Independence.

① $x \sim N_p(\mu, V)$ $\quad BVA = 0 \Rightarrow x^TAx$ 与 Bx 独立

$x_1 \perp x_2 \Rightarrow f(x_1) \perp g(x_2)$

$x^TAx = f(\underbrace{A^{1/2}x}_{y})$ $\qquad A^{1/2}x \quad Bx \sim N(\_,\_)$

$= y^Ty.$ $\qquad \mathrm{Cov}(A^{1/2}x, Bx) = 0$

A p.d $\quad x^TAx = (A^{1/2}x)^T(A^{1/2}x)$

A symmetric. $\quad A = Q\Sigma Q^T = (Q_1 \ \ Q_2)\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}Q_1^T\\ Q_2^T\end{bmatrix}$

$$x^TAx = [x^TQ_1 \ \ x^TQ_2]\begin{bmatrix}\Sigma_1 & \\ & 0\end{bmatrix}\begin{bmatrix}Q_1^Tx\\ Q_2^Tx\end{bmatrix}$$

$$= (Q_1^Tx)^T\underset{\uparrow\, R^{r\times r}}{\Sigma_1}(Q_1^Tx) \in \mathbb{R}^r \qquad r = \mathrm{rank}(A)$$

② $x \sim N(\mu, V) \quad x^TA_1x, x^TA_2x, \ldots, x^TA_mx$

$i \neq j \quad A_iVA_j = 0 \Rightarrow$ quadratic form independence.

$$\overset{R^{p\times p}}{A_i} = \underset{R^{p\times r}}{Q_i}\,\underset{\uparrow R^{r\times r}}{\Sigma_i}\,Q_i^T \qquad \mathrm{rank}(A_i) = r_i$$

$\boxed{\mathrm{Cov}(Q_i^Tx, Q_j^Tx) = Q_i^TVQ_j} = 0$

Σ_i 可逆 $\quad Q_i^TQ_i = I$

$A_iVA_j = 0 \Rightarrow Q_i^TA_iVA_jQ_j = 0 \Rightarrow \Sigma_iQ_i^TVQ_j\Sigma_j = 0$

$\Rightarrow Q_i^TVQ_j = 0$

$Q_i \in \mathbb{R}^{p\times r}$ $(r<p)$ $\quad Q_i^TQ_i = I_r \not\Rightarrow \underset{\mathrm{rank}(r)}{Q_iQ_i^T} \neq \underset{\mathrm{rank}(p)}{I_p}$

$r=p \quad Q_i^T Q_i = I_p \Leftrightarrow Q_i Q_i^T = I_p$

EXAMPLE $x_1, x_2, \dots x_n \sim N(\mu, \sigma^2)$ $\quad 1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$.

$\hat{\mu} = \frac{1}{n} 1^T x = Bx \quad B = \frac{1}{n} 1^T \in \mathbb{R}^{n \times 1}$

$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 = \frac{1}{n-1} x^T \left(I - \frac{1}{n} 1 1^T\right) x = x^T A x$

$A = \frac{1}{n-1} \left(I - \frac{1}{n} 1 1^T\right)$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \sim N(\underbrace{\mu 1}_{\vec{\mu}}, \underbrace{\sigma^2 I}_{V})$

$BVA = \frac{1}{n} 1^T (\sigma^2 I) \left[\frac{1}{n-1}\left(I - \frac{1}{n} 1 1^T\right)\right] = \frac{1}{n(n-1)} 1^T \left(I - \frac{1}{n} 1 1^T\right) = 0$

$1^T 1 = n$.

$\Rightarrow \hat{\mu} \perp \hat{\sigma}^2$

③ $x \sim N(\mu, V)$ i) AV idempotent, rank $s \Rightarrow x^T A x \sim \chi_s^2(\frac{1}{2} \mu^T A \mu)$

ii) $x^T A x \sim \chi_s^2(\phi)$ for some $\phi \Rightarrow AV$ idempotent, rank s.

① AV idempotent, VA? $V^{1/2} A V^{1/2}$ idempotent?

$VAVA = VAVAVV^{-1} = VAV \cdot V^{-1} = VA$. $\quad VA$ idempotent.

$(V^{1/2} A V^{1/2})(V^{1/2} A V^{1/2}) = V^{1/2} AVAV V^{-1/2} = V^{1/2} AV V^{-1/2} = V^{1/2} A V^{1/2}$.

i) $x^T A x = (V^{-1/2} x)^T \underbrace{V^{1/2} A V^{1/2}}_{\text{idempotent. rank } s} (V^{-1/2} x) \qquad V^{-1/2} x \sim N(V^{-1/2} \mu, I)$

ii) $V = I$ $\quad A = Q \Sigma Q^T \qquad \text{rank}(A) = r \quad \Sigma \in \mathbb{R}^{r \times r}$.

