

lecture 20, the semantics of GPL and trees for GPL

phil1012 introductory logic

overview

this lecture

- the semantics of GPL and trees for GPL
- the notion of sets of ordered n-tuples as extensions of n-place predicates in GPL

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
 - explain how the truth values of atomic formulas are determined on a model in GPL
 - determine the truth value of a GPL proposition on a model
 - find a model on which a GPL proposition is true and/or a model on which a GPL proposition is false
 - construct trees using the tree rules for GPL
 - use trees to test for various logical properties of GPL formulas
 - read off (counter)models from open paths of GPL trees

required reading

- sections 12.3 and 12.3 of chapter 12

the semantics of GPL

the semantics of GPL

- recall that the semantics for MPL proceeded in two stages.
- in the first stage we say what the values of nonlogical symbols are. A *model* is then any assignment of values to nonlogical symbols (together with a domain from which these values are drawn).
- in the second stage we specify how the truth values of all propositions (closed wffs) are determined, given values for their nonlogical components.

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- in moving from MPL to GPL we have simply added many-place predicates.
 - we need to say what the value of a many-place predicate is.
 - the rest of the semantics is just like that of MPL.
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- in MPL the extensions of one-place predicates were sets.
- why can't we use a set like {Bill, Bob} for the extension of the two-place predicate L^2 ?

- because {Bill, Bob} and {Bob, Bill} are the same set and we want to be about to say that Bill loves Bob without saying that Bob loves Bill.

- **extensionality of sets:** If everything in A is in B and vice versa then $A = B$.
- $\{1, 2, 3\} = \{2, 1, 3\}$ and $\{1, 2\} = \{1, 1, 2\}$.

- to solve this problem we introduce a new notion which indicates a group of things *taken in order*.
- $\langle 1, 2, 3 \rangle \neq \langle 2, 1, 3 \rangle$ and $\langle 1, 2 \rangle \neq \langle 1, 1, 2 \rangle$.

- an ordered nn -tuple is an ordered list of things with nn positions in the list 1st through nn th.
- there need not be nn distinct things involved: the very same object might occupy more than one place in the list.
- an ordered 2-tuple is called an ordered pair and an ordered 3-tuple is called an ordered triple.

- the **extension** of an nn -place predicate is a set of ordered nn -tuples of members of the domain
- our new semantic clause looks like this:

1. $\text{Pn}_{a_1 \dots a_n} \underline{\text{P}^n} \text{ is true in } \mathcal{M} \text{ iff the ordered } n\text{-tuple consisting of the referents in } \mathcal{M} \text{ of } a_1 \text{ through } a_n \text{ in that order is in the extension in } \mathcal{M} \text{ of } \text{Pn}.$

- let's work through some examples to get a feel for the semantics of GPL

- our model:
 - domain: {Alice, Bob, Carol, Dave, Edwina}
 - referents: aa: Alice, bb: Bob, cc: Carol, dd: Dave, ee: Edwina
 - extensions: MM: {Bob, Dave}, FF: {Alice, Carol, Edwina}, TT: $\{\langle \text{Alice}, \text{Bob} \rangle, \langle \text{Alice}, \text{Carol} \rangle, \langle \text{Alice}, \text{Dave} \rangle, \langle \text{Alice}, \text{Edwina} \rangle, \langle \text{Bob}, \text{Carol} \rangle, \langle \text{Bob}, \text{Dave} \rangle, \langle \text{Bob}, \text{Edwina} \rangle, \langle \text{Carol}, \text{Dave} \rangle, \langle \text{Carol}, \text{Edwina} \rangle, \langle \text{Dave}, \text{Edwina} \rangle\}$
- are the following formulas true or false in this model?
- TabTab
 - true. the referent of aa, namely Alice, and the referent of bb, namely Bob, taken in that order, are in the extension of TT.
- TbaTba
 - false. the referent of aa, namely Alice, and the referent of bb, namely Bob, taken in that order, are not in the extension of TT.
- TacTac
 - true. the referent of aa, namely Alice, and the referent of cc, namely Carol, taken in that order, are in the extension of TT.

- TceTce
 - true. the referent of cc, namely Carol, and the referent of ee, namely Edwina, taken in that order, are in the extension of TT.
- TaaTaa
 - false. the referent of aa, namely Alice, and the referent of aa, namely Alice, taken in that order, are not in the extension of TT.
- TeeTee
 - false. the referent of ee, namely Edwina, and the referent of ee, namely Edwina, taken in that order, are not in the extension of TT.
- TbeTbe
 - true. the referent of bb, namely Bob, and the referent of ee, namely Edwina, taken in that order, are in the extension of TT.

trees for GPL

trees for GPL

- in moving from MPL to GPL, we did not add any logical operators—so there are no new tree rules.
- the only change was that we added nn-place predicates—and in the semantics, we added extensions for these (sets of nn-tuples).
- so now all we need to do is see how to read off the extension of a many-place predicate from an open (saturated) path.

