**Expectations for Exercises**

Exercises are an important part of the Geophysics lecture. They will treat some aspects of the lecture in more detail, but also cover new ground. We expect that you work on the exercises at home and we will discuss questions and solutions interactively together. Questions that are marked with 'Extra' are not required but geared to stir your further interest. We will surely support you if you tackle those as well.

# 5 Exercises for Resistivity Method 1

**Version:** June 2, 2022
**Context:** Resistivity Method 01
**Timing:** All resistivity exercises should be completed the latest by June 4th 2022

## 5.1 Resistivity Survey in a Sandbox

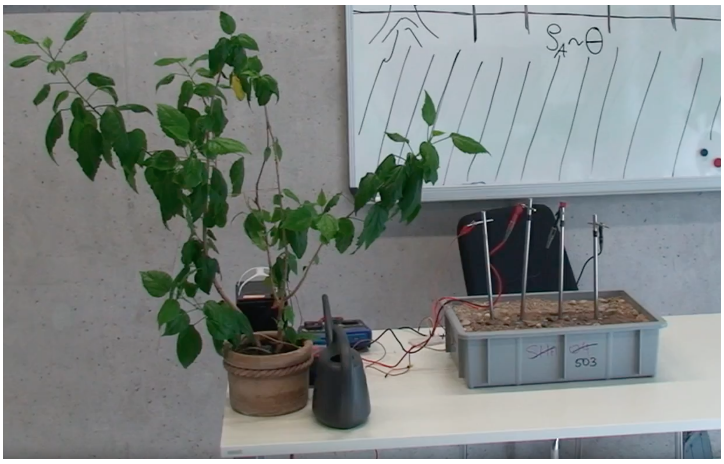


Figure 1: Toy setup for understanding the principles of the resistivity method.

**Group Work**

Self-organize with your groups to do this during our contact time on Thursdays. Ideally two to three groups should do the experiment in a total of 2h. We will assign two sessions for that. After you are done, disconnect all cables and keep everything as you found it for the next group. **When using the battery with open-end cables, be careful that you do not short them.**

(*practical*) Use a battery, four electrodes and two multimeters to determine the apparent resistivity of sand in a sandbox. You can align the electrodes in a co-linear Wenner array and also check how/if the resistivity changes with variable array types and moisture content. Post your results in the forum and compare them with the literature values. Don't overestimate the accuracy here, it is more about internalizing the measurement principle and to strengthen your cabling skills. All material (Fig. 1) will be provided on-site.

Solutions

Discussions on site and video online.

## 5.2 The LaPlace Equation

(*easy*) In the lecture we derived that the potential field of a single electrode (second electrode is located at infinity) satisfies the LaPlace equation:

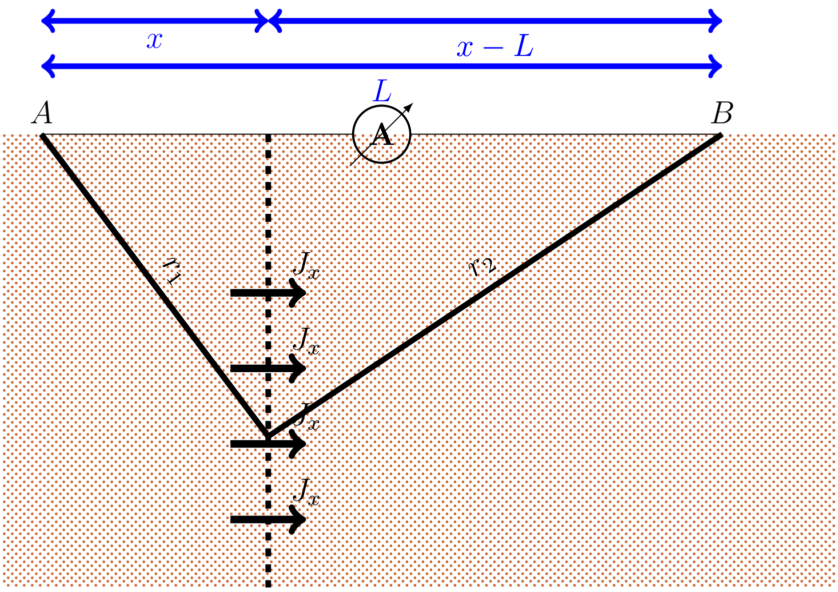
$$\vec{\nabla}^2 V = \Delta V = 0.$$

Show that our *Ansatz* $V(r) = \frac{A}{r}$ satisfies this equation in spherical coordinates. Note that the LaPlace in spherical coordinates is $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right)$.