$y = Q^T x \in \mathbb{R}^r$

$$x^T A x = y^T \Sigma y = \underline{\sum_{t=1}^{r} \sigma_t y_t^2} \quad \text{weighted } \chi^2$$

$$y = Q^T x \sim N(\underline{Q^T \mu}, \underline{I}) \qquad Var(y) = Q^T V Q = Q^T Q = I$$

$$y_t^2 \sim \chi_1^2\left((Q_t^T \mu)^2 / 2\right) \qquad Q = (Q_1\ Q_2 \ldots Q_r)$$

证 $AV = A$ idempotent rank s, $\sigma_1 = \cdots = \sigma_r = 1$

$r = s$.

MGF $u \sim \chi_p^2(\phi)$ $E(e^{tu}) = (1-2t)^{-p/2} \exp\left(\frac{2\phi t}{1-2t}\right)$

$$\prod_{t=1}^{r} E\left(e^{v \cdot \sigma_t y_t^2}\right) = \underline{\prod_{t=1}^{r}\left[(1-2v\sigma_t)^{-1/2} \exp\left(\frac{(Q_t^T \mu)^2 \cdot \sigma_t \cdot v}{1-2\sigma_t v}\right)\right]}$$

$$= \underline{(1-2v)^{-s/2} \exp\left(\frac{2\phi t}{1-2t}\right)} \qquad \forall v.$$

$\Rightarrow \sigma_1 \cdots \sigma_r = 1 \quad s = r.$

$$\exp\left[\frac{4\phi t}{1-2t} - \sum_{t=1}^{r} \frac{2(Q_t^T \mu)^2 \sigma_t v}{1-2\sigma_t v}\right] \qquad \Rightarrow \sigma_t = 1$$

$r = s.$

$$= \prod_{t=1}^{r} (1-2v\sigma_t)^{-1} (1-2v)^{s}$$

$\boxed{V \neq I}$ $$x^T A x = (V^{-1/2} x)^T \underbrace{V^{1/2} A V^{1/2}}_{A'} \underbrace{(V^{-1/2} x)}_{x'} \qquad x' \sim N(V^{-1/2}\mu, I)$$

$A' = V^{1/2} A V^{1/2}$ idempotent $\Rightarrow$ AV, VA idempotent.

$V = Q\Sigma Q^T$ $\quad V^{1/2} = Q\Sigma^{1/2}Q^T$

Cochran $\underline{A_1 + \cdots + A_k = I}$ $\in \mathbb{R}^n$

i) $A_i A_j = 0$ $\qquad A_i(A_1 + \cdots + A_k) = A_i \Rightarrow A_i^2 = A_i$

ii) $A_i^2 = A_i$

iii) $\text{rank}(A_1) + \cdots + \text{rank}(A_k) = n$ $\Leftrightarrow$ " $\sum s_i = N$ "

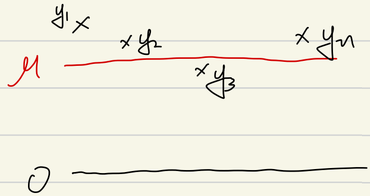
$A_i^2 = A_i$ $\quad A_i$ idempotent $\quad \text{rank}(A_i) = tr(A_i)$

$$tr(A_1 + \cdots + A_k) = tr(I) = n$$
$$= tr(A_1) + \cdots + tr(A_k)$$
$$= \text{rank}(A_1) + \cdots + \text{rank}(A_k)$$

"zero-way" ANOVA.

$$y_i = \mu + \varepsilon_i \qquad i = 1, \dots, N. \qquad E(\varepsilon_i) = 0 \qquad \varepsilon_i \sim N(0, \sigma^2)$$

$$\hat{\mu} = \frac{1}{n} \sum y_i = \bar{y}$$

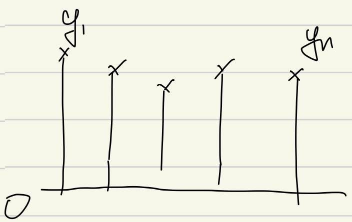


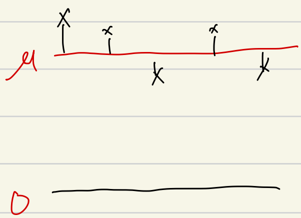
$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2 - \bar{y}^2$$

$$\frac{1}{n} \sum y_i^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 + \bar{y}^2 \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\underline{\frac{1}{n} y^T y} = \underline{\frac{1}{n} y^T (I - P_1) y} + \frac{1}{n} \underline{y^T P_1 y}. \qquad \text{"}\mu^2\text{"}$$

$$P_1 = \frac{1}{n} 1_n 1_n^T \qquad y^T P_1 y = \frac{1}{n} (1_n^T y)^2 = \frac{1}{n} \left(\sum_{i=1}^{n} y_i\right)^2 = n \cdot \bar{y}^2$$

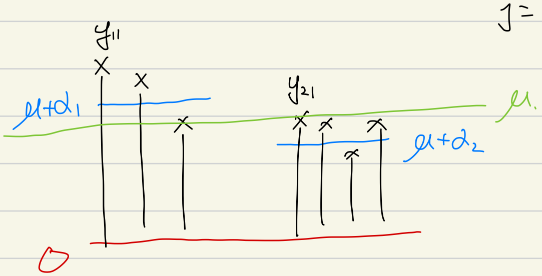




One way ANOVA.

