# lecture 06, uses of truth tables

phil1012 introductory logic

# overview

## this lecture

- the uses of truth tables
- constructing truth tables
- using truth tables to test for various logical properties

## learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - construct truth tables for arguments, single propositions, pairs of propositions and sets of propositions
  - use truth tables to determine whether an argument is valid, providing a counterexample if not
  - use truth tables to determine whether an proposition is a tautology, a contradiction, or neither
  - use truth tables to determine whether pairs of formulas are jointly satisfiable, equivalent, contradictory, or contraries.
  - use truth tables to determine whether sets of propositions are jointly satisfiable

## required reading

- all of chapter 4

# truth tables for complex propositions

## truth tables for complex propositions

- sometimes we want to know the truth value of a complex proposition for every possible assignment of truth values to basic propositions.
- or this purpose we construct a **truth table** for the complex proposition.
- each row of a truth table corresponds to a possible assignment of truth values to basic propositions

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- we set up our table so that the rows in the columns on the left correspond to the possible assignments of truth values to basic propositions
- and, we "copy over" those values to the basic propositions on the right for each row
- we proceed as in the case of a single assignment, calculating the truth value of the more complex propositions in terms of the truth value of their constituents

- until we have the values of the complex proposition for each possible assignment of values to the basic propositions

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- let's work through a simple example involving $\neg(P \wedge \neg Q)$

# truth tables for multiple propositions

## truth tables for multiple propositions

- we can also construct tables for multiple propositions
- we calculate the possible truth values for every possible assignment of values to basic propositions
- for any basic proposition which occurs in any formula

| $P$ | $Q$ | $(P \rightarrow Q)$ | $\neg(P \wedge \neg P)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | T |
| F | F | T | T |

# general points about drawing truth tables

## general points about drawing truth tables

- one distinct wff variable
- column 1 has one T and one F

| $\alpha$ | ... |
|---|---|
| T | F |
| F | T |

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- two distinct wff variables
- the previous matrix occurs in the top right-hand corner of the matrix

| $\alpha$ | $\beta$ | ... |
|---|---|---|
| T | T | ... |
| T | F | ... |
| F | T | ... |
| F | F | ... |

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- three distinct wff variables
- the previous matrix occurs in the top right-hand corner of the matrix

| $\alpha$ | $\beta$ | $\gamma$ | ... |
|---|---|---|---|
| T | T | T | ... |
| T | T | F | ... |
| T | F | T | ... |
| T | F | F | ... |
| F | T | T | ... |

| | | | |
|---|---|---|---|
| F | T | F | … |
| F | F | T | … |
| F | F | F | … |

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- four distinct wff variables
- the previous matrix occurs in the top right-hand corner of the matrix

| $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | … |
|---|---|---|---|---|
| T | T | T | T | … |
| T | T | T | F | … |
| T | T | F | T | … |
| T | T | F | F | … |
| T | F | T | T | … |
| T | F | T | F | … |
| T | F | F | T | … |
| T | F | F | F | … |
| F | T | T | T | … |
| F | T | T | F | … |
| F | T | F | T | … |
| F | T | F | F | … |
| F | F | T | T | … |
| F | F | T | F | … |
| F | F | F | T | … |
| F | F | F | F | … |

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- in general …

| number of basic components | number of rows |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| … | … |
| $n$ | $2^n$ |

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- the right-most column always alternates T, F, and so on
- then TT, FF, and so on
- then TTTT, FFFF, and so on
- then TTTTTTTT, FFFFFFFF, and so on
- always doubling the Ts and Fs
- basic propositions are sorted alphabetically

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- for the problem sets and exam . . .
  - make sure you set up your tables correctly
  - keep your rows and columns neat
  - show the truth values of all sub-formulas

# using truth tables to test for validity

## using truth tables to test for validity

- recall the definition of validity . . .
- an argument is **valid** if and only if, in virtue of the form of the argument, it is impossible for its conclusion to be false if its premises are all true
- in PL, a possibility is an assignment of truth values to the basic propositions
- a joint truth table for a group of propositions shows all of the possible ways of making the propositions or true or false

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- the question we put to a truth table in order to determine whether an argument is valid it this:
  - is there a row on which all of the premises are true and the conclusion is false?
- if so, the argument is invalid. if not, then it is valid.

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- is this argument valid?

| | |
|---|---|
| **P1.** | Either John is not a philosopher or he drinks |
| **P2.** | John is a philosopher |
| **C1.** | John drinks |

- we begin by translating the argument into PL
- here's our glossary:
  - $P$: John is a philosopher
  - $D$: John drinks
- here's our translation:
  - $(\neg P \vee D), P \therefore D$
- now consider the joint truth table for the premises and the conclusion.

| $P$ | $D$ | $(\neg P \vee D)$ | $P$ | $D$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | F | F |

---

- is there a row on which the premises are true and the conclusion false?
- if so, the argument is invalid. if not, then it is valid.

