Independence.

① $x \sim N_p(\mu, V)$ $BVA = 0 \Rightarrow x^TAx$ 与 Bx 独立

$x_1 \perp x_2 \Rightarrow f(x_1) \perp g(x_2)$

$x^TAx = f(A^{1/2}x)$ y

$= y^Ty.$

$A^{1/2}x$ $Bx \sim N(\_,\_)$

$Cov(A^{1/2}x, Bx) = 0$

A p.d $x^TAx = (A^{1/2}x)^T(A^{1/2}x)$

A symmetric. $A = Q\Sigma Q^T = (Q_1 \;\; Q_2)\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}Q_1^T\\ Q_2^T\end{bmatrix}$

$x^TAx = [x^TQ_1 \;\; x^TQ_2]\begin{bmatrix}\Sigma_1 & \\ & 0\end{bmatrix}\begin{bmatrix}Q_1^Tx\\ Q_2^Tx\end{bmatrix}$

$= (Q_1^Tx)^T\Sigma_1(Q_1^Tx) \in \mathbb{R}^r$ $r = rank(A)$

$\uparrow$ $\mathbb{R}^{r\times r}$

② $x \sim N(\mu, V)$ $x^TA_1x, x^TA_2x, \ldots, x^TA_mx$

$i \neq j$ $A_iVA_j = 0 \Rightarrow$ quadratic form independence.

$\mathbb{R}^{p\times p}$

$A_i = Q_i\Sigma_iQ_i^T$ $rank(A_i) = r_i$

$\mathbb{R}^{p\times r}$ $\uparrow \mathbb{R}^{r\times r}$

$Cov(Q_i^Tx, Q_j^Tx) = Q_i^TVQ_j = 0$

Σ_i 可逆 $Q_i^TQ_i = I$

$A_iVA_j = 0 \Rightarrow Q_i^TA_iVA_jQ_j = 0 \Rightarrow \Sigma_iQ_i^TVQ_j\Sigma_j = 0$

$\Rightarrow Q_i^TVQ_j = 0$

$Q_i \in \mathbb{R}^{p\times r}$ $(r<p)$ $Q_i^TQ_i = I_r \not\Rightarrow Q_iQ_i^T \neq I_p$

rank(r) rank(p)

$r = p \quad Q_i^T Q_i = I_p \Leftrightarrow Q_i Q_i^T = I_p$

EXAMPLE $x_1, x_2, \dots x_n \sim N(\mu, \sigma^2)$ $\quad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

$\hat{\mu} = \frac{1}{n} \mathbf{1}^T x = Bx \quad B = \frac{1}{n}\mathbf{1}^T \in \mathbb{R}^{n\times 1}$

$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 = \frac{1}{n-1} x^T\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)x = x^T A x$

$A = \frac{1}{n-1}\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \sim N(\underbrace{\mu \mathbf{1}}_{\vec{\mu}}, \underbrace{\sigma^2 I}_{V})$

$BVA = \frac{1}{n}\mathbf{1}^T(\sigma^2 I)\left[\frac{1}{n-1}\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\right] = \frac{1}{n(n-1)}\mathbf{1}^T\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = 0$

$\mathbf{1}^T\mathbf{1} = n.$

$\Rightarrow \hat{\mu} \perp \hat{\sigma}^2$

③ $x \sim N(\mu, V)$ i) AV idempotent, rank $s \Rightarrow x^T A x \sim \chi_s^2(\frac{1}{2}\mu^T A \mu)$

ii) $x^T A x \sim \chi_s^2(\phi)$ for some $\phi \Rightarrow AV$ idempotent, rank s.

① AV idempotent, VA? $V^{\frac{1}{2}} A V^{\frac{1}{2}}$ idempotent?

$VAVA = VAVAVV^{-1} = VAV \cdot V^{-1} = VA. \quad VA$ idempotent.

$(V^{\frac{1}{2}}AV^{\frac{1}{2}})(V^{\frac{1}{2}}AV^{\frac{1}{2}}) = V^{\frac{1}{2}}AVAVV^{-\frac{1}{2}} = V^{\frac{1}{2}}AVV^{-\frac{1}{2}} = V^{\frac{1}{2}}AV^{\frac{1}{2}}.$

i) $x^T A x = (V^{-\frac{1}{2}}x)^T \underbrace{V^{\frac{1}{2}}AV^{\frac{1}{2}}}_{\text{idempotent. rank } s}(V^{-\frac{1}{2}}x) \qquad V^{-\frac{1}{2}}x \sim N(V^{-\frac{1}{2}}\mu, I)$

ii) $V = I$ $\quad A = Q\Sigma Q^T \qquad \text{rank}(A) = r \quad \Sigma \in \mathbb{R}^{r\times r}.$

$y = Q^T x \in \mathbb{R}^r$

$$x^T A x = y^T \Sigma y = \underline{\sum_{t=1}^{r} \sigma_t y_t^2} \quad \text{weighted } \chi^2$$

$$y = Q^T x \sim N(\underline{Q^T \mu}, \underline{I}) \qquad Var(y) = Q^T V Q = Q^T Q = I$$

$$y_t^2 \sim \chi_1^2\left((Q_t^T \mu)^2 / 2\right) \qquad Q = (Q_1\ Q_2 \ldots Q_r)$$

证 $AV = A$ idempotent rank s, $\sigma_1 = \cdots = \sigma_r = 1$

$r = s$.

