lecture 08, functional completeness

phil1012 introductory logic

overview

this lecture

- some reflection on PL
- functional completeness: the idea that a given set of connectives is able to express all possible truth functions
- are the connectives of PL functionally complete?

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
    - explain what it means for a set of connectives to be functionally complete
    - define one set of connectives in terms of another set of connectives
    - determine whether a given set of connectives is functionally complete

required reading

- section 6.6 of chapter 6

functional completeness

functional completeness

- are our five connectives, in some sense, sufficient?
- are there formulas with truth conditions that cannot be constructed using our five connectives?
- are there any truth functions that cannot be expressed by some combination of our five connectives?

- consider the equivalence of $(A \to B)$ and $(\neg A \vee B)$

A	B	$(A \to B)$	$(\neg A \vee B)$
T	T	T	T
F	T	T	T
T	F	F	F
F	F	T	T

- we can construct a formula with the truth conditions of $(A \to B)$ using $\neg$ and $\vee$
- we can express the truth function expressed by $\to$ using $\neg$ and $\vee$
- so there's a sense in which we don't really need $\to$
- we could just use $\neg$ and $\vee$ instead of $\to$

- but now consider the truth table for a connective not included in our five, $\veebar$

A	B	$(A \veebar B)$
T	T	F
F	T	T
T	F	T
F	F	F

- can we construct a formula with the same truth conditions using connectives taken from our original five?

- it turns out that we can

A	B	$(A \veebar B)$	$\neg(A \leftrightarrow B)$
T	T	F	F
F	T	T	T
T	F	T	T
F	F	F	F

- so there a sense in which we don't need to add $\veebar$
- we can already express with our five connectives what we can express with $\veebar$

- can we do what we just did for $\rightarrow$ and $\veebar$ for any possible connective?
- can we express any possible truth function using just our five connectives?

- it turns out that we can
- with its five connectives, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, PL has the resources to construct a formula with any truth conditions whatsoever
- for any possible truth table, there is a formula of PL with that truth table
- our five connectives can express any possible truth function
- we call this feature, **functional completeness**

- a set of connectives is **functionally complete** if we can define all possible connectives from the connectives in that set
- let's prove it!
- first we'll get clearer on what it means to define one connective in terms of other connectives
- then we'll get clearer on the space of possible connectives
- then we'll prove that the set of connectives use in PL is functionally complete

defining one connective in terms of others

defining one connective in terms of others

- we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives

- recall that two forms are **equivalent** if and only if they have the same truth value on every row of the truth table
- lets consider some examples

- the connective $\rightarrow$ can be defined in terms of $\neg$ and $\vee$

α	β	$(\alpha \rightarrow \beta)$	$(\neg\alpha \vee \beta)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- the connective $\leftrightarrow$ can be defined in terms of $\rightarrow$ and $\wedge$

α	β	$(\alpha \leftrightarrow \beta)$	$((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha))$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

- the connective $\wedge$ can be defined in terms of $\vee$ and $\neg$

α	β	$(\alpha \wedge \beta)$	$\neg(\neg\alpha \vee \neg\beta)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

- the connective $\vee$ can be defined in terms of $\wedge$ and $\neg$

α	β	$(\alpha \vee \beta)$	$\neg(\neg\alpha \wedge \neg\beta)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

- so, we can define connectives in terms of other connectives
- we show that a form using the connective to be defined is equivalent to a form using only the other connectives

the range of possible connectives

the range of possible connectives

- we want to show that we all possible connectives can be defined in terms of the five connectives of PL
- in order to do so we need to understand what all the possible connectives are
- there are zero place connectives, one place connectives, two place connectives, three place connectives, and so on

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- possible zero place connectives

$\top$	$\bot$

T	F

* possible one place connectives

α				
T	T	T	F	F
F	T	F	T	F

* we use ①$_1$ for the first one-place connective
* the number in the circle represents the number of places of the connective
* the subscript represents the number of the connective

* possible one place connectives

α	①$_1$	①$_2$	①$_3$	①$_4$
T	T	T	F	F
F	T	F	T	F

* possible one place connectives

α	①$_1$	①$_2$	$\neg$	①$_4$
T	T	T	F	F
F	T	F	T	F

* ①$_3$ is $\neg$

* possible two place connectives

α	β	②$_1$	②$_2$	②$_3$	②$_4$	②$_5$	②$_6$	②$_7$	②$_8$	②$_9$	②$_{10}$	②$_{11}$	②$_{12}$	②$_{13}$	②$_{14}$	②$_{15}$	②$_{16}$
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

* possible two place connectives

α	β	②$_1$	$\vee$	②$_3$	②$_4$	$\rightarrow$	②$_6$	$\leftrightarrow$	$\wedge$	②$_9$	②$_{10}$	②$_{11}$	②$_{12}$	②$_{13}$	②$_{14}$	②$_{15}$	②$_{16}$
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

* there are also three place connectives
* they look like this: ③$_1$
* they make propositions like this: ③$_1$$(P, Q, R)$
* there are 256 three place connectives!

