lecture 08, functional completeness

phil1012 introductory logic

overview

this lecture

- some reflection on PL
- functional completeness: the idea that a given set of connectives is able to express all possible truth functions
- are the connectives of PL functionally complete?

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - explain what it means for a set of connectives to be functionally complete
  - define one set of connectives in terms of another set of connectives
  - determine whether a given set of connectives is functionally complete

required reading

- section 6.6 of chapter 6

functional completeness

functional completeness

- are our five connectives, in some sense, sufficient?
- are there formulas with truth conditions that cannot be constructed using our five connectives?
- are there any truth functions that cannot be expressed by some combination of our five connectives?

- consider the equivalence of $(A \rightarrow B)$ and $(\lnot A \lor B)$

A	B	$(A \to B)$	$(\lnot A \lor B)$
T	T	T	T
F	T	T	T
T	F	F	F
F	F	T	T

- we can construct a formula with the truth conditions of $(A \rightarrow B)$ using $\lnot$ and $\lor$
- we can express the truth function expressed by $\rightarrow$ using $\lnot$ and $\lor$
- so there's a sense in which we don't really need $\rightarrow$
- we could just use $\lnot$ and $\lor$ instead of $\rightarrow$

- but now consider the truth table for a connective not included in our five, $\veebar$

A	B	$(A \veebar B)$
T	T	F
F	T	T
T	F	T
F	F	F

- can we construct a formula with the same truth conditions using connectives taken from our original five?

- it turns out that we can

A	B	$(A \veebar B)$	$\lnot (A \leftrightarrow B)$
T	T	F	F
F	T	T	T
T	F	T	T
F	F	F	F

- so there a sense in which we don't need to add $\veebar$
- we can already express with our five connectives what we can express with $\veebar$

- can we do what we just did for $\rightarrow$ and $\veebar$ for any possible connective?
- can we express any possible truth function using just our five connectives?

- it turns out that we can
- with its five connectives, $\lnot$, $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$, PL has the resources to construct a formula with any truth conditions whatsoever
- for any possible truth table, there is a formula of PL with that truth table
- our five connectives can express any possible truth function
- we call this feature, **functional completeness**

- a set of connectives is **functionally complete** if we can define all possible connectives from the connectives in that set
- let's prove it!
- first we'll get clearer on what it means to define one connective in terms of other connectives
- then we'll get clearer on the space of possible connectives
- then we'll prove that the set of connectives use in PL is functionally complete

defining one connective in terms of others

defining one connective in terms of others

- we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives
- recall that two forms are **equivalent** if and only if they have the same truth value on every row of the truth table
- lets consider some examples

- the connective →\rightarrow can be defined in terms of ¬\lnot and ∨\lor

α\alpha	β\beta	(α(\alpha→\rightarrowβ)\beta)	(¬(\lnotα\alpha∨\lorβ)\beta)
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

- the connective ↔\leftrightarrow can be defined in terms of →\rightarrow and ∧\land

α\alpha	β\beta	(α(\alpha↔\leftrightarrowβ)\beta)	((α((\alpha→\rightarrowβ)∧(β\beta)\land(\beta→\rightarrowα))\a
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- the connective ∧\land can be defined in terms of ∨\lor and ¬\lnot

α\alpha	β\beta	(α(\alpha∧\landβ)\beta)	¬(¬\lnot(\lnotα\alpha∨\lor¬\lnotβ)\beta)
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

- the connective ∨\lor can be defined in terms of ∧\land and ¬\lnot

α\alpha	β\beta	(α(\alpha∨\lorβ)\beta)	¬(¬\lnot(\lnotα\alpha∧\land¬\lnotβ)\beta)
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

- so, we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives

the range of possible connectives

the range of possible connectives

- we want to show that we all possible connectives can be defined in terms of the five connectives of PL
- in order to do so we need to understand what all the possible connectives are
- there are zero place connectives, one place connectives, two place connectives, three place connectives, and so on

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- possible zero place connectives

⊤**\top**	⊥**\bot**
T	F

- possible one place connectives

α\alpha				
T	T	T	F	F
F	T	F	T	F

- we use ①$_1$ for the first one-place connective
- the number in the circle represents the number of places of the connective
- the subscript represents the number of the connective

- possible one place connectives

α\alpha	①$_1$	①$_2$	①$_3$	①$_4$
T	T	T	F	F
F	T	F	T	F

- possible one place connectives

α\alpha	①$_1$	①$_2$	¬\lnot	①$_4$
T	T	T	F	F
F	T	F	T	F

- ①$_3$ is ¬\lnot

- possible two place connectives

α\alpha	β\beta	②$_{1}$	②$_{2}$	②$_{3}$	②$_{4}$	②$_{5}$	②$_{6}$	②$_{7}$	②$_{8}$	②$_{9}$	②$_{10}$	②$_{11}$	②$_{12}$	②$_{13}$	②$_{14}$	②$_{15}$	②$_{16}$
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

- possible two place connectives

α\alpha	β\beta	②$_{1}$	∨\lor	②$_{3}$	②$_{4}$	→\rightarrow	②$_{6}$	↔\leftrightarrow	∧\land	②$_{9}$	②$_{10}$	②$_{11}$	②$_{12}$	②$_{13}$	②$_{14}$	②$_{15}$	②$_{16}$
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

- there are also three place connectives
- they look like this: ③$_{1}$
- they make propositions like this: ③$_{1}$(P,Q,R)(P,Q,R)
- there are 256 three place connectives!

