



Introduction to Geophysics
R. Drews

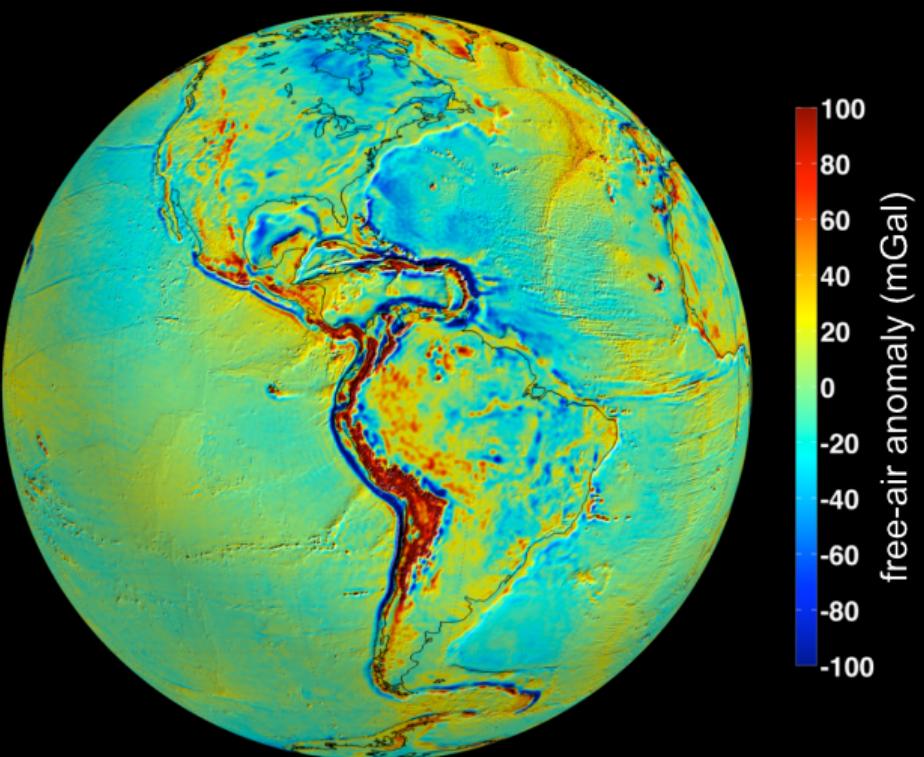
Gravimetry



Learning goals today:

- ▶ The gravitational force, its potential field, and how to measure it.
- ▶ The gravitational field of the Earth.
- ▶

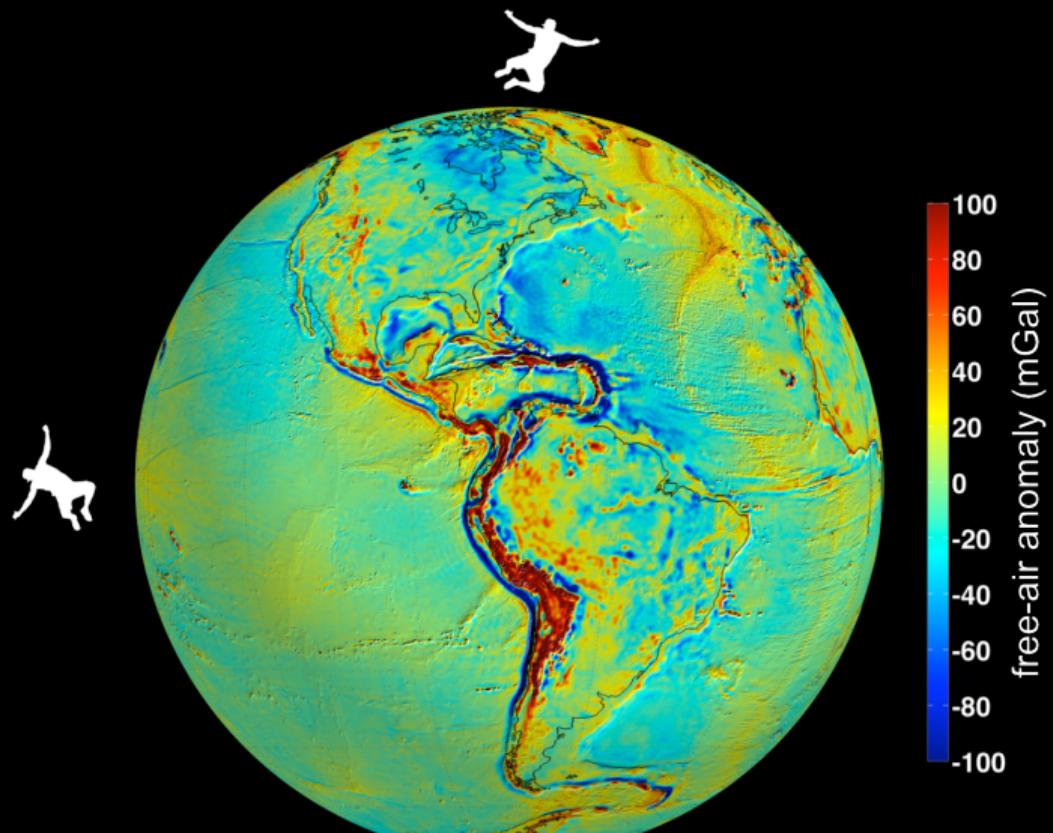
Example: Global variability



Example: Global variability



Your mass is constant but your weight is not.



What is a force?



[Newton (1642-1726) / G. Johnson.]



$$\vec{F} = m\vec{g}$$

\vec{F} : Force (N; kg m s^{-2})

\vec{g} : Acceleration (m s^{-2})

m : Mass (kg)

The gravitational force



$$\vec{F} = G \frac{mM}{r^2} \hat{r}$$

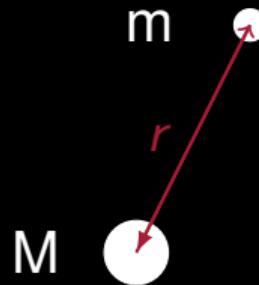
$$G = 6.674 \cdot 10^{-11} \text{ (m}^3\text{kg}^{-1}\text{s}^{-2}\text{)}$$

\hat{r} : unit vector

r : distance between point masses



[Newton (1642-1726) / G. Johnson.]



