Independence.

① $x \sim N_p(\mu, V)$ $\quad BVA = 0 \Rightarrow x^TAx$ 与 Bx 独立

$x_1 \perp x_2 \Rightarrow f(x_1) \perp g(x_2)$

$x^TAx = f(\overset{y}{A^{1/2}x})$ $\qquad A^{1/2}x \quad Bx \sim N(\_,\_)$

$= y^Ty.$ $\qquad Cov(A^{1/2}x, Bx) = 0$

A p.d $\quad x^TAx = (A^{1/2}x)^T(A^{1/2}x)$

A symmetric. $\quad A = Q\Sigma Q^T = (Q_1 \; Q_2)\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}Q_1^T\\ Q_2^T\end{bmatrix}$

$$x^TAx = [x^TQ_1 \; x^TQ_2]\begin{bmatrix}\Sigma_1 & \\ & 0\end{bmatrix}\begin{bmatrix}Q_1^Tx\\ Q_2^Tx\end{bmatrix}$$

$$= (Q_1^Tx)^T \underset{\uparrow \, R^{r\times r}}{\Sigma_1} (Q_1^Tx) \in \mathbb{R}^r \qquad r = rank(A)$$

② $x \sim N(\mu, V)$ $\quad x^TA_1x, x^TA_2x, \ldots, x^TA_mx$

$i \neq j \quad A_iVA_j = 0 \Rightarrow$ quadratic form independence.

$$A_i = \underset{R^{p\times r}}{Q_i}\, \underset{\uparrow R^{r\times r}}{\Sigma_i}\, Q_i^T \quad (A_i \in R^{p\times p}) \qquad rank(A_i) = r_i$$

$$Cov(Q_i^Tx, Q_j^Tx) = Q_i^TVQ_j = 0$$

Σ_i 可逆 $\quad Q_i^TQ_i = I$

$A_iVA_j = 0 \Rightarrow Q_i^TA_iVA_jQ_j = 0 \Rightarrow \Sigma_iQ_i^TVQ_j\Sigma_j = 0$

$\Rightarrow Q_i^TVQ_j = 0$

$Q_i \in \mathbb{R}^{p\times r}$ $(r<p)$ $\quad Q_i^TQ_i = I_r \not\Rightarrow \underset{rank(r)}{Q_iQ_i^T} \neq \underset{rank(p)}{I_p}$

$r = p \quad Q_i^T Q_i = I_p \Leftrightarrow Q_i Q_i^T = I_p$

EXAMPLE $\quad x_1, x_2, \cdots x_n \sim N(\mu, \sigma^2) \qquad 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

$$\hat{\mu} = \frac{1}{n} 1^T x = Bx \qquad B = \frac{1}{n} 1^T \in \mathbb{R}^{n \times 1}$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 = \underline{\frac{1}{n-1}}\, x^T \underline{\left(I - \frac{1}{n} 1 1^T\right)} x = x^T A x$$

$$A = \frac{1}{n-1}\left(I - \frac{1}{n} 1 1^T\right)$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \sim N(\underbrace{\mu 1}_{\vec{\mu}}, \underbrace{\sigma^2 I}_{V})$$

$$BVA = \frac{1}{n} 1^T (\sigma^2 I) \left[\frac{1}{n-1}\left(I - \frac{1}{n} 1 1^T\right)\right] = \frac{1}{n(n-1)} 1^T \left(I - \frac{1}{n} 1 1^T\right) = 0$$

$$1^T 1 = n.$$

$\Rightarrow \hat{\mu} \perp \hat{\sigma}^2$

③ $x \sim N(\mu, V)$ i) AV idempotent, rank $s \Rightarrow x^T A x \sim \chi_s^2\left(\frac{1}{2}\mu^T A \mu\right)$

ii) $x^T A x \sim \chi_s^2(\phi)$ for some $\phi \Rightarrow AV$ idempotent, rank s.

