Expectations for Exercises

Exercises are an important part of the Geophysics lecture. They will treat some aspects of the lecture in more detail, but also cover new ground. We expect that you work on the exercises at home and we will discuss questions and solutions interactively together. Questions that are marked with 'Extra' are not required but geared to stir your further interest. We will surely support you if you tackle those as well.

7 Exercises for Electromagnetic Induction

Version: June 15, 2022
Context: ElectromagneticInduction_01 + three short EMI videos
Timing: All EMI exercises should be completed the latest by July 1st 2022

7.1 Details of the Slingram Method

Parts (a)-(c) are fairly technical and geared to train your math skills (e.g. by dealing with complex numbers). This is an important skill to have because it enables you to follow more advanced textbooks and papers. They also fill some gaps that we left open in our lecture. However, it is not needed to memorize any of the specific expressions derived. Only the techniques and underlying principles matter (e.g. it is useful to memorize how the phase angle of a complex number is defined.). If you get stuck at one part, move to the next one. Explicit solutions will be provided in class.

(a) Show that the ratio of secondary over the primary voltage in the Slingram method can be expressed as:

$$\frac{V_{l3,s}}{V_{l3,p}} = -\frac{L_{12}L_{23}}{L_{13}L_{l2}}\left(\frac{i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\right)$$

by using Faraday's law of induction and expressions for for the currents in the loops derived in class.

Solutions

Using Faraday's law of induction we know that the primary voltage in coil 3 is

$$V_{l3,p} = L_{13}\frac{dI_{l1}}{dt}$$

and the secondary voltage is

$$V_{l3,s} = L_{23}\frac{dI_{l2}}{dt}.$$

We derived expressions for I_{l1} and I_{l2} in class:

$$\begin{aligned} I_{l1} &= I_1 e^{i\omega t} \\ I_{l2} &= \frac{-i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\frac{L_{12}}{L_{l2}}I_1 e^{i\omega t}. \end{aligned}$$

We now need to calculate the temporal derivatives of the electrical currents using $\frac{d}{dt}e^{i\omega t} = i\omega e^{i\omega t}$.

Then the ratio of the primary and secondary potentials is straightforwardly calculated:

$$\frac{V_{l3,s}}{V_{l3,p}} = \frac{iL_{23}\omega\frac{-i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\frac{L_{12}}{L_{l2}}I_1e^{i\omega t}}{iL_{13}\omega I_1e^{i\omega t}} = \frac{L_{23}\frac{-i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\frac{L_{12}}{L_{l2}}}{L_{13}} = -\frac{L_{12}L_{23}}{L_{13}L_{l2}}\left(\frac{i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\right).$$

(b) Show that:

$$\left(\frac{i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\right) = \left(\frac{1}{1+\alpha^2}(\alpha^2+i\alpha)\right)$$

using the induction paramter $\alpha = \omega\frac{L_{l2}}{R_{l2}}$.

Solutions

$$\begin{aligned}\left(\frac{i\alpha}{1+i\alpha}\right) &= \left(\frac{i\alpha}{1+i\alpha}\right)\left(\frac{1-i\alpha}{1-i\alpha}\right)\\ &= \left(\frac{i\alpha-i^2\alpha^2}{1-i^2\alpha^2}\right)\\ &= \left(\frac{\alpha^2+i\alpha}{1+\alpha^2}\right)\end{aligned}$$

(c) Show that:

$$\phi_p - \phi_s = \frac{\pi}{2} + atan(\frac{\omega L_{l2}}{R_{l2}})$$

whereas ϕ_p and ϕ_p are the phase differences relative to the primary loop.

Solutions

From the lecture and (a) we know that $V_{l3,p} = L_{13}\frac{dI_{l1}}{dt} = i\omega I_1 e^{i\omega t}$ and $V_{l3,p} = iL_{23}\omega\frac{-i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\frac{L_{12}}{L_{l2}}I_1e^{i\omega t}$. We use the induction parameter $\alpha = \omega\frac{L_{l2}}{R_{l2}}$ and rewrite as in the lecture:

$$V_{l3,s} = iL_{23}\omega\frac{-i\omega\frac{L_{l2}}{R_{l2}}}{1+i\omega\frac{L_{l2}}{R_{l2}}}\frac{L_{12}}{L_{l2}}I_1e^{i\omega t} = i\omega\frac{L_{12}L_{23}}{L_{l2}}I_1e^{i\omega t}\left(\frac{1}{1+\alpha^2}(\alpha^2+i\alpha)\right) = c_1\frac{1}{1+\alpha^2}(-\alpha+i\alpha^2)$$

