lecture 16, the semantics of MPL (part 2)

phil1012 introductory logic

overview

this lecture

- second of two lectures on the semantics of MPL
- the semantics of complicated quantified propositions in MPL
- analyses of logical concepts in MPL

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - explain how the truth values of complex quantified propositions are determined in MPL
  - explain how models provide analyses of various logical concepts (like validity) in MPL
  - determine whether a complex quantified proposition of MPL is true or false on a given model
  - describe a model on which an MPL proposition is true and/or a model on which a proposition is false

required reading

- sections 9.4 and 9.5 of chapter 9

complex quantified propositions

complex quantified propositions

- how are the truth values of the following determined?
  - $\forall x(\ Px \to Rx)$
  - $\exists x(\ Px \wedge Rx)$
- we do not yet have a way of determining the truth values of these propositions

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- but we do have a way of determining the truth values of the following in a model:
  - $(\ Pa \to Ra)$
  - $(\ Pa \wedge Ra)$

- so here's an idea . . .
- we can replace a question we *cannot* answer with many questions we *can* answer
- we can answer questions about the values of expressions like $(\ Pa \wedge Ra)$ in models related to $\mathscr{M}$
- it turns out that if we ask the right questions we can answer

questions about the truth values of complex quantified expressions

some new terminology

- but before we get to that, we need some new terminology
- we use $\alpha(\underset{\sim}{x})$ to stand for an arbitrary formula which has no free variables other than $\underset{\sim}{x}$
- we use $\alpha(\underset{\sim}{a}/\underset{\sim}{x})$ to stand for a formula that results from $\alpha(\underset{\sim}{x})$ by replacing all the free occurrences of $\underset{\sim}{x}$ in $\alpha(\underset{\sim}{x})$ with $\underset{\sim}{a}$

- for example
- if we have . . .
  - $\alpha(x) : (Fx \rightarrow Gx)$
- then we have . . .
  - $\alpha(a/x) : (Fa \rightarrow Ga)$
  - $\alpha(b/x) : (Fb \rightarrow Gb)$

- if we have . . .
  - $\alpha(x) : ((Fx \wedge Ga) \vee (Gx \leftrightarrow Hx))$
- then we have . . .
  - $\alpha(a/x) : ((Fa \wedge Ga) \vee (Ga \leftrightarrow Ha))$
  - $\alpha(b/x) : ((Fb \wedge Ga) \vee (Gb \leftrightarrow Hb))$

- if we have . . .
  - $\alpha(y) : \forall x((Fx \wedge Ga) \rightarrow (Gx \vee Hy))$
- then we have . . .
  - $\alpha(a/y) : \forall x((Fx \wedge Ga) \rightarrow (Gx \vee Ha))$
  - $\alpha(b/y) : \forall x((Fx \wedge Ga) \rightarrow (Gx \vee Hb))$

- now let’s put this new terminology to use

complex universally quantified propositions

- let’s see how we can work out the truth value of a complex universally quantified proposition like this:
  - $\forall x(Px \rightarrow Rx)$
- on a model like this:
  - model $\mathscr{M}$:
    - domain: {Bill, Ben, Alice}
    - extensions: P: {Bill} R: {Bill, Alice}

- here’s how we do it
- according to the semantics of MPL: $\forall \underset{\sim}{x}\alpha(\underset{\sim}{x})$ is true in $\mathscr{M}$ iff for every object o in the domain of $\mathscr{M}$, $\alpha(\underset{\sim}{a}/\underset{\sim}{x})$ is true in $\mathscr{M}^{\underset{\sim}{a}}_{o}$, where $\underset{\sim}{a}$ is some name that is not assigned a referent in $\mathscr{M}$, and $\mathscr{M}^{\underset{\sim}{a}}_{o}$ is a model just like $\mathscr{M}$ except that in it the name $\underset{\sim}{a}$ is assigned the referent o

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- so take some name that doesn’t have a referent on $\mathscr{M}$, say b
- what this means is that $\forall x(Px \rightarrow Rx)$ is true in $\mathscr{M}$ if and only if $Pb \rightarrow Rb$ is true on every model that is exactly like $\mathscr{M}$ except that it assigns a referent to b
- so to check whether our proposition is true on this model, we literally check every model which meets this condition

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- here's one such model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^1$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Bill
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(\ b/x)$:$(\ Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^1$

- here's another model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^2$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Ben
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(\ b/x)$:$(\ Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^2$

- here's another model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^3$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Alice
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(\ b/x)$:$(\ Pb \rightarrow Rb)$
- it is true in $\mathcal{M}^3$

- $\forall x(\ Px \rightarrow Rx)$ is true in $\mathcal{M}$ if and only if $Pb \rightarrow Rb$ is true on every model that is exactly like $\mathcal{M}$ except that it assigns a referent to b
- we haven't checked every such model
- but we've checked one for each object in the domain of $\mathcal{M}$
- so $\forall x(\ Px \rightarrow Rx)$ is true in $\mathcal{M}$

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complex existentially quantified propositions

- let's see how we can work out the truth value of a complex existentially quantified proposition like this:
  - $\exists x(\ Px \rightarrow Rx)$
- on a model like this:
  - model $\mathcal{M}$:
    - domain: {Bill, Ben, Alice}
    - extensions: P: {Bill} R: {Bill, Alice}

