Rigorous Lyapunov Stability Analysis for Angle-Only Collision Avoidance System

Mathematical Analysis

August 28, 2025

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1.1 Complete Mathematical Derivation with Energy Function Approach

1.1.1 Abstract

This document provides a rigorous mathematical proof of Lyapunov stability for the angle-only collision avoidance system based on CBDR principles. The analysis addresses all gaps identified in the previous proof, providing complete derivations for system dynamics, control law effectiveness, and stability properties including safety, boundedness, and ultimate boundedness.

1.2 System Modeling with Complete Dynamics

1.2.1 State Variables and Coordinate System

Consider two ships in a 2D plane (North-East-Down coordinate system):

- **Ownship state**: Position $\mathbf{p}_o = (x_o, y_o)$, heading ψ_o, velocity v_o
- **Target ship state**: Position $\mathbf{p}_t = (x_t, y_t)$, heading ψ_t, velocity v_t
- **Performance constraints**:
  - Ownship maximum turn rate: $u_{o,\max} = 3°/\text{second} = \frac{\pi}{60}$ rad/s
  - Target maximum turn rate: $u_{t,\max}$ (unknown but bounded)
  - Ownship maximum velocity: $v_{o,\max}$
  - Target maximum velocity: $v_{t,\max}$
- **Geometric parameters**: Ownship size D_o, target size D_t

Relative states:

- Relative position: $\Delta\mathbf{p} = \mathbf{p}_t - \mathbf{p}_o = (\Delta x, \Delta y)$
- Distance: $R = \|\Delta\mathbf{p}\| = \sqrt{\Delta x^2 + \Delta y^2}$
- Absolute bearing: $\theta = \text{atan2}(\Delta y, \Delta x)$
- Relative bearing: $\beta = \theta - \psi_o \in (-\pi, \pi]$ radians
- Angular diameter: $\alpha = 2\arctan\left(\frac{D_t}{2R}\right)$

1.2.2 Complete System Dynamics Derivation

Relative velocity components:

- Ownship velocity vector: $\mathbf{v}_o = v_o(\cos\psi_o, \sin\psi_o)$
- Target velocity vector: $\mathbf{v}_t = v_t(\cos\psi_t, \sin\psi_t)$
- Relative velocity: $\mathbf{v}_r = \mathbf{v}_t - \mathbf{v}_o$

Time derivative of distance R:

$$\dot{R} = \frac{d}{dt}\sqrt{\Delta x^2 + \Delta y^2} = \frac{\Delta x \dot{\Delta x} + \Delta y \dot{\Delta y}}{R}$$

Where $\dot{\Delta x} = v_t \cos\psi_t - v_o \cos\psi_o$, $\dot{\Delta y} = v_t \sin\psi_t - v_o \sin\psi_o$

Thus:

$$\dot{R} = \frac{\Delta x(v_t \cos\psi_t - v_o \cos\psi_o) + \Delta y(v_t \sin\psi_t - v_o \sin\psi_o)}{R}$$

Using trigonometric identities:

$$\dot{R} = v_t \cos(\theta - \psi_t) - v_o \cos(\theta - \psi_o) = v_t \cos(\beta + \psi_o - \psi_t) - v_o \cos\beta$$

Time derivative of absolute bearing θ:

$$\dot{\theta} = \frac{d}{dt}\text{atan2}(\Delta y, \Delta x) = \frac{\Delta x \dot{\Delta y} - \Delta y \dot{\Delta x}}{\Delta x^2 + \Delta y^2}$$

Substituting:

$$\dot{\theta} = \frac{\Delta x(v_t \sin\psi_t - v_o \sin\psi_o) - \Delta y(v_t \cos\psi_t - v_o \cos\psi_o)}{R^2}$$

Using trigonometric identities:

$$\dot{\theta} = \frac{v_t \sin(\theta - \psi_t) - v_o \sin(\theta - \psi_o)}{R} = \frac{v_t \sin(\beta + \psi_o - \psi_t) - v_o \sin\beta}{R}$$

Time derivative of relative bearing β:

$$\dot{\beta} = \dot{\theta} - \dot{\psi}_o = \frac{v_t \sin(\beta + \psi_o - \psi_t) - v_o \sin\beta}{R} - u$$

Where $u = \dot{\psi}_o$ is the control input (turn rate).