MGF $u \sim \chi_p^2(\phi)$ $E(e^{tu}) = (1-2t)^{-p/2} \exp\left(\frac{2\phi t}{1-2t}\right)$

$$\prod_{t=1}^{r} E\left(e^{v \cdot \sigma_t y_t^2}\right) = \underline{\prod_{t=1}^{r}\left[(1-2v\sigma_t)^{-1/2} \exp\left(\frac{(Q_t^T\mu)^2 \cdot \sigma_t \cdot v}{1-2\sigma_t v}\right)\right]}$$

$$= \underline{(1-2v)^{-s/2} \exp\left(\frac{2\phi t}{1-2t}\right)} \qquad \forall v.$$

$\Rightarrow \sigma_1 \cdots \sigma_r = 1 \quad s = r.$

$$\exp\left[\frac{4\phi t}{1-2t} - \sum_{t=1}^{r} \frac{2(Q_t^T\mu)^2 \sigma_t v}{1-2\sigma_t v}\right] \qquad \Rightarrow \sigma_t = 1$$

$r = s.$

$$= \prod_{t=1}^{r} (1-2v\sigma_t)^{-1} (1-2v)^{s}$$

$V \neq I$

$$x^T A x = (V^{-1/2}x)^T \underbrace{V^{1/2} A V^{1/2}}_{A'} \underbrace{(V^{-1/2}x)}_{x'} \qquad x' \sim N(V^{-1/2}\mu, I)$$

$A' = V^{1/2} A V^{1/2}$ idempotent $\Rightarrow$ AV, VA idempotent.

$V = Q\Sigma Q^T \quad V^{1/2} = Q\Sigma^{1/2}Q^T$

Cochran $\underline{A_1 + \cdots + A_k = I} \in \mathbb{R}^n$

i) $A_i A_j = 0$ $\qquad A_i(A_1 + \cdots + A_k) = A_i \Rightarrow A_i^2 = A_i$

ii) $A_i^2 = A_i$

iii) $\operatorname{rank}(A_1) + \cdots + \operatorname{rank}(A_k) = n \Leftrightarrow$ "$\sum s_i = N$"

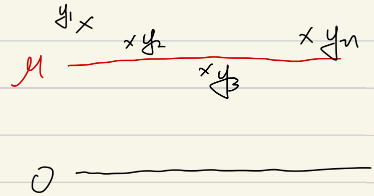
$A_i^2 = A_i \quad A_i$ idempotent $\quad \operatorname{rank}(A_i) = \operatorname{tr}(A_i)$

$$\operatorname{tr}(A_1 + \cdots + A_k) = \operatorname{tr}(I) = n$$
$$= \operatorname{tr}(A_1) + \cdots + \operatorname{tr}(A_k)$$
$$= \operatorname{rank}(A_1) + \cdots + \operatorname{rank}(A_k)$$

"zero-way" ANOVA.

$$y_i = \mu + \varepsilon_i \qquad i = 1, \dots, N. \qquad E(\varepsilon_i) = 0 \qquad \varepsilon_i \sim N(0, \sigma^2)$$

$$\hat{\mu} = \frac{1}{n} \Sigma y_i = \bar{y}$$



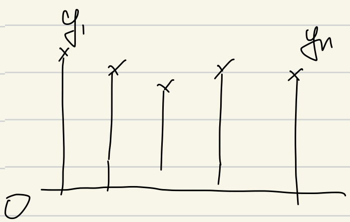
$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2 - \bar{y}^2$$

$$\frac{1}{n} \Sigma y_i^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 + \bar{y}^2 \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\underline{\frac{1}{n} y^T y} = \underline{\frac{1}{n} y^T (I - P_1) y} + \underline{\frac{1}{n} y^T P_1 y}. \qquad \text{"}\mu^2\text{"}$$

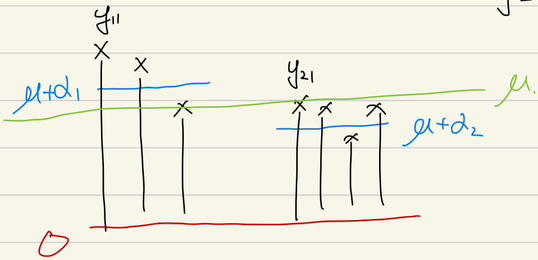
$$P_1 = \frac{1}{n} 1_n 1_n^T \qquad y^T P_1 y = \frac{1}{n} (1_n^T y)^2 = \frac{1}{n} \left(\sum_{i=1}^{n} y_i\right)^2 = n \cdot \bar{y}^2$$



One way ANOVA.

$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ $\quad i = 1, \dots, a$ Group

$\quad j = 1 \dots n_i$ obs.



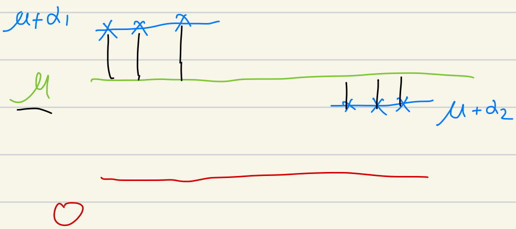
$\frac{1}{N} y^T y$

$\frac{1}{N} y^T A_1 y$.

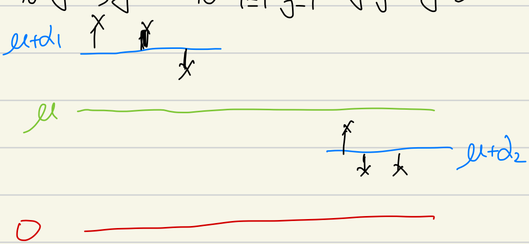
$\frac{1}{N} \bar{y}^2$ "μ"

$y_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$

$$\frac{1}{N} y^T A_2 y = \frac{1}{N} \sum_{i=1}^{a} n_i (\bar{y}_i - \bar{y})^2 = ?$$



$$\frac{1}{N} y^T A_3 y = \frac{1}{N} \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$$



$X \in \mathbb{R}^{n\times p}$

$\mathrm{rank}(X) \le p$

$\hat{\beta}$ is solution to $\underline{X^TX\beta = X^Ty}$. $\quad X^TX$

$\hat{\beta} = (X^TX)^- X^Ty$.