① $\underline{AV}$ idempotent, $\underline{VA}$? $\underline{V^{1/2} A V^{1/2}}$ idempotent?

$$VAVA = V\underline{AVAV}V^{-1} = V\underline{AV} \cdot V^{-1} = VA. \quad VA \text{ idempotent.}$$

$$(V^{1/2} A V^{1/2})\underline{(V^{1/2} A V^{1/2})} = V^{1/2} \underline{AVAV} V^{-1/2} = V^{1/2} AV V^{-1/2} = \underline{V^{1/2} A V^{1/2}}.$$

i) $x^T A x = (V^{-1/2} x)^T \underbrace{V^{1/2} A V^{1/2}}_{\text{idempotent. rank } s} (V^{-1/2} x) \qquad V^{-1/2} x \sim N(V^{-1/2}\mu, I)$

ii) $\boxed{V = I}$ $\quad \underline{A = Q \Sigma Q^T} \qquad \text{rank}(A) = r \quad \Sigma \in \mathbb{R}^{r \times r}.$

$$y = Q^T x \in \mathbb{R}^r$$

$$x^T A x = y^T \Sigma y = \underline{\sum_{t=1}^{r} \sigma_t y_t^2}$$ weighted χ^2

$y = Q^T x \sim N(\underline{Q^T \mu}, \underline{I})$ $\quad Var(y) = Q^T V Q = Q^T Q = I$

$y_t^2 \sim \chi_1^2\left(\frac{(Q_t^T \mu)^2}{2}\right) \quad Q = (Q_1\ Q_2 \ldots Q_r)$

证 $AV = A$ idempotent rank s, $\sigma_1 = \cdots = \sigma_r = 1$

$r = s$.

MGF $u \sim \chi_p^2(\phi)$ $E(e^{tu}) = (1-2t)^{-p/2} \exp\left(\frac{2\phi t}{1-2t}\right)$

$$\prod_{t=1}^{r} E\left(e^{v \cdot \sigma_t y_t^2}\right) = \underline{\prod_{t=1}^{r}\left[(1-2v\sigma_t)^{-1/2} \exp\left(\frac{(Q_t^T \mu)^2 \cdot \sigma_t \cdot v}{1-2\sigma_t v}\right)\right]}$$

$$= \underline{(1-2v)^{-s/2} \exp\left(\frac{2\phi t}{1-2t}\right)} \qquad \forall v.$$

$\Rightarrow \sigma_1 \cdots \sigma_r = 1 \quad s = r.$

$$\exp\left[\frac{4\phi t}{1-2t} - \sum_{t=1}^{r} \frac{2(Q_t^T \mu)^2 \sigma_t v}{1-2\sigma_t v}\right] \Rightarrow \sigma_t = 1$$

$r = s.$

$$= \prod_{t=1}^{r} (1-2v\sigma_t)^{-1} (1-2v)^{s}$$

$\boxed{V \neq I}$ $\quad x^T A x = (V^{-1/2} x)^T \underbrace{V^{1/2} A V^{1/2}}_{A'} \underbrace{(V^{-1/2} x)}_{x'} \qquad x' \sim N(V^{-1/2}\mu, I)$

$A' = V^{1/2} A V^{1/2}$ idempotent $\Rightarrow$ AV, VA idempotent.

$V = Q \Sigma Q^T \quad V^{1/2} = Q \Sigma^{1/2} Q^T$

Cochran $\underline{A_1 + \cdots + A_k = I} \in \mathbb{R}^n$

i) $A_i A_j = 0$ $\qquad A_i(A_1 + \cdots + A_k) = A_i \Rightarrow A_i^2 = A_i$

ii) $A_i^2 = A_i$

iii) $\text{rank}(A_1) + \cdots + \text{rank}(A_k) = n$ $\Leftrightarrow$ " $\sum s_i = N$ "

$A_i^2 = A_i \quad A_i$ idempotent $\quad \text{rank}(A_i) = \text{tr}(A_i)$

$\text{tr}(A_1 + \cdots + A_k) = \text{tr}(I) = n$

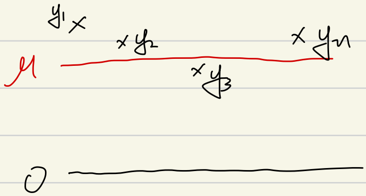
$= \text{tr}(A_1) + \cdots + \text{tr}(A_k)$

$= \text{rank}(A_1) + \cdots + \text{rank}(A_k)$

"zero-way" ANOVA.

