BIMCT Team Round Solutions

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October 2022

Team Round

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1. What is the coefficient of the $(xyz)^3$ term in the expansion of $(x + 2y + 3z)^9$?

   Solution: $\boxed{362880}$

   In the expansion of $(x + 2y + 3z)^9$, each $(xyz)^3$ term will have a coefficient of $1^3 \cdot 2^3 \cdot 3^3$, so now we find how many such terms there are. Notice that we will get an $(xyz)^3 = x^3y^3z^3$ term whenever we choose x from 3 of the 9 factors, and choose $2y$ from 3 of the remaining 6 factors, which leaves a unique way to choose the three z's; this will occur in $\binom{9}{3}\binom{6}{3}$ ways. Hence, the answer is $2^3 3^3 \binom{9}{3}\binom{6}{3} = \boxed{362800}$.
   Solution by Felix Liu

2. Let ABC be a triangle such that $AB = 5,\ AC = 6,$ and $BC = 7$. Let D be the foot of the altitude from A to BC, and let E be the midpoint of AD. Define E' as the reflection of E across C. If the area of $\triangle BEE'$ can be expressed as $a\sqrt{b}$, for an integer a and squarefree b, what is $a + b$?

   Solution: $\boxed{12}$

   By Heron's Formula, the area of the triangle is $6\sqrt{6}$. We claim that the area of ΔABC is equal to the area of $\Delta BEE'$. Note that $\Delta BEE'$ is in fact two stacked triangles with common base BC, and since E' is the reflection of E across C, both these triangles will have the same height of $AD/2$. Hence, the sum of the areas of these two triangles is equal to the area of ΔABC, which is $6\sqrt{6}$. This yields the answer of $6 + 6 = 12$. *Solution by Felix Liu*

3. An evil number is a positive integer that has an even sum of digits when expressed in binary. Find the number of evil numbers less than 2021.

   Solution: $\boxed{1009}$

   Consider the sum of the digits in the binary representation of any number when we exclude the ones digit. Then, the ones digit has two options: 0 or 1. Notice that the sums of these two different numbers include one odd sum and one even sum, meaning that every binary representation can be paired with another binary representation that differs only in the units digit and differs in parity. The only exceptions are at the endpoints: 1 and 2020, which is 11111100100. Therefore, our answer is $(2020 - 2)/2 = \boxed{1009}$.
   Solution by Derrick Liu

4. Let a, b, and c be the 3 solutions to the equation $x^3 + 9x^2 - 98x + 27 = 0$. Find the sum $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$.

Solution: $\boxed{0}$

Solution 1 We begin by adding $125x$ to both sides of the equation:

$$x^3 + 9x^2 + 27x + 27 = 125x$$

Notice that the left hand side of the equation is the expansion of $(x+3)^3$, and taking the cube root of both sides yields:

$$x + 3 = 5\sqrt[3]{x}$$

Since a, b, and c satisfy the original equation, they must satisfy this one as well. Now we just plug them in and we get:

$$a + 3 = 5\sqrt[3]{a}$$
$$b + 3 = 5\sqrt[3]{b}$$
$$c + 3 = 5\sqrt[3]{c}$$

Adding these 3 equations and rearranging, we get:

$$\frac{a+b+c+9}{5} = \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$$

By Vieta's formulas, $a+b+c = -9$, so $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} = \mathbf{0}$. *Solution by Ary Cheng*

Solution 2 From vieta's equations, we get that $a+b+c = -9$, and $abc = -27$. If we set $x = \sqrt[3]{a}$, $y = \sqrt[3]{b}$, and $z = \sqrt[3]{c}$, then $x^3 + y^3 + z^3 = -9$, and $xyz = -3$. Thus, $x^3 + y^3 + z^3 = 3xyz$, so $(x+y+z)(x^2+y^2+z^2-xy-xz-yz) = 0$. By rearrangement inequality, since x, y, and z are not all the same, $x^2+y^2+z^2 > xy+xz+yz$, so $x+y+z = 0$. *Solution by Larry Xing*

5. For how many many ordered pairs of (not necessarily distinct) positive integers $1 \le a, b \le 10000$ is $ab \equiv 2022 \pmod{10000}$?

Solution: $\boxed{8000}$

We perform casework on a.

If a is relatively prime to 10000 ($\phi(10000) = 4000$ cases), there is exactly one solution for b, which is $2022a^{-1} \pmod{10000}$. This gives 4000 total cases.

