BIMCT Individual Round Middle School Division Solutions

BIMCT Team

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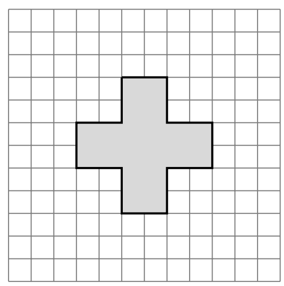
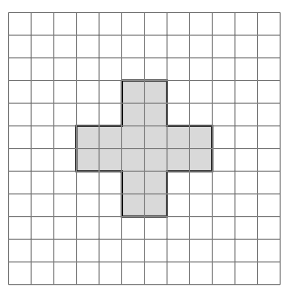
Individual Round Solutions

1. If each square on the grid in the diagram has side length 1, find the area of the shaded figure.



Problem Proposed by Derrick Liu

Solution: $\boxed{20}$



By counting, we see that the cross contains 20 squares, so our final answer is $\boxed{20}$. *Solution by Derrick Liu*

2. Define the operations $a\clubsuit b$ and $a\spadesuit b$ as follows:

$$a\clubsuit b = \frac{a^2+b^2}{a+b}$$
$$a\spadesuit b = \frac{a^2+ab}{a^2-b^2}$$

Find the value of $(4\clubsuit 12)\spadesuit 9$. *Problem Proposed by Derrick Liu*

Solution: $\boxed{10}$

The value of $(4\clubsuit 12)\spadesuit 9$ is

$$\begin{aligned}
&(4\clubsuit 12)\spadesuit 9\\
&\to (\frac{4^2+12^2}{4+12})\spadesuit 9\\
&\to (\frac{160}{16})\spadesuit 9\\
&\to 10\spadesuit 9\\
&\to \frac{10^2+10\cdot 9}{10^2-9^2}\\
&\to \frac{190}{19}=\boxed{10}
\end{aligned}$$

Solution by Derrick Liu

3. Alexis and Bryan live in cities A and B, respectively. Simultaneously, Alexis leaves A at 90mph and Bryan leaves B at 30mph, and both drive for 30 minutes. After 30 minutes, they realize that they have both arrived at the city of C. If M is the maximum distance between cities A and B and N is the minimum distance between cities A and B , find the value of $M-N$, in miles. Assume that each city is a point. *Problem Proposed by Derrick Liu*

Solution: $\boxed{30}$

The distance that Alexis travels is $\frac{90 \text{ miles}}{1 \text{ hour}}\cdot\frac{1}{2}\,\text{hour}=45$ miles, and the distance that Bryan travels is $\frac{30 \text{ miles}}{1 \text{ hour}}\cdot\frac{1}{2}\,\text{hour}=15$ miles. The maximum and minimum distance both occur if A, B, and C all lie on the same line. If C lies between A and B, then $\overline{AB}$ is equal to $\overline{AC}+\overline{BC}$, where $\overline{AB}$ denotes the distance from A to B. Therefore, $\overline{AB}=\overline{AC}+\overline{BC}=15\,\text{miles}+30\,\text{miles}=45$ miles $=M$. If B lies between A and C, then $AB=AC-BC=30\,\text{miles}-15\,\text{miles}=15\,\text{miles}=N$. Therefore, $M-N=45\,\text{miles}-15\,\text{miles}=\boxed{30}$ miles. *Solution by Derrick Liu*

4. Bob rolls two standard 6-sided dice and adds the sum of the faces. If the probability that the sum of the faces is a perfect square can be expressed as $\frac{a}{b}$, where a and b are relatively prime, find the value of $a+b$.

Solution: $\boxed{43}$

Notice that the only perfect squares that can be made are 4 and 9. There are 3 combinations that can add to 4, namely (1,3), (2,2), and (3,1). There are 4 combinations that can add to 9, namely (3,6), (4,5), (5,4), and (6,3). This means there are a total of $3+4=7$ combinations that add to a perfect square, and since there are $6\cdot 6=36$ total possible combinations, the desired probability is $7/36$, which gives us an answer of $7+36=\boxed{43}$. *Solution by Ary Cheng*

5. Amanda has a pile of n candies that she wants to distribute to 4 friends. The $i-$th friend will receive $\frac{1}{i+1}$ of the remaining candies that Amanda has and will be the $i-$th person to receive their share of the candy. What is the minimum value of n such that each friend receives an integer amount of candies? *Problem Proposed by Derrick Liu*

