Definition 2.19. *A square matrix* P *is said to be a projection onto the vector space* $\mathcal{V}$ *if and only if*

$\mathcal{V} = C(P)$

(i) $\mathbf{P}$ *is idempotent.*

(ii) $\mathbf{Px} \in \mathcal{V}$ *for any* $\mathbf{x}$. $\in C(P)$

(iii) $\mathbf{Px} = \mathbf{x}$ *for any* $\mathbf{x} \in \mathcal{V}$.

$Pp = a_1 Pp_1 + \cdots + a_n P p_n$
$= a_1 p_1 + \cdots + a_n p_n$
$= p$.

Corollary 2.20. *Any idempotent matrix is a projection onto its column space* $\mathcal{C}(\mathbf{P})$.

$p \in V \subset C(P)$

- i) $\Rightarrow$ ii), iii) 「If $V \leq C(P)$」 $\forall p \in C(P)$ $p = a_1 p_1 + \cdots + a_n p_n$.
- If $V = C(P)$, then (ii) is always true. ∴ iii) $\Rightarrow$ i).
  $\forall p_i$ $Pp_i = p_i$
  $PP = P$

If $V \neq C(P)$, According to ii). $V < C(P)$.

However, iii) $\not\Rightarrow$ i). $C(P) = V \oplus W$

$\{p_1 \cdots p_h\}$ $\{q_1, \cdots, q_l\}$ col vectors of P.

iii) only guarantees $P[p_1, \cdots, p_h] = [p_1, \cdots, p_h]$

But $P[q_1, \cdots, q_l] \neq [q_1, \cdots, q_l]$

e.g $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is a projection onto $\text{span}\{e_1, e_2, e_3\}$, not $C(P)$

To sum up, (ii) requires $V < C(P)$, (i) $\Leftrightarrow$ (iii)
$V = C(P)$.

M-P 伪逆 求法: ① SVD.

② $\exists P, Q$ s.t. $\underset{m\times m\ m\times n\ n\times n}{PAQ} = B = \begin{bmatrix} E_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_r \\ 0 \end{bmatrix} \begin{bmatrix} E_r & 0 \end{bmatrix}$

Then $A = \underbrace{P^{-1}\begin{bmatrix} E_r \\ 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} E_r & 0 \end{bmatrix} Q^{-1}}_{D} = CD$

$\underbrace{Q_m \cdots Q_1}_{Q} A \underbrace{P_1 P_2 \cdots P_n}_{P} \longrightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$A^+ = D^T(DD^T)^{-1}(C^TC)^{-1}C^T$

$(Q^{-1})^T[\quad]$

$DD^T = \begin{bmatrix} I_r & 0 \end{bmatrix} Q^{-1}(Q^{-1})^T \begin{bmatrix} I_r \\ 0 \end{bmatrix}$

Q^TQ

$(Q^TQ^T)^{-1}$

Theorem 3.9. $\mathbf{XX}^{\star} = \underbrace{\mathbf{X}(\mathbf{X}^{\intercal}\mathbf{X})^{-}\mathbf{X}^{\intercal}}_{P_x}$ *is symmetric and invariant of the choice of the generalized inverse* $(\mathbf{X}^{\intercal}\mathbf{X})^{-}$.

Since there is only one symmetric projection onto $\mathcal{C}(\mathbf{X})$, we can write such projection as $\mathbf{P}_{\mathbf{X}}$. From above, $\mathbf{P}_{\mathbf{X}} = \mathbf{XX}^{\star}$. Due to Corollary 2.27, this should coincide with $\mathbf{XX}^{+}$.

Corollary 2.27. $\mathbf{AA}^{+}$ *is the unique symmetric projector onto its* $\mathcal{C}(\mathbf{A})$.

Theorem 2.1 $\mathbf{P_X} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^g\mathbf{X}^T$ *is the projection matrix onto* $\mathcal{C}(\mathbf{X})$*, that is,* $\mathbf{P_X}$ *is*

(a) *idempotent*

(b) *projects onto* $\mathcal{C}(\mathbf{X})$,

(c) *invariant to the choice of generalized inverse,*

(d) *symmetric, and*

(e) *unique.*

$(X^TX)^gX^T = X^g$

Proof: Using Result 2.5, $\mathbf{P_X}$ can be written in the form $\mathbf{AA}^g$, so that from Result A.14 we know that $\mathbf{P_X}$ projects onto $\mathcal{C}(\mathbf{X})$, providing (a) and (b). For (c), let $\mathbf{G}_1$ and $\mathbf{G}_2$ be two generalized inverses of $(\mathbf{X}^T\mathbf{X})$, so

$$\triangle\ (\mathbf{X}^T\mathbf{X})\mathbf{G}_1(\mathbf{X}^T\mathbf{X}) = (\mathbf{X}^T\mathbf{X})\mathbf{G}_2(\mathbf{X}^T\mathbf{X}) = \mathbf{X}^T\mathbf{X}$$

and taking $\mathbf{A} = \mathbf{G}_1\mathbf{X}^T\mathbf{X}$, $\mathbf{B} = \mathbf{G}_2\mathbf{X}^T\mathbf{X}$ and applying Result 2.4 gives

$$\mathbf{XG}_1(\mathbf{X}^T\mathbf{X}) = \mathbf{XG}_2(\mathbf{X}^T\mathbf{X}).$$

$X^TXP = X^TXQ$

$\Downarrow$

$XP = XQ$

Now transpose this result to give

$$(\mathbf{X}^T\mathbf{X})\mathbf{G}_1^T\mathbf{X}^T = (\mathbf{X}^T\mathbf{X})\mathbf{G}_2^T\mathbf{X}^T$$

$X^T(XP - XQ) = 0$

and again apply Result 2.4 with $\mathbf{A} = \mathbf{G}_1^T\mathbf{X}^T$ and $\mathbf{B} = \mathbf{G}_2^T\mathbf{X}^T$ to obtain

$\in N(X^T)$

$\in C(X)$

$$\mathbf{XG}_1\mathbf{X}^T = \mathbf{XG}_2\mathbf{X}^T = \mathbf{P_X}.$$

Therefore, $\mathbf{P_X}$ is invariant to the choice of the generalized inverse of the matrix $(\mathbf{X}^T\mathbf{X})$. For symmetry, notice that if $\mathbf{G}_1$ is a generalized inverse of $\mathbf{X}^T\mathbf{X}$, so is $\mathbf{G}_1^T$ (see Exercise A.22); hence $\mathbf{P_X}$ is symmetric. Uniqueness then follows from Result A.16. ▯