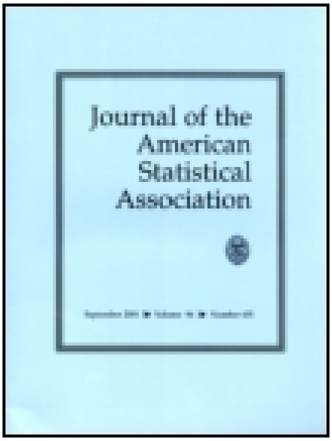

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Another Derivation of the Non-Central Chi-Square Distribution

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ANOTHER DERIVATION OF THE NON-CENTRAL CHI-SQUARE DISTRIBUTION

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A brief review of derivations of the density function of the non-central chi-square is given. Another geometrical derivation based upon the properties of spherical coordinates is then presented. The method of proof requires very little knowledge of n-dimensional geometry and does not presume that the central chi-square distribution is available.

THE probability density function of the non-central chi-square distribution is derived a number of places in the literature. Fisher [3] obtained it indirectly as a limiting case of another distribution. Tang [11] gave an analytic derivation which has been reproduced by Mann [6, pp. 65–8] and improved slightly by Anderson [1, pp. 112–3]. Graybill [4, pp. 74–6] used the moment generating function technique which works out rather easily but is non-constructive. McNolty [7] has inverted the characteristic function by integrating around a contour in the complex plane. Bose [2], Patnaik [8], Quenouille [9], and Ruben [10] all give the same geometric derivation based upon properties of solid angles and hyperspheres. Ruben has added a second geometric approach in which the density function is obtained by dividing an n-dimensional sphere by a set of parallel hyperplanes. This note adds another geometrical derivation which is based upon the properties of spherical coordinates and requires very little detailed knowledge of n-dimensional geometry. The method used does not presume knowledge of the fact that the sum of the squares of independent standard normal random variables has a central chi-square distribution.

The distribution of

$$W^2 = \sum_{i=1}^{n} (Y_i - b_i)^2,$$

where the b_i's are real constants, is non-central chi-square with n degrees of freedom and non-centrality parameter

$$\lambda^2 = \sum_{i=1}^{n} b_i^2$$

if the Y_i have independent standard normal distributions. The distribution function of W^2 is

$$H(W^2; n, \lambda^2) = \Pr\left[\sum_{i=1}^{n} (Y_i - b_i)^2 \leqq W^2\right] \tag{1}$$

$$= \int_{\sum_{i=1}^{n}(y_i - b_i)^2 \leqq W^2} \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} y_i^2\right) dy_1 \cdots dy_n.$$

This is, of course, the integral of the n-dimensional standard normal distribution over a sphere of radius W with center at $(b_1, \cdots, b_n)$. Due to the symmetry of the standard normal distribution (1) is equivalent to

$$\int_{(y_1-\lambda)^2+\sum_{i=2}^{n} y_i^2 \leqq W^2} \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} y_i^2\right) dy_1 \cdots dy_n$$

$$= \int_{\sum_{i=1}^{n} x_i^2 \leqq W^2} \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 + 2\lambda x_1 + \lambda^2\right)\right] dx_1 \cdots dx_n. \tag{2}$$

Now change to spherical coordinates, letting

$$\begin{aligned}
x_1 &= \rho \cos \theta_1 \\
x_2 &= \rho \sin \theta_1 \cos \theta_2 \\
&\vdots \\
x_{n-1} &= \rho \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\
x_n &= \rho \sin \theta_1 \cdots \qquad\qquad \sin \theta_{n-1} \\
|J| &= \rho^{n-1} (\sin \theta_1)^{n-2} (\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2}
\end{aligned}$$

so that (2) becomes

$$\int_0^W \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \frac{1}{(2\pi)^{n/2}} \cdot \exp\left[-\tfrac{1}{2}(\rho^2 + \lambda^2 + 2\lambda\rho\cos\theta_1)\right] |J| \, d\theta_{n-1} \cdots d\theta_1 d\rho.$$

Integrating out $\theta_2, \cdots, \theta_{n-1}$ yields

$$\int_0^W \frac{\rho^{n-1} \exp\left[-\frac{1}{2}(\lambda^2 + \rho^2)\right]}{2^{(n-2)/2}\pi^{1/2}\Gamma\left(\frac{n-1}{2}\right)} \left[\int_0^\pi (\sin\theta_1)^{n-2} \exp(-\lambda\rho\cos\theta_1) d\theta_1\right] d\rho. \tag{3}$$

It is known [12, p. 79] that the integral in the bracket is equal to

$$\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{\lambda\rho}{2}\right)^{(n-2)/2}} I_{(n-2)/2}(\lambda\rho). \tag{4}$$

