lecture 18, uses of trees for MPL

phil1012 introductory logic

overview

this lecture

- uses of trees for MPL
- how to set up trees to test for various logical properties in MPL
- how to read off models from completed trees
- identifying infinite trees and reading models off the open paths of infinite trees

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
    - use trees to test for various logical properties of MPL formulas
    - read off (counter)models from open paths of MPL trees
    - identify infinite trees and read models of infinite paths

required reading

- sections 10.2 and 10.3 of chapter 10

using trees

using trees

- MPL trees, like PL trees, test for satisfiability, in the first instance.
- but we can use them to test for much more
- again, you need to know how to set up the tree, and how to interpret its results
- in the case of MPL trees, we read off models on which the initial propositions are jointly satisfiable
- these are called **countermodels** in the case of arguments.

- we set up and use MPL trees to test for various properties in much the same way as we did for PL trees.
- let's look at an example of testing some proposition to see whether it is a tautology

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To prove: whether $((\exists xFx \wedge \forall x(Fx \rightarrow Gx)) \rightarrow \exists xGx)$ is a tautology.

To prove: whether $((\exists xFx \wedge \forall x(Fx \rightarrow Gx)) \rightarrow \exists xGx)$ is a tautology.

1. $\neg(\exists xFx \wedge \forall x(Fx \rightarrow Gx)) \rightarrow \exists xGx)$ Assumption

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To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx))$	Negated Conditional
3.	$\neg\exists xGx$	Negated Conditional

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx)) \checkmark$	Negated Conditional
3.	$\neg\exists xGx$	Negated Conditional
4.	$\exists xFx$	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx)) \checkmark$	Negated Conditional
3.	$\neg\exists xGx \checkmark$	Negated Conditional
4.	$\exists xFx$	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction
6.	$\forall x\neg Gx$	Negated Existential

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx)) \checkmark$	Negated Conditional
3.	$\neg\exists xGx \checkmark$	Negated Conditional
4.	$\exists xFx \checkmark a$	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction
6.	$\forall x\neg Gx$	Negated Existential
7.	Fa	Existential

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx)) \checkmark$	Negated Conditional
3.	$\neg\exists xGx \checkmark$	Negated Conditional
4.	$\exists xFx \checkmark a$	Conjunction
5.	$\forall x(Fx \to Gx) \backslash a$	Conjunction
6.	$\forall x\neg Gx$	Negated Existential
7.	Fa	Existential
8.	$Fa \to Ga$	Universal

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx) \checkmark$	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx)) \checkmark$	Negated Conditional
3.	$\neg\exists xGx \checkmark$	Negated Conditional
4.	$\exists xFx \checkmark a$	Conjunction
5.	$\forall x(Fx \to Gx) \backslash a$	Conjunction
6.	$\forall x\neg Gx$	Negated Existential
7.	Fa	Existential
8.	$Fa \to Ga \checkmark$	Universal
9.	$\neg Fa \qquad Ga$	Conditional
	$\otimes$	

To prove: whether $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology.

1.	$\neg(\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ ✓	Assumption
2.	$(\exists xFx \wedge \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists xGx$ ✓	Negated Conditional
4.	$\exists xFx$ ✓a	Conjunction
5.	$\forall x(Fx \to Gx)$ $\backslash a$	Conjunction
6.	$\forall x\neg Gx$ $\backslash a$	Negated Existential
7.	Fa	Existential
8.	$Fa \to Ga$ ✓	Universal
9.	$\neg Fa$ $\quad$ Ga	Conditional
10.	$\otimes$ $\quad$ $\neg Ga$	Universal
	$\quad\quad$ $\otimes$	

- we can conclude that $((\exists xFx \wedge \forall x(Fx \to Gx)) \to \exists xGx)$ is a tautology, since its negation is not satisfiable.

- to take another example, suppose we want to test whether some argument is a valid argument

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To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$	Assumption
2.	$\exists xGx$	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$	Negated Conclusion

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$	Assumption
2.	$\exists xGx$	Assumption
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To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$	Assumption
2.	$\exists xGx$	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx)$	Negated Existential

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx)$	Negated Existential
5.	Fa	Existential

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx)$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga)$	Universal

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx \checkmark a$		Assumption
2.	$\exists xGx \checkmark b$		Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$		Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a$		Negated Existential
5.	Fa		Existential
6.	Gb		Existential
7.	$\neg(Fa \wedge Ga) \checkmark$		Universal
8.	$\neg Fa$	$\neg Ga$	Negated Conjunction
	$\otimes$		

