Measuring the electrical conductivity of the earth

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Measuring the electrical conductivity of the earth

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We describe an undergraduate experiment for measuring the electrical conductivity of the earth with a four-electrode Wenner array, at scales approaching tens of meters. When analyzed in the context of a simple two-layer model of the earth, such measurements yield information about what is underground. In our case, this is the depth of the water table and the electrical conductivity of both the upper dry layer and the lower water-saturated layer. We also performed conductivity measurements in a water tank, to test the theory in a known situation. The experiments are discussed in the context of several boundary value problems in electricity and magnetism.

I. INTRODUCTION

What is the electrical conductivity of the earth? This question, of much interest in geophysics and geology, is rarely addressed in the undergraduate physics curriculum. This is regrettable since it involves some rather fundamental problems in the theory of electricity and magnetism. We discuss this topic in the context of an undergraduate experiment involving a four-electrode Wenner array. In the analysis we will find that, as is frequently the case, the serious consideration of a simple question leads to results beyond what one might expect, in this case, the possibility of some detailed information of what is underground.

This article should be of interest to students and faculty looking for boundary value problems related to electric current. Some of the problems here are appropriate for a junior level electricity and magnetism course. We have found problems in geophysics almost entirely absent from undergraduate electricity and magnetism texts.[1] This article should also prove of interest to students seeking thesis projects connecting the subject of electricity and magnetism to the real world.

The earth's electrical conductivity is key to its interaction with both terrestrial and extraterrestrial electric and magnetic fields. Because the primary constituents of the earth's crust are insulating silicon oxides,[2] one might think that the crust is a poor conductor of electricity. However, large quantities of surface and underground water put the earth's outer electrical conductivity within a few orders of magnitude of 1 $(\Omega\text{m})^{-1}$. In addition, pockets of underground minerals sometimes conduct readily, and this frequently offers the key to their detection.

Among the many cases where the electrical conductivity of the earth is important is in lightning discharges, where charges on the order of coulombs are transported over distances of several kilometers in times of tens of microseconds.[3] The earth conducts electricity so well that it suffices to use the electrostatic method of images during this process!

Before launching into our presentation, we emphasize that, although we slant this article towards higher-level students in physics, geophysical exploration with four-electrode arrays is accessible at lower levels as well. An understanding of boundary value problems certainly helps, but one may work instead with simple rules of thumb and simple methods of interpretation. Likewise, there are commercially available special purpose four-electrode instruments which should simplify the experimental method presented here. Our aim in this paper is to present the theory in some sophistication, and to show how it applies in the context of instrumentation commonly available in a physics lab.

II. THEORY

This section is organized around several boundary value problems involving electric current, and their solutions. These may be found, for example, in Wait[4] or Smythe,[5] and form the basis for our experiment.

Problem 1: Consider a homogeneous conducting half-space, with constant resistivity ρ, under an insulating medium; see Fig. 1. Let there be a point electrode at the surface injecting a steady current I. The current flows to a distributed sink at infinity (at zero potential). Find the electrostatic potential everywhere in the conducting medium.

Solution 1: Start with the equation of continuity, expressing charge conservation,[6]

$$\nabla \cdot \mathbf{J} + \frac{\partial n}{\partial t} = 0, \tag{1}$$

where $\mathbf{J}$ is the current density, n is the charge density, and t is the time. For a steady-state current, n does not depend on time, and

$$\nabla \cdot \mathbf{J} = 0. \tag{2}$$

Ohm's law relates $\mathbf{J}$ to the electric field $\mathbf{E}$:

$$\mathbf{J} = \frac{1}{\rho}\,\mathbf{E}. \tag{3}$$

Since the resistivity ρ is constant everywhere except across the surface, the above two equations lead to Laplace's equation for the electrostatic potential ϕ:

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = 0, \tag{4}$$

valid everywhere except possibly at the surface.