Solutions

$$\begin{aligned}
\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{A}{r}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2 \frac{A}{r^2}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r} A = 0
\end{aligned}$$

## 5.3 Depth of Investigation




(*tricky*) In this exercise we would like to approximate which depth interval the resistivity method is sensitive to. We already understand intuitively that a larger electrode spacing between A and B increases the depth penetration. Here we aim to do it quantitatively. For this we consider a vertical plane located at distance $x$ away from electrode A, and calculate how much of the current flow density crosses that plane. This means we would like to calculate $J_x(z)$. From the lecture we know that:

$$\vec{j} = \sigma \vec{E} = \sigma \nabla V \rightarrow J_x = -\frac{I}{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Adopting the coordinate system from the Figure above we choose $r_1 = \sqrt{x^2+y^2+z^2}$ with z point along depth and y into the plane. First calculate $J_x$ for any point P, then show that

$$J_x(x = L/2) = \frac{I}{2\pi}\frac{L}{(z^2+y^2+\frac{L^2}{4})^{3/2}}$$

at the center point. Sketch this function roughly on paper for a fixed L. Now consider a fixed layer at constant depth $z$. How does $J_x$ change as $L$ is varied? Show that $J_x(x = L/2, z)$ has a maximum at $L = \sqrt{2}z$. This shows that depth penetration increases as we increase the distance between the current electrodes A and B. Still, most of the current flow will be near the surface. In order to define a depth of investigation we will need to apply some thresholding and include placement of the M,N electrodes. This varies from array to array.

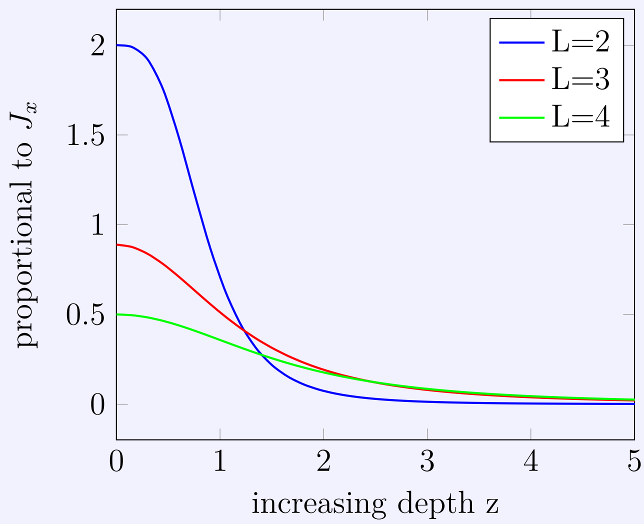
Solutions

Calculation of the derivative is:

$$-\frac{I}{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) =$$
$$-\frac{I}{2\pi}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{(x-L)^2+y^2+z^2}}\right) =$$
$$\frac{I}{4\pi}\left(\frac{2x}{(x^2+y^2+z^2)^{3/2}} - \frac{2(x-L)}{((x-L)^2+y^2+z^2)^{3/2}}\right) =$$
$$\frac{I}{2\pi}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}} - \frac{(x-L)}{((x-L)^2+y^2+z^2)^{3/2}}\right)$$

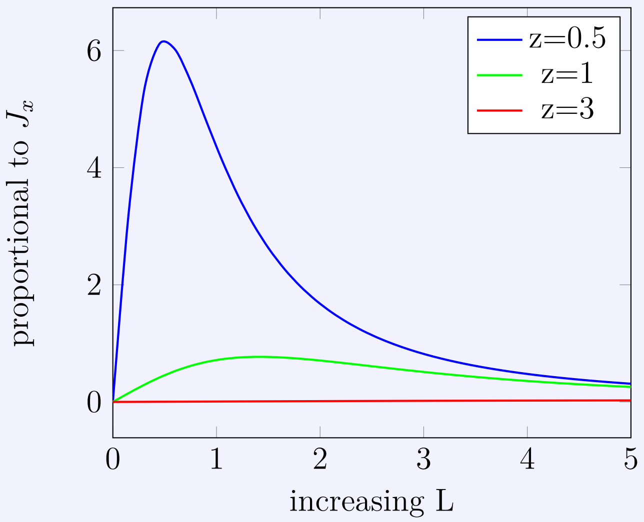
Now consider this for the center at $x = \frac{L}{2}$ where $r_1 = r_2 = \sqrt{L^2/4+y^2+z^2}$:

$$J_x(x = L/2) =$$
$$\frac{I}{2\pi}\left(\frac{L/2}{(L^2/4+y^2+z^2)^{3/2}} + \frac{(L/2)}{(L^2/4+y^2+z^2)^{3/2}}\right) =$$
$$\frac{I}{2\pi}\left(\frac{L}{(L^2/4+y^2+z^2)^{3/2}}\right)$$