---

- is this argument valid?

| | |
|---|---|
| **P1.** | Either John is a philosopher or he drinks |
| **P2.** | John drinks |
| **C1.** | John is a philosopher |

---

- we begin by translating the argument into PL
- here's our glossary:
  - $P$: John is a philosopher
  - $D$: John drinks
- here's our translation:
  - $(P \vee D), D \therefore P$
- now consider the joint truth table for the premises and the conclusion

| $P$ | $D$ | $(P \vee D)$ | $D$ | $P$ |
|---|---|---|---|---|
| T | T | T | T | T |

| | | | | |
|---|---|---|---|---|
| T | F | T | F | T |
| F | T | T | T | F |
| F | F | F | F | F |

- is there a row on which the premises are true and the conclusion false?
- if so, the argument is invalid. if not, then it is valid.

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- if an argument is invalid, then it is possible for all the premises to be true and for the conclusion to be false.
- a **counterexample** is an assignment of truth values to basic propositions on which the premises are true and the conclusion is false.
- look back at the matrix. $P$ is true and $D$ is false on the relevant row.

| $P$ | $D$ | $(\neg P \vee D)$ | $D$ | $P$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | F | T |
| **F** | **T** | **T** | **T** | **F** |
| F | F | F | F | F |

- we write a counterexample as follows:
  - $P$: F, $D$: T

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- what if there is a row where both premises are true and the conclusion is also true?

| $P$ | $D$ | $(\neg P \vee D)$ | $D$ | $P$ |
|---|---|---|---|---|
| **T** | **T** | **T** | **T** | **T** |
| T | F | T | F | T |
| F | T | T | T | F |
| F | F | F | F | F |

- this does not show that the argument is valid. (recall the definition of validity)

# using truth tables to test single propositions

## using truth tables to test single propositions

- we can use truth tables to determine whether a single propositions is:
  - a tautology
  - a contradiction
  - a non-contradiction
  - satisfiable

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- intuitively, a proposition is a tautology if it *must* be true
  - e.g. the ball is round or the ball is not round
- a proposition is a **tautology** if and only if it is true on every row of its truth table

| $P$ | $(P \vee \neg P)$ |
|---|---|

| | |
|---|---|
| T | T |
| F | T |

- we can use a truth table to determine whether a proposition is a tautology: we check whether it is true on every row of its truth table

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- intuitively, a proposition is a contradiction if it *cannot* be true
  - e.g. the ball is round and the ball is not round
- a proposition is a **contradiction** if and only if it is false on every row of its truth table

| $P$ | $(P \wedge \neg P)$ |
|---|---|
| T | F |
| F | F |

- we can use a truth table to determine whether a proposition is a contradiction: we check whether it is false on every row of its truth table

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- intuitively, a proposition is satisfiable if it *can* be true
- a proposition is **satisfiable** if and only if it is true in some or all rows of its truth table

| $P$ | $(P \to \neg P)$ |
|---|---|
| T | F |
| F | T |

- we can use a truth table to determine whether a proposition is satisfiable: we check whether it is true on some row of its truth table

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- contradictions and satisfiable propositions are opposites
  - if a proposition is not a contradiction, then it is satisfiable
  - if a proposition is not satisfiable, then it is a contradiction
- contradictions and tautologies are not opposites
  - a proposition may be neither a contradiction nor a tautology
- satisfiable propositions and tautologies are not opposites
  - a tautology is a satisfifiable proposition

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| values in truth table | type of proposition |
|---|---|
| T in every row | **tautology** |
| F in every row | **contradiction** |
| T in some or all rows | **satisfiable proposition** |
| F in some or all rows | **nontautology** |

# using truth tables to test pairs of propositions

## using truth tables to test pairs of propositions

- we can use truth tables to determine whether a pair of propositions are:
  - equivalent or not equivalent

- jointly satisfiable or not jointly satisfiable
- contrary or contradictory

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- intuitively, a pair of propositions are equivalent if it is not possible for one proposition to be true without the other being true and vice versa
- a pair of propositions are **equivalent** if and only if they have the same value in every row of the truth table

| $P$ | $Q$ | $(P \to Q)$ | $(\neg P \vee Q)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

- we can use a truth table to determine whether a pair of propositions are equivalent: we check whether they have the same value on every row of the truth table