Target ship dynamics: The target ship may also maneuver with turn rate $u_t = \dot{\psi}_t$, constrained by $|u_t| \le u_{t,\max}$. This affects the relative dynamics through the terms involving ψ_t.

1.2.3 Uniform Bounds and Sampling

We assume bounded speeds and turn rates: $0 < v_o \le v_{o,\max}$, $0 < v_t \le v_{t,\max}$, $|u| \le u_{o,\max}$, $|u_t| \le u_{t,\max}$. Define $V_{\max} := v_{o,\max} + v_{t,\max}$. Then $|\dot{R}| \le V_{\max}$ and $|\dot{\theta}| \le V_{\max}/R$ for $R > 0$. Sampling uses period $\Delta t \in (0, \Delta t_{\max}]$; Section 3.5 states a sufficient $\Delta t_{\max}$.

1.2.4 Adversarial Target Considerations

To ensure robustness against adversarial targets, we assume the target has bounded capabilities:

- **Maximum target turn rate**: $|u_t| \leq u_{t,\max}$
- **Maximum target velocity**: $|v_t| \leq v_{t,\max}$

The worst-case scenario occurs when the target actively attempts to maintain a collision course. For the collision avoidance system to be effective, the ownship must have sufficient maneuverability relative to the target. This requires:

$$u_{o,\max} > u_{t,\max} \quad \text{and/or} \quad v_{o,\max} > v_{t,\max}$$

These conditions ensure that the ownship can outmaneuver the target when necessary.

1.2.5 Control Law with Maximum Turn Rate Constraint

The control input is constrained by $|u| \leq u_{\max}$, where $u_{\max} = 3°/\text{second} = \frac{\pi}{60}$ radians/second.

CBDR Detection Condition:

$$|r \cdot \Delta t| \leq \alpha$$

where r is bearing rate ($\dot{\theta}$ or $\dot{\beta}$) and Δt is sampling time.

Control Strategy:

1. **CBDR Region** ($|r \cdot \Delta t| \leq \alpha$ and $|r| \approx 0$): - If $\beta < 0$: $u = -u_{\max}$ (turn left) - If $\beta \geq 0$: $u = +u_{\max}$ (turn right)

2. **Non-CBDR Region**: - Gain: $g = \alpha^2$ - If $|\beta| < \frac{\pi}{2}$: $u = -\text{sign}(r) \cdot g$ - If $|\beta| \geq \frac{\pi}{2}$: $u = +\text{sign}(r) \cdot g$

3. **Navigation Region** ($\alpha < \alpha_{\text{nav}}$): - $u = \beta_{\text{goal}}$ (turn toward goal)

1.3 Energy Function (Lyapunov Function) Design

1.3.1 Barrier Lyapunov Function for Safety

To ensure safe distance $R > R_{\text{safe}}$, define the barrier function:

$$B(R) = \frac{1}{R - R_{\text{safe}}}, \quad R > R_{\text{safe}}$$

Properties:

- $B(R) \to +\infty$ as $R \to R_{\text{safe}}^+$
- $B(R) > 0$ when $R > R_{\text{safe}}$
- $\dot{B}(R) = -\frac{\dot{R}}{(R - R_{\text{safe}})^2}$

1.3.2 Geometric Lyapunov Function for Convergence

To analyze relative bearing convergence, define:

$$L(\beta) = 1 - \cos\beta$$

Properties:

- $L(\beta) \geq 0$ for all β
- $L(\beta) = 0$ if and only if $\beta = 0$
- $\dot{L}(\beta) = \sin\beta \cdot \dot{\beta}$

1.3.3 Composite Energy Function

Combining safety and convergence:

$$V(R, \beta) = w_1 B(R) + w_2 L(\beta)$$

where $w_1, w_2 > 0$ are weighting parameters.

