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William C. Guenther <sup>a</sup>

<sup>a</sup> University of Wyoming , USA

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# ANOTHER DERIVATION OF THE NON-CENTRAL CHI-SQUARE DISTRIBUTION

WILLIAM C. GUENTHER

*University of Wyoming*

A brief review of derivations of the density function of the non-central chi-square is given. Another geometrical derivation based upon the properties of spherical coordinates is then presented. The method of proof requires very little knowledge of  $n$ -dimensional geometry and does not presume that the central chi-square distribution is available.

THE probability density function of the non-central chi-square distribution is derived a number of places in the literature. Fisher [3] obtained it indirectly as a limiting case of another distribution. Tang [11] gave an analytic derivation which has been reproduced by Mann [6, pp. 65–8] and improved slightly by Anderson [1, pp. 112–3]. Graybill [4, pp. 74–6] used the moment generating function technique which works out rather easily but is non-constructive. McNolty [7] has inverted the characteristic function by integrating around a contour in the complex plane. Bose [2], Patnaik [8], Quenouille [9], and Ruben [10] all give the same geometric derivation based upon properties of solid angles and hyperspheres. Ruben has added a second geometric approach in which the density function is obtained by dividing an  $n$ -dimensional sphere by a set of parallel hyperplanes. This note adds another geometrical derivation which is based upon the properties of spherical coordinates and requires very little detailed knowledge of  $n$ -dimensional geometry. The method used does not presume knowledge of the fact that the sum of the squares of independent standard normal random variables has a central chi-square distribution.

The distribution of

$$W^2 = \sum_{i=1}^n (Y_i - b_i)^2,$$

where the  $b_i$ 's are real constants, is non-central chi-square with  $n$  degrees of freedom and non-centrality parameter

$$\lambda^2 = \sum_{i=1}^n b_i^2$$

if the  $Y_i$  have independent standard normal distributions. The distribution function of  $W^2$  is

$$\begin{aligned} H(W^2; n, \lambda^2) &= \Pr \left[ \sum_{i=1}^n (Y_i - b_i)^2 \leq W^2 \right] \\ &= \int \cdots \int_{\sum_{i=1}^n (y_i - b_i)^2 \leq W^2} \frac{1}{(2\pi)^{n/2}} \exp \left( -\frac{1}{2} \sum_{i=1}^n y_i^2 \right) dy_1 \cdots dy_n. \end{aligned} \quad (1)$$

This is, of course, the integral of the  $n$ -dimensional standard normal distribution over a sphere of radius  $W$  with center at  $(b_1, \dots, b_n)$ . Due to the symmetry of the standard normal distribution (1) is equivalent to

$$\begin{aligned} & \int_{(y_1 - \lambda)^2 + \sum_{i=2}^n y_i^2 \leq W^2} \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^n y_i^2\right) dy_1 \cdots dy_n \\ &= \int \cdots \int_{\sum_{i=1}^n x_i^2 \leq W^2} \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n x_i^2 + 2\lambda x_1 + \lambda^2\right)\right] dx_1 \cdots dx_n. \end{aligned} \quad (2)$$

Now change to spherical coordinates, letting

$$\begin{aligned} x_1 &= \rho \cos \theta_1 \\ x_2 &= \rho \sin \theta_1 \cos \theta_2 \\ &\vdots \\ x_{n-1} &= \rho \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\ x_n &= \rho \sin \theta_1 \cdots \sin \theta_{n-1} \\ |J| &= \rho^{n-1} (\sin \theta_1)^{n-2} (\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2} \end{aligned}$$

so that (2) becomes

$$\int_0^W \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \frac{1}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(\rho^2 + \lambda^2 + 2\lambda\rho \cos \theta_1)\right] |J| d\theta_{n-1} \cdots d\theta_1 d\rho.$$

Integrating out  $\theta_2, \dots, \theta_{n-1}$  yields

$$\int_0^W \frac{\rho^{n-1} \exp\left[-\frac{1}{2}(\lambda^2 + \rho^2)\right]}{2^{(n-2)/2} \pi^{1/2} \Gamma\left(\frac{n-1}{2}\right)} \left[ \int_0^\pi (\sin \theta_1)^{n-2} \exp(-\lambda\rho \cos \theta_1) d\theta_1 \right] d\rho. \quad (3)$$