Example: The gravitational constant

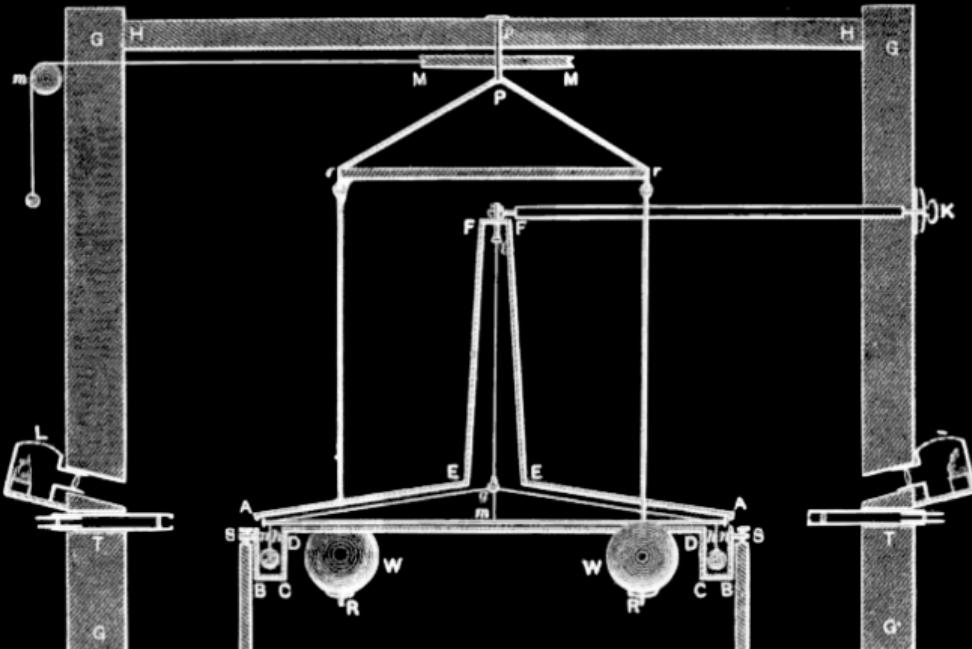
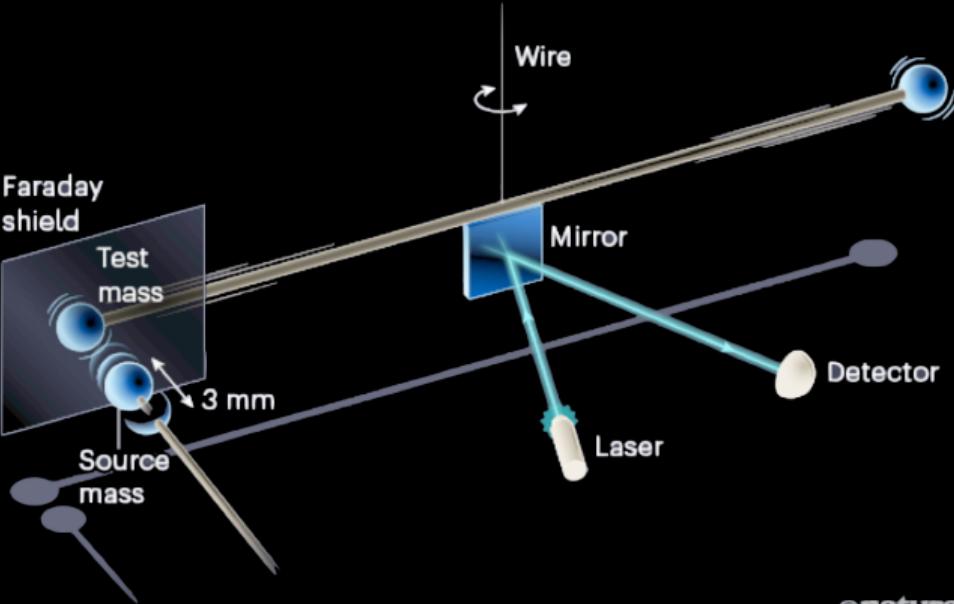


Fig. 1

Cavendish, PNAS, 1798

Example: The gravitational constant



©nature

Westphal et al., Nature, 2021

G is the worst known constant in physics.
Why?

Example: Measuring acceleration



$$\vec{F} = m\vec{g}$$

$$\vec{F} = G \frac{mM}{r^2} \hat{r}$$

$$\rightarrow \vec{g} = G \frac{M}{r^2} \hat{r}$$

$$\rightarrow \frac{d^2\vec{x}}{dt^2} = G \frac{M}{r^2} \hat{r}$$

This is a differential equation.

