lecture 18, uses of trees for MPL

phil1012 introductory logic

overview

this lecture

- uses of trees for MPL
- how to set up trees to test for various logical properties in MPL
- how to read off models from completed trees
- identifying infinite trees and reading models off the open paths of infinite trees

learning outcomes

- after doing the relevant reading for this lecture, listening to the lecture, and attending the relevant tutorial, you will be able to:
  - use trees to test for various logical properties of MPL formulas
  - read off (counter)models from open paths of MPL trees
  - identify infinite trees and read models of infinite paths

required reading

- sections 10.2 and 10.3 of chapter 10

using trees

using trees

- MPL trees, like PL trees, test for satisfiability, in the first instance.
- but we can use them to test for much more
- again, you need to know how to set up the tree, and how to interpret its results
- in the case of MPL trees, we read off models on which the initial propositions are jointly satisfiable
- these are called **countermodels** in the case of arguments.

- we set up and use MPL trees to test for various properties in much the same way as we did for PL trees.
- let's look at an example of testing some proposition to see whether it is a tautology

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To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1. $\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ Assumption

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To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$	Negated Conditional
3.	$\neg\exists x Gx$	Negated Conditional

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$	Negated Conditional
4.	$\exists x Fx$	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$ ✓	Negated Conditional
4.	$\exists x Fx$	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction
6.	$\forall x \neg Gx$	Negated Existential

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$ ✓	Negated Conditional
4.	$\exists x Fx$ ✓a	Conjunction
5.	$\forall x(Fx \to Gx)$	Conjunction
6.	$\forall x \neg Gx$	Negated Existential
7.	Fa	Existential

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$ ✓	Negated Conditional
4.	$\exists x Fx$ ✓a	Conjunction
5.	$\forall x(Fx \to Gx) \backslash a$	Conjunction
6.	$\forall x \neg Gx$	Negated Existential
7.	Fa	Existential
8.	$Fa \to Ga$	Universal

To prove: whether $((\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \to Gx)) \to \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \to Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$ ✓	Negated Conditional
4.	$\exists x Fx$ ✓a	Conjunction
5.	$\forall x(Fx \to Gx) \backslash a$	Conjunction
6.	$\forall x \neg Gx$	Negated Existential
7.	Fa	Existential
8.	$Fa \to Ga$ ✓	Universal
9.	$\neg Fa \qquad Ga$	Conditional
	$\otimes$	

To prove: whether $((\exists x Fx \land \forall x(Fx \rightarrow Gx)) \rightarrow \exists x Gx)$ is a tautology.

1.	$\neg(\exists x Fx \land \forall x(Fx \rightarrow Gx)) \rightarrow \exists x Gx)$ ✓	Assumption
2.	$(\exists x Fx \land \forall x(Fx \rightarrow Gx))$ ✓	Negated Conditional
3.	$\neg\exists x Gx$ ✓	Negated Conditional
4.	$\exists x Fx$ ✓a	Conjunction
5.	$\forall x(Fx \rightarrow Gx) \setminus a$	Conjunction
6.	$\forall x \neg Gx \setminus a$	Negated Existential
7.	Fa	Existential
8.	$Fa \rightarrow Ga$ ✓	Universal
9.	$\neg Fa$ Ga	Conditional
10.	$\otimes$ $\neg Ga$	Universal
	$\otimes$	

- we can conclude that ((∃xFx∧∀x(Fx→Gx))→∃xGx)((\exists x Fx \land \forall x (Fx \rightarrow Gx)) \rightarrow \exists xGx) is a tautology, since its negation is not satisfiable.

- to take another example, suppose we want to test whether some argument is a valid argument

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To prove: whether $\exists x Fx, \exists x Gx \therefore \exists x(Fx \land Gx)$ is a valid argument.

To prove: whether $\exists x Fx, \exists x Gx \therefore \exists x(Fx \land Gx)$ is a valid argument.

