Beltrami flow

In fluid dynamics, **Beltrami flows** are flows in which the vorticity vector $\boldsymbol{\omega}$ and the velocity vector $\mathbf{v}$ are parallel to each other. In other words, Beltrami flow is a flow where Lamb vector is zero. It is named after the Italian mathematician Eugenio Beltrami due to his derivation of the Beltrami vector field, while initial developments in fluid dynamics were done by the Russian scientist Ippolit S. Gromeka in 1881.[1][2]

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Mathematical description

Since the vorticity vector $\boldsymbol{\omega}$ and the velocity vector $\mathbf{v}$ are parallel to each other, we can write

$$\boldsymbol{\omega} \times \mathbf{v} = 0, \quad \boldsymbol{\omega} = \alpha(\mathbf{x}, t)\mathbf{v},$$

where $\alpha(\mathbf{x}, t)$ is some scalar function. One immediate consequence of Beltrami flow is that it can never be a planar or axisymmetric flow because in those flows, vorticity is always perpendicular to the velocity field. The other important consequence will be realized by looking at the incompressible vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = \nu \nabla^2 \boldsymbol{\omega} + \nabla \times f,$$

where $\mathbf{f}$ is an external body forces such as gravitational field, electric field etc., and ν is the kinematic viscosity. Since $\boldsymbol{\omega}$ and $\mathbf{v}$ are parallel, the non-linear terms in the above equation are identically zero $(\mathbf{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{v} = 0$. Thus Beltrami flows satisfies the linear equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\omega} + \nabla \times f.$$

When $\mathbf{f} = 0$, the components of vorticity satisfies a simple heat equation.

Trkalian flow

Viktor Trkal considered the Beltrami flows without any external forces in 1919[3] for the scalar function $\alpha(\mathbf{x}, t) = c = \text{constant}$, i.e.,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \nabla^2 \boldsymbol{\omega}, \quad \boldsymbol{\omega} = c\mathbf{v}.$$

Introduce the following separation of variables

$$\mathbf{v} = e^{-c^2 \nu t} \mathbf{g}(\mathbf{x}),$$

then the equation satisfied by $\mathbf{g}(\mathbf{x})$ becomes

$$\nabla \times \mathbf{g} = c\mathbf{g}.$$

Berker's solution

Ratip Berker obtained the solution in Cartesian coordinates for $\mathbf{g}(\mathbf{x})$ in 1963,[4]

$$\mathbf{g} = \cos\left(\frac{cx}{\sqrt{2}}\right) \sin\left(\frac{cy}{\sqrt{2}}\right) \left[-\frac{1}{\sqrt{2}} \mathbf{e_x} + \frac{1}{\sqrt{2}} \mathbf{e_y} + \mathbf{e_z}\right].$$

Generalized Beltrami flow

The generalized Beltrami flow satisfies the condition[5]

$$\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0$$

which is less restrictive than the Beltrami condition $\mathbf{v} \times \boldsymbol{\omega} = 0$. Unlike the normal Beltrami flows, the generalized Beltrami flow can be studied for planar and axisymmetric flows.

Steady planar flows

For steady generalized Beltrami flow, we have $\nabla^2 \boldsymbol{\omega} = 0$, $\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0$ and since it is also planar we have $\mathbf{v} = (u, v, 0)$, $\boldsymbol{\omega} = (0, 0, \zeta)$. Introduce the stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \Rightarrow \quad \nabla^2 \psi = -\zeta.$$

Integration of $\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0$ gives $\zeta = -f(\psi)$. So, complete solution is possible if it satisfies all the following three equations

$$\nabla^2 \psi = -\zeta, \quad \nabla^2 \zeta = 0, \quad \zeta = -f(\psi).$$

A special case is considered when the flow field has uniform vorticity $f(\psi) = C = \text{constant}$. Wang (1991)[6] gave the generalized solution as

$$\zeta = \psi + A(x, y), \quad A(x, y) = ax + by$$

assuming a linear function for $A(x, y)$. Substituting this into the vorticity equation and introducing the separation of variables $\psi(x, y) = X(x)Y(y)$ with the separating constant λ^2 results in

$$\frac{d^2X}{dx^2} + \frac{b}{\nu}\frac{dX}{dx} - \lambda^2 X = 0, \quad \frac{d^2Y}{dy^2} - \frac{a}{\nu}\frac{dY}{dy} + \lambda^2 Y = 0.$$

