

lecture 14, syntax of MPL

phil1012 introductory logic

overview

this lecture

- the syntax of MPL
- formal specification of the vocabulary and syntax of MPL
- how formulas of MPL are constructed
- the notions of logical operators, the scope of a quantifier, free and bound variables, open and closed formulas

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
 - identify well-formed formulas of MPL
 - construct construction tables for formulas of MPL
 - identify the main operator of a formula of MPL
 - identify the scope of a quantifier
 - identify free and bound variables and open and closed formulas of MPL

required reading

- section 8.4 of chapter 8

syntax of MPL

syntax of MPL

- we can give a precise specification of the syntax of MPL as we did for PL.

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- the **symbols** of MPL are:
 - names:
 - a, b, c, \dots, t
 - variables:
 - $x, y, z, u, v, w, x, y, z, u, v, w$
 - predicates:
 - $A, B, C, \dots, X, Y, Z, A, B, C, \dots, X, Y, Z$
 - five connectives:
 - $\neg, \forall, \exists, \rightarrow, \leftrightarrow$
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- the **symbols** of MPL (continued) are:
 - two quantifier symbols:
 - \forall, \exists
 - two punctuation symbols (parentheses):
 - $(,)$

- **terms** are defined as follows:

- a name is a term
- a variable is a term
- nothing else is a term

- **wffs** of MPL are defined as follows:

- (3i) where \underline{P} is a predicate and \underline{t} is a term, the following is a wff:
 - $\underline{P}(\underline{t})$
- (3ii) where α and β are wffs and \underline{x} is a variable, the following are wffs:
 - $\neg \alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $\forall \underline{x} \alpha$, $\exists \underline{x} \alpha$
- nothing else is a wff

constructing wffs

constructing wffs

- given this syntax for MPL we can show how any well formed formula of MPL is constructed.
- suppose we want to construct
 - $(\forall x (Rx \rightarrow \exists x Px))$
- we might construct it as follows

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	/(3i)
2	$PxPx$	/(3i)

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1	$RxRx$	/(3i)
2	$PxPx$	/(3i)
3	$\forall x Rx$	1, /(3ii \forall)
4	$\exists x Px$	2, /(3ii \exists)

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	/(3i)
2	$PxPx$	/(3i)
3	$\forall x Rx$	1, /(3ii \forall)
4	$\exists x Px$	2, /(3ii \exists)
5	$(\forall x Rx \rightarrow \exists x Px)$	3,4 /(3ii \rightarrow)

- a **logical operator** is a connective or a quantifier
- the main operator (c.f. main connective) is the last operator added in the construction of the wff
- any wff constructed along the way is a subformula

- suppose we want to construct
 - $\forall x (Rx \rightarrow \exists x Px)$
- we might construct it as follows

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$
3	$\exists xPx \backslash \text{exists } x Px$	2, $/(3ii \exists \backslash \text{exists})$

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$
3	$\exists xPx \backslash \text{exists } x Px$	2, $/(3ii \exists \backslash \text{exists})$
4	$(Rx \rightarrow \exists xPx) (Rx \rightarrow \backslash \text{exists } x Px)$	1,3 $/(3ii \rightarrow \backslash \text{exists})$

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$
3	$\exists xPx \backslash \text{exists } x Px$	2, $/(3ii \exists \backslash \text{exists})$
4	$(Rx \rightarrow \exists xPx) (Rx \rightarrow \backslash \text{exists } x Px)$	1,3, $/(3ii \rightarrow \backslash \text{exists})$
5	$\forall x(Rx \rightarrow \exists xPx) \backslash \text{forall } x (Rx \rightarrow \backslash \text{exists } x Px)$	4, $/(3ii \forall \backslash \text{forall})$

quantifier scope

quantifier scope

- if a wff has a quantifier in it, then it must have got there by being stuck on the front of some sub-formula α (by 3ii $\forall \backslash \text{forall}$ or 3ii $\exists \backslash \text{exists}$) at some stage in the construction
- for any quantifier appearing in a wff, we call this subformula α the **scope** of the quantifier

- in $(\forall xRx \rightarrow \exists xPx) (\backslash \text{forall } x Rx \rightarrow \backslash \text{exists } x Px)$, the scope of the quantifier $\forall x \backslash \text{forall } x$ is the wff $RxRx$
- to see this, consider how we would construct $(\forall xRx \rightarrow \exists xPx) (\backslash \text{forall } x Rx \rightarrow \backslash \text{exists } x Px)$