Example: Measuring acceleration



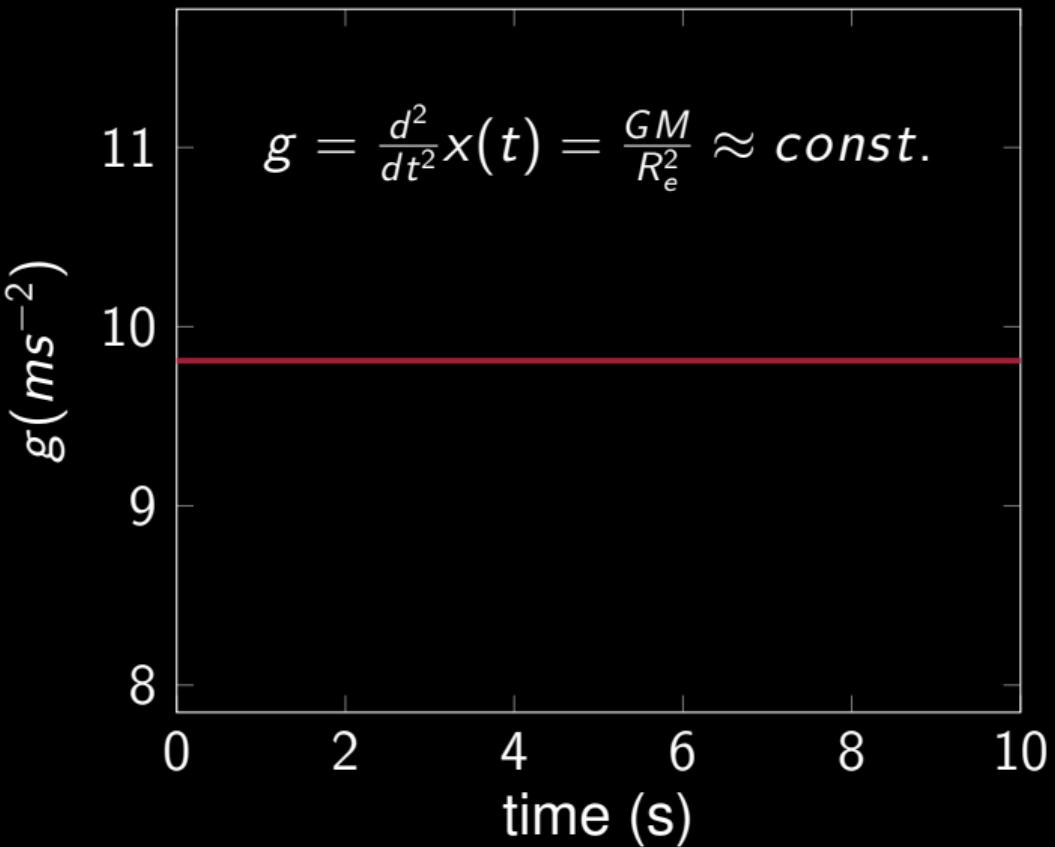
$$\frac{d^2 \vec{x}}{dt^2} = G \frac{M}{R_E^2} \approx \text{const.}$$

At the Earth's surface (R_E) g is close to constant and only vertical. (Later we will see that none of this is not quite true).