$$y_i = \mu + \varepsilon_i \qquad i = 1, \dots, N. \qquad E(\varepsilon_i) = 0 \qquad \varepsilon_i \sim N(0, \sigma^2)$$

$$\hat{\mu} = \frac{1}{n} \sum y_i = \bar{y}$$

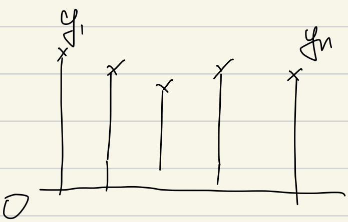


$$S^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2 - \bar{y}^2$$

$$\frac{1}{n} \sum y_i^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2 + \bar{y}^2 \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

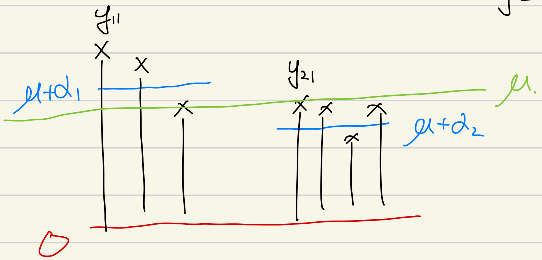
$$\underline{\frac{1}{n} y^T y} = \underline{\frac{1}{n} y^T (I - P_1) y} + \underline{\frac{1}{n} y^T P_1 y}.$$ "μ^2"

$$P_1 = \frac{1}{n} 1_n 1_n^T \qquad y^T P_1 y = \frac{1}{n} (1_n^T y)^2 = \frac{1}{n} \left(\sum_{i=1}^{n} y_i\right)^2 = n \cdot \bar{y}^2$$



One way ANOVA.

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad i = 1, \dots, a \text{ Group}$$
$$j = 1 \dots n_i \text{ obs.}$$



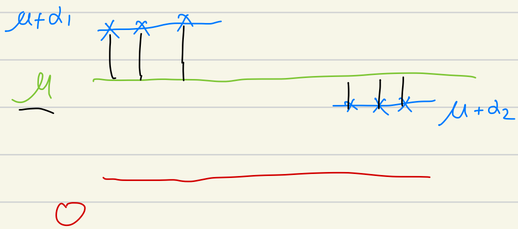
$\frac{1}{N} y^T y$

$\frac{1}{N} y^T A_1 y.$

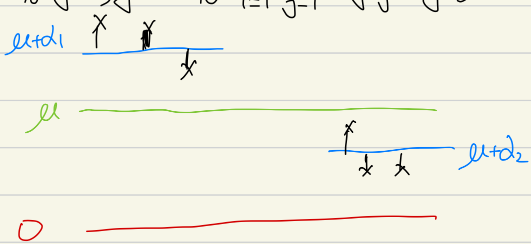
$\frac{1}{N} \bar{y}^2$ "μ"

$$\bar{y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} y_{ij}$$

$$\frac{1}{N} y^T A_2 y = \frac{1}{N} \sum_{i=1}^{a} n_i (\bar{y}_i - \bar{y})^2 = ?$$



$$\frac{1}{N} y^T A_3 y = \frac{1}{N} \sum_{i=1}^{a} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$$



$X \in \mathbb{R}^{n\times p}$.

$\text{rank}(X) \le p$

$\hat{\beta}$ is solution to $\underline{X^TX\beta = X^Ty}$. $\quad X^TX$

$\hat{\beta} = (X^TX)^- X^Ty$. $\quad k^T\beta$ estimable $\Leftrightarrow k \in \mathcal{C}(X^T)$

$\Leftrightarrow k = X^T Q$

① $E(k^T\hat{\beta}) = k^T(X^TX)^-X^T\cdot X\beta$

$= Q^T \underbrace{X(X^TX)^-X^T}_{P_X} X\beta = \underline{Q^TX}\beta = \underline{k^T}\beta$

$P_X X = X$

② $\text{Var}(k^T\hat{\beta}) = k^T(X^TX)^-X^T \underbrace{\text{Var}(y)}_{\sigma^2 I} X(X^TX)^-k$

$= k^T\underbrace{(X^TX)^-}_{G_1} X^TX \underbrace{(X^TX)^-}_{G_2} k \times \sigma^2 \qquad G_1^T \qquad P_X\cdot P_X$

$= Q^T \underbrace{XG_1X^T}_{P_X} \cdot \underbrace{XG_2X^T}_{P_X} Q. \qquad - \; k^T \in \mathbb{R}^{s\times p}$

$= Q^TP_XQ = Q^TXG_3X^TQ = k^T\underbrace{(X^TX)^-}_{G_3}k \; \in \mathbb{R}^{s\times s}$.

