Expectations for Exercises

Exercises are an important part of the Geophysics lecture. They will treat some aspects of the lecture in more detail, but also cover new ground. We expect that you work on the exercises at home and we will discuss questions and solutions interactively together. The joint meetings may start with randomly chosen students presenting their approach. It is ok if the solution is not available at this stage and there will be no interrogation. However, please don't show up unprepared, because this will inevitably be awkward.

1 Exercises for Gravity Method 1

1.1 Detection of a spherical object in the sub-surface

(a) Consider a spherical object with radius R located at depth z below the surface, and a gravimeter that is moved along the surface across the anomaly (Fig.1). Derive an analytical expression for the expected *vertical* anomaly g_z as a function of the lateral distance x and depth z. Calculate the maximum anomalies that you would expect for some realistic settings (e.g., a limestone cave.)

(b) Use a piece of paper or a software of your choice (e.g., Excel, SciDAVis, Matlab, Python) and visualize the expected vertical gravity anomaly for a specific setting as a function of lateral distance x. Show how this profile changes as you vary the depth of the object.

(c) Sometimes results in the gravity method are expressed in *mGal* with 1 Gal = 0.01 m s^{-2}. Transfer your results to those units and post a picture of your graph in the forum.

(d) We now investigate how we can estimate the depth of the object from an anomaly as derived in (a,b). For this you have to derive an expression that links the half-width of the anomaly (i.e. $g_z = \frac{1}{2} g_{z,max}$ for $x = x_{1/2}$) to the depth z of the object. Derive this relationship which has the form $z = c x_{1/2}$. Why is such a relationship be useful?

1.2 Gravitational acceleration inside the Earth

Knowing the gravitational acceleration *inside* the Earth is important, e.g., for understanding processes related to mantel convection.

(a) Use the consequences of the shell theorem to predict the gravitational acceleration $g(r)$ inside a spherical Earth. First assume that the Earth's density is constant, then that it decreases linearly with distance from the center. Assume that the density in the core is approximately $\rho_{core} = 13 \frac{g}{cm^3}$ and in the crust approximately $\rho_{crust} = 2.7 \frac{g}{cm^3}$.

(b) Draw the results on a x-y graph on paper or using a software of your choice.

(c) The PREM model provides an observationally constrained estimate of the density distribution inside the Earth. Knowing the shape of this profile, what is your guess of the gravitational acceleration inside the Earth? Why is it more difficult for you to calculate this quantitatively compared to the previous cases of constant and linearly varying density?

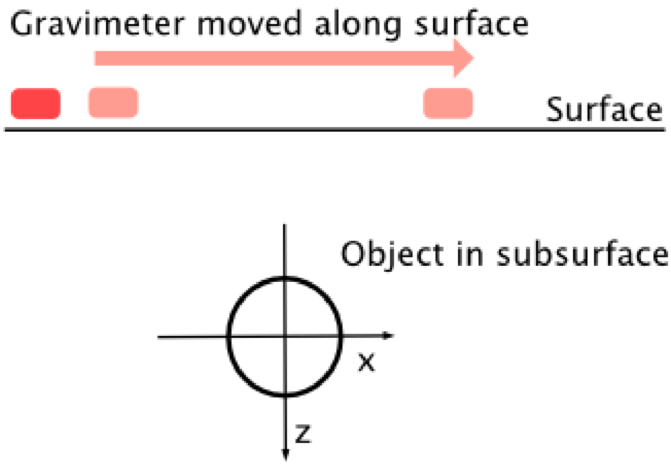
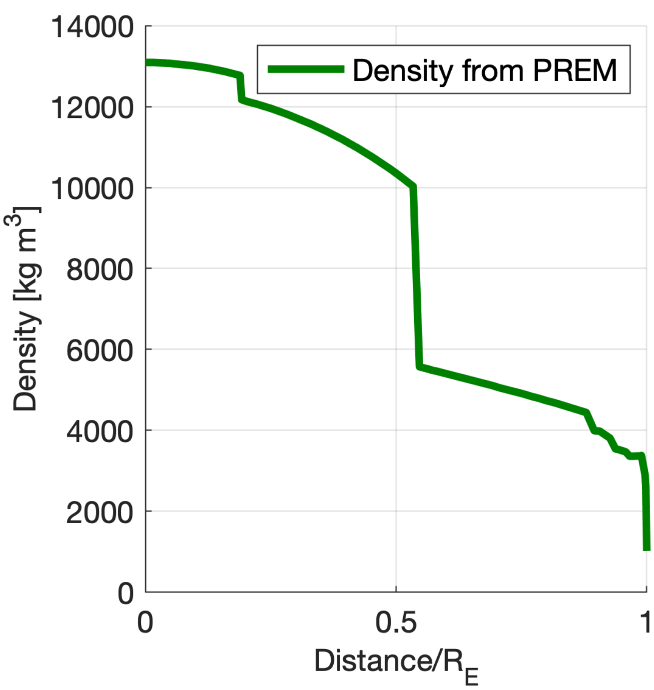


