lecture 17, trees for MPL

phil1012 introductory logic

overview

this lecture

- an introduction to trees for MPL
- motivation for MPL truth trees
- motivation for new rules
- constructing MPL truth trees
- finished trees, closed paths, and saturation

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
    - explain the general rationale behind trees for MPL
    - explain the particular rationale behind each tree rule for MPL
    - explain what it means to say that a path is saturated
    - construct trees using the tree rules for MPL
    - use trees to test for various logical properties of MPL formulas
    - read off (counter)models from open paths of MPL trees

required reading

- section 10.1 of chapter 10

tables and trees

tables and trees

- recall that in PL we had two methods of proof: truth tables and truth trees
- in MPL we have only one: truth trees
- the tree rules for MPL are similar to those for PL
- but there are additional rules for the quantifiers
- and the rationale appeals to models
- let’s start by running through the rationale for the rules

rationale for tree rules

rationale for tree rules

- the general rationale for the tree rules:
    - the rules prescribe propositions that must be true, given that what we have already written down is true

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- in the non-branching case:
- if there is a model in which every proposition on some old path is

true, then there is a model in which every proposition on the new path is true.

- in the branching case:
- if there is a model in which every proposition on some old path is true, then there is a model in which every proposition on at least one new path is true.

- all of the rules from PL have the relevant property. Take the rules for negated disjunction (a non-branching rule) and disjunction (a branching rule) for example.

the rationale for the rule for negated disjunction

- rule for negated disjunction:

$$\begin{array}{l} \neg(\alpha \vee \beta) \checkmark \\ \quad \neg\alpha \\ \quad \neg\beta \end{array}$$

- rationale:
  - suppose there is some model in which $\neg(\alpha \vee \beta)$ is true
  - it follows by rule (2) that $(\alpha \vee \beta)$ is false on this model
  - it follows by rule (4) that α and β are both false on this model
  - it follows by rule (2) that $\neg\alpha$ and $\neg\beta$ are both true on this model
  - so if there is a model in which every proposition on some old path is true *before* this rule is applied, then there is a model in which every proposition on the new path is true *after* this rule has been applied

the rationale for the rule for disjunction

- rule for disjunction:

$$\begin{array}{c} (\alpha \vee \beta) \checkmark \\ \swarrow \quad \searrow \\ \alpha \qquad \beta \end{array}$$

- rationale:
  - suppose there is a model in which $(\alpha \vee \beta)$ is true
  - it follows by rule (4) that either α is true or β is true in this model
  - so if there is a model in which every proposition on some old path is true before this rule is applied, then there is a model in which every proposition on at least one new path is true after this rule has been applied

rules for quantifiers

rules for quantifiers

- we now introduce four new rules for the quantifiers.

- the rule for negated existential quantifier.
  - the rule for negated universal quantifier.
  - the rule for existential quantifier.
  - the rule for universal quantifier.
- the application of the rules for the negated quantifiers is the same as for the other rules.
- the unnegated quantifiers require special treatment.

rule for negated existential quantifier

$\neg\exists\underline{x}\alpha(\underline{x})$ ✓
$\forall\underline{x}\neg\alpha(\underline{x})$

- rationale:
  - suppose that $\neg\exists\underline{x}\alpha(\underline{x})$ is true in some model $\mathcal{M}$
  - it follows from this that $\exists\underline{x}\alpha(\underline{x})$ is false in $\mathcal{M}$
  - it follows from this that there is no model just like $\mathcal{M}$ except that it also assigns a referent to $\underline{a}$—where is $\underline{a}$ is some name to which $\mathcal{M}$ assigns no referent—in which $\alpha(\underline{a}/\underline{x})$ is true
  - in other words, $\alpha(\underline{a}/\underline{x})$ is false in every model just like $\mathcal{M}$ except that it also assigns a referent to $\underline{a}$
  - it follows from this that $\neg\alpha(\underline{a}/\underline{x})$ is true on every model just like $\mathcal{M}$ except that it also assigns a referent to $\underline{a}$
  - it follows from this that $\forall\underline{x}\neg\alpha(\underline{x})$ is true in $\mathcal{M}$

rule for negated universal quantifier

$\neg\forall\underline{x}\alpha(\underline{x})$ ✓
$\exists\underline{x}\neg\alpha(\underline{x})$

- rationale:
  - suppose that $\neg\forall\underline{x}\alpha(\underline{x})$ is true in some model $\mathcal{M}$
  - it follows from this that $\forall\underline{x}\alpha(\underline{x})$ is false in $\mathcal{M}$
  - it follows from this that there is some model just like $\mathcal{M}$ except that it also assigns a referent $\underline{a}$—where $\underline{a}$ is some name to which $\mathcal{M}$ assign no referent—in which $\alpha(\underline{a}/\underline{x})$ is false
  - it follows from this that $\neg\alpha(\underline{a}/\underline{x})$ is true in $\mathcal{M}$. It follows that $\exists\underline{x}\neg\alpha(\underline{x})$ is true in $\mathcal{M}$

rule for existential quantifier

$\exists\underline{x}\alpha(\underline{x})$ ✓$\underline{a}$
$\alpha(\underline{a}/\underline{x})$

- note 1: at the time of applying the rule at the bottom of some path, the name $\underline{a}$ used in applying the rule must be one that has not yet appeared anywhere on the path.
- note 2: when applying the rule and checking off the formula, we write the name we have used in applying the rule next to the check mark.
- let’s look at a couple of examples . . .