Example: Measuring acceleration



$$g = \frac{d^2}{dt^2}x(t) = \frac{GM}{R_e^2} \approx \text{const.}$$



Example: Measuring acceleration



$$v = \int g dt = \frac{d}{dt} x(t) = \frac{GM}{R_e^2} t + c_1$$

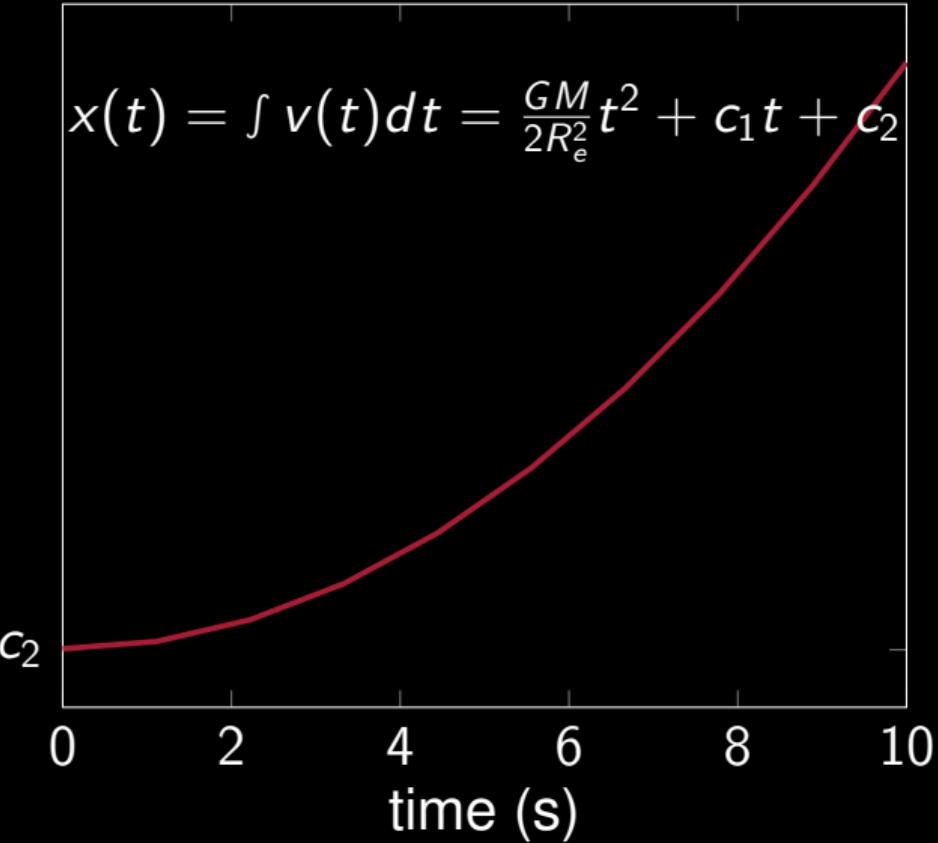


Example: Measuring acceleration



$$x(t) = \int v(t)dt = \frac{GM}{2R_e^2}t^2 + c_1t + c_2$$

$x(m)$



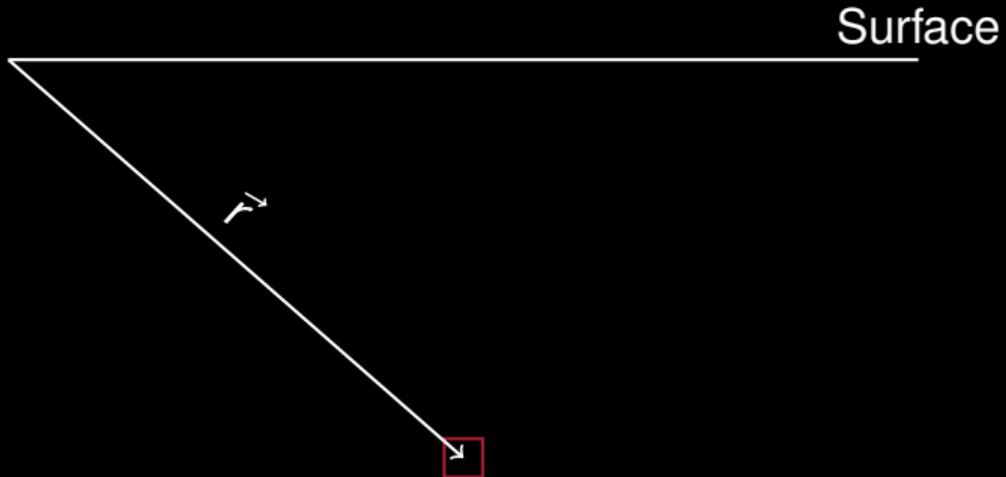


$$x(t) = \frac{GM}{2R_e^2}t^2 + c_1t + c_2$$

- ▶ Setting, e.g., $c_1 = 0$ (initial velocity) and $c_2 = 0$ (initial position) is quite convenient.
- ▶ This is the principle of a free-fall gravimeter.

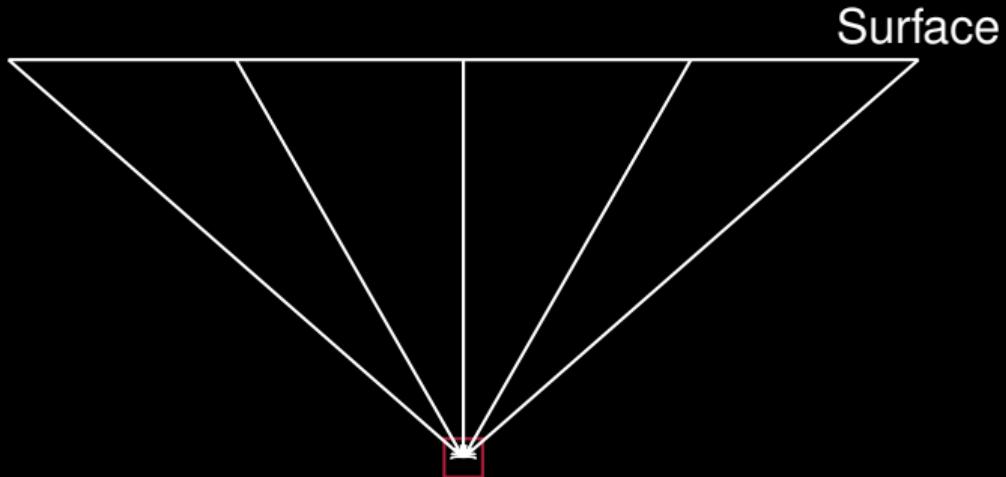


- ▶ Thanks to the Greeks we know the radius R_E for the Earth. However, its mass was unknown for a while.
- ▶ Go ahead and determine the mass of the Earth M with your Smartphone!
- ▶ There is an important first-order finding in Earth Sciences that you can (re-) discover. Which one?



$$\vec{F} = G \frac{dM}{r^2} \hat{r}$$

For a small mass dM the point mass approximation holds.

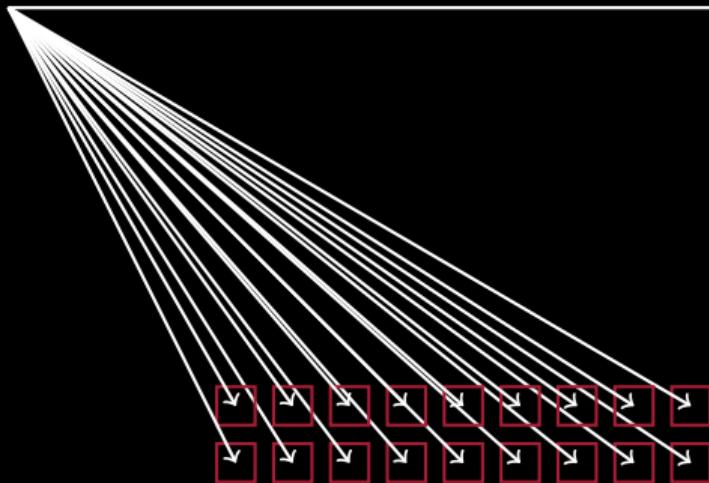


$$\vec{F} = G \frac{dM}{r^2} \hat{r}$$

Profiling across a sub-surface target results in a gravity anomaly (\rightarrow Exercises).



Surface



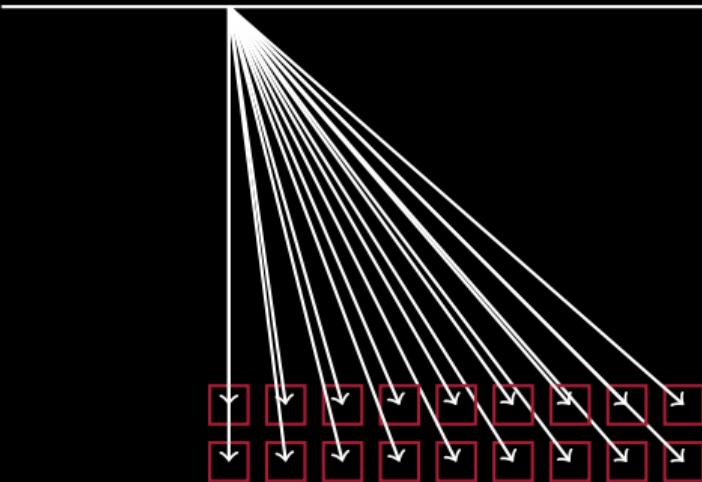
$$\vec{F}(\vec{r}) = \sum_i G \frac{dM_i}{r_i^2} \hat{r}_i$$

For i point masses the effect adds up.

