Independence.

① $x \sim N_p(\mu, V)$ $\quad BVA = 0 \Rightarrow x^TAx$ 与 Bx 独立

$x_1 \perp x_2 \Rightarrow f(x_1) \perp g(x_2)$

$x^TAx = f(\underbrace{A^{\frac{1}{2}}x}_{y})$ $\quad A^{\frac{1}{2}}x \quad Bx \sim N(\_,\_)$

$= y^Ty.$ $\quad Cov(A^{\frac{1}{2}}x, Bx) = 0$

A p.d $\quad x^TAx = (A^{\frac{1}{2}}x)^T(A^{\frac{1}{2}}x)$

A symmetric. $\quad A = Q\Sigma Q^T = (Q_1 \;\; Q_2)\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}$

$$x^TAx = [x^TQ_1 \;\; x^TQ_2]\begin{bmatrix} \Sigma_1 & \\ & 0 \end{bmatrix}\begin{bmatrix} Q_1^Tx \\ Q_2^Tx \end{bmatrix}$$

$$= (Q_1^Tx)^T \underset{\uparrow\, \mathbb{R}^{r\times r}}{\Sigma_1} (Q_1^Tx) \in \mathbb{R}^r \qquad r = rank(A)$$

② $x \sim N(\mu, V)$ $\quad x^TA_1x, x^TA_2x, \ldots, x^TA_mx$

$i \neq j \quad A_iVA_j = 0 \Rightarrow$ quadratic form independence.

$\mathbb{R}^{p\times p}$

$A_i = \underset{\mathbb{R}^{p\times r}}{Q_i}\; \underset{\uparrow\,\mathbb{R}^{r\times r}}{\Sigma_i}\; Q_i^T \qquad rank(A_i) = r_i$

$Cov(Q_i^Tx, Q_j^Tx) = Q_i^TVQ_j = 0$

Σ_i 可逆 $\quad Q_i^TQ_i = I$

$A_iVA_j = 0 \Rightarrow Q_i^TA_iVA_jQ_j = 0 \Rightarrow \Sigma_iQ_i^TVQ_j\Sigma_j = 0$

$\Rightarrow Q_i^TVQ_j = 0$

$Q_i \in \mathbb{R}^{p\times r}$ $(r<p)$ $\quad Q_i^TQ_i = I_r \not\Rightarrow \underset{rank(r)}{Q_iQ_i^T} \neq \underset{rank(p)}{I_p}$

$r = p \quad Q_i^T Q_i = I_p \Leftrightarrow Q_i Q_i^T = I_p$

EXAMPLE $x_1, x_2, \cdots x_n \sim N(\mu, \sigma^2)$ $\quad 1 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

$\hat{\mu} = \frac{1}{n} 1^T x = Bx \quad B = \frac{1}{n} 1^T \in \mathbb{R}^{n \times 1}$

$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 = \underline{\frac{1}{n-1}} x^T \underline{(I - \frac{1}{n} 1 1^T)} x = x^T A x$

$A = \frac{1}{n-1} (I - \frac{1}{n} 1 1^T)$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \sim N(\underbrace{\mu 1}_{\vec{\mu}}, \underbrace{\sigma^2 I}_{V})$

$BVA = \frac{1}{n} 1^T (\sigma^2 I) \left[\frac{1}{n-1} (I - \frac{1}{n} 1 1^T) \right] = \frac{1}{n(n-1)} 1^T (I - \frac{1}{n} 1 1^T) = 0$

$1^T 1 = n.$

$\Rightarrow \hat{\mu} \perp \hat{\sigma}^2$

③ $x \sim N(\mu, V)$ i) AV idempotent, rank $s \Rightarrow x^T A x \sim \chi_s^2(\frac{1}{2}\mu^T A \mu)$

ii) $x^T A x \sim \chi_s^2(\phi)$ for some $\phi \Rightarrow AV$ idempotent, rank s.

① $\underline{AV}$ idempotent, $\underline{VA}$? $\underline{V^{1/2} A V^{1/2}}$ idempotent?

