\sum_{i=1}^n \widehat{\mathrm{var}}_\varepsilon(\hat{\boldsymbol{\theta}}_i)$ is positive definite, by the extension of the principal axis theorem (Arnold 1981, theorem A.7, or Graybill 1983, theorem 12.2.13), $|\mathbf{M}_a - \lambda\mathbf{M}_b| = 0$, has p_θ nonnegative roots $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p_\theta}$. Let $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_{p_\theta}$ be $p_\theta \times 1$ vectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_{p_\theta}$ such that

$$\mathbf{M}_a\boldsymbol{\omega}_k = \lambda_k\mathbf{M}_b\boldsymbol{\omega}_k, \qquad k = 1, 2, \ldots, p_\theta$$

$$\boldsymbol{\omega}_k'\mathbf{M}_b\boldsymbol{\omega}_{k'} = 1 \quad \text{if } k = k'$$

$$= 0 \quad \text{if } k \neq k'.$$

The λ_k and $\boldsymbol{\omega}_k$ are also called the characteristic roots and vectors, respectively, of $\mathbf{M}_a$ in the metric of $\mathbf{M}_b$ (Amemiya 1985).

If $\boldsymbol{\Omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \ldots, \boldsymbol{\omega}_{p_\theta})$ and $\boldsymbol{\Gamma} = (\boldsymbol{\Omega}')^{-1}$, we have $\boldsymbol{\Omega}'\mathbf{M}_a\boldsymbol{\Omega} = \boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{p_\theta}\}$, $\boldsymbol{\Omega}'\mathbf{M}_b\boldsymbol{\Omega} = \mathbf{I}$, and $\mathbf{M}_a - \mathbf{M}_b = \boldsymbol{\Gamma}(\boldsymbol{\Lambda} - \mathbf{I})\boldsymbol{\Gamma}'$.

For the cases of all the nonnegative roots $\lambda_k \geq 1$ and all $\lambda_k < 1$, the estimator of $\boldsymbol{\Sigma}_\theta$ is clearly given by

$$\hat{\boldsymbol{\Sigma}}_\theta = \mathbf{M}_a - \mathbf{M}_b \quad \text{if all } \lambda_k \geq 1$$

$$= \mathbf{0} \quad \text{if all } \lambda_k < 1.$$

For the case that $\lambda_k < 1$ for some k, $\mathbf{M}_a - \mathbf{M}_b$ is no longer nonnegative definite. Then, let $\boldsymbol{\Lambda} = (\boldsymbol{\Lambda}_+, \boldsymbol{\Lambda}_-)$, where $\boldsymbol{\Lambda}_+ = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{p_{\theta+}}\}$, $\lambda_{p_{\theta+}} \geq 1$, and $\boldsymbol{\Lambda}_- = \mathrm{diag}\{\lambda_{p_{\theta+}+1}, \lambda_{p_{\theta+}+2}, \ldots, \lambda_{p_\theta}\}$ ($\lambda_{p_{\theta+}+1} < 1$). We can express $\mathbf{M}_a - \mathbf{M}_b$ as

$$\begin{aligned}\mathbf{M}_a - \mathbf{M}_b &= \boldsymbol{\Gamma}(\boldsymbol{\Lambda} - \mathbf{I})\boldsymbol{\Gamma}' \\ &= \boldsymbol{\Gamma}_+(\boldsymbol{\Lambda}_+ - \mathbf{I})\boldsymbol{\Gamma}_+' + \boldsymbol{\Gamma}_-(\boldsymbol{\Lambda}_- - \mathbf{I})\boldsymbol{\Gamma}_-' \\ &= (\mathbf{M}_a - \mathbf{M}_b)_+ + (\mathbf{M}_a - \mathbf{M}_b)_-,\end{aligned}$$

say, where $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}_+, \boldsymbol{\Gamma}_-)$, $\boldsymbol{\Gamma}_+$ is $p_\theta \times p_{\theta+}$, following the same partition as $\boldsymbol{\Lambda}$.

That is, $\mathbf{M}_a - \mathbf{M}_b$ can be decomposed as the sum of a nonnegative definite (actually, positive semidefinite) matrix $(\mathbf{M}_a - \mathbf{M}_b)_+$ and a negative definite matrix $(\mathbf{M}_a - \mathbf{M}_b)_-$. Therefore, as suggested by Amemiya (1985), the $(\mathbf{M}_a - \mathbf{M}_b)_+$ can be interpreted as the nonnegative definite portion of $(\mathbf{M}_a - \mathbf{M}_b)$, and hence $(\mathbf{M}_a - \mathbf{M}_b)_+$ is the nonnegative definite matrix "closest" to $(\mathbf{M}_a - \mathbf{M}_b)$. So, the modified estimator of $\boldsymbol{\Sigma}_\theta$ is

$$\hat{\boldsymbol{\Sigma}}_\theta = \mathbf{M}_a - \mathbf{M}_b \quad \text{if all } \lambda_k \geq 1$$

$$= \mathbf{0} \quad \text{if all } \lambda_k < 1$$

$$= (\mathbf{M}_a - \mathbf{M}_b)_+ \quad \text{otherwise,}$$

where $(\mathbf{M}_a - \mathbf{M}_b)_+ = \boldsymbol{\Gamma}_+(\boldsymbol{\Lambda}_+ - \mathbf{I})\boldsymbol{\Gamma}_+'$.

The roots of $|\mathbf{M}_a - \lambda\mathbf{M}_b| = 0$ can be computed as follows. Since $\mathbf{M}_b$ is positive definite, there exists a nonsingular matrix $\mathbf{C}$, which can be obtained by, for example, Cholesky decomposition or spectral decomposition, such that $\mathbf{C}'\mathbf{M}_b\mathbf{C} = \mathbf{I}$. Thus we can rewrite the equation $|\mathbf{M}_a - \lambda\mathbf{M}_b| = 0$ as follows:

$$\begin{aligned}0 = |\mathbf{M}_a - \lambda\mathbf{M}_b| &= |\mathbf{C}'|\,|\mathbf{M}_a - \lambda\mathbf{M}_b|\,|\mathbf{C}| \\ &= |\mathbf{C}'\mathbf{M}_a\mathbf{C} - \lambda\mathbf{I}|.\end{aligned}$$

Then the eigenvalues λ_k of $\mathbf{C}'\mathbf{M}_a\mathbf{C}$ are the elements of the diagonal matrix $\mathbf{\Lambda}$ and the matrix $\mathbf{\Gamma}$ can be obtained by $\mathbf{\Gamma} = (\mathbf{C}')^{-1}\mathbf{Q}$, where $\mathbf{Q}$ is the matrix of corresponding eigenvectors of $\mathbf{C}'\mathbf{M}_a\mathbf{C}$.

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