1.	$\exists x Fx$	Assumption
2.	$\exists x Gx$	Assumption
3.	$\neg\exists x(Fx \land Gx)$	Negated Conclusion

To prove: whether $\exists x Fx, \exists x Gx \therefore \exists x(Fx \land Gx)$ is a valid argument.

1.	$\exists x Fx$	Assumption
2.	$\exists x Gx$	Assumption
3.	$\neg\exists x(Fx \land Gx)$	Negated Conclusion

To prove: whether $\exists x Fx, \exists x Gx \therefore \exists x(Fx \land Gx)$ is a valid argument.

1.	$\exists x Fx$	Assumption
2.	$\exists x Gx$	Assumption
3.	$\neg\exists x(Fx \land Gx)$ ✓	Negated Conclusion
4.	$\forall x \neg(Fx \land Gx)$	Negated Existential

To prove: whether $\exists x Fx, \exists x Gx \therefore \exists x(Fx \land Gx)$ is a valid argument.

1.	$\exists x Fx$ ✓a	Assumption
2.	$\exists x Gx$	Assumption
3.	$\neg\exists x(Fx \land Gx)$ ✓	Negated Conclusion
4.	$\forall x \neg(Fx \land Gx)$	Negated Existential
5.	Fa	Existential

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$ ✓b	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx)$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$ ✓b	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga)$	Universal

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$ ✓b	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga)$ ✓	Universal
8.	$\neg Fa \qquad \neg Ga$	Negated Conjunction
	$\otimes$	

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \wedge Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a	Assumption
2.	$\exists xGx$ ✓b	Assumption
3.	$\neg\exists x(Fx \wedge Gx)$ ✓	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga)$ ✓	Universal
8.	$\neg Fa \qquad \neg Ga$	Negated Conjunction
9.	$\otimes \qquad \neg(Fb \wedge Gb)$	Universal

To prove: whether $\exists xFx, \exists xGx \therefore \exists x(Fx \land Gx)$ is a valid argument.

1.	$\exists xFx$ ✓a			Assumption
2.	$\exists xGx$ ✓b			Assumption
3.	$\lnot\exists x(Fx \land Gx)$ ✓			Negated Conclusion
4.	$\forall x\lnot(Fx \land Gx)$ $\backslash a, b$			Negated Existential
5.	Fa			Existential
6.	Gb			Existential
7.	$\lnot(Fa \land Ga)$ ✓			Universal
8.	$\lnot Fa$	$\lnot Ga$		Negated Conjunction
9.	$\otimes$	$\lnot(Fb \land Gb)$ ✓		Universal
10.		$\lnot Fb$	$\lnot Gb$	Negated Conjunction
			$\otimes$	

- we can conclude that ∃xFx,∃xGx∴∃x(Fx∧Gx)\exists xFx, \exists xGx \therefore \exists x(Fx \land Gx) is not a valid argument, since the premises, ∃xFx,∃xGx,\exists xFx, \exists xGx, and the negated conclusion, ¬∃x(Fx∧Gx)\lnot \exists x(Fx \land Gx) are jointly satisfiable.
- we want to be able to read off a countermodel from our tree.
- how?

how to read off models from open paths

how to read off models from open paths

- a model consists of:
  - a domain
  - a referent for each name which appears on the path
  - an extension for each predicate which appears on the path

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- where there are *n* names in the path, we write our domain as follows:
  - domain: {1, …, *n*}
- so if there are 3 names in the path, we write our domain as follows:
  - domain: {1,2,3}
- we then assign each name in the path to an object in the domain as follows:
  - referents: aa:1, bb:2, cc:3, …

- we then assign an extension to each predicate which makes atomic formulas involving the predicate true.
- if FaFa, GaGa, and GbGb, are all on an open path, then we assign the following extensions to the predicates:
  - extensions: FF: {1}, GG: {1,2}.
- if the predicate HH is on the open path but does not occur in an atomic formula, then we assign the following extension to the predicate:
  - extensions: HH: ∅\emptyset (Not HH: {∅\emptyset}).

