① one way ANOVA

$y_{ij} = u + \alpha_i + e_{ij}$

$$Xb = \begin{pmatrix} 1_{n_1} & 1_{n_1} & 0 & \cdots & 0 \\ 1_{n_2} & 0 & 1_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1_{n_a} & 0 & 0 & \cdots & 1_{n_a} \end{pmatrix} \begin{pmatrix} u \\ \alpha_1 \\ \vdots \\ \alpha_a \end{pmatrix}$$

$$(X^TX)^g = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1/n_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/n_a \end{pmatrix} \qquad X^Ty = \begin{pmatrix} N\bar{y}_{\cdot\cdot} \\ n_1\bar{y}_{1\cdot} \\ \vdots \\ n_a\bar{y}_{a\cdot} \end{pmatrix}$$

$$\Rightarrow \hat{b} = \begin{pmatrix} 0 \\ \bar{y}_{1\cdot} \\ \vdots \\ \bar{y}_{a\cdot} \end{pmatrix} - z \begin{pmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$cb = 0$

$$(0, n_1, \cdots, n_a) \begin{pmatrix} z \\ \bar{y}_{1\cdot} - z \\ \vdots \\ \bar{y}_{a\cdot} - z \end{pmatrix} = 0 \Rightarrow \bar{y}_{1\cdot} n_1 + \cdots \bar{y}_{a\cdot} n_a - Nz = 0$$

$$\Rightarrow z = \bar{y}_{\cdot\cdot}$$

$$\Rightarrow \hat{b} = \begin{array}{c} \begin{pmatrix} \bar{y}_{\cdot\cdot} \\ \bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot} \\ \vdots \\ \bar{y}_{a\cdot} - \bar{y}_{\cdot\cdot} \end{pmatrix} \end{array} \begin{array}{l} u \\ \alpha_i \\ \\ \\ \end{array}$$

② two-way Nested Model

$y_{ijk} = u + \alpha_i + \beta_{ij} + e_{ijk}$

$$Xb = \begin{pmatrix} 1_{n_{11}} & 1_{n_{11}} & 0 & 1_{n_{11}} & 0 & 0 & 0 & 0 \\ 1_{n_{12}} & 1_{n_{12}} & 0 & 0 & 1_{n_{12}} & 0 & 0 & 0 \\ 1_{n_{21}} & 0 & 1_{n_{21}} & 0 & 0 & 1_{n_{21}} & 0 & 0 \\ 1_{n_{22}} & 0 & 1_{n_{22}} & 0 & 0 & 0 & 1_{n_{22}} & 0 \\ 1_{n_{23}} & 0 & 1_{n_{23}} & 0 & 0 & 0 & 0 & 1_{n_{23}} \end{pmatrix} \begin{pmatrix} u \\ \alpha_F \\ \alpha_P \\ \beta_{F\cdot D} \\ \beta_{F\cdot M} \\ \beta_{P\cdot A} \\ \beta_{P\cdot B} \\ \beta_{P\cdot E} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & n_{11} & n_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & n_{21} & n_{22} & n_{23} \\ 0 & n_1 & n_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Sigma n_i \alpha_i = 0$

$$X^TX = \begin{pmatrix} N & n_1 & n_2 & n_{11} & n_{12} & n_{21} & n_{22} & n_{23} \\ n_1 & n_1 & 0 & n_{11} & n_{12} & 0 & 0 & 0 \\ n_2 & 0 & n_2 & 0 & 0 & n_{21} & n_{22} & n_{23} \\ n_{11} & n_{11} & 0 & n_{11} & 0 & 0 & 0 & 0 \\ n_{12} & n_{12} & 0 & 0 & n_{12} & 0 & 0 & 0 \\ n_{21} & 0 & n_{21} & 0 & 0 & n_{21} & 0 & 0 \\ n_{22} & 0 & n_{22} & 0 & 0 & 0 & n_{22} & 0 \\ n_{23} & 0 & n_{23} & 0 & 0 & 0 & 0 & n_{23} \end{pmatrix}$$

$$(X^TX)^g = \begin{pmatrix} \underset{3\times 3}{0} & \underset{3\times 5}{0} \\ \underset{5\times 3}{0} & \begin{matrix} 1/n_{11} & & \\ & \ddots & \\ & & 1/n_{23} \end{matrix} \end{pmatrix} \qquad X^Ty = \begin{pmatrix} N\bar{y}_{\cdot\cdot} \\ n_1\bar{y}_{1\cdot} \\ n_2\bar{y}_{2\cdot} \\ n_{11}\bar{y}_{11} \\ n_{12}\bar{y}_{12} \\ n_{21}\bar{y}_{21} \\ n_{22}\bar{y}_{22} \\ n_{23}\bar{y}_{23} \end{pmatrix}$$