$k^T\beta$ estimable $\Leftrightarrow k \in \mathcal{C}(X^T)$

$\Leftrightarrow k = X^TQ$

① $E(k^T\hat{\beta}) = k^T(X^TX)^- X^T \cdot X\beta$

$= Q^T \underbrace{X(X^TX)^-X^T}_{P_X} X\beta = \underline{Q^TX}\beta = \underline{k^T}\beta$

$P_X X = X$

② $Var(k^T\hat{\beta}) = k^T(X^TX)^- X^T \underbrace{Var(y)}_{\sigma^2 I} X(X^TX)^- k$

$= k^T\underbrace{(X^TX)^-}_{G_1} X^TX\underbrace{(X^TX)^-}_{G_2} k \times \sigma^2 \qquad P_X \cdot P_X$

$= Q^T \underbrace{X G_1 X^T}_{P_X} \cdot \underbrace{X G_2 X^T}_{P_X} Q. \qquad k^T \in \mathbb{R}^{s\times p}$

$= Q^TP_XQ = Q^TXG_3X^TQ = k^T\underbrace{(X^TX)^-}_{G_3} k. \quad \in \mathbb{R}^{s\times s}$

$AGA = A. \qquad A$ 对称 $A^T = A.$

$\Downarrow$

$A^TG^TA^T = A^T$

$\Downarrow$

$AG^TA = A.$

③ $\mathrm{rank}(k^T(X^TX)^-k) = s.$

k^Tb estimable $\Leftrightarrow \exists\, a' \quad k = X^Ta'.$

$\Leftrightarrow \exists\, Q \quad k = X^TXQ.$

$$K = X^T Q' = X^T P_X Q' + \underbrace{X^T (I - P_X) Q'}_{0}$$

$$= X^T \underbrace{X (X^T X)^- X^T Q'}_{Q}.$$

$$Var(K^T \hat{b}) = K^T (X^T X)^- K = Q^T X^T \underbrace{X (X^T X)^- X^T}_{P_X} X Q.$$

$$= \underline{Q^T X^T} \, \underline{X Q}.$$

$$rank(K^T (X^T X)^- K) = rank(Q^T X^T X Q) = \underline{rank(XQ) = s}.$$

$$s = rank(K) = rank(X^T X Q) \le rank(XQ) \le s.$$

$$rank(AB) \le rank(B)$$
$$\le rank(A)$$

$$\underline{K^T \hat{b}} = K^T (X^T X)^- X^T y \sim N(K^T b, \sigma^2 \underbrace{K^T (X^T X)^- K}_{H})$$

H_0 $\underline{K^T b = C}$ $\qquad$ H_1 $K^T b \ne C$.

$$\underline{K^T \hat{b} - C} \sim N(\underline{K^T b - C}, \sigma^2 H)$$

~~? $\|K^T \hat{b} - C\|_2^2 > c$ reject H_0.~~

$$\underline{(K^T \hat{b} - C)^T (\sigma^2 H)^{-1} (K^T \hat{b} - C)} \sim \underline{\chi_s^2(\phi)}$$ "test 1"

$$\phi = \tfrac{1}{2} \underbrace{(K^T b - C)(\sigma^2 H)^{-1} (K^T b - C)}_{=0}$$

Under $H_0 \Rightarrow K^T b - c = 0 \Rightarrow \phi = 0$

"test 2" $$T = \frac{\frac{1}{\sigma^2}(K^T\hat{b} - c)^T H^{-1}(K^T\hat{b} - c)/s}{\frac{1}{\sigma^2}\,\frac{y^T(I-P_x)y}{n-r}} \sim F_{s,n-r}(\phi)$$

$r = \text{rank}(X)$

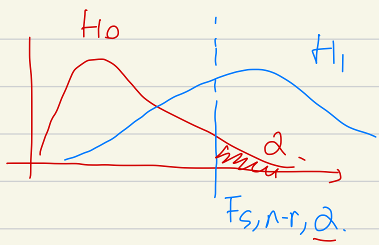
分母 $y^T(I-P_x)y = y^T(I-P_x)(I-P_x)y = f(\overbrace{(I-P_x)y}^{A})$

分子 $= g(K^T\hat{b}) = g(\underbrace{K^T(X^TX)^- X^T y}_{B})$

分子 ⊥ 分母 $\Leftarrow A \perp B \Leftarrow \text{Cov}(A,B) = 0$

$$\text{Cov}(A,B) = \underbrace{(I-P_x)(\sigma^2 I)X}_{0}(X^TX)^- K = 0$$

$T \sim F_{s,n-r}(\phi)$ $H_0\ \phi = 0$ $H_1\ \phi \neq 0$



reject H_0

Lemma 7.6. $S_1 = \{b : K^T b = c\}$ $S_2 = \{b : K_*^T b = c_*\}$

$S_1 = S_2 \iff \exists$ 可逆 Q $K_* = KQ$ $c_* = Q^T c$.

"$\Leftarrow$" $K^T b = c \iff Q^T K^T b = Q^T c \iff K_*^T b = c_*$.

"$\Rightarrow$" $K^T b = c \iff b = (K^T)^- c + (I - (K^T)^- K^T) z \quad \forall z$

$\mathbb{R}^{s\times p}$.

$S_1 = S_2$ $K_*^T b = c_*$

$$K_*^T (K^T)^- c + \underbrace{K_*^T (I - (K^T)^- K^T) z}_{Q^T} = c_* \quad \forall z.$$

$=0$

① $z = 0$

$$K_*^T (K^T)^- c = c_*.$$

$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

② $K_*^T (I - (K^T)^- K^T) = 0$

$$\Rightarrow K_*^T = \underbrace{K_*^T (K^T)^-}_{Q^T} K^T$$

K^T $K_*^T \in \mathbb{R}^{s\times p}$

$(K^T)^- \in \mathbb{R}^{p\times s}$.

$Q \in \mathbb{R}^{s\times s}$.