$AGA = A$. $\quad A$ 对称 $A^T = A$.

$\Downarrow$

$A^TG^TA^T = A^T$

$\Downarrow$

$AG^TA = A$.

③ $\text{rank}(k^T(X^TX)^-k) = s$.

k^Tb estimable $\Leftrightarrow \exists\, Q' \; k = X^TQ'$.

$\Leftrightarrow \underline{\exists\, Q \; k = X^TXQ}$.

$$k = X^T Q' = X^T P_X Q' + \underbrace{X^T (I - P_X) Q'}_{0}$$

$$= X^T X \underbrace{(X^T X)^- X^T Q'}_{Q}$$

$$Var(k^T \hat{b}) = k^T (X^T X)^- k = Q^T X^T \underbrace{X (X^T X)^- X^T}_{P_X} X Q$$

$$= Q^T X^T X Q$$

$$\text{rank}(k^T (X^T X)^- k) = \text{rank}(Q^T X^T X Q) = \text{rank}(XQ) = s$$

$$s = \text{rank}(k) = \text{rank}(X^T X Q) \le \text{rank}(XQ) \le s$$

$$\text{rank}(AB) \le \text{rank}(B)$$
$$\le \text{rank}(A)$$

$$k^T \hat{b} = k^T (X^T X)^- X^T y \sim N(k^T b, \sigma^2 \underbrace{k^T (X^T X)^- k}_{H})$$

H_0 $k^T b = c$ $\qquad$ H_1 $k^T b \ne c$

$$k^T \hat{b} - c \sim N(k^T b - c, \sigma^2 H)$$

~~? $\|k^T \hat{b} - c\|_2^2 > c$ reject H_0~~

$$(k^T \hat{b} - c)^T (\sigma^2 H)^{-1} (k^T \hat{b} - c) \sim \chi^2_s(\phi)$$ "test 1"

$$\phi = \frac{1}{2} (\underbrace{k^T b - c}_{=0}) (\sigma^2 H)^{-1} (k^T b - c)$$

Under $H_0 \Rightarrow K^T b - c = 0 \Rightarrow \phi = 0$

"test 2" $T = \dfrac{\frac{1}{\sigma^2}(K^T\hat{b} - c)^T H^{-1} (K^T\hat{b} - c)/s}{\frac{1}{\sigma^2}\frac{y^T(I-P_x)y}{n-r}} \sim F_{s,n-r}(\phi)$

$r = \mathrm{rank}(X)$

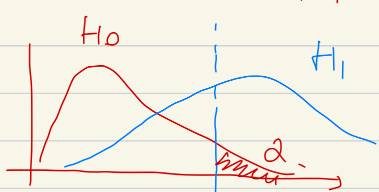
分母 $y^T(I-P_x)y = y^T(I-P_x)(I-P_x)y = f(\overset{A}{(I-P_x)y})$

分子 $= g(K^T\hat{b}) = g(\underbrace{K^T(X^TX)^- X^T y}_{B})$

分子 $\perp$ 分母 $\Leftarrow A \perp B \Leftarrow \mathrm{Cov}(A,B) = 0$

$$\mathrm{Cov}(A,B) = \underbrace{(I-P_x)(\sigma^2 I)}_{0} X (X^TX)^- K = 0$$

$T \sim F_{s,n-r}(\phi)$ H_0 $\phi = 0$ H_1 $\phi \neq 0$



$F_{s,n-r,\alpha}$

reject H_0

Lemma 7.6. $S_1 = \{b : K^T b = c\}$ $S_2 = \{b : K_*^T b = c_*\}$

$S_1 = S_2 \iff$ ∃可逆 Q $K_* = KQ$ $c_* = Q^T c$.