At the surface, there might conceivably be a buildup of surface charge. Let us argue that this does not happen. As Fig. 1 indicates, the normal component of $\mathbf{J}$ at the surface is zero because no current flows across it. By Ohm's law, the normal component of $\mathbf{E}$ will then likewise be zero just *below* the surface. However, Ohm's law does not rule out the possibility that the normal component of $\mathbf{E}$ just *above* the surface could differ from zero, corresponding to a surface charge. For this to happen, however, another set of charges would be required elsewhere to contribute a canceling electric field just below the surface. But, these extra charges could only be at infinity, where it is unlikely that they could contribute significantly to the electric field in the vicinity of

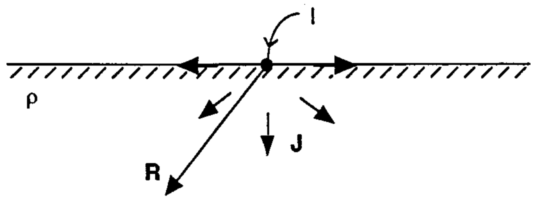

Fig. 1. A wire carrying current I to a point electrode at the planar surface between two homogeneous media, the lower (earth) conductive and the upper (air) insulating. The current flows to an imaginary distributed sink at infinity, and the arrows give the direction of the current density **J**. The displacement vector from the electrode is **R**.

the electrode. We conclude that the surface charge density must be zero, and that Laplace's equation will be satisfied everywhere, except the electrode.

Laplace's equation, the boundary condition of zero normal component of **E** at the surface, and the condition of zero potential at infinity can all be satisfied by the potential

$$\phi=\frac{A}{R}, \tag{5}$$

where R is the distance from the electrode and A is a constant. The definition of electric current and Ohm's law now yield

$$\phi=\frac{\rho I}{2\pi R}. \tag{6}$$

Problem 2: What is the electrostatic potential if there are two point electrodes, 1 and 2, at the surface, a current source I and a current sink-I, respectively (see Fig. 2)?

Solution 2: The solution follows immediately from the previous problem, using the superposition principle. For some point a distance R_1 from the current source and a distance R_2 from the current sink,

$$\phi=\frac{\rho I}{2\pi R_1}-\frac{\rho I}{2\pi R_2}. \tag{7}$$

Problem 3: For the four equally spaced electrodes, with separation distance a, pictured in Fig. 3 (the Wenner array), find the potential difference $\Delta\phi$ between the two middle electrodes A and B, neither of which draws current.

Solution 3: A double application of Eq. (7) yields

$$\Delta\phi=(\phi_A-\phi_B)=\frac{\rho I}{2\pi a}. \tag{8}$$

The next set of problems repeats those above, except with a two-layer lower conductor, with upper and lower resistivities ρ_1 and ρ_2, respectively. The upper layer has thickness h, and the lower layer extends down to infinity.

Problem 4: For the two-layer lower conductor shown in Fig. 4, and the single current injecting electrode at the surface, what is the potential at some point on the surface a distance r from the electrode?

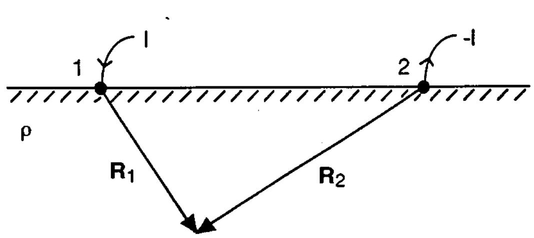

Fig. 2. Two point electrodes, 1 a current source, and 2 a current sink. The displacements from the two electrodes are denoted by $\mathbf{R}_1$ and $\mathbf{R}_2$.

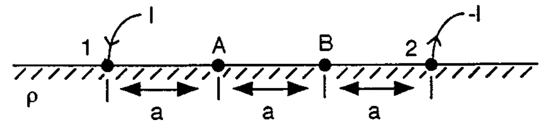

Fig. 3. Two interior electrodes A and B, drawing no current, added for measuring potentials. All of the electrodes are equally spaced, with separation distance a. Such a setup constitutes a Wenner array.

Solution 4: This is our most difficult problem. Physically, the complicating factor is surface charge at the lower interface which modifies the potential. Such charge results since the normal component of **J** must be continuous at the lower interface, by conservation of charge. With dissimilar media, Ohm's law Eq. (3) then forces a discontinuity of the normal component of **E**, and this corresponds to a surface charge density, by Gauss' law. We must then solve Laplace's equation in two pieces, and join them with boundary conditions at the lower interface. Cylindrical coordinates (r,z) are used. The origin is at the electrode, the z axis points straight down, and r measures the displacement from the z axis; see Fig. 4. The polar angle is suppressed because of azimuthal symmetry.

