lecture 16, the semantics of MPL (part 2)

phil1012 introductory logic

overview

this lecture

- second of two lectures on the semantics of MPL
- the semantics of complicated quantified propositions in MPL
- analyses of logical concepts in MPL

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - explain how the truth values of complex quantified propositions are determined in MPL
  - explain how models provide analyses of various logical concepts (like validity) in MPL
  - determine whether a complex quantified proposition of MPL is true or false on a given model
  - describe a model on which an MPL proposition is true and/or a model on which a proposition is false

required reading

- sections 9.4 and 9.5 of chapter 9

complex quantified propositions

complex quantified propositions

- how are the truth values of the following determined?
  - $\forall x(Px \rightarrow Rx)$
  - $\exists x(Px \land Rx)$
- we do not yet have a way of determining the truth values of these propositions

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- but we do have a way of determining the truth values of the following in a model:
  - $(Pa \rightarrow Ra)$
  - $(Pa \land Ra)$

- so here’s an idea . . .
- we can replace a question we *cannot* answer with many questions we *can* answer
- we can answer questions about the values of expressions like $(Pa \land Ra)$ in models related to $\mathcal{M}$
- it turns out that if we ask the right questions we can answer

questions about the truth values of complex quantified expressions

some new terminology

- but before we get to that, we need some new terminology
- we use α(x_)\alpha(\underline{x}) to stand for an arbitrary formula which has no free variables other than x_\underline{x}
- we use α(a_/x_)\alpha(\underline{a}/ \underline{x}) to stand for a formula that results from α(x_)\alpha(\underline{x}) by replacing all the free occurrences of x_\underline{x} in α(x_)\alpha(\underline{x}) with a_\underline{a}

- for example
- if we have . . .
  - α(x):(Fx→Gx)\alpha(x): (Fx \rightarrow Gx)
- then we have . . .
  - α(a/x):(Fa→Ga)\alpha(a/x): (Fa \rightarrow Ga)
  - α(b/x):(Fb→Gb)\alpha(b/x): (Fb \rightarrow Gb)

- if we have . . .
  - α(x):((Fx∧Ga)∨(Gx↔Hx))\alpha(x): ((Fx \land Ga) \lor (Gx \leftrightarrow Hx))
- then we have . . .
  - α(a/x):((Fa∧Ga)∨(Ga↔Ha))\alpha(a/x):((Fa \land Ga) \lor (Ga \leftrightarrow Ha))
  - α(b/x):((Fb∧Ga)∨(Gb↔Hb))\alpha(b/x): ((Fb \land Ga) \lor (Gb \leftrightarrow Hb))

- if we have . . .
  - α(y):∀x((Fx∧Ga)→(Gx∨Hy))\alpha(y): \forall x ((Fx \land Ga) \rightarrow (Gx \lor Hy))
- then we have . . .
  - α(a/y):∀x((Fx∧Ga)→(Gx∨Ha))\alpha(a/y): \forall x ((Fx \land Ga) \rightarrow (Gx \lor Ha))
  - α(b/y):∀x((Fx∧Ga)→(Gx∨Hb))\alpha(b/y): \forall x ((Fx \land Ga) \rightarrow (Gx \lor Hb))

- now let’s put this new terminology to use

complex universally quantified propositions

- let’s see how we can work out the truth value of a complex universally quantified proposition like this:
  - ∀x(Px→Rx)\forall x(Px \rightarrow Rx)
- on a model like this:
  - model 𝓜\mathcal{M}:
    - domain: {Bill, Ben, Alice}
    - extensions: PP: {Bill} RR: {Bill, Alice}

- here’s how we do it
- according to the semantics of MPL: ∀x_α(x_)\forall \underline{x} \alpha (\underline{x}) is true in 𝓜\mathcal{M} iff for every object oo in the domain of 𝓜\mathcal{M}, α(a_/x_)\alpha(\underline{a}/\underline{x}) is true in 𝓜oa_\mathcal{M}_{o}^{\underline{a}}, where a_\underline{a} is some name that is not assigned a referent in 𝓜\mathcal{M}, and

$\mathcal{M}_{o}^{\underline{a}}$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o

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- so take some name that doesn't have a referent on $\mathcal{M}$, say b
- what this means is that $\forall x(Px \rightarrow Rx)$ is true in $\mathcal{M}$ if and only if $Pb \rightarrow Rb$ is true on every model that is exactly like $\mathcal{M}$ except that it assigns a referent to b
- so to check whether our proposition is true on this model, we literally check every model which meets this condition

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- here's one such model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^1$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Bill
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(b/x)$: $(Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^1$

- here's another model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^2$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Ben
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(b/x)$: $(Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^2$

- here's another model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^3$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Alice
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(b/x)$: $(Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^3$

- $\forall x(Px \rightarrow Rx)$ is true in $\mathcal{M}$ if and only if $Pb \rightarrow Rb$ is true on every model that is exactly like $\mathcal{M}$ except that it assigns a referent to b
- we haven't checked every such model
- but we've checked one for each object in the domain of $\mathcal{M}$
- so $\forall x(Px \rightarrow Rx)$ is true in $\mathcal{M}$

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complex existentially quantified propositions

- let's see how we can work out the truth value of a complex existentially quantified proposition like this:

        - ∃x(Px→Rx)\exists x(Px \rightarrow Rx)
- on a model like this:
    - model 𝓜\mathcal{M}:
        - domain: {Bill, Ben, Alice}
        - extensions: PP: {Bill} RR: {Bill, Alice}