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx \checkmark a$		Assumption
2.	$\exists xGx \checkmark b$		Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$		Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$		Negated Existential
5.	Fa		Existential
6.	Gb		Existential
7.	$\neg(Fa \wedge Ga) \checkmark$		Universal
8.	$\neg Fa$	$\neg Ga$	Negated Conjunction
9.	$\otimes$	$\neg(Fb \wedge Gb)$	Universal

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$ ✓b	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga)$ ✓	Universal
8.	$\neg Fa \qquad \neg Ga$	Negated Conjunction
9.	$\otimes \qquad \neg(Fb \wedge Gb)$ ✓	Universal
10.	$\neg Fb \qquad \neg Gb$	Negated Conjunction
	$\otimes$	

- we can conclude that $\exists xFx, \exists xGx \therefore \exists x(\ Fx \wedge Gx)$ is not a valid argument, since the premises, $\exists xFx, \exists xGx$, and the negated conclusion, $\neg\exists x(\ Fx \wedge Gx)$ are jointly satisfiable.
- we want to be able to read off a countermodel from our tree.
- how?

how to read off models from open paths

how to read off models from open paths

- a model consists of:
  - a domain
  - a referent for each name which appears on the path
  - an extension for each predicate which appears on the path

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- where there are n names in the path, we write our domain as follows:
  - domain: {1, …, n}
- so if there are 3 names in the path, we write our domain as follows:
  - domain: {1,2,3}
- we then assign each name in the path to an object in the domain as follows:
  - referents: a:1, b:2, c:3, …

- we then assign an extension to each predicate which makes atomic formulas involving the predicate true.
- if Fa, Ga, and Gb, are all on an open path, then we assign the following extensions to the predicates:
  - extensions: F: {1}, G: {1,2}.
- if the predicate H is on the open path but does not occur in an atomic formula, then we assign the following extension to the predicate:
  - extensions: H: $\emptyset$ (Not H: {$\emptyset$}).

- let's consider an example …

- suppose we want to read a model off of this tree:

1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \land Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \land Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \land Ga) \checkmark$	Universal
8.	$\neg Fa \quad \neg Ga$	Negated Conjunction
9.	$\otimes \quad \neg(Fb \land Gb) \checkmark$	Universal
10.	$\neg Fb \quad \neg Gb$	Negated Conjunction
	$\otimes$	

- `first step: find the number of names on the open path`

1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \land Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \land Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \land Ga) \checkmark$	Universal
8.	$\neg Fa \quad \neg Ga$	Negated Conjunction
9.	$\otimes \quad \neg(Fb \land Gb) \checkmark$	Universal
10.	$\neg Fb \quad \neg Gb$	Negated Conjunction
	$\otimes$	

- `there are two, so we have the following domain:`
  - `domain: {1, 2}`

- `second step: assign each name in the path to an object in the domain`

1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \land Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \land Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \land Ga) \checkmark$	Universal
8.	$\neg Fa \quad \neg Ga$	Negated Conjunction
9.	$\otimes \quad \neg(Fb \land Gb) \checkmark$	Universal
10.	$\neg Fb \quad \neg Gb$	Negated Conjunction
	$\otimes$	

- we assign referents in the most natural manner:
  - referents: a: 1, b: 2

- third step: assign an extension to each predicate which makes atomic formulas involving the predicate true

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1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	$\boxed{Fa}$	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa$ $\neg Ga$	Negated Conjunction
9.	$\otimes$ $\neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb$ $\neg Gb$	Negated Conjunction
	$\otimes$	

- we assign an extension to F which makes Fa true:
  - extensions: F: {1}

1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	$\boxed{Fa}$	Existential
6.	$\boxed{Gb}$	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa$ $\neg Ga$	Negated Conjunction
9.	$\otimes$ $\neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb$ $\neg Gb$	Negated Conjunction
	$\otimes$	

- we assign an extension to G which makes Gb true:
  - extensions: F: {1}, G: {2}

- here is our completed model:
  - domain: {1, 2}
  - deferents: a: 1, b: 2
  - extensions: F: {1}, G: {2}

oh no! infinite trees

oh no! infinite trees

- unlike PL trees, MPL trees have an interesting feature: they can be infinitely long
- see pp. 372-373 of the textbook for how to avoid infinite trees in MPL
- let's consider an example of an infinite tree

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy)$

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy) \setminus a$
2.	$(Fa \wedge \exists yGy)$

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy) \setminus a$
2.	$(Fa \wedge \exists yGy)$ ✓
3.	$\exists yGy$
4.	Fa

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy) \setminus a$
2.	$(Fa \wedge \exists yGy)$ ✓
3.	$\exists yGy$ ✓ b
4.	Fa
5.	Gb