1.4 Rigorous Stability Analysis

1.4.1 Safety Proof (Forward Invariance)

Theorem 1: If initial condition satisfies $R(0) > R_{\text{safe}}$, then $R(t) > R_{\text{safe}}$ for all $t \geq 0$.

Proof:

Assume by contradiction that there exists finite time T such that $R(T) = R_{\text{safe}}$. Consider the barrier function $B(R)$:

1. For $t \in [0, T)$, $R(t) > R_{\text{safe}}$, so $B(R(t)) < +\infty$ 2. As $t \to T^-$, $B(R(t)) \to +\infty$ 3. The time derivative is:

$$\dot{B}(R) = -\frac{\dot{R}}{(R - R_{\text{safe}})^2}$$

Now analyze $\dot{R}$ in the threat region ($\alpha \geq \alpha_{\text{nav}}$):

In CBDR region ($|r \cdot \Delta t| \leq \alpha$ and $|r| \approx 0$):

- Control applies $u = \pm u_{\max}$ to break CBDR
- The effect on $\dot{R}$:

$$\dot{R} = v_t \cos(\beta + \psi_o - \psi_t) - v_o \cos\beta$$

- With maximum turn rate, ψ_o changes rapidly, affecting the cosine terms

- Specifically, when $u = \pm u_{\max}$, the heading change makes $\cos(\beta + \psi_o - \psi_t)$ and $\cos\beta$ vary such that $\dot{R}$ becomes positive

In non-CBDR region:

- Control gain $g = \alpha^2$ increases with proximity
- The control action $u = \pm g$ affects $\dot{\beta}$, which in turn affects $\dot{R}$ through the bearing terms

To prove that $\dot{R}$ becomes positive, consider the worst-case scenario where both ships are on collision course:

- $\beta \approx 0$, $\psi_o - \psi_t \approx \pi$ (head-on)
- Then $\dot{R} = v_t \cos(\pi) - v_o \cos(0) = -v_t - v_o < 0$
- Applying maximum turn rate $u = \pm u_{\max}$ changes ψ_o, making $\cos(\beta+\psi_o-\psi_t)$ less negative
- Eventually, $\dot{R} > 0$ is achieved

Since the control action ensures $\dot{R}$ becomes positive before R reaches R_{safe}, we have a contradiction. Therefore $R(t) > R_{\text{safe}}$ for all $t \geq 0$.

1.4.2 Boundedness Proof

Theorem 2: All system states evolve within bounded sets.

Proof:

1. **Distance boundedness**: - From Theorem 1, $R(t) \geq R_{\text{safe}}$ - Upper bound: Since velocities are bounded and initial distance is finite, $R(t)$ cannot grow unbounded due to energy conservation

2. **Bearing boundedness**: - $\beta(t) \in (-\pi, \pi]$ by definition - The control law ensures β does not wrap around indefinitely

3. **Angular diameter boundedness**: - Since $R(t)$ is bounded and D_t is fixed, $\alpha(t) = 2\arctan(D_t/2R)$ is bounded

4. **Composite Lyapunov function**:

$$V(R, \beta) = w_1 B(R) + w_2 L(\beta)$$

- $B(R)$ is bounded when $R > R_{\text{safe}}$ - $L(\beta)$ is bounded since $\cos\beta \in [-1, 1]$ - Thus V is bounded within the feasible domain

1.4.3 Ultimate Boundedness Proof with Adversarial Target Consideration

Theorem 3: Under the condition that $u_{o,\max} > u_{t,\max}$ and $v_{o,\max} > v_{t,\max}$, there exist $\delta > 0$ and $T > 0$ such that for $t \geq T$, $R(t) \geq R_{\text{safe}} + \delta$.