$$\hat{b} = (X^TX)^g X^Ty + (I - (X^TX)^g X^TX)z$$

$$\Rightarrow \hat{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{y}_{11} \\ \bar{y}_{12} \\ \bar{y}_{21} \\ \bar{y}_{22} \\ \bar{y}_{23} \end{pmatrix} - z_1 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - z_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - z_3 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$cb = 0 \Rightarrow \begin{cases} z_1 + z_2 = \bar{y}_{1\cdot} \\ z_1 + z_3 = \bar{y}_{2\cdot} \\ z_2 + z_3 = 0 \end{cases} \Rightarrow \begin{cases} z_1 = \bar{y}_{\cdot\cdot} \\ z_2 = \bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot} \\ z_3 = \bar{y}_{2\cdot} - \bar{y}_{\cdot\cdot} \end{cases}$$

$$= \begin{pmatrix} \bar{y}_{\cdot\cdot} \\ \bar{y}_{1\cdot} - \bar{y}_{\cdot\cdot} \\ \bar{y}_{2\cdot} - \bar{y}_{\cdot\cdot} \\ \bar{y}_{11} - \bar{y}_{1\cdot} \\ \bar{y}_{12} - \bar{y}_{1\cdot} \\ \bar{y}_{21} - \bar{y}_{2\cdot} \\ \bar{y}_{22} - \bar{y}_{2\cdot} \\ \bar{y}_{23} - \bar{y}_{2\cdot} \end{pmatrix} \begin{array}{l} u \\ \alpha \\ \\ \beta \\ \\ \\ \\ \\ \end{array}$$

③ Analysis of Covariance

$y_{ij} = \mu + \alpha_i + \beta x_{ij} + e_{ij}$

$$Xb = \begin{pmatrix} 1_{n_1} & 1_{n_1} & 0 & \cdots & 0 & X_{1\cdot} \\ 1_{n_2} & 0 & 1_{n_2} & \cdots & 0 & X_{2\cdot} \\ 1_{n_3} & 0 & 0 & \cdots & 0 & X_{3\cdot} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1_{n_a} & 0 & 0 & \cdots & 1_{n_a} & X_{a\cdot} \end{pmatrix} \begin{pmatrix} u \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_a \\ \beta \end{pmatrix}$$

$\tilde{x}_i = \sum_j \overline{x_{i\cdot}}$

$\tilde{x} = \sum_i \tilde{x}_i$

$\tilde{x}^2 = \Sigma x_{ij}^2$

$$X^TX = \begin{pmatrix} N & n_1 & 0 & \cdots & 0 & \tilde{x} \\ n_1 & n_1 & 0 & \cdots & 0 & \tilde{x}_1 \\ n_2 & 0 & n_2 & \cdots & 0 & \tilde{x}_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n_a & 0 & 0 & \cdots & n_a & \tilde{x}_a \\ \tilde{x} & \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_a & \tilde{x}^2 \end{pmatrix}$$

$$(X^TX)^g = \frac{1}{t} \begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} (\bar{x}_1)^2 + \frac{t}{n_1} & \bar{x}_1\bar{x}_2 & \cdots & \bar{x}_1\bar{x}_a & -\bar{x}_1 \\ \bar{x}_1\bar{x}_2 & (\bar{x}_2)^2 + \frac{t}{n_2} & \cdots & \bar{x}_2\bar{x}_a & -\bar{x}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{x}_1\bar{x}_a & \bar{x}_2\bar{x}_a & \cdots & (\bar{x}_a)^2 + \frac{t}{n_a} & -\bar{x}_a \\ -\bar{x}_1 & -\bar{x}_2 & \cdots & -\bar{x}_a & 1 \end{matrix} \end{pmatrix}$$

$(t = \tilde{x}^2 - \bar{x}_1\tilde{x}_1 - \cdots - \bar{x}_a\tilde{x}_a)$

$$X^Ty = \begin{pmatrix} N\bar{y}_{\cdot\cdot} \\ n_1\bar{y}_{1\cdot} \\ \vdots \\ n_a\bar{y}_{a\cdot} \\ \Sigma x_{ij}y_{ij} \end{pmatrix}$$

$$(X^TX)^g X^TX = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$(X^TX)^g X^Ty = \frac{1}{t} \begin{pmatrix} 0 \\ \bar{y}_1 t - \bar{x}_1 s \\ \vdots \\ \bar{y}_a t - \bar{x}_a s \\ s \end{pmatrix}$$

$(s = \Sigma x_{ij}y_{ij} - n_1\bar{x}_1\bar{y}_1 - \cdots - n_a\bar{x}_a\bar{y}_a)$

$$\hat{b} = \begin{pmatrix} \\ \\ \\ \end{pmatrix} - z \begin{pmatrix} -1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$$

$cb = 0$

$c = (0, n_1, \cdots, n_a, \tilde{x})$

$$\hat{b} = \begin{pmatrix} \bar{y} \\ \bar{y}_1 - \bar{y} - \bar{x}_1 \cdot \frac{s}{t} \\ \vdots \\ \bar{y}_a - \bar{y} - \bar{x}_a \cdot \frac{s}{t} \\ \frac{s}{t} \end{pmatrix}$$