Substituting (4) in (3) gives

$$\begin{aligned}
H(W^2; n, \lambda^2) &= \int_0^W \frac{\rho^{n/2}}{\lambda^{(n-2)/2}} I_{(n-2)/2}(\lambda\rho) \exp\left[-\tfrac{1}{2}(\lambda^2 + \rho^2)\right] d\rho \\
&= \int_0^{W^2} \frac{1}{2}\left[\frac{\rho}{\lambda}\right]^{(n-2)/2} I_{(n-2)/2}(\lambda\rho) \exp\left[-\tfrac{1}{2}(\lambda^2 + \rho^2)\right] d\rho^2.
\end{aligned}$$

Consequently the density of W^2 is

$$h(w^2; n, \lambda^2) = \frac{1}{2}\left(\frac{w}{\lambda}\right)^{(n-2)/2} I_{(n-2)/2}(\lambda w) \exp\left[-\tfrac{1}{2}(\lambda^2 + w^2)\right]. \tag{5}$$

Alternatively, the integral in the bracket of (3) is equal to

$$\int_0^\pi \sum_{i=0}^\infty \frac{(-1)^i(\lambda\rho)^i}{i!} (\cos\theta_1)^i (\sin\theta_1)^{n-2} d\theta_1$$

$$= \sum_{j=0}^\infty \frac{(\lambda^2\rho^2)^j}{(2j)!} \int_0^\pi (\cos^2\theta_1)^j (\sin\theta_1)^{n-2} d\theta_1. \tag{6}$$

Since $(2j)! = 2^{2j} j! \Gamma(j+\frac{1}{2})/\Gamma(\frac{1}{2})$ and

$$\int_0^\pi (\cos^2\theta_1)^j (\sin\theta_1)^{n-2} d\theta_1 = \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+j\right)},$$

(3) reduces to

$$H(W^2; n, \lambda^2) = \int_0^W \frac{1}{2^{n/2}} \sum_{j=0}^\infty \frac{(\lambda^2)^j(\rho^2)^{(n/2)+j-1} \exp\left[-\frac{1}{2}(\lambda^2+\rho^2)\right]}{j! 2^{2j} \Gamma\left(\frac{n}{2}+j\right)} (2\rho d\rho)$$

so that

$$h(w^2; n, \lambda^2) = \frac{\exp\left[-\frac{1}{2}(\lambda^2+w^2)\right]}{2^{n/2}} \sum_{j=0}^\infty \frac{(\lambda^2)^j (w^2)^{(n/2)+j-1}}{j! 2^{2j} \Gamma\left(\frac{n}{2}+j\right)},$$

the well known series result.

Of course, if we are willing to assume that we already know

$$\sum_{i=2}^n Y_i^2 = \chi^2_{n-1}$$

is central chi-square with $n-1$ degrees of freedom, then (2) is $\Pr[(Y_1-\lambda)^2 + \chi^2_{n-1} \leqq W^2]$

$$= \iint_{v^2+u \leqq W^2} \frac{u^{(n-3)/2} \exp\left(-\frac{1}{2}u\right)}{2^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)} \frac{1}{(2\pi)^{1/2}} \exp\left[-\tfrac{1}{2}(v+\lambda)^2\right] dv du. \tag{7}$$

Letting $u = \rho^2 \sin^2\theta$, $v = \rho\cos\theta$ converts (7) to (3). This is the method used by Tang [11] to complete his derivation. The use of spherical coordinates to derive the density of the central chi-square is not new and can be found in Kendall and Stuart [5, Sec. 11.2].

REFERENCES

[1] Anderson, T. W., *An Introduction to Multivariate Statistics*. New York: John Wiley and Sons, 1958.

[2] Bose, R. C., "On the exact distribution of the D^2-statistic," *Sankhyā,* 2 (1936), 143–54.

[3] Fisher, R. A., "The general sampling distribution of the multiple correlation coefficient," *Proceedings of the Royal Society, Series A,* 121 (1928), 654–73.

[4] Graybill, Franklin A., *An Introduction to Linear Models, Volume 1*. New York: McGraw-Hill Publishing Company, 1961.

[5] Kendall, M. G., and Stuart, A., *The Advanced Theory of Statistics, Volume 1*. London: Charles Griffin and Company, 1958.

[6] Mann, H. B., *The Analysis and Design of Experiments*. New York: Dover Publications, 1949.

[7] McNolty, Frank, "A contour-integral derivation of the non-central chi-square distribution," *Annals of Mathematical Statistics,* 33 (1962), 796–800.

[8] Patnaik, P. B., "The non-central chi square and F distributions and their applications, *Biometrika,* 36 (1949), 202–32.

[9] Quenouille, M. H., "The evaluation of probabilities in a normal multivariate distribution, with special reference to the correlation ratio," *Proceedings of the Edinburgh Mathematical Society,* 8 (1949), 95–100.

[10] Ruben, Harold, "Probability content of regions under spherical normal distributions, I," *Annals of Mathematical Statistics,* 31 (1960), 598–618.

[11] Tang, P. C., "The power function of the analysis of variance tests with tables and illustrations for their use," *Statistical Research Memoirs,* 2 (1938), 126–46.

[12] Watson, G. N., *A Treatise on the Theory of Bessel Functions, Second Edition.* Cambridge: Cambridge University Press, 1952.