* there are also four place connectives
* they look like this: ④$_1$
* they make propositions like this: ④$_1$$(P, Q, R, S)$
* there are 65,536 four place connectives!!

- and so on . . .

- obviously, we won't be proving that we can define any connective in terms of our five one by one!
- we better find a method that can obviously be extended to show that we can define any connective in terms of our five

defining any connective using $\neg$, $\wedge$, and $\vee$

defining any connective using $\neg$, $\wedge$, and $\vee$

- okay, we are finally ready to show that any possible connective can be defined in terms of our five
- in fact, we will show that any possible connective can be defined in terms of $\neg$, $\wedge$, and $\vee$
- obviously we aren't going to define them one by one
- we'll define the zero-place connectives first, one by one
- then we will develop a method for defining any n-place connective in terms of $\neg$, $\wedge$, and $\vee$

- the zero-place connectives $\top$ and $\perp$ can be defined in terms of the connectives $\{\neg, \wedge, \vee\}$

α	$\top$	$(\alpha \vee \neg\alpha)$	$\perp$	$(\alpha \wedge \neg\alpha)$
T	T	T	F	F
F	T	T	F	F

- so there's a sense in which we don't need the zero place connectives
- remember that to define one connective in terms of others, it is enough to show that formulas using them are equivalent

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- now we show that any n-place connective can be defined in terms of $\{\neg, \wedge, \vee\}$ by the following procedure

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- take some n-place connective. call it '*'
- and take any function from n truth values to truth values

α	β	$(\alpha * \beta)$
T	T	T
T	F	F
F	T	T
F	F	F

- now take the conjunctions which 'describe' the rows in which $(\alpha * \beta)$ is true
- in this case: $(\neg\alpha \wedge \beta)$, $(\alpha \wedge \beta)$

α	β	$(\alpha * \beta)$
T	T	T
T	F	F
F	T	T

F	F	F

- in other words
  - if α is true on the row on which $(\alpha * \beta)$ is true, then make α the first conjunct
  - if it is false, make $\neg\alpha$ the first conjunct
  - do the same for β, and then do the same for each row on which $(\alpha * \beta)$ is true
- by this method we get: $(\neg\alpha \wedge \beta)$, $(\alpha \wedge \beta)$

- now form a disjunction from the conjunctions you got from the previous step: $((\neg\alpha \wedge \beta) \vee (\alpha \wedge \beta))$
- and you are done!

- to see that you are done, you can put the disjunction into the table, and you will see that it is equivalent to the formula using '*'

α	β	$(\alpha * \beta)$	$((\neg\alpha \wedge \beta) \vee (\alpha \wedge \beta))$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	F	F

- any n-place connective can be defined in terms of $\{\neg, \wedge, \vee\}$ by this procedure!
- it should be obvious that for any truth table whatsoever, we only need $\neg$, $\wedge$, and $\vee$ to carry out the procedure of describing the row on which some proposition is true and making a disjunction of these conjunctions
- by this method we prove that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives!

functionally complete sets of connectives

functionally complete sets of connectives

- okay, we just proved that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives
- now let's consider the general case of functionally complete sets of connectives
- let's do so by considering some consequences of the fact that $\{\neg, \wedge, \vee\}$ is a functionally complete set of connectives

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- **fact**: the set of connectives $\{\neg, \wedge, \vee\}$ is functionally complete
- to show that some set of connectives is functionally complete, it suffices to show that $\neg$, $\wedge$, and $\vee$, can be defined using members of that set
  - if $\{\neg, \wedge, \vee\}$ is functionally complete, and $\neg$, $\wedge$, and $\vee$ can be defined in terms of some other set of connectives, then that set must be functionally complete too

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- **fact**: the set of connectives $\{\neg, \wedge\}$ is functionally complete
- **proof**: $\neg$, $\wedge$, and $\vee$ can be defined in terms of $\neg$ and $\wedge$ alone, and

{¬, ∧, and ∨} is functionally complete

- **fact**: the set of connectives {¬, ∨} is functionally complete
- **proof**: ¬, ∧, and ∨ can be defined in terms of ¬ and ∨ alone, and {¬, ∧, and ∨} is functionally complete

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- **fun fact**: {②$_9$} is a functionally complete set of connectives

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- to sum up, then
- to show that some set of connectives is functionally complete, it suffices to show that ¬, and either ∧ or ∨, can be defined using members of that set (for then you can rely on the proof that {¬, ∧, ∨} is functionally complete above)
- to show that a set of connectives is not functionally complete, we need to show that there is *some* connective that cannot be defined in terms of those in the set

wrapping up

this lecture

- with its five connectives, ¬, ∧, ∨, →, and ↔, PL has the resources to construct a formula with any truth conditions whatsoever
- in other words, the set of connectives in PL are **functionally complete**
- you (probably) will not be required to prove that {¬, ∧, ∨} is functionally complete
- but you will be required to prove that some given set of connectives is functionally complete
- to do so, you need only show that ¬, ∧ and ∨ can be defined in terms of the connectives you are given

next lectures

- lecture 09, issues in translation: conjunction
- lecture 10, trees for PL