If $\gcd(a, 10000) = 2$, then we may write $a = 2m$, where $1m \le 5000$ and m is relatively prime to 5000. There are $\phi(5000) = 2000$ such cases, and there are 2 solutions for b (separated by 5000) for each of these a. This gives an additional $2000 \cdot 2$=4000 cases.

If $\gcd(a, 10000) > 2$, $ab \pmod{10000}$ will be divisible by something that 2022 is not, so there are no more cases. Therefore, our answer is then $\boxed{8000}$. *Solution by Adam Tang*

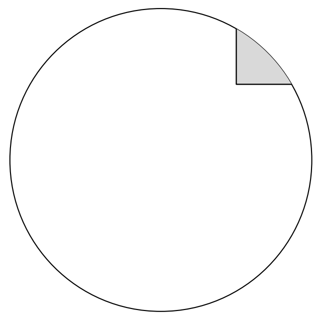
6. Real numbers x and y are chosen uniformly at random from the interval $[-1, 1]$. If the probability that they satisfy:

$$x^2 + 2|x| + y^2 + 2|y| \le 2$$

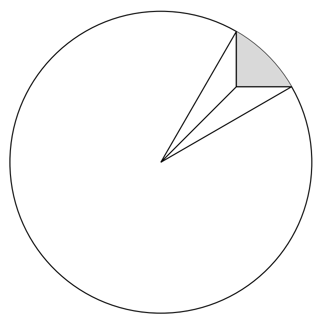
is p, p can be expressed as $\frac{a\pi}{b} + c\sqrt{d} + e$, where a, b, c, d, and e are integers, $\gcd(a, b) = 1$, and d is not divisible by the square of any prime number. Find $a+b+c+d+e$.

Solution: $\boxed{7}$

Notice that we can complete the square to see that $(|x| + 1)^2 + (|y| + 1)^2 \leq 4$. If x and y are both positive, then the solution set lies in the union of a circle centered at (-1, -1) with radius 2 and the first quadrant. The solution set looks something like this:



We can calculate the area of the shaded region by noticing that it is the area of an arc minus the area of two congruent triangles, as such:



Additionally, notice that the arc is a 30 degree angle. The area of the triangles is $2\cdot(\frac{1}{2}\cdot 1\cdot(\sqrt{3}-1)) = \sqrt{3}-1$, and the area of the arc is $\frac{1}{12} \cdot 4\pi$, so the area of the shaded area is $\frac{\pi}{3} - \sqrt{3} + 1$.

A similar argument can be applied to the other 3 cases (x positive, y negative; x negative, y positive; and x negative, y negative). Therefore, the total area of the solution set is $\frac{4\pi}{3} - 4\sqrt{3} + 4$, so the probability is $\frac{\pi}{3} - \sqrt{3} + 1$. Our answer is then $1 + 3 - 1 + 3 + 1 = \boxed{7}$. *Solution by Derrick Liu*

7. Real numbers w, x, y, and z satisfy $wxyz = 36$. Over all combinations of w, x, y, and z, what is the minimum value of $6w^2 + 3x^4 + y^{12} + 2z^6$?

Solution: $\boxed{432}$

In order to solve this problem, we have to utilize a clever trick of AM-GM. If we rewrite the problem as the minimum value of $w^2 + w^2 + w^2 + w^2 + w^2 + w^2 + x^4 + x^4 + x^4 + y^{12} + z^6 + z^6$, then using AM-GM, we see that $\frac{w^2+w^2+w^2+w^2+w^2+w^2+x^4+x^4+x^4+y^{12}+z^6+z^6}{12} \geq \sqrt[12]{w^{12}x^{12}y^{12}z^{12}} = wxyz = 36$. Therefore, the minimum value of $6w^2 + 3x^4 + y^{12} + 2z^6$ is $36 \cdot 12 = \boxed{432}$. *Solution by Derrick Liu*

8. Evan has a paper icosahedron. He fully cuts the icosahedron along a series of edges, laying it out on a flat table all in one piece. How many edges did he cut? Recall that an icosahedron is a regular polyhedron with equilateral triangular faces, with 20 faces, 30 edges, and 12 vertices.