$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$ $i = 1, \dots, a$ Group

$j = 1 \dots n_i$ obs.



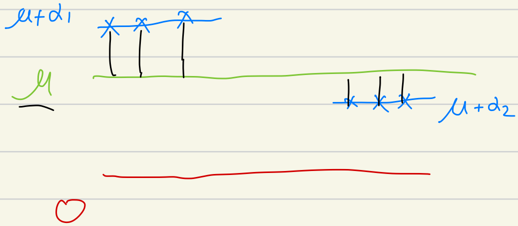
$$\frac{1}{N}\, y^T y$$

$\frac{1}{N} y^T A_1 y.$

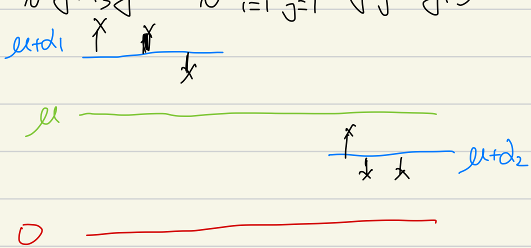
$\frac{1}{N} \bar{y}^2$ "μ"

$$\bar{y}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$$

$$\frac{1}{N} y^T A_2 y = \underline{\frac{1}{N} \sum_{i=1}^{a} \underline{n_i} (\bar{y}_i - \bar{y})^2} = ?$$



$$\frac{1}{N} y^T A_3 y = \frac{1}{N} \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$$



$X \in \mathbb{R}^{n\times p}$

$\operatorname{rank}(X) \le p$

$\hat\beta$ is solution to $\underline{X^TX\beta = X^Ty}$. $\quad X^TX$

$\hat\beta = (X^TX)^- X^Ty$.

$k^T\beta$ estimable $\Leftrightarrow k \in \mathcal{C}(X^T)$

$\Leftrightarrow k = X^T Q$

① $E(k^T\hat\beta) = k^T(X^TX)^-X^T \cdot X\beta$

$= Q^T \underbrace{X(X^TX)^-X^T}_{P_X} X\beta = Q^TX\beta = k^T\beta$

$P_X X = X$

② $Var(k^T\hat\beta) = k^T(X^TX)^-X^T \underbrace{Var(y)}_{\sigma^2 I} X(X^TX)^-k$

$= k^T \underbrace{(X^TX)^-}_{G_1} X^TX \underbrace{(X^TX)^-}_{G_2} k \times \sigma^2 \qquad P_X \cdot P_X$

$G_1 \quad G_1^T$

$= Q^T \underbrace{XG_1X^T}_{P_X} \cdot \underbrace{XG_2X^T}_{P_X} Q. \qquad k^T \in \mathbb{R}^{s\times p}$

$= Q^TP_XQ = Q^TXG_3X^TQ = k^T\underbrace{(X^TX)^-}_{G_3}k. \in \mathbb{R}^{s\times s}$

$AGA = A. \qquad A$ 对称 $A^T = A$.

$\Downarrow$

$A^TG^TA^T = A^T$

$\Downarrow$

$AG^TA = A$.

③ $\operatorname{rank}(k^T(X^TX)^-k) = s$.

k^Tb estimable $\Leftrightarrow \exists\, Q' \quad k = X^TQ'$

$\Leftrightarrow \exists\, Q \quad k = X^TXQ$

$$K = X^T Q' = X^T P_X Q' + \underbrace{X^T(I - P_X)Q'}_{0}$$

$$= X^T \underbrace{X(X^TX)^- X^T Q'}_{Q}.$$

$$Var(K^T\hat{b}) = K^T(X^TX)^- K = Q^T X^T \underbrace{X(X^TX)^- X^T}_{P_X} X Q.$$

$$= \underline{Q^T X^T} \underline{X Q}.$$

$$rank(K^T(X^TX)^- K) = rank(Q^T X^T X Q) = \underline{rank(XQ) = S}.$$

$$S = rank(K) = rank(\underline{X^T} \underline{X Q}) \le rank(XQ) \le S.$$

$$rank(AB) \le rank(B)$$
$$\le rank(A)$$

$$\underline{K^T\hat{b}} = K^T(X^TX)^- X^T y \sim N(K^Tb, \sigma^2 \underbrace{K^T(X^TX)^- K}_{H})$$

H_0 $\underline{K^T b = C}$ $\qquad$ H_1 $K^T b \ne C$.

$$\underline{K^T\hat{b} - C} \sim N(\underline{K^Tb - C}, \sigma^2 H)$$

~~? $\|K^T\hat{b} - C\|_2^2 > c$ reject H_0.~~

$$\underline{(K^T\hat{b} - C)^T (\sigma^2 H)^{-1} (K^T\hat{b} - C)} \sim \underline{\chi_S^2(\phi)} \qquad \text{"test 1"}$$

$$\underline{\phi = \frac{1}{2}(\underbrace{K^Tb - C}_{=0})(\sigma^2 H)^{-1}(K^Tb - C)}$$