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1. Fa
 2. $\neg Fb$
 3. Gc
 4. Ga

- domain: {1, 2, 3}
- referents: a: 1, b: 2, c: 3
- extensions: F: {1}, G: {1, 3}

-
1. Taa
 2. $\neg Tbc$
 3. Tab

- domain: {1, 2, 3}
- referents: a: 1, b: 2, c: 3
- extensions: T: { $\langle 1, 1 \rangle$, $\langle 1, 2 \rangle$ }

-
1. $Babc$
 2. $\neg Bccc$
 3. $Bbcd$
 4. $Baaa$

- domain: {1, 2, 3, 4}
- referents: a: 1, b: 2, c: 3, d: 4
- extensions: B: { $\langle 1, 2, 3 \rangle$, $\langle 2, 3, 4 \rangle$, $\langle 1, 1, 1 \rangle$ }

example 1

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy$
2. $\neg \forall y \exists x Lxy$

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy$
2. $\neg \forall y \exists x Lxy \checkmark$
3. $\exists y \neg \exists x Lxy$

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
2. $\neg \forall y \exists x Lxy \checkmark$
3. $\exists y \neg \exists x Lxy$
4. $\forall y Lay$

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
2. $\neg \forall y \exists x Lxy$
3. $\exists y \neg \exists x Lxy \checkmark b$
4. $\forall y Lay$
5. $\neg \exists y Lxb$

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
2. $\neg \forall y \exists x Lxy \checkmark$
3. $\exists y \neg \exists x Lxy \checkmark b$
4. $\forall y Lay$
5. $\neg \exists y Lxb \checkmark$
6. $\forall x \neg Lxb$

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
2. $\neg \forall y \exists x Lxy \checkmark$
3. $\exists y \neg \exists x Lxy \checkmark b$
4. $\forall y Lay \setminus b$
5. $\neg \exists y Lxb \checkmark$
6. $\forall x \neg Lxb$
7. Lab

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
 2. $\neg \forall y \exists x Lxy \checkmark$
 3. $\exists y \neg \exists x Lxy \checkmark b$
 4. $\forall x Lay \setminus b$
 5. $\neg \exists y Lxb \checkmark$
 6. $\forall x \neg Lxb \setminus a$
 7. Lab
 8. $\neg Lab$
-

To prove: whether $\exists x \forall y Lxy \therefore \forall y \exists x Lxy$ is valid.

1. $\exists x \forall y Lxy \checkmark a$
 2. $\neg \forall y \exists x Lxy \checkmark$
 3. $\exists y \neg \exists x Lxy \checkmark b$
 4. $\forall x Lax \setminus b$
 5. $\neg \exists y Lxb \checkmark$
 6. $\forall x \neg Lxb \setminus a$
 7. Lab
 8. $\neg Lab$
- \otimes

example 2

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

-
1. $\exists y \forall x Lxy$
 2. $\neg \forall x \exists y Lxy$

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy$
2. $\neg \forall x \exists y Lxy \checkmark$
3. $\exists x \neg \exists y Lxy$

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
2. $\neg \forall x \exists y Lxy \checkmark$
3. $\exists x \neg \exists y Lxy$
4. $\forall x Lxa$

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
2. $\neg \forall x \exists y Lxy \checkmark$
3. $\exists x \neg \exists y Lxy \checkmark b$
4. $\forall x Lxa$
5. $\neg \exists y Lby$

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
 2. $\neg \forall x \exists y Lxy \checkmark$
 3. $\exists x \neg \exists y Lxy \checkmark b$
 4. $\forall x Lxa$
 5. $\neg \exists y Lby \checkmark$
 6. $\forall y \neg Lby$
-

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
2. $\neg \forall x \exists y Lxy \checkmark$
3. $\exists x \neg \exists y Lxy \checkmark b$
4. $\forall x Lxa \setminus b$
5. $\neg \exists y Lby \checkmark$
6. $\forall y \neg Lby$
7. Lba

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
2. $\neg \forall x \exists y Lxy \checkmark$
3. $\exists x \neg \exists y Lxy \checkmark b$
4. $\forall x Lxa \setminus b$
5. $\neg \exists y Lby \checkmark$
6. $\forall y \neg Lby \setminus a$
7. Lba
8. $\neg Lba$

To prove: whether $\exists y \forall x Lxy \therefore \forall x \exists y Lxy$ is valid.

