EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Introduction to Geophysics

R. Drews

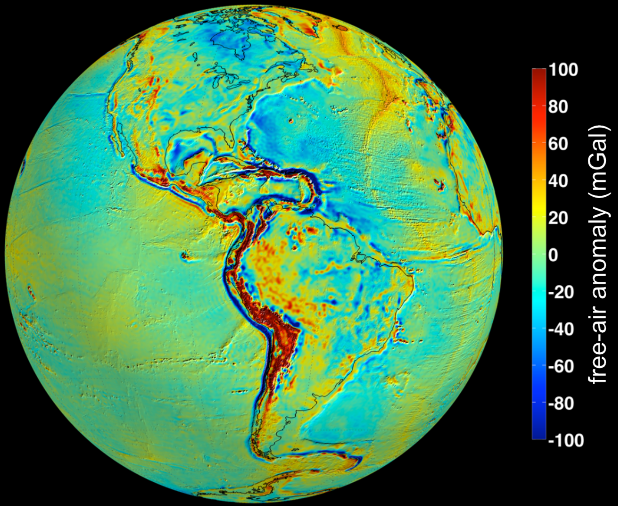
Gravity Method

Learning Goals

Learning goals today:

- Understand that gravity methods map sub-surface density variability
- Understand the gravitational force, its potential field, and one underlying measurement principle.
- Understand the Earth's geoid and reference ellipsoid

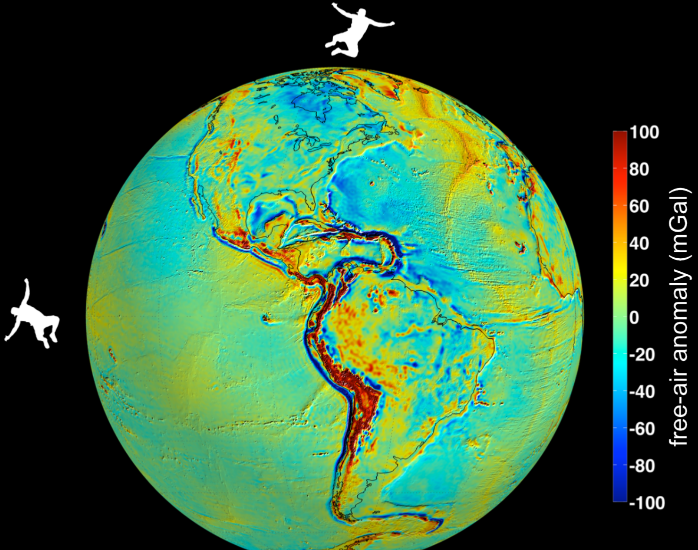
Example: Global variability



[JPL/Caltech, GRACE-FO (DT-10°)]

Example: Global variability

Your mass is constant but your weight is not.



What is a force?

[Newton (1642-1726) / G. Johnson.]

$$\vec{F} = m\vec{g}$$

$\vec{F}$: Force (N; kg m s^{-2})

$\vec{g}$: Acceleration (m s^{-2})

m : Mass (kg)

The gravitational force

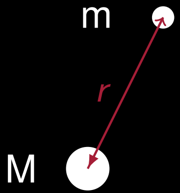
[Newton (1642-1726) / G. Johnson.]