Under $H_0 \Rightarrow K^T b - C = 0 \Rightarrow \phi = 0$

"test 2" $T = \dfrac{\frac{1}{\sigma^2}(K^T\hat{b} - c)^T H^{-1}(K^T\hat{b} - c)/s}{\frac{1}{\sigma^2}\frac{y^T(I-P_x)y}{n-r}} \sim F_{s,n-r}(\phi)$

$r = \text{rank}(X)$

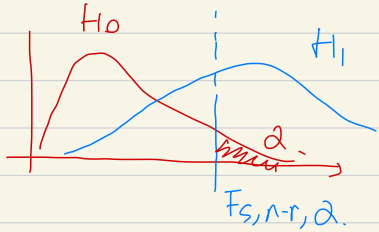
分母 $y^T(I-P_x)y = y^T(I-P_x)(I-P_x)y = f(\underbrace{(I-P_x)y}_{A})$

分子 $= g(K^T\hat{b}) = g(\underbrace{K^T(X^TX)^- X^T y}_{B})$

分子 ⊥ 分母 $\Leftarrow A \perp B \Leftarrow Cov(A, B) = 0$

$Cov(A, B) = \underbrace{(I-P_x)(\sigma^2 I)X}_{0}(X^TX)^- K = 0$

$T \sim F_{s,n-r}(\phi)$ $\quad H_0\ \phi = 0 \quad H_1\ \phi \neq 0$



reject H_0

Lemma 7.6. $S_1 = \{b : K^T b = c\}$ $S_2 = \{b : K_*^T b = c_*\}$

$S_1 = S_2 \Leftrightarrow \exists$ 可逆 Q $K_* = KQ$ $c_* = Q^T c$.

"$\Leftarrow$" $K^T b = c \Leftrightarrow Q^T K^T b = Q^T c \Leftrightarrow K_*^T b = c_*$.

"$\Rightarrow$" $K^T b = c \Leftrightarrow b = (K^T)^- c + (I - (K^T)^- K^T) z \quad \forall z$

$\mathbb{R}^{s \times p}$.

$S_1 = S_2$ $\quad K_*^T b = c_*$

$=0$

$K_*^T (K^T)^- c + K_*^T (I - (K^T)^- K^T) z = c_* \quad \forall z$.

Q^T

① $z = 0$

$K_*^T (K^T)^- c = c_*$.

$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

② $K_*^T (I - (K^T)^- K^T) = 0$

$\Rightarrow K_*^T = K_*^T (K^T)^- K^T$

Q^T

$K^T \; K_*^T \in \mathbb{R}^{s \times p}$

$(K^T)^- \in \mathbb{R}^{p \times s}$.

$Q \in \mathbb{R}^{s \times s}$.

③ $Q^T = K_*^T (K^T)^-$ full rank?

$s = \operatorname{rank}(K_*^T) = \operatorname{rank}(Q^T K^T) \le \underline{\operatorname{rank}(Q^T)} \le \operatorname{rank}(K_*^T) = s$

$$\text{Thm} \quad Q(\hat\beta_0) - Q(\hat\beta) = \underbrace{(k^T\hat\beta - c)}_{k^T\hat\beta - k^T\hat\beta_0}{}^T \left(k^T(X^TX)k\right)^{-1}(k^T\hat\beta - m)$$

(∗) $\hat\beta$: is solution to $X^TX\hat\beta = X^Ty$.

$\hat\beta_0$: is solution to $\begin{bmatrix} X^TX & k \\ k^T & 0 \end{bmatrix}\begin{bmatrix} \hat\beta_0 \\ \theta \end{bmatrix} = \begin{bmatrix} X^Ty \\ m \end{bmatrix}$ $\qquad k^T\hat\beta_0 = m.$

(∗∗) $\quad X^TX\hat\beta_0 + k\theta = X^Ty.$

(∗) − (∗∗) $\Rightarrow \quad X^TX(\hat\beta - \hat\beta_0) = k\theta$. (Δ)

$$Q(\hat\beta_0) - Q(\hat\beta) = \|y - X\hat\beta_0\|_2^2 - \|y - X\hat\beta\|_2^2$$

$$= (\cancel{y^Ty} - 2y^TX\hat\beta_0 + \hat\beta_0^TX^TX\hat\beta_0) - (\cancel{y^Ty} - 2y^TX\hat\beta + \hat\beta^TX^TX\hat\beta)$$

$$\overset{*}{=} -2\hat\beta^TX^TX\hat\beta_0 + \hat\beta_0^TX^TX\hat\beta_0 - 2\hat\beta^TX^TX\hat\beta + \hat\beta^TX^TX\hat\beta$$

$$= (\hat\beta - \hat\beta_0)^TX^TX(\hat\beta - \hat\beta_0)$$

$$\overset{(\Delta)}{=} \theta^Tk^T(\hat\beta - \hat\beta_0)$$

$$= \theta^T(k^T\hat\beta - k^T\hat\beta_0)$$

$$= \theta^T(k^T\hat\beta - m)$$

k^Tb estimable $\Leftrightarrow k \in \mathcal{C}(X^T) \Leftrightarrow \exists\, a \quad k = X^TXa.$

