BIMCT Tiebreaker Problems

BIMCT Team

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Tiebreaker Round

1. What value x is such that if Bob has an infinite number of coins with values 5 and x, 71 would be the largest value he could not make? *Problem Proposed by Larry Xing*

   Solution: $\boxed{19}$

   This problem is a direct application of the Chicken McNugget Theorem, which states that for relatively prime a, b, the largest integer N that cannot be expressed in the form $ax + by$, where x and y are nonnegative integers, is given by:

$$N = ab - a - b$$

   In this case, $N = 71$ and $a = 5$. Now we just have to solve for b:

$$71 = 5b - 5 - b \implies 4b = 76 \implies b = 19$$

   Finally, check that our answer satisfies the condition of the Chicken McNugget theorem. Since 5 and 19 are relatively prime, the theorem holds, and our answer is $\boxed{19}$ *Solution by Ary Cheng*

2. In a convex decagon, all diagonals are drawn. Given that no three diagonals intersect at one point, find the number of different triangles with vertices at the intersection points of the diagonals or vertices of the decagon. *Problem Proposed by Felix Liu*

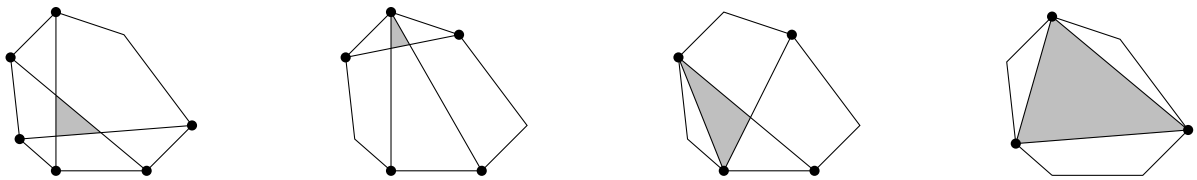
   Solution: $\boxed{2430}$
   Consider the more general case of an n-gon. We casework on the number of vertices that lie on the side of the decagon. For every 6 vertices chosen on the decagon, there is one way to draw lines to form a triangle with no vertices on the decagon (Fig 1 (a)). Continuing in this fashion, for every 5 vertices chosen on the decagon, there are 5 ways to draw lines that determine a triangle with one vertex on the decagon (Fig 1 (b)). These cases are represented in the figure below.

   Summing up the cases, we see the number of triangles is

$$\binom{n}{6} + 5\binom{n}{5} + 4\binom{n}{4} + \binom{n}{3}.$$

   For the decagon, this becomes

$$\binom{10}{6} + 5\binom{10}{5} + 4\binom{10}{4} + \binom{10}{3} = \boxed{2430}.$$



(a) 1 way to draw gray triangle

(b) 5 ways to draw gray triangle

(c) 4 ways to draw gray triangle

(d) 1 way to draw gray triangle

Figure 1: casework on the number of vertices that lie on the n-gon

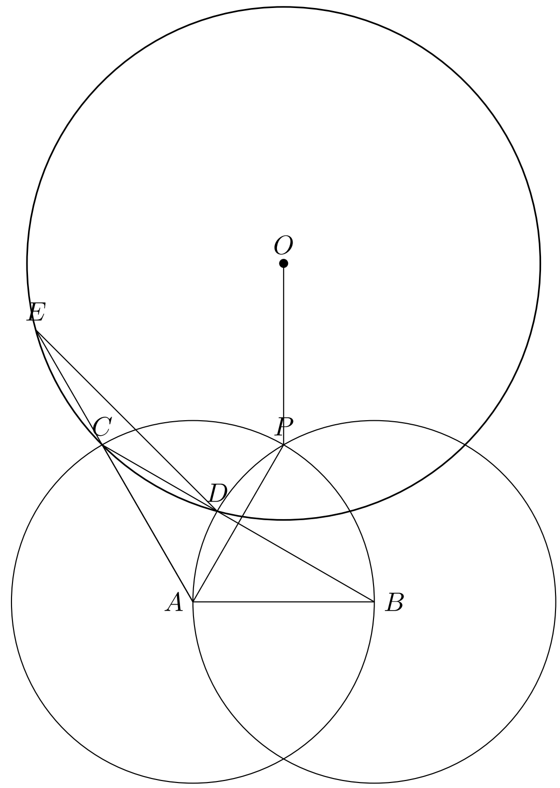
3. How many ways are there to place two pieces on an 8 by 8 chessboard, such that they are not in the same row or column? Two configurations are considered the same if a rotation of the board can make the pieces on the same positions. *Problem Proposed by Larry Xing*