③ $Q^T = K_*^T (K^T)^-$ full rank?

$$s = \operatorname{rank}(K_*^T) = \operatorname{rank}(Q^T K^T) \le \underline{\operatorname{rank}(Q^T)} \le \operatorname{rank}(K_*^T) = s$$

$$\vec{k}^T\vec{\beta} - \vec{k}^T\vec{\beta}_0$$

Thm $Q(\hat{\beta}_0) - Q(\hat{\beta}) = (k^T\hat{\beta} - c)^T (k^T(X^TX)^- k)^{-1} (k^T\hat{\beta} - m)$

(*) $\hat{\beta}$: is solution to $X^TX\hat{\beta} = X^Ty$.

$\hat{\beta}_0$: is solution to $\begin{bmatrix} X^TX & k \\ k^T & 0 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \theta \end{bmatrix} = \begin{bmatrix} X^Ty \\ m \end{bmatrix}$ $\qquad k^T\hat{\beta}_0 = m$.

(**) $\quad X^TX\hat{\beta}_0 + k\theta = X^Ty$.

$(*) - (**) \Rightarrow X^TX(\hat{\beta} - \hat{\beta}_0) = k\theta$. (Δ)

$$Q(\hat{\beta}_0) - Q(\hat{\beta}) = \|y - X\hat{\beta}_0\|_2^2 - \|y - X\hat{\beta}\|_2^2$$

$$= (\cancel{y^Ty} - 2y^TX\hat{\beta}_0 + \hat{\beta}_0^TX^TX\hat{\beta}_0) - (\cancel{y^Ty} - 2y^TX\hat{\beta} + \hat{\beta}^TX^TX\hat{\beta})$$

$$\overset{*}{=} -2\hat{\beta}^TX^TX\hat{\beta}_0 + \hat{\beta}_0^TX^TX\hat{\beta}_0 - 2\hat{\beta}^TX^TX\hat{\beta} + \hat{\beta}^TX^TX\hat{\beta}$$

$$= (\hat{\beta} - \hat{\beta}_0)^TX^TX(\hat{\beta} - \hat{\beta}_0)$$

$$\overset{(\Delta)}{=} \theta^Tk^T(\hat{\beta} - \hat{\beta}_0)$$

$$= \theta^T(k^T\hat{\beta} - k^T\hat{\beta}_0)$$

$$= \theta^T(k^T\hat{\beta} - m)$$

k^Tb estimable $\Leftrightarrow k \in \mathcal{C}(X^T) \Leftrightarrow \exists a \;\; k = X^TXa$.

$\Downarrow$

$a^T \times (\Delta) \Rightarrow \underbrace{a^TX^TX}_{k^T}(\hat{\beta} - \hat{\beta}_0) = a^Tk\theta$. $\qquad a = (X^TX)^-k$.

$$(k^T\hat{\beta} - m) = a^Tk\theta = k^T(X^TX)^-k\theta$$

$$\theta = [k^T(X^TX)^-k]^{-1}(k^T\hat{\beta} - m)$$

$y \sim N(Xb, \sigma^2 I)$ $\Lambda \in \mathbb{R}^{p \times s}$ $\text{rank}(\Lambda) = s$

$\hat{b}$ is solution to $X^T X b = X^T y$.

$$\Rightarrow \Lambda^T \hat{b} \sim N(\Lambda^T b, \sigma^2 \underbrace{\Lambda^T (X^T X)^- \Lambda}_{\text{可逆}})$$

1-α level Confidence set $R(y)$: $P_\theta(\theta \in R(y)) = 1 - \alpha$.

Fixed. Random

θ varies under H_0 H_1

$\lambda \in \mathbb{R}^p$.

Univariate Case. $\lambda^T \hat{b} = \lambda^T (X^T X)^- X^T y \sim N(\lambda^T b, \sigma^2 \lambda^T (X^T X)^- \lambda)$

① σ^2 is known. $\dfrac{\lambda^T(\hat{b} - b)}{\sigma(\lambda^T X^T X \lambda)^{1/2}} \sim N(0,1)$ z_α : $P(Z > z_\alpha) = \alpha$.

"$\hat{\sigma}$"

$$\left| \frac{\lambda^T(\hat{b} - b)}{\sigma(\lambda^T X^T X \lambda)^{1/2}} \right| < z_{\alpha/2}$$

$$\lambda^T b \in \underbrace{\lambda^T \hat{b} \pm z_{\alpha/2} \cdot \sigma (\lambda^T X^T X \lambda)^{1/2}}_{\text{interval.}}$$

② σ^2 unknown. Noncentral t-dist. $v = X / \sqrt{u/k}$, $X \perp u$, $X \sim N(\varphi, 1)$

$u \sim \chi^2_k$

$$\Rightarrow v \sim t_k(\varphi)$$

i) $v \sim t_k(\varphi) \Rightarrow v^2 \sim F_{1,k}(\varphi^2/2)$

ii) $t_{k,\alpha/2}$ $P(v > t_{k,\alpha/2}) = \alpha/2$ $t^2_{k,\alpha/2} = F_{1,k,\alpha}$.

$$\frac{\hat{\sigma}^2}{\sigma^2} = \frac{1}{N-r} \cdot \frac{y^T(I - P_X)y}{\sigma^2} \sim \frac{1}{N-r} \chi^2_{N-r} \quad \perp \hat{b}$$

$$t = \frac{\cancel{\frac{1}{\sigma}}\ \lambda^T(\hat{b}-b)/(\lambda^T(X^TX)^-\lambda)^{1/2}}{\sqrt{\hat{\sigma}^2/\cancel{\sigma^2}}} \sim t_{N-r}$$

$R(y)$ $\quad \lambda_i^T\hat{b} \pm t_{N-r,\alpha/2}\,\hat{\sigma}\,(\underbrace{\lambda_i^T(X^TX)^-\lambda_i}_{h_{ii}})^{1/2}$ $\quad i = 1, \dots, s.$