Next we calculate the derivative with respect to $L$ (at x=L/2 and y=0):

$$\frac{\partial}{\partial L}\left(\frac{L}{(L^2/4+z^2)^{3/2}}\right)=0$$
$$(L^2/4+z^2)^{-3/2}-\frac{3}{2}L(L^2/4+z^2)^{-5/2}L/2=0$$
$$1-\frac{3}{4}L^2(L^2/4+z^2)^{-1}=0$$
$$\rightarrow L^2/4+z^2=\frac{3}{4}L^2$$
$$\rightarrow -L^2/2+z^2=0$$
$$\rightarrow L^2=2z^2$$
$$\rightarrow L=\sqrt{2}z$$




Normalized in Telford chapter 8.

# 6 Exercises for Resistivity Method and Induced Polarization

**Version:** June 2, 2022
**Context:** Resistivity Method 02, Induced Polarization 01 & Self-Potential 01
**Timing:** All resistivity exercises should be completed the latest by June 13th 2022

## 6.1 Forward Modelling of a vertical electrical sounding survey

(a) Use the attached file *RD_VES_ForwardModel_Ex6.ipynb* which contains a Jupyter Notebook with Py-Gimli as discussed in the latest video. Set-up a sub-surface model with three different resistivities for cases:

- $\rho_1 > \rho_2 > \rho_3$
- $\rho_1 < \rho_2 < \rho_3$

- $\rho_1 < \rho_2, \rho_2 > \rho_3$

Also explore possibilities of four layers cases. This type of forward modelling can be useful for your applied exercises in geoelectrics.
(b) Change the Jupyter notebook code so that you can visualize output from two forward runs (use a copy and past where you can). Illustrate to different sub-surface models which result in a similar vertical electrical sounding observations. Memorize that does ambiguities (same as, e.g., for gravity surveys) are very prevalent in geophysical applications.

Solutions

../../Src/Resistivity/RD_VES_ForwardModel_Ex6Solution.ipynb

## 6.2 Geoelectric Array Types

(a) Show explicitly that for a Wenner ($\alpha$) array the geometry factor is $K = 2\pi a$ where $a$ is the distance between between all electrodes.
(b) In preparation of the applied exercises, make a table with the different array types (Wenner $\alpha$, Schlumberger, and half-Schlumberger (or pole-dipole), and dipole-dipole). Without going into too much detail, mention two importantpoints that require consideration when choosing a specific type.