Solution: $\boxed{60}$

Consider how many of the original n candies each friend receives. We can write each amount out as such:

$$\begin{aligned}
i = 1 &\to \frac{1}{2}n \\
i = 2 &\to (1 - \frac{1}{2})\frac{1}{3}n = \frac{1}{6}n \\
i = 3 &\to (1 - \frac{1}{2} - \frac{1}{6})\frac{1}{4}n = \frac{1}{12}n \\
i = 4 &\to (1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12})\frac{1}{5}n = \frac{1}{20}n
\end{aligned}$$

We see that n is both a multiple of 20 and 12, so our answer is $\text{LCM}(20, 12) = \boxed{60}$ *Solution by Derrick Liu.*

6. How many trailing zeros are in the base-12 representation of $\binom{100}{49}$? *Problem Proposed by Larry Xing*

   Solution: $\boxed{2}$

   This question is essentially asking how many factors of 12 exist in $\binom{100}{49}$, as each factor of 12 will contribute 1 trailing zero. $12 = 2^2 \cdot 3$, so we just need to figure out how many factors of 2 and 3 exist in $\binom{100}{49}$:

$$\binom{100}{49} = \frac{100!}{51! \cdot 49!} = \frac{100 \cdot 99 \cdot 98 \cdot \dots \cdot 53 \cdot 52}{49 \cdot 48 \cdot 47 \cdot \dots \cdot 2 \cdot 1}$$

   In the numerator, 25 terms are divisible by 2, 13 terms are divisible by $4 = 2^2$, 6 terms are divisible by $8 = 2^3$, 3 terms are divisible by $16 = 2^4$, 2 terms are divisible by $32 = 2^5$, and 1 term is divisible by $64 = 2^6$. This gives us $25 + 13 + 6 + 3 + 2 + 1 = 50$ factors of 2 in the numerator. A similar method can be used to find the number of factors of 2 in the denominator and factors of 3 in both the numerator and denominator. We get 46 factors of 2 in the denominator, 25 factors of 3 in the numerator, and 22 factors of 3 in the denominator. This leaves us with $50 - 46 = 4$ factors of 2 and $25 - 22 = 3$ factors of 3, which gives us 2 factors of 12. Thus our final answer is $\boxed{2}$. *Solution by Ary Cheng*

7. Zack is walking along a number line from 0 to 10. He starts at position 4. Every minute, he flips a coin. If the coin shows heads, he moves forward by 1, and if the coin shows tails, he moves backward by 1. If the probability that he arrives at 10 before he arrives at 0 can be expressed as $\frac{a}{b}$, where a and b are relatively prime integers, find $10a + b$. *Problem Proposed by Rohan Das*

   Solution: $\boxed{25}$

   Solution 1: Let p_i denote the probability of reaching position 10 before position 0 when he starts from position i. Clearly, we have $p_0 = 0$ and $p_{10} = 1$. Moreover, for $1 \leq i \leq 9$, the probability of moving to position $i - 1$ is equal to the probability of moving to position $i + 1$, which implies $p_i = \frac{1}{2}(p_{i-1} + p_{i+1})$ or

$$p_{i+1} - 2p_i + p_{i-1} = 0$$

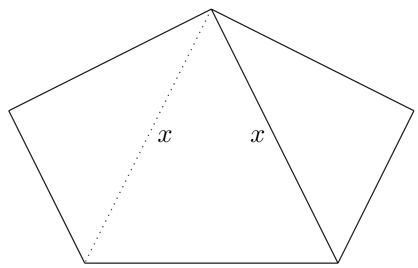
   This is a linear recurrence, so we can solve the characteristic equation $r^2 - 2r + 1 = 0$ for the general solution. Since the equation has a repeated root of 1, the general solution will be of the form $p_n = (an + b)(1)^n = an + b$ for arbitrary constants a and b. Applying our edge conditions of $p_0 = 0$ and $p_{10} = 1$, we obtain the equations $0 = b$ and $1 = 10a$ respectively, so we have $p_i = i/10$. The problem asks for p_4, which is $\frac{2}{5}$. Therefore, our final answer is $10 \cdot 2 + 5 = \boxed{25}$. *Solution by Felix Liu*