The solution obtained for different choices of a, b, λ can be interpreted differently, for example, $a = 0$, $b = -U, \lambda^2 > 0$ represents a flow downstream a uniform grid, $a = -U$, $b = 0, \lambda^2 = 0$ represents a flow created by a stretching plate, $a = -U$, $b = U, \lambda^2 = 0$ represents a flow into a corner, $a = -V$, $b = -U, \lambda^2 = 0$ represents a Asymptotic suction profile etc.

Unsteady planar flows

Here,

$$\nabla^2\psi = -\zeta, \quad \frac{\partial\zeta}{\partial t} = \nabla^2\zeta, \quad \zeta = -f(\psi, t).$$

Taylor's decaying vortices

G. I. Taylor gave the solution for a special case where $\zeta = K\psi$, where K is a constant in 1923.[7] He showed that the separation $\psi = e^{-K\nu t}\Psi(x, y)$ satisfies the equation and also

$$\nabla^2\Psi = -K\Psi.$$

Taylor also considered an example, a decaying system of eddies rotating alternatively in opposite directions and arranged in a rectangular array

$$\Psi = A\cos\frac{\pi x}{d}\cos\frac{\pi y}{d}$$

which satisfies the above equation with $K = \dfrac{2\pi^2}{d^2}$, where d is the length of the square formed by an eddy. Therefore, this system of eddies decays as

$$\psi = A\cos\left(\frac{\pi x}{d}\right)\cos\left(\frac{\pi y}{d}\right)e^{-\frac{2\pi^2}{d^2}\nu t}.$$

Steady axisymmetric flows

Here we have $\mathbf{v} = (u_r, 0, u_z)$, $\boldsymbol{\omega} = (0, \zeta, 0)$. Integration of $\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = 0$ gives $\zeta = rf(\psi)$ and the three equations are

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial z^2} = -\zeta, \quad \nabla^2\zeta = 0, \quad \zeta = rf(\psi)$$

The first equation is the Hicks equation. Marris and Aswani (1977)[8] showed that the only possible solution is $f(\psi) = C = \mathbf{constant}$ and the remaining equations reduce to

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + Cr^2 = 0$$

A simple set of solutions to the above equation is

$$\psi(r, z) = c_1 r^4 + c_2 r^2 z^2 + c_3 r^2 + c_4 r^2 z + c_5 \left(r^6 - 12 r^4 z^2 + 8 r^2 z^4 \right), \quad C = -\left(8 c_1 + 2 c_2 \right)$$

$c_1, c_4 \neq 0, \ c_2, c_3, c_5 = 0$ represents a flow due to two opposing rotational stream on a parabolic surface, $c_2 \neq 0, c_1, c_3, c_4, c_5 = 0$ represents rotational flow on a plane wall, $c_1, c_2, c_3 \neq 0, \ c_4, c_5 = 0$ represents a flow ellipsoidal vortex (special case – Hill's spherical vortex), $c_1, c_3, c_5 \neq 0, \ c_2, c_4 = 0$ represents a type of toroidal vortex etc.

The homogeneous solution for $C = 0$ as shown by Berker[9]

$$\psi = r \left[A_k J_1 (kr) + B_k Y_1 (kr) \right] \left(C_k e^{kz} + D_k e^{-kz} \right)$$

where $J_1(kr), Y_1(kr)$ are the Bessel function of the first kind and Bessel function of the second kind respectively. A special case of the above solution is Poiseuille flow for cylindrical geometry with transpiration velocities on the walls. Chia-Shun Yih found a solution in 1958 for Poiseuille flow into a sink when $C = 2, \ c_1 = -1/4, \ c_3 = 1/2, \ c_2 = c_4 = c_5 = B_k = C_k = 0$.[10]

See also

- Gromeka–Arnold–Beltrami–Childress (GABC) flow

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