Solution: $\boxed{11}$

All equilateral triangle cycles (closed loops of equilateral triangles connected by edges) on an icosahedron are not flat, so the resulting paper cannot have any cycles. Thus, the graph of the equilateral triangles (vertices on faces, edges between touching faces) after a valid cutting forms a tree. Since there are 20 faces, we have 19 remaining edges. A pentagon has 30 edges, so Evan must have cut 30-19=$\boxed{11}$ edges. *Solution by Adam Tang*

9. Let x and y be numbers such that

$$x + y = \sqrt{x^2 - 8x + 19} + \sqrt{y^2 - 8y + 19} = \sqrt{x^2 - 24x + 147} + \sqrt{y^2 - 24y + 147}$$

Then, $\max(x, y) = a + \sqrt{b}$, where a and b are integers. Find $a + b$.

Solution: $\boxed{68}$

Solution 1 By completing the square, we obtain

$$x + y = \sqrt{(x-4)^2 + 3} + \sqrt{(y-4)^2 + 3} = \sqrt{(x-12)^2 + 3} + \sqrt{(y-12)^2 + 3}.$$

Let $f(x) = \sqrt{x^2 + 3}$, which is an even function. Using this yields the equality $f(x-4) + f(y-4) = f(x-12) + f(y-12) = f(12-y) + f(12-x)$, which motivates us to set $x - 4 = 12 - y$ or $y - 4 = 12 - x$ to take advantage of the evenness of f. Both of these yield $x + y = 16$. From here, we can substitute $y = 16 - x$ to obtain the equation

$$16 = \sqrt{(x-4)^2 + 3} + \sqrt{(12-x)^2 + 3}.$$

To make solving easier, use the substitution $x = z + 8$, which yields the equation

$$16 = \sqrt{(z+4)^2 + 3} + \sqrt{(z-4)^2 + 3},$$

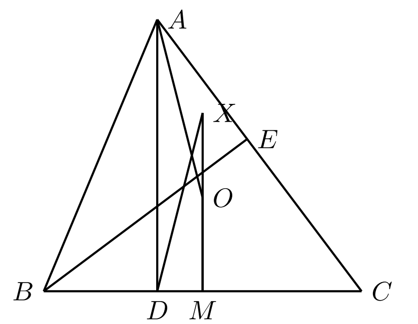
which rearranges into $3z^2 = 16^2 - 4 \cdot 19$ or $z = \pm\sqrt{60}$; one way to do this is to move a radical to one side then square both sides of the resulting equation. This means that $x = 8 \pm \sqrt{60}$ and $y = 8 \mp \sqrt{60}$, which implies the answer is $60 + 8 = \boxed{68}$. *Solution by Felix Liu*

Solution 2 Once we complete the square, we notice that if $(x, 0)$ and $(y, 0)$ are the foci of an ellipse, the equation means that $(0, 0)$, $(4, \sqrt{3})$, and $(12, \sqrt{3})$ are on the ellipse. Now, we want to "unstretch" these three points to make them on a circle, and it turns out that if we just divide each x coordinate by 4, we end up with a circle of radius 2 centered at $(2, 0)$. Then, we can just stretch the ellipse back, and we would get that $xy = 4$ and $x + y = 16$. Solving, we get $x, y = 8 \pm \sqrt{60}$, so our answer is $8 + 60 = \boxed{68}$. *Solution by Larry Xing*

10. Let acute triangle ABC be such that $AB = 13$, $AC = 15$, and $BC = 14$. Let D be the foot of the altitude from A, let M be the midpoint of BC, and let O be the circumcenter of $\triangle ABC$. Then, let X be the unique point within $\triangle ABC$ such that X is on line OM and $\overline{AO} = \overline{DX}$. If E is the foot of the altitude from B, XE can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Find $a + b$.

Solution: $\boxed{131}$

The diagram looks as such:



The circumradius of triangle ABC can be found using the formula $A = \frac{abc}{4R}$, where R is the circumradius and a, b, c are the sidelengths. The area, by Heron's formula is $\sqrt{21(7)(6)(8)} = 84$, so $R = \frac{abc}{4A} = \frac{13(14)(15)}{4(84)} = \frac{65}{8}$. Let point $B = (0,0)$, $A = (5,12)$, and $C = (14,0)$. Then, $D = (5,0)$, $M = (7,0)$, and $O = (7, \frac{33}{8})$. To find point E, we notice that the slope of AC is $\frac{-4}{3}$, so the slope of BE is $\frac{3}{4}$. Therefore, $\frac{3x}{4} = \frac{-4(x-14)}{3}$, so $x = \frac{224}{25}$ and $y = \frac{168}{25}$. Because $DX = AO = \frac{65}{8}$, $X = (7, \frac{63}{8})$. Therefore, $XE = \sqrt{(7 - \frac{224}{25})^2 + (\frac{63}{8} - \frac{168}{25})^2} = \frac{91}{40}$, so our final answer is $91 + 40 = \boxed{131}$. *Solution by Derrick Liu*