Beyond point masses



Surface

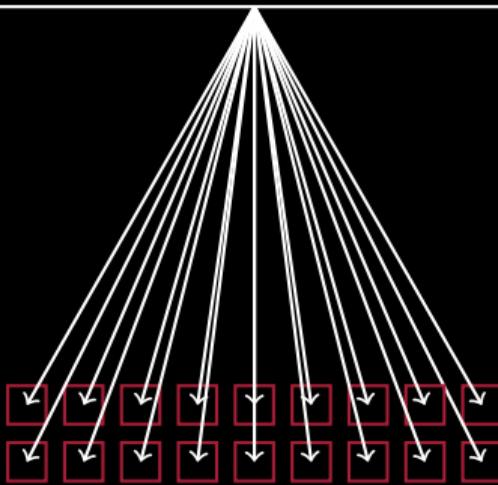


$$\vec{F}(\vec{r}) = \sum G \frac{dM_i}{r_i^2} \hat{r}_i$$

Beyond point masses

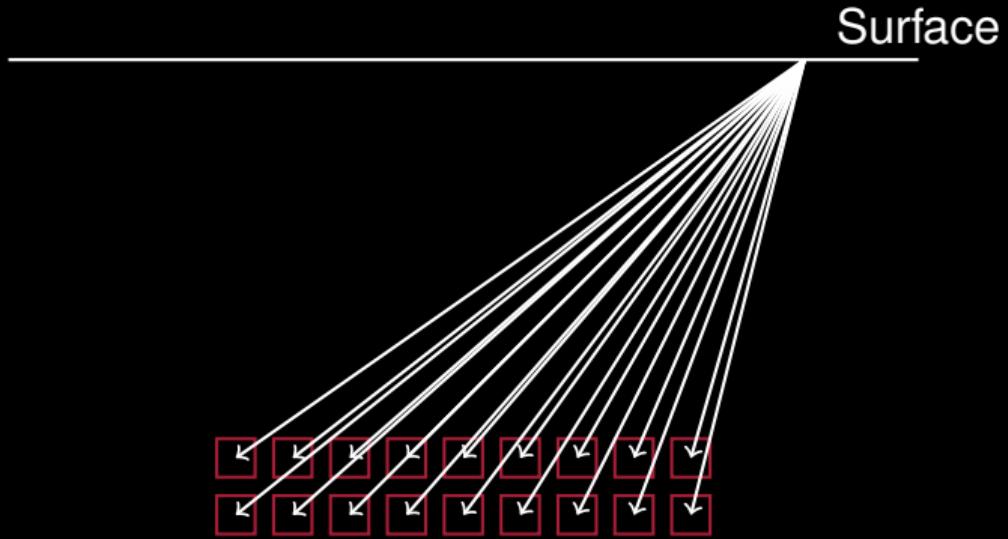


Surface



$$\vec{F}(\vec{r}) = \sum G \frac{dM_i}{r_i^2} \hat{r}_i$$

Beyond point masses

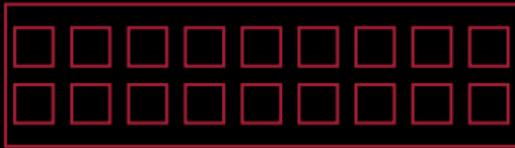


$$\vec{F}(\vec{r}) = \sum G \frac{dM_i}{r_i^2} \hat{r}_i$$



Surface

$$\vec{F}(\vec{r}) = G \int \rho \frac{1}{r^2} \hat{r} dV$$

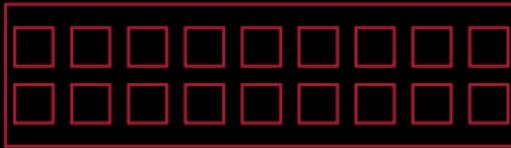


The summation can be replaced by an integration over a volume enclosing a continuous density.



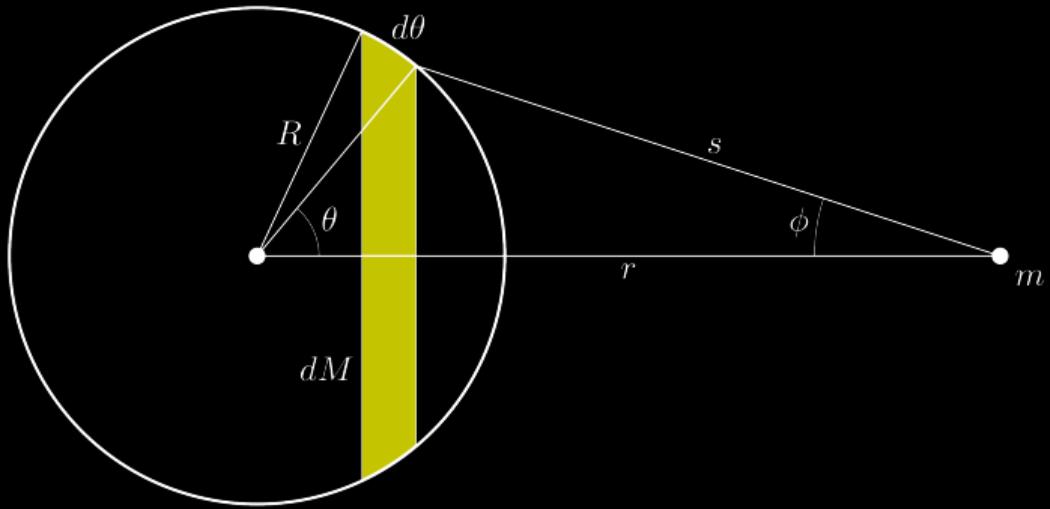
Surface

$$\vec{F}(\vec{r}) = G \int \rho \frac{1}{r^2} \hat{r} dV$$



The integration is a triple integral. Integration limits and coordinates depend on the viewpoint. Example is a Bouger plate, in general not easy to solve (\rightarrow Exercises).