Figure 1: Sketch for problem 1.1. It is easiest to choose a coordinate system with the origin inside the subsurface object.

(d) Extra: Calculate the gravitational acceleration inside the Earth based on the PREM model using Matlab/Python/Excel. The data can be found on Ilias.



1.3 Determining the mean density of the Earth with your own gravimeter

Group Work

Self-organize in groups with 5-6 team members. Choose a group name and a team captain that communicates with the instructors. It is ideal if this group stays together throughout the term for the applied exercises. Make sure that your are inclusive during the group formation.

The vertical gravitational acceleration can be determined by measuring the traveltime of a freely falling object for a known distance. Assuming that the radius of the Earth and the gravitational constant are known ($G \approx 6.674 \cdot 10^{-11}$ m^{3}kg^{-1}s^{-2}, $R_E \approx 6370$) the mass of the Earth can be determined with the basic principles derived in class.

(a) Your task is to design a home-built free-fall gravimeter that does the job. You will quickly realize that the time measurement (i.e. the time from the start to the end of the fall) is one critical aspect in the system design. A helper tool that we suggest is the *acoustic stopwatch* that can be accessed via a smartphone and the *phyphox* app. Figure out an efficient way how the traveltime of a freely falling object can be determined with this type of acoustic trigger. (Tip: You may need a hammer, a metal bar and a bar clamp for your acoustic trigger. Be creative.)

(b) Collect a dataset of traveltimes and derive the mass of the earth. Provide an error estimate. Can you identify a measurement bias? How does your result compare to literature values?

(c) Given your result, what is the mean density of the Earth? How does that compare to rock densities found at the Earth's surface? What is a main conclusion that you can draw from this?

(d) Post your result, your dataset and a picture of your system setup in the ILIAS forum.

(e) Extra: Are you a tinkerer/Bastler? If so, feel free to improve the system design, e.g., by using light barriers in combination with a raspberry pi nano. We are more than happy to buy material for you, the only constraint is that it works reliably and that it remains cheapish (let's say <100 Euro). **If you succeed, you will win a gift certificate of that can be used in a Tübingen pub of your choice to celebrate your victory with your peers.** Moreover, your system will then be used for eternity for the following classes giving you much honor within the geo- and environmental student community.

2 Exercises for Gravity Method 2

2.1 The shell theorem

[Consider including a sub-step. Quite technical.]

2.2 Potential of an infinite plate (Bouger plate)

2.3 Forward modelling and non-uniqueness in potential field methods

Matlab (or Python)

Basic programming (Matlab/Python/R) will likely be part of your study experience when you move to MSc level courses. It is a useful skill to have, but here we do not cover any introduction. What we do is that we start with codes that need little user interaction to give you a feel for what programming can be about. In order to run this exercise you should have a working Matlab version on your Computer, please follow the installation instructions provided by the ZDV. Alternatively, we can also give you a laptop for the joint meeting.