Solution 2: At each step, the expected value of Zack's displacement is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$. Thus the expected value of Zack's placement at any time is always 4. Consider the time when Zack reaches position 0 or 10. Let p be the probability that he is at position 10. Then the expected value is $10p + 0(1-p) = 10p = 4$, so $p = \frac{2}{5}$. Then the final answer is $10 \cdot 2 + 5 = \boxed{25}$. *Solution by Rohan Das*

8. A rectangular sheet of paper has dimensions 1 inch by 2 inches. The paper is then folded along a line so that two diagonally opposite points coincide. The folded paper is then laid flat on a table. The perimeter of the region of the table under two layers of paper is p_1 inches, and the perimeter of the region of the table under at least one layer of paper is p_2 inches. If $p_1 + p_2$ can be expressed as $a + \sqrt{b}$, find $a + b$. *Problem Proposed by Adam Tang*

   Solution: $\boxed{11}$

   When we fold the paper, we are left with a pentagon consisting of two congruent right triangles and an isosceles triangle, as shown below:

   Let the edge length of the folded edge be x. Then, we see that the folded edge forms a right triangle with two legs of length 1 and $2 - x$. Using the Pythagorean theorem, we see that $x^2 = 1 + (4 - 4x + x^2)$, so $x = \frac{5}{4}$. The base of the isosceles triangle is $\sqrt{1^2 + (2 - 2 \cdot \frac{3}{4})^2} = \frac{\sqrt{5}}{2}$, so the height of the isosceles triangle is $\sqrt{\frac{25}{16} - \frac{5}{4}} = \frac{\sqrt{5}}{4}$. Therefore, $p_1 = \frac{\sqrt{5}}{2} + 2 \cdot \frac{5}{4}$, and $p_2 = 2 \cdot 1 + 2 \cdot \frac{3}{4} + \frac{\sqrt{5}}{2}$, so $p_1 + p_2 = \sqrt{5} + 6$, and our final answer is $\boxed{11}$. *Solution by Derrick Liu*

9. Let n be the smallest positive integer such that 7^n ends in 00001 when expressed in base 26. Find n. *Problem Proposed by Larry Xing*

   Solution: $\boxed{59}$

   Because 7^n ends in 00001 when expressed in base 26, $7^n - 1$ is divisible by 26^5. We can use the Chinese Remainder Theorem to determine the smallest n. It is easy to see that $7^4 \equiv 1 \pmod{2^5}$. By Euler's Totient Theorem, $7^{12 \cdot 13^4} \equiv 1 \pmod{13^5}$. Therefore, the smallest n is $12 \cdot 13^4$, and sum of the not necessarily distinct prime factors is $2 + 2 + 3 + 13 \cdot 4 = \boxed{59}$.

   To show that this is indeed the minimum, we will use Lifting the Exponent. Define $o_n(k)$ to return the smallest i such that $k^i \equiv 1 \pmod{n}$. First, we show that $o_{13}(7) = 12$ and $169 \nmid 7^{12} - 1$. Notice that $7 \equiv 7 \pmod{13}$, $7^2 \equiv -3 \pmod{13}$, $7^3 \equiv 5 \pmod{13}$, $7^4 \equiv -4 \pmod{13}$, $7^5 \equiv -2 \pmod{13}$, $7^6 \equiv -14 \equiv -1 \pmod{13}$. Since $7^6 \equiv -14 \equiv -1 \pmod{13}$, $o_{13}(7) = 12$, since $(-1)^2 \equiv 1 \equiv (7^6)^2 \equiv 7^{12}$. To show that $169 \nmid 7^{12} - 1$, we need to show that $7^{12} \neq 1 \pmod{169}$. We see that $7^3 \equiv 343 \equiv 5 \pmod{169}$, so $7^{12} \equiv 5^4 \equiv 625 \equiv 118 \neq 1 \pmod{169}$. Define $v_p(k)$ to be the largest power of a prime p that divides k. Now, we need to find the smallest k such that $v_{13}(7^{12k} - 1) \geq 5$ (We know the power of 7 (n) must be a multiple of 12 because, if it wasn't, $7^n - 1 \neq 0 \pmod{13}$). By Lifting the Exponent, $v_{13}(7^{12k} - 1) = v_{13}(7^{12} - 1) + v_{13}(12k) = v_{13}(7^{12} - 1) + v_{13}(k) \geq 5$. Thus, $v_{13}(k) \geq 4$, so the minimum value of k is 13^4. Thus, our value of n is $12 \cdot 13^4$, confirming our answer above. *Solution by Rohan Das*