Example: Shell



[Xaononl CC BY-SA 4.0]

Newton's shell theorem solves the volume integral inside and outside spherical objects (→ Ex.-Discussion)

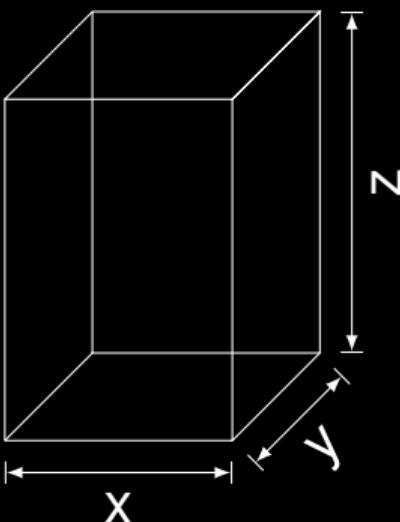


- ▶ The field outside a shell is the same as the one from an equivalent point mass
- ▶ The field inside a shell is zero. Everywhere.

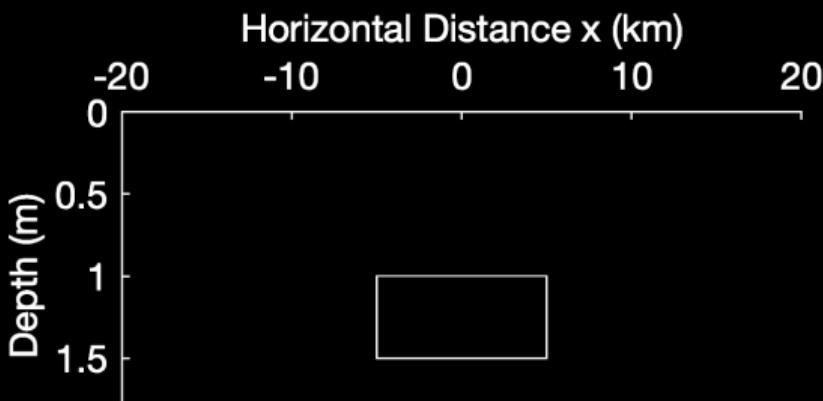
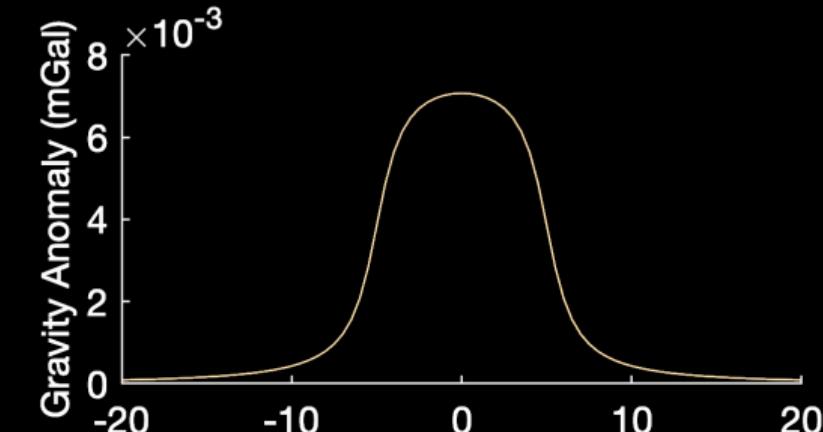


- ▶ There are analytical solutions for other shapes (e.g., Nagy 1966 for Prism).

Surface



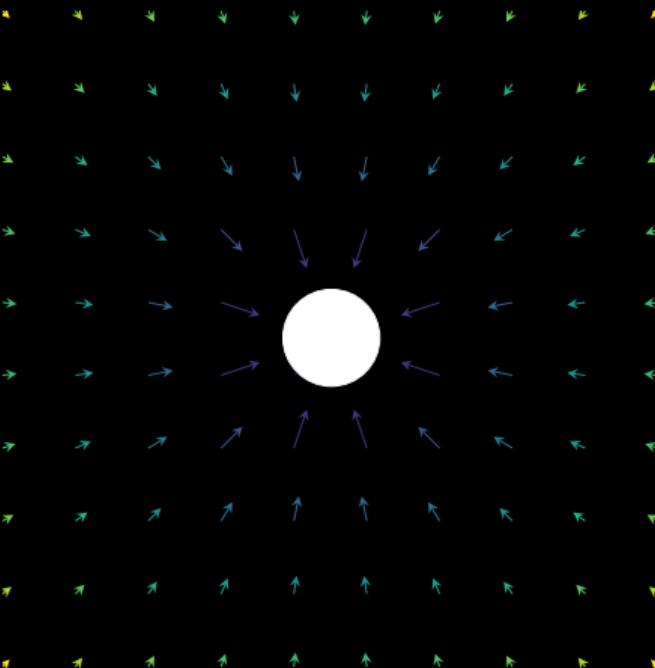
Numerical forward modelling (\rightarrow Ex)