---

- intuitively, a pair of propositions are jointly satisfiable if they can both be true together
- a pair of propositions are **jointly satisfiable** if and only if there is some row of the truth table in which they are both true

| $P$ | $Q$ | $\neg(P \to Q)$ | $(P \vee Q)$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | T |
| F | F | F | F |

- we can use a truth table to determine whether a pair of propositions are jointly satisfiable: we check whether there is some row of the truth table on which they are both true

---

- intuitively, a pair of propositions are jointly unsatisfiable if they cannot both be true together
- a pair of propositions are **jointly unsatisfiable** if and only if there is some row of the truth table in which they are both true

| $P$ | $Q$ | $\neg(P \vee Q)$ | $(P \vee Q)$ |
|---|---|---|---|
| T | T | F | T |
| T | F | F | T |
| F | T | F | T |
| F | F | T | F |

- we can use a truth table to determine whether a pair of propositions are jointly unsatisfiable: we check whether they have the same value on every row of the truth table
- note: as the example above shows, two satisfiable propositions may be jointly unsatisfiable

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- when a pair or propositions are jointly unsatisfiable they can be classified either as contradictories or contraries
- let’s consider the two cases now

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- intuitively, a pair of propositions are contradictory when if one is

true the other is false and vice versa
- a pair of propositions are **contradictory** if and only if they are jointly unsatisfiable and there is no row of the truth table in which they are both false

| $P$ | $Q$ | $(P \wedge Q)$ | $\neg(P \wedge Q)$ |
|---|---|---|---|
| T | T | T | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

- we can use a truth table to determine whether a pair of propositions are contradictory: we check whether their truth values are opposed on every row of the truth table

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- if a pair of propositions are contradictory, we can always infer from the falsity of one proposition to the truth of the other
- but this will not always be the case when a pair of propositions are jointly unsatisfiable
- why?
- they might both be false

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- intuitively, a pair of propositions are (merely) contraries if they are jointly unsatisfiable, but aren't contradictory: they can't be true together, but they can be false together
- a pair of propositions are **contraries** if and only if they are jointly unsatisfiable and there is some row in which both are false

| $P$ | $Q$ | $(P \wedge \neg P)$ | $\neg(P \wedge Q)$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

- we can use a truth table to determine whether a pair of propositions are contraries: we check whether there is no row of the truth table on which they are both true and some row on which they are both false

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| **values in truth table** | **type of propositions** |
|---|---|
| same value in every row | **equivalent** |
| some row in which both T | **jointly satisfiable** |
| no row in which both T | **jointly unsatisfiable** |
| j-unsat and no row in which both F | **contradictory** |
| j-unsat and some row in which both F | **contraries** |

# using truth tables to test sets of propositions

## using truth tables to test sets of propositions

- equivalence and joint satisfiability can be generalised from pairs of propositions to sets of propositions in obvious ways
- I will leave equivalence as an exercise for you
- but let us look at the satisfiability and unsatisfiability of sets of

propositions

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- intuitively, a set of propositions is satisfiable if they can all be true together
- a set of propositions is **satisfiable** if and only if there is some row in the truth table on which all the members of the set are true
- intuitively, a set of propositions is unsatisfiable if they cannot all be true together
- set of propositions is **unsatisfiable** if and only if there is no row in the truth table on which all the members of the set are true

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- we can use truth tables to test for the satisfiability of sets of propositions
- suppose we want to know whether {$( P \to Q)$ , $( P \wedge Q)$ , $( P \vee Q)$ } is satisfiable
- we draw up a joint truth table as follows:

| $P$ | $Q$ | $( P \to Q)$ | $( P \wedge Q)$ | $( P \vee Q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | F |

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- now, the notion of the satisfiability of a set of propositions will play a central role in the course going forward
- notice that the truth table not only tells us that the set of propositions are satisfiable, it also provides assignments on which the set of propositions are satisfiable
- as noted earlier, truth tables get unwieldy very quickly: what if we have 5 basic propositions?
- might there be a more efficient method for testing the satisfiability of sets of propositions?
- might there be a more efficient search procedure for assignments on which the propositions are satisfiable?
- yes: truth trees

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| **values in truth table** | **set of propositions** |
|---|---|
| some row on which all T | **satisfiable** |
| no row on which all T | **unsatisfiable** |

# wrapping up

## this lecture

- truth tables for complex propositions and multiple propositions
- using truth tables to test for various logical properties
- single propositions
  - tautologies
  - contradictions
  - satisfiable propositions
  - noncontradictions
- pairs of propositions
  - equivalence
  - joint satisfiability
  - joint unsatisfiability

    - contradictories
    - contraries
- sets of propositions
    - joint satisfiability
    - joint unsatisfiability

## next lecture

- lecture 07, validity and logical form