In order to predict how any kind of object will appear in a gravity survey, we need to solve the volume integral:

$$\vec{g}(r) = G \int \frac{1}{r^3} \rho(r) \vec{r} dV$$

which simplifies slightly to:

$$g_z(r) = G \int \frac{1}{r^2} \cos(\phi) \rho(r) dV = G \int \frac{z}{r^3} \rho(r) dV$$

because often only the vertical component is of interest (Ex. cf. 1.1). However, the problem remains complicated as the integration bounds depend on the object's geometry and the integral needs to be solved for every r along the gravimetry profile. Some solutions for special shapes you already know (e.g.sphere, bouger plate). Here we use the solution for a rectangular prism which fortunately others have already calculated for us (*Naggy 1966, Geophysics VOL. XXX, SO. 2*). Using this solution, we can build up more complicated shapes out of individual prisms.
In the specific model applied individual prisms are defined with their widths in the horizontal (wx, wy) and the vertical (wz), together with the positions in the subsurface. The key is that the position coordinates (dx1, dx2, dy1, dy2, dz1, dz2) need to be prescribed relative to the measurement position which changes along the profile. The expected anomaly is then calculated based on the analytical solution.

(a) This exercises uses Matlab. However, only minimal Matlab skills are required to follow along. Download the files *Gravimetry02_ForwardModelling.m* and *gravprism.m* into the same folder on your computer. Check out case 1 which simulates a rectangular object in the subsurface. Change it's location and size so that you know what is going on.

(b) Switch to case 2. This one treats the combined effect of two prisms. See what's different compared to case 1. Play around with positions to see what is going on.

(c) Switch to case 3. This one treats the individual effects of two prisms meaning that it doesn't sum them up. This one will not run until you fill out the parts marked with XXX. Use this case to illustrate that multiple situations in the sub-surface (e.g. a shallow prism with low density contrast vs. a lower prism with larger density contrast) can result in similar anomalies. This is an important finding. Forward models are often not unique, and therefore your interpretation won't be either. This situation occurs in many geophysical situations. Remember that.

Solutions

```
clear all;
close all;

%% This code quantifies the gravitational potential of a rectangular
%% Prism located in the sub-surface using an analytical solution of
%% Nagy 1966 (Geophysics)



%%%%%%%%%%%%%%%%%%%%%%%%
% Case Numbers
% RectangleCentered: 1 ;
% Two Rectangles: 2;
% Ambiguities: 3
%%%%%%%%%%%%%%%%%%%%%%%%
CaseNumber = 1;

switch CaseNumber
    case 1
        display('Calculating centered rectangle.')
        %This is the density contrast
        drho = 400;

        % Location and geometry of Prisms.
        % -----------------------------------------------
        % Width height and depth of the prism.
        wx = 10;wy = 100;wz = 0.5;
        % Offset in depth z and lateral direction x.
        % The y-dimension is (but doesn not have to)
        % is symetric to the profile direction

        offsetz=1;offsetx=0;

        % Sample Points along profile in x-direction.
        % Coordinates are symetric with origin in center.
        % -----------------------------------------------
        dx=0.5;xp = -20:dx:20;yp = xp*0;zp=xp*0;

        % Coordinates of the prism relative to sample points
        % -----------------------------------------------
        dx1=fliplr(min(xp)-wx/2:dx:max(xp)-wx/2);dx2=dx1+wx;
        dy1=dx1*0-wy/2;dy2=dy1+wy;
        dz1=dx1*0+offsetz;dz2=dz1+wz;

        % This applies the analysitcal solution of Nagy 1966.
        % -----------------------------------------------
        dg = gravprism(drho,dx1,dx2,dy1,dy2,dz1,dz2);

        % Here we visualize the results.
        fig = figure()
        subplot(3,1,1)
        plot(xp,dg)
        ylabel('Gravity Anomaly (mGal)');box off;set(gcf, 'color', 'none');set(gca,
```

```
'color', 'none');
        subplot(3,1,2)%[x y w h]
        rectangle('Position',[-wx/2,offsetz,wx,wz]);
        set(gca,'XAxisLocation','top','YAxisLocation','left','ydir','reverse');
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('Depth (m)');ylim([0,2]);set(gcf, 'color', 'none');set(gca, 'color', 'none');
        subplot(3,1,3)%[x y w h]
        rectangle('Position',[-wx/2,-wy/2,wx,wy]);
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('Horizontal Distance y (km)');
        %Export to a png. (This can be done much better.)
        set(gcf, 'color', 'none');set(gca, 'color', 'none');
        set(gcf,'PaperUnits','centimeters','PaperPosition',[0 0 10 20])
        set(findall(fig, '-property', 'FontSize'), 'FontSize', 12)
        print('-dpng','-r300','../../../LatexSlidesLectures/Figures/Gravity/Exported/ForwardModelPrism.png')
    case 2
        display('Calculating the effect of multiple rectangles.')
        % This is the density contrast
        % ---------------------------------------------
        drho = [400];
        % Width, height and depth of the two prisms.
        % ---------------------------------------------
        wx = [10 8];
        wy = [100 100];
        wz = [0.5 0.25];

        % Offset in depth z and lateral direction x.
        % The y-dimension is (but doesn not have to)
        % is symetric to the profile direction
        % ---------------------------------------------
        offsetz=[1 0.25];offsetx=[-10 12]

        % Number of prisms in this example
        % ---------------------------------------------
        np = length(offsetz);

        % Sample Points along profile in x-direction.
        % Coordinates are symetric with origin in center.
        % ---------------------------------------------
        dx=0.5;xp = -20:dx:20;yp = xp*0;zp=xp*0;

        % Coordinates of the prisms relative to sample points
        % ---------------------------------------------
        for kk=1:np
            dx1(kk,:)=fliplr(min(xp)-wx(kk)/2:dx:max(xp)-wx(kk)/2)+offsetx(kk);dx2(kk,:)=dx1(kk,:)+wx(kk);
            dy1(kk,:)=dx1(kk,:)*0-wy(kk)/2;dy2(kk,:)=dy1(kk,:)+wy(kk);
            dz1(kk,:)=dx1(kk,:)*0+offsetz(kk);dz2(kk,:)=dz1(kk,:)+wz(kk);
        end
        dg = gravprism(drho,dx1,dx2,dy1,dy2,dz1,dz2);

        % Visualization of the combined effect
        % ---------------------------------------------
```