Vector fields



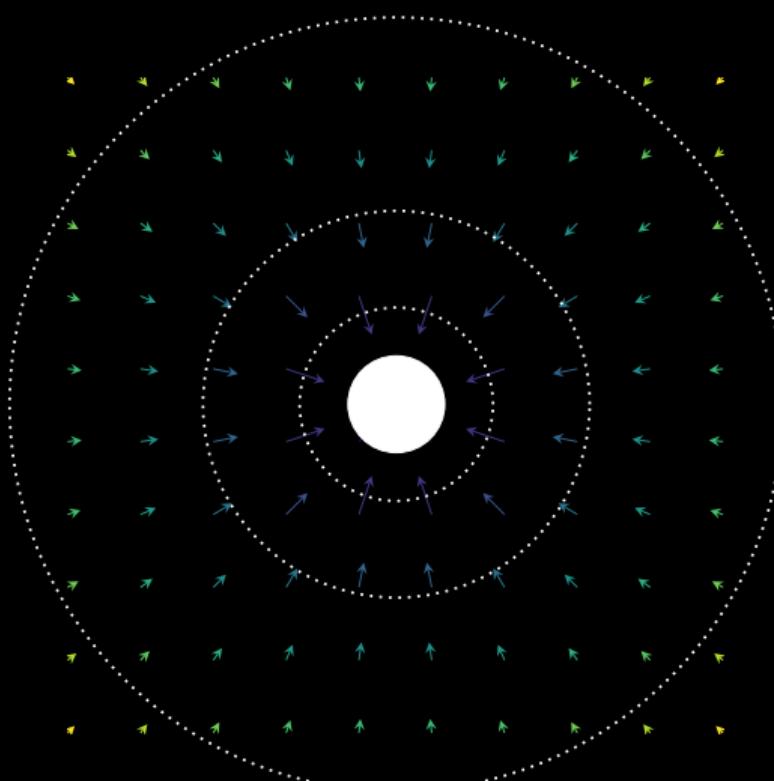
$$\vec{g} = G \frac{M}{r^2} \hat{r}$$



Potential Field

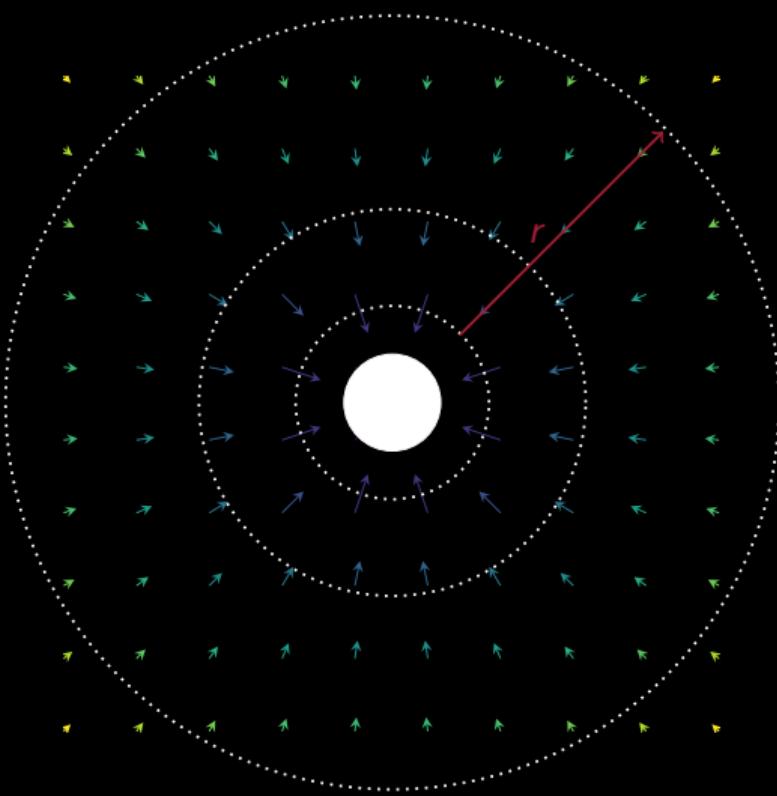


$$\vec{g} = G \frac{M}{r^2} \hat{r}$$





What is the amount of work required?





$$\begin{aligned} U(r) &= - \int_{\infty}^r \vec{g} d\vec{r} \\ &= - \int_{\infty}^r g dr \\ &= -GM \int_{\infty}^r \frac{1}{r^2} dr \\ &= -GM \left[-\frac{1}{r} \right]_{\infty}^r \\ &= GM \frac{1}{r} \end{aligned}$$

Potential for a point mass.



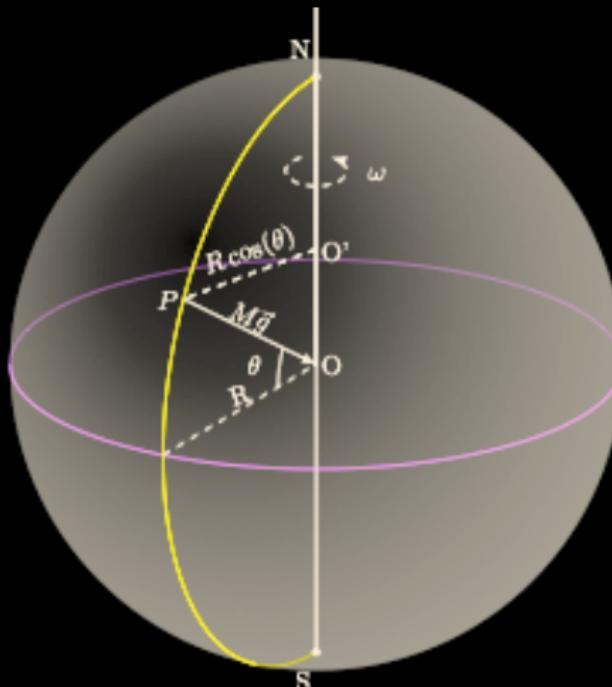
$$\vec{g}(r) = -\nabla U(r)$$

- ▶ It is sometimes easier to calculate the potential of an anomaly and to infer the acceleration via the gradient.
- ▶ Equipotential lines are perpendicular to the field direction.
- ▶ Equipotential lines are in general NOT lines of equal field strength (cf. with down-hill slope force in landscape)

Gravitational field of a rotating Earth



The Earth's rotation minimizes gravitational acceleration at the equator. At the poles it does nothing.