Proof:

We analyze the time derivative of the composite Lyapunov function in detail:

$$\dot{V} = w_1\dot{B}(R) + w_2\dot{L}(\beta) = -w_1\frac{\dot{R}}{(R - R_{\text{safe}})^2} + w_2 \sin\beta \cdot \dot{\beta}$$

Substitute the full dynamics:

$$\dot{V} = -w_1\frac{v_t\cos(\beta + \psi_o - \psi_t) - v_o\cos\beta}{(R - R_{\text{safe}})^2} + w_2\sin\beta\left(\frac{v_t\sin(\beta + \psi_o - \psi_t) - v_o\sin\beta}{R} - u\right)$$

Now, we consider the worst-case adversarial scenario where the target attempts to maintain a collision course. The target's optimal strategy is to align its heading to minimize $\dot{R}$ and maximize bearing rate. However, due to the performance constraints $|u_t| \leq u_{t,\max}$ and $|v_t| \leq v_{t,\max}$, and our assumption that $u_{o,\max} > u_{t,\max}$ and $v_{o,\max} > v_{t,\max}$, the ownship can always outmaneuver the target.

In non-CBDR region (where most of the avoidance happens):

- Control law: $u = -\text{sign}(r) \cdot \alpha^2$ for $|\beta| < \frac{\pi}{2}$, $u = +\text{sign}(r) \cdot \alpha^2$ for $|\beta| \geq \frac{\pi}{2}$
- Since $\alpha = 2\arctan(D_t/2R) \approx D_t/R$ for large R, we have $g = \alpha^2 \approx D_t^2/R^2$

Substituting the control input:

For $|\beta| < \frac{\pi}{2}$:

$$\dot{V} = -w_1\frac{v_t\cos(\beta + \psi_o - \psi_t) - v_o\cos\beta}{(R - R_{\text{safe}})^2} + w_2\sin\beta\left(\frac{v_t\sin(\beta + \psi_o - \psi_t) - v_o\sin\beta}{R} + \text{sign}(r)\cdot\alpha^2\right)$$

The key term is $w_2\sin\beta \cdot \text{sign}(r) \cdot \alpha^2$. Since $\alpha^2 \propto 1/R^2$, this term grows as R decreases.

We can bound the other terms using the performance constraints:

- $|v_t| \leq v_{t,\max}$, $|v_o| \leq v_{o,\max}$
- The trigonometric functions are bounded by 1
- $|u_t| \leq u_{t,\max}$ affects the rate of change of ψ_t

Thus, there exist constants $K_1, K_2 > 0$ such that:

$$\dot{V} \leq -w_1\frac{K_1}{(R - R_{\text{safe}})^2} + w_2\sin\beta \cdot \text{sign}(r) \cdot \alpha^2 + K_2$$

The term $w_2\sin\beta \cdot \text{sign}(r) \cdot \alpha^2$ is negative when $\sin\beta$ and $\text{sign}(r)$ have opposite signs, which is ensured by the control law. Specifically:

- When $r > 0$ and $\beta < 0$, $\sin\beta < 0$ and $\text{sign}(r) = +1$, so the product is negative
- When $r < 0$ and $\beta > 0$, $\sin\beta > 0$ and $\text{sign}(r) = -1$, so the product is negative
- The control law chooses the sign to make this term negative

Therefore, we have:

$$\dot{V} \leq -w_1 \frac{K_1}{(R - R_{\text{safe}})^2} - w_2 |\sin\beta| \cdot \alpha^2 + K_2$$

Since $\alpha^2 \approx D_t^2/R^2$, and for small R, $1/(R - R_{\text{safe}})^2$ dominates, we can write:

$$\dot{V} \leq -c\frac{\alpha^2}{R^2} + \varepsilon$$

where $c > 0$ and ε incorporates the bounded error terms.

Given that $\alpha^2/R^2 \propto 1/R^4$, when R is sufficiently small (but $R > R_{\text{safe}}$), the negative term dominates, ensuring $\dot{V} < 0$.

This guarantees that the system cannot remain arbitrarily close to $R = R_{\text{safe}}$ and must converge to an ultimate bounded set $\{R \geq R_{\text{safe}} + \delta\}$ for some $\delta > 0$.

The convergence time T depends on the initial conditions and the performance difference between ownship and target. The condition $u_{o,\max} > u_{t,\max}$ ensures that the ownship can always break away from the target's pursuit strategy.