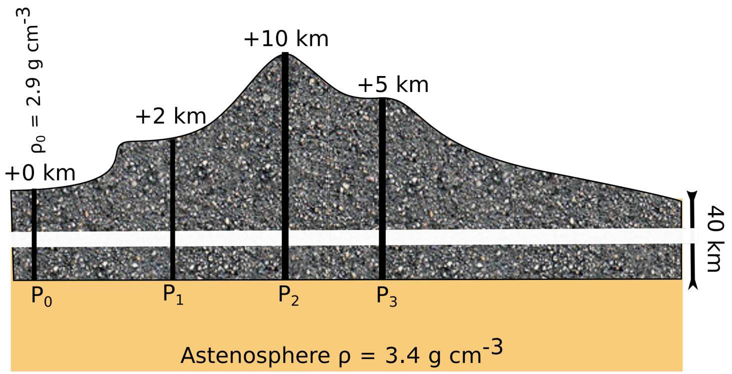
```
        figure()
        subplot(3,1,1)
        % Here we show the combined effect by summing the effects if individual
    prisms
        plot(xp,sum(dg,1))
        xlabel('Horizontal Distance x (km)');ylabel('Gravity Anomaly (mGal)');
        subplot(3,1,2)
        hold on;
        for kk=1:np
            rectangle('Position',[-wx(kk)/2+offsetx(kk),offsetz(kk),wx(kk),wz(kk)]);
        end
        set(gca,'XAxisLocation','top','YAxisLocation','left','ydir','reverse');
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('Depth (
    m)');
        subplot(3,1,3)%[x y w h]
        for kk=1:np
            rectangle('Position',[-wx(kk)/2+offsetx(kk),-wy(kk)/2,wx(kk),wy(kk)]);
        end
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('
    Horizontal Distance y (km)');
    case 3
        display('Showcasing ambiguity.')
        % Those are density contrasts. We now choose two.
        % ---------------------------------------------------------------
        drho1 = 400;drho2=950;
        % Width, height and depth of the two prisms.
        % ---------------------------------------------------------------
        wx = [10 8.7];
        wy = [100 100];
        wz = [0.5 0.25];
        %Offset in depth z and lateral direction x.
        %The y-dimension is (but doesn not have to)
        %is symetric to the profile direction
        offsetz=[1 2];offsetx=[0 0]

        % Number of prisms
        % ---------------------------------------------------------------
        np = length(offsetz);

        % Sample Points along profile in x-direction.
        % Coordinates are symetric with origin in center.
        % ---------------------------------------------------------------
        dx=0.5;xp = -20:dx:20;yp = xp*0;zp=xp*0;

        % Coordinates of prisms relative to sample points
        % ---------------------------------------------------------------
        for kk=1:np
            dx1(kk,:)=fliplr(min(xp)-wx(kk)/2:dx:max(xp)-wx(kk)/2)+offsetx(kk);dx2(
    kk,:)=dx1(kk,:)+wx(kk);
            dy1(kk,:)=dx1(kk,:)*0-wy(kk)/2;dy2(kk,:)=dy1(kk,:)+wy(kk);
            dz1(kk,:)=dx1(kk,:)*0+offsetz(kk);dz2(kk,:)=dz1(kk,:)+wz(kk);
        end

        % Now calculate the effects with variable densities.
```

```
        % For this we calculate all prisms for all densities.
        % It is a bit silly, but it works.
        % ------------------------------------------------
        dg1 = gravprism(drho1,dx1,dx2,dy1,dy2,dz1,dz2);
        dg2 = gravprism(drho2,dx1,dx2,dy1,dy2,dz1,dz2);

        % Visualization
        % ------------------------------------------------
        figure()
        subplot(3,1,1)
        hold on;
        plot(xp,dg1(1,:),'r') %This is prism 1 with density 1
        plot(xp,dg2(2,:),'m') %This is prism 2 with density 2
        xlabel('Horizontal Distance x (km)');ylabel('Gravity Anomaly (mGal)');
        subplot(3,1,2)
        hold on;
        plot(xp,0*xp,'b-x')
            kk=1
            rectangle('Position',[-wx(kk)/2+offsetx(kk),offsetz(kk),wx(kk),wz(kk)],'
    FaceColor','r');
            kk=2
            rectangle('Position',[-wx(kk)/2+offsetx(kk),offsetz(kk),wx(kk),wz(kk)],'
    FaceColor','m');


        set(gca,'XAxisLocation','top','YAxisLocation','left','ydir','reverse');
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('Depth (
    m)');
        subplot(3,1,3)%[x y w h]
          kk=1;
            rectangle('Position',[-wx(kk)/2+offsetx(kk),-wy(kk)/2,wx(kk),wy(kk)],'
    FaceColor','r');
            kk=2;
            rectangle('Position',[-wx(kk)/2+offsetx(kk),-wy(kk)/2,wx(kk),wy(kk)],'
    FaceColor','m');
        xlim([min(xp),max(xp)]);xlabel('Horizontal Distance x (km)');ylabel('
    Horizontal Distance y (km)');

end
```