With azimuthal symmetry, Laplace's equation reads[6]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}=0. \tag{9}$$

To solve this partial differential equation, proceed with separation of variables:

$$\phi(r,z)=\mathsf{R}(\mathsf{r})Z(z), \tag{10}$$

where $\mathsf{R}(\mathsf{r})$ is a function only of r, and $Z(z)$ is a function only of z. Substitution into Eq. (9) yields two second-order ordinary differential equations:

$$\frac{d^2\mathsf{R}}{dr^2}+\frac{1}{r}\frac{d\mathsf{R}}{dr}+\lambda^2 R=0 \tag{11}$$

and

$$\frac{d^2Z}{dz^2}-\lambda^2 Z=0, \tag{12}$$

where λ is the separation constant.

The first of these equations is the zeroth-order Bessel equation, with solution

$$\mathsf{R}(\lambda r)=a_\lambda J_0(\lambda r). \tag{13}$$

where $J_0(\lambda r)$ is the regular zeroth-order Bessel function,[7] and a_λ is an arbitrary constant. In addition, there is a linearly

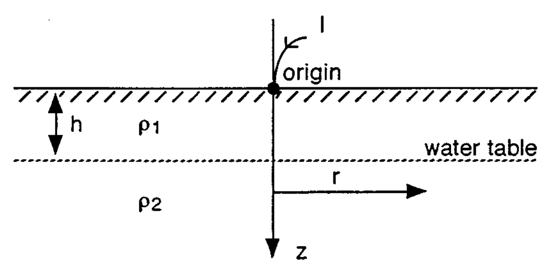

Fig. 4. The homogeneous earth has been replaced by a two-layer earth. The upper layer has thickness h and resistivity ρ_1. The lower layer extends down to infinity, and has resistivity ρ_2. The electrode is at the origin of the vertical z axis, which points straight down, and r measures the distance from the z axis.

independent irregular Bessel function solution, suppressed because it diverges (unphysically) in the limit $\lambda r \to 0$. The equation for $Z(z)$ has solution

$$Z(z) = A_\lambda e^{-\lambda z} + B_\lambda e^{\lambda z}, \tag{14}$$

where A_λ and B_λ are arbitrary constants.

Both $R(r)$ and $Z(z)$ are regular and well behaved inside each distinct medium. They represent the contribution to the potential from the surface charges at $z=h$. Summing over all solutions with different values for λ leaves the result a solution to Laplace's equation. Not yet included in the potential is the irregular part from the point charge at the electrode. By the superposition principle, this may be put in by hand. Putting it all together for $0<z<h$ results in

$$\phi_{0<z<h}(r,z) = \frac{\rho_1 I}{2\pi R} + \frac{\rho_1 I}{2\pi} \int_{\lambda=0}^{\infty} [A_\lambda e^{-\lambda z} + B_\lambda e^{\lambda z}] J_0(\lambda r)\, d\lambda, \tag{15}$$

where, again, R is the distance from the electrode. The constant a_λ has been absorbed in A_λ and B_λ, which have been made dimensionless with the constant in front of the integral.

In matching boundary conditions, it proves convenient to use the mathematical identity[4,8]

$$\frac{1}{R} = \frac{1}{(r^2+z^2)^{1/2}} = \int_{\lambda=0}^{\infty} J_0(\lambda r) e^{-\lambda z}\, d\lambda, \tag{16}$$

valid for $z>0$. This allows us to write Eq. (15) as

$$\phi_{0<z<h}(r,z) = \frac{\rho_1 I}{2\pi} \int_{\lambda=0}^{\infty} [(1+A_\lambda) e^{-\lambda z} + B_\lambda e^{\lambda z}] J_0(\lambda r)\, d\lambda, \tag{17}$$

for $z>0$.

For $h<z$,

$$\phi_{h<z}(r,z) = \frac{\rho_1 I}{2\pi} \int_{\lambda=0}^{\infty} (1+C_\lambda) e^{-\lambda z} J_0(\lambda r)\, d\lambda, \tag{18}$$

where, again, the ''1'' in the integrand corresponds to the ''bare'' $1/R$ electrode potential. The constants C_λ represent the contribution from the surface charges. We have dropped the positive exponential term to keep the potential from diverging as $z \to \infty$.