1. $\exists y \forall x Lxy \checkmark a$
 2. $\neg \forall x \exists y Lxy \checkmark$
 3. $\exists x \neg \exists y Lxy \checkmark b$
 4. $\forall x Lxa \setminus b$
 5. $\neg \exists y Lby \checkmark$
 6. $\forall y \neg Lby \setminus a$
 7. Lba
 8. $\neg Lba$
- ⊗

example 3

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax))$

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax)$
3. $\neg \exists x(Ax \wedge Tax)$

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax)$
3. $\neg \exists x(Ax \wedge Tax) \checkmark$
4. $\forall x \neg(Ax \wedge Tax)$

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
 2. $\exists x(Hx \wedge Tax) \checkmark b$
 3. $\neg \exists x(Ax \wedge Tax) \checkmark$
 4. $\forall x \neg(Ax \wedge Tax)$
 5. $Hb \wedge Tab$
-

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
 2. $\exists x(Hx \wedge Tax) \checkmark b$
 3. $\neg \exists x(Ax \wedge Tax) \checkmark$
 4. $\forall x \neg(Ax \wedge Tax)$
 5. $Hb \wedge Tab \checkmark$
 6. Hb
 7. Tab
-

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
 2. $\exists x(Hx \wedge Tax) \checkmark b$
 3. $\neg \exists x(Ax \wedge Tax) \checkmark$
 4. $\forall x \neg(Ax \wedge Tax) \setminus b$
 5. $Hb \wedge Tab \checkmark$
 6. Hb
 7. Tab
 8. $\neg(Ab \wedge Tab)$
-

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
 2. $\exists x(Hx \wedge Tax) \checkmark b$
 3. $\neg \exists x(Ax \wedge Tax) \checkmark$
 4. $\forall x \neg(Ax \wedge Tax) \setminus b$
 5. $Hb \wedge Tab \checkmark$
 6. Hb
 7. Tab
 8. $\neg(Ab \wedge Tab) \checkmark$
 9. $\neg Ab \quad \neg Tab$
-

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
 2. $\exists x(Hx \wedge Tax) \checkmark b$
 3. $\neg \exists x(Ax \wedge Tax) \checkmark$
 4. $\forall x \neg(Ax \wedge Tax) \setminus b$
 5. $Hb \wedge Tab \checkmark$
 6. Hb
 7. Tab
 8. $\neg(Ab \wedge Tab) \checkmark$
 9. $\neg Ab \quad \neg Tab$
- \otimes
-

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax) \checkmark b$
3. $\neg\exists x(Ax \wedge Tax) \checkmark$
4. $\forall x\neg(Ax \wedge Tax) \setminus b, a$
5. $Hb \wedge Tab \checkmark$
6. Hb
7. Tab
8. $\neg(Ab \wedge Tab) \checkmark$
9. $\neg Ab \quad \neg Tab$
10. $\neg(Aa \wedge Taa) \quad \otimes$

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax) \checkmark b$
3. $\neg\exists x(Ax \wedge Tax) \checkmark$
4. $\forall x\neg(Ax \wedge Tax) \setminus b, a$
5. $Hb \wedge Tab \checkmark$
6. Hb
7. Tab
8. $\neg(Ab \wedge Tab) \checkmark$
9. $\neg Ab \quad \neg Tab$
10. $\neg(Aa \wedge Taa) \checkmark \quad \otimes$
11. $\neg Aa \quad \neg Taa$

- domain:
- referents:
- extensions:

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax) \checkmark b$
3. $\neg\exists x(Ax \wedge Tax) \checkmark$
4. $\forall x\neg(Ax \wedge Tax) \setminus b, a$
5. $Hb \wedge Tab \checkmark$
6. Hb
7. Tab
8. $\neg(Ab \wedge Tab) \checkmark$
9. $\neg Ab \quad \neg Tab$
10. $\neg(Aa \wedge Taa) \checkmark \quad \otimes$
11. $\neg Aa \quad \neg Taa$

- domain: {1, 2}
- referents: aa: 1, bb:2
- extensions:

To prove: whether $\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)$ a tautology.

1. $\neg(\exists x(Hx \wedge Tax) \rightarrow \exists x(Ax \wedge Tax)) \checkmark$
2. $\exists x(Hx \wedge Tax) \checkmark b$
3. $\neg \exists x(Ax \wedge Tax) \checkmark$
4. $\forall x \neg(Ax \wedge Tax) \setminus b, a$
5. $Hb \wedge Tab \checkmark$
6. \boxed{Hb}
7. \boxed{Tab}
8. $\neg(Ab \wedge Tab) \checkmark$
9. $\neg Ab \quad \neg Tab$
10. $\neg(Aa \wedge Taa) \checkmark \quad \otimes$
11. $\neg Aa \quad \neg Taa$

- domain: $\{1, 2\}$
- referents: aa: 1, bb:2
- extensions: HH: $\{2\}$, TT: $\{ \langle 1, 2 \rangle \}$, AA: $\{\}$
- note: this handout contains a correction from the lecture regarding the extension of AA

wrapping up

this lecture

- trees for GPL
- reading models off open paths

next lecture

- lecture 21, the formal language GPLI