$$\vec{F} = G\,\frac{mM}{r^2}\,\hat{r}$$

$G = 6.674 \cdot 10^{-11}\,(m^3 kg^{-1} s^{-2})$

$\hat{r}$: unit vector

r : distance between point masses



Example: The gravitational constant

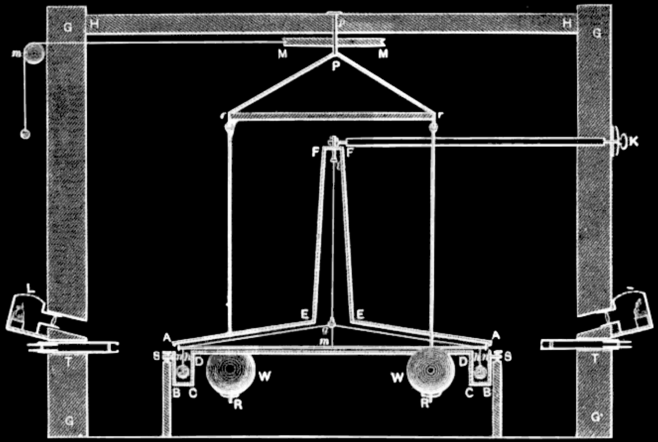
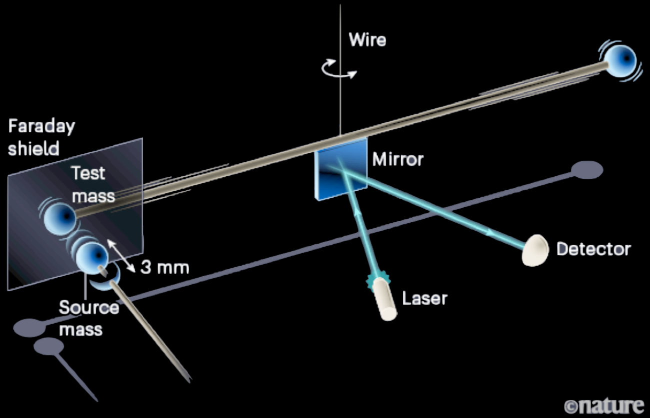


Fig. 1

Cavnedish, PNAS, 1798

Example: The gravitational constant



Westphal et al., Nature, 2021

G is the worst known constant in physics. Why?

Example: Measuring acceleration

$$\vec{F} = m\vec{g}$$

$$\vec{F} = G\frac{mM}{r^2}\hat{r}$$

Example: Measuring acceleration

$$\vec{F} = m\vec{g}$$

$$\vec{F} = G\frac{mM}{r^2}\hat{r}$$

$$\rightarrow \vec{g} = G\frac{M}{r^2}\hat{r}$$

$$\rightarrow \frac{d^2\vec{x}}{dt^2} = G\frac{M}{r^2}\hat{r}$$

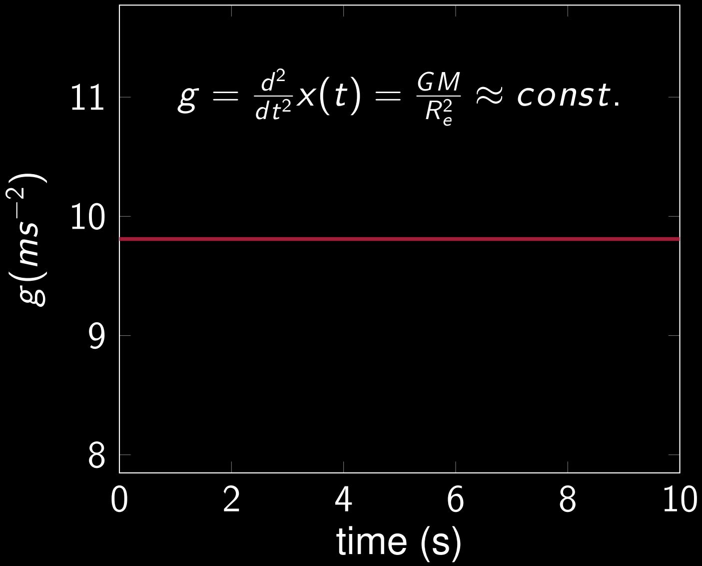
This is a differential equation.

Example: Measuring acceleration

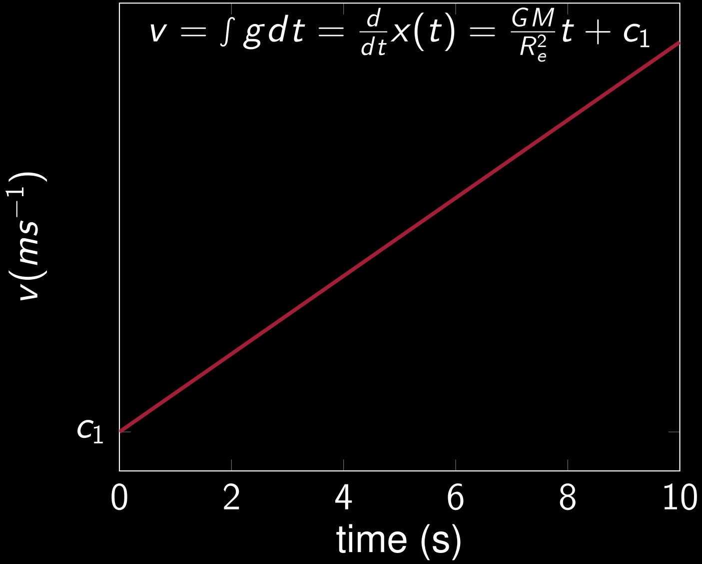
$$\frac{d^2\vec{x}}{dt^2} = G\frac{M}{R_E^2} \approx const.$$