The rest of the solution involves the application of boundary conditions. At the surface $z=0$ the normal component of **J**, and hence that of **E**, is zero, and Eq. (15) yields

$$A_\lambda = B_\lambda. \tag{19}$$

The potential must be continuous across the interface at $z=h$, and Eqs. (17) and (18) yield

$$(1+A_\lambda) e^{-\lambda h} + B_\lambda e^{\lambda h} = (1+C_\lambda) e^{-\lambda h}. \tag{20}$$

Finally, we require that the normal component of **J** be continuous across the lower interface $z=h$. Ohm's law Eq. (3), and Eqs. (17) and (18) lead to

$$\frac{1}{\rho_1} [-(1+A_\lambda) e^{-\lambda h} + B_\lambda e^{\lambda h}] = \frac{1}{\rho_2} [-(1+C_\lambda) e^{-\lambda h}]. \tag{21}$$

The linear equations (19), (20) and (21) may be solved for the coefficients, yielding

$$A_\lambda = B_\lambda = \frac{K e^{-2\lambda h}}{1-K e^{-2\lambda h}}. \tag{22}$$

Here,

$$K = \left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right), \tag{23}$$

which obeys the inequality $-1 \leqslant K \leqslant 1$. Substituting into Eq. (15) results in the potential

$$\phi(r,0) = \frac{\rho_1 I}{2\pi r} G(r,K,h), \tag{24}$$

where

$$G(r,K,h) = 1 + 2Kr \int_{\lambda=0}^{\infty} \left(\frac{e^{-2\lambda h}}{1-K e^{-2\lambda h}} \right) J_0(\lambda r)\, d\lambda. \tag{25}$$

Problem 5: For the four equally spaced electrodes in Fig. 3, and with the two-layer earth in Fig. 4, find the potential difference $\Delta\phi$ between the two middle electrodes A and B.

Solution 5: The solution follows from the superposition principle and a quadruple application of Eq. (24). It may be written in the convenient form[4]

$$\frac{\rho_u}{\rho_1} = 2G(a,K,h) - G(2a,K,h), \tag{26}$$

where

$$\rho_u = \frac{2\pi a \Delta\phi}{I} \tag{27}$$

is the resistivity in the case of the uniform earth. The quantity ρ_u plays a central role in the analysis of the data, as we shall see.

The central quantity $G(r,K,h)$ may be written in the series form[4]

$$G(r,K,h) = 1 + 2Kr \sum_{n=0}^{\infty} K^n I_n, \tag{28}$$

where

$$I_n = \int_{\lambda=0}^{\infty} e^{-2\lambda h(n+1)} J_0(\lambda r)\, d\lambda = \frac{1}{\{r^2 + [2h(n+1)]^2\}^{1/2}}. \tag{29}$$

The later equality follows from Eq. (16). The series converges for $|K|<1$.

When confronted with some equation, insight usually results from checking limiting cases. If $\rho_1 \to \rho_2$, $G(r,K,h) \to 1$, and Eq. (26) reduces to that for the homogeneous earth. The same applies in the limit $a \to 0$, where the current flows entirely in the upper layer. For $a \to \infty$, $G(a,K,h) \to \rho_2/\rho_1$, and $\rho_u \to \rho_2$. This corresponds to a homogeneous earth, but with a resistivity characteristic of the *lower* layer. For a increasing from zero, ρ_u makes a gradual transition from ρ_1 to ρ_2, with no local maximas or minimas. We show some specific cases in Sec. IV.

The last problem establishes a limiting case used in analyzing our water tank experiment.

Problem 6: For the infinite conducting slab with thickness h and resistivity ρ_1, shown in Fig. 5, find the potential difference $\Delta\phi$ between the inner electrodes. The electrodes penetrate the entire slab.

Solution 6: Note that we are not using point surface electrodes in this problem. If we had, we could attempt to invoke Solution 5 in the limiting case $\rho_2 \to \infty$. But this solution fails with $K=1$, since neither the integral in Eq. (25) nor the series in Eq. (28) converge. In this limit, all of the current flows in the upper layer and the single current electrode potential becomes logarithmic at long distance. A logarithmic term was not included in the solution to Problem 5.

