Ordinary:

$\hat{b}_{OLS}$: To minimize $(y-xb)^T(y-Xb)$. 「for Aitken model, $y_i \sim var(y_i)$ 大 入 $\hat{b}_{OLS}$ 考虑相关关系」

Generalized:

$\hat{b}_{GLS}$: To minimize $(y-Xb)^T V^{-1}(y-Xb)$

当 $V=\sigma^2 I$ 时，$\hat{b}_{GLS}$ 和 $\hat{b}_{OLS}$ 等价。

大多数情况，$\hat{b}_{OLS}$ 解决的问题为 Gauss Markov Model

(i.e. 不同变量之间无关且等方差的情况)

$\hat{b}_{GLS}$ 解决的问题为 Aitken Model.

(V^{-1} 对原始的 $(y-Xb)^T(y-Xb)$ 进行加权，消除尺度和相关性的影响)

直接从 $(y-xb)^T V(y-Xb)$ 来看：

e.g. $y_1 = x_1 b + e_1$

$y_2 = x_2 b + e_2$

① $V = var(e) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$V^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

$(y-xb)^T V^{-1}(y-xb) = \underbrace{\tfrac{1}{2}(y_1-x_1 b)^2}_{\text{消除模型的尺度差异}} + 1\,(y_2-x_2 b)^2$

② $V = var(e) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$

$V^{-1} = \frac{4}{3}\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$

$(y-Xb)^T V^{-1}(y-Xb) = \frac{4}{3}(e_1 \;\; e_2)\begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

$= \frac{4}{3}\left(e_1 - \frac{1}{2}e_2 \quad -\frac{1}{2}e_1 + e_2\right)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

$= \frac{4}{3}\left(\underbrace{e_1^2 - \frac{1}{2}e_1 e_2}_{(y_1-x_1 b)^2} + \underbrace{e_2^2 - \frac{1}{2}e_1 e_2}_{\text{减去相关性}}\right)$

Outline:

Gauss-Markov Model : $y = Xb + e$

$\left\{\begin{array}{l} ① \text{ Expectation \& Variance for } y, \hat{b}, \hat{y}, \sigma^2 \\ ② \text{ BLUE} \longleftrightarrow \lambda^T \hat{b}_{OLS} \end{array}\right.$

$\Big\downarrow$ $RVR^T = I$ (How to find?)

$Ry = R(Xb + e)$

Aitken Model : $z = Ub + f$

$\lambda^T \hat{b}_{OLS} \longrightarrow \hat{b}_{GLS}.$

Finally, what if $\lambda^T \hat{b}_{OLS}$ is the one we care about?

Section 5.3

$$X^T X b = X^T y \; ?$$

$$\Updownarrow$$

$$U^T U b = U^T y$$

由于已知 $\lambda^T \hat{b}_{GLS}$ is BLUE, section 5.3 考虑的问题有何时 $X^T X b = X^T y$ 的解为 $U^T U b = U^T y$ 的子集