10. How many ordered pairs of (not necessarily distinct) permutations (σ_1, σ_2) of the set $S = \{1, 2, 3, 4\}$ satisfy

$$\sigma_1(\sigma_2(i)) = \sigma_2(\sigma_1(i))$$

for all $i \in S$? (A permutation σ is a function that associates each $i \in S$ to some $j \in S$, such that no two $i \in S$ are associated with the same j. An example permutation σ of the set $S = \{1, 2, 3, 4\}$ is $\{1, 2, 3, 4\} \rightarrow \{2, 1, 3, 4\}$. In this case $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3, \sigma(4) = 4$. *Problem Proposed by Adam Tang*

Solution: $\boxed{120}$

Solution 1 First, define the cycle length of a permutation to be the number of times the permutation must be applied in order for the starting number to return to itself. For example, the permutation $\sigma = \{1, 2, 3, 4\} \rightarrow \{2, 1, 4, 3\}$ has two cycles of cycle length 2. The numbers 1 and 2 form one cycle of cycle length 2 ($1 \rightarrow 2 \rightarrow 1$ and $2 \rightarrow 1 \rightarrow 2$), and the numbers 3 and 4 form one cycle of cycle length 2.

Now, we can use casework on σ_1 by breaking down all possible sums of cycle lengths. Notice that the sum of the cycle lengths must equal 4.

Case 1: If σ_1 has only one cycle of length 4. There are 6 permutations σ_1 that have only one cycle of cycle length 4. To verify this, 1 has 3 distinct numbers to map to, 2 had 2 distinct numbers to map to (namely, the two numbers other than itself and the number 1 maps to), and the numbers that 3 and 4 are uniquely determined at this point. For each σ_1 of only one cycle of cycle length 4, there are exactly 4 permutations σ_2 that satisfy $\sigma_1(\sigma_2(i)) = \sigma_2(\sigma_1(i))$. Consider the number that 1 maps to after the operation $\sigma_2(\sigma_1(1))$. There are 4 choices for which number σ_2 maps to. After we determine this number, the rest of the mappings can be uniquely determined. Therefore, for this case, there are $6 \cdot 4 = 24$ pairs of permutations σ_1 and σ_2.

Case 2: If σ_1 has one cycle of cycle length 3 and one cycle of cycle length 1. There are 8 permutations in this case: 4 to choose which number maps to itself, and 2 choices for the cycle of cycle length 3, following the logic above. If we consider the operation $\sigma_2(\sigma_1(i))$ for the number that maps to itself, then $\sigma_2(i) = \sigma_1(\sigma_2(i))$. To satisfy this equation, $\sigma_2(i)$ must equal i. Then, by the same logic above, σ_2 has 3 choices for the three numbers that do not map to themselves, meaning that we have $8 \cdot 3 = 24$ pairs of permutations σ_1 and σ_2 for this case.

Case 3: If σ_1 has two cycles of cycle length 2. There are 3 permutations in this case: 1 has 3 choices to map to, which determines one cycle of cycle length 2, leaving the other two numbers as the other cycle of cycle length 2. WLOG, consider the permutation $\sigma_1 = \{1, 2, 3, 4\} \rightarrow \{3, 4, 1, 2\}$. Then $\sigma_2(\sigma_1(1)) = \sigma_2(3) = \sigma_1(\sigma_2(1))$ and $\sigma_2(\sigma_1(3)) = \sigma_2(1) = \sigma_1(\sigma_2(3))$. We have 4 choices for $\sigma_2(3)$, and now $\sigma_2(1)$ is uniquely determined. Now, $\sigma_2(2)$ has 2 choices, and $\sigma_2(4)$ has one choice, meaning that for this case, there are $3 \cdot 4 \cdot 2 = 24$ pairs of permutations σ_1 and σ_2.

Case 4: If σ_1 has two cycles of cycle length 1 and one cycle of cycle length 2. For this case, there are $\binom{4}{2} = 6$ ways to choose which two numbers map to themselves, leaving the other two numbers as a unique cycle of cycle length 2. Again, consider the two numbers that map to themselves. There are two cases: Either the two numbers map to themselves in σ_2, or they map to each other. In both cases, the other two numbers have 2 possibilities for their permutation in σ_2: either the two numbers map to each other or to themselves. Therefore, there are $6 \cdot 2 \cdot 2 = 24$ pairs of permutations σ_1 and σ_2 for this case.