- here’s how we do it
- according to the semantics of MPL: ∃x_α(x_)\exists\underline{x} \alpha (\underline{x}) is true in 𝓜\mathcal{M} iff there is at least one object oo in the domain of 𝓜\mathcal{M} such that α(a_/x_)\alpha(\underline{a}/\underline{x}) is true in 𝓜oa_\mathcal{M}_{o}^{\underline{a}}, where a_\underline{a} is some name that is not assigned a referent in 𝓜\mathcal{M}, and 𝓜oa_\mathcal{M}_{o}^{\underline{a}} is a model just like 𝓜\mathcal{M} except that in it the name a_\underline{a} is assigned the referent oo

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- so take some name that doesn’t have a referent on 𝓜\mathcal{M}, say bb
- what this means is that ∃x(Px→Rx)\exists x(Px \rightarrow Rx) is true in 𝓜\mathcal{M} if and only if Pb→RbPb \rightarrow Rb is true on *some* model that is exactly like 𝓜\mathcal{M} except that it assigns a referent to bb
- so to check whether our proposition is true on this model, we check to see if there is such a model which meets this condition

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- here’s one such model which is exactly like 𝓜\mathcal{M} except that it assigns a referent to bb:
    - model 𝓜1\mathcal{M}^1:
        - domain: {Bill, Ben, Alice}
        - referents: bb: Bill
        - extensions: PP: {Bill}, RR: {Bill, Alice}
- we ask whether the following is true on the model:
    - α(b/x):(Pb→Rb)\alpha(b/x): (Pb \rightarrow Rb)
- it is true on 𝓜1\mathcal{M}^1

- ∃x(Px→Rx)\exists x(Px \rightarrow Rx) is true in 𝓜\mathcal{M} if and only if Pb→RbPb \rightarrow Rb is true on *some* model that is exactly like 𝓜\mathcal{M} except that it assigns a referent to bb
- we have just seen such a model
- so ∃x(Px→Rx)\exists x(Px \rightarrow Rx) is true in 𝓜\mathcal{M}

the semantics of MPL

the semantics of MPL, stated again

- we are now, finally, in a position to state complete semantics of MPL . . .
- see handout “The Semantics of MPL”

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1. P_a_\underline{P}\underline{a} is true in 𝓜\mathcal{M} iff the referent of a_\underline{a} in 𝓜\mathcal{M} is in the extension of P_\underline{P} in 𝓜\mathcal{M}

2. ¬α\lnot \alpha is true in 𝓜\mathcal{M} iff α\alpha is false in

$\mathcal{M}$

3. $(\alpha \land \beta)$ is true in $\mathcal{M}$ iff α and β are both true in $\mathcal{M}$
4. $(\alpha \lor \beta)$ is true in $\mathcal{M}$ iff one or both of α and β is true in $\mathcal{M}$
5. $(\alpha \rightarrow \beta)$ is true in $\mathcal{M}$ iff α is false in $\mathcal{M}$ or β is true in $\mathcal{M}$ or both
6. $(\alpha \leftrightarrow \beta)$ is true in $\mathcal{M}$ iff α and β are both true in $\mathcal{M}$ or both false in $\mathcal{M}$

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7. $\forall \underline{x} \alpha (\underline{x})$ is true in $\mathcal{M}$ iff for every object o in the domain of $\mathcal{M}$, $\alpha(\underline{a}/\underline{x})$ is true in $\mathcal{M}_{o}^{\underline{a}}$, where $\underline{a}$ is some name that is not assigned a referent in $\mathcal{M}$, and $\mathcal{M}_{o}^{\underline{a}}$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o
8. $\exists\underline{x} \alpha (\underline{x})$ is true in $\mathcal{M}$ iff there is at least one object o in the domain of $\mathcal{M}$ such that $\alpha(\underline{a}/\underline{x})$ is true in $\mathcal{M}_{o}^{\underline{a}}$, where $\underline{a}$ is some name that is not assigned a referent in $\mathcal{M}$, and $\mathcal{M}_{o}^{\underline{a}}$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o

analyses of logical concepts

analyses of logical concepts

- we now get analyses of our core logical notions
- they are just what you would expect given that a model plays the same role in MPL as truth table rows played in PL

- an argument is **valid** iff there is no model in which the premises are all true and the conclusion is false
- an argument is **invalid** iff there is a model in which the premises are all true and the conclusion is false
- such a model is a **counterexample** or **countermodel** to the argument

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- a proposition is a **tautology** iff there is no model in which it is false
- a proposition is a **contradiction** iff there is no model in which it is true
- a proposition is **satisfiable** if and only if there is at least one model in which it is true

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- two propositions are **equivalent** iff there is no model in which one is true and the other is false
- two propositions are **contradictory** iff there is no model in which they have the same truth value
- two propositions are **jointly satisfiable** iff there is at least one model in which they are both true

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- a set of propositions is **satisfiable** iff there is at least one model in which all the propositions in the set are true

analyses and tests

analyses and tests

- in PL truth tables provided an analysis of validity and a method of testing for validity
- in MPL, models give analyses—but *not* tests
- our analysis of validity *fixes the facts* concerning which arguments in MPL are valid
- however, we have no way of finding out whether an argument is valid How do you know that there is *no* model on which the premises are all true and the conclusion is false? There is an infinite number of models. you can't check them all!
- trees for MPL to the rescue!

wrapping up

this lecture

- the semantics of complicated quantified propositions in MPL
- analyses of central logical notions in terms of models

next lecture

- lecture 17, trees for MPL