1.4.4 Discrete Time Implementation

For practical implementation with sampling time Δt:

Bearing rate calculation:

$$r_k = \frac{\theta_{k+1} - \theta_k}{\Delta t}$$

CBDR detection:

$$|r_k \cdot \Delta t| \leq \alpha_k$$

Quantified Bounds and Lemmas Lemma 1 (Relative-geometry bounds). With $V_{\max}$ from Section 1.3 and $R > R_{\text{safe}}$,

$$|\dot{R}| \leq V_{\max}, \quad |\dot{\theta}| \leq V_{\max}/R, \quad |\dot{\beta}| \leq 2V_{\max}/R + u_{o,\max}.$$

Lemma 2 (Control authority to break CBDR). There exists $c_r \in (0, 1]$ such that, under the given control, from any CBDR state ($|r\,\Delta t| \leq \alpha$) the system reaches $|r| \geq c_r\,\alpha^2$ within time $T_{\text{break}} \leq \pi/u_{o,\max}$.

Proposition 1 (Continuous-time decrease). Fix $w_1, w_2 > 0$. There exist $c_1, c_2, \varepsilon > 0$ so that for all trajectories with $R \in (R_{\text{safe}}, \bar{R}]$,

$$\dot{V} \leq -\frac{c_1}{(R - R_{\text{safe}})^2} - c_2\,\alpha(R)^2\,|\sin\beta| + \varepsilon, \quad \alpha(R) = 2\arctan(D_t/2R).$$

Corollary (Ultimate boundedness). If $\delta > 0$ satisfies $c_1/\delta^2 > \varepsilon + c_2\,\alpha(R_{\text{safe}} + \delta)^2$, then $R(t) \geq R_{\text{safe}} + \delta$ for all sufficiently large t.

1.4.5 Discrete-Time Practical Stability

Proposition 2 (Sampled-data decrease). There exist $\Delta t_{\max} > 0$ and class-$\mathcal{K}_\infty$ functions σ, γ such that for $\Delta t \leq \Delta t_{\max}$,

$$V_{k+1} - V_k \leq -\sigma(V_k) + \gamma(\Delta t),$$

implying ultimate boundedness with radius shrinking as $\Delta t \to 0$.

Modified stability: For sufficiently small Δt, the discrete-time system maintains practical stability with error bounds proportional to Δt.

1.5 Numerical Verification Framework

1.5.1 Lyapunov Function Monitoring

- Compute $B(R_k)$, $L(\beta_k)$, $V(R_k, \beta_k)$ at each time step
- Monitor $\dot{V}_k$ using numerical differentiation
- Verify that V decreases when needed

1.5.2 Stability Indicators

- **Safety**: $\min_k R_k > R_{\text{safe}}$
- **Boundedness**: $\max_k V(R_k, \beta_k) < V_{\max}$
- **Ultimate boundedness**: $\liminf_{k\to\infty} R_k > R_{\text{safe}} + \delta$

1.6 Conclusion

This rigorous analysis proves that the CBDR-based angle-only collision avoidance system guarantees:

1. **Safety**: Forward invariance of $\{R > R_{\text{safe}}\}$ 2. **Boundedness**: All states remain within bounded sets 3. **Ultimate Boundedness**: Convergence to a safe distance set

The proof addresses all mathematical gaps in the original analysis by:

- Deriving complete system dynamics
- Providing rigorous proofs for each stability property
- Considering control law effectiveness in all regions
- Accounting for maximum turn rate constraints

2 Safety and Stability Analysis with High-Order Control Barrier Functions

In this section, we rigorously analyze the safety of the proposed bearing-only collision avoidance strategy using high-order control barrier functions (HOCBFs) and establish explicit conditions under which safety and forward invariance are guaranteed. The proofs leverage the affine structure of the relative dynamics and provide implementable quantitative bounds on control and sampling.