At the Earth's surface (R_E) g is close to constant and only vertical. (Later we will see that none of this is quite true).

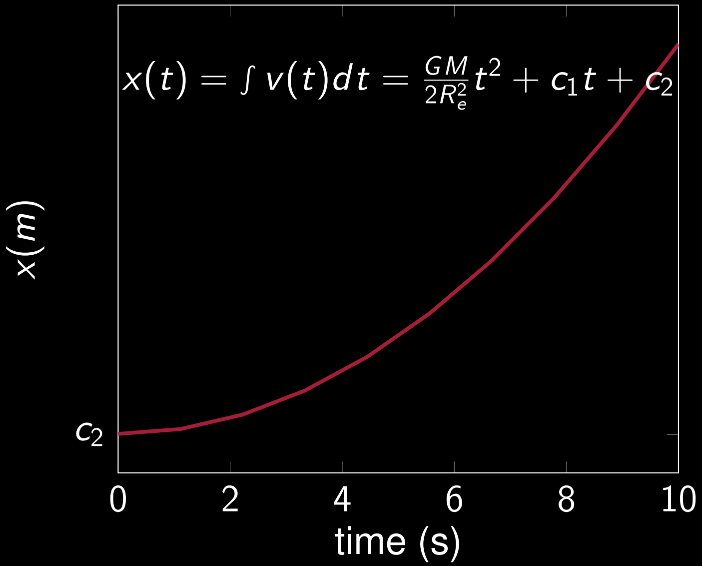
Example: Measuring acceleration



Example: Measuring acceleration



Example: Measuring acceleration



Example: Measuring acceleration

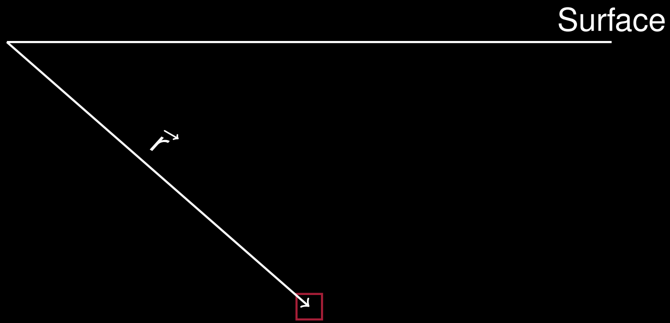
$$x(t) = \frac{GM}{2R_e^2} t^2 + c_1 t + c_2$$

- Setting, e.g., $c1 = 0$ (initial velocity) and $c_2 = 0$ (initial position) is quite convenient.
- This is the principal of a free-fall gravimeter.

Exercises: Group-Work Thursdays

- Thanks to the Greeks we know the radius R_E for the Earth. However, its mass was unknown for a while.
- Go ahead and determine the mass of the Earth M with your Smartphone!
- There is an important first-order finding in Earth Sciences that you can (re-) discover. Which one?

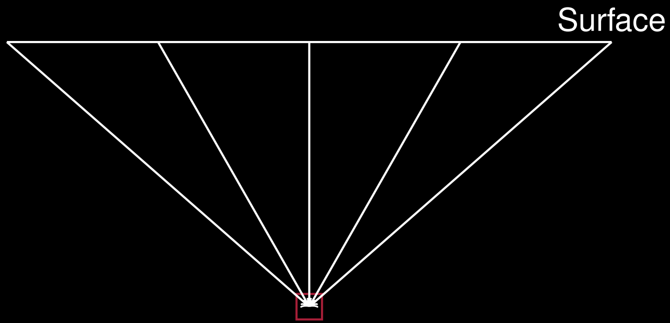
Beyond point masses



$$\vec{F} = G\frac{dM}{r^2}\hat{r}$$

For a small mass dM the point mass approximation holds.