- let’s consider an example …

- suppose we want to read a model off of this tree:

1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa \qquad \neg Ga$	Negated Conjunction
9.	$\otimes \qquad \neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb \qquad \neg Gb$	Negated Conjunction
	$\otimes$	

- first step: find the number of names on the open path

1.	$\exists xFx \checkmark a$	Assumption
2.	$\exists xGx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x\neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa \qquad \neg Ga$	Negated Conjunction
9.	$\otimes \qquad \neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb \qquad \neg Gb$	Negated Conjunction
	$\otimes$	

- there are two, so we have the following domain:
  - domain: {1, 2}

- second step: assign each name in the path to an object in the domain

1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x \neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	Fa	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa$ $\neg Ga$	Negated Conjunction
9.	$\otimes$ $\neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb$ $\neg Gb$	Negated Conjunction
	$\otimes$	

- we assign referents in the most natural manner:
  - referents: aa: 1, bb: 2

- third step: assign an extension to each predicate which makes atomic formulas involving the predicate true

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1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x \neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	$\boxed{Fa}$	Existential
6.	Gb	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa$ $\neg Ga$	Negated Conjunction
9.	$\otimes$ $\neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb$ $\neg Gb$	Negated Conjunction
	$\otimes$	

- we assign an extension to FF which makes FaFa true:
  - extensions: FF: {1}

1.	$\exists x Fx \checkmark a$	Assumption
2.	$\exists x Gx \checkmark b$	Assumption
3.	$\neg\exists x(Fx \wedge Gx) \checkmark$	Negated Conclusion
4.	$\forall x \neg(Fx \wedge Gx) \setminus a, b$	Negated Existential
5.	$\boxed{Fa}$	Existential
6.	$\boxed{Gb}$	Existential
7.	$\neg(Fa \wedge Ga) \checkmark$	Universal
8.	$\neg Fa$ $\neg Ga$	Negated Conjunction
9.	$\otimes$ $\neg(Fb \wedge Gb) \checkmark$	Universal
10.	$\neg Fb$ $\neg Gb$	Negated Conjunction
	$\otimes$	

- we assign an extension to GG which makes GbGb true:
  - extensions: FF: {1}, GG: {2}

- here is our completed model:
  - domain: {1, 2}
  - deferents: aa: 1, bb: 2
  - extensions: FF: {1}, GG: {2}

oh no! infinite trees

oh no! infinite trees

- unlike PL trees, MPL trees have an interesting feature: they can be infinitely long
- see pp. 372-373 of the textbook for how to avoid infinite trees in MPL
- let's consider an example of an infinite tree

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy)$

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a$
2. $(Fa \wedge \exists yGy)$

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a$
2. $(Fa \wedge \exists yGy)$ ✓
3. $\exists yGy$
4. Fa

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a$
2. $(Fa \wedge \exists yGy)$ ✓
3. $\exists yGy$ ✓ b
4. Fa
5. Gb

To prove: whether $\forall x(Fx \wedge \exists yGy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists yGy) \setminus a, b$
2. $(Fa \wedge \exists yGy)$ ✓
3. $\exists yGy$ ✓ b
4. Fa
5. Gb
6. $(Fb \wedge \exists yGy)$ ✓
7. $\exists yGy$ ✓ c
8. Fb
9. Gc

To prove: whether $\forall x(Fx \land \exists yGy)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \backslash a, b, c$
2. $(Fa \land \exists yGy)$ ✓
3. $\exists yGy$ ✓ b
4. Fa
5. Gb
6. $(Fb \land \exists yGy)$ ✓
7. $\exists yGy$ ✓ c
8. Fb
9. Gc
10. $(Fc \land \exists yGy)$ ✓
11. $\exists yGy$ ✓ d
12. Fc
13. Gd

- this tree is going to go on like this forever
- it will never be complete
- it has an infinite path