$\mathrm{diag}(H) = \lambda_{11} \quad \lambda_{22} \ \cdots \ \lambda_{ss}$

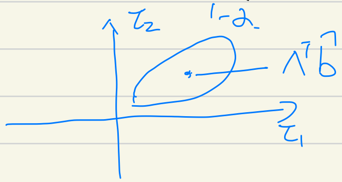
$H = \Lambda^T(X^TX)^-\Lambda$ $\qquad \dfrac{(\Lambda^T\hat{b} - \Lambda^Tb)\,H^{-1}(\overset{\hat{\tau}}{\Lambda^T\hat{b}} - \overset{\tau}{\Lambda^Tb})/s}{\hat{\sigma}^2} \sim F_{s,N-r}.$

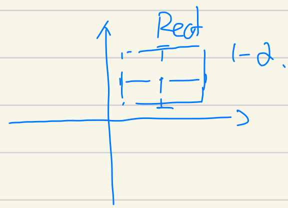
$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$R(y) = \{\tau : (\tau - \Lambda^T\hat{b})H^{-1}(\tau - \Lambda^T\hat{b}) \le s \cdot \hat{\sigma}^2\, \underline{F_{s,N-r,\alpha}}\}$

$$X = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ \vdots & \vdots \end{bmatrix}$$

$\tau \in \mathbb{R}^2$





$P((\tau_1, \tau_2) \in \text{Rect}) = 1-\alpha.$

INTERPRETABLE

$\Lambda = [\lambda_1 \cdots \lambda_s] \in \mathbb{R}^{p \times s}.$

For $i = 1, \dots, s$ $\qquad I_i = [\lambda_i^T\hat{b} \pm t_{N-r,\alpha/2}\,\hat{\sigma}\sqrt{h_{ii}}]$

event $\quad A_i = \{\lambda_i^T b \in I_i\}.$ $\quad \underline{P(A_i) = 1-\alpha}.$

$P(\Lambda^T b \in \text{Rect}) = P\left(\begin{pmatrix}\tau_1 \\ \vdots \\ \tau_s\end{pmatrix} \in \underline{I_1 \otimes \cdots \otimes I_s}\right)$

$= P(\bigcap_{i=1}^{s} A_i)$ $\quad$ ① $\underline{A_1 = \cdots = A_s}$ $\quad P(A_1) = 1-\alpha.$

② $\underline{A_1 \perp A_2 \perp \cdots \perp A_s}$ $\quad \prod P(A_i) = (1-\alpha)^s < 1-\alpha.$

Bonferroni t-interval

$P(A \cup B) \le P(A) + P(B)$

$$P(\cap A_i) = 1 - P((\bigcap_{i=1}^{S} A_i)^c) = 1 - P(\bigcup_{i=1}^{S} A_i^c)$$

$$\ge 1 - \sum_{i=1}^{S} P(A_i^c) \ge 1 - \alpha.$$

$\Rightarrow$ Set $P(A_i^c) = \alpha/s$.

$$\Rightarrow P\{\lambda_i^T b \in [\lambda_i^T \hat{b} \pm t_{N-r, \frac{\alpha}{2s}} \hat{\sigma} \sqrt{h_{ii}}]\ i = 1, \dots, s\} \ge 1-\alpha.$$

Scheffe's Method

$$\max \frac{(u^T v)^2}{u^T W u} = v^T W^{-1} v.$$

$\|u\|_W^2 = u^T W u \quad \|v\|_{W^{-1}}^2 = v^T W^{-1} v.$

$$\Leftrightarrow \max_{\|u\|_W^2 \le 1} (u^T v)^2 = \|v\|_{W^{-1}}^2$$

dual norm.

$$\underbrace{(\hat{\tau} - \tau)}_{v} H^{-1} (\hat{\tau} - \tau) \ge \frac{u^T (\hat{\tau} - \tau)}{u^T H u} \quad \forall u$$

$(\hat{\tau} - \tau)H^{-1}(\hat{\tau}-\tau) = \|v\|_{H^{-1}}^2$, $u^T(\hat{\tau}-\tau) = u^T v$

"=" $u \propto H^{-1} v$.

$$1 - \alpha = P\left(\frac{(\hat{\tau} - \tau) H^{-1} (\hat{\tau} - \tau)/s}{\hat{\sigma}^2} \le F_{s, N-r, \alpha}\right\}$$

$$= P\left[\underbrace{\frac{1}{s\hat{\sigma}^2} \cdot \frac{(u^T(\hat{\tau} - \tau))^2}{u^T H u} \le F_{s, N-r, \alpha}}_{A_u},\ \forall u\right\} = P\left(\bigcap_{u \in \mathbb{R}^s} A_u\right)$$

$$= P\left(u^T \tau \in u^T \hat{\tau} \pm \sqrt{s\hat{\sigma}^2 F_{s, N-r, \alpha}}\ (\forall u)\right)$$

Turkey's Interval.

: s linearly independent $\tau_1 \cdots \tau_s$
least square est $\hat{\tau}_1 \cdots \hat{\tau}_s$

$l = \frac{s(s-1)}{2}$ all possible differences $\tau_i - \tau_j \ \forall i \neq j$

$$Var(\hat{\tau}_1 \cdots \hat{\tau}_s)^T = c^2 \sigma^2 I. \quad \text{for some } c.$$

Studentized Range distribution.

$$q = \frac{\max z_i - \min z_i}{\sqrt{u/h}} \geq 0.$$

$z \perp u \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix} \sim N(0, I) \qquad u \sim \chi^2_h$