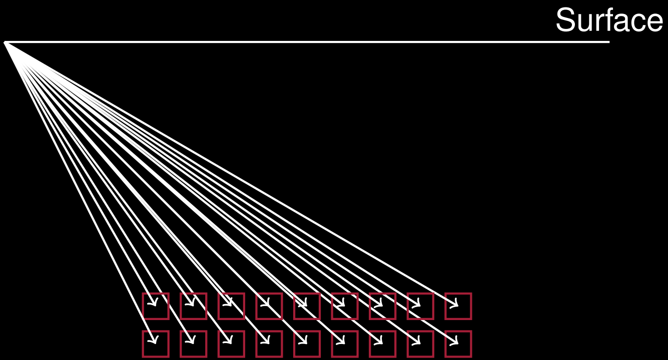
Beyond point masses



$$\vec{F} = G\frac{dM}{r^2}\hat{r}$$

Profiling across a sub-surface target results in a gravity anomaly ($\rightarrow$ Exercises).

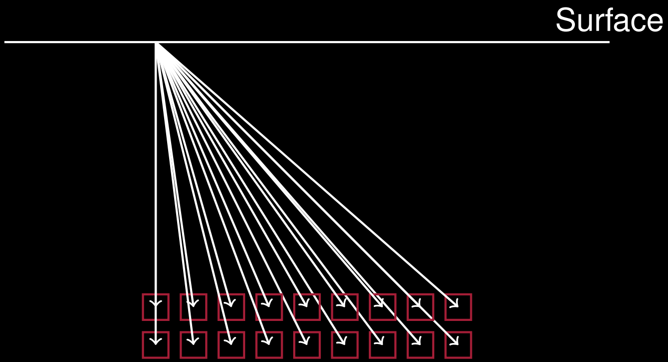
Beyond point masses



$$\vec{F}(\vec{r}) = \sum_i G \frac{dM_i}{r_i^2} \hat{r}_i$$

For i point masses the effect adds up.

Beyond point masses



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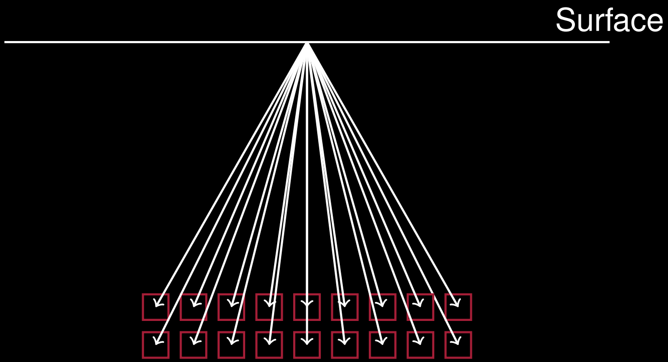
Beyond point masses



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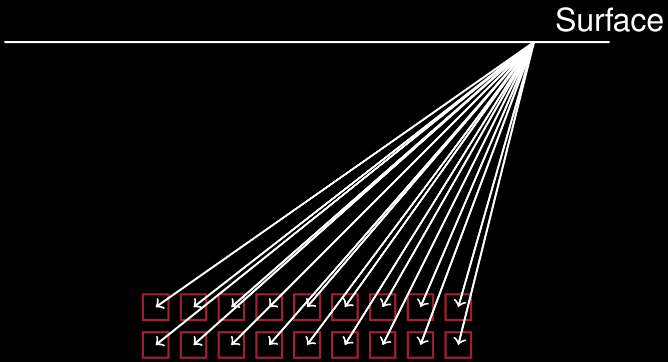
Beyond point masses



$$\vec{F}(\vec{r}) = \sum G \frac{dM_i}{r_i^2} \hat{r}_i$$

Beyond point masses

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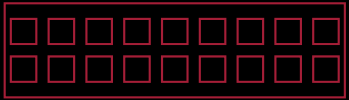
Surface

$$\vec{F}(\vec{r}) = G \int \rho \frac{1}{r^2} \hat{r} dV$$

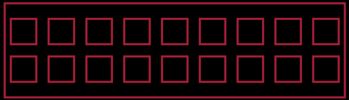


The summation can be replaced by an integration over a volume enclosing a continuous density.

