The problem

Assume that for all $(x, A) \in \mathsf{X} \times \mathcal{X}$, we have $P_1(x, A \setminus \{x\}) \geq P_2(x, A \setminus \{x\})$. According to Tierney's result, if both Markov kernels are π-reversible, the asymptotic variance of $\frac{1}{n} \sum_{i=1}^{n} f(X_n^1)$ is smaller than that of $\frac{1}{n} \sum_{i=1}^{n} f(X_n^2)$, where $\left\{X_n^1 ; n \in \mathbb{N}\right\}$ and $\left\{X_n^2 ; n \in \mathbb{N}\right\}$ are Markov chains with transitions P_1 and P_2, respectively.

I wonder if this result still holds without the reversibility assumption.

Some reflections

The assumption can be interpreted as follows: there exists a (normalized) kernel Q such that

- $Q(x, \{x\}) = 0$
- $P_2(x, A) = \bar{\beta}(x) Q(x, A) + \beta(x) \delta_x(A)$
- $P_1(x, A) = \bar{\beta}(x) Q(x, A) + \beta(x) R(x, A)$,
  with $\bar{\beta} = 1 - \beta$.

Since $\pi P_2 = \pi$, it follows that the measure $\bar{\beta}\, \mathrm{d}\pi$ is invariant for Q. Together with $\pi P_1 = \pi$, this implies that $\beta\, \mathrm{d}\pi$ is invariant for R.

Now, define:

- $\bar{P}_1(x, u; \mathrm{d}x'\, \mathrm{d}u') = \left[\mathbf{1}_{\bar{C}}(x, u) Q(x, \mathrm{d}x') + \mathbf{1}_C(x, u) \delta_x(\mathrm{d}x')\right] \mathbf{1}_{[0,1]}(u') \mathrm{d}u'$
- $\bar{P}_2(x, u; \mathrm{d}x'\, \mathrm{d}u') = \left[\mathbf{1}_{\bar{C}}(x, u) Q(x, \mathrm{d}x') + \mathbf{1}_C(x, u) R(x, \mathrm{d}x')\right] \mathbf{1}_{[0,1]}(u') \mathrm{d}u'$,

where $C = \{(x, u) \,; u \leq \beta(x)\}$ and $\bar{C}$ is the complement of C. In what follows, we use the generic notation $y = (x, u) \in \mathsf{Y}$ with $\mathsf{Y} = \mathsf{X} \times [0, 1]$. Note that P_1 and P_2 are Markov chains on $\mathsf{Y} \times \mathcal{Y}$ and we let $\left\{Y_n^1 ; n \in \mathbb{N}\right\}$, resp $\left\{Y_n^2 : n \in \mathbb{N}\right\}$, be a Markov chain with transition kernel P_1, resp. P_2. Write σ^k the k-th visit to the set C, with the convention that $\sigma^1 = \sigma_C$ is the first return time to the set C.