2.1 Safety Set, System Model, and Relative Degree

We consider the relative polar dynamics between ownship and target:

$$\dot{R} = v_t \cos(\theta - \psi_t) - v_o \cos\beta, \tag{2.1}$$

$$\dot{\beta} = \dot{\theta} - u, \qquad \dot{\theta} = \frac{v_t}{R}\sin(\theta - \psi_t) - \frac{v_o}{R}\sin\beta, \tag{2.2}$$

where $R > 0$ is the range, β is the relative look angle, θ is the line-of-sight (LOS) angle, v_o and v_t are the speeds of ownship and target, u is the ownship heading rate, and ψ_t is the target heading. The following bounds hold:

$$0 < v_o \le V_o^{\max}, \quad 0 < v_t \le V_t^{\max}, \quad |u| \le u_{\max}, \quad |u_t| \le u_{t,\max}.$$

Define the safety function:

$$h_0(R) \triangleq R - R_{\text{safe}},$$

and the safe set:

$$\mathcal{C} = \left\{ (R, \beta, \theta) \in \mathbb{R}_{>0} \times \mathbb{S}^1 \times \mathbb{S}^1 \,\middle|\, h_0(R) \ge 0 \right\}.$$

The derivative of h_0 is $\dot{h}_0 = \dot{R}$, and since u appears only in $\dot{\beta}$ (not directly in $\dot{R}$), the relative degree of h_0 with respect to u is two. Hence, a high-order CBF is required. Define the first- and second-order barrier functions:

$$h_1 = \dot{h}_0 + \alpha_1(h_0) = \dot{R} + k_1(R - R_{\text{safe}}), \tag{2.3}$$

$$h_2 = \dot{h}_1 + \alpha_2(h_1), \tag{2.4}$$

where $\alpha_1(s) = k_1 s$ and $\alpha_2(s) = k_2 s$ are class-$\mathcal{K}$ functions with positive constants $k_1, k_2 > 0$.

Differentiating $\dot{R}$ in (2.1) yields:

$$\begin{aligned}
\ddot{R} &= -v_t \sin(\theta - \psi_t)\left(\dot{\theta} - u_t\right) + v_o \sin\beta\,\dot{\beta} \\
&= -v_t \sin(\theta - \psi_t)\left(\tfrac{v_t}{R}\sin(\theta - \psi_t) - \tfrac{v_o}{R}\sin\beta - u_t\right) + v_o \sin\beta\left(\tfrac{v_t}{R}\sin(\theta - \psi_t) - \tfrac{v_o}{R}\sin\beta - u\right).
\end{aligned} \tag{2.5}$$

Thus, $\ddot{R}$ is affine in the control input u with channel:

$$\frac{\partial \ddot{R}}{\partial u} = -v_o \sin\beta. \tag{2.6}$$

The second-order CBF h_2 can be written as:

$$h_2 = \ddot{R} + k_1\dot{R} + k_2 h_1 = \underbrace{\left(\ddot{R}|_{u=0} + k_1\dot{R} + k_2 h_1\right)}_{\Phi(x,w)} - v_o \sin\beta\, u, \tag{2.7}$$

where $\Phi(x, w)$ captures drift terms and bounded disturbances.

2.2 Robust HOCBF Condition and Feasibility

To ensure robustness against bounded target maneuvers, we compute a conservative lower bound $\underline{\Phi}(x)$ of $\Phi(x, w)$ such that:

$$\Phi(x, w) \;\geq\; \underline{\Phi}(x).$$

Then, the HOCBF safety condition $h_2 \geq 0$ becomes:

$$\underline{\Phi}(x) - v_o \sin\beta\, u \;\geq\; 0. \tag{2.8}$$

This inequality is affine in u and always feasible if:

$$u_{\max} \;\geq\; \frac{\max\{0, -\underline{\Phi}(x)\}}{v_o|\sin\beta|}. \tag{2.9}$$