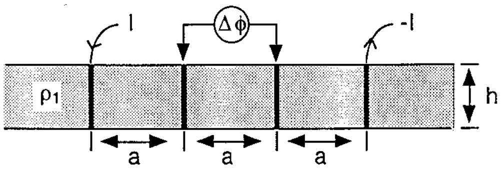

Fig. 5. An infinite conducting slab of conductivity ρ_1 and height h, with long electrodes penetrating completely. This example corresponds to the simple limiting case of the two-layer geometry with point surface electrodes, $\rho_2 \to \infty$, and large a.

The best that we can readily do in the limit $\rho_2 \to \infty$ is Problem 6. The electrodes going completely through the slab remove the z dependence from the potential, resulting in simplification. We follow the same basic motif as above, and omit the details. By conservation of charge, **J** is radial for a single current electrode, with magnitude decreasing as inverse distance. Ohm's law then leads to a logarithmic potential, and

$$\Delta \phi = \frac{\rho_1 I}{\pi h} \ln 2. \tag{30}$$

Writing this in terms of the resistivity defined in Eq. (27) results in

$$\frac{\rho_u}{\rho_1} = \frac{(2 \ln 2) a}{h}. \tag{31}$$

III. EXPERIMENT IN THE GEOPHYSICAL SETTING

In this section, we describe our setup for measuring the electrical conductivity of the earth with a Wenner array. Other four-electrode configurations are, of course, possible, each with advantages and disadvantages.[1] They are all amenable to the basic analysis methods of the previous section. The Wenner array is simple in conception and analysis, and is a good general-purpose choice. Figure 6 shows the full experimental setup, including the electronics.

An essential factor motivating the Wenner array is that of contact resistance between electrodes and earth. When current flows to the earth from an electrode, there will be an abrupt, unmeasured, change in potential because of this contact resistance. This must be avoided at the voltage measuring electrodes with extremely high impedance inputs to the voltmeter, which draw essentially no current. We used FET Operational Amplifier followers.[9] The voltage drop at the current electrodes is irrelevant since it is only the current which actually flows (readily measured) which enters into the analysis.

We used currents in the general neighborhood of 5 mA. We noted that at a given site and distance the current could change by as much as a factor of 2 on shifting all the electrodes to adjacent holes. This reflects varying degrees of contact between electrodes and earth. However, the effect on the ratio of voltage to current was much less. This leads us to believe that our FET Op Amp followers prevented current from being drawn by the voltage measuring electrodes.

To produce signals as large as possible, it is desirable to maximize the current. One obvious method is with high driving voltages, produced by a transformer. We tried this, but were dissatisfied with the resulting signal quality. Another method is with large electrodes, driven deep into the ground.

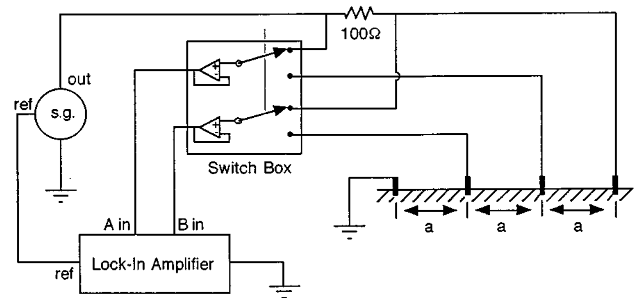

Fig. 6. The experimental setup. The sinusoidal signal was produced by a signal generator (s.g.). Current is measured through the potential difference it produces across a known (100 Ω) resistor. The switch box allowed us to change from measuring I to measuring $\Delta \phi$. This switch box contains FET operational amplifier followers at the inputs for extremely high impedance. The lock-in amplifier serves as our voltmeter, and is operated in the differential amplifier mode, with A and B inputs.

We likewise avoided this since the theory developed in the previous section is based on point electrodes. Yet another method wets the electrodes with salt water, to facilitate electrical contact. The drawbacks are that the electrical measurements drift in time as the water disperses, and the danger that the added water could provide an easy conducting path to the lower layer. We did runs both wetting and not wetting the electrodes, and offer no clear conclusions on this issue.

Unless one uses large currents, this experiment requires the measurement of small voltages (the smallest we encountered were around 10^{-5} V). These are easily overshadowed by noise from the ambient surroundings. To separate signals from noise, we operated the experiment with a sinusoidal signal at a frequency around 20 Hz. Voltages were measured with a lock-in amplifier keyed to that frequency.