$q_{l,h,\alpha}$: $P(q \geq q_{l,h,\alpha}] = \alpha.$

$$1-\alpha = P\left\{ \frac{\max \frac{\hat{\tau}_i - \tau_i}{c\sigma} - \min \frac{\hat{\tau}_i - \tau_i}{c\sigma}}{\sqrt{\hat{\sigma}^2 / \sigma^2}} \leq q_{s,N-r,\alpha} \right\}$$

$$= P\{ \underline{\max}(\hat{\tau}_i - \tau_i) - \underline{\min}(\hat{\tau}_i - \tau_i] \leq c\hat{\sigma}\, q_{s,N-r,\alpha} \}.$$

$$= P(| (\hat{\tau}_i - \tau_i) - (\hat{\tau}_j - \tau_j) | \leq c\hat{\sigma}\, q_{s,N-r,\alpha} \ \forall i,j \}.$$

$$= P((\tau_i - \tau_j) \in (\hat{\tau}_i - \hat{\tau}_j \pm c\hat{\sigma}\, q_{s,N-r,\alpha}) \ \forall i,j) \}$$

Unique Solution

$X \in \mathbb{R}^{n \times p}$ $\quad \operatorname{rank}(X) = r < p$

$$\begin{bmatrix} X^TX \\ C \end{bmatrix} b = \begin{bmatrix} X^Ty \\ 0 \end{bmatrix}$$

$Cb = 0 \quad C \in \mathbb{R}^{s \times p} \quad \operatorname{rank}(C) = s = p - r.$

$\mathcal{C}(X^T) \cap \mathcal{C}(C^T) = \{0\}.$

$$\Leftrightarrow \begin{bmatrix} X \\ C \end{bmatrix} b = \begin{bmatrix} P_x y \\ 0 \end{bmatrix}$$

"$\Rightarrow$" 右乘 $X(X^TX)^-$

$$\underbrace{X(X^TX)^- X^T}_{P_x} Xb = \underbrace{X(X^TX)^- X^T}_{P_x} y.$$

$$Xb = P_x y.$$

"$\Leftarrow$" 右乘 X^T

$$X^TXb = X^TP_x y.$$

$$\Leftrightarrow \begin{bmatrix} X^TX \\ C^TC \end{bmatrix} b = \begin{bmatrix} X^Ty \\ 0 \end{bmatrix} \begin{matrix} (1) \\ (2) \end{matrix}$$

$Cb = 0 \Leftrightarrow C^TCb = 0.$

$$C^TCb = 0 \Rightarrow \underbrace{b^TC^T}_{v^T}\underbrace{Cb}_{v} = 0 \Rightarrow Cb = 0$$

$\Leftarrow.$

$$v^Tv = \|v\|_2^2 = \|Cb\|_2^2 = 0$$

$$\Leftrightarrow (X^TX + C^TC)b = X^Ty.$$

$$\hat{b} = (X^TX + C^TC)^{-1}X^Ty.$$

"$\Rightarrow$" $(1) + (2)$

"$\Leftarrow$" $$\underbrace{X^T(Xb - y)}_{\substack{=0 \\ \in \mathcal{C}(X^T)}} = \underbrace{C^TCb}_{\substack{0 \\ \in \mathcal{C}(C^T)}} = 0$$

One Way ANOVA. $\quad y_{ij} = \mu + \alpha_i + e_{ij} \quad i = 1, \cdots, a.$

a Groups. $\quad j = 1, \cdots, n_i$

$(\mu + 1) + (\alpha_i - 1)$ $\quad$ Constraint $\sum n_i \alpha_i = 0$ $\quad \hat{\mu}, \hat{\alpha}_i = ?$

$C = (0 \; n_1 \cdots n_a)$ $\quad \beta = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix}$ $\quad C\beta = 0$

$$X^TX + C^TC = (X^T \; C^T)\begin{bmatrix} X \\ C \end{bmatrix} \qquad a=3$$

$$\begin{bmatrix} X \\ \hline C \end{bmatrix} = \left[\begin{array}{cccc} 1_{n_1} & 1_{n_1} & & \\ 1_{n_2} & 0 & 1_{n_2} & \\ 1_3 & 0 & & 1_3 \\ \hline 0 & n_1 & n_2 & n_3 \end{array}\right] \text{ "X"}$$

$$X^TX + C^TC = \begin{bmatrix} N & \overset{1\times n_1}{n_1} & n_2 & n_3 \\ n_1 & n_1 + \underset{}{\overset{n_1 \times n_1}{n_1^2}} & \underline{n_1 n_2} & n_1 n_3 \\ n_2 & \underline{n_1 n_2} & n_2 + \underline{n_2^2} & n_2 n_3 \\ n_3 & n_1 n_3 & n_2 n_3 & n_3 + n_3^2 \end{bmatrix}$$

$$= \underbrace{\mathrm{diag}(N-1, \; n_1, \; n_2, \; n_3)}_{D} + \underbrace{\begin{bmatrix} 1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{v} \underbrace{(1 \; n_1 \; n_2 \; n_3)}_{v^T}$$

$$(X^TX + C^TC)^{-1} = D^{-1} - D^{-1}v\underbrace{(1 + v^TD^{-1}v)}_{\in \mathbb{R}}{}^{-1}v^TD^{-1} \qquad N = n_1 + n_2 + n_3$$

$$1 + v^TD^{-1}v = 1 + \frac{1}{N-1} + n_1 + n_2 + n_3 = N + 1 + \frac{1}{N-1}$$

(homework) . . . . . .

$$(X^TX + C^TC)^{-1}X^T = \begin{bmatrix} \frac{1}{N}1_{n_1}^T & \frac{1}{N}1_{n_2}^T & \frac{1}{N}1_{n_3}^T \\ \frac{1}{n_1}1_{n_1}^T & & \\ & \frac{1}{n_2}1_{n_2}^T & \\ & & \frac{1}{n_3}1_{n_3}^T \end{bmatrix} - \frac{1}{N}\begin{bmatrix} 0 \\ 1_N^T \\ 1_N^T \\ 1_N^T \end{bmatrix}$$

$$\hat{\beta} = (X^TX + C^TC)^{-1}X^Ty = \begin{bmatrix} \bar{y} \\ \bar{y}_{1\cdot} - \bar{y} \\ \bar{y}_{2\cdot} - \bar{y} \\ \bar{y}_{3\cdot} - \bar{y} \end{bmatrix} \begin{matrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \end{matrix}$$

Aitken Model.

