lecture 14, syntax of MPL

phil1012 introductory logic

overview

this lecture

- the syntax of MPL
- formal specification of the vocabulary and syntax of MPL
- how formulas of MPL are constructed
- the notions of logical operators, the scope of a quantifier, free and bound variables, open and closed formulas

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - identify well-formed formulas of MPL
  - construct construction tables for formulas of MPL
  - identify the main operator of a formula of MPL
  - identify the scope of a quantifier
  - identify free and bound variables and open and closed formulas of MPL

required reading

- section 8.4 of chapter 8

syntax of MPL

syntax of MPL

- we can give a precise specification of the syntax of MPL as we did for PL.

- the **symbols** of MPL are:
  - names:
    - $a, b, c, \ldots, t$
  - variables:
    - x, y, z, u, v, w
  - predicates:
    - $A, B, C, \ldots, X, Y, Z$
  - five connectives:
    - $\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

- the **symbols** of MPL (continued) are:
  - two quantifier symbols:
    - $\forall \quad \exists$
  - two punctuation symbols (parentheses):
    - $(\quad)$

- **terms** are defined as follows:
  - a name is a term
  - a variable is a term
  - nothing else is a term

- **wffs** of MPL are defined as follows:
  - (3i) where $\underline{P}$ is a predicate and $\underline{t}$ is a term, the following is a wff:
    - $\underline{P}\underline{t}$
  - (3ii) where α and β are wffs and $\underline{x}$ is a variable, the following are wffs:
    - $\neg\alpha$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $\forall \underline{x}\alpha$, $\exists \underline{x}\alpha$
  - nothing else is a wff

constructing wffs

constructing wffs

- given this syntax for MPL we can show how any well formed formula of MPL is constructed.
- suppose we want to construct
  - $(\forall xRx \rightarrow \exists xPx)$
- we might construct it as follows

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)

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1	Rx	/(3i)
2	Px	/(3i)
3	$\forall xRx$	1, /(3ii $\forall$)
4	$\exists xPx$	2, /(3ii $\exists$)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\forall xRx$	1, /(3ii $\forall$)
4	$\exists xPx$	2, /(3ii $\exists$)
5	$(\forall xRx \rightarrow \exists xPx)$	3,4 /(3ii $\rightarrow$)

- a **logical operator** is a connective or a quantifier
- the main operator (c.f. main connective) is the last operator added in the construction of the wff
- any wff constructed along the way is a subformula

- suppose we want to construct
  - $\forall x(Rx \rightarrow \exists xPx)$.
- we might construct it as follows

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\exists x Px$	2, /(3ii $\exists$)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\exists x Px$	2, /(3ii $\exists$)
4	$(Rx \to \exists x Px)$	1,3 /(3ii $\to$)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\exists x Px$	2, /(3ii $\exists$)
4	$(Rx \to \exists x Px)$	1,3, /(3ii $\to$)
5	$\forall x(Rx \to \exists x Px)$	4/(3ii $\forall$)

quantifier scope

quantifier scope

- if a wff has a quantifier in it, then it must have got there by being stuck on the front of some sub-formula α (by 3ii $\forall$ or 3ii $\exists$) at some stage in the construction
- for any quantifier appearing in a wff, we call this subformula α the **scope** of the quantifier

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- in $(\forall x Rx \to \exists x Px)$, the scope of the quantifier $\forall x$ is the wff Rx
- to see this, consider how we would construct $(\forall x Rx \to \exists x Px)$

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\forall x Rx$	1, /(3ii $\forall$)
4	$\exists x Px$	2, /(3ii $\exists$)

- in $\forall x(Rx \to \exists x Px)$, the scope of the quantifier $\forall x$ is the wff $(Rx \to \exists x Px)$.
- to see this, consider the construction table for $\forall x(Rx \to \exists x Px)$

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)
2	Px	/(3i)
3	$\exists x Px$	2, /(3ii $\exists$)
4	$(Rx \to \exists x Px)$	1,3, /(3ii $\to$)

Step	Wff constructed at this step	From steps/by clause
1	Rx	/(3i)

2	Px	/(3i)
3	$\exists xPx$	2, /(3ii $\exists$)
4	$(Rx \rightarrow \exists xPx)$	1,3, /(3ii $\rightarrow$)
5	$\forall x(Rx \rightarrow \exists xPx)$	4, /(3ii $\forall$)

free and bound variables

free and bound variables

- an occurrence of a variable in a wff is **bound** if it is in the scope of a quantifier that contains that variable.
- an occurrence of a variable that is *not* bound in a wff is **free**.

- consider:
  - $(Fx \rightarrow \exists xGx)$
- the first occurrence of x is free.
- the second—the one in the quantifier—is bound.
- the third is bound.

- if a variable falls within the scope of multiple quantifiers containing that variable, it is bound by the one added first (in the construction of the wff).
- consider:
  - $\forall x(Fx \rightarrow \exists xGx)$
- the second occurrence of x is bound by $\forall x$.
- the fourth occurrence of x is bound by $\exists x$.

- an occurrence of a quantifier is **vacuous** if the variable in the quantifier does not occur free within the scope of the quantifier.
- consider:
  - $\exists xPy$
- this occurrence of $\exists x$ is vacuous.

- we distinguish between the quantifier symbols $\forall$ and $\exists$ and quantifiers $\forall x$ and $\exists x$.
- a quantifier consists of a quantifier symbol and a variable.
- the variable which is a constituent of a quantifier is not free. (is it bound?)
- this variable is sometimes called the **operator variable**.

open and closed wffs

open and closed wffs

- corresponding to the notion of free and bound variables is the notion of open and closed wffs
- a wff with no free occurrences of variables is a **closed** wff.
  - e.g. $\forall x(Fx \rightarrow \exists xGx)$
- a wff with one or more free occurrences of variables is an **open** wff.
  - e.g. $(F\underline{x} \rightarrow \exists xGx)$

- open and closed wffs are equally well-formed.
- but open wffs do not express propositions—they cannot themselves be

true or false.
- consider: Rx
  - this doesn't express a proposition
  - $\exists x Rx$ does, however

wrapping up

this lecture

- the syntax of MPL
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- how formulas of MPL are constructed
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next lecture

- lecture 15, the semantics of MPL, part 1