- here's how we do it
- according to the semantics of MPL: $\exists \underline{x} \alpha(\ \underline{x})$ is true in $\mathcal{M}$ iff there is at least one object o in the domain of $\mathcal{M}$ such that $\alpha(\ \underline{a}/\underline{x})$ is true in $\mathcal{M}^{\underline{a}}_{o}$, where $\underline{a}$ is some name that is not assigned a referent in $\mathcal{M}$, and $\mathcal{M}^{\underline{a}}_{o}$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o

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- so take some name that doesn't have a referent on $\mathcal{M}$, say b

- what this means is that $\exists x(Px \rightarrow Rx)$ is true in $\mathcal{M}$ if and only if $Pb \rightarrow Rb$ is true on *some* model that is exactly like $\mathcal{M}$ except that it assigns a referent to b
- so to check whether our proposition is true on this model, we check to see if there is such a model which meets this condition

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- here's one such model which is exactly like $\mathcal{M}$ except that it assigns a referent to b:
  - model $\mathcal{M}^1$:
    - domain: {Bill, Ben, Alice}
    - referents: b: Bill
    - extensions: P: {Bill}, R: {Bill, Alice}
- we ask whether the following is true on the model:
  - $\alpha(b/x) :(Pb \rightarrow Rb)$
- it is true on $\mathcal{M}^1$

- $\exists x(Px \rightarrow Rx)$ is true in $\mathcal{M}$ if and only if $Pb \rightarrow Rb$ is true on *some* model that is exactly like $\mathcal{M}$ except that it assigns a referent to b
- we have just seen such a model
- so $\exists x(Px \rightarrow Rx)$ is true in $\mathcal{M}$

the semantics of MPL

the semantics of MPL, stated again

- we are now, finally, in a position to state complete semantics of MPL . . .
- see handout "The Semantics of MPL"

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1. $\underline{P}\underline{a}$ is true in $\mathcal{M}$ iff the referent of $\underline{a}$ in $\mathcal{M}$ is in the extension of $\underline{P}$ in $\mathcal{M}$

2. $\neg\alpha$ is true in $\mathcal{M}$ iff α is false in $\mathcal{M}$

3. $(\alpha \wedge \beta)$ is true in $\mathcal{M}$ iff α and β are both true in $\mathcal{M}$

4. $(\alpha \vee \beta)$ is true in $\mathcal{M}$ iff one or both of α and β is true in $\mathcal{M}$

5. $(\alpha \rightarrow \beta)$ is true in $\mathcal{M}$ iff α is false in $\mathcal{M}$ or β is true in $\mathcal{M}$ or both

6. $(\alpha \leftrightarrow \beta)$ is true in $\mathcal{M}$ iff α and β are both true in $\mathcal{M}$ or both false in $\mathcal{M}$

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7. $\forall \underline{x}\alpha(\underline{x})$ is true in $\mathcal{M}$ iff for every object o in the domain of $\mathcal{M}$, $\alpha(\underline{a}/\underline{x})$ is true in $\mathcal{M}^{\underline{a}}_o$, where $\underline{a}$ is some name that is not assigned a referent in $\mathcal{M}$, and $\mathcal{M}^{\underline{a}}_o$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o

8. $\exists \underline{x}\alpha(\underline{x})$ is true in $\mathcal{M}$ iff there is at least one object o in the domain of $\mathcal{M}$ such that $\alpha(\underline{a}/\underline{x})$ is true in $\mathcal{M}^{\underline{a}}_o$, where $\underline{a}$ is some name that is not assigned a referent in $\mathcal{M}$, and $\mathcal{M}^{\underline{a}}_o$ is a model just like $\mathcal{M}$ except that in it the name $\underline{a}$ is assigned the referent o

analyses of logical concepts

analyses of logical concepts

- we now get analyses of our core logical notions
- they are just what you would expect given that a model plays the same role in MPL as truth table rows played in PL

- an argument is **valid** iff there is no model in which the premises are all true and the conclusion is false
- an argument is **invalid** iff there is a model in which the premises are all true and the conclusion is false
- such a model is a **counterexample** or **countermodel** to the argument

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- a proposition is a **tautology** iff there is no model in which it is false
- a proposition is a **contradiction** iff there is no model in which it is true
- a proposition is **satisfiable** if and only if there is at least one model in which it is true

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- two propositions are **equivalent** iff there is no model in which one is true and the other is false
- two propositions are **contradictory** iff there is no model in which they have the same truth value
- two propositions are **jointly satisfiable** iff there is at least one model in which they are both true

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- a set of propositions is **satisfiable** iff there is at least one model in which all the propositions in the set are true

analyses and tests

analyses and tests

- in PL truth tables provided an analysis of validity and a method of testing for validity
- in MPL, models give analyses—but *not* tests
- our analysis of validity *fixes the facts* concerning which arguments in MPL are valid
- however, we have no way of finding out whether an argument is valid How do you know that there is *no* model on which the premises are all true and the conclusion is false? There is an infinite number of models. you can't check them all!
- trees for MPL to the rescue!

wrapping up

this lecture

- the semantics of complicated quantified propositions in MPL
- analyses of central logical notions in terms of models

next lecture

- lecture 17, trees for MPL