- is the proposition satisfiable or not?
- is the path saturated or not?
- it is saturated
- so it is satisfiable
- we can read off a model

- will the proposition always be satisfiable if we have an infinite tree?
- will infinite paths always be saturated?
- no and no.
- here is an example

To prove: whether $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$

To prove: whether $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ ✓
2. $\forall x(Fx \land \exists yGy)$
3. $(Ga \land \neg Ga)$

To prove: whether $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ ✓
2. $\forall x(Fx \land \exists yGy) \backslash a$
3. $(Ga \land \neg Ga)$
4. $Fa \land \exists yGy$

To prove: whether $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ ✓
2. $\forall x(Fx \land \exists yGy) \backslash a$
3. $(Ga \land \neg Ga)$
4. $Fa \land \exists yGy$ ✓
5. Fa
6. $\exists yGy$

To prove: whether $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy) \land (Ga \land \neg Ga)$ ✓
2. $\forall x(Fx \land \exists yGy)$ \a, b
3. $(Ga \land \neg Ga)$
4. $Fa \land \exists yGy$ ✓
5. Fa
6. $\exists yGy$ ✓ b
7. Gb
8. $Fb \land \exists yGy$
9. Fb
10. $\exists yGy$ ✓ c
11. Gc

- here is another infinite path (if we ignore Ga∧¬GaGa \land \lnot Ga).
- but the proposition is not satisfiable.
- and the path is not saturated.
- and if we were to saturate it, by applying the rule for ∧\land to Ga∧¬GaGa \land \lnot Ga, it would close straight away.

reading models off infinite trees

reading models off infinite trees

- I said we could read a model off of our infinite tree above.
- let's look at how.
- here is our tree again:

To prove: whether $\forall x(Fx \land \exists yGy)$ is satisfiable.

1. $\forall x(Fx \land \exists yGy)$ \a, b, c
2. $(Fa \land \exists yGy)$ ✓
3. $\exists yGy$ ✓ b
4. Fa
5. Gb
6. $(Fb \land \exists yGy)$ ✓
7. $\exists yGy$ ✓ c
8. Fb
9. Gc
10. $(Fc \land \exists yGy)$ ✓
11. $\exists yGy$ ✓ d
12. Fc
13. Gd

- what is the domain?
- how many names occur on the open path?
- an infinite number of names occur
- so here is our domain:
  - domain: {1, 2, 3, . . . }

- how shall we assign referents to each of the infinite number of names?
- like this of course:
  - referents: aa: 1, bb: 2, cc: 3 . . .

- and what about extensions for the predicates?
- well, there are only two predicates FF and GG.
- but we need an assignment which makes them true whenever they appear in atomic propositions on the path.
- let’s look at our tree and think about it . . .

To prove: whether $\forall x(Fx \wedge \exists y Gy)$ is satisfiable.

1. $\forall x(Fx \wedge \exists y Gy) \setminus a, b, c$
2. $(Fa \wedge \exists y Gy)$ ✓
3. $\exists y Gy$ ✓ b
4. Fa
5. Gb
6. $(Fb \wedge \exists y Gy)$ ✓
7. $\exists y Gy$ ✓ c
8. Fb
9. Gc
10. $(Fc \wedge \exists y Gy)$ ✓
11. $\exists y Gy$ ✓ d
12. Fc
13. Gd

- the pattern for GG is: GbGb, GcGc, GdGd, …
- the pattern for FF is: FaFa, FbFb, FcFc, …
- our complete model, then:
  - domain: {1, 2, 3, . . . }
  - referents: aa: 1, bb: 2, cc: 3 . . .
  - extensions: FF: {11, 22, 33, . . .}, GG: {22, 33, . . .}.

- will you have to read off a model from an infinite tree in the problem sets or the exam?
- possibly. but if so, it won’t be a difficult pattern to identify.

wrapping up

this lecture

next lecture

- lecture 19, the formal language GPL