$a^T \times (\Delta) \Rightarrow \underbrace{a^TX^TX}_{k^T}(\hat\beta - \hat\beta_0) = a^Tk\theta. \qquad a = (X^TX)^-k.$

$$(k^T\hat\beta - m) = a^Tk\theta = k^T(X^TX)^-k\theta$$

$$\theta = [k^T(X^TX)^-k]^{-1}(k^T\hat\beta - m)$$

$y \sim N(Xb, \sigma^2 I)$ $\Lambda \in \mathbb{R}^{p \times s}$ $\text{rank}(\Lambda) = s$

$\hat{b}$ is solution to $X^T X b = X^T y$.

$$\Rightarrow \Lambda^T \hat{b} \sim N(\Lambda^T b, \sigma^2 \underline{\Lambda^T (X^T X)^- \Lambda})$$

可逆.

1-α level Confidence set $R(y)$: $P_\theta(\theta \in \underline{R(y)}) = 1 - \alpha$.

Fixed. Random

θ varies under H_0 H_1

Univariate Case. $\lambda \in \mathbb{R}^p$. $\lambda^T \hat{b} = \lambda^T (X^T X)^- X^T y \sim N(\lambda^T b, \sigma^2 \lambda^T (X^T X)^- \lambda)$

① σ^2 is known. $\dfrac{\lambda^T(\hat{b} - b)}{\sigma(\lambda^T X^T X \lambda)^{1/2}} \sim N(0,1)$ z_α : $P(Z > z_\alpha) = \alpha$.

$$\left| \frac{\lambda^T(\hat{b} - b)}{\sigma(\lambda^T X^T X \lambda)^{1/2}} \right| < z_{\alpha/2}$$

"$\hat{\sigma}$"

$$\lambda^T b \in \underline{\lambda^T \hat{b} \pm z_{\alpha/2} \cdot \sigma (\lambda^T X^T X \lambda)^{1/2}}$$

interval.

② σ^2 unknown. Noncentral t-dist. $v = \dfrac{x}{\sqrt{u/k}}$, $x \perp u$, $x \sim N(\varphi, 1)$

$u \sim \chi^2_k$

$$\Rightarrow v \sim t_k(\varphi)$$

i) $v \sim t_k(\varphi) \Rightarrow v^2 \sim F_{1,k}(\varphi^2/2)$

ii) $t_{k,\alpha/2}$ $P(v > t_{k,\alpha/2}) = \alpha/2$ $t^2_{k,\alpha/2} = F_{1,k,\alpha}$.

$$\frac{\hat{\sigma}^2}{\sigma^2} = \frac{1}{N-r} \cdot \frac{y^T(I - P_X)y}{\sigma^2} \sim \frac{1}{N-r} \chi^2_{N-r} \quad \perp \hat{b}$$

$$t = \frac{\cancel{\frac{1}{\sigma}}\ \lambda^T(\hat{b}-b)/(\lambda^T(X^TX)^-\lambda)^{1/2}}{\sqrt{\hat{\sigma}^2/\cancel{\sigma^2}}} \sim t_{N-r}$$

$$R(y) \quad \lambda_i^T\hat{b} \pm t_{N-r,\alpha/2}\,\hat{\sigma}\,(\underbrace{\lambda_i^T(X^TX)^-\lambda_i}_{h_{ii}})^{1/2} \qquad i=1,\dots,s.$$

$\mathrm{diag}(H) = \lambda_{11} \quad \lambda_{22} \quad \cdots \lambda_{ss}$

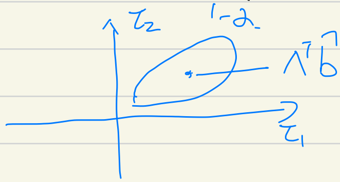
$$H = \Lambda^T(X^TX)^-\Lambda \qquad \frac{(\Lambda^T\hat{b}-\Lambda^Tb)\,H^{-1}(\overset{\hat{\tau}}{\Lambda^T\hat{b}}-\overset{\tau}{\Lambda^Tb})/s}{\hat{\sigma}^2} \sim F_{s,N-r}.$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

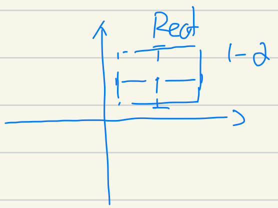
$$R(y) = \{\tau : (\tau-\Lambda^T\hat{b})H^{-1}(\tau-\Lambda^T\hat{b}) \le s\cdot\hat{\sigma}^2\,F_{s,N-r,\alpha}\}$$

$$X = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \\ \vdots & \vdots \end{bmatrix}$$

$\tau \in \mathbb{R}^2$



INTERPRETABLE



$P((\tau_1,\tau_2) \in \text{Rect}) = 1-\alpha.$

$\Lambda = [\lambda_1 \cdots \lambda_s] \in \mathbb{R}^{p\times s}.$

For $i=1,\dots,s$ $\quad I_i = [\lambda_i^T\hat{b} \pm t_{N-r,\alpha/2}\,\hat{\sigma}\sqrt{h_{ii}}]$

event $\quad A_i = \{\lambda_i^T b \in I_i\}. \quad P(A_i) = 1-\alpha.$

$$P(\Lambda^T b \in \text{Rect}) = P\left(\begin{pmatrix}\tau_1\\ \vdots \\ \tau_s\end{pmatrix} \in I_1 \otimes \cdots \otimes I_s\right)$$

$$= P\left(\bigcap_{i=1}^{s} A_i\right)$$

① $A_1 = \cdots = A_s$ $\quad P(A_1) = 1-\alpha.$

② $A_1 \perp A_2 \perp \cdots \perp A_s$ $\quad \prod P(A_i) = (1-\alpha)^s < 1-\alpha.$

Bonferroni t-interval.

