

# lecture 10, trees for PL

phil1012 introductory logic

## overview

### this lecture

- an introduction to **truth trees** for PL
- general motivation for the use of truth trees for PL
- particular motivations for each tree rule

### next lecture

- how to construct truth trees in general
- how to use truth trees to test for particular logical properties

## learning objectives

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - explain the motivation behind truth trees for PL
  - write down the tree rules for PL from memory

## required reading

- all of chapter 7

## the motivation for truth trees

### the motivation for truth trees

- the limitations of truth tables as a proof method
- proof trees are the only method of proof we will examine for MPL, GPL, and GPLI

### the limitations of truth tables as a proof method

- recall the maths concerning truth tables
  - 2 basic propositions: 4 rows
  - 3 basic propositions: 8 rows
  - 4 basic propositions: 16 rows
  - 5 basic propositions: 32 rows
  - 6 basic propositions: 64 rows
  - . . .

### the limitations of truth tables as a proof method

- truth tables (or complete ones at least) involve a kind of

enumerative search

- if an argument is valid, you've got to fill in every row to be sure
- truth tables may eventually provide an evaluation which is a counterexample
- might we work in a more direct fashion towards a counterexample?
- yes: truth trees

## the basic idea behind truth trees

### re-inventing truth trees

- let's re-invent truth trees
- suppose you have some proposition or other, and you want to know whether it is satisfiable

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B))$

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- suppose we begin by assuming that it is satisfiable
- that is, suppose there is some assignment on which it is true
- what do we know?
- well, we know that the assignment is one on which both conjuncts must be true
- for now, let's just write out a list like this to keep track:

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(A \rightarrow B) (A \rightarrow B), (A \wedge \neg B) (A \wedge \neg B)$

---

- okay, now we've got another conjunction in there.
- if there is an assignment on which all these propositions are true together, it must be an assignment on which both conjuncts are true
- so let's expand our list:

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(A \rightarrow B) (A \rightarrow B), (A \wedge \neg B) (A \wedge \neg B), AA, \neg B \neg B$

---

- okay, now comes the tricky part. how do we deal with the conditional?
- well, let's ask what kind of assignment we need in order to make it true?
- here it helps to remind ourselves that  $(A \rightarrow B) (A \rightarrow B)$  is equivalent to  $(\neg A \vee B) (\neg A \vee B)$
- so we can either take an assignment on which  $\neg A$  is true, or an assignment on which  $B$  is true
- take the first first

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(A \rightarrow B) (A \rightarrow B), (A \wedge \neg B) (A \wedge \neg B), AA, \neg B \neg B,$   
 $\neg A \neg A$

---

- uh oh. what is wrong with this list of propositions?

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(A \rightarrow B) (A \rightarrow B), (A \wedge \neg B) (A \wedge \neg B), AA, \neg B \neg B,$

$B, \neg A \wedge \neg A$

- not all of the propositions can be true at once!  $AA$  and  $\neg A \wedge \neg A$  cannot be true at once.
  - this path isn't going to make our initial assumption about satisfiability work out
- 

- what about the other assignment, on which  $BB$  is true?

$((A \rightarrow B) \wedge (A \wedge \neg B)) ((A \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(A \rightarrow B) (A \rightarrow B), (A \wedge \neg B) (A \wedge \neg B), AA,$   
 $\neg B \wedge \neg B, BB$

- uh oh. not all of the propositions can be true at once!  $BB$  and  $\neg B \wedge \neg B$  cannot be true at once
- 

- our initial assumption was mistaken. the proposition is not satisfiable!
  - this method looks like a good one for refuting the initial assumption of satisfiability
  - if we reach a contradiction on a path, we can reject that path
  - if we reach a contradiction on all paths, we know the proposition isn't satisfiable
- 

- what if we don't read a contradiction?
- what if had started with a slightly different proposition?
- what if we had arrived at the following after splitting the conditional?

$((C \rightarrow B) \wedge (A \wedge \neg B)) ((C \rightarrow B) \wedge (A \wedge \neg B)),$   
 $(C \rightarrow B) (C \rightarrow B), (A \wedge \neg B) (A \wedge \neg B), AA,$   
 $\neg B \wedge \neg B, \neg C \wedge \neg C$

---

- well, assuming that we have broken down every proposition once
  - we know that all of the propositions can be true together if and only if the basic propositions and negations of basic propositions can be true together
  - and we can just see that  $AA, \neg B \wedge \neg B$  and  $\neg C \wedge \neg C$  can all be true together
  - so we can conclude that the original proposition is satisfiable
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- even better
- we have derived an assignment on which the proposition is true:

$AA: T, BB: F, CC: F$

- unlike truth tables, truth trees can take us directly to the relevant assignment
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- essential to the procedure is the idea of "splitting" or "branching"
  - we must check every path we create after splitting or branching
  - else we won't know if there is *some* path without a contradiction on it
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- drawing this method as a tree going down the page is even better

# the tree method

## the tree method described

- here is the tree proof of the unsatisfiability of  $((A \rightarrow B) \wedge (A \wedge \neg B)) \rightarrow (A \rightarrow B) \wedge (A \wedge \neg B)$  from above