- there are also four place connectives
- they look like this: ④$_{1}$
- they make propositions like this: ④$_{1}$(P,Q,R,S)(P,Q,R,S)
- there are 65,536 four place connectives!!

- and so on . . .

- obviously, we won't be proving that we can define any connective in terms of our five one by one!
- we better find a method that can obviously be extended to show that we can define any connective in terms of our five

defining any connective using ¬\lnot, ∧\land, and ∨\lor

defining any connective using ¬\lnot, ∧\land, and ∨\lor

- okay, we are finally ready to show that any possible connective can be defined in terms of our five
- in fact, we will show that any possible connective can be defined in terms of ¬\lnot, ∧\land, and ∨\lor
- obviously we aren't going to define them one by one
- we'll define the zero-place connectives first, one by one
- then we will develop a method for defining any n-place connective in terms of ¬\lnot, ∧\land, and ∨\lor

- the zero-place connectives ⊤\top and ⊥\bot can be defined in terms of the connectives {¬\lnot, ∧\land, ∨\lor}

α\alpha	⊤\top	(α∨¬α)(\alpha \lor \lnot \alpha)	⊥\bot	(α∧¬α)(\alpha \land \lnot \alpha)
T	T	T	F	F
F	T	T	F	F

- so there's a sense in which we don't need the zero place connectives
- remember that to define one connective in terms of others, it is enough to show that formulas using them are equivalent

- now we show that any n-place connective can be defined in terms of {¬\lnot, ∧\land, ∨\lor} by the following procedure

- take some n-place connective. call it '*'
- and take any function from n truth values to truth values

α\alpha	β\beta	(α(\alpha * β)\beta)
T	T	T
T	F	F
F	T	T
F	F	F

- now take the conjunctions which 'describe' the rows in which (α\alpha * β\beta) is true
- in this case: (¬α∧β)(\lnot \alpha \land \beta), (α∧β)(\alpha \land \beta)

α\alpha	β\beta	(α(\alpha * β)\beta)
T	T	T
T	F	F
F	T	T
F	F	F

- in other words
  - if α\alpha is true on the row on which (α(\alpha * β)\beta) is true, then make α\alpha the first conjunct
  - if it is false, make ¬α\lnot \alpha the first conjunct
  - do the same for β\beta, and then do the same for each row on which (α\alpha * β\beta) is true
- by this method we get: (¬α∧β)(\lnot \alpha \land \beta), (α∧β)(\alpha \land \beta)

- now form a disjunction from the conjunctions you got from the previous step: ((¬α∧β)∨(α∧β))((\lnot \alpha \land \beta) \lor (\alpha \land \beta))
- and you are done!

- to see that you are done, you can put the disjunction into the table, and you will see that it is equivalent to the formula using '*'

α	β	$(\alpha * \beta)$	$(\lnot \alpha \land \beta) \lor (\alpha \land \beta)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	F	F

- any n-place connective can be defined in terms of $\{\lnot, \land, \lor\}$ by this procedure!
- it should be obvious that for any truth table whatsoever, we only need $\lnot$, $\land$, and $\lor$ to carry out the procedure of describing the row on which some proposition is true and making a disjunction of these conjunctions
- by this method we prove that $\{\lnot, \land, \lor\}$ is a functionally complete set of connectives!

functionally complete sets of connectives

functionally complete sets of connectives

- okay, we just proved that $\{\lnot, \land, \lor\}$ is a functionally complete set of connectives
- now let's consider the general case of functionally complete sets of connectives
- let's do so by considering some consequences of the fact that $\{\lnot, \land, \lor\}$ is a functionally complete set of connectives

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- **fact**: the set of connectives $\{\lnot, \land, \lor\}$ is functionally complete
- to show that some set of connectives is functionally complete, it suffices to show that $\lnot$, $\land$, and $\lor$, can be defined using members of that set
  - if $\{\lnot, \land, \lor\}$ is functionally complete, and $\lnot$, $\land$, and $\lor$ can be defined in terms of some other set of connectives, then that set must be functionally complete too

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- **fact**: the set of connectives $\{\lnot, \land\}$ is functionally complete
- **proof**: $\lnot$, $\land$, and $\lor$ can be defined in terms of $\lnot$ and $\land$ alone, and $\{\lnot, \land, \lor\}$ is functionally complete

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- **fact**: the set of connectives $\{\lnot, \lor\}$ is functionally complete
- **proof**: $\lnot$, $\land$, and $\lor$ can be defined in terms of $\lnot$ and $\lor$ alone, and $\{\lnot, \land, \lor\}$ is functionally complete

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- **fun fact**: $\{②_9\}$ is a functionally complete set of connectives

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- to sum up, then
- to show that some set of connectives is functionally complete, it suffices to show that $\lnot$, and either $\land$ or $\lor$, can be defined using members of that set (for then you can rely on the proof that $\{\lnot, \land, \lor\}$ is functionally complete above)
- to show that a set of connectives is not functionally complete, we need to show that there is *some* connective that cannot be defined in terms of those in the set

wrapping up

this lecture

- with its five connectives, $\lnot$, $\land$, $\lor$, $\rightarrow$, and $\leftrightarrow$, PL has the resources to construct a formula with any truth conditions whatsoever
- in other words, the set of connectives in PL are **functionally complete**
- you (probably) will not be required to prove that $\{\lnot, \land, \lor\}$ is functionally complete
- but you will be required to prove that some given set of connectives is functionally complete
- to do so, you need only show that $\lnot$, $\land$ and $\lor$ can be defined in terms of the connectives you are given

next lectures

- lecture 09, issues in translation: conjunction
- lecture 10, trees for PL