$P(A \cup B) \le P(A) + P(B)$

$$P(\cap A_i) = 1 - P((\bigcap_{i=1}^{s} A_i)^c) = 1 - P(\bigcup_{i=1}^{s} A_i^c)$$

$$\ge 1 - \sum_{i=1}^{s} P(A_i^c) \ge 1-\alpha.$$

$\Rightarrow$ Set $P(A_i^c) = \alpha/s$.

$$\Rightarrow P\{\lambda_i^T b \in [\lambda_i^T \hat{b} \pm t_{N-r,\frac{\alpha}{2s}}\, \hat{\sigma}\sqrt{h_{ii}}]\ \ i=1,\dots,s\} \ge 1-\alpha.$$

Scheffe's Method.

$$\max \frac{(u^T v)^2}{u^T W u} = v^T W^{-1} v.$$

$\|u\|_W^2 = u^T W u \qquad \|v\|_{W^{-1}}^2 = v^T W^{-1} v.$

$$\Leftrightarrow \max_{\underbrace{\|u\|_W^2 \le 1}_{t}} (u^T v)^2 = \|v\|_{W^{-1}}^2$$

dual norm.

"=" $u \propto H^{-1} v.$

$$\underbrace{(\hat{\tau}-\tau)}_{v} H^{-1} (\hat{\tau}-\tau) \ge \frac{(u^T\overbrace{(\hat{\tau}-\tau)}^{v})^2}{u^T H u} \quad \forall u$$

$\|v\|_{H^{-1}}^2$

$$1-\alpha = P\left\{ \frac{(\hat{\tau}-\tau)H^{-1}(\hat{\tau}-\tau)/s}{\hat{\sigma}^2} \le F_{s,N-r,\alpha} \right\}$$

$$= P\left[\underbrace{\frac{1}{s\hat{\sigma}^2} \cdot \frac{(u^T(\hat{\tau}-\tau))^2}{u^T H u} \le F_{s,N-r,\alpha}}_{A_u},\ \forall u \right\} = P\left(\bigcap_{u \in \mathbb{R}^s} A_u\right)$$

$$= P\left(u^T \tau \in u^T \hat{\tau} \pm \sqrt{s\hat{\sigma}^2 F_{s,N-r,\alpha}}\ \ \forall u \right)$$

Turkey's Interval.

: s linearly independent $\tau_1 \cdots \tau_s$
least square est $\hat{\tau}_1 \cdots \hat{\tau}_s$

$l = \frac{s(s-1)}{2}$ all possible differences $\tau_i - \tau_j \ \forall i \neq j$

$$Var(\hat{\tau}_1 \cdots \hat{\tau}_s)^T = c^2\sigma^2 I. \quad \text{for some } c.$$

Studentized Range distribution.

$$q = \frac{\max z_i - \min z_i}{\sqrt{u/h}} \geq 0.$$

$z \perp u \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix} \sim N(0, I) \qquad u \sim \chi_h^2$

$q_{l,h,\alpha}$: $P(q \geq q_{l,h,\alpha}) = \alpha.$

$$1-\alpha = P\left\{ \frac{\max \frac{\hat{\tau}_i - \tau_i}{c\sigma} - \min \frac{\hat{\tau}_i - \tau_i}{c\sigma}}{\sqrt{\hat{\sigma}^2/\sigma^2}} \leq q_{s,N-r,\alpha} \right\}$$

$$= P\{ \underline{\max}(\hat{\tau}_i - \tau_i) - \underline{\min}(\hat{\tau}_i - \tau_i) \leq c\hat{\sigma}\, q_{s,N-r,\alpha} \}.$$

$$= P\{ |(\hat{\tau}_i - \tau_i) - (\hat{\tau}_j - \tau_j)| \leq c\hat{\sigma}\, q_{s,N-r,\alpha} \ \forall i,j \}.$$

$$= P\{ (\tau_i - \tau_j) \in (\hat{\tau}_i - \hat{\tau}_j \pm c\hat{\sigma}\, q_{s,N-r,\alpha}) \ \forall i,j \}$$

Unique Solution

$X \in \mathbb{R}^{n\times p}$ $\quad \mathrm{rank}(X) = r < p$

$$\begin{bmatrix} X^TX \\ C \end{bmatrix} b = \begin{bmatrix} X^Ty \\ 0 \end{bmatrix}$$