$y \sim N(X\beta, \sigma^2 V)$ V pos. def.

ubiased linear estimator $c + a^T y$ of $\lambda^T \beta$.

$$E(c + a^T y) = c + a^T X\beta = \lambda^T \beta \quad \forall \beta.$$

$$\Rightarrow c = 0 \quad \lambda = \underset{p \times n,\ n > p}{\underline{\underline{X}}^T} a.$$

a infinitely many.

$\hat\beta_{GLS}$: solution to $X^T V^{-1} X \beta = X^T V^{-1} y$.

$\hat\beta_{GLS}$ is BLUE.

$\hat\beta_{OLS}$ solution to $X^T X \beta = X^T y$.

$\hat\beta_{OLS}$ is BLUE of $\lambda^T\beta$

$t^T y$ is BLUE for $E(t^T y)$ $\forall t \in \mathcal{C}(X)$

To find BLUE, we find best a such that.

$$\min \ Var(a^T y) = \sigma^2 a^T V a.$$

subject to $(E(a^T y) = a^T X \beta = \lambda^T \beta \ (\forall \beta)) \rightarrow \lambda = X^T a.$

$$\Leftrightarrow \min \ \tfrac{1}{2} a^T V a.$$

subject to $\lambda = X^T a.$

Lagrangian. $L(a, l) = \frac{1}{2} a^T V a + l^T (X^T a - \lambda)$

$$\frac{\partial L}{\partial a} = Va + Xl = 0$$

$$\frac{\partial L}{\partial l} = X^T a - \lambda = 0$$

$$\Rightarrow \begin{bmatrix} V & X \\ X^T & 0 \end{bmatrix} \begin{bmatrix} a \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \quad (*)$$

Quiz 1. ii) $b^T y$ is another unbiased estimator of $\lambda^T \beta$.

$$Var(b^T y) = Var(\underline{a^T y}) + Var(\underbrace{(b-a)^T y}_{\geq 0})$$

$$\geq Var(a^T y)$$

Result: a is solution to (*) $\Rightarrow a^T y$ is BLUE.

$\Rightarrow Va + X\lambda = 0 \qquad Va \in \mathcal{C}(X)$

(Note 5)

OLS: $\lambda^T \hat{\beta}_{OLS} = \underbrace{\lambda^T (X^T X)^- X^T}_{t^T} y$. is BLUE $\Leftrightarrow Vt \in \mathcal{C}(X)$

$$Var((b-a)^T y) = (b-a)^T \underset{p.d.}{V} (b-a) = 0 \Rightarrow V^{1/2}(b-a) = 0$$
$$\Rightarrow b-a = 0$$

$$Q^T X^T X b = Q^T X^T y$$
$$X^T V^{-1} X b = X^T V^{-1} y.$$

Note 5 $\lambda^T \hat{b}_{OLS}$ is BLUE

$\Leftrightarrow VX = XQ.$

$\Leftrightarrow X = V^{-1} X Q.$

$\Rightarrow X^T X = X^T V^{-1} X Q.$

$y^T X = y^T V^{-1} X Q$

$b^T y$ is unbiased of $\lambda^T \beta$.

$$b^T y = \underbrace{a^T y}_{\text{Best}} + \underbrace{(b-a)^T y}_{\text{estimating zero}}. \qquad Cov(a^T y, \text{zero}) = 0$$

$\Leftrightarrow a^T y$ best.

$\Rightarrow Var(b^Ty) = Var(a^Ty) + Var((b-a)^Ty) + \underbrace{2cov(a^Ty, (b-a)^Ty)}_{=0}$

$\Rightarrow a^Ty$ is BLUE.

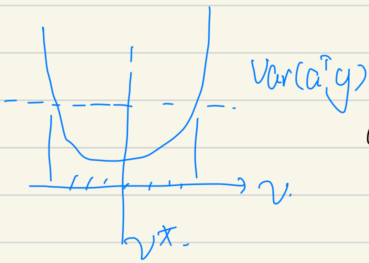
"$\Leftarrow$": a^Ty is BLUE $\Rightarrow Cov(a^Ty, zero) = 0$

Suppose $Cov(a^Ty, zero) \neq 0$

Construct another estimator $\underline{a^Ty + v \cdot zero}$ ① v 待定 ② still unbiased.

Find v such that $Var(a^Ty + v \cdot zero) < Var(a^Ty)$?

$$\| \; Var(a^Ty) + 2v \cdot \underbrace{Cov(a^Ty, zero)}_{\neq 0} + v^2 \cdot Var(zero).$$



optimal $v^* = \dfrac{-Cov(a^Ty, zero)}{Var(zero)} = \text{"} -\dfrac{b}{2a} \text{"}$

One Way Fixed Model

$$X = \begin{bmatrix} 1_{n_1} & 1_{n_1} & & & \\ 1_{n_2} & & 1_{n_2} & & \\ \vdots & & & \ddots & \\ 1_{n_a} & & & & 1_{n_a} \end{bmatrix}$$

$\text{rank}(X) = a.$

$N = n_1 + n_2 + \cdots + n_a$

$$P_X = X(X^TX)^- X^T = \begin{bmatrix} \frac{1}{n_1}J_1 & & \\ & \ddots & \\ & & \frac{1}{n_a}J_a \end{bmatrix}$$

$$X^TX = \begin{bmatrix} N & n_1 & \cdots & n_a \\ n_1 & n_1 & & \\ \vdots & & \ddots & \\ n_a & & & n_a \end{bmatrix}$$

$$(X^TX)^- = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1/n_1 & & \\ \vdots & & \ddots & \\ 0 & & & 1/n_a \end{bmatrix}$$

$$J_a = 1_a 1_a^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & & 1 \end{bmatrix} \in \mathbb{R}^{a \times a}.$$

$$P_1 = 1_N (\underbrace{1_N^T 1_N}_{N})^{-1} 1_N^T = \frac{1}{N} 1_N 1_N^T = \frac{1}{N} J_N.$$

$$\eta = E(y) = \begin{bmatrix} (\mu+\alpha_1)1_{n_1} \\ (\mu+\alpha_2)1_{n_2} \\ \vdots \\ (\mu+\alpha_a)1_{n_a} \end{bmatrix} = Xb$$