It is known [12, p. 79] that the integral in the bracket is equal to

$$\frac{\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\left(\frac{\lambda\rho}{2}\right)^{(n-2)/2}} I_{(n-2)/2}(\lambda\rho). \quad (4)$$

Substituting (4) in (3) gives

$$\begin{aligned} H(W^2; n, \lambda^2) &= \int_0^W \frac{\rho^{n/2}}{\lambda^{(n-2)/2}} I_{(n-2)/2}(\lambda\rho) \exp\left[-\frac{1}{2}(\lambda^2 + \rho^2)\right] d\rho \\ &= \int_0^{W^2} \frac{1}{2} \left[\frac{\rho}{\lambda}\right]^{(n-2)/2} I_{(n-2)/2}(\lambda\rho) \exp\left[-\frac{1}{2}(\lambda^2 + \rho^2)\right] d\rho^2. \end{aligned}$$

Consequently the density of  $W^2$  is

$$h(w^2; n, \lambda^2) = \frac{1}{2} \left( \frac{w}{\lambda} \right)^{(n-2)/2} I_{(n-2)/2}(\lambda w) \exp \left[ -\frac{1}{2}(\lambda^2 + w^2) \right]. \quad (5)$$

Alternatively, the integral in the bracket of (3) is equal to

$$\begin{aligned} \int_0^\pi \sum_{i=0}^\infty \frac{(-1)^i (\lambda \rho)^i}{i!} (\cos \theta_1)^i (\sin \theta_1)^{n-2} d\theta_1 \\ = \sum_{j=0}^\infty \frac{(\lambda^2 \rho^2)^j}{(2j)!} \int_0^\pi (\cos^2 \theta_1)^j (\sin \theta_1)^{n-2} d\theta_1. \end{aligned} \quad (6)$$

Since  $(2j)! = 2^{2j} j! \Gamma(j + \frac{1}{2}) / \Gamma(\frac{1}{2})$  and

$$\int_0^\pi (\cos^2 \theta_1)^j (\sin \theta_1)^{n-2} d\theta_1 = \frac{\Gamma\left(j + \frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2} + j\right)},$$

(3) reduces to

$$H(W^2; n, \lambda^2) = \int_0^W \frac{1}{2^{n/2}} \sum_{j=0}^\infty \frac{(\lambda^2)^j (\rho^2)^{(n/2)+j-1} \exp \left[ -\frac{1}{2}(\lambda^2 + \rho^2) \right]}{j! 2^{2j} \Gamma\left(\frac{n}{2} + j\right)} (2\rho d\rho)$$

so that

$$h(w^2; n, \lambda^2) = \frac{\exp \left[ -\frac{1}{2}(\lambda^2 + w^2) \right]}{2^{n/2}} \sum_{j=0}^\infty \frac{(\lambda^2)^j (w^2)^{(n/2)+j-1}}{j! 2^{2j} \Gamma\left(\frac{n}{2} + j\right)},$$

the well known series result.

Of course, if we are willing to assume that we already know

$$\sum_{i=2}^n Y_i^2 = \chi_{n-1}^2$$

is central chi-square with  $n-1$  degrees of freedom,

then (2) is  $\Pr[(Y_1 - \lambda)^2 + \chi_{n-1}^2 \leq W^2]$

$$= \int \int_{v^2 + u \leq W^2} \frac{u^{(n-3)/2} \exp(-\frac{1}{2}u)}{2^{(n-1)/2} \Gamma\left(\frac{n-1}{2}\right)} \frac{1}{(2\pi)^{1/2}} \exp \left[ -\frac{1}{2}(v + \lambda)^2 \right] dv du. \quad (7)$$

Letting  $u = \rho^2 \sin^2 \theta$ ,  $v = \rho \cos \theta$  converts (7) to (3). This is the method used by Tang [11] to complete his derivation. The use of spherical coordinates to derive the density of the central chi-square is not new and can be found in Kendall and Stuart [5, Sec. 11.2].

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