   Solution: $\boxed{400}$

   We will use Burnside's lemma to solve this problem. First, let's consider all of the configurations such that 1 rotation would map it back to itself. It turns out, this is actually impossible to achieve. Now, let's consider all of the configurations that 2 rotations would map back to itself. If you consider the middle of the board $(0, 0)$, then a piece on (a, b) would map to $(-a, -b)$, so you would have to have your points add up to $(0, 0)$. There are 32 ways to do this. Now, the number of configurations that takes 4 rotations to map back to itself is just $\frac{64 \cdot 49}{2} - 32 = 1536$. Thus, our final answer is $\frac{32}{2} + \frac{1536}{4} = \boxed{400}$. *Solution by Larry Xing*

4. Points A and B are such that $AB = d$. Circle α has radius 240 and is centered at A, and circle β has radius 240 and is centered at B. Point P is an intersection between α and β, and point C is on α such that $\angle CAP = 60°$ and CB is maximized. then, D is the intersection between BC and β, and E is on ray AC such that $AE = BC$. Then, O is the circumcenter of CDE. Over all $0 < d < 480$, find the minimum length of OP. *Problem Proposed by Larry Xing*

   Solution: $\boxed{240}$



First, we know that $OC \cong OD$ and $CA \cong DB$. In addition, because $EC \cong CD$, $\angle OCE \cong \angle ODC$, so $\angle OCA \cong \angle ODB$, so $\triangle OCA \cong \triangle ODB$. Then, we know that $\angle OBD \cong \angle OAC$, so $\angle OBC \cong \angle OAC$, so $OBAC$ is cyclic. Now, because $AP = BP = CP = 240$, P is the circumcenter of $\triangle ABC$, so $OP = \boxed{240}$. *Solution by Larry Xing*

5. Let S contain all positive integers x and y such that

$$((x+2y)^4 + (x-2y)^4)((y+2x)^4 + (y-2x)^4) = (x-y)^9.$$

Assume that we have polynomials $P_x(x)$ and $P_y(x)$ such that the set containing $(P_x(i), P_y(i))$ for all positive integers i is equivalent to S. Find $P_x(0)$. *Problem Proposed by Larry Xing*

Solution: $\boxed{64}$

Let's write $x = da$ and $y = db$ where $\gcd(a, b) = 1$. Then, we get that

$$\frac{((a+2b)^4 + (a-2b)^4)((b+2a)^4 + (b-2a)^4)}{(a-b)^9} = d$$

, so $(a-b)^9 | ((a+2b)^4 + (a-2b)^4)((b+2a)^4 + (b-2a)^4)$. Now, taking the right hand side mod $a-b$, we get that $a - b | 82^2 a^4$, so $a - b | 82^2$. If $a - b$ is even, then both a and b are odd, which makes the original expression false. If $a - b$ is a multiple of 41, the right hand side must be divisible by 41^9. Because this is extremely unlikely, it is safe to assume that 41 does not work (This assumption holds up when checked with code, and can be noticed by the fact that the coefficients of each term on the right hand side cannot all be a multiple of 41^9). Then, $a - b = 1$. Thus, $d = ((3b+1)^4 + (b-1)^4)((3b+2)^4 + (b+2)^4)$, so $x = da = (b+1)((3b+1)^4 + (b-1)^4)((3b+2)^4 + (b+2)^4)$. Thus, our final answer is $\boxed{64}$. *Solution by Larry Xing*