Our electrodes were aluminum rods, diameter about $\frac{1}{2}$ in., hammered to a depth of 3–6 in. Our electronics were taken out into the field, though no further than we could run a long extension cord. We found it essential to plan and design ahead, as the field under the hot sun is no place to be fumbling with finicky equipment! Shading the electronics prevented overheating; aluminum foil reflectors performed very well in this capacity. As was clear in the solutions of the problems in the previous section, electrical properties at the surface of the earth are very important, and we were careful to avoid contact between the earth and metal objects, particularly wires. This at least eliminates the possibility of spurious current flows.

We measured both the amplitudes and phases of the two voltages. The phases should be the same, and unequal phases can indicate inadequate electrical contact with the earth, since in such a case capacitance between electrodes and earth plays a larger role. Unequal phases (differences greater than about 10°) served as a reason for discarding several data points from the analysis. The practical range of this basic ac circuit is limited by the emf induced around the voltage loop by time changing magnetic fields produced by the current. Telford *et al.*[1] give a rough rule of thumb for this limiting range, and it was not a factor in our work.

IV. DATA AND ANALYSIS IN THE GEOPHYSICAL SETTING

The first step in any set of measurements was picking a site. We selected flat, open fields, which are easy to work in.

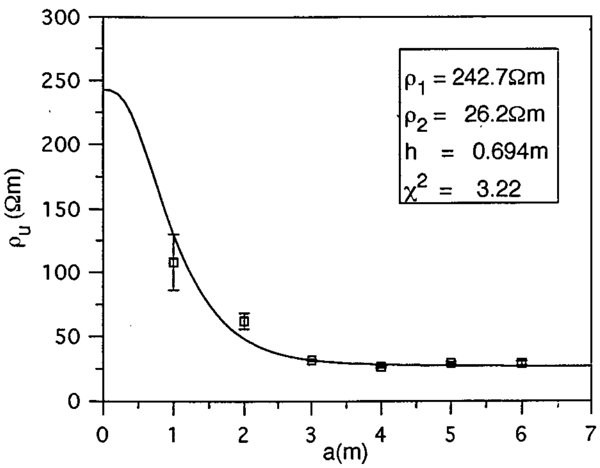

Fig. 7. Data for ρ_u taken at the first experimental site. Shown with the data points are error bars, as well as a least χ^2 fit to the theory. The least value for $\chi^2 = 3.22$ shows an acceptable fit between theory and data.

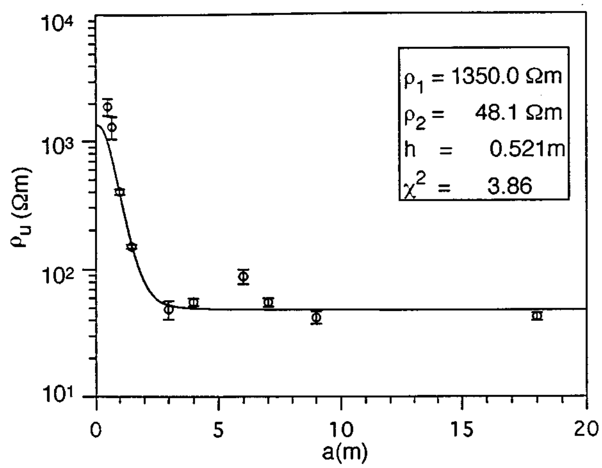

Fig. 9. Data for ρ_u taken at the second experimental site. Shown with the data points are error bars, as well as a least χ^2 fit to the theory. The least value for $\chi^2 = 3.86$ shows an acceptable fit between theory and data.

For each site we made measurements at several electrode spacings along a given straight line. At each electrode spacing, we made at least two pairs of current and voltage readings, with all the electrodes displaced by several inches from the last readings. This allowed an estimate of local errors. All current and voltage measurements were reduced to ρ_u, using Eq. (27).