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B)) \checkmark \\
 (A \rightarrow B) \checkmark \\
 (A \wedge \neg B) \checkmark \\
 A \\
 \neg B \\
 \swarrow \quad \searrow \\
 \neg A \quad B \\
 x \quad x
 \end{array}$$

- 
- how do we construct such a tree?
  - well, we begin by writing down the proposition (or propositions) we want to test for satisfiability

$$((A \rightarrow B) \wedge (A \wedge \neg B))$$

- think of this proposition as the “root” of the tree if you like (the tree is upside down)

- 
- now we write down other things which must be true assuming these are true (according to the tree rules)
  - if it follows from the assumption that  $\alpha$  is true that  $\beta$  and  $\gamma$  are true, we write  $\beta$  and  $\gamma$  at the bottom of every open path on our tree
  - in this case we only have one path

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B)) \\
 (A \rightarrow B) \\
 (A \wedge \neg B)
 \end{array}$$

- 
- in a moment, we will discuss the rules for building trees (the tree rules)
  - these just encapsulate our claims about “what must be true” if some proposition is true
  - each time we apply a tree rule to a proposition we “check it off” with a tick
  - this is just a way of keeping track what we’ve “dealt with” that proposition

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B)) \checkmark \\
 (A \rightarrow B) \\
 (A \wedge \neg B)
 \end{array}$$

- 
- here is the step applied to the conjunction

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B)) \checkmark \\
 (A \rightarrow B) \\
 (A \wedge \neg B) \checkmark \\
 A \\
 \neg B
 \end{array}$$

- 
- if it follows from the assumption that  $\alpha$  is true that either  $\beta$  and  $\gamma$  are true, or both are true, then we “split” the tree and write them down on either side of the new branch
  - we split on every open path the proposition appears on

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B))\checkmark \\
 (A \rightarrow B)\checkmark \\
 (A \wedge \neg B)\checkmark \\
 A \\
 \neg B \\
 \swarrow \quad \searrow \\
 \neg A \quad B
 \end{array}$$

- 
- note that each time we apply a rule, we write down propositions which are simpler than those we began with
  - the process ends when we cannot apply any more rules
  - whenever we encounter a contradiction on a path we immediately “close” the path/branch with a cross
  - there’s no point continuing with that path

- 
- in the case at hand, both paths close

$$\begin{array}{c}
 ((A \rightarrow B) \wedge (A \wedge \neg B))\checkmark \\
 (A \rightarrow B)\checkmark \\
 (A \wedge \neg B)\checkmark \\
 A \\
 \neg B \\
 \swarrow \quad \searrow \\
 \neg A \quad B \\
 \times \quad \times
 \end{array}$$

- 
- because propositions always get simpler, we eventually end up with only basic propositions and negations of basic propositions
  - if a path contains a contradiction, the propositions on that path cannot all be true at once
  - if a finished path does not contain a contradiction, the propositions can all be true together
  - we can “read off” an assignment on which all the propositions are true from the basic propositions and negated propositions on such a path

## the tree rules

### the tree rules stated

- rather than thinking about “what must be true” every time we build a tree, we can write down some rules that capture the relevant claims for each connective
- we’ll go over the motivation here
- if you forget a tree rule, it is helpful to know the motivation
- if is also helpful to think of the truth table for the connectives (and connectives in the immediate scope of negation).

### disjunction

- here is the tree rule for disjunction:

$$(\alpha \vee \beta) \checkmark$$

$$\alpha \quad \beta$$

- let's think through the motivation for the rule in terms of truth tables
- ask: what must be *true* if the disjunction is true?

$\alpha$	$\beta$	$(\alpha \vee \beta)$
T	T	T
T	F	T
F	T	T
F	F	F

- reading off from the rows on which  $(\alpha \vee \beta)$  is true . . .

$$(\alpha \vee \beta) \checkmark$$

$$\begin{array}{ccc} \alpha & \alpha & \neg \alpha \\ \beta & \neg \beta & \beta \end{array}$$

- then simplifying . . .

$$(\alpha \vee \beta) \checkmark$$

$$\begin{array}{ccc} \alpha & \alpha & \beta \\ \beta & & \end{array}$$

- and simplifying again . . .

$$(\alpha \vee \beta) \checkmark$$

$$\alpha \quad \beta$$

- what must be *true* if the disjunction is true?
- answer: either  $\alpha$  or  $\beta$
- the tree rule for disjunction is easy to remember even though there's a bit involved in reading the tree rule off the truth table

## negated disjunction

- here is the tree rule for negated disjunction:

$$\neg(\alpha \vee \beta) \checkmark$$

$$\neg \alpha$$

$$\neg \beta$$

- let's think through the motivation for the rule in terms of truth tables
- ask: what must be *true* if the negated disjunction is true?