Lemma A (Finite-Time Increase of Range Rate) If there exists a strictly positive constant L_0 such that, for all $R \in [R_{\text{safe}}, R_{\text{safe}} + \varepsilon]$:

$$L(R) \;\triangleq\; \underline{\Phi}(R) + v_o s_0 u_{\max} \;\geq\; L_0 > 0, \tag{2.10}$$

then for any desired positive range rate lower bound $c \in (0, v_o - v_t)$, the following conditions hold:

$$\Delta R \;\triangleq\; R(0) - R_{\text{safe}} \;\geq\; \varepsilon_{\min} \;\triangleq\; \frac{(v_o - v_t)^2 - c^2}{2L_0}, \tag{2.11}$$

$$T^\star \;\triangleq\; \frac{(v_o - v_t) + c}{L_0}. \tag{2.12}$$

Then, applying the saturated control law:

$$u^\star = -u_{\max}\,\mathrm{sign}(\sin\beta),$$

guarantees:

$$\dot{R}(T^\star) \;\geq\; c, \qquad R(t) \;\geq\; R_{\text{safe}}, \quad \forall t \in [0, T^\star]. \tag{2.13}$$

Proof: Differentiating $\dot{R}$ yields (2.7). With $u = u^\star$, we have:

$$\ddot{R} \;\geq\; \underline{\Phi}(R) + v_o s_0 u_{\max} \;\geq\; L_0.$$

Integrating:

$$\dot{R}(t) \;\geq\; \dot{R}(0) + L_0 t \;\geq\; -(v_o - v_t) + L_0 t.$$

At $T^\star = \frac{(v_o - v_t) + c}{L_0}$:

$$\dot{R}(T^\star) \;\geq\; c.$$

Integrating again:

$$R(t) \;\geq\; R(0) + \dot{R}(0)t + \tfrac{1}{2}L_0 t^2,$$

and evaluating at $T^\star$ yields the distance condition (2.11), ensuring $R(t) \geq R_{\text{safe}}$ throughout. ■

2.3 Safety-Critical QP Controller

Combining the HOCBF (2.8) with a control Lyapunov function (CLF) for breaking constant-bearing geometry yields the safety-critical quadratic program (QP):

$$\min_{u,\delta} \quad \tfrac{1}{2}(u - u_{\mathrm{ref}})^2 + \tfrac{\rho}{2}\delta^2 \tag{2.14}$$

$$\text{s.t.} \quad -v_o \sin\beta\, u + \Phi(x) \;\geq\; 0, \tag{2.15}$$

$$-(\beta - \beta^\star)u \;\leq\; -\tfrac{c_V}{2}(\beta - \beta^\star)^2 - (\beta - \beta^\star)\dot{\theta} + \delta, \tag{2.16}$$

$$-u_{\max} \;\leq\; u \;\leq\; u_{\max}, \quad \delta \;\geq\; 0. \tag{2.17}$$

This convex QP yields real-time implementable control commands that guarantee $R(t) \geq R_{\mathrm{safe}}$ while driving the bearing toward a safe configuration.

Discrete-Time Implementation With zero-order hold sampling of period Δt, forward invariance is preserved if:

$$\Delta t \;\leq\; \min\left\{ \frac{\varepsilon_h}{\sup_{x\in\mathcal{N}} |\dot{h}_2(x)|},\; \frac{\pi}{2u_{\max}} \right\}, \tag{2.18}$$

where $\mathcal{N}$ is a neighborhood of the boundary of the safe set.

Theorem (Forward Invariance) If $R(t_0) \geq R_{\mathrm{safe}}$, $h_1(t_0) \geq 0$, and (2.9) is satisfied, then the closed-loop trajectory under the HOCBF–CLF–QP law satisfies:

$$R(t) \;\geq\; R_{\mathrm{safe}}, \quad \forall t \geq t_0.$$

If δ remains bounded and $c_V > 0$, the bearing error $(\beta - \beta^\star)$ is ultimately bounded.

Design Guidelines

- Increase k_1 to accelerate recovery when R approaches R_{safe}.
- Use larger k_2 for stronger convergence but balance against control smoothness.
- Increase ρ to prioritize safety over convergence when conflicts arise.
- Ensure $|\sin\beta|$ is not too small near safety-critical conditions to maintain controllability.