We fit our data to the two-layer model of the earth in Eq. (26). Generally, fitting data to models involves two questions. First, does the model offer a reasonable depiction of reality? Second, if so, what values of the model's parameters best fit the data? To address these questions, we selected the chi-squared method.[10] The quantity χ^2, defined by

$$\chi^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{[\rho_u(a_i)_{\text{theory}} - \langle \rho_u \rangle_i]^2}{\langle (\Delta \rho_u)^2 \rangle_i}, \tag{32}$$

is a measure of how well the theory fits the data. Here, the index i ranges over the N electrode spacings a_i, and the symbols $\langle \rho_u \rangle_i$ and $\langle (\Delta \rho_u)^2 \rangle_i$ represent the average and variance of ρ_u at the electrode spacing a_i. A good fit of the data to the theory would be a value of χ^2 in the neighborhood of unity. A much larger value would indicate that the theory misses most of the error bars. A much smaller value would indicate that we have either taken too few data points relative to the number of theory parameters, or that we have somehow overestimated the true experimental error.

We report results from two sites. The first was near Sarasota Bay, with data taken in a line parallel to the shore, fixed by a sea wall, about 20 m away. An issue of interest in these measurements was the question of whether the subsurface water was fresh or brackish. Fresh water was expected, but contamination by saltwater was certainly possible, so there was some uncertainty in the outcome.

Figure 7 shows our results. To find the theory parameters [ρ_1, ρ_2, and h in Eq. (26)] which minimize χ^2, we wrote a simple program in the computer language Mathematica.[11] The best fit, with $\chi^2 = 3.22$, is shown in Fig. 7. While this χ^2 indicates that the model is basically right-minded, it is large enough to raise questions about deviations. There is certainly the possibility that our model is insufficient on the long scale, which is plausible given the proximity to salt water. There is also the possibility that small-scale inhomogeneities in the neighborhood of the electrodes affected the data. The upper layer, with resistivity 242.7 Ωm, is clearly relatively dry. The water table is at depth $h = 0.694$ m, and the resis-

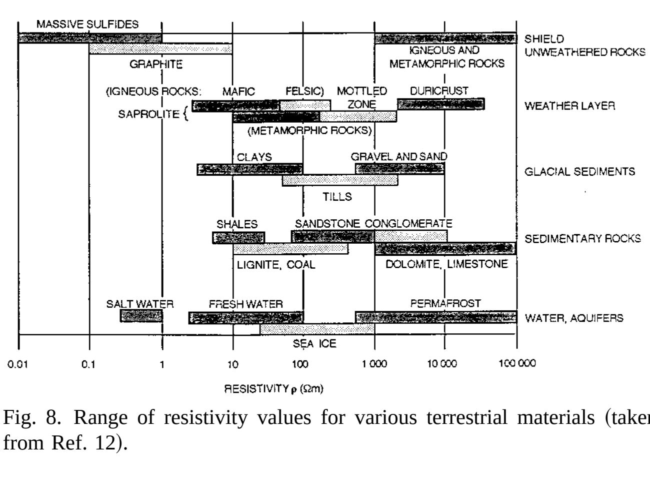

Fig. 8. Range of resistivity values for various terrestrial materials (taken from Ref. 12).

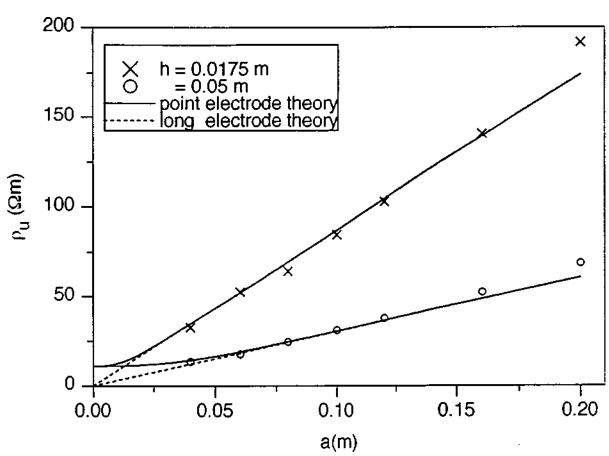

Fig. 10. The water tank data at two depths. The solid curves show the theory corresponding to point surface electrodes [Eq. (26)], with a very large value of ρ_2. The dotted lines show the theory computed with the long electrodes, and a nonconducting lower layer [Eq. (31)]. The theory used $\rho_1 = 11.0$ Ωm, which fit the data well. Edge effects become evident for larger values of a.

tivity of the water-saturated lower layer is 26.2 Ωm. To interpret this value, look at Fig. 8.[12] Clearly the resistivity value is in the range of sand saturated with fresh water, the basic picture we expected.

