1 Maxwell Equation and Electromagnetic Induction

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Motivation: Ten most important equations that changed the world. Among those:

$$
\begin{aligned}
a^2 + b^2 = c^2 \qquad & \text{(Pythagoras)} \\
e^{i\pi} + 1 = 0 \qquad & \text{(Euler)} \\
\vec{F} = m\vec{a}; \vec{F} = G\tfrac{mM}{r^2} \qquad & \text{(Newton)} \\
\text{(...Einstein, Schrödinger)} &
\end{aligned}
$$

and Maxwell:

$$
\begin{aligned}
\nabla \cdot \vec{\mathbf{D}} &= \rho \qquad \text{(Gauss)} \\
\nabla \cdot \vec{\mathbf{B}} &= 0 \qquad \text{(Gauss)} \\
\nabla \times \vec{\mathbf{E}} &= -\frac{\partial \vec{\mathbf{B}}}{\partial t} \qquad \text{(Faraday)} \\
\nabla \times \vec{\mathbf{H}} &= \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t} \qquad \text{(Ampére-Maxwell)} \\
\vec{\mathbf{D}} &= \varepsilon\varepsilon_0 \vec{\mathbf{E}} \qquad \text{(materials: electric field, dielectric field)} \\
\vec{\mathbf{H}} &= \mu\mu_0 \vec{\mathbf{B}} \qquad \text{(materials: magnetizing field, magnetic induction)} \\
\vec{j} &= \sigma \vec{\mathbf{E}} \qquad \text{(Ohm's law)}
\end{aligned}
$$

and Integral:

$$
\begin{aligned}
\int\int_{\partial\Omega} \vec{\mathbf{D}} \cdot d\vec{\mathbf{S}} &= \int\int\int_{\Omega} \rho \qquad \text{(Gauss)} \\
\int\int_{\partial\Omega} \vec{\mathbf{B}} \cdot d\vec{\mathbf{S}} &= 0 \qquad \text{(Gauss)} \\
\int_{\partial\Sigma} \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} &= -\frac{\partial}{\partial t}\int\int_{\Sigma} \vec{\mathbf{B}} \cdot d\vec{S} \qquad \text{(Faraday)} \\
\int_{\partial\Sigma} \vec{\mathbf{H}} \cdot d\vec{\mathbf{l}} &= \int\int_{\Sigma} \vec{\mathbf{J}} \cdot d\vec{\mathbf{S}} + \frac{\partial}{\partial t}\int\int_{\Sigma} \vec{\mathbf{D}} \cdot d\vec{\mathbf{S}} \qquad \text{(Ampére-Maxwell)} \\
\Omega &: \text{Volume} \\
\Sigma &: \text{Surface} \\
\partial\Omega &: \text{Surface of volume} \\
\partial\Sigma &: \text{Edge of surface}
\end{aligned}
$$

Have a picture in mind for each one of them: (1) Source of a static E-field. (2) Dipole b-Field. (3) Induction with Magnet. (4) Displacement Currents and Bio-Savart law

Integral vs. Differential form: EMF, loops, LENZ law

Principle of self-induction

Table resistance frequency dependency

$Z_L = j\omega L$ V lags I $Z_c = \frac{1}{i\omega C}$ I lags V $Z_R = R$ in phase

Classification of electrical methods

Principles of the Slingram Method

$I_{l1} = I_1 e^{i\omega t}$
$V_{l3} = L_{13}\frac{dI_{l1}}{dt} = i\omega L_{13} I_1 e^{i\omega t}$ $V_{l2} = L_{12}\frac{dI_{l1}}{dt} = i\omega L_{12} I_1 e^{i\omega t}$
What is I_2 with a R-L Subsurface model? Ohms law:
$Z_{l2} = R + i\omega L_{l2}$ $V_{l2} = (R + i\omega L_{l2}) I_2$
Induced voltage is balanced by inductance and resistance:
$i\omega L_{12} I_1 e^{i\omega t} + (R + i\omega L_{l2}) I_2 = 0$
$I_{l2} = I_2 e^{i\omega} = \frac{-i\omega L_{12}}{R_{l2}+i\omega L_{l2}} I_1 e^{i\omega t}$ $I_{l2} = I_2 e^{i\omega t} = \frac{-i\omega \frac{L_{l2}}{R_{l2}}}{1+i\omega \frac{L_{l2}}{R_{l2}}} \frac{L_{12}}{L_{l2}} I_1 e^{i\omega t}$

Il2 will produce a secondary B. How does that appear in Loop 3:
$V_{l3} = L_{13}\frac{dI_l 1}{dt}$ primary $V_{l3} = L_{23}\frac{dI_l 2}{dt}$ secondary
$\frac{U_s}{U_p} = -\frac{L_{12}L_{23}}{L_{13}L_{l2}} \left(\frac{i\omega \frac{L_{l2}}{R_{l2}}}{1+i\omega \frac{L_{l2}}{R_{l2}}} \right)$
induction number: $\alpha = \omega \frac{L_{l2}}{R_{l2}}$ helps to write the complex number in standard form: $\frac{U_s}{U_p} == -\frac{L_{12}L_{23}}{L_{13}L_{l2}} \left(\frac{1}{1+\alpha^2} (\alpha^2 + i\alpha) \right)$

2 Seismics

$$\frac{\partial^2 \Psi}{\partial t^2 \Psi} = v^2 \frac{\partial^2 \Psi}{\partial x^2 \Psi}$$

- This is an (undamped) wave equation.
- This is a second order (hyperbolic) partial differential equation.
- It is a function of space x and time t.