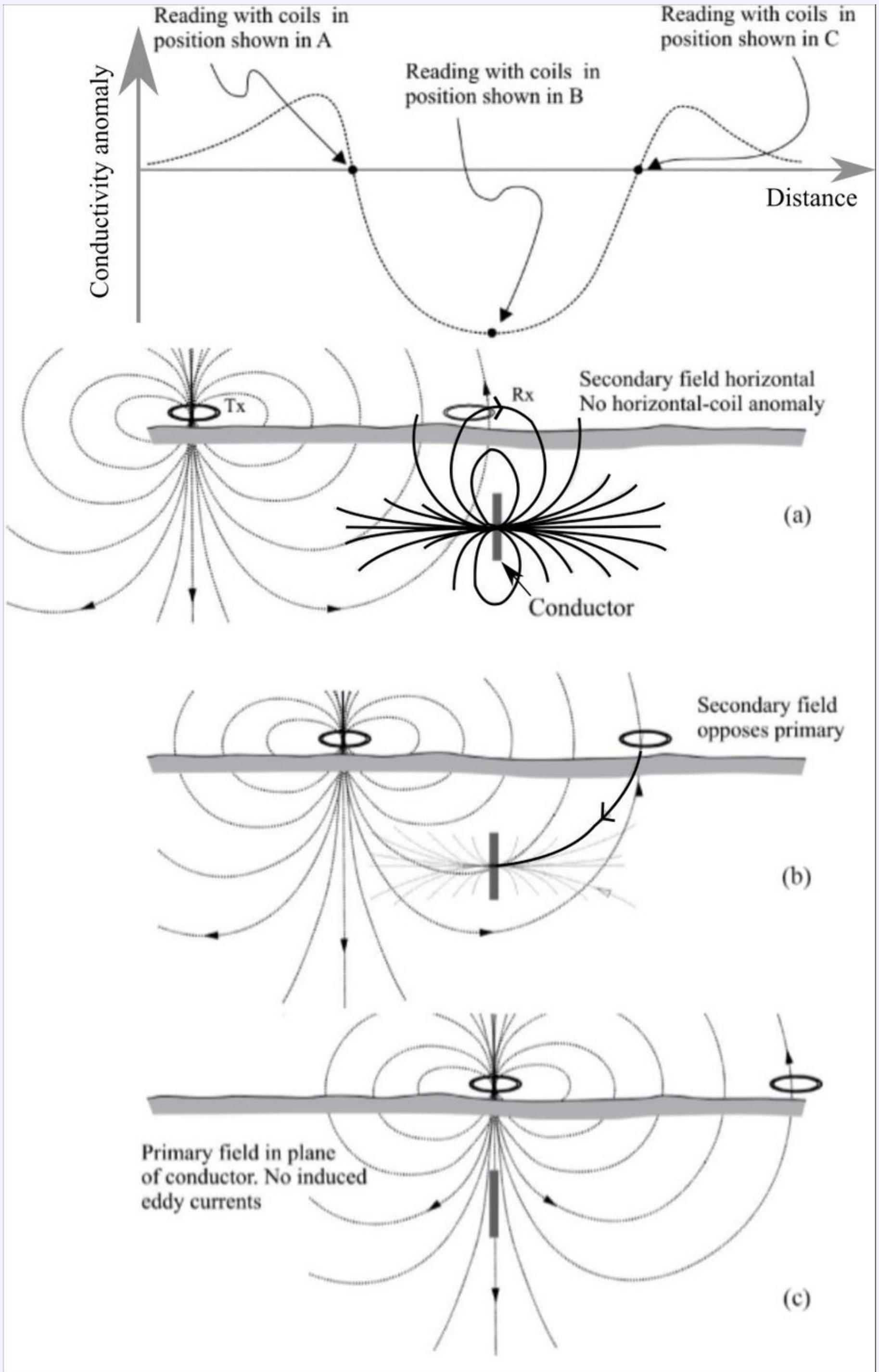
It is now clear what the real and imaginary components are. We can neglect the $e^{i\omega t}$ which is common in both primary and secondary induced voltage and will not lead to phase shifts in itself . Per definition (reassure yourself in the real/complex plane that this is the case): $\theta_s = atan(\frac{Im(V_{l3,s})}{Re(V_{l3,s})})$:

$$\theta_s = atan\left(\frac{\frac{\alpha^2}{1+\alpha^2}}{\frac{-\alpha}{1+\alpha^2}}\right) = -atan(\alpha)$$