$Cb = 0$ $\quad C \in \mathbb{R}^{s\times p}$ $\quad \mathrm{rank}(C) = s = p - r.$

$\mathcal{C}(X^T) \cap \mathcal{C}(C^T) = \{0\}.$

$$\Leftrightarrow \begin{bmatrix} X \\ C \end{bmatrix} b = \begin{bmatrix} P_x y \\ 0 \end{bmatrix}$$

"$\Rightarrow$" 左乘 $X(X^TX)^-$

$$\underbrace{X(X^TX)^- X^T}_{P_x} Xb = \underbrace{X(X^TX)^- X^T}_{P_x} y.$$

$$Xb = P_x y.$$

"$\Leftarrow$" 左乘 X^T

$$X^T \underline{Xb} = X^T \underline{P_x y}.$$

$$\Leftrightarrow \begin{bmatrix} X^TX \\ C^TC \end{bmatrix} b = \begin{bmatrix} X^Ty \\ 0 \end{bmatrix} \begin{matrix} (1) \\ (2) \end{matrix}$$

$Cb = 0 \Leftrightarrow C^TCb = 0.$

$$C^TCb = 0 \Rightarrow \underbrace{b^TC^T}_{v^T} \underbrace{Cb}_{v} = 0 \Rightarrow \underset{\Leftarrow}{Cb = 0}.$$

$$v^Tv = \|v\|_2^2 = \|Cb\|_2^2 = 0$$

$$\Leftrightarrow (X^TX + C^TC) b = X^Ty.$$

$$\hat{b} = \underline{(X^TX + C^TC)^{-1}X^T} y.$$

"$\Rightarrow$" $(1) + (2)$

"$\Leftarrow$"

$$X^T \underbrace{(Xb - y)}_{=0} = \underbrace{C^TCb}_{0} = 0$$

$\in \mathcal{C}(X^T)$ $\qquad \in \mathcal{C}(C^T)$

One Way ANOVA. $\quad y_{ij} = \mu + \alpha_i + e_{ij}$ $\quad i = 1, \cdots, a.$

a Groups. $\qquad j = 1, \cdots, n_i$

$(\mu+1) + (\alpha_i - 1)$ $\quad$ Constraint $\underline{\sum n_i \alpha_i = 0}$ $\quad \hat{\mu}, \hat{\alpha}_i = ?$

$C = (0 \;\; n_1 \cdots n_a)$ $\quad \beta = \begin{bmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix}$ $\quad \underline{C\beta = 0}$

$$X^TX + C^TC = (X^T \ C^T)\begin{bmatrix} X \\ C \end{bmatrix} \qquad a=3$$

$$\left[\begin{array}{c} X \\ \hline C \end{array}\right] = \begin{array}{c} \\ \left[\begin{array}{cccc} 1_{n_1} & 1_{n_1} & & \\ 1_{n_2} & 0 & 1_{n_2} & \\ 1_3 & 0 & & 1_3 \\ \hline 0 & n_1 & n_2 & n_3 \end{array}\right] \end{array} \text{"}X\text{"}$$

$$\begin{array}{cc} & \quad\quad 1\times n_1 \\ X^TX + C^TC = & \begin{bmatrix} N & n_1 & n_2 & n_3 \\ n_1 & n_1 + n_1^2 & n_1 n_2 & n_1 n_3 \\ n_2 & n_1 n_2 & n_2 + n_2^2 & n_2 n_3 \\ n_3 & n_1 n_3 & n_2 n_3 & n_3 + n_3^2 \end{bmatrix} \end{array}$$

$n_1 \times n_1$

$$= \underbrace{\mathrm{diag}(N-1,\ n_1,\ n_2,\ n_3)}_{D} + \underbrace{\begin{bmatrix} 1 \\ n_1 \\ n_2 \\ n_3 \end{bmatrix}}_{v} \underbrace{(1 \ n_1 \ n_2 \ n_3)}_{v^T}$$

$$(X^TX + C^TC)^{-1} = D^{-1} - D^{-1} v \underbrace{(1 + v^T D^{-1} v)}_{\in \mathbb{R}}{}^{-1} v^T D^{-1} \qquad N = n_1 + n_2 + n_3$$

$$1 + v^T D^{-1} v = 1 + \frac{1}{N-1} + n_1 + n_2 + n_3 = N + 1 + \frac{1}{N-1}$$

(homework) . . . . . .

$$(X^TX + C^TC)^{-1} X^T = \begin{bmatrix} \frac{1}{N} 1_{n_1}^T & \frac{1}{N} 1_{n_2}^T & \frac{1}{N} 1_{n_3}^T \\ \frac{1}{n_1} 1_{n_1}^T & & \\ & \frac{1}{n_2} 1_{n_2}^T & \\ & & \frac{1}{n_3} 1_{n_3}^T \end{bmatrix} - \frac{1}{N} \begin{bmatrix} 0 \\ 1_N^T \\ 1_N^T \\ 1_N^T \end{bmatrix}$$

$$\hat{\beta} = (X^TX + C^TC)^{-1} X^T y = \begin{array}{cc} \begin{bmatrix} \bar{y} \\ \bar{y}_{1\cdot} - \bar{y} \\ \bar{y}_{2\cdot} - \bar{y} \\ \bar{y}_{3\cdot} - \bar{y} \end{bmatrix} & \begin{array}{c} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \end{array} \end{array}$$

Aitken Model.