$VAVA = V\underline{AVAV} V^{-1} = V\underline{AV} \cdot V^{-1} = VA.$ $\quad VA$ idempotent.

$(V^{1/2} A V^{1/2})(\underline{V^{1/2} A V^{1/2}}) = V^{1/2} \underline{AVAV} V^{-1/2} = V^{1/2} AV V^{-1/2} = \underline{V^{1/2} A V^{1/2}}.$

i) $x^T A x = (V^{-1/2} x)^T \underbrace{V^{1/2} A V^{1/2}}_{\text{idempotent. rank } s} (V^{-1/2} x) \quad V^{-1/2} x \sim N(V^{-1/2}\mu, I)$

ii) $V = I$ $\quad \underline{A = Q \Sigma Q^T}$ $\quad \text{rank}(A) = r \quad \Sigma \in \mathbb{R}^{r \times r}.$

$y = Q^T x \in \mathbb{R}^r$

$$x^T A x = y^T \Sigma y = \underline{\sum_{t=1}^{r} \sigma_t y_t^2}$$ weighted χ^2

$y = Q^T x \sim N(\underline{Q^T\mu}, \underline{I})$ $\quad Var(y) = Q^T V Q = Q^T Q = I$

$y_t^2 \sim \chi_1^2\left((Q_t^T\mu)^2/2\right)$ $\quad Q = (Q_1\ Q_2 \ldots Q_r)$

证 $AV = A$ idempotent rank s. , $\sigma_1 = \cdots = \sigma_r = 1$

$r = s$.

MGF $u \sim \chi_p^2(\phi)$ $\quad E(e^{tu}) = (1-2t)^{-p/2}\exp\left(\frac{2\phi t}{1-2t}\right)$

$$\prod_{t=1}^{r} E\left(e^{v \cdot \sigma_t y_t^2}\right) = \underline{\prod_{t=1}^{r}\left[(1-2v\sigma_t)^{-1/2}\exp\left(\frac{(Q_t^T\mu)^2 \cdot \sigma_t \cdot v}{1-2\sigma_t v}\right)\right]}$$

$$= \underline{(1-2v)^{-s/2}\exp\left(\frac{2\phi t}{1-2t}\right)} \quad \forall v.$$

$\Rightarrow \sigma_1 \cdots \sigma_r = 1 \quad s = r.$

$$\exp\left[\frac{4\phi t}{1-2t} - \sum_{t=1}^{r}\frac{2(Q_t^T\mu)^2\sigma_t v}{1-2\sigma_t v}\right] \Rightarrow \sigma_t = 1$$

$r = s$.

$$= \prod_{t=1}^{r}(1-2v\sigma_t)^{-1}\ (1-2v)^{s}$$

$\boxed{V \neq I}$ $\quad x^T A x = (V^{-1/2}x)^T \underbrace{V^{1/2} A V^{1/2}}_{A'} \underbrace{(V^{-1/2}x)}_{x'}$ $\quad x' \sim N(V^{-1/2}\mu, I)$

$A' = V^{1/2} A V^{1/2}$ idempotent $\Rightarrow$ AV, VA idempotent.

$V = Q \Sigma Q^T$ $\quad V^{1/2} = Q \Sigma^{1/2} Q^T$

Cochran $\underline{A_1 + \cdots + A_k = I}$ $\in \mathbb{R}^n$

i) $A_i A_j = 0$ $\qquad A_i(A_1 + \cdots + A_k) = A_i \Rightarrow A_i^2 = A_i$

ii) $A_i^2 = A_i$

iii) $\text{rank}(A_1) + \cdots + \text{rank}(A_k) = n$

$A_i^2 = A_i$ $\quad A_i$ idempotent $\quad \text{rank}(A_i) = \text{tr}(A_i)$

$$\begin{aligned} &\text{tr}(A_1 + \cdots + A_k) = \text{tr}(I) = n \\ &= \text{tr}(A_1) + \cdots + \text{tr}(A_k) \\ &= \text{rank}(A_1) + \cdots + \text{rank}(A_k) \end{aligned}$$