- suppose we have:

$$\begin{array}{c} Fa \\ Gb \\ \exists x Hx \end{array}$$

- the following is a correct application of the rule for the existential quantifier:

$$\begin{array}{c} Fa \\ Gb \\ \exists x Hx \checkmark c \\ Hc \end{array}$$

- the name c has not been used on the path

- the following is an incorrect application of the rule for the existential quantifier:

$$\begin{array}{c} Fa \\ Gb \\ \exists x Hx \checkmark b \\ Hb \end{array}$$

- the name b has already been used on the path

- okay, now consider another example

- suppose we have:

$$\begin{array}{c} (\exists x Fx \vee \exists x Gx) \checkmark \\ \begin{array}{cc} / & \backslash \\ \exists x Fx & \exists x Gx \end{array} \end{array}$$

- we apply the rule for the existential quantifier once:

$$\begin{array}{c} (\exists x Fx \vee \exists x Gx) \checkmark \\ \begin{array}{cc} / & \backslash \\ \exists x Fx \checkmark a & \exists x Gx \\ Fa & \end{array} \end{array}$$

- we apply the rule again:

$$\begin{array}{c} (\exists x Fx \vee \exists x Gx) \checkmark \\ \begin{array}{cc} / & \backslash \\ \exists x Fx \checkmark a & \exists x Gx \checkmark a \\ Fa & Ga \end{array} \end{array}$$

- this is okay because the name has not been used *on this path.*

- okay, now let's look at the rationale for the rule for the existential quantifier

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- rationale:
  - suppose there is a model $\mathscr{M}$ in which $\exists \underline{x} \alpha(\underline{x})$ is true.

- it follows from this that there is at least one object o in the domain of $\mathscr{M}$ such that $\alpha(\underline{a}/\underline{x})$ is true in $\mathscr{M}^{a}_{o}$, where $\underline{a}$ is some name not assigned a referent in $\mathscr{M}$, and $\mathscr{M}^{a}_{o}$ is a model that is just like $\mathscr{M}$ except that in it the name $\underline{a}$ is assigned the referent o.
- so, if there is a model $\mathscr{M}$ in which $\exists\underline{x}\alpha(\underline{x})$ is true, then there is a different model $\mathscr{M}^{a}_{o}$ in which $\alpha(\underline{a}/\underline{x})$ is true $\exists\underline{x}\alpha(\underline{x})$ will be true on $\mathscr{M}^{a}_{o}$.
- so, if there is a model $\mathscr{M}$ in which $\exists\underline{x}\alpha(\underline{x})$ is true, then there is a different model $\mathscr{M}^{a}_{o}$ in which $\alpha(\underline{a}/\underline{x})$ is true, and $\exists\underline{x}\alpha(\underline{x})$ is true.

rule for universal quantifier:

$\forall\underline{x}\alpha(\underline{x})\ \backslash\underline{a}$
$\alpha(\underline{a}/\underline{x})$

- note 1: the name $\underline{a}$ does not have to be new on the path
- note 2: when applying the rule, we write a backslash, not a check mark—and we write the name used in applying the rule next to the backslash

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- note 3: we can apply the rule multiple times

$\forall\underline{x}\alpha(\underline{x})\ \backslash\underline{a},\underline{b},\underline{c}$
$\alpha(\underline{a}/\underline{x})$
$\alpha(\underline{b}/\underline{x})$
$\alpha(\underline{c}/\underline{x})$

- okay, let’s go over the rationale for the rule for the universal quantifier

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- rationale (case 1):
  - suppose that the name $\underline{a}$ used in applying the rule is new to the path.
  - if there is a model $\mathscr{M}$ in which $\forall\underline{x}\alpha(\underline{x})$ is true, then for every object o in the domain of $\mathscr{M}$, $\alpha(\underline{a}/\underline{x})$ is true in $\mathscr{M}^{a}_{o}$ where $\underline{a}$ is our new name (which is not assigned a referent in $\mathscr{M}$), and $\mathscr{M}^{a}_{o}$ is a model that is just like $\mathscr{M}$ except that in it the name $\underline{a}$ is assigned the referent o
  - if $\forall\underline{x}\alpha(\underline{x})$ is true in $\mathscr{M}$, then $\forall\underline{x}\alpha(\underline{x})$ is true in $\mathscr{M}^{a}_{o}$.
  - it follows that if there is a model in which $\forall\underline{x}\alpha(\underline{x})$ is true, then there is a model on which $\forall\underline{x}\alpha(\underline{x})$ is true and $\alpha(\underline{a}/\underline{x})$ is true.