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1	$RxRx$	$/(3i)$
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Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$
3	$\forall xRx \backslash \text{forall } x Rx$	1, $/(3ii \forall \backslash \text{forall})$
4	$\exists xPx \backslash \text{exists } x Px$	2, $/(3ii \exists \backslash \text{exists})$

- in $\forall x(Rx \rightarrow \exists xPx) \backslash \text{forall } x (Rx \rightarrow \backslash \text{exists } x Px)$, the scope of the quantifier $\forall x \backslash \text{forall } x$ is the wff $(Rx \rightarrow \exists xPx) (Rx \rightarrow \backslash \text{exists } x Px)$.

- to see this, consider the construction table for $\forall x(Rx \rightarrow \exists x Px) \backslash \text{forall } x (Rx \rightarrow \text{exists } x Px)$

Step	Wff constructed at this step	From steps/by clause
1	$RxRx$	$/(3i)$
2	$PxPx$	$/(3i)$
3	$\exists x Px \backslash \text{exists } x Px$	2, $/(3ii \exists \backslash \text{exists})$
4	$(Rx \rightarrow \exists x Px) (Rx \rightarrow \text{exists } x Px)$	1, 3, $/(3ii \rightarrow \backslash \text{exists})$

Step	Wff constructed at this step	From steps/by clause
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4	$(Rx \rightarrow \exists x Px) (Rx \rightarrow \text{exists } x Px)$	1, 3, $/(3ii \rightarrow \backslash \text{exists})$
5	$\forall x(Rx \rightarrow \exists x Px) \backslash \text{forall } x (Rx \rightarrow \text{exists } x Px)$	4, $/(3ii \forall \backslash \text{forall})$

free and bound variables

free and bound variables

- an occurrence of a variable in a wff is **bound** if it is in the scope of a quantifier that contains that variable.
- an occurrence of a variable that is *not* bound in a wff is **free**.

- consider:
 - $(Fx \rightarrow \exists x Gx) (Fx \rightarrow \text{exists } x Gx)$
- the first occurrence of xx is free.
- the second—the one in the quantifier—is bound.
- the third is bound.

- if a variable falls within the scope of multiple quantifiers containing that variable, it is bound by the one added first (in the construction of the wff).
- consider:
 - $\forall x(Fx \rightarrow \exists x Gx) \backslash \text{forall } x (Fx \rightarrow \text{exists } x Gx)$
- the second occurrence of xx is bound by $\forall x \backslash \text{forall } x$.
- the fourth occurrence of xx is bound by $\exists x \backslash \text{exists } x$.

- an occurrence of a quantifier is **vacuous** if the variable in the quantifier does not occur free within the scope of the quantifier.
- consider:
 - $\exists x Py \backslash \text{exists } x P y$
- this occurrence of $\exists x \backslash \text{exists } x$ is vacuous.

- we distinguish between the quantifier symbols $\forall \backslash \text{forall}$ and $\exists \backslash \text{exists}$ and quantifiers $\forall x \backslash \text{forall } x$ and $\exists x \backslash \text{exists } x$.
- a quantifier consists of a quantifier symbol and a variable.
- the variable which is a constituent of a quantifier is not free. (is it bound?)
- this variable is sometimes called the **operator variable**.

open and closed wffs

open and closed wffs

- corresponding to the notion of free and bound variables is the notion of open and closed wffs
 - a wff with no free occurrences of variables is a **closed** wff.
 - e.g. $\forall x(Fx \rightarrow \exists x Gx)$ \forall x(Fx \rightarrow \exists x Gx)
 - a wff with one or more free occurrences of variables is an **open** wff.
 - e.g. $(Fx \rightarrow \exists x Gx)$ (\underline{F} \rightarrow \exists x Gx)
-

- open and closed wffs are equally well-formed.
- but open wffs do not express propositions—they cannot themselves be true or false.
- consider: $RxRx$
 - this doesn't express a proposition
 - $\exists x Rx$ \exists x R x does, however

wrapping up

this lecture

- the syntax of MPL
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- how formulas of MPL are constructed
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next lecture

- lecture 15, the semantics of MPL, part 1