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy) \setminus a, b$
2.	$(Fa \wedge \exists yGy)$ ✓
3.	$\exists yGy$ ✓ b
4.	Fa
5.	Gb
6.	$(Fb \wedge \exists yGy)$ ✓
7.	$\exists yGy$ ✓ c
8.	Fb
9.	Gc

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1.	$\forall x(Fx \wedge \exists yGy) \setminus a, b, c$
2.	$(Fa \wedge \exists yGy)$ ✓
3.	$\exists yGy$ ✓ b
4.	Fa
5.	Gb
6.	$(Fb \wedge \exists yGy)$ ✓
7.	$\exists yGy$ ✓ c
8.	Fb
9.	Gc
10.	$(Fc \wedge \exists yGy)$ ✓
11.	$\exists yGy$ ✓ d
12.	Fc
13.	Gd

- this tree is going to go on like this forever
- it will never be complete
- it has an infinite path

- is the proposition satisfiable or not?
- is the path saturated or not?
- it is saturated
- so it is satisfiable

- we can read off a model

- will the proposition always be satisfiable if we have an infinite tree?
- will infinite paths always be saturated?
- no and no.
- here is an example

To prove: whether $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$

To prove: whether $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga) \checkmark$
2. $\forall x(Fx \wedge \exists yGy)$
3. $(Ga \wedge \neg Ga)$

To prove: whether $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga) \checkmark$
2. $\forall x(Fx \wedge \exists yGy) \backslash a$
3. $(Ga \wedge \neg Ga)$
4. $Fa \wedge \exists yGy$

To prove: whether $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga) \checkmark$
2. $\forall x(Fx \wedge \exists yGy) \backslash a$
3. $(Ga \wedge \neg Ga)$
4. $Fa \wedge \exists yGy \checkmark$
5. Fa
6. $\exists yGy$

To prove: whether $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \wedge (Ga \wedge \neg Ga) \checkmark$
2. $\forall x(Fx \wedge \exists yGy) \backslash a, b$
3. $(Ga \wedge \neg Ga)$
4. $Fa \wedge \exists yGy \checkmark$
5. Fa
6. $\exists yGy \checkmark b$
7. Gb
8. $Fb \wedge \exists yGy$
9. Fb
10. $\exists yGy \checkmark c$
11. Gc

- here is another infinite path (if we ignore $Ga \wedge \neg Ga$).
- but the proposition is not satisfiable.
- and the path is not saturated.
- and if we were to saturate it, by applying the rule for $\wedge$ to $Ga \wedge \neg Ga$, it would close straight away.

reading models off infinite trees

reading models off infinite trees

- I said we could read a model off of our infinite tree above.
- let's look at how.
- here is our tree again:

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a, b, c$
2. $(Fa \wedge \exists yGy)$ ✓
3. $\exists yGy$ ✓b
4. Fa
5. Gb
6. $(Fb \wedge \exists yGy)$ ✓
7. $\exists yGy$ ✓c
8. Fb
9. Gc
10. $(Fc \wedge \exists yGy)$ ✓
11. $\exists yGy$ ✓d
12. Fc
13. Gd

- what is the domain?
- how many names occur on the open path?
- an infinite number of names occur
- so here is our domain:
  - domain: {1, 2, 3, . . . }

- how shall we assign referents to each of the infinite number of names?
- like this of course:
  - referents: a: 1, b: 2, c: 3 . . .

- and what about extensions for the predicates?
- well, there are only two predicates F and G.
- but we need an assignment which makes them true whenever they appear in atomic propositions on the path.
- let's look at our tree and think about it . . .

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a, b, c$
2. $(Fa \wedge \exists yGy)$ ✓
3. $\exists yGy$ ✓b
4. Fa
5. Gb
6. $(Fb \wedge \exists yGy)$ ✓
7. $\exists yGy$ ✓c
8. Fb
9. Gc
10. $(Fc \wedge \exists yGy)$ ✓
11. $\exists yGy$ ✓d
12. Fc
13. Gd

- the pattern for G is: Gb, Gc, Gd, …
- the pattern for F is: Fa, Fb, Fc, …
- our complete model, then:
  - domain: {1, 2, 3, . . . }
  - referents: a: 1, b: 2, c: 3 . . .

  - extensions: F: $\{1, 2, 3, . . .\}$, G: $\{2, 3, . . .\}$.

- will you have to read off a model from an infinite tree in the problem sets or the exam?
- possibly. but if so, it won't be a difficult pattern to identify.

wrapping up

this lecture

next lecture

- lecture 19, the formal language GPL