- rationale (case 2):
  - suppose that the name $\underline{a}$ used in applying the rule is not new to the path.
  - if there is a model $\mathscr{M}$ in which $\forall\underline{x}\alpha(\underline{x})$ is true, then for every object o in the domain of $\mathscr{M}$, $\alpha(\underline{d}/\underline{x})$ is true in $\mathscr{M}^{d}_{o}$ where $\underline{d}$ is a new name (which is not assigned a referent in $\mathscr{M}$), and $\mathscr{M}^{d}_{o}$ is a model that is just like $\mathscr{M}$ except that in it the name $\underline{d}$ is assigned the referent o
  - in this case, $\underline{a}$ is (already) assigned to a referent in $\mathscr{M}$—suppose it is assigned to the object k
  - we have just seen that $\alpha(\underline{d}/\underline{x})$ is true in every model just like

$\mathscr{M}$ except that it assigns a referent to $\underline{d}$
  - so $\alpha(\,\underline{d}/\underline{x})$ is true in $\mathscr{M}^{\underline{d}}_{k}$, the model that assigns $\underline{d}$'s referent to k
  - but then $\alpha(\,\underline{a}/\underline{x})$ must be true in $\mathscr{M}$, because $\underline{a}$'s referent is the same object k

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- okay, so much for the motivation for the rules for the existential quantifier and the universal quantifier, now let's take a look at trees for MPL in their entirety

tree rules for MPL

tree rules for MPL

- the tree rules for MPL consist of:
  - the tree rules from PL.
  - tree rules for disjunction, negated disjunction, conjunction, negated conjunction, conditional, negated conditional, biconditional, negated biconditional, and double negation are the same as in PL
- and the tree rules for the quantifiers:

- rule for existential quantifier

$$\begin{array}{ll} \exists \underline{x}\alpha(\underline{x}) \checkmark \underline{a} & \quad \text{(new } \underline{a}\text{)} \\ \alpha(\underline{a}/\underline{x}) & \end{array}$$

- rule for negated existential quantifier

$$\begin{array}{l} \neg\exists \underline{x}\alpha(\underline{x}) \checkmark \\ \forall \underline{x}\neg\alpha(\underline{x}) \end{array}$$

- rule for universal quantifier

$$\begin{array}{ll} \forall \underline{x}\alpha(\underline{x}) \setminus\underline{a} & \quad \text{(any } \underline{a}\text{)} \\ \alpha(\underline{a}/\underline{x}) & \end{array}$$

- rule for negated universal quantifier

$$\begin{array}{l} \neg\forall \underline{x}\alpha(\underline{x}) \checkmark \\ \exists \underline{x}\neg\alpha(\underline{x}) \end{array}$$

recommended order of application

- there's a recommended order of application for the rules:
  - rules from PL. non-branching first
  - rules for negated quantifiers
  - rule for (unnegated) existential quantifier
  - rule for (unnegated) universal quantifier

- that's it. those are the four new rules, and notes on their application.
- now we've got to say something about finished trees

finished trees, saturated paths

finished trees, saturated paths

- because we can go on applying the rule for the universal quantifier as many times as we like, we need to say something about when our trees are finished
- to do so we appel to the idea of a saturated path
- a tree is **finished** if each of its paths is either **closed** or **saturated.**

- a path is **saturated** if and only if
  - every formula on it—apart from atomic formulas, negations of atomic formulas, and formulas whose main operator is a universal quantifier—has had the relevant rule applied; and
  - every formula on it whose main operator is a universal quantifier
    - has had the universal quantifier rule applied to it at least once, and
    - has had the rule applied to it once for each name that appears on the path.

- let's consider an example

- suppose we have:

$$\begin{array}{c} \exists x Fx \\ \forall x Hx \\ Gb \end{array}$$

- we apply the existential quantifier rule to get:

$$\begin{array}{c} \exists x Fx \checkmark a \\ \forall x Hx \\ Gb \\ Fa \end{array}$$

- then we apply the universal quantifier rule to get:

$$\begin{array}{c} \exists x Fx \checkmark a \\ \forall x Hx \setminus a \\ Gb \\ Fa \\ Ha \end{array}$$

- this tree is unfinished!
- why? because there is a name no the path, b in this case, which we have not yet used in applying the universal quantifier rule

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- so we apply the universal quanfitier rule using the name b to get:

$\exists x Fx$ ✓ a
$\forall x Hx \setminus a, b$
Gb
Fa
Ha
Hb

- this tree is finished!
- why? because we've applied the universal quantifier rule at least once, and we have applied it using every name on the path

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- don't worry if you haven't fully understood the idea of saturating a path
- we'll do plenty of practice in the live lecture and in the tutorials

wrapping up

this lecture

- introducting the tree rules for MPL
- the rationale for the rules
- finished trees, saturated paths

next lecture

- lecture 18, uses of trees for MPL