This is the phase shift of the secondary component. Verify that this lead to the bounds derived in the lectures for the resistive and conductive limits. In $V_{l3,p}$ there is only an imaginary component and the real part is zero. Taking the limit this results in the $\frac{\pi}{2}$ term.

(d) (*Already sketch out in class. This is a repetition for yourself.*) Suppose you have an idealised loop in the sub-surface (as sketched out in lecture) located in the center of an arbitrary profile. Sketch out the shape of the slingram anomaly with distance on the x-axis and the ratio $\frac{V_s}{V_p}$ on the y-axis.

Solutions



Solutions [Zylberman, 2018]: In order to sketch out the anomaly you need to worry most about the coupling coefficients L_{xx}. They change in magnitude and sign as you go across this anomaly.

With Rx and Tx on the left side of the anomaly the secondary magnetic field has the same direction in the receiver coil as the primary magnetic field. Hence the anomaly is positive. With the receiver coil directly above the target, the coupling coefficient turns to zero as field lines are parallel to the loop (i.e. no induction). With transmit and receiver coils bracketing the target, the secondary magnetic field is in opposite direction to the primary field and the anomaly is negative. The rest can be argued with symmetry reasons.

7.2 Details of the Skin Effect

Same as in 7.1 (a) is fairly technical. It is helfpul for (b).

(a) Show that

$$m^2 = i\alpha \rightarrow m = -(1+i)\sqrt{\frac{\alpha}{2}}$$

Solutions

One way is to square the right hand side of the equation:

$$\left(-(1+i)\sqrt{\frac{\alpha}{2}}\right)^2 = (1-2i+i^2)\frac{\alpha}{2} = -i\alpha$$

The more formal way suppose that so that with m = a +ib:

$$\begin{aligned} m^2 &= a^2 + 2iab + (ib)^2 \\ &= a^2 - b^2 + 2iab = i\alpha \end{aligned}$$

This equation holds if $a^2 = b^2$ (meaning that $a = \pm b$) and

$$2ia^2 = i\alpha \rightarrow a = \sqrt{\frac{a}{2}} = \pm b$$

so that

$$m = (\sqrt{\frac{a}{2}} + i\sqrt{\frac{a}{2}}) = \pm(1+i)\sqrt{\frac{a}{2}}$$

(b) Show that

$$Re(E) = E_0 e^{-kz}\cos(\omega t - kz)$$

is the real part of a solution satisfying

$$\frac{\partial^2}{\partial z^2}E = i\omega\mu\sigma E$$

with the Ansatz: $E = E_0 e^{i\omega t + mz}$. What type of differential equation is this? What process does it describe?

Solutions

Solution: Pluggin the Ansatz into the differential equation yields:

$$
\begin{aligned}
m^2 E_0 e^{i\omega t+mz} &= i\omega\mu\sigma E_0 e^{i\omega t+mz} \\
m^2 &= i\omega\mu\sigma \\
m &= -(1+i)\sqrt{\frac{\omega\mu\sigma}{2}}
\end{aligned}
$$

This means that

$$
\begin{aligned}
E &= E_0 e^{i\omega t-(1+i)\sqrt{\frac{\omega\mu\sigma}{2}}z} \\
E &= E_0 \underbrace{e^{-\sqrt{\frac{\omega\mu\sigma}{2}}z}}_{damping} \underbrace{e^{i(\omega t-\sqrt{\frac{\omega\mu\sigma}{2}}z)}}_{oscillation}
\end{aligned}
$$

which describes a damped harmonic oscillation. The damping factor $\sqrt{\frac{\omega\mu\sigma}{2}}$ can be linked to the skin depth (cf. lecture).