

lecture 08, functional completeness

phil1012 introductory logic

overview

this lecture

- some reflection on PL
- functional completeness: the idea that a given set of connectives is able to express all possible truth functions
- are the connectives of PL functionally complete?

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
 - explain what it means for a set of connectives to be functionally complete
 - define one set of connectives in terms of another set of connectives
 - determine whether a given set of connectives is functionally complete

required reading

- section 6.6 of chapter 6

functional completeness

functional completeness

- are our five connectives, in some sense, sufficient?
- are there formulas with truth conditions that cannot be constructed using our five connectives?
- are there any truth functions that cannot be expressed by some combination of our five connectives?

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- consider the equivalence of $(A \rightarrow B)$ and $(\neg A \vee B)$

A	B	$(A \rightarrow B)$	$(\neg A \vee B)$
T	T	T	T
F	T	T	T
T	F	F	F
F	F	T	T

- we can construct a formula with the truth conditions of $(A \rightarrow B)$ using \neg and \vee
- we can express the truth function expressed by \rightarrow using \neg and \vee
- so there's a sense in which we don't really need \rightarrow
- we could just use \neg and \vee instead of \rightarrow

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- but now consider the truth table for a connective not included in our five, \leftrightarrow

A	B	$(A \leftrightarrow B)$
T	T	T
F	T	F
T	F	F
F	F	T

- can we construct a formula with the same truth conditions using connectives taken from our original five?

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- it turns out that we can

A	B	$(A \leftrightarrow B)$	$(\neg(A \rightarrow \neg B))$
T	T	T	F
F	T	F	T
T	F	F	T
F	F	T	F

- so there a sense in which we don't need to add \leftrightarrow
- we can already express with our five connectives what we can express with \leftrightarrow

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- can we do what we just did for \rightarrow and \leftrightarrow for any possible connective?
 - can we express any possible truth function using just our five connectives?

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- it turns out that we can
 - with its five connectives, \neg , \wedge , \vee , \rightarrow , and \leftrightarrow , PL has the resources to construct a formula with any truth conditions whatsoever
 - for any possible truth table, there is a formula of PL with that truth table
 - our five connectives can express any possible truth function
 - we call this feature, **functional completeness**

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- a set of connectives is **functionally complete** if we can define all possible connectives from the connectives in that set
 - let's prove it!
 - first we'll get clearer on what it means to define one connective in terms of other connectives
 - then we'll get clearer on the space of possible connectives
 - then we'll prove that the set of connectives use in PL is functionally complete

defining one connective in terms of others

defining one connective in terms of others

- we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives
- recall that two forms are **equivalent** if and only if they have the same truth value on every row of the truth table
- lets consider some examples

- the connective \rightarrow can be defined in terms of \neg and \vee

α	β	$(\alpha \rightarrow \beta)$	$(\neg \alpha \vee \beta)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- the connective \leftrightarrow can be defined in terms of \rightarrow and \wedge

α	β	$(\alpha \leftrightarrow \beta)$	$((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

- the connective \wedge can be defined in terms of \vee and \neg

α	β	$(\alpha \wedge \beta)$	$(\neg(\neg \alpha \vee \neg \beta))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

- the connective \vee can be defined in terms of \wedge and \neg

α	β	$(\alpha \vee \beta)$	$(\neg(\neg \alpha \wedge \neg \beta))$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

- so, we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives

the range of possible connectives

the range of possible connectives

- we want to show that we all possible connectives can be defined in terms of the five connectives of PL
- in order to do so we need to understand what all the possible connectives are
- there are zero place connectives, one place connectives, two place connectives, three place connectives, and so on

- possible zero place connectives

\top	\bot
T	F

- possible one place connectives

α				
T	T	T	F	F
F	T	F	T	F

- we use $\textcircled{1}_1$ for the first one-place connective
- the number in the circle represents the number of places of the connective
- the subscript represents the number of the connective

- possible one place connectives

α	$\textcircled{1}_1$	$\textcircled{1}_2$	$\textcircled{1}_3$	$\textcircled{1}_4$
T	T	T	F	F
F	T	F	T	F

- possible one place connectives

α	$\textcircled{1}_1$	$\textcircled{1}_2$	\neg	$\textcircled{1}_4$
T	T	T	F	F
F	T	F	T	F

- $\textcircled{1}_3$ is \neg

- possible two place connectives

$\alpha \backslash \beta$	β	\neg	\vee	\wedge	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow
T	T	F	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	F
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T
F	F	T	F	F	F	F	F	F	F	F	F	F	F	F	F	F

- possible two place connectives

$\alpha \backslash \beta$	β	\neg	\vee	\wedge	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow	\rightarrow	\leftrightarrow
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T
F	F	T	F	F	F	F	F	F	F	F	F	F	F	F	F	F

- there are also three place connectives
- they look like this: \mathcal{C}_1
- they make propositions like this: $\mathcal{C}_1(P, Q, R)$
- there are 256 three place connectives!

- there are also four place connectives
- they look like this: \mathcal{C}_1
- they make propositions like this: $\mathcal{C}_1(P, Q, R, S)$
- there are 65,536 four place connectives!!