11. Let a sequence a_n be defined by $a_{n+2} = 9a_n - 3a_{n+1}$. If $a_{2021} = 1$ and $a_{2022} = 2$, $\sum_{n=0}^{2020} a_n$ can be expressed as $a.bc\cdots$. Find $100a + 10b + c$.

Solution: $\boxed{119}$

Let's start by reversing this sequence. If $a_0 = 2$ and $a_1 = 1$, then $a_n = \frac{1}{3}a_{n-1} + \frac{1}{9}a_{n-2}$. Then, if $b_n = 3^n a_n$, then $b_n = b_{n-1} + b_{n-2}$, and $b_0 = 2$ and $b_1 = 3$. Thus, $b_i = F_{i+3}$. So $a_i = \frac{F_{i+3}}{3^i}$. If we recall the generating function for the Fibonacci numbers, we find that $\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + \cdots$. Thus, $\sum a_i = (\frac{\frac{1}{3}}{1-\frac{1}{3}-\frac{1}{9}} - \frac{1}{3} - \frac{1}{9}) \cdot 27 - 2 - 1 = \frac{21}{5} - 3 = 1.2$

Since the desired value is slightly less, our answer is $\boxed{119}$. *Solution by Larry Xing*

12. Two identical regular hexagons are inscribed within a rectangle with side lengths 116 and $36\sqrt{3}$ without overlapping. If the maximum possible area of one of the hexagons is can be expressed as $a\sqrt{b}$, find $a + b$.

Solution: $\boxed{2325}$

To begin, we draw a quadrilateral between the two main diagonals of the hexagons. If the side length of a hexagon is x, the area of this quadrilateral can be expressed two different ways as $36\sqrt{3}y = 2\sqrt{3}x^2$, where y is the length of the side between the two main diagonals. Solving, we get $y = \frac{x^2}{18}$.

Now, we can get rid of this quadrilateral so that we are left with a rectangle of length and width $116 - \frac{x^2}{18}$ and $36\sqrt{3}$. Now, our inscribed figure is just a hexagon with side length x. We can finish this problem with complex numbers.

We have two equations, $\omega \cdot (p - 18\sqrt{3}i) = 58 - \frac{x^2}{36} - qi$, and $x^2 = (18\sqrt{3})^2 + p^2$ (p, q are the lengths along the sides). Solving, we get $p = 62 - \frac{x^2}{18}$, so $x^2 = 972 + (62 - \frac{x^2}{18})^2$. We can solve a quadratic in x^2 to get $x^2 = 1548$, so the area of one hexagon is $\frac{3\sqrt{3}}{2}x^2 = 2322\sqrt{3}$, so our answer is $2322 + 3 = \boxed{2325}$ *Solution by Larry Xing*

13. Let a list s of $2k$ integers be considered $k - pretty$ if it has median k, mode k, mean k, and standard deviation k. If S is the set of all $2023 - pretty$ lists, what is $E(x^2)$, where $x \in s$ and $s \in S$ is chosen uniformly at random between all lists within S? (Note that the set S is finite, since the standard deviation puts an upper bound on the largest value within s).

Solution: $\boxed{8185048}$

Let's first experiment around with $E(x^2)$ and try to find it in terms of $\bar{x}$. First, note that

$$E(x^2) = E(((x - \bar{x}) + \bar{x})^2)$$

Then, by linearity of expectation, we can expand to arrive at

$$E((x - \bar{x})^2 + 2(x - \bar{x})\bar{x} + \bar{x}^2) = E((x - \bar{x})^2) + E(2(x - \bar{x})\bar{x}) + E(\bar{x}^2)$$