$\alpha$	$\beta$	$\neg(\alpha \vee \beta)$
T	T	F
T	F	F
F	T	F
F	F	T

- answer:  $\neg \alpha$  and  $\neg \beta$  must be true

- the tree rule for negated disjunction is easy to read off its truth table

## conjunction

- here is the tree rule for conjunction:

$$\begin{array}{c}
 (\alpha \wedge \beta) \checkmark \\
 \alpha \\
 \beta
 \end{array}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$(\alpha \wedge \beta)$
T	T	T
T	F	F
F	T	F
F	F	F

- the tree rule for conjunction is easy to read off its truth table

## negated conjunction

- here is the tree rule for negated conjunction:

$$\begin{array}{c}
 \neg(\alpha \wedge \beta) \checkmark \\
 \swarrow \quad \searrow \\
 \neg\alpha \quad \neg\beta
 \end{array}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$\neg(\alpha \wedge \beta)$
T	T	F
T	F	T
F	T	T
F	F	T

- reading off the tree rule for negated conjunction works a bit like the case for disjunction . . .

$$\begin{array}{c}
 \neg(\alpha \wedge \beta) \checkmark \\
 \swarrow \quad \downarrow \quad \searrow \\
 \alpha \quad \neg\alpha \quad \neg\alpha \\
 \neg\beta \quad \beta \quad \neg\beta
 \end{array}$$

- then simplifying . . .

$$\begin{array}{c}
 \neg(\alpha \wedge \beta) \checkmark \\
 \swarrow \quad \downarrow \quad \searrow \\
 \neg\beta \quad \neg\alpha \quad \neg\alpha \\
 \quad \quad \neg\beta
 \end{array}$$

- then simplifying again . . .

$$\begin{array}{c}
 \neg(\alpha \wedge \beta) \checkmark \\
 \swarrow \quad \searrow \\
 \neg\alpha \quad \neg\beta
 \end{array}$$

## conditional

- here is the tree rule for conditional:

$$\frac{(\alpha \rightarrow \beta) \checkmark}{\neg \alpha \quad \beta}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$(\alpha \rightarrow \beta)$
T	T	T
T	F	F
F	T	T
F	F	T

- reading the tree rule of the truth table for conditional is a little trickier than the others . . .
- we start with . . .

$$\frac{(\alpha \rightarrow \beta) \checkmark}{\begin{array}{ccc} \alpha & \neg \alpha & \neg \alpha \\ \beta & \beta & \neg \beta \end{array}}$$

- simplifying first on  $\beta$  we get . . .

$$\frac{(\alpha \rightarrow \beta) \checkmark}{\begin{array}{cc} \beta & \neg \alpha \\ & \neg \beta \end{array}}$$

- then simplifying on  $\neg \beta$  we get . . .

$$\frac{(\alpha \rightarrow \beta) \checkmark}{\begin{array}{cc} \beta & \neg \alpha \end{array}}$$

- in the case of conditional it is probably easier to just remember that  $(\alpha \rightarrow \beta)$  is equivalent to  $(\neg \alpha \vee \beta)$
- the tree rule (and the motivation for the rule) will then be obvious!

## negated conditional

- here is the tree rule for negated conditional:

$$\frac{\neg(\alpha \rightarrow \beta) \checkmark}{\begin{array}{c} \alpha \\ \neg \beta \end{array}}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$\neg(\alpha \rightarrow \beta)$
T	T	F
T	F	T



F		T		F
F		F		F

## biconditional

- here is the tree rule for biconditional:

$$\begin{array}{c}
 (\alpha \leftrightarrow \beta) \checkmark \\
 \wedge \\
 \begin{array}{cc}
 \alpha & \neg \alpha \\
 \beta & \neg \beta
 \end{array}
 \end{array}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$(\alpha \leftrightarrow \beta)$
T	T	T
T	F	F
F	T	F
F	F	T

## negated biconditional

- here is the tree rule for negated biconditional:

$$\begin{array}{c}
 \neg(\alpha \leftrightarrow \beta) \checkmark \\
 \wedge \\
 \begin{array}{cc}
 \alpha & \neg \alpha \\
 \neg \beta & \beta
 \end{array}
 \end{array}$$

- let's think through the motivation for the rule in terms of truth tables

$\alpha$	$\beta$	$\neg(\alpha \leftrightarrow \beta)$
T	T	F
T	F	T
F	T	T
F	F	F

## double negation

- here is the tree rule for double negation:

$$\begin{array}{c}
 \neg \neg \alpha \checkmark \\
 \alpha
 \end{array}$$

- let's think through the motivation for the rule

$\alpha$	$\neg \neg \alpha$
T	T
F	F

## closure

- finally, here is the closure rule:

$\alpha$   
|  
...  
|  
 $\neg\alpha$   
x

- let's think through the motivation for the rule
- if a proposition and its negation both appear on a path, then it is not the case that all the propositions on that path can be true at the same time
- so we close that path

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- a note on closure: the closure rule applies to *all* propositions and their negations, not just basic propositions and their negations

## wrapping up

### this lecture

- an introduction to **truth trees** for PL
- general motivation for the use of truth trees for PL
- particular motivations for each tree rule

### next lecture

- how to construct truth trees in general
- how to use truth trees to test for particular logical properties

### next lectures

- lecture 11, uses of trees for PL
- lecture 12, issues in translation: conditional
- lecture 13, the formal language MPL