$\eta^T(P_X - P_1)\eta.$

$$\eta^T P_X \eta = \left((\mu+\alpha_1)1_{n_1}^T \;\cdots\; (\mu+\alpha_a)1_{n_a}^T\right) \begin{bmatrix} \frac{1}{n_1}J_{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_a}J_{n_a} \end{bmatrix} \begin{bmatrix} (\mu+\alpha_1)1_{n_1} \\ \vdots \\ (\mu+\alpha_2)1_{n_a} \end{bmatrix}$$

$$= \sum_{i=1}^{a} \frac{(\mu+\alpha_i)^2}{n_i} \underbrace{1_{n_i}^T \overset{1_{n_i}1_{n_i}^T}{J_{n_i}} 1_{n_i}}_{\underbrace{1_{n_i}^T 1_{n_i}}_{n_i} \underbrace{1_{n_i}^T 1_{n_i}}_{n_i}} = \sum_{i=1}^{a} n_i(\mu+\alpha_i)^2$$

$$\eta^T P_1 \eta = \frac{1}{N} \eta^T 1_N 1_N^T \eta = \frac{1}{N}(1_N^T \eta)^2 = \frac{1}{N}\left[\sum_{i=1}^{a} n_i(\mu+\alpha_i)\right]^2.$$

σ_G^2 $\qquad\qquad\qquad\qquad$ σ^2

$$\underline{SSA} = y^T(\overset{A_1}{\underline{P_X - P_1}})y \qquad\qquad SSE = y^T(I-P_X)y.$$

$$E\, \underset{\in \mathbb{R}}{y^T A y} = E\,tr(y^T A y) = E\,tr(A y y^T) = tr(A\, \underline{E(yy^T)})$$

$$= tr\left[A(\Sigma + \mu\mu^T)\right]$$

$$= tr(A\Sigma) + \mu^T A \mu.$$

$$E(SSA|\alpha) = tr(A_1 \underbrace{Var(y|\alpha)}_{\sigma^2 I}) + \underline{[E(y|\alpha)]}^T A_1 \underline{[E(y|\alpha)]}$$

$$\eta = \begin{bmatrix} (\mu+\alpha_1)1_{n_1} \\ \vdots \\ (\mu+\alpha_a)1_{n_a} \end{bmatrix} \qquad E(\eta) = \mu 1_N.$$

$$\underline{Var(\eta)} = \begin{bmatrix} \sigma_a^2 J_{n_1} & 0 & \\ 0 & \sigma_a^2 J_{n_2} & \\ & & \ddots \end{bmatrix}$$

$$= \sigma^2 tr(P_X - P_1) + \underline{\sum n_i(\alpha_i - \bar{\alpha})^2}.$$

$$= \sigma^2(a-1) + \sum n_i(\alpha_i - \bar{\alpha})^2$$

$$E(SSA) = E\left[E(SSA|\alpha)\right] = \sigma^2(a-1) + \underline{E(\eta^T A_1 \eta)}$$

$$E(\eta^T A_1 \eta) = tr(A_1 \cdot Var(\eta)) + \underbrace{(E\eta)^T \overset{P_X - P_1}{A_1} (E\eta)}_{0}$$

$$\begin{pmatrix} 1_N \in \mathcal{C}(X) \\ 1_N \in \mathcal{C}(1) \end{pmatrix}.$$

$$\begin{pmatrix} P_X E\eta = E\eta \\ P_1 E\eta = E\eta \end{pmatrix}$$

$$= tr((P_X - P_1)Var(\eta))$$

$$= N\sigma_a^2 - \frac{\sigma_a^2}{N}\sum_{i=1}^{a} n_i^2$$

$$\underline{P_X}\,Var(\eta) = \begin{bmatrix} \frac{1}{n_1}J_{n_1} & & \\ & \ddots & \\ & & \frac{1}{n_a}J_{n_a} \end{bmatrix} \begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_a} \end{bmatrix} \times \sigma_a^2$$

$$tr(P_X\,Var(\eta)) = \sum_{i=1}^{a} \frac{\sigma_a^2}{n_i} \underline{tr(J_{n_i}^2)} = \sum_{i=1}^{a} n_i \sigma_a^2 = N\sigma_a^2.$$

$$\underline{1_{n_i}1_{n_i}^T 1_{n_i}}1_{n_i}^T = n_i\, 1 1_{n_i}^T$$

$$tr(\underset{\frac{1}{N}1_N 1_N^T}{\underline{\underline{P_1}}}\,Var(\eta)) = \frac{\sigma_a^2}{N} 1_N^T \begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_a} \end{bmatrix} 1_N.$$

$$= \frac{\sigma_a^2}{N} 1_N^T \begin{bmatrix} \underline{n_1 1_{n_1}} \\ \vdots \\ n_a 1_{n_a} \end{bmatrix} = \frac{\sigma_a^2}{N} \underline{\sum_{i=1}^{a} n_i^2}$$

BALANCED MIXED.

$$P_X = \begin{bmatrix} \frac{1}{n}J_n & & \\ & \ddots & \\ & & \frac{1}{n}J_n \end{bmatrix} = \underline{I_a} \otimes \underline{\frac{1}{n}J_n}.$$

$$P_1 = \frac{1}{N}\underset{\mathbb{R}^{N\times N}}{\underline{\underline{J_N}}} = \frac{1}{N} J_a \otimes J_n.$$