Beyond point masses

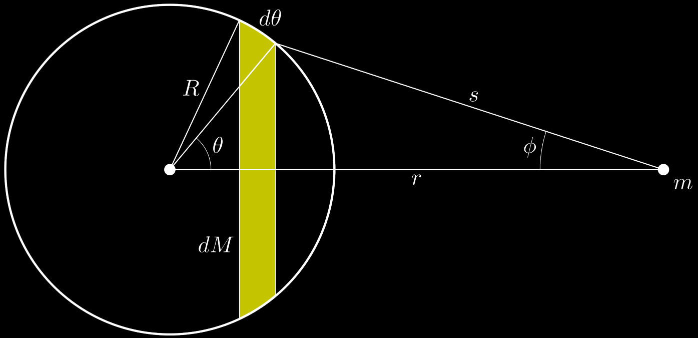
Surface

$$\vec{F}(\vec{r}) = G \int \rho \frac{1}{r^2} \hat{r} dV$$



The integration is a triple integral. Integration limits and coordinates depend on the viewpoint. Example is a Bouger plate, in general not easy to solve ($\rightarrow$ Exercises).

Example: Shell



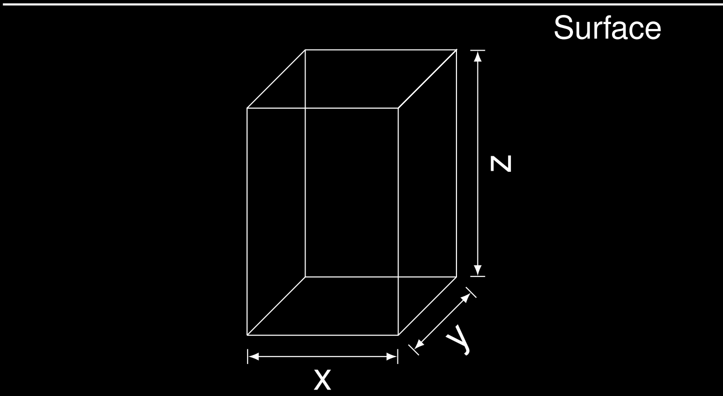
Newton's shell theorem solves the volume integral inside and outside spherical objects ($\rightarrow$ Ex.-Discussion)

Newton's Shell Theorem

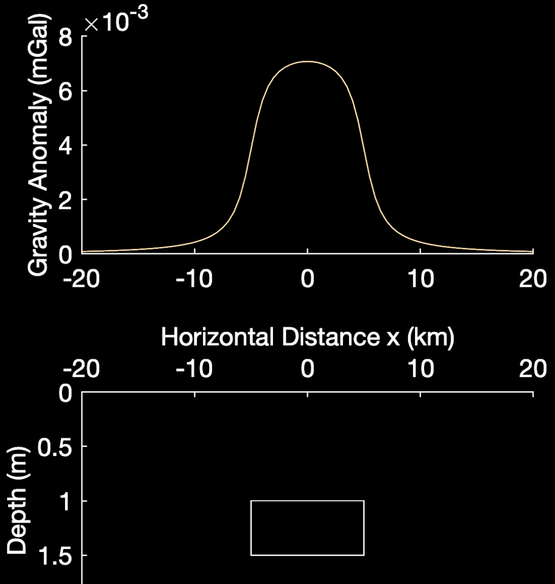
- The field outside a shell is the same as the one from an equivalent point mass
- The field inside a shell is zero. Everywhere.

Other shapes

- There are analytical solutions for other shapes (e.g., Nagy 1966 for Prism).