Because $\bar{x}$ is a constant, we can take out $2\bar{x}$ from the middle term to arrive at

$$E((x - \bar{x})^2) + 2\bar{x}E(x - \bar{x}) + \bar{x}^2$$

Because $\bar{x}$ (by definition) is the average of x, the expected value of $x - \bar{x}$ is 0, so our expression simplifies down to $E((x-\bar{x})^2)+\bar{x}^2$. Now, all that remains is to calculate $E((x-\bar{x})^2)$. If we let the standard deviation be s, then $s^2 = \frac{(x-\bar{x})^2}{n} = E((x-\bar{x})^2)$, so our final answer is $s^2 + \bar{x}^2 = 2 \cdot 2023^2 = \boxed{8185048}$. It remains to show that there exists such a set (or else the value would be undefined). Through some trial and error, we can find that the set consisting of $-288 * 2023, \cdots, -288 * 2023, 2023, \cdots, 2023, 290 * 2023, \cdots, 290 * 2023$, where there are 7 $-288 * 2023$s, 7 $290 * 2023$s, and 4032 2023s. *Solution by Larry Xing*

14. Bob has a fair 6-sided die, numbered 1 through 6. Bob rolls the die until he rolls the sequence, $123456123456 \cdots 123456$, where 123456 is repeated 2022 times. If the expected number of rolls it takes for Bob to roll that sequence is E, find $E \pmod{1000}$. *Problem Proposed by Larry Xing*

Solution: $\boxed{512}$

Solution 1 Let's try to find an invariant to solve this problem. Here is one nice construction:

Consider a game. In this game, the goal is to roll the desired sequence. Now, consider a series of invisible gamblers. Before every flip, a gambler appears with \$1. Then, the gambler bets on 1, 2, 3, $\cdots$ until the desired sequence is rolled. Every time, if the gambler wins, he sextuples his money, and otherwise, he loses all his money, and we can ignore him.

We can see that after performing such a flip, the expected amount of money within the system will stay constant, but the before each flip, the expected amount of money increases by exactly \$1. Thus, the expected number of rolls that it takes is equal to the amount of money that the gamblers have at the end.

So how much money do the gamblers have at the end? Well, we only need to consider the gamblers that still have money at the end, so essentially places within the string where the prefix is equal to the suffix. If we just sum these up, we get that our answer is $6^{6\cdot 2022} + 6^{6\cdot 2021} + \cdots + 6^{12} + 6^6 = 6^6 \cdot \frac{6^{6\cdot 2022}-1}{6^6-1}$.

Now, we just need to evaluate this mod 1000. If we use Chinese Remainder Theorem, it is clear that this number is $0 \pmod 8$, so we will evaluate it mod 125. By Euler's Totient Theorem, $6^{100} \equiv 1 \pmod{125}$. We also find that $6^6 \equiv 31 \pmod{125}$. Using these facts, $6^{6\cdot 2022} + 6^{6\cdot 2021} + \cdots + 6^{12} + 6^6 \equiv 31 \cdot \frac{31^{22}-1}{30} \pmod{125}$. Now, all we need to find is $31^{22} \pmod{625}$ (in essence, we just need the last 4 digits). We can simply bash this; $31^{22} \equiv 336^{11} \equiv 336 \cdot 396^5 \equiv 566^2 \cdot 556 \equiv 436 \pmod{625}$. Thus, $\frac{31^{22}-1}{30} \equiv \frac{87}{6} \equiv \frac{29}{2} \equiv 29 \cdot 63 \equiv 77 \pmod{125}$. Thus, the answer is $12 \pmod{125}$, so our final answer is $\boxed{512}$. *Solution by Derrick Liu and Larry Xing*

Solution 2 We will use Markov chains to solve this problem. If we let $E_k(i)$ denote the expected number of rolls until we are done if we have already rolled 123456 i times, and we end on $12\cdots k$. Then, we can find that

$$E_{k+1}(i) = 1 + \frac{2}{3}E(0) + \frac{1}{6}E_{k+1}(i) + \frac{1}{6}E_1(0)$$

and

$$E_1(i+1) = 1 + \frac{5}{6}E(0) + \frac{1}{6}E_6(i)$$

In addition, because we know that $E(0) = E_1(0) + 6$, we can write $E_{x+1}(i) = \frac{5}{6}E(0) + \frac{1}{6}E_{x+1}(i)$. Thus, $E_1(i) = \frac{1}{6^5}E_6(i) + \frac{5+5\cdot 6+5\cdot 6^2+\cdots+5\cdot 6^4}{6^5}E(0) = (1-\frac{1}{6^5})E(0) + \frac{1}{6^5}E_6(i) = \frac{1}{6^5} + (1-\frac{1}{6^6})E(0) + \frac{1}{6^6}E_1(i+1)$. Now, we can just solve for $E(0)$:

$$\begin{aligned}
E_1(0) &= (\frac{1}{6^5} + (1-\frac{1}{6^6})E(0))(\frac{1-(\frac{1}{6^6})^{2021}}{1-\frac{1}{6^6}}) + \frac{1}{6^{6\cdot 2021}}E_1(2021) \\
&= (\frac{1}{6^5} + (1-\frac{1}{6^6})E(0))(\frac{1-(\frac{1}{6^6})^{2021}}{1-\frac{1}{6^6}}) + \frac{1}{6^{6\cdot 2021}}(1-\frac{1}{6^5})E(0) \\
&= \frac{1}{6^5}\frac{1-(\frac{1}{6^6})^{2021}}{1-\frac{1}{6^6}} + E(0) - \frac{1}{6^{6\cdot 2021}}E(0) + \frac{1}{6^{6\cdot 2021}}E(0) - \frac{1}{6^{6\cdot 2021+5}}E(0)
\end{aligned}$$

Thus, we can find $E(0)$:

$$\begin{aligned}
E(0) &= 6 + \frac{1}{6^5}\frac{1-(\frac{1}{6^6})^{2021}}{1-\frac{1}{6^6}} + E(0) - \frac{1}{6^{6\cdot 2021}}E(0) + \frac{1}{6^{6\cdot 2021}}E(0) - \frac{1}{6^{6\cdot 2021+5}}E(0) \\
E(0) &= 6^{6\cdot 2022} + 6^{6\cdot 2021+5}\frac{1}{6^5}\frac{1-\frac{1}{6^{6\cdot 2021}}}{1-\frac{1}{6^6}} \\
&= 6^{6\cdot 2022} + 6^6\frac{6^{6\cdot 2021}-1}{6^6-1} \\
&= 6^6\frac{6^{6\cdot 2022}-1}{6^6-1}
\end{aligned}$$

Finding this value mod 1000 proceeds as above. *Solution by Larry Xing*

15. John is in charge of a math class with 99 perfectly logical students, numbered 1 through 99, with all students knowing each other's numbers. One day, he thinks of a number x, and comes up with the following challenge: To each student, John tells them $x \pmod{i+1}$, where i is the number corresponding to that student. Then, John tells the entire class that $x \le 150$. John then performs the following operation until at least half the class (50 students) raise their hands. He asks the entire class, "Raise your hand if you know what the value of x is!" Let $f(x)$ be the number of times John needs to perform the operation until half the class raises their hands, if the value he thought of was x. What is $\sum_{i=1}^{150} f(i)$?

Solution: $\boxed{500}$

We can split the problem into multiple cases.

Case 1: $51 \le x \le 100$. In this case, everyone will see that person 99 will know x after the first operation. Then, every student from 49 to 98 will figure it out, so the answer is 2.

Case 2: $1 \le x \le 50$ or $101 \le x \le 150$. In this case, everyone will see that person 99 will not know x after the first operation. Then, at the second operation, they will turn their eyes to person 74. If he knows the answer, then $1 \le x \le 25$ or $126 \le x \le 150$. Otherwise, $26 \le x \le 50$ or $101 \le x \le 125$. We can then split it into two further cases.

Case 2-1: 74 knows x. First, it is easy to see that no matter what, it is impossible for 50 students to discern the value of x. Thus, we now try to prove that all students from 49 to 99 will be able to tell after the third operation. Consider a student a such that student a could tell, but student $a+1$ could not (this student is guaranteed to exist because 49 is able to tell, and either 61 or 62 will not be able to tell). Thus, we can see that $49 \leq a \leq 61$. Then, everyone can discern that x is either $122-2a, 123-2a, 28+2a$, or $29+2a$. Once it is narrowed down to these, at least 50 students will be able to discern the correct answer. Thus, the answer will be 4.

Case 2-2: 74 does not know x. In this case, 74 will definitely not know x after the third operation, and since anything less than 49 will also not know, it is impossible for 3 operations. Thus, we will prove it is true for 4 operations. Again, consider the person $a \geq 50$ such that a raised their hand and $a+1$ did not. Then, everyone can discern that x is one of 4 options, and from that, everyone from 49 to 99 can figure out the answer. Thus, the answer will be 4.

Thus, over all of the cases, the total sum is $50 \cdot 2 + 50 \cdot 4 + 50 \cdot 4 = \boxed{500}$. *Solution by Larry Xing*