../../Src/Gravimetry/PrismForwardModel/GravForwardModelPrismRD.m

2.4 Airy and Pratt hypothesis for mountain ranges

(a)



The figure above illustrates a crust with with a inhomogenous density and a mountain change floating on the asthenosphere. Consider this as an idealised case in which every vertical slice is locally balanced (i.e. everything is in hydrostatic equilibrium which reduces to an effective 1D problem). Calculate the required densities in the vertical slices at $P_1 - -P_3$. (Tip: Below the crust the pressure is equal everywhere $P_1 - -P_3$)
The figure above illustrates a crust with a homogenous density and a mountain change floating on the asthenosphere. Consider this as an idealised case in which every vertical slice is locally balanced (i.e. everything is in hydrostatic equilibrium which reduces to an effective 1D problem). Calculate the thickness differences between $P_1 - -P_3$.
(c) Draw an approximate profile for the free-air and the Bouger anomalies. How would this profile change if the mountain change is not in hydrostatic equilibrium? Which conclusions regarding the temporal evolution of the mountain chain would you draw from that? In which areas along this profile do you think is the assumption of local hydrostatic equilibrium most unlikely and how would this be reflected in the free-air anomaly?

3 Some general questions to reflect on

1. Why does the mean sea level follow the shape of the Geoid? 2. How can you describe the gravitational attraction between two point masses if none of them is located in the origin of the coordinate system applied? 3. How does and equipotential line change by crossing an area of (a) mass deficit, and (b) mass excess? 4. Why do we have Earth & Ocean tides? To understand the principle focus on the Moon's effect only. 5. Discuss whether the sun or the moon is more important for tides.
6. Discuss the Airy and Pratt hypothesis.