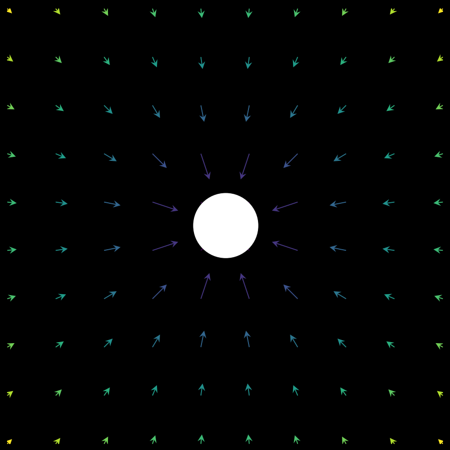


Numerical forward modelling (→ Ex)



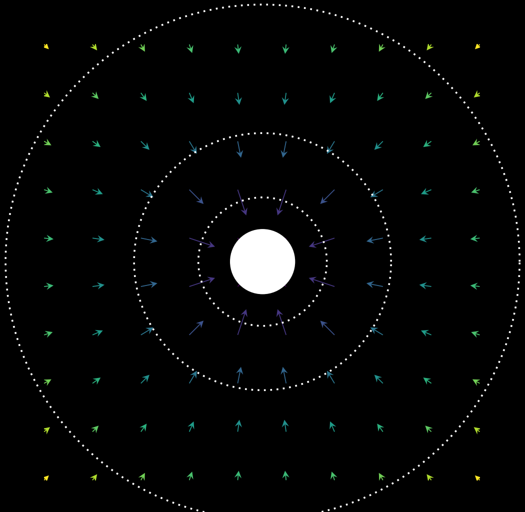
Vector fields

$$\vec{g} = G\frac{M}{r^2}\hat{r}$$

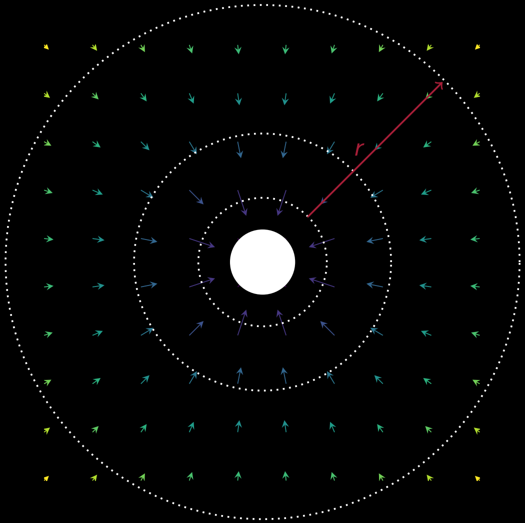


Potential Field

$$\vec{g} = G\frac{M}{r^2}\hat{r}$$



What is the amount of work required?



Potential Fields

$$
\begin{aligned}
U(r) &= -\int_{\infty}^{r} \vec{g}\, d\vec{r} \\
&= -\int_{\infty}^{r} g\, dr \\
&= -GM \int_{\infty}^{r} \frac{1}{r^2} dr \\
&= -GM \left[-\frac{1}{r} \right]_{\infty}^{r} \\
&= GM \frac{1}{r}
\end{aligned}
$$

Potential for a point mass.

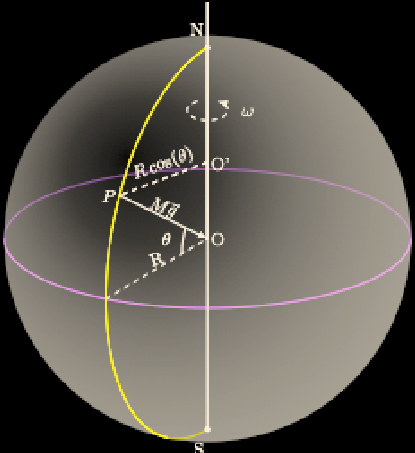
Potential Fields

$$\vec{g}(r) = -\nabla U(r)$$

- It is sometimes easier to calculate the potential of an anomaly and to infer the acceleration via the gradient.
- Equipotential lines are perpendicular to the field direction.
- Equipotential lines are in general NOT lines of equal field strength (cf. with down-hill slope force in landscape)

Gravitational field of a spherical Earth

The Earth's rotation minimizes gravitational acceleration at the equator. At the poles it does nothing.