- and so on . . .

- obviously, we won't be proving that we can define any connective in terms of our five one by one!
- we better find a method that can obviously be extended to show that we can define any connective in terms of our five

defining any connective using \neg , \wedge , and \vee

defining any connective using \neg , \wedge , and \vee

- okay, we are finally ready to show that any possible connective can be defined in terms of our five
- in fact, we will show that any possible connective can be defined in terms of \neg , \wedge , and \vee
- obviously we aren't going to define them one by one
- we'll define the zero-place connectives first, one by one
- then we will develop a method for defining any n-place connective in terms of \neg , \wedge , and \vee

- the zero-place connectives \top and \bot can be defined in terms of the connectives $\{\neg, \wedge, \vee\}$

$\alpha \backslash \beta$	\top	\bot
T	T	F
F	T	F

- so there's a sense in which we don't need the zero place connectives
- remember that to define one connective in terms of others, it is enough to show that formulas using them are equivalent

- now we show that any n-place connective can be defined in terms of $\{\neg, \wedge, \vee\}$ by the following procedure

- take some n-place connective. call it \star
- and take any function from n truth values to truth values

$\alpha \backslash \beta$	β	$(\alpha \star \beta)$
T	T	T
T	F	F
F	T	T
F	F	F

- now take the conjunctions which 'describe' the rows in which $(\alpha \star \beta)$ is true
- in this case: $(\neg \alpha \wedge \beta)$, $(\alpha \wedge \neg \beta)$

$\alpha \backslash \beta$	β	$(\alpha \star \beta)$
T	T	T
T	F	F
F	T	T
F	F	F

- in other words
 - if α is true on the row on which $(\alpha \star \beta)$ is true, then make α the first conjunct
 - if it is false, make $\neg \alpha$ the first conjunct
 - do the same for β , and then do the same for each row on which $(\alpha \star \beta)$ is true
- by this method we get: $(\neg \alpha \wedge \beta) \vee (\alpha \wedge \neg \beta)$

- now form a disjunction from the conjunctions you got from the previous step: $((\neg \alpha \wedge \beta) \vee (\alpha \wedge \neg \beta))$
- and you are done!

- to see that you are done, you can put the disjunction into the table, and you will see that it is equivalent to the formula using `'*`

$\alpha \backslash \beta$	β	$\alpha \backslash \beta$	$(\neg \alpha \wedge \beta) \vee (\alpha \wedge \neg \beta)$	$(\neg \alpha \wedge \beta) \vee (\alpha \wedge \neg \beta) \vee (\alpha \wedge \beta)$
T	T	T	T	T
T	F	F	F	F
F	T	T	T	T
F	F	F	F	F

- any n-place connective can be defined in terms of $\{\neg, \wedge, \vee\}$ by this procedure!
- it should be obvious that for any truth table whatsoever, we only need \neg , \wedge , and \vee to carry out the procedure of describing the row on which some proposition is true and making a disjunction of these conjunctions
- by this method we prove that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives!

functionally complete sets of connectives

functionally complete sets of connectives

- okay, we just proved that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives
- now let's consider the general case of functionally complete sets of connectives
- let's do so by considering some consequences of the fact that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives

- **fact:** the set of connectives $\{\neg, \wedge, \vee\}$ is functionally complete
- to show that some set of connectives is functionally complete, it suffices to show that \neg , \wedge , and \vee can be defined using members of that set
 - if $\{\neg, \wedge, \vee\}$ is functionally complete, and \neg , \wedge , and \vee can be defined in terms of some other set of connectives, then that set must be functionally complete too

- **fact:** the set of connectives $\{\neg, \wedge\}$ is functionally complete
- **proof:** \neg , \wedge , and \vee can be defined in terms of \neg and \wedge alone, and $\{\neg, \wedge, \vee\}$ is functionally complete

- **fact:** the set of connectives $\{\neg, \vee\}$ is functionally complete
- **proof:** \neg , \wedge , and \vee can be defined in terms of \neg and \vee alone, and $\{\neg, \wedge, \vee\}$ is functionally complete

- **fun fact:** $\{\oplus\}$ is a functionally complete set of connectives

- to sum up, then
- to show that some set of connectives is functionally complete, it suffices to show that \neg , and either \wedge or \vee , can be defined using members of that set (for then you can rely on the proof that $\{\neg, \wedge, \vee\}$ is functionally complete above)
- to show that a set of connectives is not functionally complete, we need to show that there is *some* connective that cannot be defined in terms of those in the set

wrapping up

this lecture

- with its five connectives, \neg , \wedge , \vee , \rightarrow , and \leftrightarrow , PL has the resources to construct a formula with any truth conditions whatsoever
- in other words, the set of connectives in PL are **functionally complete**
- you (probably) will not be required to prove that $\{\neg, \wedge, \vee\}$ is functionally complete
- but you will be required to prove that some given set of connectives is functionally complete
- to do so, you need only show that \neg , \wedge and \vee can be defined in terms of the connectives you are given

next lectures

- lecture 09, issues in translation: conjunction
- lecture 10, trees for PL