$y \sim N(X\beta, \sigma^2 V)$ $\quad V$ pos. def.

ubiased linear estimator $c + a^T y$ of $\lambda^T \beta$.

$$E(c + a^T y) = c + a^T X\beta = \lambda^T \beta \quad \forall \beta.$$

$$\Rightarrow c = 0 \quad \lambda = \underset{p \times n}{\underline{\underline{X}}^T} a. \quad n > p. \quad a \text{ infinitely many.}$$

$\hat\beta_{GLS}$: solution to $X^T V^{-1} X \beta = X^T V^{-1} y$.

$\hat\beta_{GLS}$ is BLUE.

$\hat\beta_{OLS}$ solution to $X^T X \beta = X^T y$. $\quad$ $\hat\beta_{OLS}$ is BLUE of $\lambda^T\beta$

$t^T y$ is BLUE for $E(t^T y)$ $\forall t \in \mathcal{C}(X)$

To find BLUE, we find best a such that.

$$\min \ Var(a^T y) = \sigma^2 a^T V a.$$

$$\text{subject to } \left(E(a^T y) = a^T X \beta = \lambda^T \beta \, (\forall \beta)\right) \rightarrow \lambda = X^T a.$$

$$\Leftrightarrow \min \ \tfrac{1}{2} a^T V a.$$

$$\text{subject to } \lambda = X^T a.$$

Lagrangian. $L(a, l) = \frac{1}{2} a^T V a + l^T (X^T a - \lambda)$

$$\frac{\partial L}{\partial a} = Va + Xl = 0$$

$$\frac{\partial L}{\partial l} = X^T a - \lambda = 0$$

$$\Rightarrow \begin{bmatrix} V & X \\ X^T & 0 \end{bmatrix} \begin{bmatrix} a \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \quad (*)$$

Quiz 1. ii) $b^T y$ is another unbiased estimator of $\lambda^T \beta$.

$$Var(b^T y) = Var(\underline{a^T y}) + \underbrace{Var((b-a)^T y)}_{\geq 0}$$

$$\geq Var(a^T y)$$

Result: a is solution to (*) $\Rightarrow a^T y$ is BLUE.

$\Rightarrow \forall a^T X l = 0 \quad \forall a \in \mathcal{C}(X)$

(Note 5)

OLS: $\lambda^T \hat{\beta}_{OLS} = \underbrace{\lambda^T (X^T X)^- X^T}_{t^T} y$. is BLUE $\Leftrightarrow Vt \in \mathcal{C}(X)$

$$Var((b-a)^T y) = (b-a)^T \underset{\text{p.d.}}{V} (b-a) = 0 \Rightarrow V^{1/2}(b-a) = 0$$

$$\Rightarrow b-a = 0$$

$$a^T X^T X b = a^T X^T y$$

$$X^T V^{-1} X b = X^T V^{-1} y.$$

Note 5 $\lambda^T \hat{b}_{OLS}$ is BLUE

$\Leftrightarrow VX = XQ.$

$\Leftrightarrow X = V^{-1} X Q.$

$\Rightarrow X^T X = X^T V^{-1} X Q.$

$y^T X = y^T V^{-1} X Q$

$b^T y$ is unbiased of $\lambda^T \beta$.

$$b^T y = \underbrace{a^T y}_{\text{Best}} + \underbrace{(b-a)^T y}_{\text{estimating zero}}. \qquad Cov(a^T y, \text{zero}) = 0$$

$\Leftrightarrow a^T y$ best.

$$\Rightarrow Var(b^T y) = Var(a^T y) + Var((b-a)^T y) + \underbrace{2cov(a^T y, (b-a)^T y)}_{=0}$$

$\Rightarrow a^T y$ is BLUE.

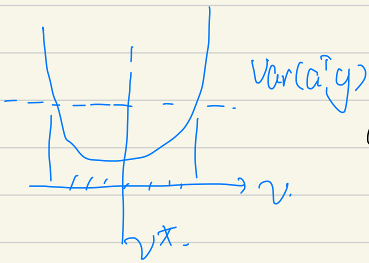
"$\Leftarrow$": $a^T y$ is BLUE $\Rightarrow Cov(a^T y, zero) = 0$

Suppose $Cov(a^T y, zero) \neq 0$

Construct another estimator $\underline{a^T y + v \cdot zero}$ ① v 待定 ② still unbiased.

Find v such that $Var(a^T y + v \cdot zero) < Var(a^T y)$?

$$\| \quad Var(a^T y) + 2v \cdot \underbrace{Cov(a^T y, zero)}_{\neq 0} + v^2 \cdot Var(zero)$$



$$\text{optimal } v^* = \frac{-Cov(a^T y, zero)}{Var(zero)} = \text{"} -\frac{b}{2a} \text{"}$$