Gravitational field of a spherical Earth

Centripedal acceleration at P perpendicular to rotation axis parallel to O'-P:

$$g_{r.} = \omega^2 R \cos(\theta)$$

Centripedal acceleration at P perpendicular to rotation axis parallel to O'-P:

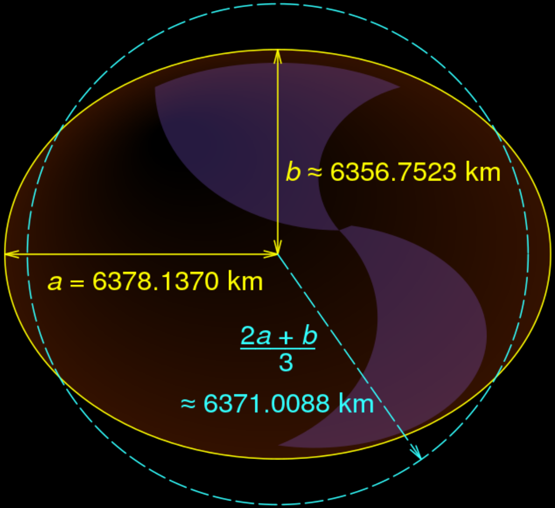
$$g_{r.,proj.} = \omega^2 R \cos^2(\theta)$$

Angular Frequency: ω

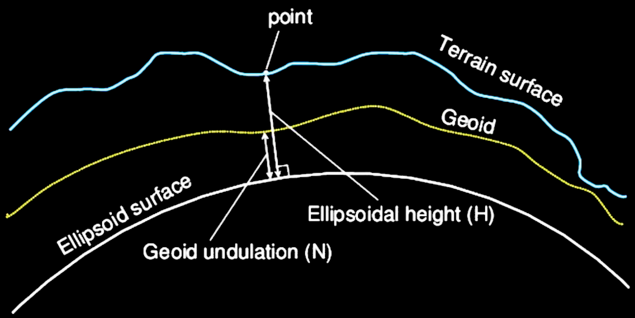
Angular Velocity: $\vec{v}_r = \vec{\omega} \times \vec{R} \cos(\theta)$

Angular Acceleration: $\vec{g}_r = \dot{\vec{v}}_r = \vec{\omega} \times \vec{\omega} \times \vec{R} \cos(\theta)$

An ellipsoidal Earth



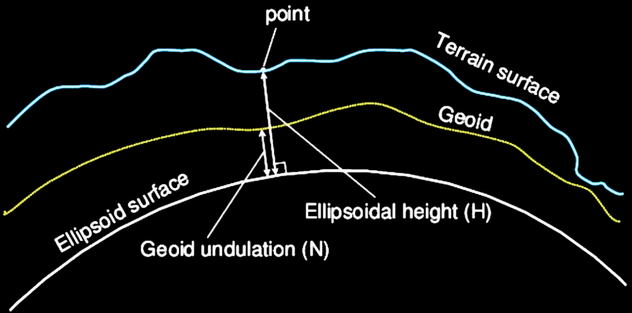
An ellipsoidal Earth



[Ziebart et al., 2004]

- Geoid is a real-world equipotential line approximating sea level.
- It is referenced to the geometric ellipsoid.

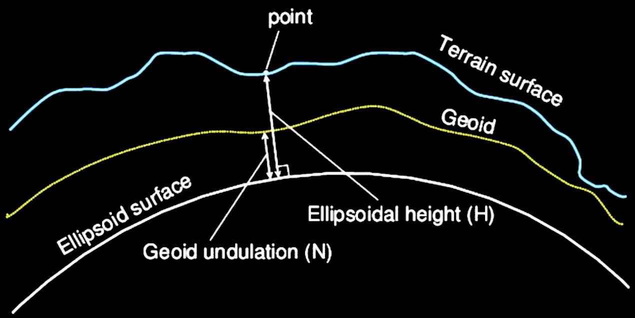
An ellipsoidal Earth



[Ziebart et al., 2004]

- The reference of elevation is a constant source of confusion.
- The geoid defines the local vertical direction.

An ellipsoidal Earth



[Ziebart et al., 2004]

- Upwarping of geoid indicates mass excess.
- Downwarping of geoid indicates mass deficit.

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