Our second experimental site was a soccer field about a mile inland from Sarasota Bay. Figure 9 shows our data. Evident again is the characteristic transition of ρ_u from the resistivity of the upper dry layer to that of the lower water-saturated layer as the electrode spacing is increased. The least $\chi^2(=3.86)$ fit yields a water table at depth $h=0.521$ m. The upper dry layer has resistivity of 1350.0 Ωm. The lower saturated layer has resistivity of 48.1 Ωm, which is, again, consistent with fresh water. Both upper and lower resistivities are larger than those near Sarasota Bay, indicating either fresher water (plausible) or a larger proportion of sand to water.

A water table so close to the surface requires reliable short-distance data. For distances less than about 0.5 m, we only hammered the electrodes to a depth of 3 in. This shallower depth made good electrical contact more problematic, and is reflected in increased scatter in this regime.

For this site, we measured the depth of the water table directly with an auger. Physical inspection of the removed soil showed a dry upper layer, with the soil lightly colored and sandy. At depth 0.55–0.65 m we encountered a color change, and saturation with water. This measured water table depth was in fair agreement with our electrical measurements reported above.

We conclude this section by saying that, despite reasonable agreement between our simple model and experiment, it is always difficult to be sure in a geophysical setting that one's efforts really give an accurate picture of what is down there. Possible complications include buried objects, soil of uneven consistency, or additional soil or rock layers near the surface. If one were making a geological study of a region, additional measurements with four-electrode arrays would be called for, and supplemented with magnetic (eddy current) measurements, georadar, and possible test digs. Nevertheless, Nature does frequently lend herself to description by simple models, and this would appear to have been the case here.

V. EXPERIMENT IN A WATER TANK

To test the theory in a controlled situation, we made measurements with our Wenner array in a water tank 1.5 m in diameter. Such a tank models a two-layer geometry, with a nonconducting lower medium $(\rho_2 \to \infty)$. (Note that the case $\rho_2 \to \infty$ offers the only way to confine this problem to a laboratory.) Electrodes were immersed as little as needed for good electrical contact (about 1 mm). Our biggest problem with this setup was that for the larger electrode spacings, the edges of the tank significantly limited the flow of current, and increased the measured value of ρ_u above that predicted by our simple two-layer theory. Since we were able to get reproducibility in the data to 1%, the edge effects dominated the deviation between theory and experiment. Hence, the χ^2 analysis of the previous section would not be appropriate with this theory, and we did not try it.

Figure 10 shows our results at two different depths of tap water. Depth was measured with a ruler, and current and potential difference with our Wenner array. We used Eq. (27) to compute the experimental ρ_u. The theoretical curve required a value for ρ_1. We determined it by matching the theory visually to the data; $\rho_1 = 11.0$ Ωm worked well at both depths. For ρ_2 we used an arbitrary very large value ($\rho_2 = 10^7$ Ωm) in the series Eq. (28) with 200 terms. An effective test of whether or not this ρ_2 was close enough to infinity to model our situation was offered by comparing its results with that of the limiting (large a) expression Eq. (31) from Problem 6. Results are plotted in Fig. 10.

Generally, the agreement between theory and experiment is good. The biggest deviation comes at large a, where the presence of edge effects becomes evident. The value of the resistivity of the tap water is well in line with that seen in the table Fig. 8, and about a factor of 4 less than what we measured in the underground water, where soil impedes the flow of current.

An interesting variation of this experiment would be with wet sand, to measure the relationship between the grain size and electrical conductivity. Theoretically, this relationship is given by Archie's law.[13] A three-layer geometry could be modeled with wet sand under a layer of water.

VI. CONCLUSION

This paper presents problems and solutions concerned with induced electrical currents in the earth. They connect the subject of electricity and magnetism with topics that are usually omitted in undergraduate physics courses. The theoretical exposition is amplified by actual experiments measuring the electrical conductivity of the earth.

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[1]In addition, a systematic search of the *American Journal of Physics* back to 1972 turned up no articles concerning electrical measurements on the earth, and, indeed, very little in the area of geophysics. The theory of the subject is well represented in the geophysics literature [W. M. Telford, L. P. Geldart, and R. E. Sheriff, *Applied Geophysics* (Cambridge U.P., Cambridge, 1990)], but actual applications tend to be reported separately in the geology literature.

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