Case 5: If σ_1 has 4 cycles of cycle length 1. Because each number in σ_1 maps to itself (i.e, the identity permutation), it does not matter which permutation σ_2 is: the equality will always be satisfied, because by definition, σ_2 multiplied by identity = identity multiplied by σ_2. Since there are $4! = 24$ total permutations (1 can map to 4 choices, then 2 can map to 3 choices, etc..), we have 24 pairs of permutations

σ_1 and σ_2 for this case.

Thus, our final answer is $24 + 24 + 24 + 24 + 24 = \boxed{120}$. *Solution by Derrick Liu*

Solution 2: Group Theory For those familiar with the group theory, the key idea is to prove that number of ordered pairs of elements that commute in group is the number of conjugacy classes times the size of the group, then use the fact that each conjugacy class of S_n corresponds to a different partition of n. This is shown in detail below.

Let S_4 be the set of all permutations on $\{1, 2, 3, 4\}$. Restating the question, we wish the find the size of the set $K = \{(\sigma_1, \sigma_2) \in S_4 \times S_4 \mid \sigma_1\sigma_2 = \sigma_2\sigma_1\}$, where the operation is function composition. Let $C(\sigma) = \{\tau \in S_4 \mid \sigma\tau = \tau\sigma\}$; that is, the set of elements that commute with σ. Since $(\sigma_1, \sigma_2) \in K$ if and only if $\sigma_2 \in C(\sigma_1)$, it follows that

$$|K| = \sum_{\sigma_1 \in S_4} |C(\sigma_1)|.$$

Define the conjugacy class of an element σ to be the set $\mathrm{cl}(\sigma) = \{\tau\sigma\tau^{-1} \mid \tau \in S_4\}$. It can be shown that the conjugacy classes partition S_4. By mapping each coset $\tau C(\sigma) = \{\tau\pi \mid \pi \in C(\sigma)\}$ to the conjugate $\tau\sigma\tau^{-1}$, it follows that the number of distinct cosets $\tau C(\sigma)$ is equal to number of conjugates of σ. Because all cosets have the same size, this implies that $|\mathrm{cl}(\sigma)| = |S_4|/|C(\sigma)|$ or $|S_4| = |\mathrm{cl}(\sigma)|\ |C(\sigma)|$, so if σ_1 and σ_2 are in the same conjugacy class, it follows that $|C(\sigma_1)| = |C(\sigma_2)|$. For more rigor, look for the properties of cosets. Now consider an arbitrary conjugacy class $\mathrm{cl}(\sigma) = \{\tau_1, \tau_2, \ldots, \tau_n\}$ so $|\mathrm{cl}(\sigma)| = n$. We now have

$$|C(\tau_1)| + |C(\tau_2)| + \cdots + |C(\tau_n)| = n|C(\sigma)| = |S_4|.$$

Let the number of conjugacy classes of S_4 be m. Taking one representative from each conjugacy class in our above equation for $|K|$, we have

$$|K| = m \cdot |S_4| = 24m.$$

Hence, we will have succeeded if we can find the number of conjugacy classes of $|S_4|$. We claim that the number of conjugacy classes in $|S_4|$ is the number of ways to partition 4. To do this, it suffices to show that two permutations are conjugates (in the same conjugacy class) if and only if they have the same cycle type; that is, if two permutations are decomposed into the product of disjoint cycles, then the number of cycles of a given length for each permutation is the same. There are proofs for this online, but it can intuitively be understood as follows: if σ_1 is a conjugate of σ_2, then $\sigma_2 = \tau\sigma_1\tau^{-1}$ for some permutation τ, which implies $\sigma_2\tau = \tau\sigma_1$. In particular, if σ_1 maps i to j, then σ_2 maps $\tau(i)$ to $\tau(j)$ because $\sigma_2(\tau(i)) = \tau(\sigma_1(i)) = \tau(j)$, so σ_2 is essentially the same permutation as σ_1, the only difference being it acts on the set $\{\tau(1), \tau(2), \tau(3), \tau(4)\}$ instead of $\{1, 2, 3, 4\}$.

With this, we know the number of conjugacy classes of S_4 is simply the number of ways to partition 4. There are 5 ways to partition 4, represented by $1+1+1+1 = 1+1+2 = 1+3 = 2+2 = 4$, so there are 5 conjugacy classes in S_4. Hence, the answer is $5 \cdot 24 = \boxed{120}$. *Solution by Felix Liu*