"⇐" $K^T b = c \iff Q^T K^T b = Q^T c \iff K_*^T b = c_*$.

"⇒" $K^T b = c \iff b = (K^T)^- c + (I - (K^T)^- K^T) z \quad \forall z$

$\mathbb{R}^{s\times p}$

$S_1 = S_2$ $K_*^T b = c_*$

$$K_*^T (K^T)^- c + \underbrace{K_*^T (I - (K^T)^- K^T) z}_{Q^T} = c_* \quad \forall z.$$

$=0$

① $z = 0$

$$K_*^T (K^T)^- c = c_*.$$

$z = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

② $K_*^T (I - (K^T)^- K^T) = 0$

$$\Rightarrow K_*^T = \underbrace{K_*^T (K^T)^-}_{Q^T} K^T$$

K^T $K_*^T \in \mathbb{R}^{s\times p}$

$(K^T)^- \in \mathbb{R}^{p\times s}$.

$Q \in \mathbb{R}^{s\times s}$.

③ $Q^T = K_*^T (K^T)^-$ full rank?

$$s = \operatorname{rank}(K_*^T) = \operatorname{rank}(Q^T K^T) \le \underline{\operatorname{rank}(Q^T)} \le \operatorname{rank}(K_*^T) = s$$

$$K^T\vec{\beta} - K^T\vec{\beta}_0$$

Thm $Q(\hat{\beta}_0) - Q(\hat{\beta}) = (K^T\hat{\beta} - c)^T\,(K^T(X^TX)K)^{-1}\,(K^T\hat{\beta} - m)$

(*) $\hat{\beta}$: is solution to $X^TX\hat{\beta} = X^Ty$.

$\hat{\beta}_0$: is solution to $\begin{bmatrix} X^TX & K \\ K^T & 0 \end{bmatrix}\begin{bmatrix} \hat{\beta}_0 \\ \theta \end{bmatrix} = \begin{bmatrix} X^Ty \\ m \end{bmatrix}$ $\quad K^T\hat{\beta}_0 = m$.

(**) $\quad X^TX\hat{\beta}_0 + K\theta = X^Ty$.

(*) − (**) $\Rightarrow$ $X^TX(\hat{\beta} - \hat{\beta}_0) = K\theta$. ($\Delta$)

$$Q(\hat{\beta}_0) - Q(\hat{\beta}) = \|y - X\hat{\beta}_0\|_2^2 - \|y - X\hat{\beta}\|_2^2$$

$$= (\cancel{y^Ty} - 2y^TX\hat{\beta}_0 + \hat{\beta}_0^TX^TX\hat{\beta}_0) - (\cancel{y^Ty} - 2y^TX\hat{\beta} + \hat{\beta}^TX^TX\hat{\beta})$$

$$\overset{*}{=} -2\hat{\beta}^TX^TX\hat{\beta}_0 + \hat{\beta}_0^TX^TX\hat{\beta}_0 - 2\hat{\beta}^TX^TX\hat{\beta} + \hat{\beta}^TX^TX\hat{\beta}$$

$$= (\hat{\beta} - \hat{\beta}_0)^TX^TX(\hat{\beta} - \hat{\beta}_0)$$

$$\overset{(\Delta)}{=} \theta^TK^T(\hat{\beta} - \hat{\beta}_0)$$

$$= \theta^T(K^T\hat{\beta} - K^T\hat{\beta}_0)$$

$$= \theta^T(K^T\hat{\beta} - m)$$

K^Tb estimable $\Leftrightarrow$ $K \in \mathcal{C}(X^T)$ $\Leftrightarrow$ $\exists\, Q$ $\;K = X^TXQ$.

$\Downarrow$

$Q^T \times (\Delta) \Rightarrow \underbrace{Q^TX^TX}_{K^T}(\hat{\beta} - \hat{\beta}_0) = Q^TK\theta$. $\qquad Q = (X^TX)^-K$.

$$(K^T\hat{\beta} - m) = Q^TK\theta = K^T(X^TX)^-K\theta$$

$$\theta = [K^T(X^TX)^-K]^{-1}(K^T\hat{\beta} - m)$$