Solutions

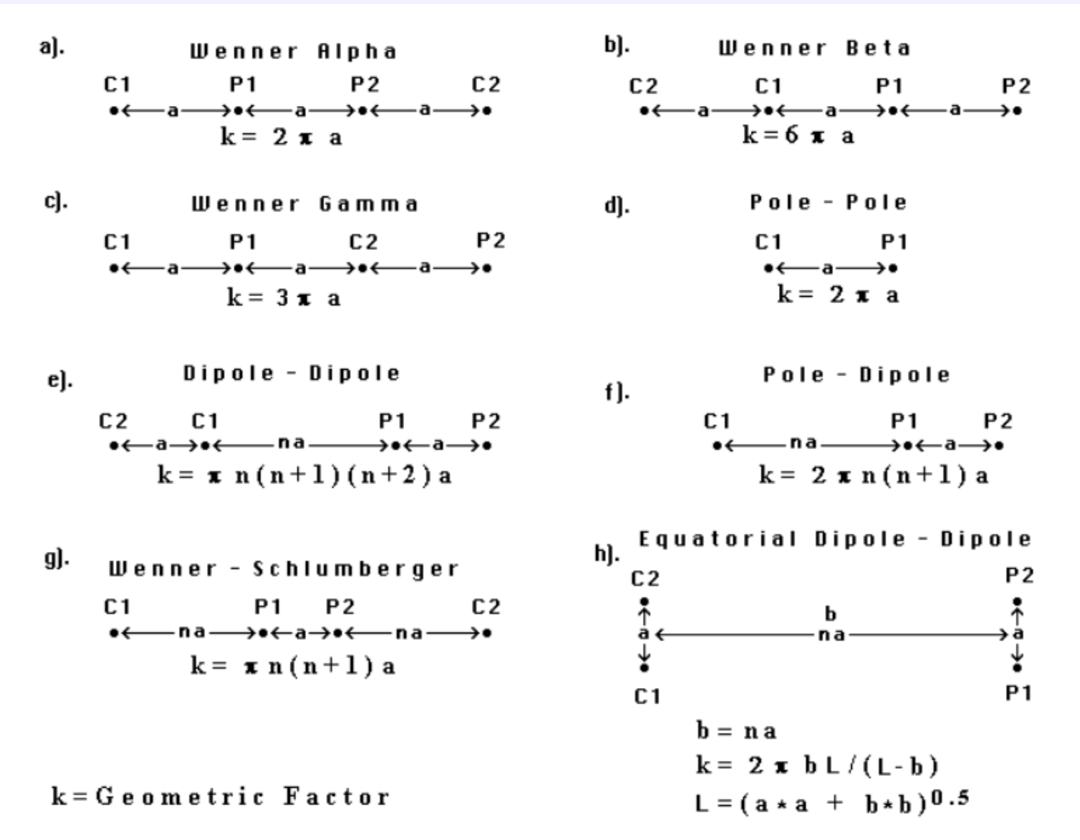


Table 7.3 Comparison of dipole-dipole, Schlumberger, square and Wenner electrode arrays.

| Criteria | Wenner | Schlumberger | Dipole-dipole | Square |
|---|---|---|---|---|
| Vertical resolution | √√√ | √√ | √ | √√ |
| Depth penetration | √ | √√ | √√√ | √√ |
| Suitability to VES | √√ | √√√ | √ | × |
| Suitability to CST | √√√ | × | √√√ | √√√ |
| Sensitivity to orientation | Yes | Yes | Moderate | No |
| Sensitivity to lateral inhomogeneities | High | Moderate | Moderate | Low |
| Labour intensive | Yes (no*) | Moderate (no*) | Moderate (no*) | Low |
| Availability of interpretational aids | √√√ | √√√ | √√ | √ |

√ = poor; √√= moderate; √√√ = good; × = unsuitable.
* When using a multi-core cable and automated electrode array.

## 6.3 Geoelectric Depth Of Investigation 2

(tricky) In a previous exercises we have shown that the horizontal component of the two current density in the center between the current electrodes is given by:

$$J_x(x = L/2) = \frac{I}{2\pi} \frac{L}{(z^2 + y^2 + L^2/4)^{3/2}}$$

We now want to calculate the cumulative signal contribution from a given depth interval between $z_1$ and $z_2$. This will give as an estimate for the depth of investigation as a function of the electrode spacing. The cumulative signal contribution can be calculated via integrating:

$$\frac{I_x}{I} = \frac{L}{2\pi} \int_{z_1}^{z_2} dz \int_{-\infty}^{\infty} dy \frac{1}{(z^2 + y^2 + L^2/4)^{3/2}}$$

Note that like this we calculate the ratio of current $I_X$ (not current density) in that depth interval relative to the overall injected current $I$. Show that for $z_2 \to \infty$:

$$\frac{I_x}{I} = 1 - \frac{2}{\pi} \tan^{-1}(\frac{2z_1}{L})$$

What does this mean, e.g., for $z_1 = 0$ or $z_1 = L/2$ ?

Hint 1:

$$\int \frac{1}{(c + z^2)^{3/2}} = \frac{z}{c\sqrt{c + z^2}}$$

Hint 2:

$$\int \frac{1}{a^2 + z} = \frac{1}{a} \tan^{-1}(\frac{z}{a})$$

Solutions

First the y-integration we we set $c = z^2 + L^2/4$ and use hint 1 then

$$\int_{-\infty}^{\infty} dy \frac{1}{(z^2 + y^2 + L^2/4)^{3/2}} =$$
$$\left[\frac{y}{(z^2 + L^2/4)\sqrt{z^2 + L^2/4 + y^2}}\right]_{-\infty}^{\infty} =$$
$$\frac{1}{(z^2 + L^2/4)}\left[\frac{y}{\sqrt{z^2 + L^2/4 + y^2}}\right]_{-\infty}^{\infty} =$$
$$\frac{1}{(z^2 + L^2/4)}\left[\frac{y}{|y|\sqrt{z^2/y^2 + L^2/4/y^2 + 1}}\right]_{-\infty}^{\infty} = \frac{2}{(z^2 + L^2/4)}$$