Centripetal acceleration at P perpendicular to rotation axis parallel to O'-P:

$$g_r = \omega^2 R \cos(\theta)$$

Centripetal acceleration at P perpendicular to rotation axis parallel to O'-P:

$$g_{r,proj.} = \omega^2 R \cos^2(\theta)$$

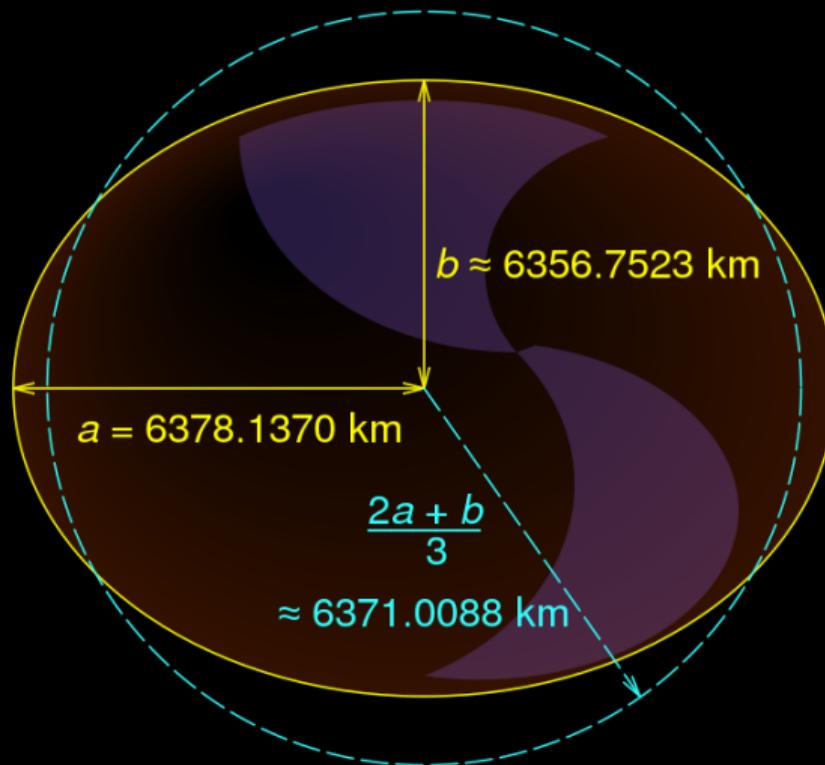
Angular Frequency: ω

Angular Velocity: $\vec{\nu}_r = \vec{\omega} \times \vec{R} \cos(\theta)$

Angular Acceleration: $\vec{g}_r = \dot{\vec{\nu}}_r = \vec{\omega} \times \vec{\omega} \times \vec{R} \cos(\theta)$

$$\vec{g} = \vec{g}_{gravitation} - \vec{g}_{rot.,proj.}$$

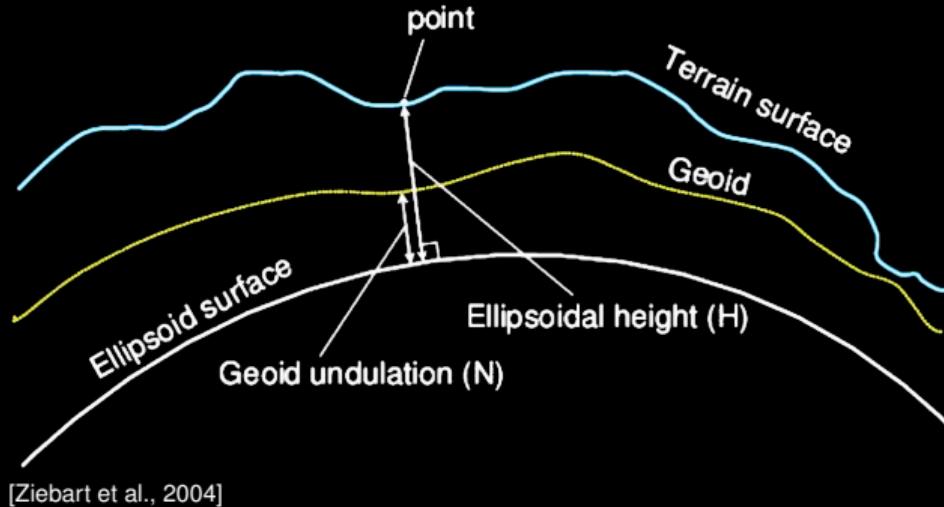
An ellipsoidal Earth





- ▶ Rotation induces ellipsoidal shape approximate with reference ellipsoid.
- ▶ Latitudinal correction of gravitation is adjusted accordingly.

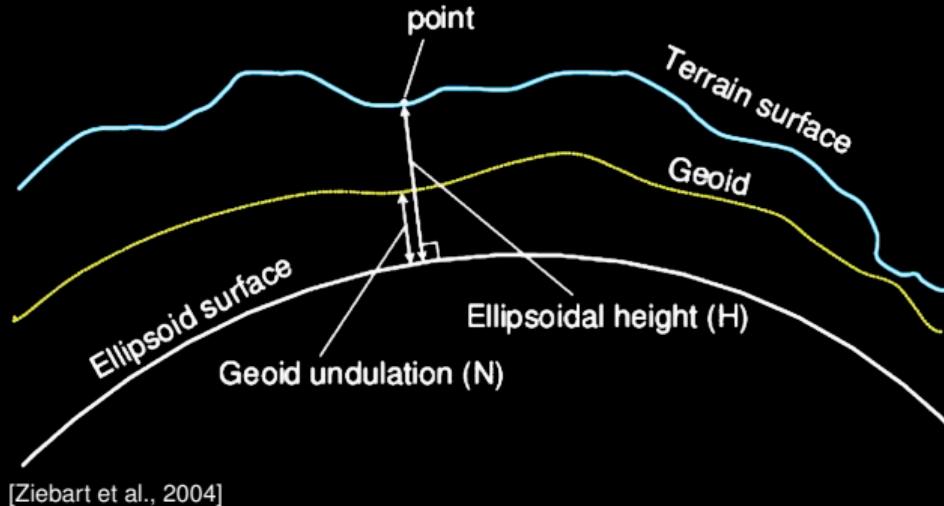
An ellipsoidal Earth



[Ziebart et al., 2004]

- ▶ Geoid is a real-world equipotential line approximating sea level.
- ▶ It is referenced to the geometric ellipsoid.

An ellipsoidal Earth



[Ziebart et al., 2004]

- ▶ Heights can be referenced to geoids or ellipsoids (common source of error).
- ▶ The direction of \vec{g} is perpendicular to the geoid, so the local normal does not point to the Earth's center.

The GEOID determined by GRACE