2.1 General solution

$$\Psi = f(x + vt) + f(x - vt)$$

Exampel: $\Psi(x,t) = e^{k(x-vt)}, \sin(x - vt), (x - vt)^3$.
However, only one solution complies with the given initial and boundary conditions. What is v?
Draw Picture 4.4 Telford of Ψ of wave form Ψ on y-axis with an x_0 and $t_0 + \Delta t$ marked.

$$t_0 + \Delta t : \Psi_1(x_0 + \Delta_x, t + t_0) = f(x_0 + \Delta_x - v(t_0 + \Delta t))$$

$$t_0 : \Psi_1(x_0, t) = f(x_0 + \Delta_x - vt_0)$$
$$x_0 - vt_0 = x_0 + \Delta x - v(t_0 + \Delta t)$$

- $x \pm vt$ is the phase
- $x \pm vt = const.$ wavefronts
- normal to the wavefront is the raypath

2.2 Specific solutions

2.2.1 Spherical waves

$$\frac{\partial^2 \Psi}{\partial t^2 \Psi} = \frac{v^2}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right)$$

Solution:

$$\psi(x, t) = \frac{1}{r} f(r - vt)$$

Spherical waves the circles are wavefronts (i.e. lines of constant phase). Raypath are thus along r. In the farfield spherical waves can be approximated with plane waves.
draw Figure 4.6 plane waves and spherical waves

2.2.2 Harmonic waves

$$\psi(x, t) = A \cos(k(x - vt)) \tag{1}$$
$$\psi(x, t) = A/r \cos(k(x - vt)) \tag{2}$$

At fixed t if x increases by 2π/k then everything repeats so

$$\lambda = \frac{2\pi}{k} \text{wavelength}$$

At fixed x it varies harmonically with time from -A to A (or -A/r to A/r). A is the amplitude.
At fixed x repetition by T if $kVT = 2\pi = 2\pi(vT/\lambda)$

$$T = \lambda/v \tag{3}$$
$$f = 1/T \tag{4}$$
$$v = \lambda f \tag{5}$$

With T period, f frequency, v phase velocity. Commonly used is the angular frequency $\omega = 2\pi f$ so that

$$\Psi(x, t) = A \cos(k(x - vt)) = A \cos(kx - \omega t) \tag{6}$$
$$\tag{7}$$

Sometimes and additional phase offset is used:

$$\Psi(x, t) = A \cos(k(x - vt)) = A \cos(kx - \omega t + \varepsilon) \tag{8}$$
$$\tag{9}$$

2.3 Some critical steps in dipped layer refraction seismics

Goal: write t^- as a function of velocities and dip. Isolate slope and simplfy y-offset. $t^- = \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} + \frac{x\cos(\theta)-(d^-+d^+)\tan(i_c)}{v_2}$

$$
\begin{aligned}
t^- &= \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} + \frac{x\cos(\theta)-(d^-+d^+)\tan(i_c)}{v_2} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} - \frac{(d^-+d^+)\tan(i_c)}{v_2} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} - \frac{(d^-+d^+)\sin(i_c)}{\cos(i_c)v_2} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} - \frac{(d^-+d^+)v_1}{\cos(i_c)v_2^2} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{d^-}{v_1\cos(i_c)} + \frac{d^+}{v_1\cos(i_c)} - \frac{(d^-+d^+)v_1^2}{v_1\cos(i_c)v_2^2} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{1}{v_1\cos(\imath_c)}(d^-+d^+-(d^-+d^+)\frac{v_1^2}{v_2^2}) \\
&= \frac{x\cos(\theta)}{v_2} + \frac{1}{v_1\cos(\imath_c)}((d^-+d^+)(1-\frac{v_1^2}{v_2^2})) \\
&= \frac{x\cos(\theta)}{v_2} + \frac{(d^-+d^+)\cos(\imath_c)}{v_1}
\end{aligned}
$$

eliminate d^+ using use: $d^+ = d^- + x\sin(\theta)$ which is clear from geometry when you put θ at the surface.

$$
\begin{aligned}
t^- &= \frac{x\cos(\theta)}{v_2} + \frac{(d^-+d^+)\cos(i_c)}{v_1} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{(2d^-+x\sin(\theta))\cos(i_c)}{v_1} \\
&= \frac{x\cos(\theta)}{v_2} + \frac{2d^-\cos(\imath_c)}{v_1} + \frac{x\sin(\theta)\cos(i_c)}{v_1} \\
&= \frac{x\cos(\theta)\sin(i_c)}{v_2\sin(i_c)} + \frac{2d^-\cos(\imath_c)}{v_1} + \frac{x\sin(\theta)\cos(i_c)}{v_1} \\
&= \frac{x\cos(\theta)\sin(i_c)}{v_1} + \frac{2d^-\cos(\imath_c)}{v_1} + \frac{x\sin(\theta)\cos(i_c)}{v_1} \\
&= \frac{x\sin(\theta+i_c)}{v_1} + \frac{2d^-\cos(i_c)}{v_1}
\end{aligned}
$$

θ and i_c are yet unknown but can be derived from the arcsin relation straightforward.

$$
\frac{1}{v_2^-} + \frac{1}{v_2^+} = \frac{1}{v_1}\sin(i_c)\cos(\theta) = \frac{2}{v_2}\cos(\theta)
$$