Next the z-integration and using a = L/2:

$$\int_{z_1}^{z_2} dz \frac{2}{(z^2 + L^2/4)} =$$
$$\frac{4}{L}\left[\tan^{-1}(2z/L)\right]_{z_1}^{z_2}$$

Taken together this results in:

$$\frac{I_x}{I} = \frac{2}{\pi}\tan^{-1}(\frac{2z_2}{L}) - \frac{2}{\pi}\tan^{-1}(\frac{2z_1}{L})$$

For $z_2 \to \infty$:

$$\frac{I_x}{I} = 1 - \frac{2}{\pi}\tan^{-1}(\frac{2z_1}{L})$$

For $z_1 = 0$ $\frac{I_x}{I} = 1$ which is logicial as it means all of the current flows through the interface at some stage as we integrate over the entire depth. For $z_1 = L/2$ half the currents flows above $z_1$ and half the current flows below $z_1$ as $\frac{I_x}{I} = 0.5$. This gives as good estimate for the depth of investigation as a function of the electrode spacing. However, keep in mind that the larger $L$ the small the current and potential differences involved, and at some stage you will not be able to detect this anymore.

## 6.4 Induced Polarization

(a) Explain explicitly why an oscillating input current can be described with

$$V(t) = U_0 e^{j(\omega t + \phi_0)}$$

where $j = \sqrt{-1}$ is the imaginary number. Assign the terms "Amplitude", "Phase Offset", and "Frequency" to the variables involved and memorize that those three parameters are always required to describe any type of waves and oscillations. Sketch the real part of V(t).
(b) Why does multiplication with "j" correspond to a phase shift of 90 degrees (or $\pi/2$)?

(c) Explain why the $I - V_c$ relationship for a capacitor is given by:

$$I = C\frac{dV_c}{dt}$$

and calculate the ratio (or the impedance) of $\frac{I}{U}$ for an ac potential V(t).
(d) Explain in a few words as to why this can be understood as the resistance of a capacitor in an AC circuit. How does the impedance change with lower and higher frequencies?
Induced Polarization surveys can also be done in the frequency-domain. Instead of measuring the chargeability (defined in class) the frequency effect of the apparent resistitivy is obtained:

$$FE = \frac{\rho_{a,dc}}{\rho_{a,ac}} - 1$$

In practice this is done by measuring the dc-resistivity at very low frequencies and the ac-resistivity at intermediate frequencies.
(e) Why are in an induced polarization survey the induced current and measured voltage out of phase? Is this also the case if the sub-surface as no polarization characteristics?

**Solutions**

(a)

$$V(t) = U_0 e^{j()\omega t + \phi_0)} = U_0(\cos(\omega t + \phi_0) + i\sin(\omega t + \phi_0))$$

The real part of this equation corresponds to a *normal* cos with amplitude $U_0$, angular frequency $\omega$ and phase offset $\phi_0$. (Drawing)
(b)

$$V(t) = U_0 e^{j()\omega t + \phi_0 + \pi/2)} \quad (1)$$
$$= U_0 e^{j(\omega t + \phi_0)} e^{j\pi/2} \quad (2)$$
$$= U_0 e^{j(\omega t + \phi_0)}(cos(\pi/2) + i\sin(\pi/2)) \quad (3)$$
$$= iU_0 e^{j(\omega t + \phi_0)} \quad (4)$$

(c) From lecture we know that $V_c = \frac{q}{C}$:

$$\frac{dV_c}{dt} = \frac{1}{c}\frac{ds}{dt} = \frac{1}{c}I$$

Hence:

$$\frac{dV_c}{dt} = \frac{d}{dt}V_0 e^{j\omega t} = j\omega V_c(t)$$

and:

$$\frac{V_c}{I} = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

(d) This is an extension to Ohm's law in the ac case. At low frequencies the impedance is large, and at high frequencies the impedance is high.
(e) The multiplication with "-j" indicates that current leads the voltage by 90 degrees if the subsurface is polarized. This phase shift can also be analyzed and will be important for other geophysical methods as well. If there is no polarization in the sub-surface that we "only" have resistive properties in which (according to Ohm's law) the potential difference and currents are always in-phase with each other.